We have been discussing what a forward-complete solution $t \to \phi_t(p)$ of $\dot{x} = f(x)$ “converges to,” what we call its omega limit set $\omega(p)$.

Whatever we prove for the omega limit set is also true for the alpha limit set, so we will focus just on the omega limit set.

Last time we proved that the omega limit set is closed and invariant. Under an additional hypothesis, we can say more about its topological properties.

We say a subset $V \subset \mathbb{R}^n$ has **compact closure** if its closure $\overline{V}$ is compact. In particular, every bounded set has compact closure.

Recall that in $\mathbb{R}^n$ compact is the same as bounded and closed, which we will use tacitly.

**Proposition 1.168.** Suppose for $p \in \mathbb{R}^n$ that the solution $\phi_t(p)$ is forward complete. If \( \{ \phi_t(p) : t \geq 0 \} \) has compact closure, then $\omega(p)$ is nonempty, compact, and connected.

**Proof.** First we show that $\omega(p)$ is nonempty.

By the compact closure hypothesis, the sequence \( \{ \phi_n(p) \}_{n=1}^{\infty} \) lies in a compact set, and therefore has a convergent subsequence \( \{ \phi_{n_i}(p) : i \in \mathbb{N} \} \) with $n_1 < n_2 < \cdots$ and limit $x$.

This says that $x \in \omega(p)$, so that $\omega(p)$ is nonempty.

Second we show that $\omega(p)$ is compact.

By Proposition 1.167, the nonempty $\omega(p)$ is closed, and being a subset of a the compact closure of the forward orbit, it is compact.

Last we prove that $\omega(p)$ is connected by way of contradiction: suppose there are two disjoint open sets $U$ and $V$ whose union contains $\omega(p)$ with $\omega(p) \cap U \neq \emptyset$ and $\omega(p) \cap V \neq \emptyset$.

Thus there is some $t_1 \geq 0$ such that $\phi_{t_1}(p) \in U$, and there is some $t_2 > t_1$ such that $\phi_{t_2}(p) \in V$.

The set $K = \{ \phi_t(p) : t_1 \leq t \leq t_2 \}$ is the continuous image of an connected interval, and hence connected.

Could $K$ belong to $U \cup V$? No, because $U$ and $V$ are disjoint while $K$ is connected with one endpoint in $U$ and the other endpoint in $V$. [Sketch a picture of this.]

So there exist $\tau_1 > 0$ such that $\phi_{\tau_1}(p) \notin U \cup V$.

Now we repeat the argument, starting with $t_1 \geq \tau_1 + 1$ such that $\phi_{t_1} \in U$ (because $\omega(p) \cap U \neq \emptyset$) and $t_2 > t_1$ such that $\phi_{t_2}(p) \in V$ (because $\omega(p) \cap V \neq \emptyset$).

Thus there is $\tau_2 > \tau_1 + 1$ such that $\phi_{\tau_2}(p) \notin U \cup V$.

Continuing this argument generates a strictly increasing sequence \( \{ \tau_i \} \) going to $\infty$ where $\phi_{\tau_i}(p) \notin U \cup V$ for all $i \in \mathbb{N}$.

The sequence $\{ \phi_{\tau_i}(p) \}$ lies in a compact set, and so has a convergence subsequence with limit point $y \notin U \cup V$. 

But this limit point belongs to $\omega(p)$, contradicting $\omega(p) \subset U \cup V$. □

The omega limit set $\omega(p)$ for a point $p$ in a (complete) flow $\phi$ in $\mathbb{R}^n$ can be very complicated when $n \geq 3$; it can indeed be fractal.

But in $\mathbb{R}^2$, we have a complete understanding of the possibilities for $\omega(p)$ because of the following deep fact from Algebraic Topology.

The Jordan Curve Theorem 1.173. A simple closed continuous curve in the plane divides the plane into two connected components, one bounded (the “inside” of the curve) and one unbounded (the “outside” of the curve), each with the curve as its boundary.

If a (complete) planar ODE $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, has a periodic solution starting at $p$, then the orbit $\{\phi_t(p) : t \in \mathbb{R}\}$ is simple closed continuous curve, so that any solution starting inside the periodic orbit must stay inside it forever by Uniqueness of Solutions of IVPs.

The following result gives sufficient conditions for the existence of a periodic orbit for a planar ODE.

Poincaré-Bendixson Theorem 1.174. Let $\phi_t$ be the flow of $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, and suppose $p \in \mathbb{R}^2$ is forward complete. If $\omega(p)$ is compact and does not contain an equilibrium of $\dot{x} = f(x)$, then $\omega(p)$ is a periodic orbit.

This begs the question: what happens if $\omega(p)$ contains an equilibrium?

To answer this we need some additional notions.

A set $S \subset \mathbb{R}^n$ is positively invariant for (a complete) $\dot{x} = f(x)$ if $S$ contains the forward orbit $\{\phi_t(p) : t \geq 0\}$ of each of its elements $p \in S$.

An orbit $\{\phi_t(x) : t \in \mathbb{R}\}$ for a complete solution $\phi_t(x)$ whose $\alpha$-limit set is an equilibrium $p$ and whose $\omega$-limit set is an equilibrium $q$ is said to connect $p$ to $q$.

When $p = q$ and $x \neq p$, we call $\{\phi_t(x) : t \in \mathbb{R}\}$ a homoclinic orbit.

When $p \neq q$ and $x \notin \{p, q\}$, we call $\{\phi_t(x) : t \in \mathbb{R}\}$ a heteroclinic orbit.

[Sketch pictures of these.]

These connecting orbits, when they exist, form part of the “skeleton” of the phase portrait.

Theorem 1.175. Suppose $\phi_t$ is a (complete) flow on $\mathbb{R}^2$, and $S \subset \mathbb{R}^2$ is a positively invariant set with compact closure. If $p \in S$ and $\phi_t$ has at most finitely many equilibria in $S$, then $\omega(p)$ is

(a) an equilibrium,
(b) a periodic orbit, or
(c) a union of finitely many equilibria and a nonempty finite or countable infinite set of connecting orbits.

We will not give a proof of this Theorem, although you do have a homework problem (Exercise 1.176) concerning part (c).

We will give a proof of the Poincaré-Bendixson Theorem in a future Lecture.

Special attention is given to a periodic orbit that is the $\omega$-limit set of a non-periodic point.
Definition 1.178. A limit cycle $\Gamma$ is a periodic orbit that is either the $\omega$-limit set or the $\alpha$-limit set of some point in phase space not in $\Gamma$.

Extensions of the Poincaré Bendixson Theorem gives conditions sufficient for the existence of a periodic orbit and of a limit cycle.

An annular region of the plane is a set $V \subset \mathbb{R}^2$ homeomorphic to the set

$$\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}.$$ 

Note that an annular region is a compact set.

Theorem 1.179. If the (complete) flow $\phi_t$ in $\mathbb{R}^2$ has a positively invariant annular region $S$ that is equilibrium-free, then $S$ contains at least one periodic orbit. If, in addition, some point in the interior of $S$ is in the forward orbit of a point on the boundary of $S$, then $S$ contains at least one limit cycle.

Remark. The additional condition is to prevent the boundary of $S$ from being a periodic orbit and to prevent every orbit in $S$ from being periodic.

[Sketch a picture of this Theorem.]

Proof. The positively invariant annular (closed) region $S$ is compact, and so has compact closure.

Thus for any $x \in S$ we have by Proposition 1.168 that $\omega(x)$ is nonempty and compact.

Fix $x \in S$.

By the compactness of $S$, every $\omega$-limit point of the forward orbit of $x$ belongs to $S$.

Thus $\omega(x) \subset S$.

Since $S$ is equilibrium-free, we have by the Poincaré-Bendixson Theorem that $\omega(x)$ is a periodic orbit in $S$.

Now assume in addition that $x$ is in the interior of $S$ and is also in the forward orbit of a point $y$ on the boundary of $S$: there exist $t_* > 0$ such that $\phi_{t_*}(y) = x$.

We will show that the orbit through $x$ is not periodic.

To this end, suppose there exists $t > t_*$ such that $\phi_t(y)$ is on the boundary of $S$.

Since $S$ is equilibrium-free, the point $\phi_t(y)$ is regular, and the Rectification Lemma applies to give orbits near $\phi_t(y)$ in $S$ that leave $S$, contradicting the positive invariance of $S$.

Thus $\phi_t(y)$ belongs to the interior of $S$ for all $t \geq t_*$.

It follows that $\phi_{t+t_*}(y) = \phi_t(\phi_{t_*}(y)) = \phi_t(x)$ belongs to the interior of $S$ for all $t \geq 0$.

If the orbit through $x$ were periodic, then as $y$ belongs to the orbit through $x$, there would be a time $t > 0$ such that $y = \phi_t(x)$, but $\phi_t(x)$ belongs to the interior of $S$ for all $t > 0$, a contradiction.

Therefore the orbit passing through $x$ is not periodic, and hence the periodic orbit $\omega(x)$ is a limit cycle. \qed
Homework. Exercises 1.169, 1.170 on p.93; Exercise 1.176 on p.95.