We continue towards a proof that the $L^p$ “norms” satisfy the triangle inequality. Two elements $p, q \in [1, \infty]$ are said to be conjugate if

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where if $p = 1$ then $q = \infty$ and $1/q = 0$ because as $p \to 1$, we have $q \to \infty$, and if $q = 1$, then $p = \infty$ and $1/p = 0$ because as $q \to 1$ we have $p \to \infty$.

For example, $p = 2$ and $q = 2$ are conjugate.

**Proposition 2.1.** If $p, q \in [1, \infty]$ are conjugate, then for all $a, b \in \mathbb{R}$ we have

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

**Proof.** The inequality holds if $|a| = 0$ or $|b| = 0$ because $|ab| = 0$.

So suppose that $|a| > 0$ and $|b| > 0$.

The inequality holds if $p = 1$ because as $p \to 1$ we have $q \to \infty$ so that when $|b| \leq 1$ we have $|b|^q/q \to 0$ as $q \to \infty$, so that

$$|ab| = |a| |b| \leq |a| \leq \frac{|a|^1}{1} + \frac{|b|^\infty}{\infty},$$

and when $|b| > 1$ we have $|b|^q/q \to \infty$ as $q \to \infty$ (by L’Hospital’s Rule), so that

$$|ab| < \infty = \frac{|a|^1}{1} + \frac{|b|^\infty}{\infty};$$

we have a similar conclusion when $q \to 1$.

So we suppose that $1 < p, q < \infty$.

The function

$$s \to \left( \frac{s^p}{p} + \frac{1}{q} - s \right), \ s \geq 0,$$

has an absolute minimum at $s = 1$ because its derivative

$$\frac{ps^{p-1}}{p} - 1 = s^{p-1} - 1$$

has a zero at $s = 1$, and its second derivative

$$(p - 1)s^{p-2}$$

is positive on $s \geq 0$. 

Hence for all $s \geq 0$ we have
\[
\frac{1}{p} + \frac{1}{q} - 1 \leq \frac{s^p}{p} + \frac{1}{q} - s,
\]
with equality holding only when $s = 1$. Since $p$ and $q$ are conjugate we have
\[
\frac{1}{p} + \frac{1}{q} - 1 = 0
\]
so that
\[
0 \leq \frac{s^p}{p} + \frac{1}{q} - s.
\]
This rearranges to give
\[
s \leq \frac{s^p}{p} + \frac{1}{q}
\]
with equality holding only when $s = 1$. Choosing
\[
s = \frac{|a|}{|b|^{q/p}}
\]
in the inequality gives
\[
\frac{|a|}{|b|^{q/p}} \leq \left( \frac{|a|}{|b|^{q/p}} \right)^p + \frac{1}{q} = \frac{|a|^p}{p|b|^q} + \frac{1}{q}.
\]
Multiplying the inequality through by $|b|^q$ gives
\[
\frac{|a|}{|b|^{q/p}} \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.
\]
Here
\[
\frac{|b|^q}{|b|^{q/p}} = |b|^{q-q/p}
\]
where, because $p$ and $q$ are conjugate we have
\[
\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{q}{p} + 1 = q \Rightarrow q - \frac{q}{p} = 1,
\]
we have $|b|^{q-q/p} = |b|$. Therefore we obtain the inequality. \qed

Proposition 2.2 (Hölder’s Inequality). If $f \in L^p(E)$ and $g \in L^q(E)$ for conjugate $p$ and $q$, then $fg \in L^1(E)$ and
\[
\int_E |fg| \, d\mu \leq ||f||_p ||g||_q.
\]
Moreover, equality holds only if there is a constant $c$ such that $|f(x)|^p = c|g(x)|^q$ for a.e. $x \in E$.

Proof. If either $f = 0$ a.e. in $E$ or $g = 0$ a.e. in $E$, there is nothing to show.

We assume WLOG that $f \geq 0$ and $g \geq 0$ with neither equal to 0 a.e. in $E$, so that $\|f\|_p \neq 0$ and $\|g\|_q \neq 0$.

For $p = 1$ and $q = \infty$ (similarly for $p = \infty$ and $q = 1$) we have

$$|fg| = |f||g| \leq \|f\|_\infty \|g\|_\infty,$$

so that

$$\int_E |fg| \, d\mu \leq \int_E |f| \|g\|_\infty \, d\mu = \|g\|_\infty \int_E |f| \, d\mu = \|g\|_\infty \|f\|_1 = \|f\|_1 \|g\|_\infty.$$

For $p, q \in (1, \infty)$, if we set

$$a = \frac{f}{\|f\|_p}, \quad b = \frac{g}{\|g\|_q}$$

and substitute these into the inequality

$$|ab| \leq \left| \frac{a}{p} \right|^p + \left| \frac{b}{q} \right|^q,$$

we get

$$\frac{fg}{\|f\|_p \|g\|_q} \leq \frac{f^p}{p \|f\|_p^p} + \frac{g^q}{q \|g\||_q^q} \text{ a.e. in } E.$$

Integrating over $E$ gives

$$\frac{1}{\|f\|_p \|g\|_q} \int_E fg \, d\mu \leq \frac{1}{p \|f\|_p^p} \int_E f^p \, d\mu + \frac{1}{q \|g\|_q^q} \int_E g^q \, d\mu$$

$$= \frac{\|f\|_p^p}{p \|f\|_p^p} + \frac{\|g\|_q^q}{q \|g\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Multiplication by $\|f\|_p \|g\|_q$ gives the inequality

$$\int_E fg \, d\mu \leq \|f\|_p \|g\|_q,$$

from which it follows that $fg \in L^1(E)$ when $f \in L^p(E)$ and $g \in L^q(E)$.

This inequality is derived from the inequality

$$s \leq \frac{s^p}{p} + \frac{1}{q}$$

for which equality holds only when $s = 1$. 

Hence
\[ 1 = s = \frac{|a|}{|b|^{q/p}} \Rightarrow |a| = |b|^{q/p}, \]
and since \( a = f/\|f\|_p \) and \( b = g/\|g\|_q \) we obtain
\[ \frac{|f|}{\|f\|_p} = \frac{|g|^{q/p}}{\|g\|_q^{q/p}}. \]
Applying the \( p^{th} \) power to both sides gives
\[ \frac{|f|^p}{\|f\|^p_p} = \frac{|g|^q}{\|g\|_q^q}. \]
and hence that
\[ |f|^p = \frac{\|f\|^p_p}{\|g\|_q^q} |g|^q. \]
Therefore equality holds only when \( |f|^p = c|g|^q \) for \( c = \|f\|^p_p/\|g\|_q^q \). \( \square \)