Singular Periodic Brake Orbits in the Planar Symmetric Planar Four Body Problem

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Nai-Chia Chen (2013): topological existence of singular periodic brake orbits in the planar isosceles three-body problem.
Four bodies with equal mass 1 and symmetric positions \((x_1, x_2), (x_1, -x_2), (-x_1, x_2), \text{ and } (-x_1, -x_2)\).
System of Equations and a Hamiltonian for the PFS4BP

For $x_1 > 0$, $x_2 > 0$, and $\dot{\cdot} = d/dt$, the equations of motion are

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} < 0,$$

$$\ddot{x}_2 = -\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}} < 0.$$

With $y_i = \dot{x}_i$, $i = 1, 2$, a Hamiltonian for the system is

$$H = \frac{y_1^2 + y_2^2}{2} - \frac{1}{4} \left( \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{(x_1^2 + x_2^2)^{1/2}} \right).$$
Brakes occur only for negative values of $H$. 
Regularization

Let $Q_1, Q_2, P_1, P_2$ be new symplectic coordinates related to the symplectic original coordinates $x_1, x_2, y_1, y_2$ by the generating function

$$F = y_1 Q_1^2 + y_2 Q_2^2.$$ 

Then

$$x_1 = \frac{\partial F}{\partial y_1} = Q_1^2, \quad x_2 = \frac{\partial F}{\partial y_2} = Q_2^2,$$

$$P_1 = \frac{\partial F}{\partial Q_1} = 2y_1 Q_1, \quad P_2 = \frac{\partial F}{\partial Q_2} = 2y_2 Q_2.$$ 

Therefore

$$y_1 = \frac{P_1}{2Q_1}, \quad y_2 = \frac{P_2}{2Q_2}.$$
Regularization

The new Hamiltonian is

\[ \hat{H} = \frac{1}{2} \left( \frac{P_1^2}{4Q_1^2} + \frac{P_1^2}{4Q_1^2} \right) - \frac{1}{4} \left( \frac{1}{Q_1^2} + \frac{1}{Q_2^2} + \frac{1}{(Q_1^4 + Q_2^4)^{1/2}} \right). \]

We define a new time variable \( s \) by

\[ \frac{dt}{ds} = Q_1^2 Q_2^2. \]

Thus

\[ \frac{dt}{ds} \hat{H} = \frac{1}{8} \left( P_1^2 Q_2^2 + P_2^2 Q_1^2 \right) - \frac{1}{4} \left( Q_2^2 + Q_1^2 + \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}} \right). \]
Regularization

In extended phase space and for a given initial Hamiltonian value $E$, the regularized Hamiltonian is the $C^1$ function

$$\Gamma = \frac{dt}{ds}(\hat{H} - E).$$

That is

$$\Gamma = \frac{1}{8} \left( P_1^2 Q_2^2 + P_2^2 Q_1^2 \right) - \frac{1}{4} \left( Q_2^2 + Q_1^2 + \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}} \right) - EQ_1^2 Q_2^2.$$

On the level set $\Gamma = 0$ and with $' = d/ds$, the Hamiltonian system

$$Q_1' = \frac{d\Gamma}{P_1}, \quad Q_2' = \frac{d\Gamma}{P_2}, \quad P_1' = -\frac{d\Gamma}{Q_1}, \quad P_2' = -\frac{d\Gamma}{Q_2},$$

is equivalent to the original Hamiltonian system.
Explicitly, the regularized equations of motion are

\[ Q'_1 = \frac{1}{4} P_1 Q_2^2, \]

\[ Q'_2 = \frac{1}{4} P_2 Q_1^2, \]

\[ P'_1 = -\frac{1}{4} P_1^2 Q_1 + \frac{1}{2} Q_1 + \frac{1}{2} \frac{Q_1 Q_2^6}{(Q_1^4 + Q_2^4)^{3/2}} + 2EQ_1 Q_2^2, \]

\[ P'_2 = -\frac{1}{4} P_2^2 Q_2 + \frac{1}{2} Q_2 + \frac{1}{2} \frac{Q_1^6 Q_2}{(Q_2^4 + Q_1^4)^{3/2}} + 2EQ_2 Q_1^2. \]
Proving the Existence of a Periodic Brake Orbit

$x_1 x_2 - plane$

$Q_1 Q_2 - plane$
Proving the Existence of a Periodic Brake Orbit

The topological shooting method.

Left graph: going down ($\dot{x}_2 < 0$) when $x_1 = 0$.
Right graph: going up ($\dot{x}_2 > 0$) when $x_1 = 0$.
Use Intermediate Value Theorem to get existence of orbit in the middle ($\dot{x}_2 = 0$) when $x_1 = 0$. 
The condition $\dot{x}_2 = y_2 = 0$ implies $P_2 = 2y_2 Q_2 = 0$, and the orbit extends by symmetries to a periodic brake orbit.
The Going Down Orbit

The conditions $\dot{x}_2 = y_2 < 0$ and $Q_2 < 0$ imply $P_2 = 2y_2 Q_2 > 0$ and $Q'_2 = P_2 Q_1^2 / 4 \geq 0$. 

![Graph showing the relationship between $Q_2^2$ and $Q_1$]
The Going Up Orbit

The conditions $\dot{x}_2 = y_2 > 0$ and $Q_2 < 0$ imply $P_2 = 2y_2Q_2 < 0$ and $Q'_2 = P_2 Q_1^2 / 4 \leq 0$. 
Existence of Going Down Orbit

Starting at a brake, choose the initial $x_1$ value to be sufficiently large so the orbit bounces off the $x_1$ axis at least twice.
Existence of Going Down Orbit

As $x_1$ decreases, the negative $\dot{x}_1$ also decreases. Thus while moving the initial value of $x_1$ to the left (while holding the initial value of $x_2$ constant), the orbit shifts to the left, passing through a total collapse.

The value of $\dot{x}_2$ is negative when $x_1 \to 0$. 
Existence of Going Down Orbit

After the orbit passes the line \( x_2 = x_1 \) from the right to left with \( x_1' < 0 \) and \( x_2' < 0 \), the negative \( \ddot{x}_2 \) implies the negative \( \dot{x}_2 \) stays negative.
Existence of Going Up Orbit

Part 1: starts at a brake, close to the total collapse homographic orbit, and ends when $x_2$ goes to 0.
Part 2: crosses the line $x_1 = x_2$ line with $\dot{x}_2 > 0$.
Part 3: estimate the time it takes for $\dot{x}_2 \to 0$.
Part 4: estimate the time it takes for $x_1 \to 0$.
Existence of Going Up Orbit: Part 1

Use the characterization $x_2 < x_1 \iff \ddot{x}_2 < \ddot{x}_1$. 
Existence of Going Up Orbit: Part 2

Numerical simulations for the orbit as initial position approaches $x_1 = x_2$. 
Use moment of inertia:

\[ I = \frac{x_1^2 + x_2^2}{2} = \frac{Q_1^4 + Q_2^4}{2}, \quad I' = \frac{Q_1^2 Q_2^2 (P_1 Q_1 + P_2 Q_2)}{2}. \]
Existence of Going Up Orbit: Parts 3 and 4

Scaling: for solution $x_1(t)$ and $x_2(t)$ with energy $E$, the function

$$s_1(t) = ax_1(a^{-\frac{3}{2}} t), \quad s_2(t) = ax_2(a^{-\frac{3}{2}} t),$$

for any $a > 0$ is also a solution with energy $E/a$.

Take orbit from Part 2, and scale it so that it crosses the $x_1 = x_2$ line at $x_1 = 1 = x_2$. 

![Graph showing scaled orbits](image-url)
The scaling moves the initial position far away from the origin \((a \gg 0)\), and pushes the negative \(E\) to \(E/a\) which is closer to 0.
Define the times $t_0$ (time at which $x_1 = 0$), $t_1$ (time at which $x_1 = x_2$), and $t_m$ (time at which $\dot{x}_2 = 0$).

A lower bound on $t_m - t_1$ is 1.155, while an upper bound on $t_0 - t_1$ is 1.14.
Existence of a Periodic Brake Orbit in the PFS4BP

Intermediate Value Theorem

Going Down

Going Up
Three singular periodic brake orbits with the same negative value of $E$. 