HOMEWORK 8

(1) Prove the rational roots test:

**Theorem:** If \( f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in \mathbb{Z}[x] \) with \( a_n, a_0 \neq 0 \), then any rational root \( \alpha \in \mathbb{Q} \) of \( f \) must have the form

\[
\alpha = \frac{r}{s}
\]

with \( r, s \in \mathbb{Z}, s \neq 0, (r, s) = 1, r|a_0 \) and \( s|a_n \).

(2) Let \( F \) be a finite field of order \( q \). Prove that every element \( \alpha \in F \) is a root of the polynomial \( x^q - x \). (Use Lagrange’s theorem and the fact that the nonzero elements of \( F \) form a group under multiplication.)

(3) Prove that the polynomial \( f(x) = x^5 + 5x^4 + 10x^3 + 10x^2 + 7x + 5 \) is irreducible over \( \mathbb{Q} \).

(4) Let \( \alpha \in \mathbb{Q}(\sqrt{2}) \). Prove that \( \alpha \) is algebraic over \( \mathbb{Q} \).