14.2: On the Preimage of a Measurable Set. Recall that a subset $E$ of $[0,1]$ is relatively open if there is an open subset $O$ of $\mathbb{R}$ such that $E = O \cap [0,1]$.

For $f : [0,1] \to [0,1]$ continuous, the preimage $f^{-1}(O)$ of a relatively open set $O$ in $[0,1]$ is a relatively open set in $[0,1]$, and hence the preimage is Lebesgue measurable.

Similarly the preimage of a relatively closed set is relatively closed, and hence the preimage of a relatively closed set is Lebesgue measurable.

More generally we consider the collection $F$ of subsets $E$ of $[0,1]$ for which the preimage $f^{-1}(E)$ (a subset of $[0,1]$) is Lebesgue measurable.

Proposition. The collection $F$ is a $\sigma$-algebra of subsets of $[0,1]$ that contains Borel subsets of $[0,1]$.

Proof. We are to show that the relative complement of any element of $F$ (relative to $[0,1]$) is in $F$, and that the union of a countable collection of elements in $F$ is in $F$.

For $E \in F$ we have that $f^{-1}(E)$ is Lebesgue measurable.

For the relative complement $[0,1] - E$ to belong to $F$, we are to show that $f^{-1}([0,1] - E)$ is Lebesgue measurable.

Using properties of preimages of functions on intersections and complements we have

$$f^{-1}([0,1] - E) = f^{-1}([0,1] \cap E^c)$$

$$= f^{-1}([0,1]) \cap f^{-1}(E^c)$$

$$= [0,1] \cap (f^{-1}(E))^c$$

$$= [0,1] - f^{-1}(E).$$

Since $f^{-1}(E)$ is Lebesgue measurable, so then is $[0,1] - f^{-1}(E) = f^{-1}([0,1] - E)$, and hence $[0,1] - E \in F$.

Now take a countable collection $\{E_n\}$ of elements in $F$.

Then for each $n$ we have $f^{-1}(E_n)$ is Lebesgue measurable, so that

$$f^{-1}\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(E_n)$$

is Lebesgue measurable, being the countable union of Lebesgue measurable sets.

Thus $F$ is $\sigma$-algebra.

Each relatively open subset of $[0,1]$ belongs to $F$ because its preimage is relatively open.

Since the $\sigma$-algebra $B$ of Borel subsets of $[0,1]$ is the smallest $\sigma$-algebra containing the relatively open subsets of $[0,1]$, it follows that $B \subseteq F$. □

Proposition. If $E$ is a Borel subset of $[0,1]$, then $f^{-1}(E)$ is a Borel subset of $[0,1]$. 

Proof. We will show that \( \Omega = \{ E \subseteq [0, 1] : f^{-1}(E) \in \mathcal{B} \} \) contains \( \mathcal{B} \).

By an argument similar to that used in the proof of the previous Proposition, we show that \( \Omega \) is a \( \sigma \)-algebra in \([0, 1]\).

Each relatively open set \( O \) in \([0, 1]\) belongs to \( \Omega \) because, by the continuity of \( f \), we have \( f^{-1}(O) \) is relatively open and hence in \( \mathcal{B} \).

Since \( \Omega \) is a \( \sigma \)-algebra containing all of the relatively open subsets of \([0, 1]\), we have \( \mathcal{B} \subseteq \Omega \).

Thus, each every Borel subset \( E \) of \([0, 1]\) belongs to \( \Omega \), and it has the property that \( f^{-1}(E) \in \mathcal{B} \).

\[\square\]

14.3: Proofs of Two Propositions. We have now developed enough to prove, using the continuous, strictly increasing function \( f : [0, 1] \to [0, 1] \), the existence of a Lebesgue measurable subset of \( \mathbb{R} \) that is not a Borel set, and the existence of a Borel measure that is not complete.

Proposition 14.1. There exists a Lebesgue measurable subset \( D \) of \([0, 1]\) which is not a Borel set, and whose preimage under \( f \) is not Lebesgue measurable.

Proof. Recall that there is a Lebesgue measurable subset \( S \) of \([0, 1]\) with Lebesgue measure 0 whose image \( f(S) \) is Lebesgue measurable with Lebesgue measure 1.

Furthermore, the function \( f \) maps the Lebesgue measurable set \([0, 1] - S\) of Lebesgue measure 1 to the Lebesgue measurable set \([0, 1] - f(S)\) of Lebesgue measure zero.

Since Lebesgue measure is complete, every subset of \( S \) is Lebesgue measurable and has Lebesgue measure zero.

Likewise, every subset of \([0, 1] - f(S)\) is Lebesgue measurable and has Lebesgue measure zero.

Let \( E \) be the Vitali subset of \([0, 1]\) that is not Lebesgue measurable.

The set \( E \cap S \) is Lebesgue measurable because Lebesgue measure is complete: the set \( E \cap S \) is a subset of a Lebesgue measurable set of measure zero.

The set \( E - S \) is not Lebesgue measurable, because if it were, then \( E \) would be the (disjoint) union of the Lebesgue measurable sets \( E - S \) and \( E \cap S \).

The set \( D = f(E - S) \) is contained in \([0, 1] - f(S)\) because \( f(E) \subset [0, 1] \) and by the injectivity of \( f \) we have

\[
 f(E - S) = f(E \cap S^c) = f(E) \cap f(S^c) = f(E) \cap [f(S)]^c = f(E) - f(S) .
\]

Since \([0, 1] - f(S)\) is a set of Lebesgue measure zero, \( f(E) - f(S) \subset [0, 1] - f(S) \), and Lebesgue measure is complete, the set \( D \) is Lebesgue measurable with Lebesgue measure zero.

The preimage of \( D \) is not Lebesgue measurable because \( f \) is invertible so that

\[
 f^{-1}(D) = f^{-1}(f(E - S)) = E - S ,
\]

which is not measurable.
If the Lebesgue measurable set $\mathcal{D}$ were a Borel set, then by the previous Proposition, the preimage $f^{-1}(\mathcal{D})$ would be a Borel set, and hence Lebesgue measurable.

By this contradiction, the set $\mathcal{D}$ is not a Borel set.  

Proposition 14.2. The restriction of Lebesgue measure on $\mathbb{R}$ to the $\sigma$-algebra of Borel sets in $\mathbb{R}$ is not a complete measure.

Proof. Let $\mathcal{D}$ be the Lebesgue measurable set of Lebesgue measure zero, as given in the proof of Proposition 14.1.

By Proposition 12.3, there is a set $\mathcal{D}_\delta$ of type $G_\delta$ such that $\mathcal{D} \subset \mathcal{D}_\delta$ and 

$$
\mu(\mathcal{D}_\delta) = \mu(\mathcal{D}_\delta) - \mu(\mathcal{D}) = \mu(\mathcal{D}_\delta - \mathcal{D}) = 0.
$$

The set $\mathcal{D}_\delta$ is a Borel set that has Lebesgue measure zero, but it contains the subset $\mathcal{D}$ that is not a Borel set.

Thus the restriction of Lebesgue measure $\mu$ to the $\sigma$-algebra of Borel sets in $\mathbb{R}$ is not a complete measure. □

Recall that $\mathcal{F}$ is the $\sigma$-algebra of subsets $E$ of $[0,1]$ whose preimages $f^{-1}(E)$ are Lebesgue measurable.

By way of a slight abuse of notation, let $\mathcal{M}$ denote the $\sigma$-algebra of Lebesgue measurable subsets of $[0,1]$.

Then 

$$
\mathcal{F} = \{ E \subset [0,1] : f^{-1}(E) \in \mathcal{M} \}.
$$

Proposition. For the function $f$, there holds $\mathcal{M} \not\subset \mathcal{F}$.

Proof. By Proposition 14.1, we have the existence of $\mathcal{D} \in \mathcal{M}$ for which $f^{-1}(\mathcal{D}) \not\in \mathcal{M}$.

Thus $\mathcal{D} \not\in \mathcal{F}$, which implies that $\mathcal{M} \not\subset \mathcal{F}$. □