II.6: Constructing measures from outer measures

6. The Carathéodory Procedure. We shall now describe the Carathéodory procedure of constructing a complete measure from an outer measure.

Let $\mu_e$ be an outer measure on a set $X$.

For any $A, E \in \mathcal{P}(X)$ we have the set identity

$$A = (A \cap E) \cup (A - E)$$

for a union of disjoint sets, which by countable subadditivity of $\mu_e$ gives

$$\mu_e(A) \leq \mu_e(A \cap E) + \mu_e(A - E).$$

We say an element $E \in \mathcal{P}(X)$ is $\mu_e$-measurable if for all $A \in \mathcal{P}(X)$ there holds

$$\mu_e(A) = \mu_e(A \cap E) + \mu_e(A - E).$$

We denote by $\mathcal{A}$ the collection of all $\mu_e$-measurable elements of $\mathcal{P}(X)$.

**Proposition 6.1.** For an outer measure $\mu_e$, the collection $\mathcal{A}$ has the following properties.

1. $\emptyset \in \mathcal{A}$.
2. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.
3. If $E \in \mathcal{P}(X)$ satisfies $\mu_e(E) = 0$, then $E \in \mathcal{A}$.
4. If $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$.
5. If $E_1, E_2 \in \mathcal{A}$, then $E_1 - E_2 \in \mathcal{A}$.
6. If $E_1, E_2 \in \mathcal{A}$, then $E_1 \cap E_2 \in \mathcal{A}$.
7. If $\{E_n\}$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then for all $A \in \mathcal{P}(X)$ there holds

$$\lim_{m \to \infty} \mu_e \left( A \cap \bigcup_{n=1}^{m} E_n \right) = \mu_e \left( A \cap \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu_e(A \cap E_n).$$

8. A countable union of sets in $\mathcal{A}$ is in $\mathcal{A}$.

**Proof.** (i) Since $\mu_e(A \cap \emptyset) + \mu_e(A - \emptyset) = \mu_e(\emptyset) + \mu_e(A) = \mu_e(A)$ for all $A \in \mathcal{P}(X)$, we have that $\emptyset \in \mathcal{A}$.

(ii) For $E \in \mathcal{A}$ we have $\mu_e(A) = \mu_e(A \cap E) + \mu_e(A - E) = \mu_e(A \cap E) + \mu_e(A \cap E^c)$ for all $A \in \mathcal{P}(X)$.

Since $A \cap E = A - E^c$ we obtain $\mu_e(A) = \mu_e(A - E^c) + \mu_e(A \cap E^c)$ for all $A \in \mathcal{P}(X)$, which implies that $E^c \in \mathcal{A}$.

(iii) If $\mu_e(E) = 0$, then for all $A \in \mathcal{P}(X)$ we have $0 \leq \mu_e(A \cap E) \leq \mu_e(E)$ and $\mu(A - E) \leq \mu_e(A)$ by monotonicity, so that for all $A \in \mathcal{P}(X)$ there holds

$$\mu_e(A \cap E) + \mu_e(A - E) \leq \mu_e(E) + \mu_e(A) = 0 + \mu_e(A) = \mu_e(A).$$
Thus \( \mu_e(A) = \mu_e(A \cap E) + \mu_e(A - E) \) and so \( E \in \mathcal{A} \).

(iv) For \( E_1, E_2 \in \mathcal{A} \) we have for any \( A \in \mathcal{P}(X) \) that

\[
\mu_e(A) \geq \mu_e(A \cap E_1) + \mu_e(A - E_1),
\]

\[
\mu_e(A - E_1) \geq \mu_e((A - E_1) \cap E_2) + \mu_e((A - E_1) - E_2),
\]

where the second inequality holds because \( A - E_1 \in \mathcal{P}(X) \).

Because of the common summand \( \mu_e(A - E_1) \) in these inequalities we get

\[
\mu_e(A) \geq \mu_e(A \cap E_1) + \mu_e((A - E_1) \cap E_2) + \mu_e((A - E_1) - E_2).
\]

By the subadditivity of \( \mu_e \) we obtain

\[
\mu_e(A) \geq \mu_e((A \cap E_1) \cup ((A - E_1) \cap E_2)) + \mu_e((A - E_1) - E_2).
\]

By the set identities \((A \cap E_1) \cup ((A - E_1) \cap E_2) = A \cap (E_1 \cup E_2)\) and \((A - E_1) - E_2 = A - (E_1 \cup E_2)\), we have

\[
\mu_e(A) \geq \mu_e(A \cap (E_1 \cup E_2)) + \mu_e(A - (E_1 \cup E_2)).
\]

Thus \( E_1 \cup E_2 \in \mathcal{A} \).

(v) Using the set identity \( E_1 - E_2 = E_1 \cap E_2^c = (E_1^c \cup E_2)^c \) and (ii) and (iv), we have that \( E_1 - E_2 \in \mathcal{A} \) whenever \( E_1, E_2 \in \mathcal{A} \).

(vi) Using the set identity \( E_1 \cap E_2 = (E_1^c \cup E_2^c)^c \) and (ii) and (iv), we have that \( E_1 \cap E_2 \in \mathcal{A} \) whenever \( E_1, E_2 \in \mathcal{A} \).

(vii) For a countable collection \( \{E_n\} \) of pairwise disjoint sets in \( \mathcal{A} \), set \( B_k = \bigcup_{j=1}^{k} E_j \).

By pairwise disjointness of \( \{E_n\} \) we have that \( B_{k+1} - B_k = E_{k+1} \) for all \( k \).

Let \( A \in \mathcal{P}(X) \).

For \( k = 1 \), we have \( \mu_e(A \cap B_1) = \sum_{j=1}^{1} \mu_e(A \cap E_j) \).

Suppose that for \( k \geq 1 \) we have \( \mu_e(A \cap B_k) = \sum_{j=1}^{k} \mu_e(A \cap E_j) \).

By (iv) we have \( B_k \in \mathcal{A} \), so that

\[
\mu_e(A \cap B_{k+1}) = \mu_e((A \cap B_{k+1}) \cap B_k) + \mu_e((A \cap B_{k+1}) - B_k)
\]

\[
= \mu_e(A \cap B_k) + \mu_e(A \cap E_{k+1})
\]

By induction there holds \( \mu_e(A \cap B_k) = \sum_{j=1}^{k} \mu_e(A \cap E_j) \) for all \( k \in \mathbb{N} \).

By subadditivity and monotonicity of \( \mu_e \), and the induction above, we have for all \( m \in \mathbb{N} \) that

\[
\sum_{n=1}^{\infty} \mu_e(A \cap E_n) \geq \mu_e \left( \bigcup_{n=1}^{\infty} (A \cap E_n) \right) = \mu_e \left( A \cap \left( \bigcup_{n=1}^{\infty} E_n \right) \right)
\]

\[
\geq \mu_e \left( A \cap \left( \bigcup_{n=1}^{m} E_n \right) \right) = \sum_{n=1}^{m} \mu_e(A \cap E_n).
\]
Letting \( m \to \infty \) forces the inequalities to be equalities, giving the result.

(viii) We may assume that \( \{E_n\} \) are pairwise disjoint by replacing \( \{E_n\} \) by \( \{D_n\} \) where \( D_1 = E_1 \) and \( D_n = E_{n+1} - \bigcup_{j=1}^n E_n \) if needed since each \( D_n \in \mathcal{A} \) and \( \bigcup D_n = \bigcup E_n \).

Each finite union \( \bigcup_{n=1}^m E_n \) belongs to \( \mathcal{A} \) so that for all \( A \in \mathcal{P}(X) \) there holds

\[
\mu_e(A) = \mu_e\left(A \cap \left( \bigcup_{n=1}^m E_n \right)\right) + \mu_e\left(A - \bigcup_{n=1}^m E_n \right).
\]

Since \( A - \bigcup_{n=1}^m E_n \supset A - \bigcup_{n=1}^\infty E_n \), we have by monotonicity that

\[
\mu_e\left(A - \bigcup_{n=1}^m E_n \right) \geq \mu_e\left(A - \bigcup_{n=1}^\infty E_n \right).
\]

By (vii) we have that

\[
\lim_{m \to \infty} \mu_e\left(A \cap \left( \bigcup_{n=1}^m E_n \right)\right) = \mu_e\left(A \cap \left( \bigcup_{n=1}^\infty E_n \right)\right).
\]

Thus

\[
\mu_e(A) \geq \mu_e\left(A \cap \left( \bigcup_{n=1}^\infty E_n \right)\right) + \mu_e\left(A - \bigcup_{n=1}^\infty E_n \right).
\]

This implies that \( \bigcup E_n \) belongs to \( \mathcal{A} \).

**Proposition 6.2 (Carathéodory).** The restriction of \( \mu_e \) to \( \mathcal{A} \) is a complete measure.

**Proof.** The nonempty set \( \mathcal{A} \) is a algebra by parts (i), (ii), and (iv) of Proposition 6.1. The algebra \( \mathcal{A} \) is a \( \sigma \)-algebra by part (viii) of Proposition 6.1.

The outer measure \( \mu_e \) restricted to \( \mathcal{A} \) is a measure, where countable additivity follows from part (vii) of Proposition 6.1 with \( A = \bigcup E_n \) for \( \{E_n\} \) in \( \mathcal{A} \) being pairwise disjoint:

\[
\mu_e\left(\bigcup_{n=1}^\infty E_n\right) = \mu_e\left(\left(\bigcup_{m=1}^\infty E_m\right) \cap \left(\bigcup_{n=1}^\infty E_n\right)\right)
= \sum_{n=1}^\infty \mu_e\left(\left(\bigcup_{m=1}^\infty E_m\right) \cap E_n\right)
= \sum_{n=1}^\infty \mu_e(E_n).
\]

The completeness of \( \mu_e \) restricted to \( \mathcal{A} \) is by part (iii) of Proposition 6.1.

\[\square\]