Chapter VIII

Cardinality

…it's very much like your trying to reach infinity. You know that it's there, you just don't know where—but just because you can never reach it doesn't mean that it's not worth looking for. Norton Juster, The Phantom Tollbooth

In the very first chapter of this book we defined the cardinality of a finite set to equal the number of elements it contains. Thus, for instance, the sets \( \{a, b, c\} \) and \( \{1, 2, 3\} \) have the same cardinality, which is 3. For infinite sets we cannot define the cardinality to be the number of elements, because such sets do not have any (finite) number of elements.

However, there is a reason we do not just define the cardinality of an infinite set to be the symbol \( \infty \); there is a better way to measure the size of sets! This came as a shock to mathematicians in the late 1800’s, who expected all infinite sets to have the same size. This theory was developed by Cantor, who showed that the set of real numbers \( \mathbb{R} \) has bigger cardinality than \( \mathbb{N} \). In this chapter, we develop Cantor’s theory of cardinality, which has become an important part of modern mathematics.
28 Definitions regarding cardinality

In this section, we will define what it means for two sets to have the same cardinality, discuss what it means for sets to have the same size as the natural numbers, and prove some basic facts about cardinality.

28.A How do we measure the size of sets?

The first question we must answer is: How do we measure the size of a set? For finite sets we simply count out the elements. For instance, if \( S = \{68, 5, 39291, 90\} \), then we just count off each element, and we see that \(|S| = 4\). Another way to think about counting is forming a bijection \( f : S \rightarrow \{1, 2, 3, 4\} \) given by

\[
    f(68) = 1, \quad f(5) = 2, \quad f(39291) = 3, \quad f(90) = 4.
\]

Intuitively, we can think of sets having the same “size” if there is a bijection between them. This motivates the following definition.

**Definition 28.1.** Let \( S \) and \( T \) be sets. We say that \( S \) and \( T \) have the same cardinality if there exists a bijection \( f : S \rightarrow T \). If this holds, we write \(|S| = |T|\).

If there is no bijection from \( S \) to \( T \), we say that they have different cardinalities, and write \(|S| \neq |T|\).

**Remark 28.2.** We will prove, shortly, that this relation is an equivalence relation, which will justify our use of an equality sign for the relation. ▲

The following are some examples and non-examples of sets with the same cardinality.

**Example 28.3.**

(1) Let \( A = \{a, b, c, d, e, f\} \), \( B = \{1, 2, 3, 4, 5, 6\} \), and \( C = \{0, 1, 2, 3, 4, 5\} \). It is easy to construct a bijection from \( A \) to \( B \) (since both sets have exactly six elements). So \(|A| = |B|\). There are also bijections from \( A \) to \( C \), and from \( B \) to \( C \) (since \( C \) also has six elements), so \(|A| = |C|\), and \(|B| = |C|\).

(2) Let \( S = \{1, 2, 3\} \) and \( T = \mathbb{N} \). There is no bijection from \( S \) to \( T \) (since \( T \) has more than three elements). Thus \(|S| \neq |T|\).

(3) It can be tricky when working with infinite sets to tell whether they have the same cardinality. For instance, does \( \mathbb{N} \) have the same cardinality as \( 2\mathbb{N} \)? The answer is yes! There is a bijection \( f : \mathbb{N} \rightarrow 2\mathbb{N} \), given by \( f(n) = 2n \). In other words,

\[
    f(1) = 2, \quad f(2) = 4, \quad f(3) = 6, \quad f(4) = 8, \ldots
\]

is a bijection from \( \mathbb{N} \) to \( 2\mathbb{N} \). Thus, we do have

\(|\mathbb{N}| = |2\mathbb{N}|\). △

The following example is so important that we’ll call it a theorem.
Theorem 28.4. The cardinalities of $\mathbb{N}$ and $\mathbb{Z}$ are the same.

Proof. We need to construct a bijection $f : \mathbb{N} \to \mathbb{Z}$; or, in other words, we need to “count” the elements of $\mathbb{Z}$ using the natural numbers. The idea is to first count 0, and then successively count the positive and negative integers, as in the following table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
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<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>-3</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>-4</td>
</tr>
<tr>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>

which yields the needed bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Remark 28.5. The previous proof was very informal. First, we didn’t prove that the function $f$ is injective and surjective. We’ll leave that as an exercise to be verified later.

Second, the definition of the function $f$ is sloppy. To be more precise we should define $f$ as a piecewise function on $n \in \mathbb{N}$ by the rule:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Amazingly, it turns out that this function can be expressed by a (somewhat complicated) single formula:

$$f(n) = 1 + (-1)^n(2n-1)/4.$$

(It is not expected that a student would be able to come up with this formula without a lot of help!)

Advice 28.6. To prove that two sets have the same cardinality, you are required to find a bijection between the two sets. In general there are usually lots of different bijections. Try to look for a simple one.

We end this subsection with one more example.

Example 28.7. We will prove that the open interval $A = (0, 1)$ and the open interval $B = (1, 4)$ have the same cardinality. We thus want to construct a bijection between these two sets. The most obvious option would be to send 0 to 1, and stretch by a factor of 3. So we define

$$g : (0, 1) \to (1, 4), \quad g(x) = 1 + 3x.$$  

It is straightforward to check that $g$ is a function from $A$ to $B$, and that $g$ is both one-to-one and onto.
28.B Basic Results and a Picture

In the previous subsection, we defined what it means for two sets to have the same cardinality. To really justify that definition, we should show that the relation of “having the same cardinality” is an equivalence relation on sets.

**Theorem 28.8.** The relation of “having the same cardinality” as given in Definition 28.1 is an equivalence relation on the collection of sets.

**Proof.** We first prove this relation is reflexive. Let \( X \) be any set. The identity function \( 1_X : X \to X \) is a bijection. Thus \( X \) is related to \( X \).

Next, we prove this relation is symmetric. Let \( X \) and \( Y \) be any sets, and assume \( X \) relates to \( Y \). In other words, assume there is a bijection \( f : X \to Y \). Then \( f \) has an inverse function \( f^{-1} : Y \to X \) which is also a bijection. Hence \( Y \) relates to \( X \).

Finally, we prove transitivity. Let \( X \), \( Y \), and \( Z \) be any sets, and assume there are bijections \( f : X \to Y \) and \( g : Y \to Z \). The composite function \( g \circ f : X \to Z \) is a bijection, as needed.

The equivalence classes of this equivalence relation are precisely the collections of sets with the same cardinality.

The observant reader will have noticed that we defined when two sets \( S \) and \( T \) have equal cardinality, \( |S| = |T| \), but that we have not defined what the cardinality of an individual set is. We will rectify that situation now. The cardinality of a set, denoted \( |S| \), is the equivalence class of \( S \) under the relation “having the same cardinality.” This is an extremely complicated object, so many mathematicians instead simply choose a symbol to represent the cardinality. For finite sets, that symbol is just the actual size of the set. Thus, we still have \(|\{2, 79, -4\}| = 3\).

For infinite sets, things are much more complicated. (Did you expect otherwise?) The smallest infinite cardinal \( |\mathbb{N}| \) is written as \( \aleph_0 \). The next infinite cardinal is \( \aleph_1 \), and so forth. The diagram below gives some perspective to this chain. (We put question marks in places where we do not yet have any examples.)

28.C Definition of Countable Sets

The easiest infinite set to understand is the set of natural numbers. It’s cardinality is given a special symbol \( |\mathbb{N}| = \aleph_0 \). We think of the natural numbers as “counting numbers” which motivates the following definition.

**Definition 28.9.** A set \( S \) is countably infinite if \( |S| = |\mathbb{N}| \).

A set is countable if it is either finite or countably infinite.

The following are some examples and non-examples involving these definitions.

**Example 28.10.** (1) The empty set is countable, since it is finite. It is not countably infinite (since it isn’t infinite).
(2) The set \(\{1, 2, 93828283928\}\) is countable and finite, but not infinite, and hence not countably infinite.

(3) Theorem 28.4 tells us that the integers are a countably infinite set. Similarly, Example 28.3 tells us that \(2\mathbb{N}\) is a countably infinite set.

(4) The set \(\{n^2 : n \in \mathbb{Z}\}\) is infinite. We will see shortly that it is countably infinite.

(5) Are there any sets which are infinite, but not countably infinite? These would be sets which occur strictly above \(\aleph_0\) in the diagram below. We will prove in Section 30 that, yes, there are such sets!

\[
\begin{align*}
\text{Cardinalities} & \quad \text{Examples} \\
\vdots & : \\
\aleph_2 & \quad ? \\
\aleph_1 & \quad ? \\
\aleph_0 & \quad \mathbb{N}, 2\mathbb{N}, \mathbb{Z}, \ldots \\
\vdots & : \\
2 & \quad \{1, 2\}, \{1, 3\}, \ldots \\
1 & \quad \{1\}, \{2\}, \ldots \\
0 & \quad \emptyset
\end{align*}
\]

**Advice 28.11.** To show that a set \(A\) is countably infinite, you just need to arrange its elements in a non-repeating, infinite list

\[A = \{a_1, a_2, a_3, \ldots\}.\]

This is precisely what we did when we proved that \(\mathbb{Z}\) is countably infinite, we put its elements in the list \(0, 1, -1, 2, -2, 3, -3, \ldots\).

**Warning 28.12.** If you are proving that a set is countably infinite by putting its elements into a list, do not skip elements, and do not repeat elements. Otherwise, you didn’t really create a bijection.

**Question:** Which of the following lists proves that \(\mathbb{Z}\) is countably infinite?

(a) \(\{0, 1, 2, 3, \ldots\}\).
(b) \(\{0, 1, 0, -1, 0, 2, 0, -2, \ldots\}\).
(c) \(\{1, 0, 2, -1, 3, -2, 4, -3, \ldots\}\).
**Answer:** Only (c) works. The list in (a) skips the negative integers. [However, it does prove that the non-negative integers are countably infinite.] The list in (b) repeats 0. Of the choices, only (c) lists every integer exactly once, hence gives the bijection with \( \mathbb{N} \).

### 28.D Subsets of Countable Sets

Many operations involving countable sets yield new countable sets. One of the most useful results is the following:

**Theorem 28.13.** Any infinite subset of a countably infinite set is countably infinite.

**Proof.** Let \( A \) be a countably infinite set. We can write the elements of \( A \) in an infinite list \( a_1, a_2, a_3, \ldots \). Let \( B \) be an infinite subset of \( A \).

Let \( n_1 \) be the smallest natural number with \( a_{n_1} \in B \), which exists since \( B \neq \emptyset \) as \( B \) is infinite. Put \( b_1 = a_{n_1} \).

Next let \( n_2 \) be the smallest natural number with \( n_2 > n_1 \) and \( a_{n_2} \in B \), which exists since \( B - \{b_1\} \neq \emptyset \) as \( B \) is infinite. Put \( b_2 = a_{n_2} \).

Repeating this process (by induction) we create an infinite list \( b_1, b_2, \ldots \). Clearly there are no repetitions in this list. This new list covers every element of \( B \) because we can also prove (by induction) that \( n_i \geq i \) for every \( i \in \mathbb{N} \); hence, we have worked all the way through the list of elements of \( A \).

**Example 28.14.** Not every subset of \( \mathbb{N} \) is countably infinite. For instance \( \{3, 7, 19\} \) is a subset but not countably infinite.

However, every infinite subset of \( \mathbb{N} \) is countably infinite by Theorem 28.13. For instance, once we prove there are infinitely many primes, then we’ll know the set of all primes

\[
\{2, 3, 5, 7, 11, 13, 17, \ldots \}
\]

is countably infinite.

Is \( S = \{x^3 : x \in \mathbb{Z}\} = \{\ldots, -27, -8, -1, 0, 1, 8, 27, 64, \ldots\} \) countably infinite? Yes! First, it is an infinite set since \( f(x) = x^3 \) is a strictly increasing function. As \( S \) is an infinite subset of the countably infinite set \( \mathbb{Z} \), we know \( S \) is countably infinite by Theorem 28.13.

**Example 28.15.** Is \( |2\mathbb{Z}| = \aleph_0 \)? Yes, since \( 2\mathbb{Z} \) is an infinite subset of the countably infinite set \( \mathbb{Z} \).

Alternatively, you could create an explicit bijection \( f : \mathbb{N} \to 2\mathbb{Z} \), although this is more difficult. We see that we can list the elements of \( 2\mathbb{Z} \) as \( 0, 2, -2, 4, -4, 6, -6, \ldots \). From this pattern, one possible bijection is given by the rule

\[
f(n) = \frac{1 + (-1)^n (2n - 1)}{2}.
\]
28. E Exercises

Exercise 28.1. Declare whether the following statements are true or false, with proof/reason or counter-example:
(a) All finite sets have the same cardinality.
(b) If \( f : A \to B \) is a function between two sets, then \( |f| = |A| \) (thinking of \( f \) as a set of ordered pairs).
(c) Every subset of \( \mathbb{N} \) is countably infinite.
(d) Every subset of an infinite set has cardinality \( \aleph_0 \).
(e) If \( f : A \to B \) is a surjective function then \( |f| = |B| \) (thinking of \( f \) as a set of ordered pairs).

Exercise 28.2. Define \( h : (0, \infty) \to (0, 1) \) by the rule
\[
h(x) = \frac{x}{x + 1}.
\]
Verify that \( h \) is a bijection. What does this say about the cardinality of these open intervals?

Exercise 28.3. Prove that the set of natural numbers with exactly one of the digits equal to 7 is countably infinite. For instance, the number 103792 has exactly one of its digits equal to 7, while 8772 has two digits equal to seven.

Exercise 28.4. Consider the set
\[
S = \{x \in \mathbb{Z} : x = a^2 + b^2 \text{ for some } a, b \in \mathbb{Z}\}.
\]
Prove that \( |S| = |\mathbb{N}| \).

Exercise 28.5. Prove that the function in Theorem 28.4 is a bijection. (See the remark following the theorem for a formal definition of the function. The Pasting-together Theorem might be helpful.)

Exercise 28.6. Prove that \( |\mathbb{R}| = |(0, 1)| \). (Hint: Consider the tangent function. Alternatively, use Exercise 28.2 in pieces.)
29 More Examples of Countable Sets

Usually a combination of two countable sets is countable. For instance, in this section we will prove that unions of countable sets are countable, as well as Cartesian products. Finally, we will show that \( \mathbb{Q} \) is countably infinite.

29.A Unions

**Theorem 29.1.** If \( S \) and \( T \) are countable sets, then \( S \cup T \) is countable.

*Proof.* We will prove the case when \( S \) and \( T \) are both countably infinite and \( S \cap T = \emptyset \), leaving the other cases as an exercise for the student.

So assume that \( S \) and \( T \) are countably infinite. Thus, we can write the elements of \( S \) in an infinite list \( s_1, s_2, s_3, \ldots \). Similarly list the elements of \( T \) as \( t_1, t_2, t_3, \ldots \). We need to list the elements of \( S \cup T \) in an infinite list, which is easy to do by interlacing the two lists, as

\[
s_1, t_1, s_2, t_2, s_3, t_3, \ldots
\]

There is no repetition in this list, since \( S \cap T = \emptyset \) and the two original lists have no repetitions. Also this new list contains all the elements of \( S \cup T \).

29.B Products

Taking a union is not the only operation we can do with two sets. Another operation is intersection. When we intersect two sets, the cardinality can get much smaller. There is a third operation: the Cartesian product. Cantor came up with a very clever method for showing that the product of two countably infinite sets is still countably infinite. Thus we have:

**Theorem 29.2.** If \( A \) and \( B \) are countably infinite sets, then \( A \times B \) is as well.

*Proof.* Without loss of generality, we just need to show that \( \mathbb{N} \times \mathbb{N} \) is countably infinite. Consider the following diagram:

\[
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]
Travel along each arrow, starting at the smallest arrow, and passing to the next smallest arrow. This allows us to list the elements of \(\mathbb{N} \times \mathbb{N}\) as
\[
(1, 1), (2, 1), (1, 2), (3, 1), (2, 2), (1, 3), \ldots,
\]
according to when we pass through each ordered pair. We will hit each ordered pair exactly once.

\textbf{Remark 29.3.} The bijection \(f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) described in Theorem 29.2 was only described implicitly because the idea of the proof is much more important than the details. However, we can explicitly describe the function. It is given by the rule
\[
f(m, n) = \frac{(m + n - 1)(m + n - 2)}{2} + n.
\]
The first term \(\frac{(m + n - 1)(m + n - 2)}{2}\) is the size of the triangle covered by the previous arrows, and the last term \(n\) counts how far along the current arrow we have travelled. (The details of proving that \(f\) is a bijection are left to the over-motivated reader!)

There are other ways of showing \(|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|\). For instance, the function \(g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) given by the rule
\[
g(m, n) = 2^{m-1}(2n - 1)
\]
is also a bijection. (Proving that \(g\) is a bijection requires the lemma that every natural number can be written as a unique power of 2 times a unique odd integer.)

There are many other options. For instance, we could have used arrows pointing down and to the left, instead of up and to the right. Alternatively, we could have “snaked” back and forth along each finite diagonal. ▲

29.C Rational Numbers

We are almost ready to describe the cardinality of \(\mathbb{Q}\). First, let’s deal with the positive rational numbers \(\mathbb{Q}_+\).

\textbf{Theorem 29.4.} The set \(\mathbb{Q}_+\) is countably infinite.

\textit{Proof.} Put the elements of \(\mathbb{Q}_+\) into a diagram as below. (We put fractions which are not in lowest terms as light gray.)

\[
\begin{array}{cccccc}
\frac{1}{1} & 2 & 3 & 4 & \ldots \\
1 & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \ldots \\
\frac{1}{2} & 2 & 3 & 4 & \ldots \\
1 & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \ldots \\
\frac{1}{3} & 2 & 3 & 4 & \ldots \\
1 & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]
Now, we just list the elements as before, skipping over the elements in light gray, since they will be counted when they are in lowest terms. This counting procedure never repeats elements (since we skip those fractions not in lowest terms), and continues forever since \( \mathbb{Q}_+ \) is infinite (since it contains \( \mathbb{N} \); in other words, the top row of the diagram is infinite).

**Remark 29.5.** Here is an alternate proof of Theorem 29.4. Define a function \( f : \mathbb{Q}_+ \to \mathbb{N} \times \mathbb{N} \) by sending a rational number \( a/b \) (written in lowest terms) to the ordered pair \((a, b)\). The map \( f \) is injective, but not surjective. The set \( \text{im}(f) \) is infinite, and a subset of the countably infinite set \( \mathbb{N} \times \mathbb{N} \). Thus \( \text{im}(f) \) is countably infinite. But \( f \) is a bijection from \( \mathbb{Q}_+ \) to \( \text{im}(f) \), hence \( \mathbb{Q}_+ \) is countably infinite. ▲

**Corollary 29.6.** The set \( \mathbb{Q} \) is countably infinite.

*Proof sketch.* We have \( \mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\} \). We know \( \mathbb{Q}_+ \) is countable by Theorem 29.4. By the same reasoning, \( \mathbb{Q}_- \) is countable. Also, \( \{0\} \) is finite, hence countable. By Theorem 29.1, unions of countable sets are countable, so we see that \( \mathbb{Q} \) is countable. It is also infinite, hence countably infinite. □

We finish with one more example of how to show a set is countably infinite.

**Example 29.7.** Let \( S = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i \geq j\} \). This set is pictured below. We will prove that \( S \) is countably infinite.

\[
\begin{array}{ccccccc}
(1, 1) & (1, 2) & (1, 3) & (1, 4) & (1, 5) & \cdots \\
(2, 1) & (2, 2) & (2, 3) & (2, 4) & (2, 5) & \cdots \\
(3, 1) & (3, 2) & (3, 3) & (3, 4) & (3, 5) & \cdots \\
(4, 1) & (4, 2) & (4, 3) & (4, 4) & (4, 5) & \cdots \\
(5, 1) & (5, 2) & (5, 3) & (5, 4) & (5, 5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

First, the set \( S \) is infinite, since the left column is infinite. Since \( S \subseteq \mathbb{N} \times \mathbb{N} \) and \( \mathbb{N} \times \mathbb{N} \) is countably infinite, we know that \( S \) is countably infinite by Theorem 28.13.

Alternatively, we can list the elements of \( S \) by using the “diagonal” argument from earlier. (We can’t list the elements of \( S \) by going down columns, but could we list the elements of \( S \) by traveling across the successive rows?) △
29.D Exercises

Exercise 29.1. Prove that \( \{0,1\} \times \mathbb{N} \) is countably infinite. (Hint: Use theorems in the section.)

Exercise 29.2. Finish the proof of Theorem 29.1. (Hint: (1) Since \( S \cup T = S \cup (T - S) \), we can replace \( T \) by \( T - S \). After this change, we reduce to the case when \( S \cap T = \emptyset \). (2) There are two unfinished cases: (a) both \( S \) and \( T \) are finite, or (b) one of them is finite and the other infinite.)

Exercise 29.3. Let \( A \) and \( B \) be countable sets. Prove that \( A \times B \) is countable. (How is this different from what was proven in Theorem 29.2?)

Exercise 29.4. Let \( n \geq 2 \) be an integer, and let \( A_1, A_2, \ldots, A_n \) be countable sets. Prove that \( A_1 \times A_2 \times \cdots \times A_n \) is countable. (Hint: Induction.)

Exercise 29.5. Prove \( |\mathbb{Z} \times \mathbb{N}| = |\mathbb{Q}| \).

Exercise 29.6. Prove that if \( A_1, A_2, \ldots \) are disjoint, countably infinite sets, then \( \bigcup_{i=1}^{\infty} A_i \) is countably infinite. (Hint: Not induction. Think about diagonal arguments.)

Exercise 29.7. Prove that the set of finite subsets of \( \mathbb{N} \) is countable.
30 Uncountable Sets

The results of this section will be centered around the following definition.

**Definition 30.1.** A set $S$ which is not countable is said to be *uncountable*.

We can think of the uncountable sets as those sets which are *bigger* than the countably infinite sets, as in the following figure.

<table>
<thead>
<tr>
<th>Cardinalities</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Countably Infinite</td>
<td>$\mathbb{N}_0$</td>
</tr>
<tr>
<td>Countable</td>
<td>${0}$</td>
</tr>
<tr>
<td></td>
<td>${1}$</td>
</tr>
<tr>
<td></td>
<td>${2}$</td>
</tr>
<tr>
<td>Uncountable</td>
<td>$\mathbb{N}_1$</td>
</tr>
<tr>
<td></td>
<td>$\mathbb{N}_2$</td>
</tr>
</tbody>
</table>

As is evident from this diagram, we still don’t have any examples of uncountable sets. In this section we will see that there are many examples.

30.A How big is $\mathbb{R}$?

Before we can talk about how big the set of real numbers is, we need to explain more precisely what a real number is. We usually think of real numbers as infinite decimal expansions. For instance:

$$
\begin{align*}
1 & = 1.00000 \ldots \\
\sqrt{2} & = 1.41421 \ldots \\
-\frac{\pi}{13} & = -0.24166 \ldots \\
\exp\left(\frac{24}{7}\right) & = -3.42609 \ldots
\end{align*}
$$

However, real numbers do not always have unique infinite decimal expansions. If a
number ends in repeating 9’s, we can shift up and end in repeating 0’s. For example,

\[
0.99999\ldots = 1.00000\ldots \\
8.3929999\ldots = 8.3930000\ldots \\
-3928.8382999\ldots = -3928.8383000\ldots
\]

To avoid non-uniqueness issues, we will always avoid writing decimal expansions which end in repeating 9’s.

Our goal now is to show that \( \mathbb{R} \) is uncountable. From a previous homework problem we know that \( |(0,1)| = |\mathbb{R}| \), so it suffices to show that \( (0,1) \) is uncountable. (This set is easier to work with.) We know that \( (0,1) \) is infinite, so to prove that it is uncountable we must show that there does not exist any bijection \( f : \mathbb{N} \rightarrow (0,1) \). Cantor’s trick to do this is to show that every function \( f : \mathbb{N} \rightarrow (0,1) \) is not surjective, using what is now commonly called a “diagonalization argument.” Before we give the technical proof, we demonstrate the idea with an example.

Suppose \( f : \mathbb{N} \rightarrow (0,1) \) is the function

\[
\begin{align*}
f(1) &= 0.29838293\ldots \\
f(2) &= 0.43828183\ldots \\
f(3) &= 0.73826261\ldots \\
f(4) &= 0.20030000\ldots \\
f(5) &= 0.73724892\ldots \\
&\quad \vdots
\end{align*}
\]

Our goal is to prove that \( f \) is not surjective. Thus, we must find some element \( x \in (0,1) \) that \( f \) does not map to. We will construct \( x \) digit by digit, so that it doesn’t match any of the numbers on our list.

First, we want \( x \) to be different from \( f(1) = 0.29838293\ldots \). We can make sure this is true by guaranteeing \( x \) is different from the first digit past the decimal point of \( f(1) \). So, let’s change that first 2 to a 4, and put

\[
x = 0.4\ldots
\]

Notice that no matter what we do with the rest of the digits of \( x \), it will not match \( f(1) \).

Second, we want \( x \) to be different from \( f(2) = 0.43828183\ldots \). They do match on their first digit, but we can make their second digits different by changing the 3 to a 4. So we put

\[
x = 0.44\ldots
\]

and it will not equal \( f(1) \) or \( f(2) \).

Third, we want \( x \) to be different from \( f(3) = 0.73826261\ldots \). It already is different because of our choice of the first two digits, but we probably should continue the pattern we’ve already come up with, and make sure that the third digit is different. So we change the 8 to a 4, and put

\[
x = 0.444\ldots
\]
and we have \( x \neq f(1), f(2), f(3) \).

For the fourth number, \( f(4) = 0.20030000 \ldots \), we change the 3 to a 4, and put

\[
x = 0.4444 \ldots
\]

and we have \( x \neq f(1), f(2), f(3), f(4) \).

For the fifth number, \( f(5) = 0.73724892 \ldots \), we need to change the fifth digit 4, so we change it to 7. Put

\[
x = 0.44447 \ldots
\]

In general, we look at the \( n \)th digit of \( f(n) \), and change it to a 4, unless it is already a 4 in which case we change it to a 7. We place that new digit in the appropriate place in \( x \). After all of these changes, \( x \) cannot match any number on our list, and so \( f \) is not onto.

**Remark 30.2.** If we start with a different list of numbers, the number \( x \) we construct will be different (depending on that list). ▲

To make this work more easily, define the *digit change* function

\[
dig : \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \to \{4, 7\}
\]

by the rule

\[
dig(i) = \begin{cases} 
4 & \text{if } i \neq 4 \\
7 & \text{if } i = 4.
\end{cases}
\]

Note that because the digit change function does not use 9’s, we don’t need to worry about \( x \) ending in repeating 9’s.

**Remark 30.3.** There are many other digit change functions we could have used. This is just one option. ▲

We are now prepared to give the formal proof that \((0, 1)\) is uncountable. As discussed above, the technique used in this proof is known as Cantor’s diagonalization argument.

**Theorem 30.4.** The set \((0, 1)\) is uncountable.

**Proof.** Let \( f : \mathbb{N} \to (0, 1) \) be any function. We will show that \( f \) is not surjective.

Write \( f(n) \) using a decimal expansion \( f(n) = 0.d_{1n}d_{2n}d_{3n} \ldots \) (which doesn’t end in repeating 9’s). Let \( x \in (0, 1) \) be the number with decimal expansion \( x = 0.x_1x_2x_3 \ldots \) where \( x_n = dig(d_{nn}) \). In other words, the \( n \)th digit of \( x \) is the digit change of the \( n \)th digit of \( f(n) \). Hence \( x \neq f(n) \) for each \( n \in \mathbb{N} \). Therefore \( f \) is not onto, as it doesn’t map to \( x \). □
Corollary 30.5. The set \( \mathbb{R} \) is uncountable.

The cardinality of \( \mathbb{R} \) is called the continuum, and we write \(|\mathbb{R}| = \aleph_1\). You might ask: Where does \( \aleph_1 \) fit in the chain of cardinalities? Is it just one step up from \( \aleph_0 \)?

The answer is strange. It depends on the axioms you use! Some mathematicians do assume \( \aleph_0 = \aleph_1 \). Most mathematicians simply do not worry about this question.

We have already seen that \(|(0, 1)| = |\mathbb{R}|\), so \((0, 1)\) also has continuum cardinality. Here are some more examples of sets with continuum cardinality.

1. Any open interval \((a, b)\) with \(a, b \in \mathbb{R}\). (We can also replace \(a\) with \(-\infty\), or \(b\) with \(\infty\).)
2. Any half-open interval \([a, b)\) with \(a, b \in \mathbb{R}\). (We can replace \(b\) with \(\infty\).)
3. Any half-open interval \((a, b]\) with \(a, b \in \mathbb{R}\). (We can replace \(a\) with \(-\infty\).)
4. Any closed interval \([a, b]\) with \(a, b \in \mathbb{R}\).

To give the idea behind how to prove these facts, we will show that \((0, 1]\) has continuum cardinality.

Proof. We define a piece-wise bijection \( f : (0, 1] \rightarrow (0, 1) \). Let \( S = \{1/n : n \in \mathbb{N}\} \subseteq (0, 1] \). Now, define \( f \) by the rule

\[
f(x) = \begin{cases} 
  x & \text{if } x \notin S \\
  1/(n + 1) & \text{if } x = 1/n \in S.
\end{cases}
\]

It is easy to see that \( f \) is a bijection from \((0, 1] - S\) to the set \((0, 1) - S\) (as it is essentially the identity function on this set). It is also a bijection from \(S \rightarrow (0, 1) \cap S\). By pasting together, we have a bijection.

We end this section with one last result which can be useful to tell whether a set is uncountable.

Theorem 30.6. Let \( A \) and \( B \) be sets, with \( A \subseteq B \). If \( A \) is uncountable, then \( B \) is uncountable.

Proof. This is the contrapositive of Theorem 28.13, after noting that \( A \) and \( B \) must be infinite. \( \square \)

Example 30.7. Any subset of \( \mathbb{R} \) containing \((0, 1)\) is uncountable, by the previous theorem. In a later section we will show that any such subset has continuum cardinality.

There are lots of subsets of \( \mathbb{R} \) which are not uncountable. Can you list some? \( \triangle \)

30.B Exercises

Exercise 30.1. Let \( a, b \in \mathbb{R} \) with \( a < b \). Construct a bijection \( f : (0, 1) \rightarrow (a, b) \), and prove it is a bijection. (This shows that bounded open intervals all have the same cardinality.)
Exercise 30.2. Prove that the interval $[0, 1)$ has continuum cardinality, by creating a bijection $[0, 1) \rightarrow (0, 1)$.

Exercise 30.3. Prove that the interval $[0, 1]$ has continuum cardinality.

Exercise 30.4. Prove that the irrational numbers are uncountable. (Hint: Theorem 29.1 may be useful, along with contradiction.) Find a subset of the irrational numbers which is countably infinite.

Exercise 30.5. Prove or disprove: The complex numbers are uncountable. (Extra: Can you find exactly what their cardinality is?)

Exercise 30.6. We defined a product of two sets $A$ and $B$ to be the collection of ordered pairs from $A$ and $B$.

Let $A_1, A_2, A_3, \ldots$ be sets. Define the product $\prod_{i=1}^{\infty} A_i = A_1 \times A_2 \times A_3 \times \cdots$ to be the set of ordered sequences

\[ \{(a_1, a_2, a_3, \ldots) : a_i \in A_i \text{ for each } i \geq 1\} \]

We showed previously that a finite product of countable sets is countable. Show that the countable product $\prod_{i=1}^{\infty} \{0, 1\} = \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \cdots$ is not countable. (Hint: (1) This product is the set of infinite sequences of 0’s and 1’s. (2) Use Cantor’s diagonalization argument.)
31 Injections and Cardinalities

In the previous section we proved the amazing fact that \( \mathbb{R} \) is not a countable set. Thus, we might expect that \( |\mathbb{N}| < |\mathbb{R}| \). We have only defined when cardinalities are equal. In this section we give a method for determining inequalities between cardinalities.

31.A Injections vs. bijections

Let \( S \) and \( T \) be arbitrary sets. We would like to think of \( S \) as “smaller” than \( T \) if we can fit \( S \) inside \( T \). However, consider the set \( \mathbb{N} \) which sits properly inside \( \mathbb{Z} \). These sets have the same cardinality! Thus, we have to be extra careful about whether cardinalities are strictly smaller or not.

One way to think about fitting \( S \) inside another set \( T \) is to use an injection. This motivates the following definitions (which will only fully be justified in the next section).

Definition 31.1. Let \( A \) and \( B \) be sets. If there is an injective function \( f : A \to B \) we write \( |A| \leq |B| \). In this case, if there is no bijection from \( A \) to \( B \), then we write \( |A| < |B| \).

The following are some examples of these definitions in action.

Example 31.2. (1) What is the relation between the sets \( S = \{1, 3, 5\} \) and \( T = \{2, 4, 6, 8\} \)? There is an injection from \( S \) to \( T \), but no bijection (since \( S \) has three elements, but \( T \) has four elements). Thus \( |S| < |T| \).

(2) What is the relation between the sets \( 2\mathbb{N} \) and \( \mathbb{N} \)? The inclusion map \( f : 2\mathbb{N} \to \mathbb{N} \) defined by the rule \( f(x) = x \) is an injective function. Hence, \( |2\mathbb{N}| \leq |\mathbb{N}| \). However, both sets are countably infinite, so in fact there is a (different) bijective function between the sets, so we have \( |2\mathbb{N}| = |\mathbb{N}| \).

(3) What is the relation between the sets \( \mathbb{N} \) and \( \mathbb{R} \)? The inclusion map \( \mathbb{N} \to \mathbb{R} \) is an injective function. Hence \( |\mathbb{N}| \leq |\mathbb{R}| \). In the previous section we proved that \( \mathbb{R} \) is uncountable, so there is no bijection between these sets. Hence, we have the strict inequality \( |\mathbb{N}| < |\mathbb{R}| \). △

Advice 31.3. It is good to think of injections as giving you “half” of the information needed to construct a bijection, which is why we only get an inequality \( \leq \).

You might recall that in our tower of cardinalities (found at the beginning of the previous section) we had an infinite list of infinite cardinalities \( \aleph_0 < \aleph_1 < \aleph_2 < \ldots \). But so far, we have only found two types of infinite cardinalities; the countably infinite sets, and the sets of continuum size. In our next theorem, we will prove that for any
set $S$, we have $|S| < |\mathcal{P}(S)|$. Thus, we have an infinite chain of increasing infinite cardinalities

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots.$$ 

When $S$ is a finite set, say $|S| = n$, then we know $|S| < |\mathcal{P}(S)|$ because $n < 2^n$. But how does this process work when $S$ is an infinite set? In this situation we cannot simply count elements. Rather, we must prove that there is no bijection $g : S \rightarrow \mathcal{P}(S)$. Our approach will be similar to how we showed $\mathbb{R}$ is not countable. Start with an arbitrary function $g : S \rightarrow \mathcal{P}(S)$, and show that $g$ is not surjective by finding some set $B \in \mathcal{P}(S)$ which is not in the image of $g$. The hardest part is constructing $B$. We will give an explicit example (using finite sets), and then give the formal proof for arbitrary sets.

Fix $S = \{1, 2, 3\}$. Hence

$$\mathcal{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$ 

Consider the function $g : S \rightarrow \mathcal{P}(S)$ given by $g(1) = \{2\}$, $g(2) = \emptyset$, and $g(3) = \{1, 2, 3\}$. We want to find a set $B \in \mathcal{P}(S)$ that we can prove is not equal to $g(1)$, $g(2)$, or $g(3)$. Of course, we could just pick one of the other five sets not in the image of $g$ in this case; but we want to come up with a method that will work for any set $S$.

So, we ask the question: Is $x \in g(x)$?

- Is $1 \in g(1) = \{2\}$? The answer is no.
- Is $2 \in g(2) = \emptyset$? The answer is no.
- Is $3 \in g(3) = \{1, 2, 3\}$? The answer is yes.

We construct $B \in \mathcal{P}(S)$ by the following rule: if the answer to the question “Is $x \in g(x)$?” is no then we put $x \in B$, but if the answer is yes then we leave $x$ out of $B$. Using the answers we had above, we see that $B = \{1, 2\}$.

Notice that $B$ will not equal $g(x)$ because if $x \in g(x)$ then $x \notin B$, and vice versa. Indeed, we see that

- $1 \notin g(1)$ but $1 \in B$.
- $2 \notin g(2)$ but $2 \in B$.
- $3 \in g(3)$ but $3 \notin B$.

This forces $B$ to be different than any element in the image.

One more example is in order, to test our understanding. Suppose that $S = \{1, 2, 3\}$ as above, and suppose $h : S \rightarrow \mathcal{P}(S)$ is the function defined by the rule $h(1) = \{2, 3\}$, $h(2) = \{2\}$, $h(3) = \{2, 3\}$. If we follow the same pattern as above, asking the question “Is $x \in h(x)$?” and using the answers to define $B$, what set $B$ do we get? (Before looking at the answer, try this construction yourself.)

**Answer:** The set is $\{1\}$.

**Remark 31.4.** The set $B$ is sometimes called the **barber set**. This is because there is some connection with the following paradox: There lives a barber in a small town who always obeys the rule that he will shave everyone in town who doesn’t shave themselves, but if they shave themselves he will not shave them. Does the barber shave himself? If he does, then he cannot shave himself by his own rule. But if he doesn’t, then he must shave himself by his rule.
One way to resolve the paradox is to assume instead the barber does not live in the town. This corresponds, roughly, to the fact that \( B \) is not in the image of the map.

We are now ready to prove the theorem in general.

**Theorem 31.5.** If \( S \) is a set, then \(|S| < |\mathcal{P}(S)|\).

**Proof.** Let \( S \) be any set. First, we prove that \(|S| \leq |\mathcal{P}(S)|\), so we need to find an injective function \( f : S \to \mathcal{P}(S) \). Define \( f \) by the rule \( f(s) = \{s\} \). To prove that \( f \) is injective, let \( a, b \in S \) and assume \( f(a) = f(b) \). Hence \( \{a\} = \{b\} \). Therefore \( a = b \), since sets are equal exactly when they have the same elements.

Next, let \( g : S \to \mathcal{P}(S) \) be an arbitrary function. We will show that \( g \) is never surjective, and hence there is no bijection between \( S \) and \( \mathcal{P}(S) \). Define the (barber) set as
\[
B = \{x \in S : x \notin g(x)\}.
\]
This is a subset of \( S \), hence an element of \( \mathcal{P}(S) \). We will show that \( B \) is not in the image of \( g \).

Let \( s \in S \) be arbitrary. There are two cases.

**Case 1:** Assume \( s \in g(s) \). In this case \( s \notin B \), hence \( g(s) \neq B \).

**Case 2:** Assume \( s \notin g(s) \). Then \( s \in B \), hence \( g(s) \neq B \).

In every case \( B \neq g(s) \). Since \( s \in S \) was arbitrary, this means \( B \) cannot be in the image (since it does not equal any element of the image).

31.B How big is \( \mathcal{P}(\mathbb{N}) \)?

We now know that \(|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|\), but exactly how big is \( \mathcal{P}(\mathbb{N}) \)? In the next section we will prove it has continuum cardinality. However, we currently have the tools to find another set with the same cardinality. For the rest of this section, given a set \( S \) we let \( \mathcal{F}(S) \) be the set of all functions from \( S \) to \( \{0, 1\} \).

**Theorem 31.6.** If \( S \) is any set, then \(|\mathcal{P}(S)| = |\mathcal{F}(S)|\).

**Proof.** We must construct a bijection between the two sets \( \mathcal{P}(S) \) and \( \mathcal{F}(S) \). The map is this: send a subset to its characteristic function
\[
f : \mathcal{P}(S) \to \mathcal{F}(S), \quad f(A) = \chi_A.
\]
We first show that \( f \) is injective. Let \( A, B \in \mathcal{P}(S) \) and assume \( f(A) = f(B) \). We then have \( \chi_A = \chi_B \). Plugging in an arbitrary element \( s \in S \), we have \( \chi_A(s) = \chi_B(s) \). The left-hand-side is 1 when \( s \in A \) and 0 otherwise, and similarly for the right-hand side. Thus, \( s \in A \) if and only if \( s \in B \). In other words \( A = B \).

Finally, we show that \( f \) is surjective. Let \( \varphi : S \to \{0, 1\} \) be any function. Put \( A = \{s \in S : \varphi(s) = 1\} \). We then check directly that \( \varphi = \chi_A \). (They have the same domain and codomain, and the same rule.) Hence \( \varphi = f(A) \), so \( f \) is onto.
Corollary 31.7. \( |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})| \) is uncountable.

\begin{proof}
The equality follows from the previous theorem. We also know \( |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| \), hence it is uncountable.
\end{proof}

Remark 31.8. In Exercise 30.6, you proved that \( \prod_{i=1}^{\infty} \{0, 1\} \) is uncountable. The corollary above gives an easier way to see this, since functions \( \mathbb{N} \to \{0, 1\} \) can be thought of as infinite sequences of 0’s and 1’s.

31.C Exercises

Exercise 31.1. Answer each of the following true or false problems, proving your answer.

(a) Every uncountable set has the same cardinality as \( (0, 1) \).
(b) Let \( A \) and \( B \) be sets. If \( A \subseteq B \), then \( |A| \leq |B| \).
(c) For sets \( A \) and \( B \), if \( A \subsetneq B \), then \( |A| < |B| \).
(d) Given sets \( A \), \( B \), and \( C \), if \( A \subseteq B \subseteq C \) and both \( A \) and \( C \) are countably infinite, then \( B \) is countably infinite.
(e) No subset of \( \mathbb{R} \) has smaller cardinality.
(f) For sets \( S \) and \( T \), if \( |S| < |T| \) and \( S \) is finite, then \( T \) is infinite.
(g) For sets \( S \) and \( T \), if \( |S| < |T| \) and \( S \) is countable, then \( T \) is uncountable.
(h) For sets \( S \) and \( T \), if \( |S| < |T| \) and \( S \) is countably infinite, then \( T \) is uncountable.
(i) For any set \( S \), there exists another set \( T \), with \( |S| < |T| \).

Exercise 31.2. Let \( S = \{a, b, c, d, e\} \) and let \( g : S \to \mathcal{P}(S) \) be defined by the rule \( g(a) = \{b, d\} \), \( g(b) = \{a, c, e\} \), \( g(c) = \{a, c, d, e\} \), \( g(d) = \emptyset \), \( g(e) = \{e\} \). List the elements of the barber set \( B = \{s \in S : s \notin g(s)\} \). Why is it not in the image of \( g \)?

Exercise 31.3. Find a set with cardinality bigger than \( |\mathbb{R}| \). Then find a set with cardinality bigger than that.

Exercise 31.4. The pigeon-hole principle says that for finite sets \( A \) and \( B \), if \( |A| = |B| \) and \( f : A \to B \) is a function, then \( f \) is injective if and only if \( f \) is surjective. Prove that this fails for infinite sets, by proving the following:

(a) Find an infinite set \( S \) and a function \( f : S \to S \) that is injective but not surjective.
(b) Find an infinite set \( S \) and a function \( g : S \to S \) that is surjective but not injective.

In both parts prove that the function you construct has the requisite properties.

Exercise 31.5. Let \( A \) and \( B \) be sets with \( f : A \to B \) a bijection. Define a new map \( g : \mathcal{P}(A) \to \mathcal{P}(B) \) by the rule \( g(S) = \{f(s) : s \in S\} \), where \( S \subseteq A \) is an arbitrary element of \( \mathcal{P}(A) \). Prove that \( g \) is a bijection.

Conclude that if \( |A| = |B| \) then \( |\mathcal{P}(A)| = |\mathcal{P}(B)| \).
Exercise 31.6. (a) Define a function $f : \mathbb{R} \to \mathcal{P}(\mathbb{Q})$ by the rule
$$f(x) = \{ q \in \mathbb{Q} : q \leq x \}.$$  

Prove that $f$ is injective. (Hint: For any two real numbers $x < y$, there is a rational number strictly between them. See Exercise 11.2.)

(b) Using the previous exercise, conclude that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$.

Exercise 31.7. Let $A$ and $B$ be non-empty sets. Prove that there exists an injection $f : A \to B$ if and only if there exists a surjection $g : B \to A$. (Hint: For the backwards direction, given a surjection $g : B \to A$ define a function $f : A \to B$ by the rule $f(a) =$ one of the elements which mapped to $a$.)
32 The Schröder-Bernstein Theorem

Let \( a, b \in \mathbb{R} \). If we know \( a \leq b \) and \( b \leq a \), then we must have \( a = b \). In other words, the “less than or equal to” relation on the real numbers is antisymmetric.

We have used a similar notation for cardinalities, and it is natural to ask whether or not this relation is antisymmetric. In other words, given sets \( A \) and \( B \), if we know \(|A| \leq |B|\) and \(|B| \leq |A|\), must we have \(|A| = |B|\)? The answer is yes, and this result is called the Schröder-Bernstein Theorem.

**Theorem 32.1** (Schröder-Bernstein). Let \( A \) and \( B \) be sets. If \(|A| \leq |B|\) and \(|B| \leq |A|\), then \(|A| = |B|\).

**Remark 32.2.** The story of how this theorem came to be is long and somewhat convoluted. Cantor was the first to state the theorem, but apparently he had no proof. The first proof (that we know of) was found by Dedekind, but he did not publish his work at the time.

Schröder announced a proof, which was later shown to have an error. Finally, in 1897, Bernstein (who was only 19 years old, and a student of Cantor) presented a proof. At nearly the same time Schröder independently found a proof as well. Hence, these two mathematicians have their names attached to the theorem.

### 32.A Sketching the Proof Using Genealogy

Here we will sketch the main ideas of the proof. The proof is a direct one. Given sets \( A \) and \( B \), assume that \(|A| \leq |B|\) and that \(|B| \leq |A|\). In other words, we assume that we have two injective functions \( f : A \to B \) and \( g : B \to A \). Our goal is to show that \(|A| = |B|\), in other words we need to create a bijection \( h : A \to B \).

The main idea of the proof is that we partition both \( A \) and \( B \) into four pieces, and create a bijection between those pieces which, by the “Pasting-together Theorem” will give us the bijection \( h \):

\[
\begin{align*}
A &= A_1 \cup A_2 \cup A_3 \cup A_4 \\
B &= B_1 \cup B_2 \cup B_3 \cup B_4.
\end{align*}
\]

\( A_1 \) \( \downarrow h_1 \) \( \downarrow h_2 \) \( \downarrow h_3 \) \( \downarrow h_4 \)

The only information we have available comes from the two maps \( f \) and \( g \) that we have given to us. We must somehow use the maps \( f \) and \( g \) to make any progress on this problem. We might ask how these maps behave. Fix some element \( a_0 \in A \). Applying \( f \), we have a new element \( f(a_0) \in B \). We call this new element \( b_0 \). We can think of \( a_0 \) as the **parent** of \( b_0 \), because \( a_0 \) gives rise to \( b_0 \) (through the function \( f \)). We also call \( b_0 \) the **child** of \( a_0 \).
There are some very important facts we need to know about this parent-child relationship. First, every element in $A$ is the parent of exactly one child in $B$ because $f$ is a function. Second, not every element in $B$ is a child of some parent in $A$, because $f$ may not be surjective. Third, every element in $B$ which actually is a child has exactly one parent in $A$, because $f$ is injective.

We can also talk about elements of $B$ having children, using the function $g$, and the same facts we mentioned in the previous paragraph are still true.

Does $b_0$ have a child? Yes! We can pass back over to $A$ by applying the map $g$ to $b_0$. Set $a_1 = g(b_0) \in A$, which is the child of $b_0$. There are two cases.

**Case 1:** $a_0 = a_1$.

In this case, the maps $f$ and $g$ just send $a_0$ and $b_0$ back and forth to each other. Note that $a_0$ is its own grandparent! It seems very natural that in this case we would want $a_0$ and $b_0$ to correspond under $h$.

**Case 2:** $a_0 \neq a_1$.

We can now repeat the parent-to-child process. Let $b_1 = f(a_1) \in B$ which is the child of $a_1$. Is it possible that $b_1 = b_0$? No! We have $b_0 = f(a_0) \neq f(a_1) = b_1$, because $f$ is injective and $a_0 \neq a_1$.

Similarly, let $a_2 = g(b_1) \in A$ which is the child of $b_1$. We see that $a_1 = g(b_0) \neq g(b_1) = a_2$, because $g$ is injective and $b_0 \neq b_1$. However, we cannot tell whether $a_0$ and $a_2$ are the same or different, so we again have two cases.

**Case 2A:** $a_0 = a_2$. 

**Note:** The diagrams are not shown in the text. They are described in the narrative.
In this case, applying $f$ and $g$ sends the points $a_0, b_0, a_1, b_1$ in a loop. (In this case, each point is its own great-great-grandparent.) It seems natural for $a_0$ and $b_0$ to correspond, and for $a_1$ and $b_1$ to correspond, under $h$.

**Case 2B: $a_0 \neq a_2$.**

![Diagram](image)

At this point it is recommended that the readers workout for themselves that if we let $b_2 = f(a_2)$, then $b_2$ is different from $b_0$ and $b_1$ (by injectivity of $f$). Similarly, if we let $a_3 = g(b_2)$, then the readers should show that $a_3$ is different from $a_1$ and $a_2$ (by injectivity of $g$). Again we have two cases: $a_3$ could equal $a_0$ and we have a loop, or it is a new element.

In general, for $n \geq 0$, define $b_n = f(a_n)$ to be the child of $a_n$, and define $a_{n+1} = g(b_n)$ to be the child of $b_n$. Working by induction one can prove that there are exactly two options. First, these elements can end up in a loop with $a_{n+1} = a_0$, where $a_0, a_1, a_2, \ldots, a_n$ are distinct elements of $A$ and $b_0, b_1, b_2, \ldots, b_n$ are distinct elements of $B$. The second option is that there is no loop, and so we have an infinite chain of descendants: $a_0, a_1, a_2, \ldots$ are distinct elements of $A$ and $b_0, b_1, b_2, \ldots$ are distinct elements of $B$.

![Diagram](image)

Our discussion has concentrated around passing from a parent to a child, but we can, sometimes, reverse the process. Remember that elements can have at most one parent (because $f$ and $g$ are injective). So, if our chain of descendants ends in a loop, then when we go “backwards” up through the ancestors, we just cycle backwards through the loop.

What happens in the case where an element $a_0$ had a non-looping chain of descendants? If $a_0$ has no parent, we can say that $a_0$ is an **ultimate ancestor**, and we have the entire chain of descendants and ancestors.

However, if $a_0$ does have a parent $b_{-1} \in B$, then our chain can be extended:
In this case, \( b_{-1} \neq b_n \) for any \( n \geq 0 \), as depicted in the picture above. This is because 
\[ g(b_{-1}) = a_0 \neq a_{n+1} = g(b_n) \text{ and } g \text{ is injective.} \]

It is possible that \( b_{-1} \) has no parent, and hence is the ultimate ancestor at which the chain stops. On the other hand \( b_{-1} \) could have a parent \( a_{-1} \), and a similar argument shows that \( a_{-1} \) does not equal any of \( a_0, a_1, a_2, \ldots \).

In total, we see that there are four types for the chain of descendants and ancestors of an element:

1. The chain forms a finite loop (of the type described above).
2. The chain never loops, and has an ultimate ancestor in \( A \).
3. The chain never loops, and has an ultimate ancestor in \( B \).
4. The chain never loops, and has no ultimate ancestor. (Thus, it is doubly infinite.)

We are now ready to describe a partition of \( A \). We put \( A = A_1 \cup A_2 \cup A_3 \cup A_4 \) where

\[ A_i = \{ a \in A : \text{the chain of ancestors and descendants of } a \text{ is of type } i \} \]

Similarly, we put \( B = B_1 \cup B_2 \cup B_3 \cup B_4 \) where

\[ B_i = \{ b \in B : \text{the chain of ancestors and descendants of } b \text{ is of type } i \} \]

Technically \( \{A_1, A_2, A_3, A_4\} \) is sometimes not a partition, because some of the parts could be empty. However, if \( A_i \) is empty then so is \( B_i \) (and conversely). Thus, if one of these sets is empty, then we can just drop that case from consideration. In the remainder of the proof we will treat each case as if it occurs, leaving the contrary case unstated.

### 32.B Defining \( h \) piecewise

Now that we have a partition of \( A \) and \( B \), to construct a bijection \( h : A \to B \) it suffices to construct a bijection \( h_i : A_i \to B_i \) for \( i = 1, 2, 3, 4 \). Each of the cases is slightly different.

**Case 1: Defining \( h_1 : A_1 \to B_1 \).** In this case every element \( a_0 \in A_1 \) belongs to a loop of the form

\[ a_0 \mapsto b_0 \mapsto a_1 \mapsto b_1 \mapsto \cdots \mapsto a_n \mapsto b_n \mapsto a_0. \]

The element \( b_0 = f(a_0) \) belongs to the same loop, and hence \( b_0 \in B_1 \). We define \( h_1 = f|_{A_1} : A_1 \to B_1 \).
We need to show that $h_1$ is a bijection. First, it is injective because $f$ is injective (and the restriction of an injective function is still injective). Second, to show that $h_1$ is surjective, start by letting $b \in B_1$ be arbitrary. By definition of $B_1$, we know that $b$ lives in a loop, and so has a parent $a \in A_1$. Thus $h_1(a) = f(a) = b$.

**Case 2: Defining** $h_2 : A_2 \to B_2$. In this case every element has an ultimate ancestor in $A$. We define $h_2 = f|_{A_2} : A_2 \to B_2$. Nearly the same argument as in the previous case shows that $h_2$ is a bijection.

**Case 3: Defining** $h_3 : A_3 \to B_3$. In this case every element has an ultimate ancestor in $B$. Here we cannot define $h_3$ as $f_3|_{A_3}$, because this function would not be surjective. (We would miss the ultimate ancestors.)

However, just like in case 2, we can show that $g|_{B_3} : B_3 \to A_3$ is a bijection. Hence, it has an inverse function and we let $h_3$ equal that inverse,

$$h_3 = (g|_{B_3})^{-1} : A_3 \to B_3.$$ 

**Case 4: Defining** $h_4 : A_4 \to B_4$. We leave it to the reader to show that $h_4 = f|_{A_4}$ is a function from $A_4 \to B_4$, and is a bijection. [The argument is very similar to case 1 and case 2.]

### 32.C Examples

The Schröder-Bernstein theorem is not only beautiful symbolically, but also quite useful because it is sometimes very easy to describe injections back-and-forth between two sets $A$ and $B$, yet may be difficult to describe a bijection. Here are some standard examples.

**Example 32.3.** We will prove that the closed interval $[3, 10]$ has the same cardinality as $(0, 1)$.

Define $f : [3, 10] \to (0, 1)$ by the rule $f(x) = (x - 2)/10$. This is a linear function with $f(3) = 1/10$ and $f(10) = 8/10$. So it maps $[3, 10]$ into the interval $[1/10, 8/10] \subseteq (0, 1)$ injectively.

On the other hand, the map $g : (0, 1) \to [3, 10]$ given by $g(x) = x + 3$ is also an injection.

By the Schröder-Bernstein theorem, we are done. \(\triangle\)

This next example is so important, we will call it a theorem.

### Theorem 32.4. $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

**Proof.** In Exercise 31.6, we proved that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$. (For an alternate proof of this inequality, see Exercise 32.5 below.) By the Schröder-Bernstein theorem, it suffices to now prove $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Define $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ by the rule $f(A) = 0.X_A(1)X_A(2)X_A(3)\ldots$ (For instance, if $A = \{1, 3, 4, 7, 9, \ldots\}$ then $f(A) = 0.101100101\ldots \in \mathbb{R}$.) It just remains to show.
that this function is injective. Let $A, B \subseteq \mathbb{N}$ be arbitrary, and assume $f(A) = f(B)$. Thus
\[
0.x_A(1)x_A(2)x_A(3)\ldots = 0.x_B(1)x_B(2)x_B(3)\ldots.
\]
Since neither decimal expansion involves repeating 9's, the two expansions are equal. Hence $x_A(n) = x_B(n)$ for every $n \in \mathbb{N}$. This means that $A$ and $B$ have exactly the same elements so $A = B$, which finishes showing that $f$ is an injective function.

32.D Exercises

Exercise 32.1. Let $X, Y, Z$ be sets. Prove that if $X \subseteq Y \subseteq Z$ and $|X| = |Z|$, then $|X| = |Y|$ as well.

Exercise 32.2. Prove that $[5, 16)$ and $(0, \infty)$ have the same cardinalities.

Exercise 32.3. Given sets $A$ and $B$, prove that if there is an injection $f : A \to B$ and a surjection $g : A \to B$, then $|A| = |B|$. (Hint: A previous homework exercise might be useful.)

Exercise 32.4. Complete the proof in Case 2 (of Subsection 32.B) of the Schröder-Bernstein theorem, by showing that $f|_{A_2}$ is a function from $A_2$ to $B_2$, and also that it is bijective.

Exercise 32.5. In Exercise 31.6 we showed that $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$. Here is another way to do that.

Define a function $f : (0, 1) \to \mathcal{P}(\mathbb{N})$, by sending (the decimal expansion of) a real number $0.a_1a_2a_3\ldots$ (not ending in repeating 9’s) to the set
\[
\{a_1, 10a_2, 100a_3, \ldots\} - \{0\} \subseteq \mathbb{N}.
\]
(For instance, 0.03193\ldots maps to \{0, 30, 100, 9000, 30000, \ldots\} – \{0\}.) Prove that this is an injective function.