Math 541 Lecture #34
III.17: The Lebesgue-Radon-Nikodym Theorem

§17: The Lebesgue-Radon-Nikodym Theorem. For two measures \( \mu \) and \( \nu \) on the same \( \sigma \)-algebra \( A \), we say that \( \nu \) is absolutely continuous with respect to \( \mu \), and write \( \nu \ll \mu \), if when \( \mu(E) = 0 \) for \( E \in A \), there holds \( \nu(E) = 0 \).

Given a measurable nonnegative \( f : X \to \mathbb{R}^* \), the set function

\[
E \mapsto \nu(E) = \int_E f \, d\mu, \quad E \in A,
\]

is a measure on \( A \) that is absolutely continuous with respect to \( \mu \).

We ask for the opposite: if \( \nu \ll \mu \), is there a measurable nonnegative \( f : X \to \mathbb{R}^* \) such that

\[
\nu(E) = \int_E f \, d\mu?
\]

The partial answer is the content of the Lebesgue-Radon-Nikodym Theorem (Lebesgue proved it for Lebesgue measure on \( \mathbb{R}^N \), then Radon extended it to Radon measures, and then Nikodym extended it to general measures).

**Theorem 17.1 (Lebesgue-Radon-Nikodym).** Let \( \{X, A, \mu\} \) and \( \{X, A, \nu\} \) be \( \sigma \)-finite measure spaces. If \( \nu \ll \mu \), then there is a measurable nonnegative function \( f : X \to \mathbb{R}^* \) such that

\[
\nu(E) = \int_E f \, d\mu, \quad E \in A.
\]

The function \( f \) is unique up to a set of \( \mu \)-measure zero.

Some Remarks: (1) The function \( f \) here is called the Radon-Nikodym derivative, since formally it satisfies

\[
d\nu = f \, d\mu.
\]

(2) The Theorem does not assert that \( f \) is \( \mu \)-integrable. This occurs if and only if \( \nu \) is finite.

(3) The assumption of \( \sigma \)-finiteness on both measures cannot be removed. You have it as two homework problems to construct counterexamples.

Proof of the Lebesgue-Radon-Nikodym Theorem in the case that both \( \mu \) and \( \nu \) are finite measures.

Let \( \Phi \) be the collection of measurable nonnegative functions \( \varphi : X \to \mathbb{R}^* \) that satisfy

\[
\int_E \varphi \, d\mu \leq \nu(E) \quad \text{for all} \ E \in A.
\]

The collection \( \Phi \) is nonempty since it contains the zero function.
For two $\varphi_1, \varphi_2 \in \Phi$, the function $\max\{\varphi_1, \varphi_2\}$ also belongs to $\Phi$, because for any $E \in \mathcal{A}$, we have

$$
\int_E \max\{\varphi_1, \varphi_2\} d\mu = \int_{E \cap [\varphi_1 \geq \varphi_2]} \varphi_1 \, d\mu + \int_{E \cap [\varphi_1 < \varphi_2]} \varphi_2 \, d\mu \\
\quad \leq \nu(E \cap [\varphi_1 \geq \varphi_2]) + \nu(E \cap [\varphi_1 < \varphi_2]) \\
= \nu(E).
$$

Since $\nu$ is finite, i.e., $\nu(X) < \infty$, the quantity

$$
M = \sup_{\varphi \in \Phi} \int_X \varphi \, d\mu \leq \nu(X) < \infty.
$$

Let $\{\varphi_n\}$ be a sequence in $\Phi$ such that

$$
\lim_{n \to \infty} \int_X \varphi_n \, d\mu = M.
$$

The sequence of nonnegative measurable functions

$$
f_n = \max\{\varphi_1, \ldots, \varphi_n\}
$$

is nondecreasing and converges pointwise to a measurable nonnegative function $f : X \to \mathbb{R}^*$. This function $f$ belongs to $\Phi$ because by the Monotone Convergence Theorem we have

$$
\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{A}.
$$

To show that this $f$ satisfies $\nu(E) = \int_E f \, d\mu$, we consider the measure

$$
\eta(E) = \nu(E) - \int_E f \, d\mu, \ E \in \mathcal{A}.
$$

If this measure is not the zero measure, then there is $A \in \mathcal{A}$ such that $\eta(A) > 0$. Since $\nu \ll \mu$, then $\eta \ll \mu$.

Thus $\eta(A) > 0$ implies by absolute continuity with respect to $\mu$ that $\mu(A) > 0$ (the contrapositive of absolute continuity of $\eta$ with respect to $\mu$).

Since $\mu$ is finite, i.e., $\mu(X) < \infty$, there exists $\epsilon > 0$ such that

$$
\xi(A) = \eta(A) - \epsilon \mu(A) > 0.
$$

The function $\xi : \mathcal{A} \to \mathbb{R}^*$ defined by

$$
\xi(E) = \eta(E) - \epsilon \mu(E)
$$

is a signed measure on $\mathcal{A}$. 
By Proposition 16.2, the set \( A \) contains a positive subset \( A_0 \), so that
\[
\xi(E) = \eta(E \cap A_0) - \epsilon \mu(E \cap A_0) \geq 0 \quad \text{for all } E \in \mathcal{A}.
\]
Using the definition of the measure \( \eta \) we have for all \( E \in \mathcal{A} \) that
\[
\nu(E \cap A_0) - \int_{E \cap A_0} f \, d\mu - \epsilon \mu(E \cap A_0) \geq 0,
\]
or rewritten, that for all \( E \in \mathcal{A} \) that
\[
\int_{E \cap A_0} f \, d\mu + \epsilon \mu(E \cap A_0) \leq \nu(E \cap A_0).
\]
This implies that the measurable nonnegative function \( f + \epsilon \chi_{A_0} \) belongs to \( \Phi \) because for all \( E \in \mathcal{A} \), we have
\[
\int_{E \cap A_0} f \, d\mu + \epsilon \mu(E \cap A_0) \leq \nu(E \cap A_0).
\]
But \( f + \epsilon \chi_{A_0} \in \Phi \) contradicts the definition of \( M \) because
\[
\int_X (f + \epsilon \chi_{A_0}) \, d\mu = \int_X f \, d\mu + \epsilon \int_X \chi_{A_0} \, d\mu = M + \epsilon \mu(A_0) > M.
\]
Thus \( \eta \) is the zero measure, and hence
\[
\nu(E) = \int_E f \, d\mu \quad \text{for all } E \in \mathcal{A}.
\]
Suppose \( g : X \to \mathbb{R}^+ \) is another measurable nonnegative function for which
\[
\nu(E) = \int_E g \, d\mu, \quad \text{for all } E \in \mathcal{A}.
\]
To show that \( f = g \) a.e. with respect to \( \mu \), we consider for \( n \in \mathbb{N} \) the sets
\[
A_n = \left\{ x \in X : f(x) - g(x) \geq \frac{1}{n} \right\}.
\]
Then for all \( n \in \mathbb{N} \) we have
\[
0 = \nu(A_n) - \nu(A_n) = \int_{A_n} (f - g) \, d\mu \geq \int_{A_n} \frac{1}{n} \, d\mu = \frac{\mu(A_n)}{n}.
\]
These implies that \( \mu(A_n) = 0 \) so that \( f \geq g \) a.e. with respect to \( \mu \).
A similar argument shows that \( f \leq g \) a.e. with respect to \( \mu \), so that \( f = g \) a.e. with respect to \( \mu \). \( \square \)