5) Let \( \lambda \) be an eigenvalue of \( A \) and let \( x \) be an eigenvector belonging to \( \lambda \). Use mathematical induction to show that \( \lambda^m \) is an eigenvalue of \( A^m \) and \( x \) is an eigenvector of \( A^m \) belonging to \( \lambda^m \) for \( m = 1, 2, \ldots \).

Proof. let \( m = 1 \). Then \( \lambda \) is an eigenvalue of \( A \) by assumption with \( x \) as an eigenvector.

Now suppose that the statement holds true for \( m = n \). Then

\[
A^n x = \lambda^n x
\]

and

\[
A^{(n+1)} x = A A^n x = A \lambda^n x = \lambda^n A x = \lambda^{n+1} x
\]

thus \( \lambda^{n+1} \) is an eigenvalue of \( A^{n+1} \) with \( x \) as an eigenvector.

\( \square \)

8) Answer in the back of the book.

9) Show that \( A \) and \( A^T \) have the same eigenvalues. Do they necessarily have the same eigenvectors? Explain.

Proof. Let \( \lambda \) be an eigenvalue of \( A \). Then

\[
\det A - \lambda I = 0
\]

and thus

\[
\det A^T - \lambda I = \det (A - \lambda I)^T = \det A - \lambda I = 0
\]

They may not have the same eigenvectors, though they may be the same vector in opposite order.

\( \square \)

15) Let \( A \) be an \( n \times n \) matrix and let \( \lambda \) be an eigenvalue of \( A \). If \( A - \lambda I \) has rank \( k \), what is the dimension of the eigenspace corresponding to \( \lambda \)? Explain.

Proof. Then \( A - \lambda I \) will have nullity equal to the dimension of the eigenspace corresponding to \( \lambda \) since if a vector is in the nullspace of \( A - \lambda I \) it will be an eigenvector. Thus the dimension is \( n - k \).

\( \square \)

18) Let \( Q \) be an orthogonal matrix.

(a) Show that \( \lambda \) is an eigenvalue of \( Q \) then \( |\lambda| = 1 \).

(b) Show that \( |\det Q| = 1 \).

Proof. It is easier to do this problem backwards. Note

\[
QQ^{-1} = I
\]

\[
\det Q \det Q^{-1} = 1
\]

\[
\det Q \det Q^T = 1
\]

\[
\det Q^2 = 1
\]

\[
\det Q = \pm 1
\]
Then $Q^T = Q^{-1}$ and $\det Q - \lambda I = 0$. Thus $\det Q = \det \lambda I = \lambda = \pm 1$ by (a). Thus $\lambda = \pm 1$.

22) Let $B = S^{-1}AS$ and let $x$ be an eigenvector of $B$ belonging to an eigenvalue $\lambda$. Show that $Sx$ is an eigenvector of $A$ belonging to $\lambda$.

Proof.

\begin{align*}
Bx &= \lambda x \\
S^{-1}ASx &= \lambda x \\
ASx &= \lambda Sx
\end{align*}

by multiplying both sides on the left by $S$. 

□