

REDUCIBLE GALOIS REPRESENTATIONS AND ARITHMETIC HOMOLOGY FOR $GL(4)$

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ABSTRACT. We prove that a sum of two odd irreducible two-dimensional Galois representations with squarefree relatively prime Serre conductors is attached to a Hecke eigenclass in the homology of a subgroup of $GL(4, \mathbb{Z})$, with the level, nebentype, and coefficient module of the homology predicted by a generalization of Serre's conjecture to higher dimensions. To do this we prove along the way that any Hecke eigenclass in the homology of a congruence subgroup of a maximal parabolic subgroup of $GL(n, \mathbb{Q})$ has a reducible Galois representation attached, where the dimensions of the components correspond to the type of the parabolic subgroup. Our main new tool is a resolution of \mathbb{Z} by $GL(n, \mathbb{Q})$ -modules consisting of sums of Steinberg modules for all subspaces of \mathbb{Q}^n .

1. INTRODUCTION

Serre's conjecture [23] (now a theorem of Khare, Wintenberger, and Kisin [18, 19, 20]) gives a connection between odd irreducible Galois representations $\rho : G_{\mathbb{Q}} \rightarrow GL(2, \overline{\mathbb{F}}_p)$ and modular forms that are simultaneous eigenvectors of all the Hecke operators. Via the Eichler-Shimura isomorphism [24], it can be interpreted as giving a connection between such Galois representations and elements of a cohomology group $H^1(\Gamma_0(N), V)$ for an appropriate coefficient module V . This interpretation of Serre's conjecture was generalized by [10] to a conjecture relating odd Galois representations $\rho : G_{\mathbb{Q}} \rightarrow GL(n, \overline{\mathbb{F}}_p)$ to eigenclasses of the Hecke operators in cohomology groups $H^*(\Gamma_0(n, N), V)$, where $\Gamma_0(n, N)$ is the congruence subgroup of $SL(n, \mathbb{Z})$ that generalizes $\Gamma_0(N) \subset SL(2, \mathbb{Z})$. Refinements of the conjecture [8, 16] make more precise predictions concerning the proper coefficient modules.

Some proven cases of the conjectured connection between Galois representations and cohomology eigenclasses are known. In particular, the conjecture is known for two-dimensional Galois representations. In [12], the conjecture is proven for certain irreducible symmetric square representations of odd irreducible two-dimensional representations.

Our long-term goal is to prove the conjecture for any odd reducible Galois representation under the assumption that the conjecture holds for each constituent. In [3], odd Galois representations that are sums of characters are shown to correspond to cohomology eigenclasses. This work was extended in [5] to show that any sum of characters with a Galois representation satisfying the conjecture such that the resulting representation is odd will satisfy the conjecture. In [6], it is shown that if ρ_1 and ρ_2 are two Galois representations, each attached to a cohomology eigenclass

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with trivial coefficient module, then a twisted sum of the two representations will also be attached to a cohomology Hecke eigenclass with trivial coefficients (although not in the cohomology of a group of the form $\Gamma_0(n, N)$).

In this paper, we prove that Galois representations of the form $\rho_1 \oplus \rho_2$ of square-free level N with ρ_1 and ρ_2 two-dimensional irreducible and odd are attached to Hecke eigenclasses in the cohomology of $\Gamma_0(4, N)$. Together with our previous results cited above, we have now proven the conjecture of [8] for all odd reducible four-dimensional Galois representations, as long as the constituents have squarefree pairwise relatively prime conductors, and satisfy the conjecture themselves.

The two main new tools that we use in the proof are an exact sequence of $\mathrm{GL}(n, K)$ -modules for any field K that involves the Steinberg modules for $\mathrm{GL}(W)$ for all subspaces W of K^n (Section 4), and Theorem 11.5 on the reducibility of Galois representations attached to the cohomology of arithmetic subgroups of parabolic subgroups of $\mathrm{GL}(n, \mathbb{Q})$.

The exact sequence of Section 4 generalizes the exact sequence used in [5, 7], which only works for $n = 3$, to arbitrary n . It is a resolution of \mathbb{Z} by non-free modules which are induced from various parabolic subgroups. Using a certain Hecke equivariant spectral sequence arising from this exact sequence, we are able to construct a system of Hecke eigenvalues that have $\rho_1 \oplus \rho_2$ attached.

Our other main tool, Theorem 11.5 proves that any Hecke eigenclass attached to the cohomology of a maximal parabolic subgroup of type (n_1, n_2) has an attached Galois representation that is a sum of (possibly reducible) Galois representations $\rho_1 \oplus \rho_2$, with $\rho_i : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(n_i, \overline{\mathbb{F}}_p)$ for $i = 1, 2$.

On the more technical side, this paper includes extensions from $n = 3$ to general n of the study we made in [5, 7] of orbits of subspaces of \mathbb{Q}^n under certain congruence subgroups of $\mathrm{GL}(n, \mathbb{Z})$. We also study the action of a Levi component on the groups $H_*(\Gamma \cap U, M)$ appearing in the Hochschild-Serre spectral sequence for

$$1 \rightarrow \Gamma \cap U \rightarrow \Gamma \cap P \rightarrow \Gamma \cap L \rightarrow 1,$$

where U is the unipotent radical of a maximal parabolic subgroup $P = LU$ of $\mathrm{GL}(n, \mathbb{Q})$. We show in Theorem 7.11 that under certain hypotheses, the Hecke algebra for Γ_P acts in an equivariant fashion on the spectral sequence. This is surprisingly hard to do and we don't even know if this continues to be the case if P is not maximal. A third generalization of our earlier work concerns the detailed action of the matrices defining the Hecke operators on the homology. The last new technical point regards the interplay of the Hecke operators and the Künneth formula in section 10.

We expect to be able to apply these methods to higher dimensional Galois representations; at present, we are not able to because we have not yet been able to prove that a certain Hochschild-Serre spectral sequence, used to construct the systems of Hecke eigenvalues that we need, degenerates when $n > 4$.

We know that a system of Hecke eigenvalues appears in $H^k(\Gamma_0(n, N), V)$ if and only if it appears in $H_k(\Gamma_0(n, N), V)$. We have discussed mostly cohomology in the introduction to conform with common usage. Our theorems and proofs below will all be stated for homology.

2. GALOIS REPRESENTATIONS, HECKE OPERATORS, COHOMOLOGY AND HOMOLOGY

Let $p > 2$ be a prime number, and let $\mathbb{F} = \overline{\mathbb{F}}_p$. Throughout the paper, we use Borel-Serre duality for subgroups of $\mathrm{GL}(n, \mathbb{Z})$ [13, Section 11.4], and the fact that the Borel-Serre duality isomorphism is Hecke equivariant [9, Corollary 3]; when we do this we will require $p > n + 1$, so that no torsion element of $\mathrm{GL}(n, \mathbb{Z})$ has order divisible by p . By a Galois representation we mean a continuous homomorphism $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(n, \mathbb{F})$ for some positive integer n . For each prime ℓ , we fix a choice of Frobenius element $\mathrm{Fr}_{\ell} \in G_{\mathbb{Q}}$; we use the arithmetic Frobenius, so that if $\omega : G_{\mathbb{Q}} \rightarrow \mathbb{F}^{\times}$ is the cyclotomic character, $\omega(\mathrm{Fr}_{\ell}) = \ell$. If ρ is unramified at ℓ , then $\rho(\mathrm{Fr}_{\ell})$ is defined up to similarity. Hence, for ρ unramified at ℓ , the characteristic polynomial $\det(I - \rho(\mathrm{Fr}_{\ell})X)$ of $\rho(\mathrm{Fr}_{\ell})$ is well defined.

Definition 2.1. Let $n > 1$ and $N \in \mathbb{N}$, and let p be a prime in \mathbb{Z} .

- (1) $S_0^{\pm}(n, N)$ consists of the set of all $n \times n$ matrices with integer entries and nonzero determinant prime to pN whose first row is congruent to $(*, 0, \dots, 0)$ modulo N .
- (2) $S_0(n, N)$ consists of elements of $S_0^{\pm}(n, N)$ with positive determinant.
- (3) $\Gamma_0^{\pm}(n, N) = S_0^{\pm}(n, N) \cap \mathrm{GL}(n, \mathbb{Z})$.
- (4) $\Gamma_0(n, N) = S_0(n, N) \cap \mathrm{SL}(n, \mathbb{Z})$.

For a given prime p and positive integer N prime to p , $(\Gamma_0(n, N), S_0(n, N))$ is a Hecke pair (see [1]), and we denote the \mathbb{F} -algebra of its double cosets by $\mathcal{H}_{n, N}$. We note that $\mathcal{H}_{n, N}$ is commutative, and is generated by the double cosets

$$\Gamma_0(n, N)s(\ell, n, k)\Gamma_0(n, N),$$

where $s(\ell, n, k) = \mathrm{diag}(1, \dots, 1, \ell, \dots, \ell)$ is a diagonal matrix with k copies of ℓ on the diagonal, ℓ runs over all primes not dividing pN , and $0 \leq k \leq n$. The algebra $\mathcal{H}_{n, N}$ acts on the homology or cohomology of $\Gamma_0(n, N)$ with coefficients in any $\mathbb{F}[S_0(n, N)]$ -module M . When the double coset of $s(\ell, n, k)$ acts on homology or cohomology, we will denote it by $T_n(\ell, k)$.

Definition 2.2. Let V be any $\mathcal{H}_{n, N}$ -module, and suppose that $v \in V$ is a simultaneous eigenvector of all the $T_n(\ell, k)$ for $\ell \nmid pN$, with eigenvalues $a(\ell, k) \in \mathbb{F}$. Suppose that $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(n, \mathbb{F})$ is a Galois representation unramified outside pN . We say that ρ is *attached* to v if, for all $\ell \nmid pN$,

$$\det(I - \rho(\mathrm{Fr}_{\ell})X) = \sum_{k=0}^n (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k.$$

If ρ is attached to $v \in V$, we will also say that ρ *fits* V .

3. CONJECTURES RELATING GALOIS REPRESENTATIONS AND ARITHMETIC HOMOLOGY/COHOMOLOGY

Definition 3.1. [7] Let S be a subsemigroup of the matrices in $\mathrm{GL}(n, \mathbb{Q})$ with integer entries whose determinants are prime to pN . A (p, N) -*admissible* S -module M is an $\mathbb{F}[S]$ -module of the form $M' \otimes F_{\epsilon}$, where M' is an $\mathbb{F}[S]$ -module on which $S \cap \mathrm{GL}(n, \mathbb{Q})^+$ acts via its reduction modulo p , and ϵ is a character $\epsilon : S \rightarrow \mathbb{F}^{\times}$ which factors through the reduction of S modulo N . Here \mathbb{F}_{ϵ} is the vector space \mathbb{F} , with S acting as multiplication via ϵ . An admissible module is one which is (p, N) -admissible for some choice of p and N .

We can construct (p, N) -admissible modules by starting with $\mathrm{GL}(n, \mathbb{F}_p)$ -modules, and letting S act via reduction modulo p . We have the following parametrization of irreducible $\mathrm{GL}(n, \mathbb{F}_p)$ -modules.

Theorem 3.2. [15] *Call an n -tuple of (a_1, \dots, a_n) of integers p -restricted if $0 \leq a_n < p - 1$ and, for each $i < n$, $0 \leq a_i - a_{i+1} \leq p - 1$. Then there is a bijection between p -restricted n -tuples of integers and irreducible $\mathbb{F}[\mathrm{GL}(n, \mathbb{F}_p)]$ -modules, with the n -tuple (a_1, \dots, a_n) corresponding to the unique simple submodule of the dual Weyl module with highest weight (a_1, \dots, a_n) .*

Definition 3.3. Denote by $F(a_1, \dots, a_n)$ the irreducible $\mathbb{F}[\mathrm{GL}(n, \mathbb{F}_p)]$ -module corresponding to the p -restricted n -tuple (a_1, \dots, a_n) .

As described above, $F(a_1, \dots, a_n)$ becomes an S -module on which S acts via reduction modulo p . We will relax the condition on the value of a_n , allowing it to be an arbitrary integer; this has the effect that a given module corresponds to infinitely many n -tuples, all congruent to some p -restricted n -tuple modulo $p - 1$, and it allows flexibility in specifying modules. Given a character $\epsilon : S \rightarrow \mathbb{F}^\times$ that factors through the reduction of S modulo N , we see that $F(a_1, \dots, a_n)_\epsilon = F(a_1, \dots, a_n) \otimes \mathbb{F}_\epsilon$ is a (p, N) -admissible module.

If $S = S_0(n, N)$ and $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{F}^\times$ is a character, we will also denote by $\epsilon : S \rightarrow \mathbb{F}^\times$ the character sending $s \in S$ to the image under ϵ of the mod N reduction of the $(1, 1)$ entry of s . In this case, we call ϵ a *nebentype character*.

In order to state the main conjecture of [8], we recall the following definition.

Definition 3.4. For $n > 1$, a Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(n, \mathbb{F})$ is odd if the image of complex conjugation is similar to a matrix with alternating 1's and -1 's on the diagonal. A Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(1, \mathbb{F}) = \mathbb{F}^\times$ is odd if the image of complex conjugation is -1 , and is called even otherwise.

Conjecture 3.5. [8, Conjecture 3.1] *For any odd Galois representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(n, \mathbb{F})$, we may find an integer N (called the level), an irreducible $\mathrm{GL}(n, \mathbb{F}_p)$ -module M (called the weight), and a Dirichlet character ϵ (called the nebentype), such that ρ fits $H^k(\Gamma_0(N), M_\epsilon)$.*

In fact, [8, Conjecture 3.1] predicts the level, weight and nebentype from the structure of ρ . We do not give these definitions in detail in this paper; see [8] for detailed descriptions.

In this paper, we will generalize techniques developed in [3, 5, 7] to prove the following theorem.

Theorem 3.6. *Let $p > 5$. For $i = 1, 2$, let $\rho_i : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \mathbb{F})$ be odd, irreducible Galois representations with squarefree relatively prime levels. Then $\rho_1 \oplus \rho_2$ fits at least one of $H_6(\Gamma_0(n, N), M_\epsilon)$ or $H_2(\Gamma_0(n, N), M_\epsilon)$, with N , M , and ϵ as predicted by [8].*

4. A STEINBERG MODULE EXACT SEQUENCE

Throughout this section, fix a field K . In this section, we derive a resolution of \mathbb{Z} by $\mathrm{GL}(n, K)$ -modules that generalizes the resolution used in [5], which only works for $\mathrm{GL}(3, K)$.

For a vector space W over K , denote by $\mathbb{P}(W)$ the projective space $(W - \{0\})/K^\times$ of nonzero vectors in W modulo scalar multiplication. For any collection $w_1, \dots, w_k \in \mathbb{P}(W)$, we will denote by $\mathrm{span}(w_1, \dots, w_k)$ the subspace of W

spanned by lifts of the w_i to W (we see easily that the span is independent of the choice of lifts).

We recall the definition of the sharbly complex.

Definition 4.1. [2] Let W be a k -dimensional vector space over a field K . For $i \geq 0$, define the i -sharblies $\text{Sh}_i(W)$ to be the free \mathbb{Z} -module generated by the $(k+i)$ -tuples (w_1, \dots, w_{k+i}) for $w_i \in \mathbb{P}(W)$, modulo the \mathbb{Z} -span of the following elements:

- (1) $(w_{\sigma(1)}, \dots, w_{\sigma(k+i)}) - (-1)^\sigma (w_1, \dots, w_{k+i})$ for all permutations $\sigma \in S_{k+i}$,
- (2) (w_1, \dots, w_{k+i}) if $\{w_1, \dots, w_{k+i}\}$ does not span W .

We denote a basis element of the i -sharblies by the symbol $[w_1, \dots, w_{k+i}]$ where each $w_j \in \mathbb{P}(W)$, with (1) implying that this symbol is antisymmetric in the entries, and (2) implying that it is 0 if the entries do not span W . By considering $\text{GL}(W)$ to act on W by right multiplication, there is a natural right action of $\text{GL}(W)$ on $\text{Sh}_i(W)$. The boundary map $d_i : \text{Sh}_i(W) \rightarrow \text{Sh}_{i-1}(W)$ is given by

$$d_i([w_1, \dots, w_{k+i}]) = \sum_{j=1}^{k+i} (-1)^j [w_1, \dots, \hat{w}_j, \dots, w_{k+i}].$$

The *sharbly complex* is the complex of right $\text{GL}(W)$ -modules

$$\dots \rightarrow \text{Sh}_i(W) \xrightarrow{d_i} \text{Sh}_{i-1}(W) \rightarrow \dots \rightarrow \text{Sh}_1(W) \xrightarrow{d_1} \text{Sh}_0(W).$$

By [2] the Steinberg module $\text{St}(W)$ is isomorphic to the cokernel of the map $d_1 : \text{Sh}_1(W) \rightarrow \text{Sh}_0(W)$, and therefore we get a resolution of the Steinberg module:

$$\dots \rightarrow \text{Sh}_j(W) \rightarrow \text{Sh}_{j-1}(W) \rightarrow \dots \rightarrow \text{Sh}_1(W) \rightarrow \text{Sh}_0(W) \rightarrow \text{St}(W) \rightarrow 0.$$

We will denote the image of a 0-sharbly $[w_1, \dots, w_k]$ in $\text{St}(W)$ by $\llbracket w_1, \dots, w_k \rrbracket$. We note that if W is 0-dimensional, the Steinberg module and the 0-sharblies are isomorphic to \mathbb{Z} (with trivial $\text{GL}(W)$ -action), generated by the empty symbols $\llbracket \rrbracket$ and $\llbracket \cdot \rrbracket$. If W is 1-dimensional, we also have that the Steinberg module and the 1-sharblies are isomorphic to \mathbb{Z} (with trivial $\text{GL}(W)$ -action), generated by the symbols $\llbracket w \rrbracket$ and $\llbracket w \rrbracket$, where w is the unique element of $\mathbb{P}(W)$.

Theorem 4.2. *Let V be an n -dimensional vector space over K with $n > 0$. Then there is an exact sequence of $\text{GL}(V)$ -modules*

$$0 \rightarrow \bigoplus_{W^n} \text{St}(W^n) \xrightarrow{\delta_n} \bigoplus_{W^{n-1}} \text{St}(W^{n-1}) \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_2} \bigoplus_{W^1} \text{St}(W^1) \xrightarrow{\delta_1} \bigoplus_{W^0} \text{St}(W^0) \rightarrow 0,$$

where each W^i runs through all subspaces of V of dimension i , and the map

$$\delta_k : \bigoplus_{W^k} \text{St}(W^k) \rightarrow \bigoplus_{W^{k-1}} \text{St}(W^{k-1})$$

is defined by

$$\delta_k(\llbracket w_1, \dots, w_k \rrbracket) = \sum_{j=1}^k (-1)^j \llbracket w_1, \dots, \hat{w}_j, \dots, w_k \rrbracket,$$

for $\llbracket w_1, \dots, w_k \rrbracket \in \text{St}(W^k)$, with $W^k = \text{span}(w_1, \dots, w_k)$ and

$$\llbracket w_1, \dots, \hat{w}_j, \dots, w_k \rrbracket \in \text{St}(W_j^{k-1}),$$

where $W_j^{k-1} = \text{span}(w_1, \dots, \hat{w}_j, \dots, w_k)$.

Proof. For $\gamma \in \mathrm{GL}(V)$, and a generator $[[w_1, \dots, w_k]] \in \mathrm{St}(W^k)$, we define

$$[[w_1, \dots, w_k]]\gamma = [[w_1\gamma, \dots, w_k\gamma]] \in \mathrm{St}(W^k\gamma).$$

With this action, we see easily that

$$\bigoplus_{W^k} \mathrm{St}(W^k)$$

is a $\mathrm{GL}(V)$ -module and that δ_k is an equivariant map of $\mathrm{GL}(V)$ -modules.

We now check that each δ_k is a well defined map.

To begin, let W be a k -dimensional subspace of V . We may write the Steinberg module of W as $A(W)/(B(W)+C(W))$, where $A(W)$ is the free \mathbb{Z} -module generated by antisymmetric symbols (w_1, \dots, w_k) with $w_i \in \mathbb{P}(W)$, $B(W)$ is generated by all such symbols where w_1, \dots, w_k are contained in a proper subspace of W^k , and $C(W)$ is generated by elements of the form

$$\sum_{j=1}^k (-1)^j (w_1, \dots, \hat{w}_j, \dots, w_k).$$

Define $\phi : A(W) \rightarrow \bigoplus \mathrm{St}(W^{k-1})$ by

$$(w_1, \dots, w_k) \mapsto \sum_{j=1}^k (-1)^j [[w_1, \dots, \hat{w}_j, \dots, w_k]].$$

Note that in any case where the symbol $[[w_1, \dots, \hat{w}_j, \dots, w_k]]$ is not in a unique Steinberg module because the dimension of the span of $w_1, \dots, \hat{w}_j, \dots, w_k$ is less than $k-1 = \dim W^{k-1}$, the symbol vanishes, regardless of which module it is considered to lie in. Hence, since $A(W)$ is free over \mathbb{Z} , ϕ is well-defined, and if ϕ maps both $B(W)$ and $C(W)$ to 0, then the map δ_k that it induces on $A(W)/(B(W)+C(W))$ will be well defined.

Now ϕ maps $C(W)$ to 0 by the standard argument that the boundary of the boundary is 0. Further, if $(w_1, \dots, w_k) \in B(W)$, then letting W^{k-1} be a $(k-1)$ -dimensional subspace of W containing w_1, \dots, w_k , we find that

$$\sum_{j=1}^k (-1)^j (w_1, \dots, \hat{w}_j, \dots, w_k) \in C(W^{k-1}).$$

Now $C(W^{k-1})$ maps to 0 in $\mathrm{St}(W^{k-1})$, so $\phi(w_1, \dots, w_k) = 0$ and $\phi(B(W)) = 0$. Thus, we see that δ_k is well defined.

It is clear that the composition $\delta_{k-1} \circ \delta_k$ is equal to 0 for $k > 0$. In addition, δ_1 is clearly surjective, since there is only one zero-dimensional subspace of V and $\delta_1(w) = -[[]]$ for any $w \in V$.

Now we wish to prove that any element of $\ker \delta_k$ is contained in the image of δ_{k+1} .

Suppose that for some index set \mathcal{A} , and some integers c_a for $a \in \mathcal{A}$, we have

$$s = \sum_{a \in \mathcal{A}} c_a [[w_1^a, \dots, w_k^a]]$$

is in the kernel of δ_k for $0 < k < n$. We will show that it is in the image of δ_{k+1} .

Choose an arbitrary $x \in \mathbb{P}(V)$. Then

$$\sum_a c_a [[x, w_1^a, \dots, w_k^a]] \in \bigoplus_{W^{k+1}} \mathrm{St}(W^{k+1})$$

(note that some of the terms may be 0, if $x \in \text{span}(w_1^a, \dots, w_k^a)$). In any case, we have

$$\begin{aligned} \delta_{k+1} \left(\sum_a c_a \llbracket x, w_1^a, \dots, w_k^a \rrbracket \right) &= - \sum_a c_a \llbracket w_1^a, \dots, w_k^a \rrbracket \\ &\quad - \sum_a \sum_{j=1}^k (-1)^j c_a \llbracket x, w_1^a, \dots, \hat{w}_j^a, \dots, w_k^a \rrbracket. \end{aligned}$$

Adding this to s , we obtain a new element $s' \in \ker \delta_k$ that differs from s by an element of the image of δ_{k+1} . It thus suffices to prove that s' is in the image of δ_{k+1} . We note that each symbol comprising s' has as its first component the chosen x . Hence, (changing the w_i^a , the c_a , and indeed the index set \mathcal{A}), we may write

$$s' = \sum_{a \in \mathcal{A}} c_a \llbracket x, w_2^a, \dots, w_k^a \rrbracket.$$

By eliminating terms where $\{x, w_2^a, \dots, w_k^a\}$ does not span a k -dimensional space, we may also assume that for each a , we have $x \notin \text{span}(w_2^a, \dots, w_k^a)$.

Now,

$$0 = \delta_k(s') = - \sum_a c_a \llbracket w_2^a, \dots, w_k^a \rrbracket - \sum_{j=1}^k \sum_a (-1)^j c_a \llbracket x, w_2^a, \dots, \hat{w}_j^a, \dots, w_k^a \rrbracket.$$

Since, for each a , we have $x \notin \text{span}(w_2^a, \dots, w_k^a)$, we see that no terms of the double sum are in the same component of

$$\bigoplus_{W^{k-1}} \text{St}(W^{k-1})$$

as any term in the first sum. Hence, we must have

$$\sum_a c_a \llbracket w_2^a, \dots, w_k^a \rrbracket = 0.$$

For each $(k-1)$ -dimensional subspace W of V , set $\mathcal{A}_W = \{a : \text{span}(w_2^a, \dots, w_k^a) = W\}$. Then each a is in precisely one \mathcal{A}_W . We see that for each W

$$\sum_{a \in \mathcal{A}_W} c_a \llbracket w_2^a, \dots, w_k^a \rrbracket = 0.$$

For any W with \mathcal{A}_W nonempty, since the Steinberg module of W is the cokernel of $\text{Sh}_1(W) \xrightarrow{d_1} \text{Sh}_0(W)$, we see that there is an index set \mathcal{B} , integers c_b and elements $[y_1^b, \dots, y_k^b] \in \text{Sh}_1(W)$ for each $b \in \mathcal{B}$ such that

$$\sum_{a \in \mathcal{A}_W} c_a \llbracket w_2^a, \dots, w_k^a \rrbracket = d_1 \left(\sum_{b \in \mathcal{B}} c_b [y_1^b, \dots, y_k^b] \right) = \sum_{b \in \mathcal{B}} \sum_{j=1}^k (-1)^j c_b [y_1^b, \dots, \hat{y}_j^b, \dots, y_k^b].$$

Let W_x be the span of W and a lift of x . Then in $\text{Sh}_0(W_x)$ we have

$$\sum_{a \in \mathcal{A}_W} c_a \llbracket x, w_2^a, \dots, w_k^a \rrbracket = \sum_{b \in \mathcal{B}} \sum_{j=1}^k (-1)^j c_b \llbracket x, y_1^b, \dots, \hat{y}_j^b, \dots, y_k^b \rrbracket.$$

Hence, in $\text{St}(W_x)$,

$$\sum_{a \in \mathcal{A}_W} c_a \llbracket x, w_2^a, \dots, w_k^a \rrbracket = \sum_{b \in \mathcal{B}} \sum_{j=1}^k (-1)^j c_b \llbracket x, y_1^b, \dots, \hat{y}_j^b, \dots, y_k^b \rrbracket.$$

Because y_1^b, \dots, y_k^b span a $(k-1)$ -dimensional subspace, this equals

$$- \sum_b \left(c_b \llbracket y_1^b, \dots, y_k^b \rrbracket - \sum_{j=1}^k (-1)^j c_b \llbracket x, y_1^b, \dots, \hat{y}_j^b, \dots, y_k^b \rrbracket \right),$$

which is equal to

$$-\delta_{k+1} \left(\sum_b c_b \llbracket x, y_1, \dots, y_k \rrbracket \right),$$

so that

$$\sum_{a \in \mathcal{A}_W} c_a \llbracket x, w_2^a, \dots, w_k^a \rrbracket = \delta_{k+1} \left(- \sum_b c_b \llbracket x, y_1, \dots, y_k \rrbracket \right).$$

Since this is true for all W , we have that $s' = \sum_a c_a \llbracket x, w_2^a, \dots, w_k^a \rrbracket$ is in the image of δ_{k+1} .

The proof that δ_n is injective is similar, but uses $d_1 : \text{Sh}_1(V) \rightarrow \text{Sh}_0(V)$ in place of the nonexistent δ_{n+1} , and is left to the reader (see [4, Theorem 2.1] for more details). \square

5. $\Gamma_0(n, N)$ -ORBITS OF SUBSPACES OF \mathbb{Q}^n

By considering the elements of \mathbb{Q}^n as row vectors, right multiplication by elements of $\text{GL}(n, \mathbb{Q})$ yields a right action of $\text{GL}(n, \mathbb{Q})$ on \mathbb{Q}^n . This action restricts to an action of $\Gamma_0(n, N)$ on \mathbb{Q}^n . In a natural way, we may also consider $\text{GL}(n, \mathbb{Q})$ (and hence also $\Gamma_0(n, N)$) as acting on the set of k -dimensional subspaces of \mathbb{Q}^n , for $0 \leq k \leq n$. We wish to find explicit representatives of the $\Gamma_0(n, N)$ -orbits of k -dimensional subspaces. Note that for $k = 0$ and $k = n$, there is only one k -dimensional subspace, and hence only one orbit, with a unique orbit representative.

Theorem 5.1. *Let $0 < k < n$ and assume that N is squarefree. Then the $\Gamma_0(n, N)$ -orbits of k -dimensional subspaces of \mathbb{Q}^n are in one-to-one correspondence with the set of positive divisors of N , where the orbit corresponding to the divisor d contains the k -dimensional subspace spanned by*

$$e_1 + de_{k+1}, e_2, e_3, e_4, \dots, e_k,$$

where e_i denotes the standard basis element of \mathbb{Q}^n with a 1 in the i th column, and 0's elsewhere.

Multiplication by elements of $S_0^\pm(n, N)$ preserves the $\Gamma_0(n, N)$ -orbits.

Proof. Let W be a k -dimensional subspace, and let M be a $k \times n$ matrix with integer entries whose rows span W . For $\gamma \in \Gamma_0(n, N)$, $M\gamma$ has row space $W\gamma$. Left multiplication by an element of $\text{GL}(k, \mathbb{Q})$ does not change the row space of a matrix, so we wish to find a canonical element of the double coset

$$\text{GL}(k, \mathbb{Q})M\Gamma_0(n, N).$$

Integer column operations on the rightmost $n-1$ columns of M can be represented by right multiplication by an element of $\Gamma_0(n, N)$; similarly, arbitrary row

operations correspond to left multiplication by an element of $\mathrm{GL}(k, \mathbb{Q})$. Using these operations we find that the row space of M is $\Gamma_0(n, N)$ -equivalent to the row space of

$$M' = \begin{pmatrix} a & 0_{k-1}^t & b & 0 & \cdots & 0 \\ 0_{k-1} & I_{k-1} & 0_{k-1} & 0_{k-1} & \cdots & 0_{k-1} \end{pmatrix},$$

where $a, b \in \mathbb{Z}$ and $\gcd(a, b) = 1$. Here, 0_{k-1} denotes the column vector of all zeros and length $k-1$, and 0_{k-1}^t denotes its transpose.

Since $\gcd(a, b) = 1$, we may find r and s so that $ar + bs = 1$. Then for all $\ell \in \mathbb{Z}$, $a(r + b\ell) + b(s - a\ell) = 1$ and for all $d \in \mathbb{Z}$,

$$(a + d(s - a\ell))(r + b\ell) + (b - d(r + b\ell))(s - a\ell) = 1.$$

Hence, the matrix

$$S = \begin{pmatrix} r + \ell b & -(b - d(r + b\ell)) \\ s - a\ell & a + d(s - a\ell) \end{pmatrix}$$

has determinant 1. In order for it to be in $\Gamma_0(2, N)$, we need $b - d(r + \ell b) = mN$ for some $m \in \mathbb{Z}$. To guarantee this, we choose $d = \gcd(b, N)$. Since N is squarefree, we see that $\gcd(b, N/d) = 1$, so that there are integers ℓ, m with $b\ell + mN/d = b/d - r$. With this choice of ℓ , we see that $S \in \Gamma_0(2, N)$.

Form the matrix T by replacing the $(1, 1)$, $(1, k+1)$, $(k+1, 1)$ and $(k+1, k+1)$ entries of I_n by the $(1, 1)$, $(1, 2)$, $(2, 1)$, and $(2, 2)$ entries of S . Then T is clearly in $\Gamma_0(n, N)$. Thus, the row space of M is $\Gamma_0(n, N)$ -equivalent to the row space of

$$M'T = \begin{pmatrix} 1 & 0_{k-1}^t & d & 0 & \cdots & 0 \\ 0_{k-1} & I_{k-1} & 0_{k-1} & 0_{k-1} & \cdots & 0_{k-1} \end{pmatrix}.$$

Hence, every $\Gamma_0(n, N)$ -orbit of a k -dimensional subspace of \mathbb{Q}^n contains a subspace of the proper form.

Suppose $S \in S_0^\pm(n, N)$ takes a subspace in the $\Gamma_0(n, N)$ -orbit corresponding to $d|N$, to a subspace in the orbit corresponding to $d'|N$. Then, after multiplying S by an appropriate element of $\Gamma_0(n, N)$ so that it takes the representative subspace of the orbit corresponding to d to the representative corresponding to d' , for some $x \in \mathbb{Z}$, we must have

$$(1, 0_{k-1}^t, d, 0_{n-k-1}^t)S = (x, *, \dots, *, xd', 0_{n-k-1}^t).$$

If we now denote the $(1, 1)$, $(1, k+1)$, $(k+1, 1)$ and $(k+1, k+1)$ entries of S by a , bN , r , and s , respectively, and note that $\gcd(a, N) = 1$, we see that we must have $a + rd = x$ and $bN + sd = d'x$. Hence, $bN + sd = d'a + rdd'$, and we see that $d|d'a$, so that $d|d'$. Now using that $\det(S) \cdot S^{-1} = S' \in S_0^\pm(n, N)$, replacing S by S' , and reversing the roles of d and d' , we must similarly have $d'|d$. Hence, $d = d'$. \square

Now, let W_0^k be the row space of the $k \times n$ matrix

$$M_0 = \left(I_k \mid 0 \right).$$

Let g_d be equal to the $n \times n$ identity matrix I_n , with the $(1, k+1)$ entry replaced by d . Let P_0^k be the stabilizer (in $\mathrm{GL}(n, \mathbb{Q})$, acting on the right) of W_0^k . Define $P_d^k = g_d^{-1} P_0^k g_d$; it is the stabilizer of the row space W_d^k of $M_0 g_d$, or in other words the stabilizer of the canonical representative of the $\Gamma_0(n, N)$ -orbit of k -dimensional subspaces of \mathbb{Q}^n corresponding to the divisor d of N . Typically, when k is understood, we will omit it, writing P_0 , P_d , or W_d rather than P_0^k , P_d^k , or W_d^k . We will call

a subgroup P_d a *representative maximal parabolic subgroup*, and denote its unipotent radical by U_d and its Levi quotient by $L_d = P_d/U_d$. Note that $U_d = g_d^{-1}U_0g_d$, where U_0 is the unipotent radical of P_0 .

The subgroup P_0 consists of block matrices

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

in which A is an invertible $k \times k$ matrix, C is an invertible $(n-k) \times (n-k)$ matrix, B is an arbitrary $(n-k) \times k$ matrix, and the block of zeroes is $k \times (n-k)$. For a block matrix $g \in P_0$ as above, we define $\psi_0^1(g) = A$ and $\psi_0^2(g) = C$. One sees easily that $\psi_0^1 : P_0 \rightarrow \mathrm{GL}(k, \mathbb{Q})$ and $\psi_0^2 : P_0 \rightarrow \mathrm{GL}(n-k, \mathbb{Q})$ are group homomorphisms.

For $s \in P_d$, define $\psi_d^1(s) = \psi_0^1(g_d s g_d^{-1})$.

We have the following straightforward generalization of [3, Theorem 7].

Theorem 5.2. *Let d be a positive divisor of N and assume that $(d, N/d) = 1$.*

- (1) *If $s \in P_d \cap S_0(n, N)^\pm$, then $\psi_d^1(s)_{11} \equiv s_{11} \pmod{d}$ and $\psi_d^2(s)_{11} \equiv s_{11} \pmod{N/d}$.*
- (2) *$\psi_d^1(P_d \cap S_0(n, N)^\pm) \subset S_0(k, d)^\pm$.*
- (3) *$\psi_d^2(P_d \cap S_0(n, N)^\pm) \subset S_0(n-k, N/d)^\pm$.*
- (4) *There is an exact sequence of groups*

$$1 \rightarrow U_d \cap \Gamma_0^\pm(n, N) \rightarrow P_d \cap \Gamma_0^\pm(n, N) \xrightarrow{\psi_d^1 \times \psi_d^2} \Gamma_0^\pm(k, d) \times \Gamma_0^\pm(n-k, N/d) \rightarrow 1.$$

Proof. Statements (1), (2), and (3) are proven as in [3, Theorem 7].

The only question in the exactness of the sequence in (4) is whether $\psi_d^1 \times \psi_d^2$ is surjective. To show this surjectivity, suppose that $A = (a_{ij}) \in \Gamma_0^\pm(k, d)$ and $B = (b_{ij}) \in \Gamma_0^\pm(n-k, N/d)$, choose $q, r \in \mathbb{Z}$ such that $q(N/d) + r(d) = a_{11} - b_{11}$ (which can be done since $\mathrm{gcd}(d, N/d) = 1$), and let Z be the block matrix

$$Z = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

where $C = (c_{ij})$ has

$$c_{ij} = \begin{cases} 0 & \text{if } i > 1, \\ a_{1j}/d & \text{if } i = 1 \text{ and } j > 1, \\ r & \text{if } i = j = 1. \end{cases}$$

We note that since $A \in \Gamma_0^\pm(k, d)$, each of these entries is an integer. One checks easily that $g_d^{-1}Zg_d \in \Gamma_0^\pm(n, N) \cap P_d$, and that $(\psi_d^1 \times \psi_d^2)(g_d^{-1}Zg_d) = (A, B)$. \square

6. A SPECTRAL SEQUENCE

Definition 6.1. Let V be an n -dimensional vector space over a field K with $n > 0$. For $0 \leq i \leq n$, let C_i be the set of all subspaces of V of dimension i .

Definition 6.2. Let V be an n -dimensional vector space over a field K with $n > 0$. For $-1 \leq i \leq n-1$ define

$$X_i = \bigoplus_{W \in C_{i+1}} \mathrm{St}(W).$$

We note that Theorem 4.2 yields an exact sequence of $\mathrm{GL}(n, K)$ -modules,

$$(1) \quad 0 \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \cdots \rightarrow X_0 \rightarrow \mathbb{Z} \rightarrow 0,$$

since $X_{-1} \cong \mathbb{Z}$.

Let $K = \mathbb{Q}$ and let \mathbb{F} be a field of characteristic $p > n + 1$. Let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$, and let M be an admissible right S -module over \mathbb{F} . We apply the spectral sequence of [5, Section 8], to the exact sequence (1) to obtain

$$\mathcal{E}_{q,r}^1 = H_r(\Gamma, X_q \otimes M) \implies H_{q+r}(\Gamma, M).$$

As in [5] this spectral sequence is equivariant for the action of the Hecke operators. For each q with $0 \leq q \leq n - 1$, we choose a set \hat{C}_{q+1} of Γ -orbit representatives of C_{q+1} . For a $W \in \hat{C}_{q+1}$, let Γ_W be the stabilizer of W in Γ . Then we may decompose

$$X_q \otimes M = \bigoplus_{W \in \hat{C}_{q+1}} \mathrm{Ind}_{\Gamma_W}^{\Gamma} \mathrm{St}(W) \otimes M$$

into a finite direct sum of induced modules.

Now, by the Shapiro isomorphism, we obtain

$$\mathcal{E}_{q,r}^1 \cong \bigoplus_{W \in \hat{C}_{q+1}} H_r(\Gamma_W, \mathrm{St}(W) \otimes M).$$

For $q < n - 1$, we note that \hat{C}_{q+1} consists exactly of the subspaces W_d^q , as d runs through the divisors of N . In addition, $\Gamma_W = \Gamma \cap P_d$, which we will denote by Γ_{P_d} . For $q = n - 1$, we have that $\hat{C}_{q+1}^k = V$. Hence, the first page of the spectral sequence has the following terms.

$$\mathcal{E}_{q,r}^1 = \begin{cases} H_r(\Gamma, \mathrm{St}(V) \otimes M) & \text{if } q = n - 1, \\ \bigoplus_{d|N} H_r(\Gamma_{P_d^{q+1}}, \mathrm{St}(W_d^{q+1}) \otimes M) & \text{if } q < n - 1. \end{cases}$$

The remainder of the paper is devoted to studying the terms of this spectral sequence, in order to show that for reducible Galois representations $\rho = \rho_1 \oplus \rho_2$ satisfying certain conditions, ρ fits $\mathcal{E}_{q,r}^1$, and that the eigenvector with ρ attached either survives to the infinity page of the spectral sequence, showing that ρ fits $H_{q+r}(\Gamma, M)$, or is killed off in such a way as to show that ρ nevertheless fits $H_s(\Gamma, M)$ for some value of s .

In order to proceed with this argument, it is necessary to show that the terms of the spectral sequence are finite-dimensional vector spaces over \mathbb{F} . For the terms with $q = n - 1$, this follows immediately from Borel-Serre duality. For $q < n - 1$, we prove the following theorem.

Theorem 6.3. *Let $0 < k < n$. Let $\Gamma = \Gamma_0(n, N)$, and assume that $p > n + 1$. Let $P = P_d^k$ for some $d|N$, let $W = W_d^k$, and let M be a finite-dimensional Γ_P -module. Then for $r \geq 0$, the homology*

$$H_r(\Gamma_P, \mathrm{St}(W) \otimes M)$$

is a finite dimensional vector space over \mathbb{F} .

Proof. For convenience, we will write $\ker \psi_d^1$ for $\ker \psi_d^1 \cap \Gamma_P$ in this proof. We use the Hochschild-Serre spectral sequence for the exact sequence

$$1 \rightarrow \ker \psi_d^1 \rightarrow \Gamma_P \rightarrow \Gamma_P / \ker \psi_d^1 \rightarrow 1.$$

This gives us a spectral sequence

$$E_{ij}^2 = H_i(\Gamma_P / \ker \psi_d^1, H_j(\ker \psi_d^1, \text{St}(W) \otimes M)) \implies H_{i+j}(\Gamma_P, \text{St}(W) \otimes M),$$

and if each term of the E^2 page is finite dimensional, the abutment must be finite dimensional. Now $\ker \psi_d^1$ acts trivially on $\text{St}(W)$, so a term E_{ij}^2 may be written as

$$H_i(\Gamma_P / \ker \psi_d^1, \text{St}(W) \otimes H_j(\ker \psi_d^1, M)).$$

We now note that Theorem 5.2(4) implies that $\Gamma_P / \ker \psi_d^1 \cong \psi_d^1(\Gamma_P) = \Gamma_0^\pm(k, d)$, so, noting that conjugation by g_d takes the action of Γ_P on W to an action of $\psi_d^1(\Gamma_P)$ on W_0 , we have that (as a vector space), E_{ij}^2 is isomorphic to

$$H_i(\Gamma_0^\pm(k, d), \text{St}(W_0) \otimes H_j(\ker \psi_d^1, M)),$$

where the action of $\gamma \in \Gamma_0^\pm(k, d)$ on $H_j(\ker \psi_d^1, M)$ is obtained by choosing any $\gamma' \in \Gamma_P$ with $\psi_d^1(\gamma') = \gamma$ and allowing γ' to act on $H_j(\ker \psi_d^1, M)$. By Borel-Serre duality, E_{ij}^2 is then isomorphic to

$$H^i(\Gamma_0^\pm(k, d), H_j(\ker \psi_d^1, M)),$$

which is finite dimensional since $\ker \psi_d^1$ is an arithmetic group and M is finite dimensional. \square

We also have the following related theorem.

Theorem 6.4. *Let $0 < k < n$. Let $\Gamma = \Gamma_0(n, N)$, and assume that $p > n + 1$. Let $P = P_d^k$ for some $d|N$, let $W = W_d^k$, and let M be a finite-dimensional $\Gamma_L = \Gamma_P / \Gamma_U$ -module. Then for $r \geq 0$, the homology*

$$H_r(\Gamma_L, \text{St}(W) \otimes M)$$

is a finite dimensional vector space over \mathbb{F} .

Proof. Since ψ_d^1 is trivial on Γ_U , it yields a well defined function on Γ_L . The proof is then basically the same as the proof of Theorem 6.3. \square

Remark 6.5. If $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$, and if P is a representative maximal parabolic subgroup stabilizing a subspace W , then the results of [5, Section 3] show that as long as (*) given any $s \in S$ we can choose left coset representatives s_α for Γs to be in S_P , there is an isomorphism $\mathcal{H}(\Gamma, S) \cong \mathcal{H}(\Gamma_P, S_P)$ and we can view $H_r(\Gamma_P, \text{St}(W) \otimes M)$ as an $\mathcal{H}(\Gamma, S)$ -module via this isomorphism. Hence, under condition (*), in studying systems of $\mathcal{H}(\Gamma, S)$ -eigenvalues in $H_r(\Gamma_P, \text{St}(W) \otimes M)$ we may study the homology as an $\mathcal{H}(\Gamma_P, S_P)$ -module. In Theorem 8.6, we show that condition (*) holds.

7. TWO CASES OF HECKE EQUIVARIANCE OF THE HOCHSCHILD-SERRE SPECTRAL SEQUENCE

From this section on, we require definitions of additional subsets of $\text{GL}(n, \mathbb{Q})$.

Definition 7.1. Let $P = P_d^k$ be a representative maximal parabolic subgroup of $\text{GL}(n, \mathbb{Q})$ for some $d|N$ and let p be a prime in \mathbb{Z} . We define the following subgroups and subsemigroups of $\text{GL}(n, \mathbb{Q})$.

- (1) $\Gamma(N) = \{A \in \text{GL}(n, \mathbb{Z}) : A \equiv I \pmod{N}\}$.
- (2) $S_k(N)$ is the set of matrices $A \in \text{GL}(n, \mathbb{Q})$ with integer entries and positive determinant prime to pN such that $A \equiv \text{diag}(1, \dots, 1, *, 1, \dots, 1, *) \pmod{N}$, with the $*$'s in the k and n positions.

- (3) $\Gamma_P(N) = \Gamma(N) \cap P$.
 (4) $S_P(N) = \{s \in S_k(N) \cap P : \psi_d^1(s), \psi_d^2(s) \text{ both have positive determinant}\}$.

We now prove two results involving the Hecke equivariance of the Hochschild-Serre spectral sequence. The first of these results (Theorem 7.11) is closely related to section 4 of [7]; however, the results here are more general. The second (Theorem 7.15) is similar to [1, Theorem 3.1].

Definition 7.2. Let P be a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ and let U be the unipotent radical of P . We note that $U(\mathbb{Q})$ is an abelian group, and we set r equal to the \mathbb{Q} -dimension of U . Let (Γ, S) be a congruence Hecke pair of level pN (see [1, Definition 1.2]). Then Γ_U is a free abelian group of rank r . Let M be a (p, N) -admissible S -module.

Let T be the set of matrices $t \in \mathrm{GL}(n, \mathbb{Q})$ with determinant prime to pN such that all denominators of both t and t^{-1} are prime to pN and t normalizes $U(\mathbb{Q})$. Let C_\bullet be the standard resolution (also known as the bar resolution) of \mathbb{Z} over $\mathrm{GL}(n, \mathbb{Q})$. For $t \in T$, we define an action of t on $C_\bullet \otimes_{\Gamma_U} M$ by setting

$$(c \otimes m) \cdot t = \frac{1}{d^r} \sum_b cu_b t \otimes mu_b t,$$

where d is any integer prime to p such that the right conjugation action of t on Γ_U^d is contained in Γ_U , and $\{u_b : 1 \leq b \leq d^r\}$ is a set of coset representatives for Γ_U^d inside Γ_U .

We can take $d = \det(t)$, so there is a d that fits the conditions of the definition (note that the condition that d be prime to p was inadvertently omitted in [7]). In [7, Lemma 4.3], this action of t is shown to be well defined; in particular, it does not depend on the choice of d or on the choice of coset representatives. It also commutes with the boundary operator. This action of individual elements of T does not normally extend to a group action of T on $C_\bullet \otimes_{\Gamma_U} M$. However, we now show that the action does, in fact, define a semigroup action of $S_P \subseteq T$ on $H_*(\Gamma_U, M)$ under certain hypotheses, making $H_*(\Gamma_U, M)$ an S_P -module. We note also that an element of T actually has entries in $\mathbb{Z}_{(p)}$ (the localization of \mathbb{Z} at the prime ideal (p)), so there is a well defined notion of reduction modulo p for such an element.

The proof of the following lemma is clear.

Lemma 7.3. *Fix t and d as in Definition 7.2. Then the largest subgroup H of Γ_U such that $t^{-1}Ht \subset \Gamma_U$ is*

$$H = \Gamma_U \cap t\Gamma_U t^{-1},$$

and we have $\Gamma_U^d \subseteq H \subseteq \Gamma_U$. Hence, $[\Gamma_U : H] \mid d^r$, and is thus prime to p (since we may take $d = \det(t)$).

By [24, p. 51], we note that if we write $\Gamma_U t \Gamma_U = \coprod_{j=1}^{e_t} \alpha_j \Gamma_U$, then we have $e_t = [\Gamma_U : H]$. There is a natural action of double cosets on homology and cohomology; when we consider the double coset $\Gamma_U t \Gamma_U$ as a Hecke operator, we will denote it by $[\Gamma_U t \Gamma_U]$. Since $C_\bullet \otimes_{\Gamma_U} M = H_0(\Gamma_U, C_\bullet \otimes_{\mathbb{Z}} M)$, the Hecke operator $[\Gamma_U t \Gamma_U]$ acts on $C_\bullet \otimes_{\Gamma_U} M$. This action is given by

$$c \otimes m [\Gamma_U t \Gamma_U] = \sum_{j=1}^{e_t} c \alpha_j \otimes m \alpha_j.$$

Lemma 7.4. *Let $t \in T$. Then with the notation defined above, the action of t on $C_\bullet \otimes M$ given in definition 7.2 is equal to the action of $\frac{1}{e_t}[\Gamma_U t \Gamma_U]$.*

Proof. The action of the Hecke operator $[\Gamma_U t \Gamma_U]$ is independent of the choice of coset representatives $\{\alpha_j\}$. We choose the α_j as follows: fix a collection of coset representatives $\{w_j : j = 1, \dots, e_t\}$ of H inside Γ_U , so that $\Gamma_U = \coprod_{j=1}^{e_t} w_j H$. Set $\alpha_j = w_j t$. One checks easily that the α_j thus defined give distinct cosets of Γ_U inside $\Gamma_U t \Gamma_U$; since there are e_t of them, they are a complete set of coset representatives.

Now choose a collection $\{v_k : k = 1, \dots, [H : \Gamma_U^d]\}$ of coset representatives of Γ_U^d inside H , so that $H = \coprod_k v_k \Gamma_U^d$. Then the set $\{w_j v_k\}$ is a complete collection of coset representatives of Γ_U^d inside Γ_U . Since the action of t is independent of the choice of coset representatives, we may choose $\{u_b\} = \{w_j v_k\}$. Since Γ_U is abelian, we have $w_j v_k = v_k w_j$. We then obtain

$$\begin{aligned}
(c \otimes m) \cdot t &= \frac{1}{d^n} \sum_{b=1}^{d^n} c u_b t \otimes m u_b t \\
&= \frac{1}{d^n} \sum_{j,k} c v_k w_j t \otimes m v_k w_j t \\
&= \frac{1}{d^n} \sum_{j,k} c v_k \alpha_j \otimes m v_k \alpha_j \\
&= \frac{1}{d^n} \left(\sum_k c v_k \otimes m v_k \right) [\Gamma_U t \Gamma_U] \\
&= \frac{1}{d^n} \left(\sum_k c \otimes m \right) [\Gamma_U t \Gamma_U] \\
&= \frac{[H : \Gamma_U^d]}{[\Gamma_U : \Gamma_U^d]} (c \otimes m) [\Gamma_U t \Gamma_U] \\
&= \frac{1}{e_t} (c \otimes m) [\Gamma_U t \Gamma_U],
\end{aligned}$$

where the v_k vanish in the sixth line because $v_k \in \Gamma_U$ and the tensor product is over Γ_U . \square

Let T' be any subgroup of T such that every element of $T' \cap U(\mathbb{Q})$ is congruent modulo p to an element of Γ_U . See Theorem 7.10 for examples where T' may be taken to be the group generated by a semigroup S_P . We will show that for any such subgroup T' , the individual actions of elements of T' on $H_k(\Gamma_U, M)$ compile together into a group action. To do this, we need to show that the composition of the actions of $s, t \in T'$ is equal to the action of st . We will see that this can be done on the level of the homology groups $H_k(\Gamma_U, M)$, but not on the level of chains $C_\bullet \otimes_{\Gamma_U} M$. Our first step is the following lemma, from [24, p. 51] (since we are working with right modules, we have changed right cosets to left cosets).

Lemma 7.5. *Let $s, t \in T$, and suppose that $\Gamma_U s \Gamma_U = \coprod_i s_i \Gamma_U$ and $\Gamma_U t \Gamma_U = \coprod_j t_j \Gamma_U$. We may choose a finite set $\Xi \subset T$ such that*

$$\Gamma_U s \Gamma_U t \Gamma_U = \coprod_{\xi \in \Xi} \Gamma_U \xi \Gamma_U.$$

Then in the Hecke algebra we have the equality

$$[\Gamma_U s \Gamma_U][\Gamma_U t \Gamma_U] = \sum_{\xi \in \Xi} m(\xi) [\Gamma_U \xi \Gamma_U],$$

where $m(\xi) = |\{(i, j) | s_i t_j \Gamma_U = \xi \Gamma_U\}|$.

Lemma 7.6. *With notation as in Lemma 7.5, for each $\xi \in \Xi$, we have $\xi = stu(\xi)$ for some $u(\xi) \in U(\mathbb{Q})$, and*

$$\Gamma_U \xi \Gamma_U = \Gamma_U st \Gamma_U u(\xi).$$

In addition,

$$[\Gamma_U st \Gamma_U][\Gamma_U u(\xi) \Gamma_U] = [\Gamma_U \xi \Gamma_U],$$

and $e_{st} = e_\xi$.

Proof. Since T normalizes $U(\mathbb{Q})$, and $U(\mathbb{Q})$ is abelian, this is immediate. \square

Lemma 7.7. *Let $u \in T \cap U(\mathbb{Q})$, and assume that there is some $u' \in \Gamma_U$ that is congruent to u modulo pN . Let M be a (p, N) -admissible T -module. Then $[\Gamma_U u \Gamma_U]$ acts trivially on the homology groups*

$$H_*(\Gamma_U, M).$$

Proof. There is only one single coset in the double coset $\Gamma_U u \Gamma_U$, so the Hecke operator $[\Gamma_U u \Gamma_U]$ acts on the homology as u does.

Since u centralizes Γ_U , it acts on the homology via its action on M . Since M is (p, N) -admissible the action of u' and the action of u on M are the same. Hence, the action of u and the action of u' on the homology are the same. However, u' acts trivially on the homology, by [14, Proposition III.8.1]. \square

Remark 7.8. We note that this lemma fails if we apply it on the chain level. This is because the chains, $C \otimes_{\Gamma_U} M = H_0(\Gamma_U, C \otimes_{\mathbb{Z}} M)$, although acted on by the Hecke operators, are the homology with coefficients in $C \otimes M$, which is not an admissible coefficient module.

Corollary 7.9. *Let T' be any subgroup of T such that every element of $T' \cap U(\mathbb{Q})$ is congruent modulo p to an element of Γ_U . Let $s, t \in T'$. Then, with notation as in Lemma 7.5, we have, for $z \in H_*(\Gamma_U, M)$,*

$$z[\Gamma_U s \Gamma_U][\Gamma_U t \Gamma_U] = z \left(\sum_{\xi \in \Xi} m(\xi) \right) [\Gamma_U st \Gamma_U].$$

Also,

$$z \frac{1}{e_s e_t} [\Gamma_U s \Gamma_U][\Gamma_U t \Gamma_U] = z \frac{1}{e_{st}} [\Gamma_U st \Gamma_U],$$

and we see that the action of individual elements of T' on $H_*(\Gamma_U, M)$ in Definition 7.2 yields a group action of T' given by $z \cdot s = z \frac{1}{e_s} [\Gamma_U s \Gamma_U]$.

Proof. The first displayed equation follows immediately from Lemmas 7.6 and 7.7. For the second, by [24, Proposition 3.3], Lemma 7.5, and Lemma 7.6, we have

$$\begin{aligned} e_{st} \sum_{\xi \in \Xi} m(\xi) &= \sum_{\xi \in \Xi} e_\xi m(\xi) = \deg([\Gamma_U s \Gamma_U][\Gamma_U t \Gamma_U]) \\ &= \deg([\Gamma_U s \Gamma_U]) \deg([\Gamma_U t \Gamma_U]) = e_s e_t. \end{aligned}$$

Hence, by Lemma 7.4 and the above equation,

$$(z \cdot s) \cdot t = z \frac{1}{e_s e_t} [\Gamma_U s \Gamma_U] [\Gamma_U t \Gamma_U] = z \frac{\sum m(\xi)}{e_s e_t} [\Gamma_U s t \Gamma_U] = z \frac{1}{e_{st}} [\Gamma_U s t \Gamma_U] = z \cdot (st). \quad \square$$

Theorem 7.10. *Let $P = P_d^k$ for some $d|N$ and some k . Let (Γ, S) be either $(\Gamma(pN), S_k(pN))$ or $(\Gamma_0(n, N), S_0(n, N))$. Let $z \in H_*(\Gamma_U, M)$, and let $s, t \in S_P$. Then the action of individual elements of S_P on $H_*(\Gamma_U, M)$ in Definition 7.2 yields a semigroup action of S_P under which S_U acts trivially.*

Proof. By the previous lemmas, we are finished if we can show that S_P lies in a subgroup T' such that every element of $T' \cap U(\mathbb{Q})$ is congruent modulo p to an element of Γ_U . It suffices to show that every element of $S_P^{-1} S_P \cap U(\mathbb{Q})$ is congruent modulo p to an element of Γ_U .

For $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$ one checks that the intersection $S_P^{-1} S_P \cap U(\mathbb{Q})$ is contained in the set of matrices

$$\mathcal{M} = \left\{ g_d^{-1} \left(\begin{array}{c|c} I_k & 0 \\ \hline A & I_{n-k} \end{array} \right) g_d \right\},$$

where the entries of A are rational, with denominators prime to pN , and the entries in the top row of A have numerators divisible by N/d . The set $\Gamma_U = \Gamma \cap U$ consists of exactly those matrices in \mathcal{M} with integer entries. Since that any rational with denominator prime to pN lies in $\mathbb{Z}_{(p)}$, we see that it is congruent modulo p to some integer. Thus, every element of \mathcal{M} is congruent modulo p to an element of Γ_U .

For $(\Gamma, S) = (\Gamma(pN), S_k(pN))$, suppose that $u \in S_P^{-1} S_P \cap U(\mathbb{Q})$. Then $g_d u g_d^{-1} \in S_P^{-1} S_P \cap U_0(\mathbb{Q})$ (where we use the fact that g_d normalizes $S_k(pN)$). Hence, $g_d u g_d^{-1} \equiv \text{diag}(1, \dots, 1, *, 1, \dots, 1, *)$ modulo pN , and has all diagonal elements equal to 1, so $g_d u g_d^{-1} \equiv I_n \pmod{pN}$. Hence, $u \equiv I_n \pmod{pN}$, and we are finished. \square

Theorem 7.11. *Let $P = P_d^k$ be a representative maximal parabolic subgroup of $\text{GL}(n, \mathbb{Q})$ stabilizing the subspace $W = W_d^k$. Let M be a (p, N) -admissible S -module with $p > n + 1$. Let (Γ, S) equal $(\Gamma_0(n, N), S_0(n, N))$ or $(\Gamma(pN), S_k(pN))$. Then the Hecke algebra $\mathcal{H}(\Gamma_P, S_P)$ acts equivariantly on the Hochschild-Serre spectral sequence*

$$E_{ij}^2 = H_i(\Gamma_L, H_j(\Gamma_U, \text{St}(W) \otimes M)) \implies H_{i+j}(\Gamma_P, \text{St}(W) \otimes M),$$

and a given packet of Hecke eigenvalues occurs in $H_k(\Gamma_P, \text{St}(W) \otimes M)$ if and only if it appears in

$$\bigoplus_{i+j=k} E_{ij}^\infty.$$

Proof. For the given congruence subgroups, S_U and Γ_U have the same image modulo pN , so the proof of [7, Theorem 4.6] applies with only minor changes to account for the factor of $\text{St}(W)$, and with the appeal to [7, Theorem 4.4] (which requires the irreducibility of $H_*(\Gamma_U, M)$) replaced by an appeal to our Theorem 7.10, which requires that $U(\mathbb{Q})$ be abelian, which it is, since P is a maximal parabolic subgroup. Note that we need to use the finite dimensionality of each term in the E^2 page of the spectral sequence (from Theorem 6.4) for the final statement to be true. \square

We will require one other instance of the Hecke invariance of the Hochschild-Serre spectral sequence. We refer again to ([1, p. 238]) for the definition of a Hecke

pair, and give the definition of compatible Hecke pairs from [11]. Note that since we are working with left cosets rather than right cosets, we reverse the order of multiplication in this definition.

Definition 7.12. [11, Definition 1.1.2] A Hecke pair (Γ, S) is said to be compatible to the Hecke pair (Γ', S') if

- (1) $\Gamma \subseteq \Gamma'$ and $S \subseteq S'$,
- (2) $\Gamma' \cap S^{-1}S = \Gamma$,
- (3) $S\Gamma' = S'$.

If (Γ, S) is compatible to (Γ', S') , then the natural map $\mathcal{H}(\Gamma, S) \rightarrow \mathcal{H}(\Gamma', S')$ of Hecke algebras is an isomorphism.

Lemma 7.13. *Let $P = P_d^k$ for some positive $d|N$ and some k with $1 \leq k < n$ and let (Γ, S) be a Hecke pair such that $\Gamma(pN) \subset \Gamma$, $S_k(pN) \subset S$, the elements of S have positive determinant, and $S^{-1}S \cap \mathrm{SL}(n, \mathbb{Z}) = \Gamma$. Then the Hecke pairs $(\Gamma_P(pN), S_P(pN))$ and (Γ_P, S_P) are compatible.*

Proof. Note that $\Gamma \subseteq S \cap \mathrm{SL}(n, \mathbb{Z}) \subseteq S^{-1}S \cap \mathrm{SL}(n, \mathbb{Z}) = \Gamma$, so $\Gamma = S \cap \mathrm{SL}(n, \mathbb{Z})$.

- (1) Since $\Gamma_P(pN) \subset \Gamma_P$ and $S_P(pN) \subset S_P$, we have the necessary containments.
- (2) We now wish to show that $\Gamma_P \cap S_P(pN)^{-1}S_P(pN) = \Gamma_P(pN)$. Let $s_1, s_2 \in S_P(pN)$ and suppose that $s = s_1^{-1}s_2 \in \Gamma_P$. Then s has integer entries and determinant 1. We see that $\det(\psi_d^i(s)) = \pm 1$ for $i = 1, 2$; in fact, since each $\det(\psi_d^i(s)) > 0$, each $\det(\psi_d^i(s)) = 1$. We also have that $s \equiv \mathrm{diag}(1, \dots, *, 1, \dots, *) \pmod{pN}$. Hence, $g_d s g_d^{-1} \equiv \mathrm{diag}(1, \dots, *, 1, \dots, *) \pmod{pN}$ and we see that each $*$ must be 1. Hence, $s \in \Gamma_P(pN)$.
- (3) Finally, let $s \in S_P$. Then

$$g_d s g_d^{-1} = \left(\begin{array}{c|c} s_1 & 0 \\ \hline A & s_2 \end{array} \right) \in P_0.$$

We easily find

$$\gamma = \left(\begin{array}{c|c} \gamma_1 & 0 \\ \hline B & \gamma_2 \end{array} \right) \in \mathrm{SL}(n, \mathbb{Z}) \cap P_0$$

such that $g_d s g_d^{-1} \gamma \equiv \mathrm{diag}(1, \dots, *, 1, \dots, *) \pmod{pN}$. This matrix has positive determinant; if $\det(s_1 \gamma_1) < 0$, we multiply γ on the right by the matrix $\mathrm{diag}(1, \dots, -1, 1, \dots, -1)$. With this adjustment, we see that $s(g_d^{-1} \gamma g_d) \in S_P(pN)$. Then we see that $g_d^{-1} \gamma g_d \in \mathrm{SL}(n, \mathbb{Z}) \cap S^{-1}S_k(pN) \subset \mathrm{SL}(n, \mathbb{Z}) \cap S^{-1}S = \Gamma$ and is contained in P , so $g_d \gamma g_d^{-1} \in \Gamma_P$. Hence, $s \in S_P(pN)\Gamma_P$, as desired. \square

Remark 7.14. Note that $(\Gamma_0(n, N), S_0(n, N))$ is a Hecke pair that satisfies the conditions of Lemma 7.13.

Theorem 7.15. *Let (Γ, S) be a Hecke pair such that $\Gamma(pN) \subset \Gamma$, $S_k(pN) \subset S$, the elements of S have positive determinant, and $S^{-1}S \cap \mathrm{SL}(n, \mathbb{Z}) = \Gamma$. Assume that $p > n + 1$. Let $P = P_d$ with $d|N$. Any system of Hecke eigenvalues occurring in $H_k(\Gamma_P, \mathrm{St}(W) \otimes M)$ also appears in $H_j(\Gamma_P(pN), \mathrm{St}(W) \otimes M)$ for some $j \leq k$.*

Proof. We have seen in Lemma 7.13 that the Hecke pairs $(\Gamma_P(pN), S_P(pN))$ and (Γ_P, S_P) are compatible. The Hochschild Serre spectral sequence for the exact sequence $0 \rightarrow \Gamma_P(pN) \rightarrow \Gamma_P \rightarrow \Gamma_P/\Gamma_P(pN) \rightarrow 0$ computes $H_i(\Gamma_P, M)$ in two different ways (as in [14, VII.7.6] and [7, Theorem 4.6]) by using two spectral sequences to compute the total homology of the double complex

$$F_\bullet \otimes_{\Gamma_P/\Gamma_P(pN)} (C_\bullet \otimes_{\Gamma_P(pN)} (\text{St}(W) \otimes M)),$$

where F_\bullet is the standard resolution of \mathbb{Z} over the finite group $\Gamma_P/\Gamma_P(pN)$, and C_\bullet is the standard resolution of \mathbb{Z} over $\text{GL}(n, \mathbb{Q})$. We let the Hecke algebra $\mathcal{H}(\Gamma_P(pN), S_P(pN))$ (and hence, by compatibility, $\mathcal{H}(\Gamma_P, S_P)$) act on the double complex by its natural action on $C_\bullet \otimes_{\Gamma_P(pN)} (\text{St}(W) \otimes M)$ (with the trivial action on F_\bullet). This action commutes with the differentials of the double complex, and hence the spectral sequence is Hecke equivariant. Therefore, any system of Hecke eigenvalues appearing in the abutment $H_k(\Gamma_P, M)$ of the first spectral sequence must occur in the E^1 page of the other, i.e., in $F_i \otimes_{\Gamma_P/\Gamma_P(pN)} H_j(\Gamma_P(pN), \text{St}(W) \otimes M)$ for some $i + j = k$. This uses the fact that F_i has finite rank over \mathbb{Z} for each i , and the finite-dimensionality of each $H_j(\Gamma_P(pN), \text{St}(W) \otimes M)$ (by adapting the proof of Theorem 6.3). Since we have chosen to have the Hecke algebra act trivially on F_i , the desired system of eigenvalues must appear in $H_j(\Gamma_P(pN), \text{St}(W) \otimes M)$ for some $j \leq k$. \square

8. HECKE MATRICES

Throughout this section, fix a positive squarefree integer N , a positive divisor $d|N$, a positive integer n , let $0 < k < n$, and let P_0 and g_d be defined as in Section 5. Let $\Gamma = \Gamma_0(n, N)$ and let $S = S_0(n, N)$.

We have defined $s(\ell, r, n) = \text{diag}(1, \dots, 1, \ell, \dots, \ell)$ to be a diagonal $n \times n$ matrix with diagonal entries 1 and ℓ and determinant ℓ^r . The Hecke algebra $\mathcal{H}(\Gamma, S)$ is generated by the double cosets of $s(\ell, r, n)$ as ℓ varies over primes not dividing N and $0 \leq r \leq n$. The following description of left coset representatives of the double coset $\Gamma s \Gamma$ is easily confirmed.

Theorem 8.1. *Set $\Gamma = \Gamma_0(n, N)$ and $S = S_0(n, N)$. Let $s = s(\ell, r, n)$ for $\ell \nmid pN$.*

- (1) *In the decomposition $\Gamma s \Gamma = \coprod s_\alpha \Gamma$, we may take the s_α to be the set of lower triangular matrices (a_{ij}) of determinant ℓ^r , where the entries a_{ij} satisfy the following conditions:*
 - (a) *If $i < j$, then $a_{ij} = 0$.*
 - (b) *Each a_{ii} is either 1 or ℓ .*
 - (c) *If $i > j$, then $0 \leq a_{ij} < \ell$.*
 - (d) *If $a_{ii} = 1$ or $a_{jj} = \ell$, then $a_{ij} = 0$.*
- (2) *For each $i < j$, let $R_{ij} \in \mathbb{Z}$ be a complete residue system modulo ℓ such that $0 \in R_{ij}$, and assume that every element of R_{ij} is divisible by N if $i = 1$. In the decomposition $\Gamma s \Gamma = \coprod s_\alpha \Gamma$, we may take the s_α to be the set of upper triangular matrices (a_{ij}) of determinant ℓ^r , where the entries a_{ij} satisfy the following conditions.*
 - (a) *If $i > j$ then $a_{ij} = 0$.*
 - (b) *Each a_{ii} is either 1 or ℓ .*
 - (c) *if $i < j$, then $a_{ij} \in R_{ij}$.*
 - (d) *If $a_{ii} = 1$ or $a_{jj} = \ell$, then $a_{ij} = 0$.*

Remark 8.2. Typically we will take the $R_{ij} = \{0, 1, \dots, \ell - 1\}$ for $i > 1$, and $R_{ij} = \{0, N, 2N, \dots, (\ell - 1)N\}$ for $i = 1$.

Definition 8.3. Let $\mathcal{T}_n(\ell, r)$ be the set of lower triangular s_α defined above, and let $\mathcal{T}^n(\ell, r)$ be the set of upper triangular s_α defined above. Note that we suppress the dependence of the set $\mathcal{T}^n(\ell, r)$ on N and the possible choices of M .

Theorem 8.4. *Let $0 < k < n$, and let $0 \leq r \leq n$, and let $s = s(\ell, r, n)$. The following set of matrices (given in block form) is a complete set of left coset representatives of $\Gamma s \Gamma$.*

$$\mathcal{T}(s, k) = \left\{ \begin{pmatrix} t_1 & 0 \\ * & t_2 \end{pmatrix} : \max(0, r - (n - k)) \leq m \leq \min(r, k), \right. \\ \left. t_1 \in \mathcal{T}^k(\ell, m), t_2 \in \mathcal{T}_{n-k}(\ell, r - m) \right\}$$

where the entries a_{ij} in the lower left block (with $i > k$ and $j \leq k$) satisfy the following conditions:

- (1) $1 \leq a_{ij} \leq \ell$
- (2) $a_{ij} = 0$ if $a_{ii} = 1$ or $a_{jj} = \ell$

Proof. One can count the number of matrices in the set, and it is equal to the number of single coset representatives of s in the double coset $\Gamma s \Gamma$. It is a simple matter to check that no two of the matrices in the set are in the same coset. \square

Remark 8.5. Note that if we were to change the definition of $\mathcal{T}(s, k)$ so that $t_1 \in \mathcal{T}_k(\ell, m)$, then the resulting set would just be $\mathcal{T}_n(\ell, r)$.

We now prove that we can choose the coset representatives of $\Gamma s \Gamma$ to be in P_d .

Theorem 8.6. *Let $0 < k < n$, $d|N$, and $P = P_d^k$, and let $s = s(\ell, r, n)$ with $\ell \nmid pN$. Then for each element $t \in \mathcal{T}(s, k)$, there is some $\gamma \in \Gamma_0(n, N)$ such that $t\gamma \in P_d$.*

Proof. We distinguish five cases.

- (1) Assume that the $(1, 1)$ and the $(k+1, k+1)$ entries of t are both equal to 1. Then we may take $\gamma = I$, and we see that $g_d t \gamma g_d^{-1} \in P_0$, so that $t\gamma \in P_d$.
- (2) Assume that the $(1, 1)$ and the $(k+1, k+1)$ entries of t are both equal to ℓ . Then we may take $\gamma = I$, and we see that $g_d t \gamma g_d^{-1} \in P_0$, so that $t\gamma \in P_d$.
- (3) Assume that the $(1, 1)$ -entry of t is ℓ and the $(k+1, k+1)$ entry is 1. We may choose $a, b \in \mathbb{Z}$ so that

$$a(d) + b(\ell N/d) = 1 - \ell$$

(since d and $\ell N/d$ are relatively prime). Let γ be the $n \times n$ matrix which is the identity, except for the $(1, 1)$, $(1, k+1)$, $(k+1, 1)$ and $(k+1, k+1)$ entries, which are (respectively) $1 + bN/d$, bN , a , $\ell + ad$. Then $\gamma \in \Gamma$, and we check by direct computation that $g_d t \gamma g_d^{-1} \in P_0$.

- (4) Assume that the $(1, 1)$ -entry of t is 1 and the $(k+1, k+1)$ entry is ℓ . Let e be the $(k+1, 1)$ entry of t , and assume that $\ell \nmid ed + 1$. Then we may choose a and b so that $a(\ell d) + b((ed + 1)N/d) = 1 - (ed + 1)\ell$ (since ℓd and $(ed + 1)N/d$ are relatively prime). Letting γ be the $n \times n$ matrix which is the identity, except for the $(1, 1)$, $(1, k+1)$, $(k+1, 1)$ and $(k+1, k+1)$ entries, which are (respectively) $\ell + bN/d$, bN , a , $(ed + 1) + ad$, we see that $\gamma \in \Gamma$, and we check by direct computation that $g_d t \gamma g_d^{-1} \in P_0$.

- (5) Assume that the $(1, 1)$ -entry of t is 1 and the $(k+1, k+1)$ entry is ℓ . Let e be the $(k+1, 1)$ entry of t , and assume that $\ell \mid ed + 1$. Then we may choose a and b so that

$$a(d) + b \left(\frac{(ed+1)}{\ell} N/d \right) = 1 - \frac{(ed+1)}{\ell}$$

(since d and $\frac{(ed+1)N}{\ell d}$ are relatively prime). Letting γ be the $n \times n$ matrix which is the identity, except for the $(1, 1)$, $(1, k+1)$, $(k+1, 1)$ and $(k+1, k+1)$ entries, which are (respectively) $1 + bN/d$, bN , a , $\frac{(ed+1)}{\ell} + ad$, we see that $\gamma \in \Gamma$, and we check by direct computation that $g_d t \gamma g_d^{-1} \in P_0$. \square

Since any single coset representative of $\Gamma s \Gamma$ may be taken to be s , this theorem shows that for each $s = s(\ell, i, n)$, we can find an $s' \in S_P$ with $\Gamma s \Gamma = \Gamma s' \Gamma$.

The next theorem is designed to enable us to compute the Hecke operators on a tensor product of two Hecke modules. As such, it considers the Hecke operator $T_n(\ell, r)$ to be a sum in the free abelian group generated by single cosets. In order to facilitate the statement of this theorem, we make the following definitions.

Definition 8.7. Let N and k be positive integers. Set $C_{k,N}$ to be the set of left cosets of $\Gamma_0(k, N)$ inside $S_0(k, N)$. Denote by $F_{k,N}$ the free abelian group on the elements of $C_{k,N}$. For a collection \mathcal{S} of matrices in $S_0(k, N)$, we will write $\overline{\mathcal{S}}$ for the element

$$\sum_{s \in \mathcal{S}} s \Gamma_0(k, N).$$

Theorem 8.8. Let $0 < k < n$, $d \mid N$, and $P = P_d$. Choose a prime ℓ with $\ell \nmid pN$, and let $s = s(\ell, r, n)$. Then in the tensor product $F_{k,d} \otimes_{\mathbb{Z}} F_{n-k, N/d}$, we have

$$\begin{aligned} \sum_{t \in \mathcal{T}(s,k)} \psi_d^1(t) \Gamma_0(k, d) \otimes \psi_d^2(t) \Gamma_0(n-k, N/d) \\ = \sum_{m=\max(0, r-(n-k))}^{\min(r,k)} \ell^{(k-m)(r-m)} \overline{\mathcal{T}_k(\ell, m)} \otimes \overline{\mathcal{T}^{n-k}(\ell, r-m)}. \end{aligned}$$

Proof. We divide the elements of $\mathcal{T}^k(\ell, m)$ into two subsets, $A^k(\ell, m)$ and $B^k(\ell, m)$, where $A^k(\ell, m)$ consists of elements of $\mathcal{T}^k(\ell, m)$ with a 1 in the $(1, 1)$ -position, and $B^k(\ell, m)$ consists of elements with an ℓ in the $(1, 1)$ -position. Then $\mathcal{T}^k(\ell, m) = A^k(\ell, m) \cup B^k(\ell, m)$. We also note that

$$\mathcal{T}^k(\ell, m) = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & t \end{array} \right) : t \in \mathcal{T}^{k-1}(\ell, m) \right\} \cup \left\{ \left(\begin{array}{c|c} \ell & r \\ \hline 0 & t \end{array} \right) : t \in \mathcal{T}^{k-1}(\ell, m-1) \right\},$$

where the r runs through row vectors (r_2, \dots, r_k) with $r_j = 0$ if the $(j-1, j-1)$ entry of t is ℓ , and $r_j \in R_{1j}$ otherwise.

Similarly, we divide $\mathcal{T}_{n-k}(\ell, m) = A_{n-k}(\ell, m) \cup B_{n-k}(\ell, m)$, based on whether the $(1, 1)$ -entries are 1 or ℓ . Then

$$A_{n-k}(\ell, m) = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline c & t \end{array} \right) : t \in \mathcal{T}_{n-k-1}(\ell, m) \right\}$$

and

$$B_{n-k}(\ell, m) = \left\{ \left(\begin{array}{c|c} \ell & 0 \\ \hline 0 & t \end{array} \right) : t \in \mathcal{T}_{n-k-1}(\ell, m-1) \right\},$$

where c runs through appropriate column vectors.

We now evaluate $\psi_d^1(t)\Gamma_0(k, d) \times \psi_d^2(t)\Gamma_0(n-k, N/d)$ for $t \in \mathcal{T}(r, k)$ in each of the five cases in the proof of Theorem 8.6. To illustrate the method, we give details of case (1), and just give the results of the other cases.

(1) In this case, we have that

$$t = \left(\begin{array}{c|c} t_1 & 0 \\ \hline C & t_2 \end{array} \right),$$

with $t_1 \in A^k(\ell, m)$ and $t_2 \in A_{n-k}(\ell, r-m)$, with $\max(0, r-(n-k)) \leq m \leq \min(r, k)$ and the entries of C are between 0 and $\ell-1$ and an entry must be equal to 0 unless it lies beneath a 1 on the diagonal and to the left of a ℓ on the diagonal. For each choice of t_1, t_2 , there are $\ell^{(k-m)(r-m)}$ distinct C that can occur. Multiplying by $\gamma = I_n$ and conjugating by g_d fixes t , so we have

$$\psi_d^1(t)\Gamma_0(k, d) \otimes \psi_d^2(t)\Gamma_0(n-k, N/d) = t_1\Gamma_0(n-k, N/d) \otimes t_2\Gamma_0(n-k, N/d),$$

and each element of $\overline{A^k(\ell, m)} \times \overline{A_{n-k}(\ell, r-m)}$ appears $\ell^{(k-m)(r-m)}$ times. In other words, as we allow m to vary, and go through all $t \in \mathcal{T}(r, k)$ that fall into case (1), we obtain

$$\sum_{m=\max(0, r-(n-k))}^{\min(r, k-1)} \ell^{(k-m)(r-m)} \overline{A^k(\ell, m)} \otimes \overline{A_{n-k}(\ell, r-m)}.$$

(2) As t runs through all elements of $\mathcal{T}(r, k)$ that fall into case (2), we obtain

$$\sum_{m=\max(0, r-(n-k))}^{\min(k, r)} \ell^{(k-m)(r-m)} \overline{B^k(\ell, m)} \otimes \overline{B_{n-k}(\ell, r-m)}.$$

(3) As t runs through all elements of $\mathcal{T}(r, k)$ that fall into case (3), we obtain

$$\sum_{m=\max(0, r-(n-k))}^{\min(k, r)} \ell^{(k-m)(r-m)-1} \overline{A^k(\ell, m)} \otimes \overline{B_{n-k}(\ell, r-m)}.$$

(4) As t runs through all elements of $\mathcal{T}(r, k)$ that fall into case (4), we obtain

$$\sum_{m=\max(0, r-(n-k))}^{\min(k, r)} (\ell^{(k-m)(r-m)} - \ell^{(k-m)(r-m)-1}) \overline{A^k(\ell, m)} \otimes \overline{B_{n-k}(\ell, r-m)}.$$

(5) As t runs through all elements of $\mathcal{T}(r, k)$ that fall into case (5), we obtain

$$\sum_{m=\max(0, r-(n-k))}^{\min(k, r)} \ell^{(k-m)(r-m)} \overline{B^k(\ell, m)} \otimes \overline{A_{n-k}(\ell, r-m)}.$$

Then cases (1), (3), and (4) add to

$$\sum_{\max(0, r-(n-k))}^{\min(r, k)} \ell^{(k-m)(r-m)} \overline{A^k(\ell, m)} \otimes \overline{\mathcal{T}_{n-k}(\ell, r-m)},$$

and cases (2) and (5) add to

$$\sum_{\max(0, r-(n-k))}^{\min(r, k)} \ell^{(k-m)(r-m)} \overline{B^k(\ell, m)} \otimes \overline{\mathcal{T}_{n-k}(\ell, r-m)}.$$

Adding these, we obtain

$$\sum_{\max(0, r-(n-k))}^{\min(r, k)} \ell^{(k-m)(r-m)} \overline{\mathcal{T}^k(\ell, m)} \otimes \overline{\mathcal{T}_{n-k}(\ell, r-m)},$$

which is the desired result. \square

9. TWISTING THE ACTION ON THE COEFFICIENTS

Let M be a right $\mathrm{GL}(n, \mathbb{F}_p)$ -module, let N be a positive integer prime to p , and let d be a positive divisor of N . Let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{F}^\times$ be a character. Let $S = S_0(n, N)$ act on M via reduction modulo p . Then M is an admissible $\mathbb{F}_p[S_0(n, N)]$ -module. Let $\theta : S \rightarrow (\mathbb{Z}/N\mathbb{Z})^\times$ take $s \in S$ to the mod N reduction of its $(1, 1)$ entry. As before, we define the nebentype character $S \rightarrow \mathbb{F}^\times$ as the composition $\epsilon \circ \theta$. We will write $\epsilon(s)$ for this composition. Let M_ϵ be the module consisting of the elements of M , with the action of S adjusted to equal

$$m|_\epsilon s = \epsilon(s)ms.$$

Let $P_0 = P_0^k$ be a standard maximal parabolic subgroup defined as in section 5. As in that section, let $P = P_d = g_d^{-1}P_0g_d$. Let $S_P = S \cap P$. We denote by M_ϵ^d the S_P -module on which an element of S_P acts as

$$m|_\epsilon^d s = \epsilon(s)m \cdot (g_d s g_d^{-1}).$$

When using this notation, if $d = 0$ we omit it; similarly if $\epsilon = 1$ we omit it. As in [5, Section 5], we note that if $M = F(a_1, \dots, a_n)$, then M^d is isomorphic to $F(a_1, \dots, a_n)$, with g_d acting as an intertwining operator.

Let $\Gamma = \Gamma_0(n, N)$, let $U = U_d$ be the unipotent radical of P_d , $L = P/U$, and let $\Gamma_U = \Gamma \cap U$, $S_U = S \cap U$, and $S_L = S_P/S_U$. We note that $U(\mathbb{Z})$ is free abelian of rank $k(n-k)$; Γ_U is a subgroup of $U(\mathbb{Z})$ of finite index prime to p .

We now prove a special case of a version of Kostant's theorem in characteristic p in the top dimension.

Theorem 9.1. *Let N be squarefree and prime to p , let $\epsilon : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{F}^\times$ be a character, let $d|N$ and let $1 \leq k \leq n-1$. Let $P = P_d^k$ and let $U = U_d$ be the unipotent radical of P . Set $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$. Let $r = k(n-k)$ be the \mathbb{Q} -dimension of U . Then*

$$H_r(\Gamma_U, F(a_1, \dots, a_n)_\epsilon) \cong (F(a_1+(n-k), \dots, a_k+(n-k)) \otimes F(a_{k+1}-k, \dots, a_n-k))_\epsilon^d$$

as S_L -modules.

Proof. Set $M = F(a_1, \dots, a_n)$. Since ϵ is trivial on Γ_U , we see that as an S_P -module, $H_r(\Gamma_U, M_\epsilon) = H_r(\Gamma_U, M)_\epsilon$. We note (as in [14, p. 79]) that $H_r(\Gamma_U, M) \cong H_r(g_d \Gamma_U g_d^{-1}, M)$ as abelian groups, with the isomorphism defined on the chain level by taking a resolution X of \mathbb{Z} as a $\mathrm{GL}(n, \mathbb{Q})$ -module, and mapping $X \otimes_{\Gamma_U} M \rightarrow X \otimes_{g_d \Gamma_U g_d^{-1}} M$ by $x \otimes m \mapsto x g_d^{-1} \otimes m g_d^{-1}$. Under this isomorphism, the action of $s \in S_P$ on the left hand side converts to an action of $g_d s g_d^{-1}$ on the right hand

side; hence, when we consider the homology groups as S_P -modules, we see that $H_r(\Gamma_U, M_\epsilon) \cong H_r(g_d \Gamma_U g_d^{-1}, M)_\epsilon^d$.

Since $g_d \Gamma_U g_d^{-1} \cong \Gamma_U$ is free abelian of rank r , we may view $H_r(g_d \Gamma_U g_d^{-1}, M)$ as the homology of a real r -torus with fundamental group $g_d \Gamma_U g_d^{-1}$ with local coefficient system determined by M . Taking f to be the fundamental class of the n -torus, we can identify elements in $H_r(g_d \Gamma_U g_d^{-1}, M)$ with elements of the form $f \otimes m$, where $m \in M^{g_d \Gamma_U g_d^{-1}}$. Now as $L(\mathbb{Z}/p\mathbb{Z})$ -modules, we have

$$M^{g_d \Gamma_U g_d^{-1}} \cong (M^d)^{\Gamma_U} \cong M^{\Gamma_U} = M^{U(\mathbb{Z}/p\mathbb{Z})} \cong (F(a_1, \dots, a_k) \otimes F(a_{k+1}, \dots, a_n)),$$

where we use the fact that $M \cong M^d$ with g_d as an intertwining operator, that the mod p reduction of Γ_U is $U(\mathbb{Z}/p\mathbb{Z})$, and [17, Corollary 5.10]. Using definition 7.2 to compute the action of S_P , we note that the highest weight vector in $H_r(\Gamma_U, F(a_1, \dots, a_n))$ is the highest weight vector in $F(a_1, \dots, a_n)^{U(\mathbb{Z}/p\mathbb{Z})}$. The weight of this vector is the weight (a_1, \dots, a_n) twisted by a certain character on the diagonal matrices in $g_d S_P g_d^{-1}$. The key point to determine this twist is the following. The action on the fundamental class is determined by the conjugation action of $t = \text{diag}(t_1, \dots, t_n) \in g_d S_P g_d^{-1}$ on $\eta \in \bigwedge^{k(n-k)} U(\mathbb{Z})$ induced by the action $(t, \eta) \mapsto t^{-1} \eta t$ and this is given by the character

$$\text{diag}(t_1, \dots, t_n) \mapsto \frac{t_1^{n-k} \dots t_k^{n-k}}{t_{k+1}^k \dots t_n^k}. \quad \square$$

Corollary 9.2. *With notation as above, we may uniquely factor ϵ into a product of a character ϵ_1 modulo d and ϵ_2 modulo N/d . Then an element $s \in S_P$ acts on a simple tensor*

$$m_1 \otimes m_2 \in (F(a_1 + (n-k), \dots, a_k + (n-k)) \otimes F(a_{k+1} - k, \dots, a_n - k))_\epsilon^d$$

by

$$(m_1 \otimes m_2)|_\epsilon^d s = \epsilon_1(\psi_d^1(s)) m_1 \psi_d^1(s) \otimes \epsilon_2(\psi_d^2(s)) m_2 \psi_d^2(s).$$

Proof. By Theorem 5.2(1), $\psi_d^1(s)_{11} \equiv s_{11} \pmod{d}$ and $\psi_d^2(s)_{11} \equiv s_{11} \pmod{N/d}$, so $\epsilon_1(\psi_d^1(s)) \epsilon_2(\psi_d^2(s)) = \epsilon(s)$. In addition, we have

$$g_d s g_d^{-1} = \left(\begin{array}{c|c} \psi_d^1(s) & 0 \\ \hline * & \psi_d^2(s) \end{array} \right),$$

with the upper left block acting on the first component of $m_1 \otimes m_2$, and the lower right block acting on the second component. The corollary follows. \square

10. HECKE OPERATORS AND THE KÜNNETH FORMULA

For $i = 1, 2$, let G_i be a group, and let M_i be an $\mathbb{F}[G_i]$ -module. Let F_i be a resolution of \mathbb{Z} by projective $\mathbb{Z}G_i$ -modules. Then [14, p. 109], we have an isomorphism of complexes of \mathbb{F} -vector spaces

$$(F_1 \otimes_{G_1} M_1) \otimes_{\mathbb{F}} (F_2 \otimes_{G_2} M_2) \cong (F_1 \otimes F_2) \otimes_{G \times G'} (M_1 \otimes M_2),$$

given by $(f_1 \otimes m_1) \otimes (f_2 \otimes m_2) \mapsto (f_1 \otimes f_2) \otimes (m_1 \otimes m_2)$. Applying the Künneth formula to this isomorphism of complexes, we obtain an isomorphism (since we are working with modules over a field)

$$\bigoplus_{r+s=n} H_r(G_1, M_1) \otimes_{\mathbb{F}} H_s(G_2, M_2) \rightarrow H_n(G_1 \times G_2, M_1 \otimes M_2).$$

We will apply the Künneth isomorphism to the case of congruence subgroups, and study how it interacts with the action of Hecke operators.

Set $P = P_d^k = g_d^{-1}P_0g_d$, and denote by $U = U_d$ the unipotent radical of P , and by $L = L_d$ the Levi quotient P/U . The exact sequence $1 \rightarrow U_0 \rightarrow P_0 \rightarrow P_0/U_0 \rightarrow 1$ splits, giving us that P_0/U_0 is isomorphic to a subgroup $L_0^1 \times L_0^2$ of P . In this case, L_0^1 consists of block diagonal matrices $A \oplus I_{n-k}$, and L_0^2 consists of block diagonal matrices $I_k \oplus B$. Then $L_d = P_d/U_d$ is isomorphic to the subgroup $L_d^1 \times L_d^2$ of P_d , with $L_d^i = g_d^{-1}L_0^i g_d$.

Let $\Gamma = \Gamma_0(n, N)$, $S = S_0(n, N)$, $\Gamma^\pm = \Gamma_0^\pm(n, N)$, and $S^\pm = S_0^\pm(n, N)$. We note that the Hecke algebras $\mathcal{H}(\Gamma, S)$ and $\mathcal{H}(\Gamma^\pm, S^\pm)$ are easily seen to be isomorphic, as are $\mathcal{H}(\Gamma_P, S_P)$ and $\mathcal{H}(\Gamma_P^\pm, S_P^\pm)$.

By Theorem 5.2, we see that $\Gamma_L^\pm = \Gamma_P^\pm/\Gamma_U^\pm = \Gamma_{L^1}^\pm \times \Gamma_{L^2}^\pm$, where

$$\Gamma_{L^1}^\pm = (\psi_d^1 \times \psi_d^2)^{-1}(\Gamma_0^\pm(k, d) \times I_{n-k})/\Gamma_U^\pm \cong \Gamma_0^\pm(k, d)$$

and

$$\Gamma_{L^2}^\pm = (\psi_d^1 \times \psi_d^2)^{-1}(I_k \times \Gamma_0^\pm(n-k, N/d))/\Gamma_U^\pm \cong \Gamma_0^\pm(n-k, N/d).$$

Hence, by the Künneth isomorphism, if $M = M_1 \otimes M_2$, where M_i is an L^i -module, we have

$$\begin{aligned} H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M) &\cong \bigoplus_{i+j=k} H_i(\Gamma_{L^1}^\pm, \text{St}(W_d) \otimes M_1) \otimes H_j(\Gamma_{L^2}^\pm, M_2) \\ &\cong \bigoplus_{i+j=k} H_i(\Gamma_0^\pm(k, d), \text{St}(W_0) \otimes M_1^d) \otimes H_j(\Gamma_0^\pm(n-k, N/d), M_2^d) \end{aligned}$$

after conjugating by g_d .

Let F_i be the standard resolution of \mathbb{Z} by $L^i(\mathbb{Q})$ -modules. Then we may compute $H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M)$ as the homology of the complex

$$(F_1 \otimes_{\Gamma_{L^1}^\pm} \text{St}(W) \otimes M_1) \otimes (F_2 \otimes_{\Gamma_{L^2}^\pm} M_2) \cong (F_1 \otimes F_2) \otimes_{\Gamma_L^\pm} (\text{St}(W) \otimes M),$$

Since U acts trivially on $F_1 \otimes F_2$ and on $\text{St}(W) \otimes M$, we have that the Hecke algebra $\mathcal{H}(\Gamma_P, S_P)$ acts on the right hand complex, and therefore on the homology $H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M)$. Translating this action to the left-hand complex, and applying Theorem 8.8, we obtain the following theorem.

Theorem 10.1. *Let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$, and let P_d^k be a representative maximal parabolic subgroup of $\text{GL}(n, \mathbb{Q})$ for some $d|N$. Let M_1 be a $\text{GL}(k, \mathbb{F}_p)$ -module and let M_2 be a $\text{GL}(n-k, \mathbb{F}_p)$ -module. Let S_P act on $M_1 \otimes M_2$ via $(m_1 \otimes m_2)s = m_1\psi_d^1(s) \otimes m_2\psi_d^2(s)$. Then the natural action of $T_n(\ell, r) \in \mathcal{H}(\Gamma_P, S_P)$ on*

$$H_t(\Gamma_L^\pm, \text{St}(W_d) \otimes M_1 \otimes M_2)$$

is given on the component

$$H_r(\Gamma_0^\pm(k, d), \text{St}(W_0) \otimes M_1) \otimes H_{t-r}(\Gamma_0^\pm(n-k, N/d), M_2)$$

by

$$(f \otimes g)|T_n(\ell, r) = \sum_{m=\max(0, r-(n-k))}^{\min(r, k)} \ell^{(k-m)(r-m)} f|T_k(\ell, m) \otimes g|T_{n-k}(\ell, r-m).$$

The same holds true if $\text{St}(W_d)$ and $\text{St}(W)$ are removed from the formulas.

In terms of Hecke eigenvectors, we obtain the following two corollaries.

Corollary 10.2. *Let $P = P_d^k$ be a representative maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ of type $(k_1, k_2) = (k, n - k)$ with unipotent radical U and Levi quotient L , and denote the two components of the Levi quotient by L^1 and L^2 . Let M_i be an L^i -module, and $M = M_1 \otimes M_2$. For $i = 1, 2$, let $f_i \in H_{p_i}(\Gamma_{L^i}^\pm, M_i)$ be an eigenclass of all the Hecke operators $T_{k_i}(\ell, j)$, with eigenvalues $a_i(\ell, j)$. Then $f_1 \otimes f_2$, considered as an element of $H_{p_1+p_2}(\Gamma_L^\pm, M)$, is an eigenclass of the Hecke operators $T_n(\ell, r)$, with eigenvalues*

$$c(\ell, r) = \sum_{m=\max(0, r-(n-k_1))}^{\min(k_1, r)} \ell^{(k_1-m)(r-m)} a_1(\ell, m) a_2(\ell, r-m).$$

If each f_i is attached to a Galois representation ρ_i , then $f_1 \otimes f_2$ is attached to $\rho_1 \oplus \omega^{k_1} \rho_2$.

Proof. The values of the $c(\ell, r)$ follow immediately from Theorem 10.1. To see the statement about attachment, we let

$$P_i(X, \ell) = \sum_{j=0}^{k_i} (-1)^j \ell^{j(j+1)/2} a_i(\ell, j) X^j$$

be the Hecke polynomial associated with f_i . We recall that attachment implies that

$$P_i(X, \ell) = \det(I - \rho_i(\mathrm{Fr}_\ell)X).$$

We note that

$$P_1(X, \ell) P_2(\ell^{k_1} X, \ell) = \det(I - (\rho_1 \oplus \omega^{k_1} \rho_2)(\mathrm{Fr}_\ell)X).$$

Hence, in order to complete the proof, we need only show that the Hecke polynomial

$$P(X, \ell) = \sum_{i=0}^n (-1)^i \ell^{i(i+1)/2} c(\ell, i) X^i$$

is equal to $P_1(X, \ell) P_2(X, \ell)$. This is a routine algebraic computation. \square

Corollary 10.3. *Assume that $p > n + 1$. Any system of simultaneous eigenvalues of the $T_n(\ell, m)$ acting on*

$$H_r(\Gamma_0^\pm(k, d), \mathrm{St}(W_0) \otimes M_1) \otimes H_s(\Gamma_0^\pm(n-k, N/d), M_2)$$

arises as a tensor product of a simultaneous eigenvector of the $T_k(\ell, m)$ and a simultaneous eigenvector of the $T_{n-k}(\ell, m)$.

Hence, any system of simultaneous eigenvalues of the $T_n(\ell, m)$ acting on

$$H_{r+s}(\Gamma_L^\pm, \mathrm{St}(W_d) \otimes (M_1 \otimes M_2)^d)$$

appears as such a tensor product.

Proof. We can find a basis $\{v_r\}$ of $H_i(\Gamma_0^\pm(k, d), \mathrm{St}(W) \otimes M_1)$ such that the actions of the $T_k(\ell, m)$ are all upper triangular, and a basis $\{w_s\}$ of $H_j(\Gamma_0^\pm(n-k, N/d), M_2)$ with respect to which the $T_{n-k}(\ell, m)$ are all upper triangular. Then, by Theorem 10.1 and the Kronecker product, there is a basis $\{v_r \otimes w_s\}$ with respect to which all of the $T_n(\ell, m)$ are upper triangular. We can then read off the systems of simultaneous eigenvalues of the $T_n(\ell, m)$ from the diagonal elements of the matrix, and we see that they all arise from elements of the form $v \otimes w$. \square

The purpose of Theorem 10.1 is to construct eigenvectors of the Hecke algebra with predetermined eigenvalues in the homology of Γ_L . With this in mind, we now relate the homology of Γ_L^\pm to the homology of Γ_L .

Theorem 10.4. *Let M be an S^\pm -module on which S_U^\pm acts trivially, and assume that $p > n + 1$. Then any system Φ of $\mathcal{H}(\Gamma_P, S_P)$ -eigenvalues appearing in $H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M)$ appears in $H_k(\Gamma_L, \text{St}(W_d) \otimes M)$.*

Proof. Because $p > n + 1$, we may use Borel-Serre duality. An argument similar to that in Theorem 6.3 shows that $H_k(\Gamma_L, \text{St}(W_d) \otimes M)$ is finite dimensional.

As noted previously, the Hecke algebras $\mathcal{H}(\Gamma_P, S_P)$ and $\mathcal{H}(\Gamma_P^\pm, S_P^\pm)$ are isomorphic. Hence, $\mathcal{H}(\Gamma_P, S_P)$ acts on both $H_k(\Gamma_L, \text{St}(W_d) \otimes M)$ and on $H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M)$. The corestriction map $\text{cores} : H_k(\Gamma_L, \text{St}(W_d) \otimes M) \rightarrow H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M)$ is easily seen to be an $\mathcal{H}(\Gamma_P, S_P)$ homomorphism. By the adaptation of [14, Proposition III.10.4] to homology, the corestriction factors as a composite

$$H_k(\Gamma_L, \text{St}(W_d) \otimes M) \rightarrow H_k(\Gamma_L, \text{St}(W_d) \otimes M)_{\Gamma_L^\pm/\Gamma_L} \rightarrow H_k(\Gamma_L^\pm, \text{St}(W_d) \otimes M),$$

where the first map is surjective, and the second is an isomorphism. Since the corestriction map is Hecke equivariant and surjective, any system Φ of eigenvalues in the codomain must appear in the domain. \square

We now prove a similar theorem for principal congruence subgroups. Since conjugation by g_d preserves $\Gamma(N)$, we see easily that $\Gamma_L(N) = \Gamma_{L^1}(N) \times \Gamma_{L^2}(N)$ (since for instance, if $\gamma \in \text{GL}(k, \mathbb{Z})$ is congruent to the identity, then $g_d^{-1}(\gamma \oplus I_{n-k})g_d \in \Gamma_P(N)$). Hence, we may decompose

$$H_k(\Gamma_L(N), \text{St}(W) \otimes M_1 \otimes M_2) \cong \bigoplus_{i+j=k} H_i(\Gamma_{L^1}(N), \text{St}(W) \otimes M_1) \otimes H_j(\Gamma_{L^2}(N), M_2).$$

Since $(\Gamma_P(N), S_P(N))$ and (Γ_P, S_P) are compatible Hecke pairs, $\mathcal{H}(\Gamma_P, S_P)$ acts on the above module, and we obtain the following theorem.

Theorem 10.5. *Let Φ be a system of Hecke eigenvalues of all the $T_n(\ell, r)$ acting on $H_t(\Gamma_L(N), \text{St}(W) \otimes M_1 \otimes M_2)$. Assume that $p > n + 1$. Then there is an eigenvector f with eigenvalue system Φ that can be written as $f = f_1 \otimes f_2$ with $f_1 \in H_i(\Gamma_{L^1}(N), \text{St}(W) \otimes M_1)$ and $f_2 \in H_j(\Gamma_{L^2}(N), M_2)$ each a simultaneous eigenvector of the appropriate Hecke algebra, with $i + j = t$. If f_1 and f_2 have attached Galois representations ρ_1 and ρ_2 , then $f_1 \otimes f_2$ has attached Galois representation $\rho_1 \oplus \omega^k \rho_2$.*

Proof. Since $(\Gamma_P(pN), S_P(pN))$ and (Γ_P, S_P) are compatible, for $s = s(\ell, r, n) \in S_P$, we may choose $\gamma \in \Gamma_P$ and $\gamma_\alpha \in \Gamma_P$ such that

$$\Gamma_P s \Gamma_P = \Gamma_P s \gamma \Gamma_P = \prod_{s_\alpha \in \mathcal{T}(s, k)} s_\alpha \Gamma_P = \prod_{s_\alpha \in \mathcal{T}(s, k)} s_\alpha \gamma_\alpha \Gamma_P$$

and $s\gamma$ and each $s_\alpha \gamma_\alpha \in S_P(pN)$. Then, by compatibility, we have

$$\Gamma_P(pN) s \gamma \Gamma_P(pN) = \prod s_\alpha \gamma_\alpha \Gamma_P(pN).$$

We note that $\psi_d^1(s_\alpha \gamma_\alpha) = \psi_d^1(s_\alpha) \psi_d^1(\gamma_\alpha)$ and that $\psi_d^1(\gamma_\alpha) \in \Gamma_0^\pm(k, d)$. If necessary, we may multiply γ_α by $\text{diag}(1, \dots, -1, 1, \dots, -1) \in \Gamma_P$ so that $\psi_d^1(\gamma_\alpha) \in \Gamma_0(k, d)$. Hence, the $\Gamma_0(k, d)$ -coset $\overline{\psi_d^1(s_\alpha)}$ is the same as the coset of $\overline{\psi_d^1(s_\alpha \gamma_\alpha)}$.

Using these facts, and the fact that $(\Gamma_0(d, k), S_0(d, k))$ is compatible with the principal congruence Hecke pair of level pN for $\mathrm{GL}(k)$ (and analogous facts for ψ_d^2), we see that

$$\begin{aligned} \sum_{s_\alpha \in \mathcal{T}(s, k)} \overline{\psi_d^1(s_\alpha)} \otimes \overline{\psi_d^2(s_\alpha)} &= \sum_{s_\alpha \in \mathcal{T}(s, k)} \overline{\psi_d^1(s_\alpha \gamma_\alpha)} \otimes \overline{\psi_d^2(s_\alpha \gamma_\alpha)} \\ &= \sum_{s_\alpha \in \mathcal{T}(s, k)} \psi_d^1(\widetilde{s_\alpha \gamma_\alpha}) \otimes \psi_d^2(\widetilde{s_\alpha \gamma_\alpha}) \\ &= \sum_{m=\max(0, r-(n-k))}^{\min(r, k)} \ell^{k-m} \mathcal{T}'_k(\ell, m) \otimes \mathcal{T}'_{n-k}(\ell, r-m), \end{aligned}$$

where $\mathcal{T}'_k(\ell, m)$ is a collection of left coset representatives that compute $T_k(\ell, m)$ for the principal congruence Hecke pair, and the tildes represent cosets for the appropriate principal congruence subgroup. The theorem then follows exactly as in Theorem 10.1, Corollary 10.2, and Corollary 10.3. \square

11. REDUCIBILITY OF GALOIS REPRESENTATIONS ATTACHED TO HOMOLOGY OF A PARABOLIC SUBGROUP

Throughout this section, we will let P be a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$, with unipotent radical U and Levi subgroup L . We will also assume that (Γ, S) is a congruence Hecke pair such that Γ_U and S_U have the same reduction modulo p .

Lemma 11.1. *Let A be a free abelian group of finite rank r , Let P be a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ with unipotent radical U , and suppose that $L(\mathbb{Z}) = P(\mathbb{Z})/U(\mathbb{Z})$ acts on A . Assume that there is an isomorphism $\varphi : A \rightarrow \mathbb{Z}^r$ and a representation $M : P(\mathbb{Z}) \rightarrow \mathrm{GL}(r, \mathbb{Z})$ that factors through $U(\mathbb{Z})$ such that*

$$\varphi(a\lambda) = \varphi(a)M(\lambda)$$

for every $a \in A$ and $\lambda \in P(\mathbb{Z})$. Then $P(\mathbb{Z})$ acts on

$$H_k(A, \mathbb{F}) \cong \bigwedge^k (A \otimes \mathbb{F})$$

via $\bigwedge^k(M)$.

Proof. This follows immediately from the naturality of the isomorphism in [14, V.6.4 and the preceding page]. \square

Lemma 11.2. *Let P be a maximal parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$, and let $s \in S_P$. The action of the Hecke operator $[\Gamma_U s \Gamma_U]$ on $H_1(\Gamma_U, \mathbb{Z}) \cong \Gamma_U$ described by Definition 7.2 is given on an element $u \in \Gamma_U$ by*

$$u[\Gamma_U s \Gamma_U] = s^{-1}(u^{e_s})s.$$

Proof. Let $H = \Gamma_U \cap s \Gamma_U s^{-1}$ be the largest subgroup of Γ_U such that $s^{-1} H s \subseteq \Gamma_U$. We have previously denoted $[\Gamma_U : H]$ by e_s , and noted that it is prime to p . The Hecke operator $[\Gamma_U s \Gamma_U]$ is given by the composition

$$\Gamma_U \xrightarrow{\text{transfer}} H \xrightarrow{\alpha} s^{-1} H s \xrightarrow{\iota} \Gamma_U,$$

where the first map is the group-theoretic transfer, the second is conjugation by s , and the third is inclusion. By [21, Theorem 10.1.3], the transfer from Γ_U to H is just the e_s -power map. Hence,

$$u[\Gamma_U s \Gamma_U] = s^{-1}(u^{e_s})s,$$

where $u^{e_s} \in H$, since $[\Gamma_U : H] = e_s$. \square

For ease of notation, we will now consider Γ_U as a free abelian group written additively, and call it A . As above, we obtain an action of the Hecke algebra of Γ_U on $H_1(\Gamma_U, \mathbb{F}) = H_1(\Gamma_U, \mathbb{Z}) \otimes \mathbb{F} \cong A \otimes \mathbb{F}$; namely $[\Gamma_U s \Gamma_U]$ acts on an element $a \otimes 1 \in A \otimes \mathbb{F}$ by taking it to

$$(s^{-1}(e_s a)s) \otimes 1.$$

Note that on the free abelian group $e_s A \subset A$, conjugation by s is given (in terms of a basis of A) by a matrix $M(S) \in \mathrm{GL}(n, \mathbb{Q})$, all of whose denominators divide e_s (and are hence in $\mathbb{Z}_{(p)}$). This is the same matrix that describes the action of s (in terms of a \mathbb{Z} -basis of $A = \Gamma_U$) on $U(\mathbb{Q})$. Using the Pontryagin product (as in Lemma 11.1) to extend this action from $H_1(\Gamma_U, \mathbb{F})$ to $H_k(\Gamma_U, \mathbb{F})$ and invoking the last statement in Corollary 7.9, we obtain the following theorem.

Theorem 11.3. *Let $M : S_P \rightarrow \mathrm{GL}(n, \mathbb{Z}_{(p)})$ be the matrix that describes the action of S_P on $U(\mathbb{Q})$ in terms of a \mathbb{Z} -basis of Γ_U . Then the action of $s \in S_P$ on $H_k(\Gamma_U, \mathbb{F})$ is given by the mod p reduction of $\wedge^k M$. In particular, $H_k(\Gamma_U, \mathbb{F})$ is an admissible S_P -module.*

We will apply this theorem in the following special case.

Corollary 11.4. *For a maximal parabolic subgroup P of $\mathrm{GL}(n, \mathbb{Q})$, with Levi subgroup L and unipotent radical U , the homology $H_k(\Gamma_U(pN), \mathbb{F})$ is an admissible $S_P(pN)$ -module on which $\Gamma_P(pN)$ acts trivially.*

If we denote reduction modulo pN by a bar, then the map $D : S_P(pN) \rightarrow (\mathbb{Z}/pN\mathbb{Z})^\times \times (\mathbb{Z}/pN\mathbb{Z})^\times$ given by $D(s) = (\overline{\det(\psi_d^1(s))}, \overline{\det(\psi_d^2(s))})$ is a homomorphism with kernel equal to $\Gamma_P(pN)$. Hence, on any module M on which $S_P(pN)$ acts via reduction modulo p , $S_P(pN)$ acts through the image of D , which is an abelian group. Hence, it acts on any irreducible constituent of M via a character that factors through D . It follows that any irreducible constituent of M is of the form \mathbb{F}_χ , where χ is a character that factors through D .

Theorem 11.5. *Let P be a maximal \mathbb{Q} -parabolic subgroup of $\mathrm{GL}(n, \mathbb{Q})$ with unipotent radical U and Levi quotient $L = P/U$ and let $(\Gamma, S) = (\Gamma_0(n, N), S_0(n, N))$. Let W be the maximal proper P -stable subspace of \mathbb{Q}^n and let $k = \dim(W)$. Set $n_1 = k$ and $n_2 = n - k$. Assume that $p > n + 1$. Let M be an irreducible (p, N) -admissible $\mathbb{F}[S]$ -module, and let Φ be a system of $\mathcal{H}(\Gamma, S)$ -eigenvalues occurring in $H_t(\Gamma_P, \mathrm{St}(W) \otimes M)$. Then there is some reducible Galois representation $\rho = \sigma_1 \oplus \sigma_2$ with $\sigma_i : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(n_i, \mathbb{F})$ that is attached to Φ .*

Proof. We note that there is some $\gamma \in \Gamma_0(n, N)$ such that $\gamma P \gamma^{-1} = P_d = P_d^k$ for some $d|N$. Conjugation by γ then yields an isomorphism of $\mathcal{H}(\Gamma, S)$ -modules

$$H_t(\Gamma_P, \mathrm{St}(W) \otimes M) \cong H_t(\Gamma_{P_d}, \mathrm{St}(W) \otimes M),$$

so that we may, without loss of generality, assume that $P = P_d$ and $W = W_d$ for some $d|N$.

Now, by Theorem 7.15, we may assume that Φ appears in

$$H_t(\Gamma_P(pN), \text{St}(W) \otimes M).$$

Given an exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ of S_P -modules, since $\text{St}(W) \otimes M_i = (\text{St}(W) \otimes \mathbb{F}) \otimes_{\mathbb{F}} M_i$ and $\text{St}(W) \otimes \mathbb{F}$ is a free (hence flat) \mathbb{F} -module, we see that

$$0 \rightarrow \text{St}(W) \otimes M_1 \rightarrow \text{St}(W) \otimes M_2 \rightarrow \text{St}(W) \otimes M_3 \rightarrow 0$$

is again an exact sequence of S_P -modules. Then (see [11, Lemma 2.1]) Φ must appear in

$$H_t(\Gamma_P(pN), \text{St}(W) \otimes M'),$$

where M' is some irreducible constituent of M as an $S_P(pN)$ -module. Now, $S_P(pN)$ acts on M' via D composed with some character χ on $(\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$, so M' must be isomorphic to \mathbb{F}_χ , and we see that Φ must actually appear in

$$H_t(\Gamma_P(pN), \text{St}(W) \otimes \mathbb{F}_\chi).$$

Note that we may write χ as a product of two characters χ_1, χ_2 on $(\mathbb{Z}/N\mathbb{Z})^\times$, where $\chi \circ D = (\chi_1 \circ \det \circ \psi_d^1) \cdot (\chi_2 \circ \det \circ \psi_d^2)$. We see easily that $\mathbb{F}_\chi = \mathbb{F}_{\chi_1} \otimes \mathbb{F}_{\chi_2}$.

By Theorem 7.11, we see that the Hochschild-Serre spectral sequence for $1 \rightarrow \Gamma_U(pN) \rightarrow \Gamma_P(pN) \rightarrow \Gamma_L(pN) \rightarrow 1$ is Hecke equivariant, so Φ appears in some term

$$H_i(\Gamma_L(pN), H_j(\Gamma_U(pN), \text{St}(W) \otimes \mathbb{F}_\chi)) \cong H_i(\Gamma_L(pN), \text{St}(W) \otimes H_j(\Gamma_U(pN), \mathbb{F}_\chi)).$$

Now, by Corollary 11.4, we see that $H_j(\Gamma_U(pN), \mathbb{F}_\chi) \cong H_j(\Gamma_U(pN), \mathbb{F}) \otimes \mathbb{F}_\chi$ is (p, N) -admissible, and hence is acted on trivially by $\Gamma_P(pN)$. Using the fact that $\Gamma_L(pN) \cong \Gamma_{L^1}(pN) \times \Gamma_{L^2}(pN)$, where $\Gamma_{L^1}(pN)$ is isomorphic to the principal congruence subgroup of $\text{GL}(k, \mathbb{Z})$, and $\Gamma_{L^2}(pN)$ is isomorphic to the principal congruence subgroup of $\text{GL}(n-k, \mathbb{Z})$, and $\Gamma_{L^2}(pN)$ acts trivially on $\text{St}(W)$, we see that Φ must appear in

$$\begin{aligned} H_i(\Gamma_L(pN), \text{St}(W) \otimes \mathbb{F}_\chi) &\cong H_i(\Gamma_{L^1}(pN) \times \Gamma_{L^2}(pN), \text{St}(W) \otimes \mathbb{F}_\chi) \\ &\cong \bigoplus_{r+s=i} H_r(\Gamma_{L^1}(pN), \text{St}(W) \otimes \mathbb{F}_{\chi_1}) \otimes H_s(\Gamma_{L^2}(pN), \mathbb{F}_{\chi_2}) \\ &\cong \bigoplus_{r+s=i} H^{r'}(\Gamma_{L^1}(pN), \mathbb{F}_{\chi_1}) \otimes H_s(\Gamma_{L^2}(pN), \mathbb{F}_{\chi_2}), \end{aligned}$$

where $r' = k(k-1)/2 - r$ (using Borel-Serre duality).

By Theorem 10.5, we see that the system of eigenvalues Φ must have an eigenvector $f_1 \otimes f_2$, where

$$f_1 \in H^{r'}(\Gamma_{L^1}(pN), \mathbb{F}_{\chi_1})$$

and

$$f_2 \in H_s(\Gamma_{L^2}(pN), \mathbb{F}_{\chi_2})$$

are eigenvectors for the appropriate Hecke algebras.

Now [22] shows (conditional on the stabilization of a twisted trace formula; see [22, p. 949]) that each of f_i has an attached Galois representation $\sigma_i : G_{\mathbb{Q}} \rightarrow \text{GL}(n_i, \overline{\mathbb{F}}_p)$. Then by Theorem 10.5, we see that Φ has attached Galois representation $\rho = \sigma_1 \otimes \omega^k \sigma_2$, as desired. \square

12. APPLICATION TO $\mathrm{GL}(4)$ AND REDUCIBLE GALOIS REPRESENTATIONS

We now apply the theory that we have developed to the case of a Galois representation $\rho = \rho_1 \oplus \rho_2$, where ρ_1 and ρ_2 are odd irreducible two-dimensional Galois representations, $\rho_i : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \mathbb{F})$ such that the levels N_i of ρ_i are squarefree and relatively prime and $p > 5$. Denote by ϵ_i the nebentype of ρ_i . Let ρ_1 have Serre weight $F(a+2, b+2)$, and let ρ_2 have Serre weight $F(c, d)$. Note that we may adjust b modulo $p-1$ so that $0 \leq b-c \leq p-1$. One then easily checks that a predicted weight for $\rho_1 \oplus \rho_2$ is $F(a, b, c, d)$.

Now, we know (since Serre's conjecture is a theorem) that ρ_1 is attached to some Hecke eigenvector $f \in H^1(\Gamma_0^{\pm}(2, N_1), F(a+2, b+2)_{\epsilon_1})$ and (twisting by a character) that $\omega^{-2} \otimes \rho_2$ is attached to some Hecke eigenvector

$$g \in H_1(\Gamma_0^{\pm}(2, N_2), F(c-2, d-2)_{\epsilon_2}).$$

By Borel-Serre duality ρ_1 is attached to some eigenvector

$$f' \in H_0(\Gamma_0^{\pm}(2, N_1), \mathrm{St}(W) \otimes F(a+2, b+2)_{\epsilon_1}),$$

where $W = \mathbb{Q}^2$ is acted on in the natural way by $\Gamma_0^{\pm}(2, N_1), S_0^{\pm}(2, N_1) \subset \mathrm{GL}(2, \mathbb{Q})$. Hence, by Corollary 10.2 and Theorem 10.4, we see that $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenvector in

$$H_1(\Gamma_{L_{N_1}}, \mathrm{St}(W_{N_1}) \otimes (F(a+2, b+2) \otimes F(c-2, d-2))_{\epsilon_1 \epsilon_2}^{N_1}),$$

where W_{N_1} is the space stabilized by $P_{N_1}^2 \in \mathrm{GL}(4, \mathbb{Q})$.

Let ϵ be the character modulo $N_1 N_2$ defined by $\epsilon = \epsilon_1 \epsilon_2$.

Now, since U_{N_1} acts trivially on $\mathrm{St}(W_{N_1})$, we know by Theorem 9.1 that

$$H_4(\Gamma_{U_{N_1}}, \mathrm{St}(W_{N_1}) \otimes F(a, b, c, d)_{\epsilon}) \cong \mathrm{St}(W_{N_1}) \otimes (F(a+2, b+2) \otimes F(c-2, d-2))_{\epsilon}^{N_1}.$$

Hence, we see that $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenvector in

$$H_1(\Gamma_{L_{N_1}}, H_4(\Gamma_{U_{N_1}}, \mathrm{St}(W_{N_1}) \otimes F(a, b, c, d)_{\epsilon})).$$

This is the E_{14}^2 term in the Hochschild-Serre spectral sequence for the exact sequence $1 \rightarrow \Gamma_{U_{N_1}} \rightarrow \Gamma_{P_{N_1}} \rightarrow \Gamma_{L_{N_1}} \rightarrow 1$. Since the p -homological dimension of $\Gamma_{L_{N_1}}$ is two, this spectral sequence is only three columns wide; since E_{14}^2 is in the center column, everything in it survives to the infinity page. Hence, $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenvector in $H_5(\Gamma_0(n, N_1 N_2) \cap P_{N_1}, \mathrm{St}(W_{N_1}) \otimes F(a, b, c, d)_{\epsilon})$.

Hence, $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenclass in the term \mathcal{E}_{15}^1 of the spectral sequence of section 6. This class cannot be killed off by elements in \mathcal{E}_{25}^1 or \mathcal{E}_{05}^1 , since any system of Hecke eigenvalues in those terms must be attached to a Galois representation that has a character as a direct summand by Theorem 11.5 (the semi-simplification of a Galois representation attached to a system of Hecke polynomials is unique up to isomorphism by the Chebotarev Density Theorem, the fact that the Frobenius elements generate the Galois group, and Brauer-Nesbitt Theorem). The only other way that the class could be killed off would be by an eigenclass in $\mathcal{E}_{3,4}^2$, in which case there must be a class in \mathcal{E}_{34}^1 with $\rho_1 \oplus \rho_2$ attached. Hence, we see that either $\rho_1 \oplus \rho_2$ fits $H_6(\Gamma_0(N_1 N_2), F(a, b, c, d))$ or it fits

$$\mathcal{E}_{34}^1 = H_4(\Gamma_0(V), \mathrm{St}(V) \otimes F(a, b, c, d)) \cong H^2(\Gamma_0(N_1 N_2), F(a, b, c, d)).$$

Since a Galois representation fits $H_t(\Gamma, M)$ if and only if it fits $H^t(\Gamma, M)$ [7, Lemma 2.4], we see that we have proved the following.

Theorem 12.1. *Let $p > 5$. For $i = 1, 2$, let $\rho_i : G_{\mathbb{Q}} \rightarrow \mathrm{GL}(2, \mathbb{F})$ be an odd irreducible Galois representation with level N_i and nebentype ϵ_i , and let $\epsilon = \epsilon_1 \epsilon_2$. Then $\rho_1 \oplus \rho_2$ is attached to a Hecke eigenclass in either $H_6(\Gamma_0(N_1 N_2), M_{\epsilon})$ or in $H_2(\Gamma_0(N_1 N_2), M_{\epsilon})$, where M is a weight for $\rho_1 \oplus \rho_2$ predicted by the conjecture in [8].*

Remark 12.2. We think it very unlikely that $\rho_1 \oplus \rho_2$ fits $H_2(\Gamma_0(N_1 N_2), M_{\epsilon})$, but we know of no theorems that would prove that it can not.

We hope in a future paper to be able to extend this theorem to representations $\rho = \rho_1 \oplus \rho_2$ where ρ is an odd Galois representation that is a sum of two odd irreducible n -dimensional Galois representations for arbitrarily large n , assuming that each constituent is attached to an appropriate homology Hecke eigenclass. Such an extension will require a better understanding of the terms of the Hochschild-Serre spectral sequence for the exact sequence $0 \rightarrow \Gamma_U \rightarrow \Gamma_P \rightarrow \Gamma_L \rightarrow 0$, which degenerates in the case $n = 2$.

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