Proposition 1.4. Let \( \{f_n\} \) be a sequence of measurable functions defined on \( E \). Then the functions

\[
\varphi = \sup f_n, \quad \psi = \inf f_n, \quad f'' = \limsup f_n, \quad f' = \liminf f_n
\]
on \( E \) are measurable.

Proof. For \( c \in \mathbb{R} \) we have

\[
[\varphi > c] = \{ x \in E : \varphi(x) > c \} = \{ x \in E : \sup \limits_n f_n(x) > c \} = \{ x \in E : f_n(x) > c \text{ for some } n \} = \bigcup \limits_{n=1}^{\infty} \{ x \in E : f_n(x) > c \} = \bigcup \limits_{n=1}^{\infty} [f_n > c].
\]

Because each \( [f_n > c] \) is measurable, so then is \( [\varphi > c] \), and hence \( \varphi \) is a measurable function.

Similarly, for \( c \in \mathbb{R} \) we have

\[
[\psi \geq c] = \{ x \in E : \psi(x) \geq c \} = \{ x \in E : \inf \limits_n f_n(x) \geq c \} = \{ x \in E : f_n(x) \geq c \text{ for all } n \} = \bigcap \limits_{n=1}^{\infty} \{ x \in E : f_n(x) \geq c \} = \bigcap \limits_{n=1}^{\infty} [f_n \geq c].
\]

Because \( [f_n \geq c] \) is measurable, so then is \( [\psi \geq c] \), and hence \( \psi \) is a measurable function.

By what we have shown, the functions

\[
\varphi_n = \sup \limits_{j \geq n} f_j \quad \text{and} \quad \psi_n = \inf \limits_{j \geq n} f_j,
\]

are measurable, and so

\[
\limsup f_n = \inf \sup \limits_{n \geq 1} \sup \limits_{j \geq n} f_j \quad \text{and} \quad \liminf f_n = \sup \inf \limits_{n \geq 1} \inf \limits_{j \geq n} f_j
\]

are also measurable functions. \( \square \)
Definitions. Let \( \{ X, \mathcal{A}, \mu \} \) be a measure space, and let \( E \in \mathcal{A} \).

For two real-extended valued functions \( f \) and \( g \) defined on \( E \), we say that \( f \) equals \( g \) \textit{almost everywhere}, written \( f = g \) a.e. in \( E \), if there exists a measurable subset \( E \subset E \) for which \( f(x) = g(x) \) for all \( x \in E - E \) and \( \mu(E) = 0 \).

A property of an extended real-valued function \( f \) defined on a measure space \( \{ X, \mathcal{A}, \mu \} \) is said to hold almost everywhere if it holds except on a set of measure zero.

Lemma 1.5. Let \( \{ X, \mathcal{A}, \mu \} \) be a complete measure space. For \( E \in \mathcal{A} \), if \( f : E \rightarrow \mathbb{R}^* \) is measurable, and \( g : E \rightarrow \mathbb{R}^* \) satisfies \( f = g \) a.e. in \( E \), then \( g \) is measurable.

Proof. Let \( E = [f \neq g] = \{ x \in E : f(x) \neq g(x) \} \).

By hypothesis, the set \( E \) is measurable, has measure zero, and every subset of \( E \) is measurable and has measure zero.

Let \( c \in \mathbb{R} \) be arbitrary.

Because \( f = g \) a.e. in \( E \) we have that

\[
[g > c] = ([f > c] \cap (E - E)) \cup ([g > c] \cap E).
\]

The first set on the right is measurable because \([f > c]\) and \( E - E \) are measurable, and the second set on the right is measurable because it is a subset of the measurable \( E \) of measure zero.

Thus \( g \) is measurable. \( \square \)

Corollary 1.6. Let \( \{ X, \mathcal{A}, \mu \} \) be a complete measure space. For \( E \in \mathcal{A} \), let \( \{ g_n \} \) be a sequence of measurable extended real-valued functions with domain \( E \). If

\[
g(x) = \lim g_n(x) \text{ exists a.e. in } E,
\]

then \( g : E \rightarrow \mathbb{R}^* \) is measurable.

Proof. Let \( \mathcal{E} \) be the subset of \( E \) on which \( \lim g_n \) does not exist, i.e., for \( x \in \mathcal{E} \) we have \( \lim g_n(x) \) does not exist.

Since \( \lim g_n(x) \) exists a.e. in \( E \), we have \( \mathcal{E} \) is measurable and \( \mu(\mathcal{E}) = 0 \).

For \( x \in \mathcal{E} \), define \( g(x) \) arbitrarily, i.e., randomly pick the value of \( g(x) \) from \( \mathbb{R}^* \) for each \( x \in \mathcal{E} \).

Define a function \( f : E \rightarrow \mathbb{R}^* \) by \( f(x) = g(x) \) for \( x \in E - \mathcal{E} \), and \( f(x) = \infty \) for \( x \in \mathcal{E} \).

If \( f \) is indeed measurable, then by Lemma 1.5, the function \( g \) is measurable.

Let \( c \in \mathbb{R} \).

Since \( E = (E - \mathcal{E}) \cup \mathcal{E} \) disjointly, we have the disjoint union

\[
[g_n > c] = \{ x \in E - \mathcal{E} : g_n(x) > c \} \cup \{ x \in \mathcal{E} : g_n(x) > c \}.
\]

Then

\[
\{ x \in E - \mathcal{E} : g_n(x) > c \} = [g_n > c] - \{ x \in \mathcal{E} : g_n(x) > c \}.
\]
The set \([g_n > c]\) is measurable because \(g_n\) is measurable, and \(\{x \in \mathcal{E} : g_n(x) > c\}\) is measurable because it is a subset of a measurable set of measure zero.

Thus \(\{x \in E - \mathcal{E} : g_n(x) > c\}\) is measurable for every \(c \in \mathbb{R}\).

We have shown that \(\{g_n\}\) restricted to \(E - \mathcal{E}\) is a sequence of measurable extended real-valued functions.

Since \(\{g_n(x)\}\) converges for each \(x \in E - \mathcal{E}\), we have

\[
f(x) = g(x) = \lim g_n(x) = \limsup g_n(x), \quad x \in E - \mathcal{E}.
\]

By Proposition 1.4, we have the measurability of \(f\) restricted to \(E - \mathcal{E}\).

Thus for any \(c \in \mathbb{R}\) we have

\[
[f > c] = \{x \in E : f(x) > c\} = \{x \in E - \mathcal{E} : f(x) > c\} \cup \{x \in \mathcal{E} : f(x) > c\} = \{x \in E - \mathcal{E} : f(x) > c\} \cup \mathcal{E}
\]

because \(f(x) = \infty\) for \(x \in \mathcal{E}\).

Since each set is measurable, the union is measurable, and hence \(f\) is measurable.

By Lemma 1.5, the function \(g : E \to \mathbb{R}^*\) is measurable. \qed