We consider two examples where we use Theorem 1.179 to get the existence of a periodic orbit (limit cycle) by finding a positively invariant annular region that is equilibrium-free (with the additional hypothesis that a point in the interior is in the forward orbit of a point on the boundary).

Example. Consider

\[
\begin{align*}
\dot{x} &= -y + x(1 - x^2 - y^2), \\
\dot{y} &= x + y(1 - x^2 - y^2).
\end{align*}
\]

The only equilibrium of this ODE is located at \((0,0)\).

The annular region

\[ \mathcal{S} = \{(x, y) : \frac{1}{4} \leq x^2 + y^2 \leq 4\} \]

then does not contain an equilibrium of the ODE.

We will show that every solution that starts close to and on the outside of \(\mathcal{S}\) crosses the boundary of \(\mathcal{S}\) to its interior.

This implies that \(\mathcal{S}\) is positively invariant and the additional condition of Theorem 1.179.

An “outer” normal vector to \(\partial \mathcal{S}\), the boundary of \(\mathcal{S}\), is given by the vector field

\[ N(x, y) = (x, y, x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \]

evaluated at a point \((x, y) \in \partial \mathcal{S}\).

Taking the dot product with this normal vector with the vector field of the ODE at each point \((x, y) \in \partial \mathcal{S}\) gives

\[
\langle N(x, y), f(x, y) \rangle = (x, y), (-y + x(1 - x^2 - y^2), x + y(1 - x^2 - y^2))
\]

\[
= -xy + x^2(1 - x^2 - y^2) + xy + y^2(1 - x^2 - y^2)
\]

\[
= (x^2 + y^2)(1 - x^2 - y^2).
\]

For \((x, y)\) on the component \(1/4 = x^2 + y^2\) of \(\partial \mathcal{S}\) this dot product is positive, meaning the angle the outer normal \(N(x, y)\) and vector field \(f(x, y)\) makes is acute, i.e., between \(0\) and \(\pi/2\).

This means that a solution starting on this component of \(\partial \mathcal{S}\) moves into the interior of \(\mathcal{S}\).

For \((x, y)\) on the component \(x^2 + y^2 = 4\) of \(\partial \mathcal{S}\), the dot product is negative, meaning the angle the outer normal \(N(x, y)\) makes with \(f(x, y)\) lies between \(\pi/2\) and \(\pi\).

This means that a solution starting on this component of \(\partial \mathcal{S}\) moves into the interior of \(\mathcal{S}\).

All together this means that \(\mathcal{S}\) is positively invariant and that there is a point in the interior of \(\mathcal{S}\) that is on the forward orbit of a point on \(\partial \mathcal{S}\).

By Theorem 1.179, there exists a limit cycle of the ODE in \(\mathcal{S}\).
By switching to polar coordinates, we can explicitly identify the limit cycle.

The ODE in polar coordinates \( (r, \theta) \) is
\[
\begin{align*}
\dot{r} &= x \dot{x} + y \dot{y} \\
&= x(-y + x(1 - x^2 - y^2)) + y(x + (1 - x^2 - y^2)) \\
&= (x^2 + y^2)(1 - x^2 - y^2) \\
&= r^2(1 - r^2) \\
\dot{\theta} &= xy - y \dot{x} \\
&= x(x + y(1 - x^2 - y^2)) - y(-y + x(1 - x^2 - y^2)) \\
&= x^2 + y^2 \\
&= r^2.
\end{align*}
\]

The singularity at \( r = 0 \) is removable (because the origin is an equilibrium, see Proposition 1.35). Removing it gives the “decoupled” system
\[
\begin{align*}
\dot{r} &= r(1 - r^2), \\
\dot{\theta} &= 1,
\end{align*}
\]
which integrates to give the flow
\[
\phi_t(r, \theta) = \left( \left( \frac{r^2 e^{2t}}{1 - r^2 + r^2 e^{2t}} \right)^{1/2}, \theta + t \right).
\]

The limit cycle is the solution \( \phi_t(1, 0) \) which is \( 2\pi \)-periodic, i.e., \( \phi_{2\pi}(1, 0) = (1, 2\pi) \), where we view the angle modulo \( 2\pi \), or the flow on the cylinder \( \mathbb{R} \times \mathbb{T} \).

Returning the periodic solution to rectangular coordinates gives the periodic solution as
\[
t \mapsto (\cos t, \sin t).
\]

The limit cycle appears to asymptotically stable because \( \dot{r} < 0 \) when \( 1 < r < 2 \) and \( \dot{r} > 0 \) when \( 0 < r < 1 \).

We can verify this by a Poincaré map.

The postive \( x \)-axis (i.e., \( r > 0 \) and \( \theta = 0 \)) is a Poincaré section for the flow with associated Poincaré map
\[
P(x) = \left( \frac{x^2 e^{4\pi}}{1 - x^2 + x^2 e^{4\pi}} \right)^{1/2}.
\]
Here \( P(1) = 1 \) and \( P'(1) = e^{-4\pi} \) which is a number inside the unit circle, so that the periodic orbit is indeed asymptotically stable.

We call such an orbit a hyperbolic stable limit cycle.

Example. Consider the second order ODE
\[
\dot{\theta} + \lambda \dot{\theta} + \sin \theta = \mu
\]
where \( \lambda > 0 \) and \( \mu \) are parameters, and \( \theta \) is an angular variable, i.e., defined modulo \( 2\pi \).
Converting the ODE into a system of first-order ODEs by setting \( v = \dot{\theta} \) gives

\[
\begin{align*}
\dot{\theta} &= f_1(\theta, v) = v, \\
\dot{v} &= f_2(\theta, v) = -\sin \theta + \mu - \lambda v.
\end{align*}
\]

Because \( \theta \) is an angular variable, the natural phase space is the cylinder \( \mathbb{T} \times \mathbb{R} \).

We will show that if \( |\mu| > 1 \), then there exists a “globally attracting” limit cycle, where “globally attracting” means that the \( \omega \)-limit set of every point in phase space is the limit cycle.

When \( |\mu| > 1 \), the system has no equilibrium because \( v = 0 \) and \(-\sin \theta + \mu - \lambda v = -\sin \theta + \mu \neq 0\).

The quantity \(-\sin \theta + \mu - \lambda v\) is negative for all sufficiently large positive \( v \) and is positive for all sufficiently large negative \( v \).

Thus there are values \( v_+ \) and \( v_- \) such that the compact set

\[
\mathcal{A} = \{(\theta, v) : v_- \leq v \leq v_+\} \subset \mathbb{T} \times \mathbb{R}
\]

is positive invariant with every solution starting on the boundary moving immediately into the interior.

Since \( \mathcal{A} \) is diffeomorphic to an annular region of a plane, we can push forward the vector field on \( \mathcal{A} \) to a vector field on an annular region \( \mathcal{S} \) of the plane which is positively invariant and every solution starting on the boundary moves immediately into the interior.

By Theorem 1.179 there then exists a limit cycle in \( \mathcal{A} \).

To prove that there is only one limit cycle in \( \mathcal{A} \), we will use two facts (that you will prove in upcoming Homework Problems): (1) if the divergence of a vector field is negative, then the flow of the vector field contracts volume (Exercise 1.200), and (2) every periodic orbit in an open ball of the plane surrounds at least one equilibrium (Exercise 1.217).

The divergence of the vector field here is

\[
\text{div} f(x, y) = (f_1)_\theta + (f_2)_v = 0 - \lambda = -\lambda < 0,
\]

and its flow contracts area (volume in 2-dimensions).

If a periodic orbit is lies in a homeomorphic image of an open ball of the plane, then the periodic orbit contains at least one equilibrium inside the periodic orbit.

With \( |\mu| > 1 \) there are no equilibria, and so any periodic orbit must wrap around the cylinder.

If there were two periodic orbits, then the annular region \( \Omega \) between them in the cylinder would be an invariant set.

But the area of \( \Omega \) is preserved by the flow because \( \phi_t(\Omega) = \Omega \) for all \( t \geq 0 \), contradicting the contraction of area by the flow.

Thus there is only one periodic orbit, and it is globally attracting.