Name:________________________________

Student ID:______________________________

Instructor: Jason Grout

Math 343-1 (Linear Algebra with Applications)
Final Exam
22 June 2006

Instructions:

- Notes, books, and calculators are not allowed.
- For multiple choice and true/false questions, put your answer in the blank provided.
- For questions which require a written answer, show all your work. In order to earn full credit, you will need to *neatly* show the work necessary to justify your answer on these pages.
- If an answer box is provided, put your answer in it.
- Simplify your answers.
- Should you have need for more space than is provided to answer a question, use the blank sides of adjacent pages and indicate this fact. Do not attach extra pages.
- Please do not talk about the test with other students until after the last day to take the exam.

For Instructor use only.

#	Possible	Earned
MC	25	
6	6	
7	6	
8	10	
9	5	
10	17	
Sub	69	

#	Possible	Earned
11	6	
12	5	
13	15	
14	10	
15	30	
Sub	66	
Total	135	

1 Multiple Choice

_____ 1. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{vmatrix} =$

(a) -6 (b) -4 (c) -2
(d) 0 (e) 2 (f) 4
(g) 6 (h) 12 (i) none of the above

_____ 2. Which set of vectors is linearly dependent in P_4?

(a) x, x^2, x^3
(b) $1 + x^2, 1 + 2x + x^2, x - x^2, x + x^2, 1 + 5x^3$
(c) $x^3 + 2x^2, 2x^2 + x, x + 1, x^3 + 4x^2 + 4$
(d) $1 + x, 1 - x, x^3$
(e) $1 + 2x + 3x^2 + 4x^3, 1, x, x^2$
(f) $x^2 + x + 1, x + 1, 1$

_____ 3. Which matrix has rank 2?

(a) $\begin{bmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ -3 & -6 & -12 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & 1 & 2 \\ 4 & 1 & 2 & 5 \\ -3 & 2 & -1 & -3 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 & 4 \\ 1 & 0 & 1 & 1 & 3 \end{bmatrix}$

______ 4. If

$$C = \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right] \quad B = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right], \text{ and } L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-2y \end{pmatrix},$$

(where L gives a linear transformation with respect to standard coordinates). Which of the following is the matrix representing L from $\mathbb{R}^3$ with basis C to $\mathbb{R}^2$ with basis B?

(a) $\frac{1}{2}\begin{bmatrix} -3 & -5 & 1 \\ -2 & -4 & 2 \end{bmatrix}$ (b) $\frac{1}{2}\begin{bmatrix} -2 & -4 & 2 \\ -3 & -7 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & -3 & -4 \\ 0 & -4 & -6 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & -5 & -6 \\ -1 & -3 & -4 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 4 & 6 \end{bmatrix}$ (f) $\begin{bmatrix} 3 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$

(g) none of the previous answers

______ 5. If

$$S = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\},$$

what is the dimension of $S^{\perp}$?

(a) 0 (b) 1 (c) 2 (d) 3

2 Computation and Short Answer

For problems 6–8, list one characteristic or property in each answer box.

6. (6 points) Let A and B be similar matrices. List three different characteristics that are guaranteed to be shared by A and B. (for example, A and B have the same nullity, but don't use this as one of your answers). Listing trivial shared characteristics may not earn full credit.

7. (6 points) Let A and B be row-equivalent matrices. List three different characteristics that are guaranteed to be shared by A and B. Listing trivial shared characteristics may not earn full credit.

8. (10 points) Let A be an $n \times n$ matrix. List 5 different statements about A that are equivalent to the statement "A is invertible". Listing trivial equivalent statements may not earn full credit.

9. (5 points) If A and B are $n \times n$ matrices for which $\det(A) = 2$ and $\det(B) = 5$, what is $\det(A^T B^{-1})$?

10. (17 points) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Find the eigenvalues and bases for the associated eigenspaces of A. Put your eigenspace bases in the empty answer box and clearly label which eigenvalue each is associated with. As always, show your work.

$\lambda_1 =$	$\lambda_2 =$	$\lambda_3 =$

11. (6 points) Let

$$\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ be the subspace spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Find the point $\mathbf{p}$ in W that is closest to $\mathbf{y}$. You may write the point $\mathbf{p}$ in standard coordinates or in U coordinates (indicate which you did). (Hint: $U = [\mathbf{u}_1, \mathbf{u}_2]$ is an orthonormal basis for W.)

$\mathbf{p} =$

12. (5 points) Let $S = \{(x_1, x_2)^T \mid x_1 \leq x_2\}$. Prove or disprove: S is a subspace of $\mathbb{R}^2$.

3 Always/Sometimes/Never

13. (15 points) For each of the following statements, if the statement is always true, write "Always" in the blank and write a short justification. If the statement is never true, write "Never" in the blank and write a short justification. If the statement is sometimes true and sometimes false, write "Sometimes" in the blank and give an example when the statement is true and an example when the statement is false.

_____(a) If A and B are $n \times n$ matrices, then $\text{rank}(A + B) = \text{rank}(A) + \text{rank}(B)$.

_____(b) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$.

_____(c) If S is an invertible symmetric matrix, then S^{-1} is symmetric.

_____(d) The product of two orthogonal matrices is orthogonal.

_____(e) If A and B are row-equivalent, then A and B have the same eigenvalues.

4 The rest of the test

14. (10 points) Here's a useful fact from section 6.4, which you really ought to read sometime this week: Every symmetric matrix is diagonalizable, and furthermore, it is diagonalizable by an orthogonal matrix. Use this useful fact to prove one of (a) or (b). Cross out the problem you do not want graded.

    (a) Let A be a symmetric matrix for which each eigenvalue is 0 or 1. Show that $A^2 = A$.

    (b) Let A be a symmetric matrix for which each eigenvalue is -1 or 1. Show that A is an orthogonal matrix.

15. (30 points) You may work any 3 of the following 4 problems. Space is provided on the next two pages. Give careful and complete arguments for each. Spare no words in explaining exactly what you are thinking. Cross out the problem you do not want graded. Each problem is worth 10 points.

   (a) A square matrix A is called *nilpotent* if there is a positive integer k such that $A^k = \mathbf{0}$. Prove or disprove: Every nilpotent matrix is singular.

   (b) Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ be linearly independent vectors in $\mathbb{R}^n$ and let A be an invertible $n \times n$ matrix. Let $\mathbf{y}_i = A\mathbf{x}_i$ for $i = 1, \ldots, k$. Show that $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are linearly independent.

   (c) Prove or disprove: If U is an orthogonal $n \times n$ matrix, then $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for every $\mathbf{x} \in \mathbb{R}^n$.

   (d) Let A be an $m \times n$ matrix. Show that the columns of A span $\mathbb{R}^m$ if and only if the rows of A are linearly independent. (Reminder: since this is an "if and only if" statement, you need to show both the statement and its converse are true.)

Put your name in the upper right corner of this page.

1. (Bonus) Let A and B be $n \times n$ matrices. Prove that AB and BA have the same eigenvalues (Hint: Consider the cases $\lambda \neq 0$ and $\lambda = 0$ separately.)

2. (Bonus) Is it possible to find a pair of 3-dimensional subspaces V and W of R^5 such that $V \cap W = \{\mathbf{0}\}$? Prove your answer.