5) let $A$ be a nondefective $n \times n$ matrix with diagonalizing matrix $X$. Show that the matrix $Y = (X^{-1})^T$ diagonalizes $A^T$.

**Proof.**

$$(X^{-1})^T A^T (X^{-1}) = X^T A^T (X^{-1})^T = (X^{-1}AX)^T$$

which is a diagonal matrix.

\[\square\]

13) Let $A$ be a diagonalizable matrix and let $X$ be the diagonalizing matrix. Show that the column vectors of $X$ that correspond to nonzero eigenvalues of $A$ form a basis for $R(A)$.

**Proof.** They are linearly independent by construction, and clearly they are in the column space of $A$. Thus we must just show that the number of such columns is equal to the rank of $A$. Since they are linearly independent, the number must be less than or equal to the rank. Since the number of nonzero eigenvalues is equal to the rank, the number of independent columns is equal to the rank. Thus it is a basis.

\[\square\]

17) Let $A$ be a diagonalizable $n \times n$ matrix. Prove that if $B$ is any matrix that is similar to $A$, then $B$ is diagonalizable.

**Proof.** Let $X$ diagonalize $A$ and $A = S^{-1}BS$. Then

$$X^{-1}AX = X^{-1}S^{-1}BSX = (SX)^{-1}B(SX)$$

is diagonal.

\[\square\]

18) Show that if $A$ and $B$ are two $n \times n$ matrices that both have the same diagonalizing matrix $X$, then $AB = BA$.

**Proof.** First you need to show that diagonal matrices commute under multiplication.

Then if

$$X^{-1}AX = D$$

and

$$X^{-1}BX = E$$

, we have

$$X^{-1}ABX = DE = ED = X^{-1}BAX$$

thus

$$AB = BA$$

\[\square\]