Homework 35

1. Let $K = \mathbb{Q}$ and $L = K(x)$. Let $S = \{x^2 + x + 1, 2x^3 + x^2 + 3x + 1\}$. Prove that $S$ is algebraically dependent over $K$ by finding a nonzero polynomial $f \in K[x_1, x_2]$ with $f(x^2 + x + 1, 2x^3 + x^2 + 3x + 1) = 0$.

2. Let $L/K$ be a field extension, and let $S$ be a subset of $L$. Prove that if $u \in L$ is algebraic over $K(S)$, but not over $K(S - \{v\})$ for some $v \in S$, then $v$ is algebraic over $K((S - \{v\}) \cup \{u\})$.

3. Let $L/K$ be a field extension. Prove that every subset of $L$ that is algebraically independent over $K$ is contained in a transcendence base of $L/K$. Use this result to show that $L/K$ has a transcendence base.