Chapter IV

Proof by Induction

Without continual growth and progress, such words as improvement, achievement, and success have no meaning. Benjamin Franklin

Mathematical induction is a proof technique that is designed to prove statements about all natural numbers. It should not be confused with inductive reasoning in the sciences, which claims that if repeated observations support a hypothesis, then the hypothesis is probably true; mathematical induction gives a definitive proof.

The basic idea of mathematical induction is to use smaller cases to prove larger ones. For instance, if one wished to prove that the open sentence

$$P(n) : n < 2^n$$

is true for all positive integers $n$, one might first check that it is true when $n = 1$. In fact, it is easy to check it for many different values of $n$.

Suppose we could prove that whenever $P(k)$ is true for some positive integer $k$, then $P(k + 1)$ is true. We could use this to finish the problem as follows:

Since $P(1)$ is true, $P(2)$ must be true; since $P(2)$ is true, $P(3)$ must be true; since $P(3)$ is true, $P(4)$ must be true; and so on, forever.

Induction is a technique for making clear what the phrase “and so on forever” means in this last sentence. Anytime we find ourselves wanting to repeat a process infinitely often in a proof, it is probably a sign that we should think about using induction.
13 Mathematical Induction

13.A The Principle of Mathematical Induction

An important property of the natural numbers is the principle of mathematical induction. It is a basic axiom, used in the definition of the natural numbers, and as such, it has no proof. It is as much a part of the definition of the natural numbers as the fact that if we add 1 to any natural number, we obtain a natural number (although its statement is more complicated).

**Axiom 13.1** (The Principle of Mathematical Induction). Let $P(n)$ be an open sentence, where the domain of $n$ is $\mathbb{N}$. Suppose that

(i) $P(1)$ is true, and
(ii) $\forall k \in \mathbb{N}, P(k) \Rightarrow P(k + 1)$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

A proof by mathematical induction proceeds by verifying that (i) and (ii) are true, and then concluding that $P(n)$ is true for all $n \in \mathbb{N}$. We call the verification that (i) is true the *base case* of the induction, and the proof of (ii) the *inductive step*. Typically, the inductive step will involve a direct proof; in other words, we will let $k \in \mathbb{N}$, assume that $P(k)$ is true, and then prove that $P(k + 1)$ is true. If we are using a direct proof, we call $P(k)$ the *inductive hypothesis*.

A proof by induction thus has the following four steps.

- **Identify $P(n)$**: Clearly state the open sentence $P(n)$. This needs to be done, but if $P(n)$ is obvious, then it need not be a written part of the proof.
- **Base Case**: Verify that $P(1)$ is true. This will typically be done by direct computation, or by giving an example (but a harder proof may be necessary).
- **Inductive Step**: Prove the implication $P(k) \Rightarrow P(k + 1)$ for any $k \in \mathbb{N}$. Typically this will be done by a direct proof; assume $P(k)$ and prove $P(k+1)$. (Occasionally it may be done by contrapositive or contradiction.)
- **Conclusion**: Conclude that the theorem is true by induction. As with identifying $P(1)$, this may not need to be a written part of the proof.

**Remark 13.2.** An intuitive way to think of mathematical induction is as a ladder with infinitely many rungs, numbered from the bottom. Stepping on rung number $n$ corresponds to confirming that $P(n)$ is true. Hypothesis (i) of mathematical induction says that we can step onto the first rung. Hypothesis (ii) of mathematical induction says that if we can reach rung number $n$, then we can reach rung number $n + 1$. Together, these two hypotheses allow us to reach any rung of the ladder.
We begin at rung 1 (by (i)).

Since we can reach rung 1, we can reach rung 2 (by (ii)).

Since we can reach rung 2, we can reach rung 3 (by (ii)).

Since we can reach rung 3, we can reach rung 4 (by (ii)).

Since we can reach rung 4, we can reach rung 5 (by (ii)).

And so on... For any given rung, we see that we can reach it.

Warning 13.3. Note the importance of the base case. Without it, the inductive step shows that we can move from one rung of the ladder to the next higher one, but there is no evidence that we can reach the bottom of the ladder at all. Perhaps the ladder is suspended high above the ground; the base case shows that we can actually reach the bottom rung.

Remark 13.4. Summation notation comes up often in induction proofs. It is important to be familiar with it. The notation
\[ \sum_{i=1}^{n} f(i) \]
means \( f(1) + f(2) + \cdots + f(n-1) + f(n) \), where we plug in each possible value of \( i \) between 1 and \( n \) into the function \( f \), and add them together. Hence,
\[
\sum_{i=1}^{n+1} f(i) = f(1) + \ldots + f(n) + f(n+1)
\]
\[ = (f(1) + \ldots + f(n)) + f(n+1) \]
\[ = \left( \sum_{i=1}^{n} f(i) \right) + f(n+1). \]
This fact, and variations of it are often used in induction proofs involving summation.
We now proceed to give an example of proof by induction, using summation notation. We will first sketch the strategy of the proof, and afterwards write the formal proof.

**Proposition 13.5.** For all \( n \in \mathbb{N} \),

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.
\]

**Proof Strategy.** We begin by identifying the open sentence \( P(n) \). In this case, \( P(n) \) is

\[ P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \]

The base case, verifying \( P(1) \), is done by a simple computation (plugging 1 in for \( n \)).

For the inductive step, we wish to assume \( P(k) \) and prove \( P(k+1) \). Hence, we are assuming for some \( k \in \mathbb{N} \) that \( P(k) \) is true, so

\[ \sum_{i=1}^{k} i = \frac{k(k+1)}{2}, \]

and we wish to prove

\[ P(k+1): \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}. \]

Examining the sum in the statement of \( P(k+1) \), we find that we have the left-hand side (LHS) is equal to:

\[
\text{LHS} = \sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k + 1) \\
= (1 + 2 + \cdots + k) + (k + 1) = \sum_{i=1}^{k} i + (k + 1),
\]

where the sum on the right is the sum involved in \( P(k) \). We now use our assumption of \( P(k) \) to simplify the sum, and complete the proof.

Now that we have sketched the proof method, let’s write a full and formal proof.

**Proof.** Let \( P(n) \) be the open sentence

\[ P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \]
We work by induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.

**Base case:** $P(1)$ is true, since we have

$$
\sum_{i=1}^{1} i = 1 = \frac{1(1 + 1)}{2}.
$$

**Inductive step:** Let $k \in \mathbb{N}$ and assume that

$$
P(k) : \sum_{i=1}^{k} i = \frac{k(k + 1)}{2}
$$

is true. We want to show

$$
P(k + 1) : \sum_{i=1}^{k+1} i = \frac{(k + 1)(k + 2)}{2}.
$$

We find

$$
\begin{align*}
\text{LHS} &= \sum_{i=1}^{k+1} i = 1 + 2 + 3 + \cdots + k + (k + 1) \\
&= (1 + 2 + 3 + \cdots + k) + (k + 1) \\
&= \left( \sum_{i=1}^{k} i \right) + (k + 1) \\
&= \frac{k(k + 1)}{2} + (k + 1) \\
&= \frac{k(k + 1) + 2(k + 1)}{2} \\
&= \frac{(k + 1)(k + 2)}{2} = \text{RHS}.
\end{align*}
$$

So $P(k + 1)$ is true. Hence, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

**Remark 13.6.** It can be helpful to point out to the reader of your proofs where you use the inductive hypothesis, as done above. Note that if you don’t ever use the inductive hypothesis, then you could have just proven the theorem without induction. ▲

**Remark 13.7.** With practice, you will become better at seeing how $P(k)$ and $P(k+1)$ are related (especially with sums like the one above), and these proofs will go much smoother for you. For instance, with practice we could have gone directly to the equality

$$
\sum_{i=1}^{k+1} i = \left( \sum_{i=1}^{k} i \right) + (k + 1)
$$

in the proof above. ▲
Warning 13.8. A common mistake that students make is to consider \( P(k) \) as a number. It is a statement, not a number. For example, in the previous proof, students might mistakenly write

\[
P(k) = \sum_{i=1}^{k} i = k^2
\]

which is incorrect, as it says that \( P(k) \) is the number \( k^2 \). Another incorrect use of \( P(k) \) is the following

\[
\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k + 1) = P(k) + (k + 1) = k^2 + (k + 1)
\]

Note that this also arises from thinking of \( P(k) \) as equal to part of the statement that it represents.

Remark 13.9. It might appear that in an induction proof we are assuming what we are attempting to prove. For instance, if we are trying to prove

\[
\forall n \in \mathbb{N}, P(n),
\]

by induction, then in the inductive step of the proof we will need to assume \( P(k) \). It would indeed be a logical mistake to assume \( P(k) \) if our immediate goal is to prove \( P(k) \).

However, that is not the case. The goal of the inductive step is not to prove \( P(k) \), but to prove that \( P(k+1) \) follows from \( P(k) \). Hence, in fact, we are not assuming what we wish to prove (namely that \( P(n) \) is true for all \( n \in \mathbb{N} \)). Note also that proving \( \forall k \in \mathbb{N}, P(k) \implies P(k+1) \) by itself does not prove that \( P(k) \) is true for any integer; it just proves that if \( P(k) \) is true for some \( k \), then \( P(k+1) \) must be true as well (which is why we also need the base case to start the induction).

Here is another result we can prove by induction.

**Proposition 13.10.** Given \( n \in \mathbb{N} \) it happens that \( 2^n > n \).

**Proof.** We work by induction on \( n \in \mathbb{N} \).

- **Base case:** We see that \( 2^1 = 2 > 1 \).
- **Inductive step:** Let \( k \in \mathbb{N} \) and assume \( 2^k > k \). We want to prove \( 2^{k+1} > k + 1 \).
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We find
\[
\text{LHS} = 2^{k+1} = 2 \cdot 2^k > 2 \cdot k \quad \text{(by the inductive assumption)} \\
= k + k \geq k + 1 = \text{RHS} \quad \text{(since } k \geq 1) 
\]

This finishes the inductive step, so by induction we know that \(2^n > n\) for all \(n \in \mathbb{N}\).

Induction can often be used to prove facts about finite sets. In this case, the general technique is to induct on the size \(n\) of the sets. Typically a proposition will be trivial to prove for the empty set, or for sets with a single element. We may assume the proposition holds for sets of size \(k\), and let \(A\) be a set of size \(k + 1\). Removing one element from \(A\) yields a set of size \(k\), to which the inductive hypothesis applies. Then, we only need to extend the proposition to \(A\); how we do it depends on exactly what we wish to prove. The following theorem is a typical example.

**Proposition 13.11.** Let \(A\) be a finite nonempty set of real numbers. Then \(A\) has a smallest element.

**Proof.** Let \(P(n)\) be the open sentence
\[
P(n): \text{Every set of } n \text{ real numbers has a smallest element.}
\]

We work by induction to show that \(P(n)\) is true for all \(n \in \mathbb{N}\).

**Base case:** It is clear that any set containing only 1 real number contains a smallest element, so \(P(1)\) is true.

**Inductive step:** Let \(k \in \mathbb{N}\) and assume \(P(k)\). In other words, assume that every set of \(k\) real numbers has a smallest element. This is our inductive hypothesis; we want to use it to prove \(P(k + 1)\).

Let \(A\) be any set containing \(k + 1\) real numbers. Choose one of them, and call it \(a\). Let \(B = A - \{a\}\). We note that \(B\) has \(k\) elements, so by the inductive hypothesis, \(B\) has a smallest element. Call this element \(b\). By the definition of a least element, we have \(b \leq x\) for all \(x \in B\). Note that since \(b \in B = A - \{a\}\), we must have \(b \neq a\). Therefore, either \(b < a\) or \(b > a\).

**Case 1.** If \(b < a\), then \(b \leq x\) for all \(x \in B \cup \{a\} = A\). Hence, \(b\) is the smallest element of \(A\).

**Case 2.** If \(a < b\), then \(a < b \leq x\) for all \(x \in B\), and \(a \leq a\). Hence, \(a \leq x\) for all \(x \in B \cup \{a\} = A\), and so \(a\) is the smallest element of \(A\).

In either case, \(A\) has a smallest element. Since \(A\) was an arbitrary set with \(k + 1\) elements, \(P(k + 1)\) is true. This completes the inductive step.

Hence, by induction, \(P(n)\) is true for all \(n \in \mathbb{N}\). Therefore, any finite set of real numbers has a least element.

We now give an application of induction by proving a very important counting principle in mathematics; the Pigeonhole Principle. This principle may seem like common sense, hence all the more reason to prove it.
Theorem 13.12 (The Pigeonhole Principle). Let \( m \) and \( n \) be natural numbers, with \( m > n \). If \( m \) objects are placed in \( n \) bins, then two (or more) objects must share a bin.

Proof. We prove this by induction on \( n \in \mathbb{N} \).

Let \( P(n) \) be the open sentence

\[
P(n): \text{For each } m \in \mathbb{N} \text{ if } m > n \text{ and } m \text{ objects are placed in } n \text{ bins, then two (or more) objects must share a bin.}
\]

Base case: We verify that \( P(1) \) is true. If we have more than one object, and we wish to place them all in 1 bin, then all the objects must clearly share a bin.

Inductive step: Let \( k \in \mathbb{N} \) and assume \( P(k) \).

We now prove \( P(k+1) \). Let \( m \in \mathbb{N} \). Assume \( m > k + 1 \) and \( m \) objects are placed into \( k + 1 \) bins. We need to show that two objects share a bin. Choose one of the objects and call it \( x \). We divide the proof into two cases.

Case 1. Suppose that the object \( x \) shares a bin with at least one other object. In this case, two objects clearly share a bin, so we are finished.

Case 2. The object \( x \) is in a bin by itself; no other object shares the bin with \( x \). In this case, there are \( m - 1 > k \) remaining objects, none of which are in the same bin as object \( x \). Hence, these \( m - 1 \) objects must all be placed into the \( k \) remaining bins. By the induction hypothesis, (i.e. our assumption that \( P(k) \) is true) two of these objects must share a bin.

In both cases two objects must share a bin, which completes the inductive step.

Hence, by the principle of mathematical induction, \( P(n) \) is true for all \( n \in \mathbb{N} \).

Remark 13.13. The pigeonhole principle can be applied to many situations. For instance, if we choose three integers, then two of them must have the same parity. Here there are two bins; even and odd. If we choose three numbers, two of them (possibly all three) must end up in the same bin, or in other words have the same parity.

As another example, in a class of 30 people, if each person scores between 80 and 100 percent on an exam (with no fractional scores allowed), two people must have received the same score, since there are 21 possible scores (bins) which must contain the 30 people.

A variation on the pigeonhole principle occurs if we are assigning objects to bins, and we have fewer objects than bins. In this case, common sense tells us that some bin will remain empty. You will be asked to prove this “common sense” statement in Exercise 13.8.
**Warning 13.14.** It is important to note that induction cannot be used to prove “infinite” statements. It does prove infinitely many statements: for instance, we can prove that
\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]
for all \(n \in \mathbb{N}\). However, since \(\infty \notin \mathbb{N}\), induction can not be used to prove anything about
\[
\sum_{i=1}^{\infty} i.
\]

In terms of the ladder analogy, induction proves that we can reach every rung of the ladder, but it cannot be used to prove that we can reach the top of the ladder (since the ladder actually has no top).

### 13.B Exercises

**Exercise 13.1.** Prove that for every \(n \in \mathbb{N}\),
\[
\sum_{i=1}^{n} (2i - 1) = n^2.
\]

**Exercise 13.2.** Prove that for every \(n \in \mathbb{N}\),
\[
\sum_{i=1}^{n} \frac{1}{(2i - 1)(2i + 1)} = \frac{n}{2n + 1}.
\]

**Exercise 13.3.** Prove that for every \(n \in \mathbb{N}\),
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Exercise 13.4.** (a) Prove that for every \(n \in \mathbb{N}\),
\[
n < 3^n.
\]

(b) Prove that for every \(n \in \mathbb{Z}\), \(n < 3^n\). (Hint: For any \(n \in \mathbb{Z}\), we have \(3^n > 0\).)

**Exercise 13.5.** Let \(x \in \mathbb{R}\), with \(x \neq 1\). Prove that for every \(n \in \mathbb{N}\),
\[
\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.
\]

**Exercise 13.6.** Let \(x \in \mathbb{R}\) with \(x > -1\). Prove that for every \(n \in \mathbb{N}\),
\[
(1 + x)^n \geq 1 + nx.
\]
Exercise 13.7. Let $S$ be any nonempty set of natural numbers. Prove that $S$ has a smallest element. (Hint: Use Proposition 13.11 and the fact that for any $n \in \mathbb{N}$, any subset of $\{1, \ldots, n\}$ is finite. You will not need to use induction in your proof, since the induction is done in the proof of Proposition 13.11.)

The fact that any nonempty subset of the natural numbers has a smallest element is called the **Well-Ordering Principle**.

Exercise 13.8. Prove the following variation of the pigeonhole principle.

Let $m, n \in \mathbb{N}$, and suppose $m$ objects are placed in $n$ bins, and $m < n$. Conclude that some bin does not contain any object.

(Hint: Use induction on $m$.)
14 More examples of induction

In this section we will discuss two tricks related to induction.

14.A Starting induction somewhere else

Often, we wish to prove a statement of the form

$$P(n)$$ is true for all integers $$n \geq a$$

where $$a$$ is a fixed integer. Note that if $$a = 1$$, this is just a proof of a statement for all natural numbers.

Induction can be used to prove such statements. The only change is that our base case starts at $$a$$ instead of 1. We will give a proof of this fact at the end of this section, but for now we demonstrate how this changes proofs, by giving some examples.

**Proposition 14.1.** For all integers $$n \geq 10$$ we have $$2^n > n^3$$.

Before we start the proof, we make a few remarks. First, why are we restricting to integers $$n \geq 10$$? The reason is because the claim is false for some smaller integers. The inequality is false when $$n = 9$$. (Try it!) Second, what is the open sentence $$P(n)$$? It is just $$P(n) : 2^n > n^3$$. When we plug in $$k + 1$$ for $$n$$, we have

$$P(k + 1) : 2^{k+1} > (k + 1)^3.$$ 

The right hand side can be simplified a bit. We note that

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1.$$ 

In the computation in the proof below, we will slowly try to “peel off” each of the terms $$k^3$$, $$3k^2$$, $$3k$$, and 1, one at a time, so that eventually we can end up with $$(k+1)^3$$.

We are now ready for the formal proof.

*Proof.* We wish to prove that the open sentence

$$P(n) : 2^n > n^3$$

is true for all $$n \geq 10$$ with $$n \in \mathbb{N}$$. We work by induction.

**Base case:** We verify that $$P(10)$$ is true, as follows:

$$2^{10} = 1024 > 1000 = 10^3.$$ 

**Inductive step:** Let $$k \in \mathbb{N}$$ with $$k \geq 10$$. Assume that $$P(k)$$ is true. So we now know that $$2^k > k^3$$. We wish to prove $$P(k + 1)$$, which states that

$$2^{k+1} > (k + 1)^3.$$
In order to do this, we examine $2^{k+1}$ closely.

\[
\text{LHS} = 2^{k+1} = 2 \cdot 2^k \\
= 2^k + 2^k \\
> k^3 + k^3 \quad \text{(2^k > k^3, by inductive hypothesis)} \\
\geq k^3 + 10k^2 \quad \text{(since k ≥ 10)} \\
= k^3 + 3k^2 + 7k^2 \quad \text{(peeling off 3k^2)} \\
\geq k^3 + 3k^2 + 70k \quad \text{(since k ≥ 10)} \\
= k^3 + 3k^2 + 3k + 67k \quad \text{(peeling off 3k)} \\
\geq k^3 + 3k^2 + 3k + 1 \quad \text{(since 67k > 1)} \\
= (k + 1)^3 = \text{RHS}.
\]

Hence, \(P(k + 1)\) is true. Therefore by mathematical induction, \(P(n)\) is true for all \(n ≥ 10\).

\[\square\]

Remark 14.2. When trying to prove the inductive step, it can sometimes be difficult to verify \(P(k + 1)\) follows from \(P(k)\). Notice that in the previous example we wanted to prove that \(2^{k+1} > (k + 1)^3\). In order to do this, we wrote down one side of the inequality (the left-hand side), and manipulated it in order to reach the other side.

Notice that in the previous example we were aided by the knowledge that

\[
\text{RHS} = (k + 1)^3 = k^3 + 3k^2 + 3k + 1;
\]

this gave us a target to shoot for. Moreover, the right-hand side cannot be simplified much further, which is why we started with the left-hand side in the proof above. It is often useful to manipulate both sides of an equation or inequality in order to work out (on scratch paper) how to get from one side to the other. However, in the proof we must be careful that our inequalities all go the same direction.

We now introduce an important mathematical object called the factorial. Its definition is reminiscent of induction, in that for an integer \(n\), the factorial of \(n\) is defined in terms of the factorial of \((n - 1)\). It is no surprise that many theorems about factorials are proved by induction.

**Definition 14.3.** Given \(n \in \mathbb{Z}_{≥0}\), we define the factorial of \(n\), written as \(n!\) and read as “\(n\) factorial,” to be

\[
n! = \begin{cases} 
1 & \text{if } n = 0, \\
n \cdot (n - 1)! & \text{if } n > 0.
\end{cases}
\]
Example 14.4. If we wish to compute $5!$, we use the formula repeatedly as follows:

$$
5! = 5 \cdot 4!
= 5 \cdot 4 \cdot 3!
= 5 \cdot 4 \cdot 3 \cdot 2!
= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1!
= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 0!
= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 1
= 120.
$$

In general, for $n > 0$, we see that

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1. \quad \triangle$$

Proposition 14.5. Prove that for all $n \geq 4$, we have $n! > 2^n$.

Proof. We wish to prove that the open sentence

$$P(n) : n! > 2^n$$

is true for all $n \geq 4$. We work by induction.

**Base case:** Note that $4! = 24 > 16 = 2^4$. Hence, $P(4)$ is true.

**Inductive step:** Let $k \geq 4$ be an integer, and assume $P(k)$ is true. Then we know that $k! > 2^k$. Now we wish to show $(k + 1)! > 2^{k+1}$. We find

$$
\begin{align*}
\text{LHS} &= (k + 1)! = (k + 1)k! \\
&> (k + 1)2^k \\
&> 2 \cdot 2^k \\
&= 2^{k+1} = \text{RHS}.
\end{align*}
$$

Hence, $P(k + 1)$ is true. Therefore, by induction $P(n)$ is true for all $n \geq 4$. \hfill \Box

If we wish to prove facts about finite sets, it will often be convenient to start our induction with the base case being sets of size 0 (the empty set).

Proposition 14.6. If $A$ is a finite set, then $|\mathcal{P}(A)| = 2^{|A|}$.

Proof. Let $P(n)$ be the open sentence

$$P(n): \text{If } A \text{ is a set with } n \text{ elements, then } |\mathcal{P}(A)| = 2^n.$$

We will prove that $P(n)$ is true for all integers $n \geq 0$, by induction.
14. MORE EXAMPLES OF INDUCTION

**Base case:** $P(0)$ is true; the only set with 0 elements is the empty set, and its only subset is itself, so $|\mathcal{P}(\emptyset)| = 1 = 2^0$.

**Inductive step:** Assume that $P(k)$ is true for some $k \geq 0$; namely, that for any set $A$ with $k$ elements, $|\mathcal{P}(A)| = 2^k$.

We want to prove $P(k+1)$. Let $B$ be a set with $k+1$ elements. Choose an element of $B$; call it $b$. We divide the power set of $B$ into two collections of subsets. Let

$$S = \{ X \in \mathcal{P}(B) : b \in X \},$$

and let

$$T = \{ X \in \mathcal{P}(B) : b \notin X \}.$$  

We note that $T$ consists of the subsets of $B - \{b\}$; hence, $T$ is just the power set of $B - \{b\}$. Since $B - \{b\}$ has $k$ elements (one element less than $B$), our inductive hypothesis tells us that $|T| = 2^k$.

On the other hand, each element of $S$ is the union of an element of $T$ with the set $\{b\}$. Hence, $|S| = |T| = 2^k$. Since $S$ and $T$ have no elements in common, the number of elements in $\mathcal{P}(B) = S \cup T$ is $|S| + |T| = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$. Hence, $P(k+1)$ is true.

Therefore, by induction we see that $P(n)$ is true for all $n \geq 0$. □

We can often use induction to extend statements concerning two objects to statements concerning any finite number of objects. For instance, the following proposition is an extension of DeMorgan’s law, from two terms to an arbitrary (finite number) of terms.

**Proposition 14.7.** For any $n \in \mathbb{N}$, if $P_1, \ldots, P_n$ are any statements, then

$$\neg(P_1 \lor \cdots \lor P_n) \equiv (\neg P_1) \land \cdots \land (\neg P_n).$$

**Proof.** Let $Q(n)$ be the open sentence

For any $n$ statements $P_1, \ldots, P_n$, we have $\neg(P_1 \lor \cdots \lor P_n) \equiv (\neg P_1) \land \cdots \land (\neg P_n)$.

We will now work by induction for $n \geq 1$.

**Base case:** $Q(1)$ is just the statement $\neg P_1 \equiv \neg P_1$, which is true.

**Inductive step:** Let $k \in \mathbb{N}$ with $k \geq 1$, and assume that $Q(k)$ is true; i.e.

$$\neg(P_1 \lor \cdots \lor P_k) \equiv (\neg P_1) \land \cdots \land (\neg P_k).$$

Then we have

$$\neg(P_1 \lor \cdots \lor P_{k+1}) \equiv \neg((P_1 \lor \cdots \lor P_k) \lor P_{k+1}) \quad \text{(Associativity of \lor)}$$

$$\equiv \neg(P_1 \lor \cdots \lor P_k) \land (\neg P_{k+1}) \quad \text{(DeMorgan’s law)}$$

$$\equiv (\neg P_1) \land \cdots \land (\neg P_k) \land (\neg P_{k+1}) \quad \text{(Inductive hypothesis)}$$

Hence, $Q(k+1)$ is true.

Therefore, $Q(n)$ is true for all $n \in \mathbb{N}$. □
14.B Many base cases

Sometimes, when proving a statement by induction, the inductive step will not work when $k$ is small. In that situation, we might prove several cases of the statement by hand, and then pass to the inductive step. We demonstrate with an example.

**Theorem 14.8.** For all $n \in \mathbb{N}$, we have $2^{n+1} > n^2$.

**Proof.** We take $P(n)$ to be

$$P(n) : 2^{n+1} > n^2.$$  

When $n = 1$, we have $2^{1+1} = 4 > 1 = 1^2$. When $n = 2$, we have $2^{2+1} = 8 > 4 = 2^2$. When $n = 3$, we have $2^{3+1} = 16 > 9 = 3^2$. We will now use induction to prove that $P(n)$ is true for all $n \geq 3$.

**Base case:** $P(3)$ has already been shown to be true.

**Inductive step:** Now assume $P(k)$, for some $k \geq 3$. Hence, we know that $2^{k+1} > k^2$. Then

$$2^{(k+1)+1} = 2 \cdot 2^{k+1}$$

$$> 2k^2$$

$$= k^2 + k^2$$

$$\geq k^2 + 3k \quad \text{(since } k \geq 3)$$

$$= k^2 + 2k + k \quad \text{(peeling off } 2k)$$

$$\geq k^2 + 2k + 1 \quad \text{(since } k > 1)$$

$$= (k + 1)^2$$

Hence, $P(k + 1)$ is true. Therefore, by induction $P(n)$ is true for all $n \geq 3$. Since we have already demonstrated $P(1)$ and $P(2)$, we see that $P(n)$ is true for all $n \in \mathbb{N}$.

**Advice 14.9.** To decide whether or not to do extra cases, try the inductive step first (perhaps on scratch paper). If you need extra information (as we did above, to peel off $3k$) this could be a reason to do extra base cases.

Another reason to use extra cases is if you are working with a piecewise defined function. Doing small cases might help handle places where the piecewise function is different.

14.C Proof of generalized induction

Here is the promised proof that induction can start at any integer.
Theorem 14.10. Let \( a \in \mathbb{Z} \), and let \( P(n) \) be an open sentence whose domain includes the set \( S = \{ n \in \mathbb{Z} : n \geq a \} \). If

(i) \( P(a) \) is true,

(ii) \( P(k) \Rightarrow P(k + 1) \) for all \( k \in S \),

then \( P(n) \) is true for all \( n \in S \).

Proof. For \( n \in \mathbb{N} \), define \( P'(n) = P(n + a - 1) \). Then we have a correspondence between \( P \) and \( P' \):

\[
\begin{array}{cccccccc}
P(a) & P(a + 1) & P(a + 2) & P(a + 3) & P(a + 4) & P(a + 5) & P(a + 6) & \cdots \\
P'(1) & P'(2) & P'(3) & P'(4) & P'(5) & P'(6) & P'(7) & \cdots \\
\end{array}
\]

This correspondence makes it clear that if we can prove \( P'(n) \) for each \( n \in \mathbb{N} \), then we will have proved \( P(n) \) for each \( n \in S \).

Now, \( P'(1) = P(a) \) is true by (i).

Further, for each \( k \in \mathbb{N} \), \( P'(k) \Rightarrow P'(k + 1) \), since \( P(k + a - 1) \Rightarrow P(k + 1 + a - 1) \), by (ii).

Hence, by the principle of mathematical induction, \( P'(n) \) is true for every \( n \in \mathbb{N} \), so \( P(n) \) is true for every \( n \in S \).  

14.D Exercises

Exercise 14.1. Prove that \( n! > 3^n \) for all natural numbers \( n > 6 \).

Exercise 14.2. Prove that if \( n \) is any natural number greater than 5, then \( n! > n^3 \).

Exercise 14.3. Prove that for every \( n \in \mathbb{N} \), we have \( 3^n \geq n^3 \).

(Hint: Demonstrate this by direct calculation for \( n = 1, 2, 3 \). Then use induction to complete the proof for \( n \geq 3 \), with \( n = 3 \) as your base case.

Exercise 14.4. Prove that for any \( n \in \mathbb{N} \) with \( n \geq 2 \), if \( P_1, \ldots, P_n \) are statements, then

\[ \neg(P_1 \land \cdots \land P_n) \equiv (\neg P_1) \lor \ldots \lor (\neg P_n). \]

Exercise 14.5. Use induction to prove Proposition 2.3: for finite sets \( A \) and \( B \),

\[ |A \times B| = |A| \cdot |B|. \]

(Hint: Let \( A \) be a fixed set with \( |A| = m \) and \( m \in \mathbb{Z} \). Let \( P(n) \) be the open sentence:

\[ P(n): \text{For any set } B_n, \text{ if } |B_n| = n, \text{ then } |A \times B_n| = mn. \]

Prove that \( P(n) \) is true for all \( n \geq 0 \) by induction.)
Exercise 14.6. Prove that for any \( n \in \mathbb{N} \), if \( x_1, \ldots, x_n \in \mathbb{R} \), then
\[
\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|.
\]
(Note that for \( n = 2 \), this is just Theorem 8.21, the Triangle Inequality.)

Exercise 14.7. The Fibonacci Numbers are the sequence of natural numbers \( F_1, F_2, F_3, \ldots \) defined by the formula
\[
F_1 = F_2 = 1,
\]
and for \( n > 2 \),
\[
F_n = F_{n-1} + F_{n-2}.
\]
For instance, \( F_3 = F_2 + F_1 = 2 \), and \( F_4 = F_3 + F_2 = 2 + 1 = 3 \).
(a) Write down the first fifteen Fibonacci numbers.
(b) Prove by induction that for all \( n \geq 1 \),
\[
\sum_{i=1}^{n} F_i = F_{n+2} - 1.
\]
(c) Prove by induction that for all \( n \geq 1 \),
\[
\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}.
\]

Exercise 14.8. Using the definition of the Fibonacci numbers \( F_n \) from the previous problem, prove by induction that for any integer \( n > 12 \) that \( F_n > n^2 \).
(Hint: One possible idea for a proof is to let \( P(n) \) be the open sentence
\[
P(n) : F_n > n^2 \text{ and } F_{n-1} > (n - 1)^2.
\]
Use induction to prove that \( P(n) \) is true for all \( n \geq 14 \). Then this implies that \( F_n > n^2 \) for all \( n \geq 13 \). )
15 Strong Induction

Sometimes, when trying to do a proof by induction, the inductive step is not feasible because \( P(k) \) does not provide enough information to conclude \( P(k+1) \). In this case, a variation on induction called “strong induction” is often useful.

15.A Defining strong induction

The idea of strong induction is very intuitive. Recall the ladder analogy. If we can climb to the \( k \)th rung, we just need to know that we can climb to the \( k+1 \)st rung. However, we have more information available. If we have climbed up to the \( k \)th rung, then we have also climbed all the steps below! It is possible to make use of this extra information. We formalize this reasoning in the following theorem.

**Theorem 15.1.** Let \( P(n) \) be an open sentence, where the domain of \( n \) is \( \mathbb{N} \). For \( n \in \mathbb{N} \), let \( Q(n) \) be the open sentence

\[
Q(n): P(1), P(2), \ldots, P(n) \text{ are all true.}
\]

If

(i) \( P(1) \) is true,

(ii) For each \( k \in \mathbb{N} \), \( Q(k) \Rightarrow P(k+1) \),

then \( P(n) \) is true for all \( n \in \mathbb{N} \).

**Proof.** Note that \( Q(k+1) \equiv Q(k) \land P(k+1) \). Additionally, \( \neg Q(k) \Rightarrow \neg P(k+1) \Rightarrow \neg Q(k+1) \). Hence, given any \( k \in \mathbb{N} \), if we know that \( Q(k) \Rightarrow P(k+1) \) then we know that \( Q(k) \Rightarrow Q(k+1) \). Applying the principle of mathematical induction to \( Q(n) \), we see that \( Q(n) \) is true for all natural numbers. This immediately implies that \( P(n) \) is true for all natural numbers.

**Remark 15.2.** Strong induction is almost identical to standard induction; indeed it is not necessary to distinguish between the two when proving a theorem. Note that the base case in both induction and strong induction is identical, since \( Q(1) \) is the same as \( P(1) \). The only difference is that in the inductive step we have a stronger premise.

One can always replace a proof by induction with a proof by strong induction. For this reason, when writing a proof, only use the words “strong induction” if you want to emphasize the proof technique; otherwise just say “by induction.”

Our first example of strong induction will be given in the proof of the following:

**Proposition 15.3.** Every natural number can be written as a sum of distinct integers, each of which is a power of 2.
Before we begin the proof, we want to make a few remarks which will help explain what we are trying to prove.

**Remark 15.4** (Sums of one object). When mathematicians say that a number can be written as a sum, they allow the possibility of adding only one object. Hence, the number \(8 = 2^3\) can be considered as a sum of a single power of two.

**Remark 15.5** (Meaning of distinct). In English, the word “distinct” is often used to mean special, or distinguished. In mathematics, the word has a very precise meaning, which is quite different; a list of objects is called *distinct* if no two of the objects are equal.

For instance, there are several ways to write the number 6 as a sum of powers of two. We have

\[
6 = 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 + 1 \\
= 2 + 2 + 1 + 1 = 2 + 2 + 2 = 4 + 1 + 1 = 4 + 2. 
\]

Note that only the last way (6 = 4 + 2) gives 6 as a sum of *distinct* powers of 2 (all the rest have repetition).

**Remark 15.6** (Base 2). We illustrate the theorem for the first few natural numbers.

\[
1 = 1, \quad 2 = 2, \quad 3 = 2 + 1, \quad 4 = 4, \quad 5 = 4 + 1, \quad 6 = 4 + 2, \quad 7 = 4 + 2 + 1. 
\]

In each case, we have written \(n\) as a sum of distinct powers of two. Mathematicians call this representing a number in *binary*, or *base* 2. Most current cultures write numbers in base 10, but other bases can be very important (such as base 16, or hexadecimal, for computers).

We are now ready to begin the proof. Try to figure out why the proof will fail if, in the inductive step, we only assume \(P(k)\).

**Proof.** Let \(P(n)\) be the open sentence

\[
P(n): \text{ } n \text{ can be written as a sum of distinct integers, each a power of 2.}
\]

Let \(Q(n)\) be the open sentence

\[
Q(n): \ P(1) \land \cdots \land P(n).
\]

We work by (strong) induction to show \(P(n)\) is true for each \(n \in \mathbb{N}\).

**Base case:** \(P(1)\) is true, since \(1 = 2^0\).

**Inductive step:** Let \(k \in \mathbb{N}\), and assume \(Q(k)\). In other words, assume that every integer from 1 to \(k\) can be written as a sum of distinct powers of 2. We wish to use this assumption to prove \(P(k + 1)\); that \(k + 1\) can be written as a sum of distinct powers of two.

We will examine two cases.
Case 1. If $k + 1$ is odd, then $k$ is even. By our induction hypothesis, $k$ can be written as a sum of distinct powers of 2. Since only one power of two is odd (namely $2^0 = 1$) and $k$ is even, all of these powers of two must be even. Adding $2^0 = 1$ to the collection, we still have a collection of distinct powers of two, and they now add up to $k + 1$.

Case 2. If $k + 1$ is even, then $k + 1 = 2a$ for some $a \in \mathbb{N}$. Now $1 \leq a \leq k$, so $a$ can be written as a sum of distinct powers of two, by our induction hypothesis. Increasing each of these exponents by one gives us a collection of powers of two, all distinct, that add to $2a = k + 1$.

We have thus proven that $Q(k) \Rightarrow P(k + 1)$. Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

In order to further illustrate how the proof works, we give an example of the proof in action. In the proof, if we have an odd number, we subtract 1. If we have an even number, we divide by 2. We stop when we hit the base case. Thus we have:

$$11 \rightarrow 10 \rightarrow 5 \rightarrow 4 \rightarrow 2 \rightarrow 1.$$ 

Now, writing $1 = 2^0$ we work backwards through this chain, either adding 1 or multiplying by 2 (to undo what we originally did in the chain above). Eventually we end up with a binary expansion for 11, as follows:

$$2^0 \rightarrow 2^1 \rightarrow 2^2 \rightarrow 2^2 + 2^0 \rightarrow 2^3 + 2^1 \rightarrow 2^3 + 2^1 + 2^0.$$ 

Thus, we have $\boxed{11 = 2^3 + 2^1 + 2^0}$.

15.B Where to start?

Just like for induction, strong induction can start at any integer. The formal statement merges Theorems 14.10 and 15.1, as below. (The proof is also similar, and left as an exercise.)

**Theorem 15.7.** Let $a \in \mathbb{Z}$, and let $P(n)$ be an open sentence whose domain includes the set $S = \{n \in \mathbb{Z} : n \geq a\}$. For $n \in S$, let $Q(n)$ be the open sentence

$$Q(n) : P(a) \land \cdots \land P(n).$$

If

(i) $P(a)$ is true,

(ii) $Q(k) \Rightarrow P(k + 1)$ for all $k \in S$,

then $P(n)$ is true for all $n \in S$.

We demonstrate how such proofs work with two examples.

**Proposition 15.8.** Let $n$ be any integer greater than 5. Any square can be subdivided into $n$ squares.
Proof. Let $P(n)$ be the open sentence

$$P(n): \text{A square can be subdivided into } n \text{ squares.}$$

We work by induction on $n \geq 6$.

**Base case:** We can verify that $P(6)$ is true. We also verify that $P(7)$, and $P(8)$ are true in the diagrams below. (We do these extra base cases to help with the inductive step.)

In addition, we have also given a picture showing how to subdivide a square into 4 smaller squares; we will use this in our proof.

![Diagrams showing subdivision of squares](image)

**Inductive step:** Let $k \geq 6$, and assume that $P(\ell)$ is true for $6 \leq \ell \leq k$. We wish to prove that $P(k + 1)$ is true. If $k = 6$ or $k = 7$, we have already seen that $P(k + 1)$ is true, so we may assume that $k \geq 8$.

Since $k \geq 8$, we have that $k - 2 \geq 6$. Hence, by our inductive assumption, since $k - 2 \leq k$, we know that $P(k - 2)$ is true. In other words, we know that we can subdivide a square into $k - 2$ squares. Starting with this subdivision, we further subdivide the upper-rightmost square into 4 squares. This adds three squares to the subdivision. Thus, we have subdivided the square into $k - 2 + 3 = k + 1$ squares. Hence, $P(k + 1)$ is true.

Therefore, by mathematical induction, $P(n)$ is true for all $n \geq 6$. $\square$

To see this proof in action, we demonstrate how to subdivide a square into 20 squares. Start from the subdivision into 8 squares, and repeatedly divide the upper-rightmost square into 4 smaller squares at each stage.

![Diagrams showing subdivision of squares](image)

We see easily how we could extend these diagrams to demonstrate the result for $n = 23, 26, 29, 32, \ldots$ (although the upper-right square would quickly become too small to see).

For our last example, we investigate a different type of statement that can be proved by strong induction. The following proposition is a special case of the:
Postage Stamp Problem: Given several denominations of postage stamps, what possible amounts of postage can be paid precisely?

Note that only a nonnegative number of each stamp can be used. (You can’t use a negative number of stamps!) We will demonstrate this idea with a specific example. Suppose that your local post-office has stamps with two denominations: 5 cents and 7 cents. What other denominations can you get using these two types of stamps?

You can pay 21 cent postage using three 7 cent stamps. You can also pay 22 cent postage, with three 5 cent stamps and one 7 cent stamp. However, 23 cent postage is not possible using those two denominations.

In the following proposition, we prove that every postage above 23 cents is possible. The main idea will be that if we can get five consecutive denominations, then by adding enough 5 cent stamps, we can reach all other higher denominations.

**Proposition 15.9.** Prove that every integer \( n > 23 \) can be written as
\[
  n = 5x + 7y
\]
for some integers \( x, y \geq 0 \).

**Proof.** We will let

\[ P(n) : \text{For some nonnegative integers } x \text{ and } y, \text{ we have } n = 5x + 7y. \]

We have the following equations:
\[
24 = 5 \cdot 2 + 7 \cdot 2 \quad 25 = 5 \cdot 5 + 7 \cdot 0 \\
26 = 5 \cdot 1 + 7 \cdot 3 \quad 27 = 5 \cdot 4 + 7 \cdot 1 \\
28 = 5 \cdot 0 + 7 \cdot 4.
\]

Hence, we see that \( P(24), P(25), P(26), P(27), \) and \( P(28) \) are true. (The reason we did so many cases will become apparent in the inductive step below.) We proceed by (strong) induction on \( n \geq 24 \).

**Base case:** We have seen that \( P(24), \ldots, P(28) \) are all true.

**Inductive step:** Assume that for some \( k \geq 28 \), all of \( P(24), \ldots, P(k) \) are true.

Since \( k \geq 28 \), we have \( k - 4 \geq 24 \). Hence, \( P(k - 4) \) is true; in other words, we can write \( k - 4 = 5x + 7y \) with \( x, y \geq 0 \). Then (adding a 5 cent stamp) yields
\[
k + 1 = k - 4 + 5 = 5x + 7y + 5 = 5(x + 1) + 7y.
\]

Since \( x + 1, y \geq 0 \), we see that \( P(k + 1) \) is true.

Hence, by induction, \( P(n) \) is true for all \( n \geq 24 \). \( \Box \)


15.C Exercises

Exercise 15.1. Prove by induction that for any integer $n > 5$, it is possible to subdivide an equilateral triangle into $n$ equilateral triangles. (For example, a subdivision into 6 equilateral triangles is given below.)

![Dividing an equilateral triangle into 6 equilateral triangles]

Exercise 15.2. Prove that for every natural number $n > 43$, we can write

$$n = 6x + 9y + 20z$$

for some nonnegative integers $x, y, z$. Then prove that 43 cannot be written in this form.

Exercise 15.3. Find the largest amount of postage that cannot be paid exactly with 4, 10, and 15 cent stamps. Prove that your answer is correct. (This proof will include showing not only that the amount that you find cannot be achieved, but also that every larger amount can be achieved.)

Exercise 15.4. Recall the definition of the Fibonacci numbers from Exercise 14.7. Prove that every positive integer is a sum of two or more distinct Fibonacci numbers. (Hint: For every $n > 1$, we can find a $k$ so that $F_k \leq n < F_{k+1}$. Examine $n - F_k$.)

Exercise 15.5. Prove Theorem 15.7.

Exercise 15.6 (Extra credit). Let $n \in \mathbb{N}$.

(a) Prove that $n$ can be written in the form $a_m10^m + a_{m-1}10^{m-1} + \cdots + a_110^1 + a_010^0$, where each $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $m \geq 0$ is an integer. (In other words, every natural number has a base 10 expansion.)

(b) Prove that if we additionally posit that $a_m \neq 0$ (i.e. the leading digit is nonzero), then base 10 expansion is unique.
16 The Binomial Theorem

16.A Binomial coefficients and Pascal’s triangle

Binomial coefficients show up throughout mathematics. As we will see, they are the coefficients of \(x^k\) in the expansion of \((x + 1)^n\). They also allow us to count certain collections of objects in a finite set. These binomial coefficients have many amazing properties, that we will prove by use of mathematical induction.

**Definition 16.1.** Let \(n, k \in \mathbb{Z}\). We define the binomial coefficient as

\[
\binom{n}{k} = \begin{cases} 
\frac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n, \\
0 & \text{otherwise.}
\end{cases}
\]

We read the symbol \(\binom{n}{k}\) as “\(n\ choose \ k\).”

**Example 16.2.** We may compute

\[
\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 7 \cdot 5 = 35.
\]

In Exercise 16.1, you will prove that for any \(n \geq 0\),

\[
\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{1} = \binom{n}{n-1} = n.
\]

**Remark 16.3.** It is not obvious from the definition that the binomial coefficient is an integer, but it will follow from the properties that we describe below.

Reading the symbol as “\(n\ choose \ k\)” comes from the fact that the binomial coefficient \(\binom{n}{k}\) counts the number of ways to choose \(k\) objects from among \(n\) objects. See Theorem 16.6 for a proof of this fact.

We now state a fundamental property of the binomial coefficients.

**Theorem 16.4.** Let \(n, k \in \mathbb{Z}\), with \(n \geq 0\). Then

\[
\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.
\]
Before proving Theorem 16.4, we demonstrate how to use it to build Pascal’s triangle, which is a useful computational mnemonic for the binomial coefficients. See Figure 16.5. Here, row $n$ corresponds to the values of $\binom{n}{k}$, with $\binom{n}{0}$ being the leftmost dark entry in each row. The arrows indicate how $\binom{5}{1}$ and $\binom{5}{2}$ add to give $\binom{6}{2}$. Note that each entry is the sum of the two entries one row higher and to the right and left.

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 3 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 & 6 & 4 & 1 & 0 & 0 \\
0 & 0 & 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 \\
0 & 0 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 \\
0 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & 0 \\
0 & 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 \\
\end{array}
$$

Figure 16.5: Pascal’s Triangle

We now prove Theorem 16.4.

Proof. We will use properties of the factorial function here. In particular, we note that $(k + 1)! = (k + 1) \cdot k!$, and $(n - k)! = (n - k) \cdot (n - k - 1)!$ (when $0 \leq k < n$). We will use these facts to obtain a common denominator in the fractions defining the binomial coefficients $\binom{n}{k}$ and $\binom{n}{k+1}$.

We break the proof into cases, doing the easiest cases first.

Case 1. Assume that $k = n$. Then both $\binom{n}{k}$ and $\binom{n+1}{k+1}$ are 1, and $\binom{n}{k+1} = 0$.

Case 2. Assume that $k > n$. Then all three binomial coefficients in the formula are 0.

Case 3. Assume that $k = -1$. Then $\binom{n}{k} = 0$ but $\binom{n+1}{k+1} = \binom{n+1}{k+1} = 1$.

Case 4. Assume that $k < -1$. Then all three binomial coefficients in the formula are 0. □

Case 5. Assume that $0 \leq k < n$. Then $0 < k + 1 \leq n$, and all three binomial coefficients in the formula are defined by the formula involving factorials.
Hence,
\[
\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} = \frac{n!(n+1)}{(k+1)!(n-k)!} = \binom{n+1}{k+1}.
\]

We can now use mathematical induction to give an interpretation of the binomial coefficients in terms of counting subsets.

**Theorem 16.6.** Let \( n \geq 0 \). Then the binomial coefficient \( \binom{n}{k} \) counts the number of subsets of order \( k \) in a set of order \( n \).

**Proof.** Let \( P(n) \) be the open sentence

\[
P(n): \text{For all } k \in \mathbb{Z}, \text{ the binomial coefficient } \binom{n}{k} \text{ counts the number of subsets of order } k \text{ in any set of cardinality } n.
\]

We work by induction on \( n \geq 0 \).

**Base case:** We verify that \( P(0) \) is true. For \( k \neq 0 \), there are no subsets of order \( k \) in the empty set, matching the value \( \binom{0}{k} = 0 \). For \( k = 0 \) there is one subset of order \( k \) in the empty set, matching the value \( \binom{0}{0} = 1 \). Hence, \( P(0) \) is true.

**Inductive step:** Let \( m \geq 0 \) be an integer and assume that \( P(m) \) is true. (We use \( m \) here, because \( k \) already has a meaning.) Thus, for any \( k \in \mathbb{Z} \), any set of order \( m \) has \( \binom{m}{k} \) subsets of order \( k \).

Let \( S \) be a set containing \( m + 1 \) elements. Choose one of them and call it \( x \). Let \( k \in \mathbb{Z} \). If \( k \) is negative, then \( \binom{m}{k} = 0 \) is the number of \( k \) element subsets of \( S \). Similarly, if \( k = 0 \), then \( \binom{m}{k} = 1 \) is the number of \( k \) elements subsets of \( S \). Therefore, in what follows, we may assume that \( k > 0 \).

We will count subsets of order \( k \) in \( S \) by counting the subsets containing \( x \) separately from those not containing \( x \), and adding their sizes together.

Any subset of order \( k \) containing \( x \) contains \( k - 1 \) elements that are not \( x \). These elements form a subset of size \( k - 1 \) contained in \( S - \{x\} \). Since \( |S - \{x\}| = m \) the inductive hypothesis says that there are \( \binom{m}{k-1} \) such subsets.
Any subset of order $k$ not containing $x$ contains $k$ elements that are not $x$. These elements form a subset of size $k$ contained in $S - \{x\}$. Again, the inductive hypothesis applies, and says that there are $\binom{m}{k}$ such subsets.

Adding these, the number of subsets of size $k$ in $S$ is

$$\binom{m}{k - 1} + \binom{m}{k} = \binom{m + 1}{k}$$

by Theorem 16.4. This completes the inductive step.

Therefore, by mathematical induction $P(n)$ is true for all $n \geq 0$.

Theorem 16.6 gives an easy way to prove that the binomial coefficients are all integers, a fact that is not at all obvious from the definition.

**Theorem 16.7.** Let $n, k \in \mathbb{Z}$. Then the binomial coefficient $\binom{n}{k}$ is an integer.

*Proof.* If $n < 0$, then $\binom{n}{k} = 0 \in \mathbb{Z}$. If $n \geq 0$, then $\binom{n}{k}$ counts the number of $k$-element subsets in a set of size $n$. Hence, it must be an integer.

16.B Proof of the Binomial Theorem

With basic facts about the binomial coefficients established, we are now ready to prove the Binomial Theorem.

**Theorem 16.8** (Binomial Theorem). Let $x, y$ be variables and let $n \geq 0$ be in $\mathbb{Z}$. Then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

*Proof.* We prove the theorem by induction on $n \geq 0$. Let $P(n)$ be the open sentence

$$P(n): (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

We note that $P(0)$ is true, since $(x + y)^0 = 1$ and

$$\sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k = \binom{0}{0} x^0 y^0 = 1.$$

Now suppose that $P(m)$ is true for some $m \geq 0$. Then we know that

$$(x + y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k.$$
Multiplying this first by \( x \), we obtain
\[
x(x + y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k
\]
\[
= \binom{m}{0} x^{m+1} y^0 + \binom{m}{1} x^m y^1 + \ldots + \binom{m}{m} x y^m \quad (16.9)
\]
and multiplying by \( y \), we obtain
\[
y(x + y)^m = \sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^{k+1}
\]
\[
= \binom{m}{0} x^m y^1 + \ldots + \binom{m}{m-1} x y^m + \binom{m}{m} x^0 y^{m+1}. \quad (16.10)
\]

Adding equations (16.9) and (16.10), we obtain
\[
(x + y)^{m+1} = (x + y)(x + y)^m = x(x + y)^m + y(x + y)^m = \\
= \binom{m}{0} x^{m+1} y^0 + \binom{m}{1} x^m y^1 + \ldots + \binom{m}{m} x y^m \\
+ \binom{m}{0} x^m y^1 + \ldots + \binom{m}{m-1} x y^m + \binom{m}{m} x^0 y^{m+1}
\]
\[
= \binom{m+1}{0} x^{m+1} y^0 + \binom{m+1}{1} x^m y^1 + \ldots \\
+ \binom{m+1}{m} x y^m + \binom{m+1}{m+1} x^0 y^{m+1}
\]
\[
= \sum_{k=0}^{m+1} \binom{m+1}{k} x^{m+1-k} y^k
\]

Hence, \( \forall m \in \mathbb{Z}_{\geq 0}, P(m) \Rightarrow P(m + 1) \), so by the principle of mathematical induction \( P(n) \) is true for all \( n \geq 0 \).

16.C Exercises

Exercise 16.1. Use the definition of the binomial coefficient to prove that for any \( n \geq 0 \),
\[
\binom{n}{0} = \binom{n}{n} = 1, \quad \text{and} \quad \binom{n}{1} = \binom{n}{n-1} = n.
\]
(Note: You will not need induction for this problem.)
Exercise 16.2. Prove that for any \( n, k \in \mathbb{Z} \) with \( n \geq 0 \),
\[
\binom{n}{k} = \binom{n}{n-k}.
\]

Exercise 16.3. Let \( n, h, k \in \mathbb{Z} \) with \( n \geq 0 \). Using the definition of the binomial coefficient, prove that
\[
\binom{n}{h} \binom{n-h}{k} = \binom{n}{k} \binom{n-k}{h}.
\]
(Hint: Do not forget to deal with the cases where \( h < 0 \), or \( k < 0 \), or \( h + k > n \).)

Exercise 16.4. Use the fact that a set of size \( n \) has \( 2^n \) subsets and Theorem 16.6 to prove that for any integer \( n \geq 0 \),
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n.
\]

Exercise 16.5. Prove that for \( n \in \mathbb{N} \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.
\]

Exercise 16.6. Determine the coefficient of \( x^5 y^3 \) in the expansion of \( (2x + 3y)^8 \).
(Warning: It is not just \( \binom{8}{5} \).)

Exercise 16.7. Use the definition of the binomial coefficient to prove that for all \( n, k \in \mathbb{Z} \),
\[
k \binom{n}{k} = n \binom{n-1}{k-1}.
\]

Exercise 16.8. Prove that for \( n \in \mathbb{N} \), the “middle” binomial coefficient
\[
\binom{2n}{n}
\]
is an even integer.
(Hint: To get an idea how to prove it, look at Pascal’s triangle. Use problem 2.)

Exercise 16.9. Let \( n, k \in \mathbb{Z} \).
(a) Use induction to prove that for \( n > 8 \),
\[
\binom{n}{k} < 2^{n-2} \text{ for all } k \in \mathbb{Z}.
\]
(b) Use induction to prove that for \( n > 7 \),
\[
\binom{n}{k} < (n-3)! \text{ for all } k \in \mathbb{Z}.
\]