Stabilization and Blow-up of Solutions of a
Nonlinear Parabolic Equation

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INTRODUCTION

The purpose of this thesis is to describe the behavior of solutions to the initial-boundary value problem

\[ u_t = (u_x + u^q)_x \quad (x,t) \in (0,1) \times (0,T) \]
\[ u_x + u^q = 0 \quad (x,t) \in \{0,1\} \times (0,T) \]
\[ u(x,0) = u_0(x) \quad x \in [0,1], \]

where \( q > 1, u_0 \in C([0,1],(0,8)) \), and \( T \) is a positive time, possibly infinite.

In Chapter 1, we provide background on this problem and mention some similar problems recently studied by others. In Chapter 2, we describe the steady-state solutions of (0.1) and show that for general initial data this problem has a unique solution existing until blow-up occurs, i.e., until the solution becomes unbounded at some point. In Chapter 3 we establish sufficient conditions for stabilization to occur. In Chapter 4 we, likewise, establish sufficient conditions for finite-time blow-up and describe the nature of this blow-up. Finally, in Chapter 5 we list some open questions involving (0.1) which deserve further study.
CHAPTER 1
PHYSICAL MOTIVATION

Equation (0.1) belongs to a class of problems known as flux-conservative equations. As defined in [9], a flux-conservative equation is one of the form

\[ u_t = (F(u))_x, \]

where the flux $F$ may depend not only on $u$, but also on the spatial derivatives of $u$. In (0.1) we also impose no-flux boundary conditions, $F(u) = 0$, on the boundary of the space domain. Examples of flux-conservative equations on finite domains with no-flux boundary conditions are a model of population dynamics studied by Shigesada [12] and a model of the ultracentrifuge studied by Yoshikawa [14]. In equation (0.1) the flux consists of a diffusion term and an advection term. The diffusion term $u_x$ tends to smooth $u$ as time progresses by causing $u$ to increase at relative minima and decrease at relative maxima. The advection term $u\theta$ represents flow towards the left which increases as $u$ increases.

If $u$ is thought of as the density of some substance, then

\[ \int_0^1 u(x,t)dx \]

can be thought of as the mass at time $t$. An important property of (0.1) is conservation of mass, i.e.,

\[ \left( \int_0^1 u(x,t)dx \right)_t = \int_0^1 u_t(x,t)dx = \int_0^1 (u_x + u\theta)_x dx = (u_x + u\theta)|_{x=1} - (u_x + u\theta)|_{x=0} = 0. \]
This conservation motivates the definition of the function $v$, defined to be the amount of mass to the left of $x$ at time $t$, i.e.,

$$v(x,t) = \int_0^x u(s,t) \, ds.$$ 

Using conservation of mass, it is easy to calculate that $v$ is a solution to

$$v_t = v_{xx} + v_x q \quad (x,t) \in (0,1) \times (0,T)$$

$$v(0,t) = 0, \quad v(1,t) = M = \int_0^1 u_0(x) \, dx \quad t \in (0,T) \quad (1.1)$$

$$v(x,0) = v_0(x) = \int_0^x u_0(s) \, ds \quad x \in [0,1].$$

Because of its constant boundary conditions, this equation is somewhat easier to work with than (0.1), so we will often prove things about solutions to (0.1) by working with (1.1) instead.
CHAPTER 2
EXISTENCE OF SOLUTIONS

In this chapter we consider special instances of (0.1) for which the solution can be calculated explicitly. In particular, we consider those solutions which are independent of time. We also show that for general exponents and initial conditions, (0.1) has a solution that exists for all time or until it becomes unbounded.

Steady-State Solutions

The simplest solutions to (0.1), and actually the only ones we can calculate explicitly, are the steady-state solutions or equilibria. These are solutions which are independent of time, i.e., \( u(x,t) = u_0(x) \) for all \( t \in [0,8) \). Of course, these are only solutions for very special initial conditions, but they are important in giving us ideas about the behavior of solutions resulting from more general initial data, and they are used in proving those ideas. Calculating, \( u \) is an equilibrium for (0.1) when

\[
\begin{align*}
  u_t &= 0 \quad \Rightarrow \quad (u_x + u^q)_x = 0 \quad \Rightarrow \quad u_x + u^q \text{ is constant on } [0,1] \\
  &\Rightarrow \quad u_x + u^q = 0 \text{ on } [0,1], \text{ since it is } 0 \text{ on } \{0,1\} \\
  &\Rightarrow \quad u_x = -u^q \quad \Rightarrow \quad u_xu^{-q} = -1 \quad \Rightarrow \quad \frac{u^{1-q}}{1-q} = -c - x \\
  &\Rightarrow \quad u = \frac{1}{[(q-1)(x+c)]^{q-1}}. \quad (2.1)
\end{align*}
\]

Note that this only determines an equilibrium when \( c > 0 \). We shall, however, call (2.1) with \( c = 0 \) the critical equilibrium, since this is the dividing point between those equilibria which exist on the entire unit interval and those "equilibria" which
don't. Note also that the mass of (2.1) increases as \( c \) decreases to 0, so no equilibrium has mass greater than or equal to the mass of the critical equilibrium. For this reason we call the mass of the critical equilibrium the **critical mass**, written \( M_c \). If the mass \( M \) of an initial condition satisfies \( M > M_c \) we say it has **supercritical mass**, and if it satisfies \( M < M_c \) we say it has **subcritical mass**. Integrating, we see that

\[
M_c = \int_0^1 \frac{dx}{1 - [(q - 1)x]^{q-1}} = \frac{q-1}{(q-2)(q-1)}
\]

if \( q > 2 \), and \( M_c = 8 \) if \( q = 2 \). Note that the equilibria of (1.1) corresponding to (2.1) are of the form

\[
v = \frac{(q - 1)^\alpha}{q - 2}[(x + c)^\alpha - c^\alpha], \quad \alpha = \frac{q - 2}{q - 1}
\]

if \( q \neq 2 \)

\[
v = \ln\left(\frac{x + c}{c}\right)
\]

if \( q = 2 \).

To get an idea what these equilibria look like, consider the case \( q = 3 \). In this case, all of the equilibria of (0.1) are horizontal translates of the graph of the critical equilibrium \( 1/\sqrt{2x} \). In fact, we have a continuum of non-intersecting equilibria. Only those equilibria representing translation to the left exist on the whole interval. The corresponding equilibria of (1.1) are translates of the graph of
the critical equilibrium $\sqrt{2x}$. These translates are such that they pass through (0,0). Again, we have a continuum of non-intersecting equilibria, all lying below the critical equilibrium. The closer $c$ is to 0, the closer the graph of the equilibrium is to the graph of the critical equilibrium. For general $q$, the graphs of the equilibria are similar.

The conservation of mass described in the previous chapter leads to some immediate conjectures. Because of the general smoothing nature of parabolic differential equations, it appears that the solution should converge to the unique equilibrium with the same mass as the initial data. If no such equilibrium exists, the solution should become unbounded either in finite or infinite time. The verity of these conjectures will be investigated in Chapters 3 and 4.

**Existence until Blow-up**

For more general initial data, we have the following existence result:

**Theorem 2.1.** Let $u_0 \in C^1([0,1],\mathbb{R})$ satisfy the boundary conditions, and let $q > 1$ be given. Suppose (0.1) has a solution $u$ valid until some time $T^*$ and that $u$ is bounded for $(x,t) \in [0,1] \times (0,T^*)$, say by the constant $u^*$. Then $u$ can be extended to be a solution to (0.1) valid until some time $T^{**} > T^*$.

Proof. We shall apply a standard existence theorem to a slightly altered form of (1.1). More precisely, we work with the problem

$$
\begin{align*}
\frac{\partial v}{\partial t} &= v_{xx} + f(v_x) & (x,t) \in (0,1) \times (0,2T^*) \\
v(0,t) &= 0 & \text{for } (x,t) \in [0,1], \\
v(1,t) &= M \\
v(x,0) &= v_0(x) & \text{for } x \in [0,1],
\end{align*}
$$

where

$$
f(p) \equiv \begin{cases} 
p^q & \text{if } p = 2u^* \\
(2u^*)^q & \text{if } p > 2u^*.
\end{cases}
$$
By bounding the nonlinearity in this manner we can apply, for example, Theorem 6.1 in chapter 5 of [7] to get existence of a unique solution \( v \) of (2.4) which is continuous and has continuous derivatives. Now if we let \( T^{**} = \min(2T^*, \inf\{t : v_x = 2u^* \text{ for some } x\}) \) then these continuity properties imply that \( T^{**} > T^* \). Also, up until the time \( T^{**} \) the solution \( v \) of (2.4) is also a solution to (1.1), since the bounding of the nonlinearity does not take effect until after \( T^{**} \). This solution to (1.1) induces a corresponding solution \( \hat{u} \) to (0.1) valid until \( T^{**} \) which is unique since \( v \) was. Because of the uniqueness of \( \hat{u} \) as a solution on this time interval, it must actually be an extension of the solution \( u \) with which we started. Thus, we have extended \( u \) to a solution valid to some time \( T^{**} > T^* \), so we are done.

Given initial data \( u_0 \) (and \( q > 1 \)) let \( T^* \) be the maximal time such that a solution to (0.1) exists on the time interval \((0,T^*)\). By using Theorem 6.1 of [7] and bounding the nonlinearity as we did above, it is easy to show that we have local existence, i.e., \( T^* > 0 \). Theorem 2.1 tells us that \( T^* \) is well-defined and that the solution on \((0,T^*)\) is unique. It also tells us that unless \( T^* = 8 \), the solution becomes unbounded at some point \( x \) at time \( T^* \), for otherwise we could extend it to a later time. In other words, the solution blows up at time \( T^* \), so the solution exists until blow-up occurs. The motivation for proving this theorem was an article by J. M. Ball in which he showed that for general parabolic partial differential equations, it is possible for a solution to cease to exist at some time without actually blowing up (see [3]). By Theorem 2.1 we know this cannot occur for problem (0.1) if the initial condition satisfies certain simple hypotheses. Note that the initial conditions satisfying these hypotheses are dense in the continuous functions. Since the regularity of solutions is not our prime concern here, for the remainder of this thesis we will assume that all initial conditions are sufficiently
smooth to induce the smoothness of solutions we need in taking various derivatives.
 CHAPTER 3
STABILIZATION

**Definition.** We shall say that the solution \( u(x,t) \) to (0.1) **stabilizes** if the functions \( u(\cdot,t) : [0,1] \not\Subset (0,8) \) converge uniformly to an equilibrium as \( t \not\Subset 8 \).

Under certain conditions, we can show that the solution to (0.1) does stabilize. Roughly speaking, these conditions are that either \( q \) is sufficiently small or that the mass of \( u_0 \) is sufficiently small and not concentrated too much near \( x = 0 \). It was proved by Bates and Alikakos in [1] that stabilization occurs for any initial data if \( 1 < q < 2 \), so we shall consider here the case \( q = 2 \).

**Definition.** A function \( z : [0,1] \not\Subset [0,T] \not\Subset (0,8) \) will be called a **supersolution** of (1.1) if it satisfies the following conditions:

(a) \( z \) is continuous;

(b) \( z_x, z_{xx}, \) and \( z_t \) are continuous except along finitely many smooth curves;

(c) the limit of \( z_x \) as one of these curves is approached from the left is greater than the limit as it is approached from the right; similarly, the limit of \( z_t \) as one of the these curves is approached from earlier time is greater than the limit as it is approached from later time;

(d) \( z \) is a solution to

\[
\begin{align*}
  z_t &= z_{xx} + z_t^q \\
  z(0,t) &= 0, \quad z(1,t) = M \\
  z(x,0) &= v_0(x),
\end{align*}
\]

with the first inequality being satisfied in the regions where the derivatives are defined and continuous.
We will show that if the initial data in (0.1) is such that an appropriate 
supersolution of (1.1) exists for all time then stabilization occurs. First, we show 
that the existence of such a supersolution provides a bound on the solution to (0.1).

**Lemma 3.1.** Suppose \( z \) is a supersolution of (1.1) valid up to time \( T \), with \( T \) possibly \( 8 \), and suppose \( z(0,t) = 0 \). Then

\[
(d\max(sup\{z_x(0,t) : t < T\}, sup\{u_0(x) : x \in [0,1]\})) \tag{3.2}
\]

Proof. By the maximum principle for nonlinear parabolic equations, 
proved by Protter and Weinberger in [10], the value of \( u \) in the domain of interest 
is bounded above by its maximum value on the parabolic boundary \{ \( (x,t) : x = 0, 
or x = 1, or t = 0 \} \}. Because the boundary conditions in (0.1) specify that \( u_x \) is 
negative at \( x = 1 \), this maximum must occur where \( x = 0 \) or \( t = 0 \). Thus, to prove 
(3.2) it suffices to show that

\( u(0,t) = z_x(0,t) \). Now, if \( v \) is the solution of (1.1) corresponding to \( u \), this can be 
written \( v_x(0,t) = z_x(0,t) \). Moreover, since \( v(0,t) = 0 = z(0,t) \), it is enough to prove 
that \( v(x,t) = z(x,t) \) throughout the domain.

Let us define \( w \equiv (v - z)e^{-t} \). By the definition of supersolution, \( w = 0 \) on 
the parabolic boundary. If \( v > z \) somewhere on \([0,1] \equiv [0,T]\) then \( w \) must have a 
positive local maximum in this domain. Note that because of condition (c) in the 
definition of supersolution, such a local maximum cannot occur where \( z \) is not 
differentiable. At a local maximum where \( z \) is differentiable, we have

\[
w_t = 0 \quad \Rightarrow \quad (v_t - z_t)e^{-t} - (v - z)e^{-t} = 0 \quad \Rightarrow \quad v_t - z_t = v - z > 0, \tag{3.3}
\]

\[
w_x = 0 \quad \Rightarrow \quad (v_x - z_x)e^{-t} = 0 \quad \Rightarrow \quad v_x = z_x, \tag{3.4}
\]

and

\[
w_{xx} = 0 \quad \Rightarrow \quad (v_{xx} - z_{xx})e^{-t} = 0 \quad \Rightarrow \quad v_{xx} = z_{xx}. \tag{3.5}
\]

But because \( v \) is a solution and \( z \) is a supersolution of (1.1),
Combining (3.4), (3.5), and (3.6) we get \( z_t - v_t = 0 \), contradicting (3.3). This tells us that \( v = z \) throughout the domain, which completes the proof of the lemma.

We also need the following definitions and basic results:

**Definition.** A **Lyapunov functional** for a parabolic differential equation is a continuous functional on the twice-differentiable functions of \( x \) which decreases along trajectories, i.e., if \( u(x,t) \) is a solution and \( V \) is a Lyapunov functional then \( V(u(\cdot,t_2)) = V(u(\cdot,t_1)) \) for \( t_1 < t_2 \) with equality only if \( u(\cdot,t_2) = u(\cdot,t_1) \).

**Definition.** The **\( \omega \)-limit set** of a solution \( u \) to a parabolic differential equation is the set of all functions \( \hat{u}(x) \) such that there exists a sequence of times \( (t_i) \not\to 8 \) for which \( u(\cdot,t_i) \not\to \hat{u}(\cdot) \).

An important connection between these two objects is that if a Lyapunov functional \( V \) is bounded below, its time derivative is 0 on the \( \omega \)-limit set. This result can be found in [11]. Here we assume that, considering its action along trajectories, \( V \) is continuously differentiable with respect to time.

**Definition.** A family \( F \) of functions from a compact subset \( K \not\to \mathbb{R}^p \) to \( \mathbb{R}^q \) is said to be **uniformly equicontinuous** if given any \( \varepsilon > 0 \) there is some \( \delta > 0 \) such that for any \( f \in F \) and for any \( x_1,x_2 \in K \), if \( \| x_1 - x_2 \| < \delta \) then \( |f(x_1) - f(x_2)| < \varepsilon \).

**Arzela-Ascoli Theorem.** (See e.g. [4].) If \( F \) is a family of continuous functions from a compact subset \( K \not\to \mathbb{R}^p \) to \( \mathbb{R}^q \) then the following two properties are equivalent:

1. The family \( F \) is bounded and uniformly equicontinuous.

2. Every sequence from \( F \) has a uniformly convergent subsequence.

We are now ready to prove stabilization.

**Theorem 3.2.** Suppose \( z \) is a supersolution of (1.1) valid for all time, and suppose \( z(0,t) = 0 \). Then the corresponding solution \( u \) of (0.1) stabilizes.
Proof. By Lemma 3.1, \( u \) is bounded for all time. Let \( w = u_x + u^q \) and note that \( w \) is a solution to the parabolic equation

\[
  w_t = w_{xx} + [q(u(x,t))^{q-1}]w_x
\]

\[
  w(0,t) = w(1,t) = 0 \quad (3.7)
\]

\[
  w(x,0) = u_0'(x) + [u_0(x)]^q
\]

We have not assumed that the derivative of \( u_0 \) makes sense, but if \( u_0 \) is not differentiable then we can shift our initial condition to some time \( t_0 > 0 \) when the regularizing effect of (0.1) will ensure that the derivative exists and is bounded.

We can now apply Protter and Weinberger's maximum principle to (3.7) and get the fact that \( w \) is bounded. Since \( u \) is also bounded, \( u_x \) must be bounded, too. This means that the family \( F \equiv \{ u(\cdot,t) : t > 0 \} \) is uniformly equicontinuous, since given \( \varepsilon \) we can pick \( \delta = \varepsilon / \sup(u_x) \). Applying the Arzela-Ascoli theorem, any sequence from \( F \) has a uniformly convergent subsequence. In particular, the \( \omega \)-limit set of \( u \) is nonempty.

We now claim that

\[
  V(u) \equiv \int_0^1 \int_0^1 \exp(x + \frac{u^{1-q}}{1-q})d\sigma dx \quad (3.8)
\]

is a Lyapunov functional. The general form for this functional was suggested by Zelenyak in [15]. Calculating, we have

\[
  \frac{d}{dt} V(u) = \int_0^1 \exp(x + \frac{u^{1-q}}{1-q})u_t dx
\]
\[
\begin{align*}
\int_0^1 \exp(x + \frac{u^{1-q}}{1-q}(u_x + u^q))dx &= 1 \\
\int_0^1 \exp(x + \frac{u^{1-q}}{1-q})(1 + u^{-q}u_x)(u_x + u^q)dx &= -\int_0^1 \exp(x + \frac{u^{1-q}}{1-q}(1 + u^{-q}u_x)(u_x + u^q)dx \\
\int_0^1 \exp(x + \frac{u^{1-q}}{1-q})u^{-q}(u_x + u^q)^2dx &= 0
\end{align*}
\]

with equality only if \(u_x + u^q = 0\), i.e., if \(u\) is an equilibrium. This verifies that \(V\) is a Lyapunov functional.

Now the fact that Lyapunov functionals have zero time derivative on the \(\omega\)-limit set implies that in this case the \(\omega\)-limit set consists entirely of equilibria. Since mass is conserved and since there is a unique equilibrium of a given mass, there can clearly be only one equilibrium \(\hat{u}\) in the \(\omega\)-limit set, namely the one with the same mass as \(u_0\). In order to show stabilization we must show that convergence to this equilibrium occurs along any sequence of times. Suppose there were a sequence of times \((t_i) \not\subseteq 8\) for which \(u(\cdot, t_i)\) did not converge to \(\hat{u}\).

Arzela-Ascoli tells us that \(u(\cdot, t_i)\) or a subsequence must converge uniformly to something in the \(\omega\)-limit set, but \(\hat{u}\) is the only thing in the \(\omega\)-limit set. Thus, stabilization does occur.

The following concrete results depend on the existence of supersolutions to which Theorem 3.2 can be applied.

**Theorem 3.3.** If \(q = 2\) then for any initial data the solution to (0.1) stabilizes.

**Proof.** Let initial data \(u_0 \in C([0,1])\) be given, let \(v_0\) be the corresponding initial data to (1.1), and let \(M\) be \(v_0(1)\), as usual. Since \(u_0\) is continuous and,
therefore, bounded, \( v_0 \) goes through \((0,0)\) and has bounded slope. Thus, the graph of \( v_0 \) lies beneath some line through the origin, say \( y = kv \). Let \( z \) be the equilibrium of (1.1) which passes through \((1,k)\), i.e.,

\[
z(x,t) = \ln \frac{x + c}{c}, \text{ where } c = \frac{1}{e^k - 1}.
\]

Note that \( z \) is increasing and concave down, so \( z \) lies above \( y \) and, hence, above \( v_0 \). The other requirements of a supersolution are clearly satisfied by \( z \). Applying Theorem 3.2, the solution to (0.1) stabilizes.

**Theorem 3.4.** Suppose \( q > 2 \) and \( u_0 \) has subcritical mass. If the graph of the corresponding initial data \( v_0 \) for (1.1) lies beneath the graph of the critical equilibrium (except at \( x = 0 \)) then the solution to (0.1) stabilizes.

Proof. Under the given conditions, we claim there exists some equilibrium \( z \) with bounded derivative which lies beneath the graph of the critical equilibrium but above the graph of \( v_0 \). Suppose not. Then there exists a sequence \((x_k)\) such that at \( x_k \), \( v_0 \) is greater than or equal to the equilibrium with \( c = 1/k \). Let \( x_0 \) be a limit point of this sequence. We know \( x_0 \neq 0 \) since the equilibria have arbitrarily large derivatives there, while \( v_0'(0) \) is finite. Thus \( x_0 \in (0,1] \). Since our sequence of equilibria converges uniformly to the critical equilibrium, at \( x_0 \), \( v_0 \) must be greater than or equal to the critical equilibrium, which is contrary to assumption. Hence, our claim holds. Using \( z \) as a supersolution and applying Theorem 3.2, we get stabilization.
CHAPTER 4

BLOW-UP

Before proving any results on "blow-up" we will need to be more precise as to what we mean by this term. Also, we need to define what we mean by subsolutions, which play roughly the same role in the proof of blow-up in this chapter that supersolutions did in the proof of stabilization in Chapter 3.

**Definition.** We say that a solution $u$ to (0.1) **blows up** at a point $x$ at time $t$ if there is a sequence of points $(x_n)$ converging to $x$ and a sequence of times $(t_n)$ converging to $t$ from below, such that $u(x_n,t_n) \not\to 8$ as $n \not\to 8$. In this case, we also say that **blow-up** occurs at $(x,t)$.

**Supercritical Mass**

**Definition.** A function $z : [0,1] \to [0,T] \not\to [0,8)$ will be called a **subsolution** of (1.1) if it satisfies the following conditions:

(a) $z$ is continuous;

(b) $z_x$, $z_{xx}$, and $z_t$ are continuous except along finitely many smooth curves;

(c) the limit of $z_x$ as one of these curves is approached from the left is less than the limit as it is approached from the right; similarly, the limit of $z_t$ as one of the these curves is approached from earlier time is less than the limit as it is approached from later time;

(d) $z$ is a solution to

\[ z_t = z_{xx} + z_x^q \]

\[ z(0,t) = 0, \quad z(1,t) = M \]

\[ z(x,0) = v_0(x), \]

with the first inequality being satisfied in the regions where the derivatives are defined and continuous.
Lemma 4.1. Suppose \( z \) is a subsolution to (1.1) with \( z(0,t) = 0 \), and suppose that for some \( T \), \( z_x(0,T) \neq 0 \). (By this we mean that \( z_x(x,T) \not\equiv 0 \) as \( x \not\equiv 0 \).) Then the solution to (0.1) blows up at \( x = 0 \) at some time \( t = T \).

Proof. The proof is very similar to the proof of Lemma 3.1, only a little bit simpler, since we do not have to use the maximum principle for nonlinear parabolic equations. Clearly, it suffices to prove that \( u(0,t) = z_x(0,t) \). If \( v \) is the solution of (1.1) corresponding to \( u \) then this is equivalent to \( v_x(0,t) = z_x(0,t) \).

Since \( v(0,t) = 0 = z(0,t) \), it suffices to show that \( v(x,t) = z(x,t) \) for all \( x \in [0,1] \) and for all \( t \in [0,T] \).

Consider \( w = (z - v)e^{-t} \). By the definition of subsolution, \( w = 0 \) on the parabolic boundary. Suppose \( w > 0 \) somewhere in the domain. Then \( w \) has a positive local maximum somewhere on the domain. If this occurs at a point where \( z \) is differentiable then simple calculus tells us that at this point

\[
w_t = 0 \quad \Rightarrow \quad (z_t - v_t)e^{-t} - (z - v)e^{-t} = 0 \quad \Rightarrow \quad z_t - v_t = z - v > 0,
\]

\[
w_x = 0 \quad \Rightarrow \quad (z_x - v_x)e^{-t} = 0 \quad \Rightarrow \quad z_x = v_x,
\]

and

\[
w_{xx} = 0 \quad \Rightarrow \quad (z_{xx} - v_{xx})e^{-t} = 0 \quad \Rightarrow \quad z_{xx} = v_{xx}.
\]

But since \( z \) is a subsolution and \( v \) is a solution to (1.1),

\[
z_t - z_{xx} - (z_x)q - v_t + v_{xx} + (v_x)q = 0. \tag{4.5}
\]

Combining (4.3), (4.4), and (4.5) we get \( z_t - v_t = 0 \), which contradicts (4.2). Part (c) of the definition of subsolution prevents a local maximum from occurring where \( z \) is not differentiable. Thus, \( w = 0 \) everywhere, which implies that \( v = z \) throughout the domain. As we mentioned previously, this implies the lemma as stated.
**Theorem 4.2.** Suppose $q > 2$ and $u_0$ has supercritical mass. Then the solution $u$ to (0.1) blows up at $x = 0$ in finite time. In fact, if $u_0$ has mass $M$, then $u$ blows up at some time

$$t = \frac{(q - 1)q - 1}{[M(q - 2)]q - 1 - (q - 1)q - 2}.$$

Proof. The key to proving this is finding an appropriate subsolution. We try one made up of the zero function and a stretched and translated version of the critical equilibrium (2.3). Specifically, with $T > 0$ to be determined later we try $z : [0,1] \times [0,T] \setminus (0,8)$ defined by

$$z(x,t) = \begin{cases} 0 & \text{if } 0 = x = 1 - \frac{t}{T} \\ M(x + \frac{t}{T} - 1)^\alpha & \text{if } 1 - \frac{t}{T} < x = 1 \end{cases}$$

where $\alpha = \frac{q - 2}{q - 1}$.

For convenience, let us name the two triangular halves of the domain used in the definition of $z$ as

$$\Omega_1 = \{(x,t) : [0,1] \times [0,T] : 0 = x = 1 - \frac{t}{T}\}$$

$$\Omega_2 = \{(x,t) : [0,1] \times [0,T] : 1 - \frac{t}{T} < x = 1\}.$$

Clearly, $z$ satisfies the requirements of being continuous and having piecewise continuous derivatives, and the derivatives of $z$ satisfy the necessary jump conditions, since $z_x$ and $z_t$ are 0 on the lower triangular region $\Omega_1$. All that remains to do to make $z$ a subsolution is to see if we can choose $T$ so (4.1) is satisfied.
First checking the boundary we have

\[ v(0,t) = 0 = z(0,t) \]
\[ v(1,t) = M = M(t/T)^\alpha = z(1,t) \]
\[ v(x,0) = v_0(x) = 0 = z(x,0) \]

so the boundary conditions of (4.1) are satisfied with no further restrictions on \( T \).

Note also that \( z \) is 0 on the left boundary as Lemma 4.1 requires.

We next check the interior. In \( \Omega_1 \),

\[ z_t = 0, \quad z_x = 0, \quad z_{xx} = 0 \]

\[ \Rightarrow \]

\[ z_t = z_{xx} + z_x q \]

so we need make no restrictions on \( T \) because of the way \( z \) is defined on this region. Checking \( \Omega_2 \) is more complicated. Here we have

\[ z_t = \frac{\alpha M}{T} (x + \frac{t}{T} - 1)^{\alpha-1}, \quad z_x = \alpha M(x + \frac{t}{T} - 1)^{\alpha-1}, \quad z_{xx} = \alpha(\alpha - 1)M(x + \frac{t}{T} - 1)^{\alpha-2}, \]

so, using the definition of \( \alpha \), we get

\[ z_t = z_{xx} + z_x q \]

\[ \frac{\alpha M}{T} (x + \frac{t}{T} - 1)^{\alpha-1} = \alpha(\alpha - 1)M(x + \frac{t}{T} - 1)^{\alpha-2} + \left[ \alpha M(x + \frac{t}{T} - 1)^{\alpha-1} \right] q \]

\[ 1 = (\alpha - 1)T(x + \frac{t}{T} - 1)^{-1} + (\alpha M)^{-1}T(x + \frac{t}{T} - 1)^{\alpha-1}(q - 1) \]

\[ 1 = [\alpha - 1 + (\alpha M)^{-1}]T(x + \frac{t}{T} - 1)^{-1}. \]
Noting that in $\Omega_2$, $(x + \frac{t}{T} - 1) \in (0,1]$, we therefore have

$$1 = [\alpha - 1 + (\alpha M)^{q-1}]T \quad \Rightarrow \quad z_t = z_{xx} + z_x^q,$$

so, assuming $M > M_c$, $z$ will satisfy (4.1) in $\Omega_2$ if

$$T = \frac{1}{\alpha - 1 + (\alpha M)^{q-1}} = \frac{(q-1)^{q-1}}{[M(q-2)]^{q-1} - (q-1)^{q-2}}.$$

Setting $T$ equal to the expression on the right, we have shown that $z$ is a subsolution to (1.1) with $z(0,t) = 0$. Note also that $z_x(0,T) = 8$; therefore, we can apply Lemma 4.1 to obtain the blow-up of $u$ at $x = 0$ at or before the specified time $T$.

**Numerical Methods**

As we mentioned in Chapter 1, the fact that supercritical mass leads to blow-up was something we could anticipate by noting the nonexistence of equilibria for such initial data. Most of the results in the remainder of this chapter are not so intuitive and, in fact, were only discovered after the results of numerical experiments were examined. In this section we describe the various numerical techniques that were applied to this problem.

The difficulty of simulating the blow-up of solutions to nonlinear partial differential equations is well-known (see [5]). This is true, in particular, for problems (0.1) and (1.1). A number of methods were tried before one was found that gave reasonable results.

All of the methods we used were **finite difference** methods. In these, the space-time domain is replaced by a two-dimensional grid of points, and the derivatives in the equation are replaced by approximating difference quotients. These methods can be either **explicit** or **implicit** and can use either **fixed** or
adaptive grids. In explicit methods, the value of the solution $u$ at a given grid point is only dependent on its values at grid points for previous times, so once $u$ has been calculated for earlier time steps, the value of $u$ at the current time step can be calculated directly. On the other hand, implicit methods, because of interdependence between grid points at the same time step, require the solution of a system of linear or nonlinear equations at each time step in order to update $u$. With a fixed grid, the arrangement of points is the same at each time step, while with adaptive or moving grids, the location of points may be changed so as to concentrate them where the solution is ill-behaved, e.g. near the blow-up point.

The first method applied to our problem used an explicit, fixed-grid, finite difference method on (0.1) directly; however, the nonlinear boundary conditions in their discretized form caused strong instabilities to arise at early times. It was, therefore, decided to use (1.1) instead, because of its tractable boundary conditions. In order of increasing complexity, the following types of methods were applied to this problem:

(a) the standard explicit method on a fixed grid;

(b) an implicit Crank-Nicholson predictor-corrector method on a fixed grid, such as that found in [8];

(c) an implicit Crank-Nicholson predictor-corrector method on a grid adapted to concentrate points near $x = 0$ as $t$ approached the blow-up time $T$. The more sophisticated methods gave better results but were still somewhat unsatisfactory. The behavior of the approximate solutions they produced would change behavior suddenly and unexpectedly, and in some cases oscillations would appear spontaneously. These occurred no matter what difference quotients were used to approximate derivatives.

The problem appeared to be the effect that the approximations of the derivatives in (1.1) had when they became large. To remedy this problem, we
switched the roles of $x$ and $v$ as independent and dependent variables, respectively. To do this, we noted that the solution $v$ of (1.1) determines a surface in $\mathbb{R}^3$. This surface can be thought of as a level surface $F(x,t,v) = 0$ of some differentiable function $F : \mathbb{R}^3 \not\mathbb{R}$. Since, for fixed time, $v$ is a one-to-one function of $x$, we can think of this surface as representing $x$ as a function of $v$ and $t$, instead. The partial derivatives of $x(v,t)$ can be calculated by applying the implicit function theorem, such as found in [4], to $F$. After performing this calculation, we see that $x$ is a solution of the new problem

$$
x_t = \frac{x_{vv}}{x_v^2} - \frac{1}{x_v q - 1}
$$

$$
x(0,t) = 0, \quad x(M,t) = 1
$$

$$
x(v,0) = x_0(v),
$$

where $x_0$ is the inverse function of $v_0$.

We applied an explicit finite difference method to (4.6) on a fixed spatial grid, but with a time stepsize that decreased to 0 as the solution approached blow-up. For (4.6), the derivatives remain bounded. The only problem is that the $x_v$ in the denominator goes to 0 at $v = 0$ as blow-up occurs. By making sure that the time stepsize satisfied a criterion analogous to the stability criterion for standard explicit methods, as described in [2], we appeared to eliminate numerical instability for (4.6), since the behavior of our numerical solutions created by this method appeared reasonable. In any case, the numerical experiments were not made to prove anything about the problem we studied, but rather to suggest what results could be proved by other means.

Subcritical Mass
Contrary to our original conjecture in Chapter 1, it is possible for solutions with subcritical mass to blow up in finite time, even though an equilibrium with the same mass exists. This was first discovered numerically when a computer run with \( q = 3 \) and \( u_0(x) = 7.5(1 - x)^5 \) was made. From (2.2) it is easy to check that for this value of \( q \), the critical mass is \( \sqrt{2} \), while the mass of \( u_0 \) is 1.25. Nevertheless, the numerical results seemed to show blow-up occurring (fig. 1). This led to further analytic investigation of initial data with subcritical mass, and it was soon proved that blow-up can occur in this case. In fact, we can show that for any \( q > 2 \) there are initial conditions with arbitrarily small mass that blow up.

To begin with, we will need a slightly more general version of Theorem 4.2. Note that in the proof of that theorem we did not use the fact that \( v(1,t) = M \), but only that \( v(1,t) = M \), for some supercritical mass \( M \). The proof goes through as is, giving the identical upper bound on blow-up time. With \( w \) replacing \( v \), \( y \) replacing \( x \), and \( \tau \) replacing \( t \), we have the following result.

**Lemma 4.3.** Let \( w \) be a solution to

\[
    w_t = w_{yy} + w_y^q \\
    w(0,\tau) = 0, \quad w(1,\tau) = w_1(\tau) = M \\
    w(y,0) = w_0(y)
\]

with \( M \) a supercritical mass. Then \( w_y(0,\tau) \nexists \) at some time

\[
    \tau = \frac{(q - 1)q^{-1}}{[M(q - 2)]q^{-1} - (q - 1)q^{-2}}.
\]
**Theorem 4.4.** Let \( u_0 \) be a subcritical initial condition for (0.1) and let \( v_0 \) be the corresponding initial condition for (1.1). Suppose that for some \( \lambda \in (0,1] \) and some supercritical mass \( M \)

\[
v_0(\lambda) + \min(v_0'' + (v_0')^q) \frac{\lambda^2 (q-1)^{q-1}}{[M(q-2)]^{q-1} - (q-1)^{q-2}} = \lambda a M \tag{4.9}
\]

with \( a = \frac{q-2}{q-1} \). Then the solution \( u \) to (0.1) blows up at \( x = 0 \) at some time

\[
t = T \equiv \frac{\lambda^2 (q-1)^{q-1}}{[M(q-2)]^{q-1} - (q-1)^{q-2}}. \tag{4.10}
\]

Proof. To prove this theorem, we use a rescaling argument, that relates the behavior of our subcritical solution on a portion of its space domain to the behavior of a supercritical solution of a similar problem. The proof is by contradiction. Suppose \( u = v_x \) does not blow up at \( x = 0 \) by the time \( T \) defined in (4.10). Note that \( v_t \) satisfies the parabolic partial differential equation (3.7), and apply the maximum principle for nonlinear parabolic equations. This principle states that not only the maximum value but also the minimum value of a solution occurs on the parabolic boundary (see [10]). Since \( v_t = 0 \) at \( x = 0 \) and \( x = 1 \), we have

\[
v_t(x,t) = \min(v_t(x,0)) = \min(v_0'' + (v_0')^q)
\]

for all \( (x,t) \in [0,1] \times [0,T] \). We have used the fact that this minimum is negative, since if it were nonnegative, then \( u_0 \) could not have subcritical mass. Thus, in particular,
\[v(\lambda, t) = v_0(\lambda) + \int_0^t v(\lambda, s) \, ds = v_0(\lambda) + \min(v_0'' + (v_0')^q) \cdot t = \lambda^\alpha M\]

for all \( t = T \).

Now we rescale as follows:

\[y \equiv \frac{x}{\lambda^\alpha}, \quad \tau \equiv \frac{t}{\lambda^2}, \quad w(y, \tau) \equiv \lambda^{-\alpha} v(x, t).\]

Calculating the derivatives, we get

\[w_\tau = w_\tau t = \lambda^{2-\alpha} v_t, \quad v_x = \lambda^{\alpha} w_y x_x = \lambda^{\alpha-1} w_y, \quad v_{xx} = \lambda^{\alpha-2} w_{yy},\]

so

\[\frac{1}{\lambda^2} w_\tau = \lambda^{-\alpha} v_t = \lambda^{-\alpha} [v_{xx} + v_x q] = \lambda^{-\alpha} [\lambda^{\alpha-2} w_{yy} + (\lambda^{\alpha-1} w_y)^q]\]

\[= \lambda^{-2} w_{yy} + \lambda^{\alpha(q-1)} - q w_y = \lambda^{-2} (w_{yy} + w_y q).\]

Note also that for \( \tau = \lambda^{-2} T \), \( w(1, \tau) = \lambda^{-\alpha} (\lambda, \lambda^2 \tau) = \lambda^{-\alpha} \lambda^\alpha M = M \), because of our hypothesis. Thus, if we let \( w_1(\tau) = \lambda^{-\alpha} (\lambda, \lambda^2 \tau) \) and if we let \( w_0(x) = \lambda^{-\alpha} v_0(\lambda y) \) then \( w \) is a solution to (4.7) with \( M \) a supercritical mass. Applying Lemma 4.3, \( w_y \) blows up at \( y = 0 \) at some time

\[\tau = \frac{(q - 1)q^{-1}}{[M(q - 2)]q^{-1} - (q - 1)q^{-2}}.\]

But clearly this means that \( u = v_x \) must blow up at \( x = 0 \) at some time
\[ t = \frac{\lambda^2 (q-1)^{q-1}}{[M(q-2)]^{q-1} (q-1)^{q-2}}, \]

contrary to assumption. Hence, the assumption was false and this theorem is true.

This theorem by itself does not tell us that there actually exist subcritical initial conditions which give rise to blow-up. We first must find initial conditions that meet the hypotheses of the theorem. Even the example of numerical blow-up given at the start of this section does not satisfy these criteria, although we were able to show that it really does blow up by using more refined estimates which will appear in a later section. The next theorem tells us that there are initial conditions of arbitrarily small mass that meet the requirements of Theorem 4.4, and the proof gives specific examples.

**Theorem 4.5.** Let \( q > 2 \). Then there are initial conditions for (0.1) with arbitrarily small mass that give finite-time blow-up.

**Proof.** With \( \lambda \in (0, 1] \) and \( k > 0 \) to be determined later, define

\[ u_0(x) = \frac{(k\lambda)^{\alpha + 1}}{(x + \lambda)^2}, \]

with \( \alpha = \frac{q-2}{q-1} \). If this is an initial condition for (0.1), the corresponding initial condition for (1.1) is

\[ v_0(x) = \frac{k(k\lambda)^\alpha}{x + \lambda}. \]  \((4.11)\)

In order to apply Theorem 4.4, we need to find the minimum value of
\[ f(x) \equiv v_0''(x) + [v_0'(x)]' = -\frac{2(k\lambda)^{\alpha + 1}}{(x + \lambda)^3} + \frac{(k\lambda)^{(\alpha + 1)q}}{(x + \lambda)^2q}. \]

Differentiating \( f \), we get

\[ f'(x) = \frac{6(k\lambda)^{\alpha + 1}}{(x + \lambda)^4} - \frac{2q(k\lambda)^{(\alpha + 1)q}}{(x + \lambda)^{2q + 1}}, \]

\[ f''(x) = -\frac{24(k\lambda)^{\alpha + 1}}{(x + \lambda)^5} + \frac{2q(2q + 1)(k\lambda)^{(\alpha + 1)q}}{(x + \lambda)^{2q + 2}}. \]

Now \( x \) is a critical point of \( f \) iff

\[ f'(x) = 0 \quad / \quad 6(k\lambda)^{\alpha + 1} = \frac{2q(k\lambda)^{(\alpha + 1)q}}{(x + \lambda)^{2q + 1}}, \]

\[ / \quad (x + \lambda)^{2q - 3} = \frac{q}{3}(k\lambda)^{(\alpha + 1)(q - 1)} \]

\[ / \quad (x + \lambda)^{2q - 3} = \frac{q}{3}(k\lambda)^{2q - 3} \]

\[ / \quad x + \lambda = k\lambda \left( \frac{q}{3} \right)^{\frac{1}{2q - 3}}. \]

Since there is only one critical point, this will represent a local, and therefore, global minimum if the second derivative of \( f \) is positive at this point. If \( x_0 \) is the critical point

\[ f''(x_0) = -\frac{24(k\lambda)^{\alpha + 1}}{(k\lambda)^5 \left( \frac{q}{3} \right)^{2q - 3}} + \frac{2q(2q + 1)(k\lambda)^{(\alpha + 1)q}}{(k\lambda)^{2q + 2} \left( \frac{q}{3} \right)^{2q - 3}} \]

\[ = -\frac{24}{(k\lambda)^{q - 1} \left( \frac{q}{3} \right)^{2q - 3}} + \frac{2q(2q + 1)}{(k\lambda)^{q - 1} \left( \frac{q}{3} \right)^{2q - 3}}. \]
This is positive iff

\[
\frac{2q(2q + 1)}{\left(\frac{q}{3}\right)^{2q+2} - \left(\frac{q}{3}\right)^{2q-3}} > \frac{24}{5} \left(\frac{q}{3}\right)^{2q-3}
\]

\[
/ \quad 2q(2q + 1) > 24 \left(\frac{q}{3}\right)
\]

\[
/ \quad q > \frac{3}{2}.
\]

Thus, the critical point \(x_0\) represents the minimum value of \(f\) on the entire real line, so, in particular, the value of \(f\) at \(x_0\) is a lower bound for \(f\) on \([0,1]\). Calculating again,

\[
f(x_0) = -\frac{2(k\lambda)^\alpha+1}{(k\lambda)^3\left(\frac{q}{3}\right)^{2q-3}} + \frac{(k\lambda)^{(\alpha+1)}q}{(k\lambda)^{2q}\left(\frac{q}{3}\right)^{2q-3}}
\]

\[
= \left[ -\frac{2}{\left(\frac{q}{3}\right)^{2q-3}} + \frac{1}{\left(\frac{q}{3}\right)^{2q-3}} \right] (k\lambda)^{-\frac{q}{q-1}}.
\]

By Theorem 4.4, we will have finite time blow-up if

\[
l\left(\lambda, k, \alpha, \lambda^\alpha M, M\right) = \frac{k}{2}(k\lambda)^\alpha + \left[ \frac{1}{\left(\frac{q}{3}\right)^{2q-3}} - \frac{2}{\left(\frac{q}{3}\right)^{2q-3}} \right] (k\lambda)^{-\frac{q}{q-1}} \frac{\lambda^2(q-1)^{q-1}}{[M(q-2)\lambda^{-1} - (q-1)\lambda^{-2}]} = \lambda^\alpha M
\]

for some choice of \(\lambda, k, \) and supercritical mass \(M\). Note that the \(\lambda\) terms can be factored out of this, so that the blow-up criterion simplifies to
\[
\frac{k}{2}k^\alpha + \left[ \frac{1}{\left( \frac{q}{3} \right)^{2q-3}} - \frac{2}{\left( \frac{q}{3} \right)^{3q-3}} \right] k^{-\frac{q}{q-1}} \frac{(q-1)^{q-1}}{[M(q-2)]^{-1} - (q-1)^{q-2}} = M.
\]

Take \( M \) to be any supercritical mass, and take \( k \) large. Then this inequality will be satisfied if \( k \) is sufficiently large. Checking (4.11), if we let \( \lambda \not\equiv 0 \) with \( k \) fixed the mass shrinks to 0, also. Thus, by taking \( k \) sufficiently large and then taking \( \lambda \) sufficiently small, we have blow-up for arbitrarily small mass for any \( q > 2 \).

**Blow-up Time**

Recall that Theorem 4.2 gave an upper bound on the blow-up time for supercritical mass. This result can actually be improved significantly. It is also possible, using Lemma 3.1, to obtain a lower bound for the blow-up time. In this section, we present these two bounds in the form of theorems, and we compare the bounds with numerical data to determine their relative sharpness.

**Theorem 4.6.** Let \( u_0 \) be a supercritical initial condition for (0.1) and let \( v_0 \) be the corresponding initial condition for (1.1). Let \( v^* \) be the critical equilibrium of (1.1) for the value of \( q \) in question. Then the solution \( u \) to (0.1) does not blow up before time

\[
T = \frac{(v^* - v_0) \text{ at its first critical point}}{\max(v_0'' + (v_0')^q)}. \tag{4.12}
\]

Proof. To prove this, we find an appropriate supersolution that has bounded derivative for a period of time and then apply Lemma 3.1. The idea of
the supersolution we shall use is simple; we take the initial condition and move it upwards at constant velocity, adjusting it near \( x = 0 \) so that it does not go above a certain equilibrium. Take \( v_E \) to be an equilibrium such that \( v_E'(0) > v_0'(0) \). Let \( g(t) = \min \{ x \in [0,1] : v_E(x) = v_0(x) + \mu t \} \), where \( \mu = \max(v_0'' + (v_0')^q) \). Then we define

\[
z(x,t) = \begin{cases} v_E(x) & \text{if } 0 = x = g(t) \\ v_0(x) + \mu t & \text{if } g(t) < x = 1 \end{cases}
\]

We define \( z \) up to the first time \( t \) such that \( v_E'(g(t)) = v_0'(g(t)) \). Call this time \( T_E \). It is easy to see that because of the way we chose \( \mu \), \( z \) satisfies the differential inequalities necessary in each of its domains of definition. Also, \( z \) is continuous, satisfies the required boundary conditions, and for \( t = T_E \), \( z_x \) and \( z_t \) have the correct type of jump discontinuities along the curve \( x = g(t) \). Using lemma 3.1, we know that \( u \) does not blow up before time \( T_E \).

Now let \( F : [0,1] \otimes \mathbb{R} \) be defined by \( F(x) = v_E(x) - v_0(x) \). Then the definition of \( g \) implies \( F(g(T_E)) = \mu T_E \), and the definition of \( T_E \) implies \( F'(g(T_E)) = 0 \). Hence,

\[
T_E = \frac{\mu T_E}{\mu} = \frac{F(g(T_E))}{\mu} = \frac{(v_E - v_0) \text{ at its first critical point}}{\max(v_0'' + (v_0')^q)}.
\]

Taking the limit as \( v_E \otimes v^* \), where \( v^* \) is the critical equilibrium, \( T_E \otimes T \), as defined in (4.12), so we have existence at least until that time.

**Corollary.** If \( u_0 \) is the constant supercritical initial condition \( u_0(x) = M \), then the solution \( u \) to (0.1) does not blow up before time
\[ T = \frac{1}{(q-2)(q-1)M^{2(q-1)}}. \]

Proof. This is an important special case of Theorem 4.6. The corresponding initial condition for (1.1) is \( v_0(x) = Mx \), so \( \max (v_0'' + (v_0')^q) = M^q \).

Let \( F = v^* - v_0 \), using the terminology of Theorem 4.6. From (2.3),

\[ F(x) = \frac{(q-1)^\alpha}{q-2} x^\alpha - Mx, \]

so

\[ F'(x) = \alpha \frac{(q-1)^\alpha}{q-2} x^{\alpha-1} - M = [(q-1)x]^{\alpha-1} - M. \]

Hence, the only critical point of \( F \) is

\[ x_0 = \frac{1}{M^{\alpha-1}} = \frac{1}{M^{\alpha-1}(q-1)}, \]

and

\[ F(x_0) = \frac{(q-1)^\alpha}{(q-2)Mq^{-2}(q-1)^\alpha} - \frac{M}{M^{\alpha-1}(q-1)} \]

\[ = \frac{1}{(q-2)Mq^{-2}} - \frac{1}{Mq^{-2}(q-1)} = \frac{1}{(q-2)(q-1)Mq^{-2}}. \]

Using Theorem 4.6, we have the stated lower bound on the blow-up time.
**Theorem 4.7.** Let $u_0$ be a supercritical initial condition for (0.1) with mass $M$, and $v_0$ its corresponding initial condition for (1.1). Suppose that for some $b_0 \in (0,1]$ and for all $x \in [b_0,1]$,

$$v_0(x) = M(x - b_0)^\alpha, \quad \alpha = \frac{q - 2}{q - 1}.$$

Then the solution $u$ of (0.1) blows up no later than

$$T = \frac{(q - 1)q^{-1}(2b_0 - b_0^2)}{2[[M(q - 2)]q^{-1} - (q - 1)q^{-2}]}.$$

Proof. By the similarity of this time bound to that proven in Theorem 4.2, one would suspect that the two theorems are based on similar subsolutions. The differences are that here we do not restrict the boundary between the two main domains of definition to be a straight line and that we allow the boundary to start closer to $x = 0$. To make this more precise, define

$$z(x,t) = \begin{cases} 
0 & \text{if } 0 = x = b(t) \\
M(x - b(t))^{\alpha} & \text{if } b(t) < x = 1
\end{cases},$$

where $b$ is some differentiable, nonincreasing, nonnegative function with $b(0) = b_0$. We want to choose $b$ so that it goes to 0 as quickly as possible while still allowing $z$ to meet the requirements of a supersolution.

Clearly, $z$ meets the boundary conditions at $x = 0$ and at $x = 1$. The main hypothesis of our theorem implies that $z$ meets the boundary condition at $t = 0$. The jump condition is also satisfied along the internal boundary $x = b(t)$ because of the infinite slope of $z$ to the right of this boundary. It is also clear that $z$ solves the requisite differential inequality where $z = 0$. Thus, all that remains to investigate is
the portion of the interior of the domain where \( z > 0 \). This region corresponds to \( \Omega_2 \) in the proof of Theorem 4.2.

For \( b(t) < x = 1 \),

\[
z_t = -M\alpha b'(t)(x - b(t))^{\alpha - 1}, \quad z_x = M\alpha(x - b(t))^{\alpha - 1}, \quad z_{xx} = M\alpha(\alpha - 1)(x - b(t))^{\alpha - 2},
\]

so

\[
z_t = z_{xx} + z_t q
\]

\[
/ - M\alpha b'(t)(x - b(t))^{\alpha - 1} = M\alpha(\alpha - 1)(x - b(t))^{\alpha - 2} + M\alpha(q(x - b(t))^{(\alpha - 1)q}
\]

\[
/ - b'(t) = \frac{\alpha - 1}{x - b(t)} + (M\alpha)^{q - 1}(x - b(t))^{(\alpha - 1)(q - 1)}
\]

\[
/ - b'(t) = \frac{\alpha - 1 + (M\alpha)^{q - 1}}{x - b(t)}
\]

\[
\iff - b'(t) = \frac{k}{1 - b(t)}, \quad k = \alpha - 1 + (M\alpha)^{q - 1}. \quad (4.13)
\]

Since we want \( b \) to go to 0 as quickly as possible, it would be best to choose \( b \) so that equality holds in (4.13). Thus, we want \( b \) to be a solution to the ordinary differential equation

\[
- b'(t) = \frac{k}{1 - b(t)}, \quad b(0) = b_0. \quad (4.14)
\]

We can solve (4.14) easily to get

\[
b(t) = 1 - \sqrt{2kt + (1 - b_0)^2}.
\]

The time \( T \) at which \( b \) becomes 0 is the upper bound on blow-up time guaranteed by this supersolution. Calculating,
\[ 0 = 1 - \sqrt{2kT + (1 - b_0)^2} \]

\[ \therefore \quad 2kT + (1 - b_0)^2 = 1 \]

\[ \therefore \quad T = \frac{1 - (1 - b_0)^2}{2k} = \frac{2b_0 - b_0^2}{2[\alpha - 1 + (M\alpha)^{q-1}]} \]

\[ = \frac{(q - 1)^{q - 1}(2b_0 - b_0^2)}{2[(M(q - 2))^{q-1} - (q - 1)^{q-2}]} \]

as claimed.

**Corollary.** If \( u_0 \) is the constant supercritical initial condition \( u_0(x) = M \),

then the solution \( u \) to (0.1) blows up no later than

\[ T = \frac{(q - 1)^{q - 1}(2b_0 - b_0^2)}{2[(M(q - 2))^{q-1} - (q - 1)^{q-2}]} \]

where

\[ b_0 = \left( \frac{q - 2}{q - 1} \right)^{q - 2} - \left( \frac{q - 2}{q - 1} \right)^{q - 1}. \]

Proof. This is a special case of Theorem 4.7. The function \( v_0(x) = Mx \) is

the initial condition for (1.1) corresponding to the given \( u_0 \). In order to apply

Theorem 4.7, we wish to find the smallest \( b_0 \) such that

\[ Mx = M(x - b_0)^\alpha \quad \text{on} \ [b_0, 1], \quad \alpha = \frac{q - 2}{q - 1}. \quad (4.15) \]

For this choice of \( b_0 \), the graphs of the functions on each side of the inequality in

(4.15) will meet at a point of tangency, say \( x_0 \); therefore,
\[ Mx_0 = M(x_0 - b_0)^\alpha, \quad (4.16) \]

and

\[ M = M\alpha(x_0 - b_0)^{\alpha - 1}. \quad (4.17) \]

Solving (4.17) for \((x_0 - b_0)\) and substituting this into (4.16) we get

\[ x_0 = \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha - 1}}. \]

Substituting this back into (4.17) we get

\[ b_0 = \left(\frac{1}{\alpha}\right)^{\frac{\alpha}{\alpha - 1}} - \left(\frac{1}{\alpha}\right)^{\frac{1}{\alpha - 1}} = \left(\frac{q - 2}{q - 1}\right)^{\frac{1}{q - 2}} - \left(\frac{q - 2}{q - 1}\right)^{\frac{1}{q - 1}}. \]

This is the value for \(b_0\) we claimed in the statement of this corollary, so using Theorem 4.7, the corollary holds.

As can be readily seen, the upper and lower bounds we have derived in this section are very different in form. It is natural to ask which of these bounds is the tightest for simple initial conditions. In Table 1, we compare our bounds on blow-up time with numerically-observed blow-up times for various values of \(q\) and various constant initial conditions \(u_0 = M\). In each case presented in this table, the lower bound is much tighter than the upper bound. In several cases, particularly those with small \(q\) and large \(M\), the lower bound is quite good.

TABLE 1

BLOW-UP TIME FOR SOLUTIONS OF (0.1)
FOR SOME SUPERCritical, CONSTANT INITIAL CONDITIONS
<table>
<thead>
<tr>
<th>Parameters for (0.1)</th>
<th>Blow-up Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>$u_0 \equiv M$</td>
</tr>
<tr>
<td>2.25</td>
<td>20.0</td>
</tr>
<tr>
<td>2.5</td>
<td>6.0</td>
</tr>
<tr>
<td>3.0</td>
<td>2.0</td>
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<tr>
<td>3.0</td>
<td>4.0</td>
</tr>
<tr>
<td>3.5</td>
<td>1.6</td>
</tr>
<tr>
<td>4.0</td>
<td>1.2</td>
</tr>
<tr>
<td>4.5</td>
<td>1.4</td>
</tr>
<tr>
<td>5.0</td>
<td>1.2</td>
</tr>
<tr>
<td>5.5</td>
<td>1.1</td>
</tr>
<tr>
<td>6.0</td>
<td>1.0</td>
</tr>
<tr>
<td>6.5</td>
<td>0.95</td>
</tr>
</tbody>
</table>

**Asymptotic Growth Rate**

In this section we consider the rate at which $\max\{u(x,t) : x \in [0,1]\} \varnothing 8$ as $t \varnothing T$, the blow-up time. In particular we consider these questions: For fixed $q$ and $u_0$, do there exist constants $\beta$ and $k > 0$ such that $(T - t)^{\beta} \cdot \max\{u(x,t)\} \varnothing k$ as $t \varnothing 8$? How do $k$ and $\beta$ depend on $q$ and $u_0$? We provide only a partial answer to these questions. We prove one analytic result and compare it with numerical evidence to determine its relative sharpness.

**Theorem 4.8.** Let $u_0$ and $q$ be given, and let $u$ be the solution to (0.1). If $u$ blows up at time $T$, then there exists a constant $c_1 > 0$ such that

$$\frac{c_1}{(T - t)^{\beta}} = \max\{u(x,t)\}, \quad \beta = \frac{1}{q - 2}$$  \hspace{1cm} (4.18)

for all $t < T$.

**Proof.** The proof is by contradiction. Suppose there is no positive constant $c_1$ for which (4.18) holds. Then if we define $w(x,t) \equiv (T - t)^{\beta}u(x,t)$, we have $\max\{w(x,t)\} \varnothing 0$ as $t \varnothing T$. 
Now

\[ w_t = (T - t)^\beta u_t - \beta(T - t)^{\beta - 1} u = (T - t)^\beta (u_x + u^q)_x - \beta(T - t)^{\beta - 1} u \]

\[ = [w_x + w^q(T - t)^{\beta(1 - q)}]_x - \beta w(T - t)^{-1}. \]

Rescaling time as \( \tau = -\ln(T - t) \),

\[ w_\tau = w_{\tau \tau} = (T - t)[w_x + w^q(T - t)^{\beta(1 - q)}]_x - \beta w, \]

with the bracketed quantity being 0 at \( x = 0 \) and \( x = 1 \). Multiplying by \( w^{2q - 1} \) and integrating with respect to \( x \) over \([0,1]\),

\[ \frac{d}{d\tau} \int 2q dx = (T - t) \int [w_x + w^q(T - t)^{\beta(1 - q)}]_x w^{2q - 1} dx - \beta \int w^{2q} dx \]

\[ = - (T - t)(2q - 1) \int [w_x + w^q(T - t)^{\beta(1 - q)}] w^{2q - 2} w_x dx - \beta \int w^{2q} dx, \tag{4.19} \]

using integration by parts. Now as we mentioned in the proofs of Theorems 3.2 and 4.6, the quantity \( |u_x + u^q| \) is bounded by some constant \( Q \). Thus, \( |w_x + w^q(T - t)^{\beta(1 - q)}| \) is bounded by \( Q(T - t)^\beta \). This tells us that the right-hand-side of (4.19) is less than or equal to

\[ (T - t)^{\beta + 1} Q(2q - 1) \int w^{2q - 2} w_x dx - \beta \int w^{2q} dx \]

\[ = (T - t)^{2\beta + 1} Q^2(2q - 1) \int w^{2q - 2} dx + (T - t)^{1 + \beta(2 - q)} Q(2q - 1) \int w^{2q - 2} dx - \beta \int w^{2q} dx \]

\[ = (T - t)^{2q - 2} w_{\tau \tau} \sigma^2 dx + Q(2q - 1) \int w^{2q - 2} dx - \beta \int w^{2q} dx \]

\[ = (T - t)^{2q - 2} \sup \{w^2\} \int w^{2q} dx + Q(2q - 1) \sup \{w^q\} \int w^{2q} dx - \beta \int w^{2q} dx. \]
Since $u$ is bounded below, by the maximum principle, and since we assumed that $w ∅ 0$ uniformly as $t ∅ T$, we can take the coefficients of the first two integrals to be arbitrarily small if we take $t$ sufficiently close to $T$. Thus, we can summarize our string of inequalities as

$$\frac{d}{dt} \int_{V} w^{2q} dx = (\varepsilon - \beta) \int_{V} w^{2q} dx \quad \text{or} \quad \frac{d}{dt} \int_{V} w^{2q} dx = 2q(\varepsilon - \beta) \int_{V} w^{2q} dx.$$

Applying an elementary theorem concerning ordinary differential inequalities from [13], there is some constant $k$ such that

$$\int_{V} w^{2q} dx = k e^{2q(\varepsilon - \beta)t} = k(T - t)^{2q(\beta - \varepsilon)},$$

so

$$\int_{V} u^{2q} dx = \frac{k}{(T - t)^{2q\varepsilon}} \Rightarrow \|u\|_{2q} = \frac{1}{k^{2q}} \frac{1}{(T - t)^{\varepsilon}}. \quad (4.20)$$

It is true that given $\varepsilon > 0$ we only showed (4.20) to be true for $t$ close to $T$, but clearly if we take $k$ large enough, (4.20) will be true for $t$ away from $T$, also, and therefore for all $t \in [0, T]$.

It should seem strange that $\|u\|_{2q}$ can be bounded above by a function of the form shown in (4.20) for $\varepsilon > 0$ arbitrarily small. Using some advanced methods from the theory of semilinear parabolic equations, we can show that, in fact, this leads to a contradiction. To do this, we define $z(x,t) \equiv v(x,t) - Mx$, where $v$ is the solution to (1.1), as usual. Then $z$ is a solution to
\[ z_t = z_{xx} + (z_x + M)^\eta \]

\[ z(0,t) = 0 = z(1,t) \quad \text{(4.21)} \]

\[ z(x,0) = z_0(x) \equiv v_0(x) - Mx. \]

As Henry does in [6], we can write (4.21) as the corresponding variation of constants formula

\[ z(t) = e^{\Delta t}z(0) + \int_0^t e^{\Delta (t-s)} (z_x + M)^\eta ds, \quad \text{(4.22)} \]

where \( z(t) \) represents \( z(\cdot,t) \) and \( \Delta \) represents the second space derivative operator.

Through estimates from [6], (4.22) gives, for some constant \( c \),

\[ \max \{u(x,t)\} = c + ct^{\frac{3}{4}} + c \int_0^t (t-s)^{-\frac{3}{4}} \|u\|_2^\eta ds. \quad \text{(4.23)} \]

Using (4.20), the right-hand side of (4.23) becomes, as \( t \cap T \),

\[ c + cT^{\frac{3}{4}} + c \int_0^T (T-s)^{-\frac{3}{4}} \frac{k^2}{(T-s)^\eta} ds, \]

which is finite if the singularity is integrable, i.e., if \( q \epsilon + 3/4 < 1 \). Since we could take \( \epsilon \) arbitrarily small, this is the case. Thus, (4.23) and (4.20) tell us that \( u \) is uniformly bounded as \( t \cap T \). But this contradicts \( T \) being the blow-up time. This contradiction tells us there must exist some \( c_1 \) such that (4.18) holds.

Figures 2 through 11 in the appendix provide information on the asymptotic
blow-up rate from numerical experiments. These graphs plot $t$ vs. $(u(0,t))^{-(q-2)}$ for various constant initial conditions $u_0$ and various exponents $q$. Note that according to Theorem 4.8 each of these graphs should be bounded above by some line with finite negative slope passing through the graph's intersection with the $t$-axis. If the estimate given in Theorem 4.8 is tight, i.e., if $\max \{u(x,t)\}$ has an upper bound of the same form as the given lower bound with a different constant in the numerator, then the graph should be bounded below by a similar line with negative slope. For some values of $q$ and $M$ the graph does appear to be tangent to the $t$-axis, suggesting that our lower bound is not sharp.
CHAPTER 5
OPEN QUESTIONS

Research on the topic of this thesis is an ongoing project of Dr. Peter Bates, Dr. Nicholas Alikakos, and the author. This topic has been by no means exhausted. In this chapter, we briefly outline some of the more interesting open questions pertaining to (0.1).

First, what generalizations to the nonlinearity \( u^q \) in (0.1) can be made without changing the various results on existence, stabilization, and blow-up? If \( f(u) = ru^q \) for all \( u \), and if \( f \) is sufficiently smooth, it is not hard to verify that the basic blow-up theorem remains true. The only difference is that the critical mass one uses is the critical mass from the original (0.1), not the new (0.1) where \( f(u) \) replaces \( u^q \). This is somewhat undesirable. Similarly, if \( f(u) = ru^q \) for all \( u \), a version of the main stabilization theorem carries over. What other extensions of our theorems can be made for such \( f \), and if \( f \) is generalized further can significant information about the corresponding solution be obtained?

Second, let \( S \) be the set of all initial conditions to (0.1) that lead to stabilization, and let \( B \) be the set of all such initial conditions that lead to finite time blow-up. Are \( S \) and \( B \) open in \( C([0,1],(0,\infty)) \)? Are they closed? Is their union all of \( C([0,1],(0,\infty)) \)? This last question is equivalent to the question of the existence of a solution that exists for all time but becomes unbounded as \( t \not\in \mathbb{R} \). In particular, does the constant initial condition with critical mass exist for all time? Answering these questions is one of our major efforts at the present time.

Third, is the bound on the blow-up rate proven in Theorem 4.8 sharp? The discovery of any upper bound on the blow-up rate similar in form to our lower bound would be useful, even if it uses a different exponent \( \beta \). Also, can the spatial profile at the blow-up time be characterized similarly? For example, if \( T \) is the
blow-up time, does $u(x,T)$ converge, in some sense, to the critical equilibrium as $x \not\to 0$? Figure 12 shows that for $x$ not close to 0, the spatial profile at the blow-up time can retain much of the character of the initial condition. This graph was one of the things that convinced us that subcritical mass could lead to blow-up, since the left side of the space domain seemed to be slow to “learn” about what was happening on the right side, and vice versa.

Fourth and finally, what numerical schemes could be used effectively on this problem? Berger and Kohn [5] have suggested a complex method which iteratively rescales the equation near the blow-up point for modelling similar nonlinear parabolic partial differential equations. If access to a supercomputer were available, adapting this scheme to (0.1) might prove fruitful.
Fig. 1. \( y \equiv u(0,t)^{-(q - 2)}, \quad q = 3, \quad u_0 = 7.5(1 - x)^5. \)
Fig. 2. $y \equiv u(0,t)^{-q-2}$, $q = 2.25$, $u_0 = 20$.

Fig. 3. $y \equiv u(0,t)^{-q-2}$, $q = 2.5$, $u_0 = 6$. 
Fig. 4. \( y = u(0,t)^{-(q-2)}, \quad q = 3, \quad u_0 = 2. \)

Fig. 5. \( y = u(0,t)^{-(q-2)}, \quad q = 3.5, \quad u_0 = 1.6. \)
Fig. 6. \( y \equiv u(0,t)^{(q-2)}, \quad q = 4, \quad u_0 = 1.2. \)

Fig. 7. \( y \equiv u(0,t)^{(q-2)}, \quad q = 4.5, \quad u_0 = 1.4. \)
Fig. 8. \( y \equiv u(0,t)^{(q-2)} \), \( q = 5 \), \( u_0 = 1.2 \).

Fig. 9. \( y \equiv u(0,t)^{(q-2)} \), \( q = 5.5 \), \( u_0 = 1.1 \).
Fig. 10. \( y = u(0, t)^{(q - 2)}, \quad q = 6, \quad u_0 = 1. \)

Fig. 11. \( y = u(0, t)^{(q - 2)}, \quad q = 6.5, \quad u_0 = 0.95. \)
Fig. 12. Space profile at blow-up time, \( q = 3 \), \( u_0 = 4 + 3\sin(4\pi x) \).
REFERENCE LIST


Stabilization and Blow-up of Solutions of a Nonlinear Parabolic Equation

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Department of Mathematics
M. S. Degree, December 1988

ABSTRACT

In this thesis, we describe the behavior of solutions of a parabolic partial differential equation with a nonlinear advection term. The space domain is a bounded one-dimensional interval, and no-flux boundary conditions are imposed at the endpoints. Under suitable conditions, we show that solutions stabilize. Under different conditions, solutions are shown to blow up in finite time. The nature of this blow-up is partially described.

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