Math 541 F17: Homework Problems

HW 1. Due Friday September 15 at 4 p.m.

I.1 (pp49-50 Ed.1; pp48-49 Ed.2): 1.1, 1.2, 1.3, 1.7, 1.10

I.4 (pp51-52 Ed.1; p50 Ed.2): 4.1, 4.2 (the first part only where you are asked to show a collection of sets is a topology)

I.9 (p56 Ed.1; pp57 Ed.2): 9.2(v)

I:10 (p58 Ed.1; p58 Ed. 2): 10.1 (i)(ii)(iii) (assume in (i) that not both α and β are zero; ignore the piece in (ii) about $\overline{A}$ or $\overline{B}$ being compact)

I:13 From Lecture Note #2, Homework problems

  2A. Prove that the collection $\mathcal{B}$ of open balls $B_\rho(x)$ in a metric space satisfies the two conditions of Proposition 4.1.

  2B. In a metric space $\{X; d\}$ prove that each singleton set $\{x\}$ is closed.

HW 2. Due Friday September 22 at 4 p.m.

I:13 (p59 Ed.1; p59 Ed.2): 13.3, 13.5

I:16 From Lecture Note 3, the homework problems

  3A. Give a proof of Corollary 16.2 (on p.46 of Ed.1; this is Corollary 16.1 in Ed.2).

  3B. Give a proof of Proposition 16.2c (on p.63 in Ed.1, on p.65 in Ed.2).

II:2 (p110 Ed.1; pp108-109 Ed.2): 2.2, 2.3 (Ignore last part of 2.2 in Ed.2 that starts with "Set $D_1 = E_1$ and $D_{n+1} = D_n \triangle E_{n+1}$ ... ")

II:3 (p111 Ed.1) 3.3 or (p109 Ed.2) 3.2 (with $\mu(E) = \infty$ replaced by $\mu(E) = 1$ when E has countable complement, as this makes it more interesting).

II:3 From Lecture Note #5, the homework problems

  5A. For a measure μ on a σ-algebra $\mathcal{A}$, prove that if $\{E_n\}$ in $\mathcal{A}$ is monotone increasing and $E = \cup E_n$, then $\mu(E_n) \to \mu(E)$ as $n \to \infty$.

  5B. For a measure μ on a σ-algebra $\mathcal{A}$, prove that if $\{E_n\}$ in $\mathcal{A}$ is monotone decreasing, there exists $k \in \mathbb{N}$ such that $\mu(E_k) < \infty$, and $E = \cap E_n$, then $\mu(E_n) \to \mu(E)$ as $n \to \infty$; show that this is false if there is no $k \in \mathbb{N}$ such that $\mu(E_k) < \infty$.

HW 3. Due Friday September 29 at 4 p.m.

II.3: From Lecture Note #6, the homework problems

  6A. Give an example of a measure μ for which $\mu(B - A) = \mu(B) - \mu(A)$ fails when $\mu(A) = \infty$ and $A \subset B$.

  6B. If $\{\mu_\alpha : \alpha \in I\}$ is a finite or countable collection of measures on the same σ-algebra $\mathcal{A}$, then $\sum \mu_\alpha$ is a measure on $\mathcal{A}$.

II.4: (p111 in Ed.1, p112 in Ed.2), 4.1, 4.3

II:4 and II.5: From Lecture Note #7, the homework problems

  7A. For $f(x) = e^x$, find $\mu_{f,e}\big((a,b)\big)$.

  7B. For E a square of unit edge in $\mathbb{R}^2$, prove that $\mathcal{H}_{\alpha,\epsilon}(E) = 0$ for all $\alpha > 2$.

HW 4. Due Friday October 6 at 4 p.m.

II.8: From Lecture Note #10, the homework problem

10A. In proving that the Hausdorff measure μ_α is a Borel measure, we showed that every closed set E belongs to $\mathcal{A}_\alpha$. Where in the proof of Lemma 8.2 in Ed.1 (Lemma 8.1 in Ed. 2) did we use that E is closed? Explain by way of proof the necessity of E being closed.

II.9: From Lecture Note #11, the homework problems

11A. (i) Prove that the Lebesgue-Stieltjes function $\lambda_f\big((a,b)\big) = f(b) - f(a)$ is finitely additive on the sequential covering $\mathcal{Q}$ consisting of all the open subintervals of $\mathbb{R}$. (ii) Prove the function $\lambda(E) = (\operatorname{diam}(E))^\alpha$ for the Hausdorff outer measures is not finitely additive for any $\alpha \neq 1$.

11B. Prove that if λ is a measure on a semialgebra $\mathcal{Q}$, then λ is finitely additive.

HW 5 Due Friday October 13 at 4 p.m.

II.10. From Lecture Note #12, the homework problem

12A. Complete the proof of Proposition 10.1 in Lecture Note #12: show that the set $E_{\sigma\delta}$ satisfies $\mu_e(E) = \mu(E_{\sigma\delta})$.

II.11. (p113 in Ed.1) 11.1 or (pp117-118 in Ed.2) 11.7 (ignore the part about showing that $\mathcal{Q}_0 = \mathcal{P}(X)$); you are to prove (1) that the collection $\mathcal{Q}$ of finite unions of sets of the form $(a,b] \cap \mathbb{Q}$ for $0 \leq a < b \leq 1$ and the empty set is a semialgebra, (2) that λ is a measure on $\mathcal{Q}$ that is not σ-finite (you may assume that λ is defined on finite unions of sets of the form $(a,b] \cap \mathbb{Q}$ in the obvious way), (3) that μ_1 and μ_2 are indeed extensions of λ and (4) that μ_1 and μ_2 are different by exhibiting a set in $\mathcal{Q}_0 - \mathcal{Q}$ on which the two measures differ.

II.12. (p113 in Ed.1, p118 in Ed.2) 12.2, 12.4, 12.5 (you may assume that Lebesgue measure in is rotation invariant), and from Lecture Note #14, the homework problem

14A. Let $\mathcal{Q}_0$ be the smallest σ-algebra containing the semialgebra $\mathcal{Q}$ of $\frac{1}{2}$-closed dyadic cubes in $\mathbb{R}^N$. Prove that $\mathcal{Q}_0$ is the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}^N$. (See the updated Lecture Note #7 where in the $\frac{1}{2}$-closed dyadic cubes are defined.)

HW 6 Due Friday October 20 at 4 p.m.

II.13. (p115 in Ed.1, pp122-123 in Ed.2) 13.1 (look for typos), 13.2 (look for typos), 13.3 (Ignore in Ed.2 the part that starts with "Moreover ...."), and from Lecture Note #17, the homework problem

17A. If A is a Lebesgue measurable subset of $[0,1)$ and $y \in [0,1)$, then $A \bullet y$ is Lebesgue measurable and $\mu(A \bullet y) = \mu(A)$.

Mini Midterm Due Friday Oct 27 at 4 p.m. This mini midterm is to be done by you without the help of other students in the class, without the help of online resources, without the help of any books except the text and lecture notes, and without help from the instructor except to clarify what is being asked in these questions.

1. For a set X, a collection $\mathcal{M}$ in $\mathcal{P}(X)$ is called a monotone class if
   (i) for $\{E_n\} \subset \mathcal{M}$ with $E_{n+1} \subset E_n$ for all n, there holds $E = \cap E_n \in \mathcal{M}$, and
   (ii) for $\{E_n\} \subset \mathcal{M}$ with $E_n \subset E_{n+1}$ for all n, there holds $E = \cup E_n \in \mathcal{M}$.
   Let $\mathcal{A}$ be an algebra of sets in X and $\mathcal{N}$ the smallest σ-algebra containing $\mathcal{A}$. Prove that if $\mathcal{M}$ is a monotone class containing $\mathcal{A}$, then $\mathcal{N} \subset \mathcal{M}$. [Hint: The intersection of two monotone classes containing $\mathcal{A}$ is a monotone class containing $\mathcal{A}$, so you may assume WLOG that $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$. Proceed according to the following steps. First, for $A \in \mathcal{A}$ show that

   $$\{B \in \mathcal{M} : A \cup B \in \mathcal{M}\}$$

   is a monotone class containing $\mathcal{A}$. Second, for $B \in \mathcal{M}$ show that

   $$\{D \in \mathcal{M} : D \cup B \in \mathcal{M}\}$$

   is a monotone class containing $\mathcal{A}$. Third, for $A \in \mathcal{A}$ show that

   $$\{B \in \mathcal{M} : B - A \in \mathcal{M} \text{ and } A - B \in \mathcal{M}\}$$

   is a monotone class containing $\mathcal{A}$. Fourth, show that the monotone class $\mathcal{M}$ is an algebra. The last step is to show that $\mathcal{M}$ is a σ-algebra.]

2. Let $\{X, \mathcal{A}, \mu\}$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{A} : \mu(N) = 0\}$ and

   $$\overline{\mathcal{A}} = \{E \cup F : E \in \mathcal{A}, F \subset N \text{ for some } N \in \mathcal{N}\}.$$

   Prove that $\overline{\mathcal{A}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ from $\mathcal{A}$ to $\overline{\mathcal{A}}$ in which $\overline{\mu}$ is a complete measure. [Hint: an extension of μ is $\overline{\mu}(E \cup F) = \mu(E)$.]

3. Let μ_e be the Lebesgue outer measure on $\mathbb{R}^N$. For any $E \subset \mathbb{R}^n$ and a fixed $y \in \mathbb{R}^N$, let $E + y$ be the translation of E by y. Define $\nu_e(E) = \mu_e(E + y)$. DO NOT ASSUME that Lebesgue outer measure is translation invariant in answering this question.

   (i) Prove that ν_e is an outer measure on $\mathbb{R}^N$.
   (ii) By the Carathéodory procedure, there is a σ-algebra $\mathcal{N}$ in $\mathbb{R}^N$, the collection of ν_e-measurable sets, for which the restriction of ν_e to $\mathcal{N}$ is a complete measure. Prove that $\mathcal{N}$ contains the Borel σ-algebra of $\mathbb{R}^N$. [Hint: for $\mathcal{M}$, the σ-algebra of μ_e-measurable sets, show that $\mathcal{N} = \mathcal{M} - y$.]

4. Let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^N$, and μ the Lebesgue measure on $\mathbb{R}^N$.
   (i) For any $E \in \mathcal{M}$ and any $\epsilon > 0$, prove there is a closed set C and an open set O such that $C \subset E \subset O$ and $\mu(O - C) \leq \epsilon$.
   (ii) For any $E \in \mathcal{M}$, prove there is an $\mathcal{F}_\sigma$ set A and a $\mathcal{G}_\delta$ set B such that $A \subset E \subset B$ and $\mu(B - A) = 0$.

HW 8 Due Friday November 3 at 4 p.m.

II.15. From Lecture Note #23, the homework problems

23A. A Borel measure μ is called **inner regular** on a Borel set E if

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ is compact}\}.$$

Prove that if a Radon measure μ on $\mathbb{R}^N$ is inner regular on all open sets, then μ is inner regular on all Borel sets. [Hint: By the Corollary in Lecture Note #23 the Radon measure is a regular Borel measure. Consider two cases for a Borel set E: case 1 $\mu(E) < \infty$, and case 2 $\mu(E) = \infty$. For case 1 use the regularity of the Radon measure to approximate E by an open set U, and then use the assumed inner regularity of the Radon measure on open sets to find a compact subset K of E for which $\mu(K) \geq \mu(E) - \epsilon$. For case 2 use the sets $E_n = B(0, n) \cap E$ for $n \in \mathbb{N}$ and case 1, i.e., show there exist compact subsets of E with larger and larger measure.]

23B. Regularity of a Borel measure μ is determined by its inner and outer regular on Borel sets. The definition of inner regularity on a Borel set is given in Problem 23A. The definition of **outer regularity** of μ on a Borel set E is that

$$\mu(E) = \inf\{\mu(O) : O \supset E, O \text{ open}\}$$

holds. A Borel measure is called "**regular**" if it is outer regular on all Borel sets and inner regularity on all Borel sets. (This is not the definition of regular in the text.) We proved that Lebesgue measure μ on $\mathbb{R}^N$ is outer regular on every Borel set. If Lebesgue measure is inner regular on open sets, then by 23A it would be inner regular on all Borel sets, making Lebesgue measure a "regular" Borel measure. Prove that Lebesgue measure is inner regular on open sets.

HW 9 Due ***MONDAY*** Nov. 20 at 4 p.m.

III.1 (pp160-161 Ed.1, p170 in Ed.2): 1.1, 1.3, 1.4, and (Ed.1, not found in Ed.2) 1.6: The sum $f + g$ and the product fg of two measurable functions $f, g : E \to \mathbb{R}^*$ are measurable. [Hint: define $f(x) + g(x)$ to be a fixed constant when $f(x) + g(x)$ is $\infty - \infty$ or $-\infty + \infty$.]

III.5. From Lecture Note #27, the homework problem

27A. Prove that the difference of two real-valued quasi-continuous functions is quasi-continuous.

III.6. From Lecture Note #28, the homework problems

28A. For nonnegative simple functions $f, g : E \to \mathbb{R}^*$, prove that if $f \leq g$ a.e. in E, then

$$\int_E f \, d\mu \leq \int_E g \, d\mu.$$

28B. For Lebesgue integrable nonnegative simple functions $f, g : E \to \mathbb{R}^*$, prove that

$$\int_E (\alpha f + \beta g) d\mu = \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu$$

for all $\alpha, \beta \in \mathbb{R}$.

HW 10 Due Friday Dec. 1 at 4 p.m.

III.7 (pp163-164 in Ed.1): 7.2, 7.3, 7.4, 7.6, (pp175-176 in Ed.2) 7.1, 7.2, 7.3, 7.5

III.10 (pp165-166 in Ed.1): 10.1, 10.7, (pp178-180 in Ed.2) 10.1, 10.11

III.16. From Lecture Note #33, the homework problems

33A. For an integrable function f prove that $E \to \int_E f \, d\mu$ is a signed measure, i.e., prove
that $E \to \int_E f^+ d\mu$ and $E \to \int_E f^- d\mu$ are measures.

33B. Prove the four properties listed for positive sets.

HW 11 Due ***MONDAY*** Dec. 11 at 4 p.m.

III.17 (p169 in Ed.1, p187 in Ed.2) 17.1, 17.2, and from Lecture Note #34, the homework
problem

34A. Suppose μ and ν are measures on the same σ-algebra $\mathcal{A}$ in a set X. Prove that if
μ is σ-finite and ν is finite, and if $\nu \ll \mu$, then there exists a measurable function
f such that

$$\nu(E) = \int_E f \, d\mu, \ E \in \mathcal{A}.$$

Hint: Use the σ-finiteness of μ to generate a sequence of measurable subset $\{E_n\}$
that satisfy $E_n \subset E_{n+1}$, $\mu(E_n) < \infty$, and $\cup E_n = X$. For each n define $\mu_n(E) =$
$\mu(E \cap E_n)$ and $\nu_n(E) = \nu(E \cap E_n)$ for $E \in \mathcal{A}$. Show that μ_n and ν_n are finite
measures that satisfy $\nu_n \ll \mu_n$, and then apply the case of the Lebesgue-Radon-
Nikodym Theorem proved in Lecture Note #34 to get a measurable function f_n
that satisfies

$$\nu_n(E) = \int_E f_n \, d\mu, \ E \in \mathcal{A}.$$

Set $f = \lim f_n$. Show that

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu$$

and

$$\nu(E) = \lim_{n \to \infty} \nu_n(E).$$

Put the pieces together to complete the proof.

V.1 From Lecture Note #35, the homework problem

35A. For real $p \geq 1$, prove that the function $\varphi(t) = t^p$ defined on $(0, \infty)$ is convex.