The Principle of Mathematical Induction

Mathematical induction is the name given to a powerful mathematical principle used to verify statements that in some way depend on a positive integer. Examples of such statements are:

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

(b) If $A_1, \cdots, A_k$ are $n \times n$ matrices,

$$(A_1 A_2 \cdots A_k)^T = A_k^T \cdots A_2^T A_1^T.$$

(c) If $x_1, x_2, x_3, \ldots,$ are solutions of $Ax = 0$, then for each positive integer $m, \quad x_1 + \cdots + x_m$ is a solution of $Ax = 0$.

The positive integers in these statements are n, k, and m respectively, and such statements carry with them the implicit claim of validity for every positive integer. For example, letting $n = 5$ in statement (a) yields

$$1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2}$$

which is correct since both sides equal 15.

Statements such as $(a), (b), (c)$ must first be discovered somehow before they can be verified by mathematical induction. To illustrate how such a statement is discovered, suppose we must simplify $(ABC)^T$.
First regroup and write

$$(ABC)^T = ((AB)C)^T.$$

Now apply Theorem 1.4.9(d)[1], page 45, $(AB)^T = B^T A^T$, twice to obtain

$$((AB)C)^T = C^T (AB)^T = C^T B^T A^T.$$

Note that in the first application of Theorem 1.4.9(d), we regard AB as a single matrix.

[1]This handout was originally written for another textbook. The page numbers and theorem numbers are not the same in our textbook, but we do have the same theorems.

Observing how the transpose affects the product of two or three matrices

$$(AB)^T = B^T A^T$$

$$(ABC)^T = C^T B^T A^T,$$

it is easy to guess that

$$(A_1 A_2 \cdots A_k)^T = A_k^T \cdots A_2^T A_1^T$$

for any positive integer k.

Statement (c) can be derived in a straightforward manner. The identity (a) is usually derived in college algebra classes and that will not be repeated here.

We now discuss how such statements can be verified for any positive integer.

The Principle of Mathematical Induction:

If a statement involving the positive integer k can be proved to have the following properties:

(i) The statement is correct for $k = 1$ (substitute $k = 2$ if the statement is not meaningful for $k = 1$.),

(ii) If k is any integer for which the statement is true, then the statement is also true for the next integer $k + 1$,

then the statement is true for every positive integer k.

This principle is often abbreviated symbolically as

Let P_k be a statement depending on the positive integer k.

(i) If P_1 is true and (ii) P_k is true $\Rightarrow P_{k+1}$ is true,

then P_k is true for every positive integer k.

As an example, we verify statement (a) above by mathematical induction

(a) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

$1 = \frac{1(1+1)}{2}$, so the statement is correct for $n = 1$.

(Note that $1 + 2 + \cdots + n$ reduces to just 1 if $n = 1$.)

Assume $n = k$ is an integer for which statement (a) is true, that is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

In this case the procedure for showing that statement (a) is true for $n = k+1$ is to add $k + 1$ to both sides of this last equation.
We have

$$1+2+\cdots k+(k+1) = \frac{k(k+1)}{2}+(k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Equivalently,

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)[(k+1)+1]}{2},$$

which is statement (a) with $n = k + 1$.
The principle of mathematical induction can now be applied to conclude that statement (a) is true for every positive integer n.

We now verify statement (b).
(b) $(A_1 A_2 \cdots A_k)^T = A_k^T \cdots A_2^T A_1^T$

$k = 1:\quad A_1^T = A_1^T$ which is true.

$k = 2:\quad (A_1 A_2)^T = A_2^T A_1^T$ which is Theorem 1.4.9(d).

Assume $(A_1 A_2 \cdots A_k)^T = A_k^T \cdots A_2^T A_1^T$ for any $n \times n$ matrices $A_1, \ldots, A_k$.
Then $(A_1 A_2 \cdots A_k A_{k+1})^T = ((A_1 A_2 \cdots A_k) A_{k+1})^T$.
Now think of the product inside the parentheses as the product of two matrices, $A_1 A_2 \cdots A_k$ and A_{k+1}. By Theorem 1.4.9(d),
$((A_1 A_2 \cdots A_k) A_{k+1})^T = A_{k+1}^T (A_1 \cdots A_k)^T$.
By the induction hypothesis, this last expression equals $A_{k+1}^T A_k^T \cdots A_1^T$. Therefore,

$$(A_1 A_2 \cdots A_k A_{k+1})^T = ((A_1 A_2 \cdots A_k) A_{k+1})^T = A_{k+1}^T (A_1 \cdots A_k)^T = A_{k+1}^T A_k^T \cdots A_1^T$$

Thus, by induction, statement (b) is true for every positive integer m.

This proof is somewhat drawn out in order to amplify on the various steps involved.

We now verify statement (c) and include only essential steps:
Since x_1 is a solution, $Ax_1 = 0$ and statement (c) is true for $m = 1$. Assume that $x_1 + \cdots + x_k$ is a solution of $Ax = 0$, that is, $A(x_1 + \cdots + x_k) = 0$. Then

$$A(x_1 + x_2 + \cdots + x_k + x_{k+1}) = A(x_1 + x_2 + \cdots + x_k) + A(x_{k+1})$$

by Theorem 1.4.1d, which is $0 + 0$ by the induction hypothesis and the given fact that x_{k+1} is a solution. Therefore, $x_1 + x_2 + \cdots + x_k + x_{k+1}$ is a solution of $Ax = 0$, which completes the proof of (c) for all positive integers k.

Here is a final example.
Let A and B be $n \times n$ matrices that commute with one another. Prove that

$$A^k B = BA^k$$

for every positive integer k.

$k = 1:$ $A^1 B = BA^1$ or $AB = BA$ which is correct since A and B commute.

Assume $A^k B = BA^k$.

Then

$A^{k+1}B = (AA^k)B = A(A^k B) = A(BA^k) = (AB)A^k = (BA)A^k = B(AA^k) = BA^{k+1}$.

By induction $A^k B = BA^k$ for every positive integer k.

Upon first becoming acquainted with the principle of mathematical induction, some view it with suspicion. It sometimes *appears* that one is not really doing anything that merits the assertion that a given statement is true for every positive integer k. Here is an argument to support its validity.

Let P_k be a statement depending on the positive integer k.
Assume i) P_1 is true and ii) P_k is true $\Rightarrow P_{k+1}$ is true.

Now either P_k is true for every positive integer k, or there is some integer for which it is false. Assume the latter. For concreteness sake assume P_{17} is false. Then P_{16} is also false because P_{16} is true $\Rightarrow P_{17}$ is true. But since P_{16} is false, by the same argument P_{15} is false. Then P_{14} is false, P_{13} is false, etc. In 17 steps we have P_1 is false contradicting assumption i). (Similarly if P_k is false for some other k, a contradiction is reached in k steps.) So it must be the case that P_k is true for every positive integer k.