Picard-Lindelöf Theorem

Lecture 4
Math 634
9/8/99

Theorem *The space $\mathcal{C}([a,b])$ of continuous functions from $[a,b]$ to $\mathbb{R}^n$ equipped with the norm*

$$\|f\|_\infty := \sup\bigl\{|f(x)| \bigm| x \in [a,b]\bigr\}$$

is a Banach space.

Definition Two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $\mathcal{X}$ are *equivalent* if there exist constants $m, M > 0$ such that

$$m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$$

for every $x \in \mathcal{X}$.

Theorem *If $(\mathcal{X}, \|\cdot\|_1)$ is a Banach space and $\|\cdot\|_2$ is equivalent to $\|\cdot\|_1$ on $\mathcal{X}$, then $(\mathcal{X}, \|\cdot\|_2)$ is a Banach space.*

Theorem *A closed subspace of a complete metric space is a complete metric space.*

We are now in a position to state and prove the Picard-Lindelöf Existence-Uniqueness Theorem. Recall that we are dealing with the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = a. \end{cases} \tag{1}$$

Theorem (Picard-Lindelöf) *Suppose $f : [t_0 - \alpha, t_0 + \alpha] \times \overline{\mathcal{B}(a,\beta)} \to \mathbb{R}^n$ is continuous and bounded by M. Suppose, furthermore, that $f(t,\cdot)$ is Lipschitz continuous with Lipschitz constant L for every $t \in [t_0 - \alpha, t_0 + \alpha]$. Then* (1) *has a unique solution defined on $[t_0 - b, t_0 + b]$, where $b = \min\{\alpha, \beta/M\}$.*

Proof. Let $\mathcal{X}$ be the set of continuous functions from $[t_0 - b, t_0 + b]$ to $\overline{\mathcal{B}(a,\beta)}$. The norm

$$\|g\|_w := \sup\bigl\{e^{-2L|t-t_0|}|g(t)| \bigm| t \in [t_0 - b, t_0 + b]\bigr\}$$

is equivalent to the standard supremum norm $\|\cdot\|_\infty$ on $\mathcal{C}([t_0-b,t_0+b])$, so this vector space is complete under this weighted norm. The set $\mathcal{X}$ endowed with this norm/metric is a closed subset of this complete Banach space, so $\mathcal{X}$ equipped with the metric $d(x_1,x_2) := \|x_1-x_2\|_w$ is a complete metric space.

Given $x \in \mathcal{X}$, define $T(x)$ to be the function on $[t_0-b,t_0+b]$ given by the formula

$$T(x)(t) = a + \int_{t_0}^{t} f(s,x(s))\,dx.$$

Step 1: If $x \in \mathcal{X}$ then $T(x)$ makes sense.
This should be obvious.

Step 2: If $x \in \mathcal{X}$ then $T(x) \in \mathcal{X}$.
If $x \in \mathcal{X}$, then it is clear that $T(x)$ is continuous (and, in fact, differentiable). Furthermore, for $t \in [t_0-b,t_0+b]$

$$|T(x)(t)-a| = \left|\int_{t_0}^{t} f(s,x(s))\,ds\right| \le \left|\int_{t_0}^{t} |f(s,x(s))|\,ds\right| \le Mb \le \beta,$$

so $T(x)(t) \in \overline{\mathcal{B}(a,\beta)}$. Hence, $T(x) \in \mathcal{X}$.

Step 3: T is a contraction on $\mathcal{X}$.
Let $x,y \in \mathcal{X}$, and note that $\|T(x)-T(y)\|_w$ is

$$\sup\left\{ e^{-2L|t-t_0|}\left|\int_{t_0}^{t}[f(s,x(s))-f(s,y(s))]\,ds\right| \;\middle|\; t \in [t_0-b,t_0+b]\right\}.$$

For a fixed $t \in [t_0-b,t_0+b]$,

$$\begin{aligned}
&e^{-2L|t-t_0|}\left|\int_{t_0}^{t}[f(s,x(s))-f(s,y(s))]\,ds\right| \\
&\le e^{-2L|t-t_0|}\left|\int_{t_0}^{t}|f(s,x(s))-f(s,y(s))|\,ds\right| \\
&\le e^{-2L|t-t_0|}\left|\int_{t_0}^{t}L|x(s)-y(s)|\,ds\right| \\
&\le Le^{-2L|t-t_0|}\left|\int_{t_0}^{t}\|x-y\|_w e^{2L|s-t_0|}\,ds\right| \\
&= \frac{\|x-y\|_w}{2}\left(1-e^{-2L|t-t_0|}\right) \\
&\le \frac{1}{2}\|x-y\|_w.
\end{aligned}$$

Taking the supremum over all $t \in [t_0 - b, t_0 + b]$, we find that T is a contraction (with $\lambda = 1/2$).

By the contraction mapping principle, we therefore know that T has a unique fixed point in $\mathcal{X}$. This means that (1) has a unique solution in $\mathcal{X}$ (which is the only place a solution could be). $\square$