Do all five problems and show all work. Be sure that your work is done neatly and correctly. The exam is due on Wednesday, February 24, by 2:00 PM. You may use your course notes, textbook, and the algebra textbook by Dummitt and Foote. If you used a different algebra text when you took algebra, let me know which one, and I will probably allow you to use it as well. Please do not discuss the problems with anyone else except me until after 2:00 PM on the 24th.

(1) Let $K = \mathbb{Q}(\sqrt{-5})$, and let $L = K(i)$.
   (a) Find an integral basis for $L/\mathbb{Q}$ and the discriminant of $L/\mathbb{Q}$.
   (b) Determine all primes of $K$ that ramify in $L/K$.
   (c) Let $p$ be a prime in $\mathbb{Z}$. Describe the shape of the factorization of $p$ in $L$. Your description will depend on the congruence class of $p$ modulo some integer $N$.

(2) Let $K = \mathbb{Q}(\sqrt{-23})$. The class number of $\mathcal{O}_K$ is 3. Determine a representative of each ideal class.

(3) Let $R$ be a Dedekind domain, and let $I, J$ be nonzero ideals of $R$. Prove that there is an element $\alpha \in R$ such that $\gcd(IJ, \alpha R) = I + J$.

(4) Let $K = \mathbb{Q}(\sqrt{6}, \sqrt{10})$. You may assume that the discriminant of $K$ is divisible by $2^8$. Determine the discriminant of $K$ and an integral basis of $K$.

(5) Let $\alpha$ be a root of an irreducible cubic polynomial $f(x) \in \mathbb{Q}[x]$. Let $K = \mathbb{Q}(\alpha)$, and assume that 2 splits completely in $K/\mathbb{Q}$. Prove that there is no $\beta \in \mathcal{O}_K$ such that $\{1, \beta, \beta^2\}$ is an integral basis of $K$. 