A Brief Introduction to Logic

In calculus, the main thrust in our use of logic is to prove and use theorems. The purpose of a theorem is to prove a property in a general situation, to save us work in more specific circumstances. For example, we consider the case of Sophie, my father's Irish Setter. She was born in a litter of about eight puppies, each of which is also (surprisingly enough) an Irish Setter.

We would like to conclude that each of these animals is a dog, which of course they are. We could verify this by checking that each of the eight exhibits the properties that characterize dogs. However, that is a lot of work (more than I am willing to go to, anyway) and we have to do the same steps for each specimen. Rather, we prove a theorem (well, it's proved somewhere, anyway):

Theorem 1: Every Irish Setter is a dog.

Since Sophie and her siblings are all Irish Setters, we cite Theorem 1 and conclude that they are all dogs—much easier.

Though Irish Setters will not often come up in our discussions of calculus, the basic principle illustrated will be crucial. In calculus, we will frequently show that functions have certain properties that are easy to verify and then use theorems to show that they must also have other properties that are harder to show directly.

In English, we have many ways of stating a fact that are equivalent. For example, here are a few of the many ways that Theorem 1 can be stated:

1. All Irish Setters are dogs.
2. Whenever a thing is an Irish Setter, that thing is a dog.
3. If X is an Irish Setter, then X is a dog.
4. (X is an Irish Setter) $\Rightarrow$ (X is a dog)
5. "X is an Irish Setter" implies that X is a dog.
6. All non-dogs are not Irish Setters.

Each of these statements is an implication—just a way of telling us that once we show that something is an Irish Setter, we can immediately conclude that it is also a dog.

When we have an implication, it is sometimes worthwhile to ask if the statement is true if we reverse the order. Take Statement 1, for example:

Statement 1: Every dog is an Irish Setter.

This statement is exactly the same as Theorem 1, except that we have traded the places of the two subjects. When we switch the order in this way, the new statement is called the *converse* of the original statement. Usually, the converse of a statement is not true just because the statement is, as in this case. After all, everybody knows that Snoopy is a beagle—not an Irish Setter. We must always be careful to use a theorem properly and not accidentally use its converse.

Another manifestation of a statement is called its *contrapositive*. The sixth restatement of Theorem 1 (All non-dogs are not Irish Setters) is an example. It is constructed from the statement by taking the converse and negating both subjects. The contrapositive of a statement is exactly equivalent to the statement itself. That is, they are each true under exactly the same circumstances. The *inverse* is just another name for the converse of the contrapositive (Say that five times fast). The converse and the inverse are also equivalent to one another.

Example.
Statement: If a cow was flung from a catapault, it is udderly terrified.
Converse: If a cow is udderly terrified, it was flung from a catapult.
Contrapositive: If a cow is not udderly terrified, it was not flung from a catapult.
Inverse: If a cow was not flung from a catapult, it is not udderly terrified.

Just as in Statement 1, this is an example of a statement that is not equivalent to its converse. After all, there are other ways of udderly terrifying a cow than just flinging it from a catapult. You could also fling it from a trebuchet, for example.

When both the implication and its converse are true, we say that this is an *if and only if* condition. The symbol for this is $\iff$.

Example.
We know that if we run across a horse that can talk, then we must conclude that it is Mr. Ed, since he is the only such beast. So, we conclude:

If a horse can talk, it must be Mr. Ed.

On the other hand, if we meet Mr. Ed we know that he can talk (though he won't unless we are Wilbur). So:

If a horse is Mr. Ed, that horse can talk.

Since each condition implies the other, we combine these two implications:

A horse can talk if and only if that horse is Mr. Ed.

"If and only if" implications will be our favorites, since they allow us to move back and forth between equivalent things without too much thought or effort. Unfortunately, though, it is too much to ask for very many of these.

Quantifiers.

Often, it is necessary to quantify the subjects that are included in our statements. To some degree, we actually state the number of subjects to which a statement applies. The two most common quantifiers we use, in common forms, are:

1. (a) "there exists"
   (b) "there is"
2. (a) "for each"
   (b) "for every"
   (c) "for all"

When we use "there exists," we mean that there is *at least* one of whatever we are talking about. For example, when we say that there exists a number greater than 1, by no means do we mean that there is exactly one, for there are infinitely many. When we wish to say that there exists exactly one and no more than one, we will say so—probably with the word 'unique.' For example, there is exactly one number, called zero, that doesn't change sums: $a + 0 = a$ and no other number has this property.

Another quantifier we use frequently is "for every." When we say "for every," we mean *every.* That is, there is not a single one that fails to satisfy what we have proved. We now turn to rock and roll (specifically Poison) for logical statements. In the classic metal ballad "Every Rose has its Thorn," the band would have us believe that "every cowboy sings a sad, sad song." We can quantify this statement as:

For every cowboy there is a sad,
sad song that he sings.

Note that they are not saying that there is only one song for each cowboy or that each has his own and doesn't share. Perhaps some cowboys sing multiple sad, sad songs. Some pair of cowboys may sing the same song. They simply mean that there is no cowboy that doesn't sing a sad, sad song. Of course, this is not a theorem—just a stereotype. So, we can question the validity of the assertion, but if it is true that every cowboy sings a sad song, then if we meet a cowboy, we know without a doubt that he sings a sad, sad song.

Negation.

One of the more troublesome things for people to remember is what it means to negate a statement. For example, we consider a statement from Aaron Neville:

Everybody plays the fool.

What would it mean to negate this statement? It is tempting, perhaps, to say that the opposite of "for every" is "for no" so that the negation of this statement would be: *Nobody plays the fool.* However, the statement becomes false as soon as we find a single instance in which a person exists without ever playing the fool. So, the negation of "For every X we know that Y" is

For some X we know that NOT Y.

The negation of the previous statement is really:

Somebody does not play the fool.

Similarly, the phrase "There is sunshine in my soul today" can be negated in either of these ways:

There is no sunshine in my soul today

All sunshine is not in my soul today.

AND / OR

When we use the word "and," we mean that both of the conditions (before and after the "and") must be true simultaneously before we can proceed. For example,

If you are sick and tired of doing math,
then you are probably normal.

In this case, we use 'and' so that neither being sick nor being tired is sufficient to conclude that you are probably normal. Both conditions must be present simultaneously to make that conclusion.

"OR" has a different usage from what is common in English. When Glinda asks Dorothy "Are you a good witch or a bad witch?" she means that Dorothy must be one, but can't be the other. When we list two conditions separated by OR we mean for one to be true, or the other, or both.

Lava lamps are being given out at
Zions Bank to people who open a checking
account or a savings account.

So, what happens if you open both? Surely, they won't deny you your precious lava lamp for opening up both.

Exercises.

1. Negate the following statements:
   (a) Today is Thursday.
   (b) $2 + 1 = 3$
   (c) All dogs go to heaven.
   (d) For every ϵ there exists a δ.
2. Is it true that $x > 0 \iff x^2 > 0$? Explain your answer.
3. Which statements below are equivalent to the statement

   If x is positive, then x^2 is positive, too.

   (a) $x > 0 \Rightarrow x^2 > 0$
   (b) If x^2 is positive, then x is positive.
   (c) If x^2 is not positive, then x is not positive.
   (d) If x is not positive, then x^2 is not positive.
4. Assume that the following statement is true:

   (0): If I have hit a fly with a fly swatter, the fly is dead.

   In each of parts (a) through (d), you are given a fact. For each part, evaluate the fact, together with statement (0), and write down any conclusions that can be drawn. If no conclusion can be made, say so.

   (a) A fly is dead.
   (b) I hit this fly with the fly swatter.
   (c) This fly is alive.
   (d) I did not hit that fly with a fly swatter.