3.1: Finite, \( \sigma \)-finite, and complete measures. Let \( \mu \) be a measure on a \( \sigma \)-algebra \( \mathcal{A} \) in a set \( X \).

The measure \( \mu \) is **finite** if \( \mu(X) < \infty \).

The measure \( \mu \) is **\( \sigma \)-finite** if there exists a countable collection \( \{E_n\} \) in \( \mathcal{A} \) such that \( X = \bigcup_{n=1}^{\infty} E_n \) and \( \mu(E_n) < \infty \).

A measure space \( \{X, \mathcal{A}, \mu\} \) is **complete** if for each \( A \in \mathcal{A} \) with \( \mu(A) = 0 \), every subset \( E \subset A \) is in \( \mathcal{A} \).

It follows from the monotonicity of a measure, that if \( \{X, \mathcal{A}, \mu\} \) is complete, then for each \( A \in \mathcal{A} \) with \( \mu(A) = 0 \), we have for every \( E \subset A \) that \( \mu(E) = 0 \).

3.2: Some Examples. (a) For any nonempty set \( X \) and \( \mathcal{A} = \{\emptyset, X\} \), the trivial \( \sigma \)-algebra, the function \( \mu : \mathcal{A} \to \mathbb{R}^* \) defined by

\[
\mu(E) = \begin{cases} 
0 & \text{if } E = \emptyset, \\
\infty & \text{if } E = X,
\end{cases}
\]

is a measure.

(b) For a nonempty set \( X \) and \( \mathcal{A} = 2^X = \mathcal{P}(X) \) the discrete \( \sigma \)-algebra, the function \( \mu : \mathcal{A} \to \mathbb{R}^* \) defined by setting \( \mu(E) \) equal to the number of elements of \( E \) if \( E \) is a finite set, and setting \( \mu(E) = \infty \) if \( E \) is not a finite set, is a measure, called the counting measure.

For this measure \( \mu(B - A) = \mu(B) - \mu(A) \) fails when \( \mu(A) = \infty \): for \( X = \mathbb{N} \), \( B = \{n, n+1, n+2, \ldots\} \) and \( A = \{n+k, n+k+1, n+k+2, \ldots\} \) for some integer \( k \geq 1 \), we have \( B - A = \{n, n+1, \ldots, n+k-1\} \) for which \( \mu(B - A) = k \) while \( \mu(B) = \infty \) and \( \mu(A) = \infty \), making \( \mu(B) - \mu(A) \) undefined.

(c) Let \( X = \{x_n\} \) be a sequence, and \( \{\alpha_n\} \) a sequence of nonnegative real numbers.

The function

\[
\mu(E) = \sum\{\alpha_n : x_n \in E\}
\]

is a \( \sigma \)-finite measure on the discrete \( \sigma \)-algebra \( \mathcal{A} = 2^X \).

This measure is finite if \( \sum \alpha_n < \infty \).

(d) For an infinite set \( X \) (possibly uncountable), and the discrete \( \sigma \)-algebra \( \mathcal{A} = 2^X \), the function \( \mu : \mathcal{A} \to \mathbb{R}^* \) defined by \( \mu(E) = 0 \) if \( E \) is countable (including finite), and \( \mu(E) = \infty \) otherwise, is a measure.

(e) Let \( X = \mathbb{R}^n \) and \( \mathcal{A} = \mathcal{P}(\mathbb{R}^n) \).
For a fixed $x \in \mathbb{R}^n$ we define $\mu : \mathcal{A} \to \mathbb{R}^*$ by

$$
\mu(E) = \begin{cases} 
1 & \text{if } x \in E, \\
0 & \text{if } x \notin E,
\end{cases}
$$

is a finite measure, known as the **Dirac delta**-measure $\delta_x$ in $\mathbb{R}^N$ concentrated at $x$.

**Proposition A.** If $\{\mu_n\}$ are measures on the same $\sigma$-algebra $\mathcal{A}$, then $\sum \mu_n$ is a measure on $\mathcal{A}$.

The proof of this is a homework problem.

**Corollary B.** If $\mu_1, \ldots, \mu_k$ is a finite collection of measures on the same $\sigma$-algebra $\mathcal{A}$, then $\mu_1 + \cdots + \mu_k$ is a measure on $\mathcal{A}$.

Proof. If we define $\mu_n(E) = 0$ for all $E \in \mathcal{A}$ (the trivial or zero measure) for all $n \geq k + 1$, we can then apply Proposition A to $\{\mu_n\}$ to get its sum $\mu = \sum \mu_n = \mu_1 + \cdots + \mu_n + 0 + 0 + \cdots = \mu_1 + \cdots + \mu_k$ is a measure on $\mathcal{A}$. □

**Proposition C.** Let $\{X, \mathcal{A}, \mu\}$ be a measure space. If $\mathcal{B}$ is a $\sigma$-algebra in $X$ such that $\mathcal{B} \subset \mathcal{A}$, then the restriction of $\mu$ to $\mathcal{B}$ is a measure.

The proof of this is a straight-forward exercise.

**Proposition D.** If $\mathcal{A}$ is a $\sigma$-algebra in $X$, and $B \subset X$, then $\mathcal{B} = \{A \cap B : A \in \mathcal{A}\}$ is a $\sigma$-algebra in $B$.

The proof of this is a homework problem.

**Proposition E.** If $\mathcal{A}$ is a $\sigma$-algebra in $X$, $\mu$ a measure on $\mathcal{A}$, and $B \in \mathcal{A}$, then the restriction of $\mu$ to $\mathcal{B} = \{A \cap B : A \in \mathcal{A}\}$ is a measure.

Proof. Apply Propositions C and D, noting that with $B \in \mathcal{A}$ we have $\mathcal{B} \subset \mathcal{A}$. □

**4. Outer Measures.** An extended real-valued set function on $X$ is an **outer measure** if

(i) $\mu_e$ is defined for every element of $\mathcal{P}(X)$,
(ii) $\mu_e$ is nonnegative and $\mu_e(\emptyset) = 0$,
(iii) $\mu_e$ is monotone, i.e., if $A \subset B$, then $\mu_e(A) \leq \mu_e(B)$, and
(iv) $\mu_e$ is countably subadditive, i.e., for $\{A_n\} \in \mathcal{P}(X)$, there holds $\mu_e(\bigcup A_n) \leq \sum \mu_e(A_n)$.

A collection $\mathcal{Q}$ of subsets of a set $X$ is a sequential covering for $X$ if

(i) $\emptyset \in \mathcal{Q}$, and
(ii) for every $E \subset X$ there is a countable collection $\{Q_n\}$ in $\mathcal{Q}$ such that

$$
E \subset \bigcup_{n=1}^{\infty} Q_n.
$$
Example. A sequential covering of \( \mathbb{R}^n \) is the collection of all closed cubes.

We describe a general procedure by which an outer measure is constructed from a sequential covering \( Q \) of set \( X \) and an arbitrary nonnegative function \( \lambda : Q \to \mathbb{R}^* \) satisfying \( \lambda(\emptyset) = 0 \).

For each \( E \in \mathcal{P}(X) \), we define \( \mu_e : \mathcal{P}(X) \to \mathbb{R}^* \) by

\[
\mu_e(E) = \inf \left\{ \sum_{n=1}^\infty \lambda(Q_n) : Q_n \in Q, E \subset \bigcup_{n=1}^\infty Q_n \right\}.
\]

By the definition of \( \inf \), if \( \mu_e(E) < \infty \), then for every \( \epsilon > 0 \) there is a countable collection \( \{Q_{n,\epsilon}\} \) of elements in \( Q \) such that

\[
E \subset \bigcup_{n=1}^\infty Q_{n,\epsilon} \text{ and } \sum_{n=1}^\infty \lambda(Q_{n,\epsilon}) \leq \mu_e(E) + \epsilon.
\]

**Proposition 4.1.** The function \( \mu_e \) is an outer measure.

Proof. We have four properties to verify.

(i) \( \mu_e(E) \) is defined on every element of \( \mathcal{P}(X) \): this follows because the infimum is defined for each \( E \in \mathcal{P}(X) \).

(ii) \( \mu_e(E) \geq 0 \) for all \( E \in \mathcal{P}(X) \) and \( \mu(\emptyset) = 0 \): these follows because \( \lambda \) is nonnegative, and \( \lambda(\emptyset) = 0 \) and the constant sequence \( \{\emptyset\} \) is a sequential covering of \( \emptyset \).

(iii) \( \mu_e \) is monotone, i.e., \( A \subset B \) implies \( \mu_e(A) \leq \mu_e(B) \): this follows because every sequential cover of \( B \) is a sequential cover of \( A \), but not every sequential of \( A \) is a sequential cover for \( B \), so that the infimum for \( \mu_e(E) \) is smaller or equal to that for \( \mu_e(B) \).

(iv) \( \mu_e \) is countably subadditive.

We assume for a countable collection \( \{E_n\} \) of elements of \( \mathcal{P}(X) \) that \( \mu_e(E_n) < \infty \) for all \( n \) (for otherwise countable subadditivty follows trivially).

Fix \( \epsilon > 0 \).

For each \( n \in \mathbb{N} \), there is a countable collection \( \{Q_{j,n}\} \) in \( Q \) such that

\[
E \subset \bigcup_{j=1}^\infty Q_{j,n} \text{ and } \sum_{j=1}^n \lambda(Q_{j,n}) \leq \mu_e(E_n) + \frac{\epsilon}{2n}.
\]

The doubly-indexed collection \( \{Q_{j,n}\} \) is a countable collection that covers the union of the \( E_n \) so that

\[
\mu_e \left( \bigcup_{n=1}^\infty E_n \right) \leq \sum_{n=1}^\infty \sum_{j=1}^\infty \lambda(Q_{n,j}) \leq \sum_{n=1}^\infty \mu_e(E_n) + \epsilon \sum_{n=1}^\infty \frac{1}{2n} = \sum_{n=1}^\infty \mu_e(E_n) + \epsilon.
\]

Since this holds for any \( \epsilon > 0 \) we obtain the countable subadditivity of \( \mu_e \). \( \square \)
The outer measure $\mu_e$ generated by the sequential covering $\mathcal{Q}$ and the nonnegative function $\lambda$ may not coincide with $\lambda$ on elements of $\mathcal{Q}$.

By the construction of $\mu_e$ we have for all $Q \in \mathcal{Q}$ that

$$\mu_e(Q) \leq \lambda(Q),$$

and strict inequality may occur for some $Q$. [We will see some examples of this soon.]