We consider \( \dot{u} = f(u, \lambda) \) for \( u \in \mathbb{R}^2 \) and \( \lambda \in \mathbb{R} \).

**Definition 8.20.** The pair \((0,0) \in \mathbb{R}^2 \times \mathbb{R}\) consisting of an equilibrium \( u = 0 \) for \( \dot{u} = f(u, 0) \) is called a **Hopf point** if there is a curve \( c : \lambda \mapsto c(\lambda) \) defined on some open interval containing 0, such that

(i) \( c(0) = 0 \) and \( f(c(\lambda), \lambda) = 0 \) for all \( \lambda \) near 0, and

(ii) the family of linear maps \( D_u f(c(\lambda), \lambda) \) has a pair of nonzero complex conjugate eigenvalues \( \alpha(\lambda) \pm i\beta(\lambda) \) with \( \alpha(0) = 0 \), \( \alpha'(0) \neq 0 \), and \( \beta(0) \neq 0 \).

We say that a set \( S \subset \mathbb{R}^2 \) has radii \( (r_1, r_2) \) relative to a point \( p \) if \( r_1 \geq 0 \) is the radius of the smallest closed ball centered at \( p \) that contains \( S \), and \( r_2 \) is the distance from \( S \) to \( p \).

**Definition 8.21.** The ODE \( \dot{u} = f(u, \lambda) \) has a **supercritical Hopf bifurcation** at a Hopf point \((u_0, \lambda_0)\) with associated curve \( C : \lambda \mapsto c(\lambda) \) if there are positive numbers \( \epsilon_0 \), \( K_1 \), and \( K_2 \) such that for each \( 0 < \lambda < \epsilon_0 \) the ODE \( \dot{u} = f(u, \lambda) \) has a hyperbolic limit cycle with radii

\[
(K_1 \sqrt{\lambda} + O(\lambda), K_2 \sqrt{\lambda} + O(\lambda))
\]

relative to the rest point \( u = c(\lambda) \).

The Hopf bifurcation is **subcritical** if there is a similar limit cycle for \(-\epsilon_0 < \lambda < 0\).

To focus on one of several similar cases, we will assume WLOG that

\[
\alpha'(0) > 0, \quad \beta(0) > 0.
\]

That is, we assume that eigenvalue \( \alpha(\lambda) + i\beta(\lambda) \) travels in the upper half plane from the left side to the right side of the imaginary axis as \( \lambda \) increases from negative to positive.

Then for \( \lambda < 0 \), the equilibrium \( u = c(\lambda) \) of \( \dot{u} = f(u, \lambda) \) is an asymptotically stable spiral point, while for \( \lambda > 0 \), the equilibrium \( u = c(\lambda) \) of \( \dot{u} = f(u, \lambda) \) is an unstable spiral point.

Furthermore, we may assume WLOG that \( c(\lambda) = 0 \) for all \( \lambda \) close to 0 because we can replace \( \dot{u} = f(u, \lambda) \) with

\[
\dot{v} = f(v + c(\lambda), \lambda).
\]

**Proposition 8.22.** If \((0,0)\) is a Hopf point for \( \dot{u} = f(u, \lambda) \) with the associated curve \( C : \lambda \mapsto c(\lambda) \) and eigenvalues \( \alpha(\lambda) \pm i\beta(\lambda) \), then there is a smooth parameter-dependent linear change of coordinates of the form \( u = L(\lambda)z \) that transforms the system \( A(\lambda) = D_u f(0, \lambda) \) (the matrix of the linearization at the origin) into the Jordan normal form

\[
\begin{bmatrix}
\alpha(\lambda) & -\beta(\lambda) \\
\beta(\lambda) & \alpha(\lambda)
\end{bmatrix}.
\]

Using Proposition 8.22 we can, WLOG, write the ODE \( \dot{u} = f(u, \lambda) \) as

\[
\begin{align*}
\dot{x} &= \alpha(\lambda)x - \beta(\lambda)y + g(x, y, \lambda), \\
\dot{y} &= \beta(\lambda)x + \alpha(\lambda)y + h(x, y, \lambda),
\end{align*}
\]
where \( g(0, 0, 0) = 0, \ h(0, 0, 0) = 0, \ g_x(0, 0, 0) = 0, \ g_y(0, 0, 0) = 0, \ h_x(0, 0, 0) = 0, \) and \( h_y(0, 0, 0) = 0. \)

Recall that \( \alpha(0) = 0, \ \alpha'(0) > 0, \) and \( \beta(0) > 0, \) so that \( \alpha(\lambda) \pm i\beta(\lambda) \) are complex conjugate eigenvalues passing through the imaginary axis from left to right as \( \lambda \) passes through 0 from left to right.

By a reparameterization of time, we may assume WLOG that \( \beta(0) = 1. \)

We are after the existence of periodic orbit for \( \lambda \neq 0. \)

To this end we transform the ODE system to polar coordinates to get

\[
\dot{r} = \alpha(\lambda) r + p(r, \theta, \lambda), \\
\dot{\theta} = \beta(\lambda) + q(r, \theta, \lambda),
\]

where

\[
p(r, \theta, \lambda) = g(r \cos \theta, r \sin \theta, \lambda) \cos \theta + h(r \cos \theta, r \sin \theta, \lambda) \sin \theta, \\
q(r, \theta, \lambda) = \frac{h(r \cos \theta, r \sin \theta, \lambda) \cos \theta - g(r \cos \theta, r \sin \theta, \lambda) \sin \theta}{r}.
\]

Since \((x, y) \mapsto g(x, y, \lambda)\) and \((x, y) \mapsto h(x, y, \lambda)\) and their first order partials with respect to \( x \) and \( y \) vanish at \((0, 0, 0)\), the function \( q \) has a removable singularity at \( r = 0. \)

Moreover, \( p(0, \theta, \lambda) = 0 \) and \( q(0, \theta, \lambda) = \beta(\lambda). \)

For each \( \lambda \) near 0, the equilibrium at the origin in \( xy \)-coordinates has become the circle \( \{0\} \times \mathbb{T} \) in the cylinder phase space \( \mathbb{R} \times \mathbb{T} \) corresponding to \( r(t) = 0 \) and \( \theta(t) = \beta(\lambda)t \) where \( \beta(\lambda) > 0. \)

We have converted the problem of finding a periodic orbit bifurcation from an equilibrium to finding a periodic orbit bifurcating from another periodic orbit.

By our hypotheses, if \(|r|\) is small enough, then a Poincaré section for the flow in the cylinder is given by the line \( \theta = 0. \)

We will consider the displacement function associated with this Poincaré section.

Let \( t \mapsto (r(t, \xi, \lambda), \theta(t, \xi, \lambda)) \) denote the solution of the polar coordinate equations with initial conditions on the Poincaré section,

\[
r(0, \xi, \lambda) = \xi, \ \theta(0, \xi, \lambda) = 0.
\]

Because \( \theta(t, 0, 0) = \beta(\lambda)t \), we have \( \theta(2\pi, 0, 0) = 2\pi \) and \( \dot{\theta}(2\pi, 0, 0) = \beta(0) = 1 \neq 0. \)

By an application of the Implicit Function Theorem, there is a product neighbourhood \( U_0 \times V_0 \) of the origin in \( \mathbb{R} \times \mathbb{R} \) and a function \( T: U_0 \times V_0 \rightarrow \mathbb{R} \) such that \( T(0, 0) = 2\pi \) and

\[
\theta(T(\xi, \lambda), \xi, \lambda) = 2\pi
\]

for all \((\xi, \lambda) \in U_0 \times V_0.\)
The displacement function associated with the Poincaré section is
\[ \delta(\xi, \lambda) = r(T(\xi, \lambda), \xi, \lambda) - \xi. \]

We can remove the implicitly defined return time map \( T \) by another change of coordinates.

Since \( T(0, 0) = 2\pi \) and \( \dot{\theta}(t, 0, 0) = \beta(0) = 1 \neq 0 \), the continuity of \( T \) and \( \theta \) and the Implicit Function Theorem imply that there is a product neighbourhood \( U \times V \subset U_0 \times V_0 \) such that for each \((\xi, \lambda) \in U \times V\) the function \( t \to \theta(t, \xi, \lambda) \) is invertible on some bounded time interval containing \( T(\xi, \lambda) \).

If we denote the inverse function by \( s \mapsto \theta^{-1}(s, \xi, \lambda) \), then \( s = \theta(t, \xi, \lambda) \) and the function \( \rho : \mathbb{R} \times U \times V \to \mathbb{R} \) defined by
\[ \rho(s, \xi, \lambda) = r(\theta^{-1}(s, \xi, \lambda), \xi, \lambda) \]
is the solution of the IVP
\[ \frac{d\rho}{ds} = \frac{\alpha(\lambda)\rho + p(\rho, s, \lambda)}{\beta(\lambda) + q(\rho, s, \lambda)}, \quad \rho(0, \xi, \lambda) = \xi \]
for which
\[ \rho(2\pi, \xi, \lambda) = r(T(\xi, \lambda), \xi, \lambda). \]

If we rename the variables \( \rho \) and \( s \) to new variables \( r \) and \( \theta \), then the displacement function \( \delta : U \times V \to \mathbb{R} \) is given by
\[ \delta(\xi, \lambda) = r(2\pi, \xi, \lambda) - \xi \]
where \( \theta \mapsto r(\theta, \xi, \lambda) \) is the solution of the IVP
\[ \frac{dr}{d\theta} = \frac{\alpha(\lambda)r + p(r, \theta, \lambda)}{\beta(\lambda) + q(r, \theta, \lambda)}, \quad r(0, \xi, \lambda) = \xi. \]

**Definition 2.84.** Suppose that \((0, 0)\) is a Hopf point for \( \dot{u} = f(u, \lambda) \). The equilibrium \( u = 0 \) of \( \dot{u} = f(u, 0) \) is called a weak attractor (respectively a weak repeller) if the associated displacement function \( \delta \) satisfies \( \delta_{\xi\xi\xi}(0, 0) < 0 \) (respectively \( \delta_{\xi\xi\xi}(0, 0) > 0 \)).

**Hopf Bifurcation Theorem 8.25.** If \((0, 0)\) is a Hopf point for \( \dot{u} = f(u, \lambda) \) such that the equilibrium \( u = 0 \) of \( \dot{u} = f(u, 0) \) is a weak attractor (respectively, weak repeller), then there is a supercritical (respectively, subcritical) Hopf bifurcation at the Hopf point.