We have used the Implicit Function Theorem to prove that the regular level sets are submanifolds, and that a Poincaré map is smooth.

**Implicit Function Theorem 1.259.** Suppose \( X, Y, \) and \( Z \) are Banach spaces, \( U \subset X, V \subset Y \) are open set, \( F : U \times V \rightarrow Z \) is a \( C^1 \) function, and \( (x_0, y_0) \in U \times V \) satisfies \( F(x_0, y_0) = 0 \). If \( F_x(x_0, y_0) : X \rightarrow Z \) has bounded inverse, then there is a product neighbourhood \( U_0 \times V_0 \subset U \times V \) with \( (x_0, y_0) \in U_0 \times V_0 \) and a \( C^1 \) function \( \beta : V_0 \rightarrow U_0 \) such that \( \beta(y_0) = x_0 \) and \( F(\beta(y), y) = 0 \) for all \( y \in V_0 \). Moreover, if \( F(x, y) = 0 \) for \( (x, y) \in U_0 \times V_0 \) then \( x = \beta(y) \).

**Proof.** Define \( L : X \rightarrow Z \) by \( Lz = [F_x(x_0, y_0)]^{-1}z \) and \( G : U \times V \rightarrow X \) by \( G(x, y) = x - LF(x, y) \).

The function \( G \) is \( C^1 \), and \( F(x, y) = 0 \) if and only if \( G(x, y) = x \).

Also, \( G(x_0, y_0) = x_0 \) and \( G_x(x_0, y_0) = I - LF_x(x_0, y_0) = 0 \).

Since \( G \) is \( C^1 \) there is a product neighbourhood \( U_0 \times V_1 \) whose factors are metric balls, \( U_0 \subset U \) centered at \( x_0 \) and \( V_1 \subset V \) centered at \( y_0 \), for which \( \|G_x(x, y)\| < 1/2 \) for all \( (x, y) \in U_0 \times V_1 \).

Let \( \delta > 0 \) be the radius of the ball \( U_0 \).

Since the function \( y \mapsto F(x_0, y) \) is continuous and vanishes at \( y_0 \), there is a metric ball \( V_0 \subset V_1 \) centered at \( y_0 \) such that \( \|L\| \|F(x_0, y)\| < \delta/2 \) for all \( y \in V_0 \).

For \( (x, y) \in U_0 \times V_0 \), we have by the Mean Value Theorem 1.226 that

\[
\|G(x, y) - x_0\| = \|G(x, y) - G(x_0, y) + G(x_0, y) - x_0\| \\
\leq \|G(x, y) - G(x_0, y)\| + \|LF(x_0, y)\| \\
\leq \sup_{u \in U_0} \|G_x(u, y)\| \|x - x_0\| + \frac{\delta}{2} \leq \delta.
\]

This implies that \( G(x, y) \in \overline{U_0} \), and so \( G : \overline{U_0} \times V_0 \rightarrow \overline{U_0} \).

Again by the Mean Value Theorem we have

\[
\|G(x_1, y) - G(x_2, y)\| \leq \sup_{u \in U_0} \|G_x(u, y)\| \|x_1 - x_2\| \leq \frac{\|x_1 - x_2\|}{2},
\]

and hence \( G \) is a uniform contraction.

Then by the Uniform Contraction Theorem, there exists a unique \( C^1 \) function \( y \mapsto \beta(y) \) (the unique fixed point for each \( y \)) defined on the open ball \( V_0 \) with range in \( U_0 \) such that \( \beta(y_0) = x_0 \) and \( G(\beta(y), y) = \beta(y) \) for all \( y \in V_0 \).

By the definition of \( G \), we have that \( \beta(y) = \beta(y) - LF(\beta(y), y) \), and since \( L \) is has bounded inverse, we conclude that \( F(\beta(y), y) = 0 \) for all \( y \in V_0 \).

The proof of the Existence, Uniqueness, and Continuous Dependence on Parameters Theory is a consequence of this Implicit Function Theorem.
Existence, Uniqueness, and Continuous Dependence on Parameters

Theorem 1.260. Suppose the function \( f : J \times \Omega \times \Lambda \to \mathbb{R}^n \) is \( C^1 \). For \( t_0 \in J \), \( x_0 \in \Omega \), and \( \lambda_0 \in \Lambda \), there exist open sets \( J_0 \subset J \), \( \Omega_0 \subset \Omega \), and \( \Lambda_0 \subset \Lambda \) such that \((t_0, x_0, \Lambda_0) \in J_0 \times \Omega_0 \times \Lambda_0\), and a unique \( C^1 \) function \( \sigma : J_0 \times \Omega_0 \times \Lambda_0 \to \mathbb{R}^n \) given by \((t, x, \lambda) \mapsto \sigma(t, x, \lambda)\) such that \( \sigma(0, x, \lambda) = x \) and \( t \mapsto \sigma(t, x, \lambda) \) is a solution of \( \dot{x} = f(t, x, \lambda) \).

Proof. We first show that by scaling the time variable, we can assume the maximal interval of existence contains \([-1, 1]\).

Suppose \( \sigma \) is a solution of \( \dot{x} = f(t, x, \lambda) \), \( x(t_0) = x_0 \), where \( \sigma \) is defined on \([t_0 - \delta, t_0 + \delta]\) for some small \( \delta > 0 \).

For the scaled time variable \( \tau = (t - t_0)/\delta \), the function \( z(\tau) = \sigma(\delta \tau + t_0) - x_0 \) satisfies \( z(0) = 0 \) and

\[
\frac{dz}{d\tau}(\tau) = \delta \frac{d\sigma}{dt}(\delta \tau + t_0) = \delta f(\delta \tau + t_0, \sigma(\delta \tau + t_0), \lambda) = \delta f(\delta \tau + t_0, z + x_0, \lambda)
\]

for \(-1 \leq \tau \leq 1\), at least when \( z + x_0 \in \Omega \).

Conversely, if \( dz/d\tau = \delta f(\delta \tau + t_0, z + x_0, \lambda) \) has a solution defined on \(-1 \leq \tau \leq 1\), then \( \dot{x} = f(t, x, \lambda) \) has a solution defined on \([t_0 - \delta, t_0 + \delta]\).

Choose an open ball centered at the origin with radius \( r \) inside \( \Omega \) and let \( U \) denote the open ball centered at the origin of radius \( r/2 \).

Define Banach spaces

\[
X = \{ \phi \in C^1([-1, 1], \mathbb{R}^n) : \phi(0) = 0 \}, \quad Y = C([-1, 1], \mathbb{R}^n)
\]

where the norm on \( Y \) is the usual supremum norm, and the norm on \( X \) is the \( C^1 \) norm

\[
\|\phi\|_1 = \|\phi\|_0 + \|\phi'\|_0.
\]

Let \( X_0 \) denote the open subset of \( X \) consisting of those elements of \( X \) whose ranges are in the open ball \( U \) at the origin of radius \( r/2 \).

Consider the function \( F : (-1, 1) \times J \times U \times \Lambda \times X_0 \to Y \) given by

\[
F(\delta, t, x, \lambda, \phi)(\tau) = \phi'(\tau) - \delta f(\delta \tau + t, \phi(\tau) + x, \lambda).
\]

We will apply the Implicit Function Theorem to this function.

To do this we need to show that \( F \) is \( C^1 \).

The second summand in \( F \) is \( C^1 \) by the Omega Lemma (see Exercise 1.224).

For \( F \) to be \( C^1 \) it remains to show that the map \( d \) given by \( \phi \mapsto \phi' \) is \( C^1 \).

For \( \phi \in X \) we have \( \phi' \in Y \), and the map \( d \) is linear.

Because

\[
\|d\phi\|_0 = \|\phi'\|_0 \leq \|\phi'\|_0 + \|\phi\|_0 = \|\phi\|_1,
\]

the linear operator \( d \) is continuous, hence bounded.
Since the linear operator \( d : X \to Y \) is linear and bounded, it is its own derivative (property (v) of the derivative).

Thus we have that \( d \) is \( C^1 \).

If \( (t_0, x_0, \lambda_0) \in J \times \Omega \times \Lambda \), then \( F(0, t_0, x_0, \lambda_0, \phi(0))(\tau) = 0 \) where \( \phi(0) = 0 \).

If we set \( \delta = 0 \) before we compute the partial derivative of \( F \) with respect to \( \phi \), we see that

\[
F_{\phi}(0, t_0, x_0, \lambda_0, 0) = d.
\]

We will show that \( d \) has bounded inverse, and then apply the Implicit Function Theorem.

With \( y \in Y \) continuous on \([-1, 1]\), we can define a linear operator \( L : Y \to X \) by

\[
(Ly)(\tau) = \int_0^\tau y(s)ds.
\]

By the Fundamental Theorem of Calculus,

\[
((d \circ L)(y))(\tau) = \frac{d}{d\tau} \int_0^\tau y(s)ds = y(\tau),
\]

and for \( \psi \in X \) (for which \( \psi(0) = 0 \)),

\[
((L \circ d)(\psi))(\tau) = \int_0^\tau \frac{d}{ds} \psi(s)ds = \psi(\tau).
\]

Thus \( L \) is an inverse for \( d \).

Is \( L \) bounded? That is, do we have \( \|L\| = \sup\{\|Ly\|_1 : y \in Y, \|y\|_0 = 1\} < \infty \)?

Since

\[
\|Ly\|_0 = \sup_{\tau \in [-1,1]} \left| \int_0^\tau y(s)ds \right| \leq \sup_{\tau \in [-1,1]} \int_0^\tau |y(s)|ds \leq \sup_{\tau \in [-1,1]} \int_0^\tau \|y\|_0ds = \|y\|_0,
\]

then

\[
\|Ly\|_1 = \|Ly\|_0 + \|(d \circ L)y\|_0 \leq \|y\|_0 + \|y\|_0 = 2\|y\|_0,
\]

so that \( \|L\| \leq 2 \).

By the Implicit Function Theorem, there is an open set \( K_0 \times J_0 \times \Omega_0 \times \Lambda_0 \) containing \((0, t_0, x_0, \lambda_0)\) and a unique \( C^1 \) function \((\delta, t, x, \lambda) \to \beta(\delta, t, x, \lambda)\) with range in \( X_0 \) such that \( \beta(0, t_0, x_0, \lambda_0) = 0 \) and

\[
F(\delta, t, x, \lambda, \beta(\delta, t, x, \lambda)) = 0 \text{ for all } (\delta, t, x, \lambda) \in K_0 \times J_0 \times \Omega_0 \times \Lambda_0.
\]

Thus there is \( \delta > 0 \) such that \( \tau \to z(\tau, t_0, x_0, \lambda_0) = \beta(\delta, t_0, x_0, \lambda_0)(\tau) \) is the unique solution of the IVP that depends \( C^1 \) on the its initial conditions and parameters. \( \square \)