Undergraduate Spectral Theory with Computer Labs

Emily Evans – Brigham Young University
BYU’s Applied and Computational Mathematics Program

- **Freshman & Sophomore Years**
  - Minor in Mathematics (3 Calculus, Linear Algebra, ODE, proof)
  - Intro Computer Programming (C++)
  - First Semester of Real Analysis

- **Junior Year**
  - Mathematical Analysis
  - Design, Analysis & Optimization of Algorithms
  - Work on Concentration

- **Senior Year**
  - Modeling w/ Uncertainty & Data
  - Modeling w/ Dynamics and Control
  - Work in Concentration
Mathematical Analysis Sequence

- Vector Spaces
- Linear Transformations
- Inner Product Spaces
- Spectral Theory
- Metric Topology
- Differentiation
- Contraction Mappings
- Integration
- Integration on Manifolds
- Complex Analysis
- Advanced Spectral Theory
- Krylov Subspaces
- Pseudospectrum
Mathematical Analysis Sequence

- Vector Spaces
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- Krylov Subspaces
- Pseudospectra

First Semester

Second Semester
Eigenvalues and Eigenvectors - Classroom

- Review properties of eigenvalues/eigenvectors.
- Review invariant subspaces
- Introduce simple/semi-simple matrices
- Introduce a simpler version of the spectral mapping theorem that holds for semi-simple matrices and polynomials
- Introduce Schur decomposition
Practical Eigenvalue Lab

• Code the power method to find the dominant eigenvalue/eigenvector.
• Code the QR algorithm to find all the eigenvalues
• Modify the code to use Hessenberg preconditioning
• Vista: QR method with shifts.
Application Eigenvalue Lab:

- Image Segmentation
  - Students represent an image as a weighted graph
  - Weight of edge between two pixels is determined by their similarity and their distance apart
  - Using eigenvector associated with the second smallest eigenvalue students minimize the cut.
Singular Value Decomposition - Classroom

- Definition of SVD
- Calculation of the SVD
- Fundamental Subspace Theorem
- Consequences of the SVD
- Moore-Penrose pseudoinverse
- Low-rank approximate behavior
SVD Practical Lab

- Compute the singular value decomposition of a matrix
- Approximate a matrix using the truncated SVD
- Use the SVD to perform image compression

Original Image  Rank 40 Approximation  Rank 14 Approximation
SVD Application Lab: Facial Recognition

A simple facial recognition system can be implemented using the Singular Value Decomposition.

- A walkthrough of creating a facial recognition system
- Uses and explains in detail the eigenfaces method
- Teaches the concept of principal component analysis through example

(e) 75 eigenfaces, about (f) All 153 of the eigenfaces.
Advanced Spectral Theory – Topics Covered

- Projections
- Generalized Eigenvectors
- The Resolvent
- Spectral Resolution
- Spectral Decomposition
- Spectral Mapping Theorem
- The Perron-Frobenius Theorem
- The Drazin Inverse
Application Lab: PageRank and Tournaments

• Introduce the PageRank Algorithm
• Apply Perron’s Theorem to guarantee the eigenvalue 1 is simple
• Apply the power method to find the steady state configuration.
• Fun application: Rank college basketball teams to determine the most probable winner of the NCAA tournament
Krylov Subspaces—Topics Covered

• Iterative Methods
• Krylov Subspaces
• Minimal Polynomials
• Arnoldi Iteration
• GMRES
• Arnoldi for eigenvalues
GMRES Lab:

- Practical: Solve large linear systems with GMRES. Students implement the algorithm
- Practical: How fast does GMRES converge?
- Practical: Improving GMRES with restarts
- Practical: GMRES in SciPy
- Application: Solve large (interesting) problem using GMRES

![Eigenvalues of a matrix A](image1.png) ![Convergence of GMRES algorithm on A](image2.png)
Arnoldi Iteration Lab:

- Practical: Use Arnoldi Iteration to find an orthonormal basis of a Krylov subspace
- Practical: Approximate eigenvalues by finding Ritz values of upper Hessenberg matrix generated by Arnoldi Iteration

Largest 15 Ritz values for a $500 \times 500$ matrix with random entries

Largest 15 Ritz values for a $500 \times 500$ matrix with uniformly distributed eigenvalues
More Information About ACME program

• 4 Books: To be published by SIAM
  • Foundations of Applied Mathematics
    • Volume 1 Math Analysis
    • Volume 2 Algorithm Design and Optimization
    • Volume 3 Modeling with Uncertainty and Data
    • Volume 4 Modeling with Dynamics and Control

• Lab Manuals
  • 96 computing labs

• Soft Skills
  • Slides

• Supporting Materials
  • slides

https://foundations-of-applied-mathematics.github.io
Practical Lab: Pseudospectra

• Goal of the lab is to get students to better understand pseudospectra
• Practical: Plot point spectra of A+E
• Practical: Using Lanczos Method plot contour of pseudospectra

• Application: Compare pseudospectra of Nonnormal, Hermitian and Normal Matrices
Thank You!