10. **Necessary and sufficient conditions for measurability.** Recall that a measure \( \lambda \) on a semilagebra \( Q \) (that is also a sequential covering of \( X \)) generates a \( \sigma \)-algebra \( A \) and a measure \( \mu \) on \( A \) (the restriction of the outer measure \( \mu_e \) to \( A \)).

That is, the measure \( \mu \) on \( A \) is an extension of the measure \( \lambda \) on \( Q \) in that \( Q \subset A \) and \( \lambda(E) = \mu(E) \) for all \( E \in Q \).

Recall that \( A \) consists of those sets \( E \) in \( X \) for which
\[
\mu_e(A) \geq \mu_e(A \cap E) + \mu_e(A - E)
\]
for all sets \( A \) in \( X \); we say the elements of \( A \) are \( \mu \)-measurable (what we called \( \mu_e \)-measurable sets before).

We let \( \{X, A, \mu\} \) be the measure space generated by the measure \( \lambda \) on the sequential covering and semialgebra \( Q \).

We will describe sufficient and necessary conditions for the \( \mu \)-measurable sets in terms of sets derived from elements of \( Q \).

Denote by \( Q_\sigma \) the collection of all sets that are countable unions of elements of \( Q \).

Note that \( Q_\sigma \subset A \) since \( Q \subset A \) and the latter is a \( \sigma \)-algebra.

**Lemma.** For each element \( E = \cup Q_n \) of \( Q_\sigma \) there exists a countable collection of pairwise disjoint set \( \{D_n\} \) in \( A \) such that \( D_n \subset Q_n \) for all \( n \in \mathbb{N} \), and \( E = \cup D_n \), and each \( D_n \) is a finite union of pairwise disjoint elements of \( Q \).

**Proof.** Recall that we have seen before that any countable union can be rewritten as a countable union of disjoint sets: \( E = \cup D_n \) where \( D_1 = Q_1 \), and
\[
D_n = Q_n - \bigcup_{j=1}^{n-1} Q_j = \bigcap_{j=1}^{n-1} (Q_n - Q_j), \quad n \in \mathbb{N}.
\]

Here \( D_n \subset Q_n \), and since \( Q \subset A \) and the latter is a \( \sigma \)-algebra, we have \( D_n \in A \).

Because \( Q \) is a semialgebra, each difference \( Q_n - Q_j \) is the finite disjoint union of elements of \( Q \).

Then for each \( n \) and \( j \) there exist \( k(j) \) disjoint sets \( P_{j,1}, \ldots, P_{j,k(j)} \) in \( Q \) (with \( k \) depending on \( n \) as well but suppressed in the notation for the sake of clarity) such that
\[
Q_n - Q_j = \bigcup_{l=1}^{k(j)} P_{j,l}.
\]

We then have that
\[
D_n = \bigcap_{j=1}^{n-1} \left( \bigcup_{l=1}^{k(j)} P_{j,l} \right).
\]
Using the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we have

$$(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D).$$

Applying this repeatedly gives

$$D_n = (P_{1,1} \cup P_{1,2} \cup \cdots \cup P_{1,k(1)}) \cap (P_{2,1} \cup P_{2,2} \cup \cdots \cup P_{2,k(2)})$$

$$\cap \cdots \cap (P_{n-1,1} \cup P_{n-1,2} \cup \cdots \cup P_{n-1,k(n-1)})$$

$$= (P_{1,1} \cap P_{2,1} \cdots \cap P_{n-1,1}) \cup (P_{1,1} \cap P_{2,1} \cdots \cap P_{n-1,2})$$

$$\cup \cdots \cup (P_{1,k(1)} \cap P_{2,k(2)} \cdots \cap P_{n-1,k(n-1)}).$$

That is, we have a union of $k(1) \cdot k(2) \cdots k(n-1)$ sets of the form

$$\bigcap_{j=1}^{n-1} P_{j,s(j)},$$

where for each $j$ the value of $s(j)$ is chosen from $\{1, 2, \ldots, k(j)\}$.

Because a semialgebra is also “closed” under intersections, we have that each of these sets $\bigcap_{j=1}^{n-1} P_{j,s(j)}$ belongs to $Q$.

For another set $\bigcap_{j=1}^{n-1} P_{j,r(j)}$, where $r(j) \in \{1, 2, \ldots, k(j)\}$ differs from $s(j)$ for some $j = \hat{j}$, we have that $\bigcap_{j=1}^{n-1} P_{j,s(j)}$ and $\bigcap_{j=1}^{n-1} P_{j,r(j)}$ are disjoint because the first is a subset of $P_{j,s(j)}$, the second a subset of $P_{j,r(j)}$, while $P_{j,s(j)}$ and $P_{j,r(j)}$ are disjoint.

Thus each $D_n$ is a disjoint union of elements of $Q$. \qed

Denote by $Q_{\sigma,\delta}$ the collection of sets that are countable intersections of elements of $Q_\sigma$.

Note that $Q_{\sigma,\delta} \subset A$ because $Q_\sigma \subset A$ and the latter is a $\sigma$-algebra.

**Proposition 10.1.** If $E \subset X$ is of finite outer measure, then for each $\epsilon > 0$ there exists $E_{\sigma,\epsilon} \in Q_\sigma$ such that

$$E \subset E_{\sigma,\epsilon} \text{ and } \mu_\epsilon(E) \geq \mu(E_{\sigma,\epsilon}) - \epsilon.$$ 

Moreover, there exists a set $E_{\sigma,\delta} \in Q_{\sigma,\delta}$ such that

$$E \subset E_{\sigma,\delta} \text{ and } \mu_\epsilon(E) = \mu(E_{\sigma,\delta}).$$

Proof. For a given $\epsilon > 0$ there exists $\{Q_{n,\epsilon}\}$ in $Q$ such that

$$\mu_\epsilon(E) + \epsilon \geq \sum_{n=1}^{\infty} \lambda(Q_{n,\epsilon}), \quad E \subset \bigcup Q_{n,\epsilon}.$$ 

Set $E_{\sigma,\epsilon} = \bigcup Q_{n,\epsilon}$.

By the Lemma we can replace $\bigcup Q_{n,\epsilon}$ by a union of disjoint elements $D_{n,\epsilon} \in A$ satisfying $D_{n,\epsilon} \subset Q_{n,\epsilon}$ (and where each $D_{n,\epsilon}$ is a finite disjoint union of elements of $Q$ — something we will not need here).

Then $\mu(D_{n,\epsilon}) = \mu_\epsilon(D_{n,\epsilon}) \leq \mu_\epsilon(Q_{n,\epsilon}) \leq \lambda(Q_{n,\epsilon})$ because $D_{n,\epsilon} \in A$ and $D_{n,\epsilon} \subset Q_{n,\epsilon} \in Q$. 


Thus by the pairwise disjointness of \( \{D_{n,\epsilon}\} \) and the countable additivity of \( \mu \) we get

\[
\sum_{n=1}^{\infty} \lambda(Q_{n,\epsilon}) \geq \sum_{n=1}^{\infty} \mu(D_{n,\epsilon}) = \mu\left(\bigcup_{n=1}^{\infty} D_{n,\epsilon}\right) = \mu(E_{\sigma,\epsilon}).
\]

This gives

\[
\mu_e(E) + \epsilon \geq \mu(E_{\sigma,\epsilon}).
\]

Now for each \( n \in \mathbb{N} \), there is \( E_{\sigma,1/n} \in \mathcal{Q}_{\sigma} \) that satisfies

\[
\mu(E_{\sigma,1/n}) - \frac{1}{n} \leq \mu_e(E) \leq \mu_e(E_{\sigma,1/n}) = \mu(E_{\sigma,1/n}),
\]

the last inequality holds by \( E \subset E_{\sigma,1/n} \) and the monotonicity of \( \mu_e \), while the last equality holds by \( \mu = \mu_e \) on \( \mathcal{A} \).

The set \( E_{\sigma,\delta} = \cap E_{\sigma,1/n} \in \mathcal{Q}_{\sigma,\delta} \) contains \( E \) because each \( E_{\sigma,1/n} \) does.

**Homework Problem 12A.** Prove that \( E_{\sigma,\delta} \) satisfies \( \mu_e(E) = \mu(E_{\sigma,\delta}) \).  \( \square \)
Appendix. Proof that a measure $\lambda$ on semialgebra is countably subadditive.

Suppose for $\{Q_n\} \subset Q$ that

$$Q = \bigcup_{n=1}^{\infty} Q_n \in Q.$$ 

By Lemma in this Lecture Note there exist disjoint subsets $D_n \subset Q_n$ such that

$$Q = \bigcup_{n=1}^{\infty} D_n,$$

where $D_1 = Q_1$ and for each $n \geq 2$, there exists $k(n) \in \mathbb{N}$ and sets $P_{j,l} \in Q$ for all $j = 1, \ldots, n$ and $l = 1, \ldots, k(n)$ such that

$$D_n = Q_n - \bigcup_{j=1}^{n-1} Q_j = \bigcap_{j=1}^{n-1} (Q_n - Q_j) = \bigcap_{j=1}^{n-1} \left( \bigcup_{l=1}^{k(j)} P_{j,l} \right) = \bigcup_{s=1}^{n-1} \left( \bigcap_{j=1}^{n-1} P_{j,s} \right).$$

Set

$$E_{n,s} = \bigcap_{j=1}^{n-1} P_{j,s} \in Q.$$ 

For different functions $s$, the sets $E_{n,s}$ are pairwise disjoint. Then

$$Q = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} \bigcup_{s} E_{n,s}$$

is a countable union of pairwise disjoint sets. Hence by countable additivity we have

$$\lambda(Q) = \sum_{n=1}^{\infty} \sum_{s} \lambda(E_{n,s}).$$

If we can show the claim that for all $n$ there holds

$$\sum_{s} \lambda(E_{n,s}) \leq \lambda(Q_n)$$

then we obtain the desired countable subadditivity,

$$\lambda \left( \bigcup_{n=1}^{\infty} Q_n \right) \leq \sum_{n=1}^{\infty} \lambda(Q_n).$$

It remains to establish the claim. To this end we consider $Q_n - D_n$, i.e.,

$$Q_n - \bigcup_{s} E_{n,s} = Q_n \cap \left( \bigcup_{s} E_{n,s} \right)^c = Q_n \cap \left( \bigcap_{s} E_{n,s}^c \right) = \bigcap_{s} (Q_n \cap E_{n,s}^c) = \bigcap_{s} (Q_n - E_{n,s}).$$
Since $Q_n \in \mathcal{Q}$ and $E_{n,s} \in \mathcal{Q}$, there exists $m(n,s) \in \mathbb{N}$ and disjoint $A_{n,s,t} \in \mathcal{Q}$ for $t = 1, \ldots, m(n,s)$ such that

$$Q_n - E_{n,s} = \bigcup_{t=1}^{m(n,s)} A_{n,s,t}.$$ 

Thus

$$Q_n - \bigcup_s E_{n,s} = \bigcap_s \left( \bigcup_{t=1}^{m(n,s)} A_{n,s,t} \right).$$

The finite intersection of the finite disjoint union can be written as a finite disjoint union of finite intersections of the sets $A_{n,s,t}$ where these finite intersections belong to $\mathcal{Q}$. (Recall the proof of the Lemma in this Lecture Note wherein the order of the intersection and union were reversed.) That is, there exist finitely many disjoint elements $B_1, \ldots, B_u$ of $\mathcal{Q}$ such that

$$Q_n - \bigcup_s E_{n,s} = \bigcup_{i=1}^{u} B_i.$$ 

Because $D_n \subset Q_n$, we obtain

$$Q_n = \bigcup_s E_{n,s} \cup \bigcup_{i=1}^{u} B_i.$$ 

Thus by finite additivity we have

$$\lambda(Q_n) = \sum_s \lambda(E_{n,s}) + \sum_{i=1}^{u} \lambda(B_i).$$

Since $\lambda \geq 0$, we obtain

$$\lambda(Q_n) \geq \sum_s \lambda(E_{n,s}),$$

which establishes the claim.