III.8: Fatou’s Lemma and the Monotone Convergence Theorem

§8: Fatou’s Lemma and the Monotone Convergence Theorem. We will present these results in a manner that differs from the book: we will first prove the Monotone Convergence Theorem, and use it to prove Fatou’s Lemma.

Proposition. Let \( \{X, \mathcal{A}, \mu\} \) be a measure space. For \( E \in \mathcal{A} \), if \( \varphi : E \to \mathbb{R} \) is a nonnegative simple function, then

\[
A \to \int_A \varphi \, d\mu, \quad A \in \mathcal{A},
\]

is a measure on the \( \sigma \)-algebra \( \mathcal{A} \).

Proof. If \( \varphi = \sum_{i=1}^n b_i \chi_{B_i} \) is the canonical representation of the nonnegative simple \( \varphi \), then

\[
\int_A \varphi \, d\mu = \sum_{i=1}^n b_i \mu(A \cap B_i).
\]

The domain of \( A \to \int_A \varphi \, d\mu \) is the \( \sigma \)-algebra \( \mathcal{A} \), and \( A \to \int_A \varphi \, d\mu \) is nonnegative for each \( A \) because \( b_i \geq 0 \) and \( \mu(A \cap B_i) \geq 0 \) for all \( i = 1, 2, \ldots, n \).

For \( A = \emptyset \), we have \( \int_A \varphi \, d\mu = 0 \).

For a countable collection \( \{A_k\} \) of pairwise disjoint sets in \( \mathcal{A} \), we have for \( A = \bigcup A_k \) that

\[
\int_A \varphi \, d\mu = \sum_{i=1}^n a_i \mu(A \cap B_i)
\]

\[
= \sum_{i=1}^n \mu \left( \bigcup_{k=1}^{\infty} A_k \right) \cap B_i
\]

\[
= \sum_{i=1}^n \sum_{k=1}^{\infty} a_i \mu(A_k \cap B_i)
\]

\[
= \sum_{k=1}^{\infty} \sum_{i=1}^n a_i \mu(A_k \cap B_i)
\]

\[
= \sum_{k=1}^{\infty} \int_{A_k} \varphi \, d\mu.
\]

This gives countable additivity. \( \square \)

The Monotone Convergence Theorem. If \( \{f_n\} \) is a sequence of nonnegative measurable functions defined on a measurable \( E \), such that \( f_n \leq f_{n+1} \) for all \( n \), and \( f = \lim f_n = \sup f_n \) (which exists), then

\[
\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu.
\]
Proof. Because \( \{f_n\} \) is monotone nondecreasing, the sequence
\[
\left\{ \int_E f_n \, d\mu \right\}
\]
is monotone nondecreasing and so its limit exists (possibly equal to \( \infty \)).

Since \( f_n \leq \sup f_n = f \), we have for all \( n \) that
\[
\int_E f_n \, d\mu \leq \int_E f \, d\mu,
\]
and hence that
\[
\lim_{n \to \infty} \int_E f_n \, d\mu \leq \int_E f \, d\mu.
\]

To establish the reverse inequality, we fix \( \alpha \in (0, 1) \), let \( \psi \) be a simple function satisfying \( 0 \leq \psi \leq f \), and set
\[
E_n = [f_n \geq \alpha \psi] = \{ x \in E : f_n(x) \geq \alpha \psi(x) \}.
\]

Then \( \{E_n\} \) is monotone increasing sequence of measurable sets whose union is \( E \), and we have
\[
\int_E f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} \alpha \psi \, d\mu = \alpha \int_{E_n} \psi \, d\mu.
\]

By the Proposition, the function
\[
\nu(A) = \int_A \psi \, d\mu, \quad A \in \mathcal{A}
\]
is a measure on \( \mathcal{A} \).

Since \( \{E_n\} \) is monotone increasing with \( \bigcup E_n = E \), we have that
\[
\lim_{n \to \infty} \int_{E_n} \psi \, d\mu = \lim_{n \to \infty} \nu(E_n) = \nu(E) = \int_E \psi \, d\mu.
\]

Thus
\[
\lim_{n \to \infty} \int_E f_n \, d\mu \geq \alpha \lim_{n \to \infty} \int_{E_n} \psi \, d\mu = \alpha \int_E \psi \, d\mu.
\]

As this holds for all \( \alpha \in (0, 1) \), it holds also for \( \alpha = 1 \), and taking the supremum over all simple \( 0 \leq \psi \leq f \) gives
\[
\lim_{n \to \infty} \int_E f_n \, d\mu \geq \int_E f \, d\mu.
\]

This is the reverse inequality sought. \( \square \)

Fatou’s Lemma. If \( \{f_n\} \) is a sequence of nonnegative measurable functions on \( E \), then
\[
\int_E \liminf f_n \, d\mu \leq \liminf \int_E f_n \, d\mu.
\]
Proof. For each $k$ we have for all $j \geq k$ that
\[
\inf_{n \geq k} f_n \leq f_j.
\]
Hence for all $j \geq k$ we have
\[
\int_E \inf_{n \geq k} f_n \, d\mu \leq \int_E f_j \, d\mu.
\]
The left-hand side is a lower bound for the integrals on the right-hand side, so that
\[
\int_E \inf_{n \geq k} f_n \, d\mu \leq \inf_{j \geq k} \int_E f_j \, d\mu.
\]
Since the sequence \( \{\inf_{n \geq k} f_n\} \) is monotone nondecreasing in $k$, we apply the Monotone Convergence Theorem to get
\[
\int_E \liminf_{n \to \infty} f_n \, d\mu = \int_E \lim_{k \to \infty} \inf_{n \geq k} f_n \, d\mu
\]
\[
= \lim_{k \to \infty} \int_E \inf_{n \geq k} f_n \, d\mu
\]
\[
\leq \lim_{k \to \infty} \inf_{j \geq k} \int_E f_j \, d\mu
\]
\[
= \liminf_{n \to \infty} \int_E f_n \, d\mu,
\]
giving the desired result.