§14.1A: Measure Theoretic Properties of $f$. We now explore some of the measure theoretic properties of the continuous, strictly increasing function $f$, with respect to Lebesgue measure $\mu$ on $\mathbb{R}$.

Set

\[
\begin{align*}
A_n &= \bigcup \{ \text{intervals } [\alpha, \beta], \text{ where } f_n \text{ is affine and } f'_n = 2^n \}, \\
B_n &= \bigcup \{ \text{intervals } [\alpha, \beta], \text{ where } f_n \text{ is affine and } f'_n = 2^{-n} \}
\end{align*}
\]

For an endpoint $z$ of an interval $[\alpha, \beta] \in A_n$, we have $f(z) = f_n(z)$; hence

\[
\begin{align*}
[\alpha, \beta] \in A_n \Rightarrow f(\beta) - f(\alpha) &= f_n(\beta) - f_n(\alpha) = 2^n (\beta - \alpha), \\
[\alpha, \beta] \in B_n \Rightarrow f(\beta) - f(\alpha) &= f_n(\beta) - f_n(\alpha) = 2^{-n} (\beta - \alpha).
\end{align*}
\]

Since Lebesgue measure of an interval is the length of the interval, we have

\[
\begin{align*}
\mu(f([\alpha, \beta])) &= f(\beta) - f(\alpha) = 2^n (\beta - \alpha) = 2^n \mu([\alpha, \beta]), \\
\mu(f([\alpha, \beta])) &= f(\beta) - f(\alpha) = 2^{-n} (\beta - \alpha) = 2^{-n} \mu([\alpha, \beta]).
\end{align*}
\]

Since the finitely many intervals in $A_n$ are pairwise disjoint and $f$ is strictly increasing, the images of those intervals by $f$ are also pairwise disjoint.

Similarly, the finitely many intervals in $B_n$ are pairwise disjoint, and so the strict monotonicity of $f$ implies the images of those intervals in $B_n$ are also pairwise disjoint.

Thus for $A_n = [\alpha_1, \beta_1] \cup \cdots \cup [\alpha_k, \beta_k]$, we have

\[f_n(A_n) = [f(\alpha_1), f(\beta_1)] \cup \cdots \cup [f(\alpha_k), f(\beta_k)];\]

a similar statement holds for $B_n$ and $f_n(B_n)$.

Finite additivity (implied by countable additivity of $\mu$) gives

\[
\begin{align*}
\mu(f(A_n)) &= 2^n \mu(A_n), \\
\mu(f(B_n)) &= 2^{-n} \mu(B_n).
\end{align*}
\]

Since $[0, 1] = A_n \cup B_n$ and since $A_n \cap B_n$ is a finite set (the common endpoints of the intervals) and hence of Lebesgue measure 0, we have

\[
1 = \mu([0, 1]) = \mu(A_n \cup B_n) = \mu(A_n) + \mu(B_n) - \mu(A_n \cap B_n) = \mu(A_n) + \mu(B_n).
\]

Similarly, since $[0, 1] = f(A_n) \cup f(B_n)$, and since $f(A_n) \cap f(B_n)$ is a finite set, we have

\[
1 = \mu(f(A_n)) + \mu(f(B_n)).
\]

Since $\mu(f(A_n)) = 2^n \mu(A_n)$ and $\mu(f(B_n)) = 2^{-n} \mu(B_n)$, we obtain

\[
1 = 2^n \mu(A_n) + 2^{-n} \mu(B_n).
\]
Thus we have two linear equations in the two unknowns \( \mu(A_n) \) and \( \mu(B_n) \):

\[
\begin{align*}
\mu(A_n) + \mu(B_n) &= 1 \\
2^n \mu(A_n) + 2^{-n} \mu(B_n) &= 1.
\end{align*}
\]

Using Cramer’s Rule we obtain

\[
\begin{align*}
\mu(A_n) &= \frac{2^{-n} - 1}{2^{-n} - 2^n} = \frac{2^n - 1}{2^n - 2^{-n}}, \\
\mu(B_n) &= \frac{1 - 2^n}{2^{-n} - 2^n} = \frac{2^n - 1}{2^n - 2^{-n}}.
\end{align*}
\]

Since \( \mu(f(A_n)) = 2^n \mu(A_n) \) and \( \mu(f(B_n)) = 2^{-n} \mu(B_n) \), we also obtain

\[
\begin{align*}
\mu(f(A_n)) &= 2^n \frac{2^n - 1}{2^{2n} - 1} = \mu(B_n), \\
\mu(f(B_n)) &= 2^n - 1 = \mu(A_n).
\end{align*}
\]

Set

\[
S_n = \bigcup_{j=n}^{\infty} A_j, \quad S = \bigcap_{n=1}^{\infty} S_n = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_n = \limsup A_n.
\]

Since each \( A_n \) is a union of intervals, it is measurable; hence for each \( n \) the set \( S_n \) is measurable; finally \( S \) is measurable.

Since \( \emptyset \subset S \subset S_n \) for all \( n \), and since \( S_n = \bigcup_{j=n}^{\infty} A_j \) (not disjointly), we have

\[
0 \leq \mu(S) \leq \mu(S_n) \leq \sum_{j=n}^{\infty} \mu(A_j).
\]

The tail of the series goes to 0 as \( n \to \infty \) because

\[
\sum_{j=n}^{\infty} \mu(A_j) = \sum_{j=n}^{\infty} \frac{2^j - 1}{2^{2j} - 1} = \sum_{j=n}^{\infty} \frac{2^j - 1}{(2^j - 1)(2^j + 1)} = \sum_{j=n}^{\infty} \frac{1}{2^j + 1} \leq \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^n - 1} \to 0.
\]

By the Squeeze Theorem we have \( \mu(S) = 0 \).

Now we show that \( f(S) \) is measurable and determine the value of \( \mu(f(S)) \).

Each set \( f(A_n) \) is a finite union of closed intervals because \( f \) is continuous and strictly increasing.

Thus \( f(A_n) \) is measurable, making the sets

\[
f(S_n) = \bigcup_{j=n}^{\infty} f(A_j), \quad f(S) = \bigcap_{n=1}^{\infty} f(S_n) = \limsup f(A_n)
\]

measurable as well.
Since $\bigcup_{n=1}^{\infty} f(A_n) \subset [0, 1]$, we have $\mu(\bigcup f(A_n)) < \infty$.

Thus by Proposition 3.1 we have

$$\mu(f(S)) = \mu(\limsup f(A_n)) \geq \limsup \mu(f(A_n)) = \limsup 2^n \frac{2^n - 1}{2^{2n} - 1}.$$  

The sequence in the limsup is a convergent sequence with limit 1 because

$$2^n \frac{2^n - 1}{2^{2n} - 1} = \frac{2^{2n} - 2^n}{2^{2n} - 1} = 2^{2n} - 2^n \frac{2^{-2n}}{2^{2n}} = \frac{1 - 2^{-n}}{1 - 2^{-2n}} \to 1.$$  

We obtain that $1 \leq \mu(f(S))$.

On the other hand, since $f(S) \subset [0, 1]$, we have $\mu(f(S)) \leq 1$.

By the Squeeze Theorem, we obtain $\mu(f(S)) = 1$.

Therefore, the function $f$ maps the set $S$ of measure 0 to the set $f(S)$ of measure 1. Likewise, the function $f$ maps the set $[0, 1] - S$ of measure 1 to the set $[0, 1] - f(S)$ of measure 0 because (using the injectivity of $f$) we have

$$f([0, 1] - S) = f([0, 1] \cap S^c) = f([0, 1]) \cap f(S^c) = [0, 1] \cap [f(S)]^c = [0, 1] - f(S),$$

so that

$$\mu(f([0, 1] - S)) = \mu([0, 1] - f(S)) = 1 - \mu(f(S)) = 0.$$