In order to disprove the assertion that all crows are black, one white crow is sufficient.

- William James
A **theorem** is a mathematical statement that provides a conclusion, provided that a set of specific assumptions holds.

**Theorem**
If certain assumptions hold, then a specific conclusion will also hold.

The proof of a theorem consists of statements, each of which is

- an assumption,
- a conclusion, which clearly follows from an assumption or a previously proved result.
To disprove a mathematical statement, it is often enough to provide a single *counterexample* showing that the statement is false.

**Example:** To disprove the statement

\[
\text{All prime numbers are odd.}
\]

it suffices to find a single prime number which is even (e.g. 2).

Showing that a mathematical statement is true requires a *proof*.

**Example:** The statement

\[
\text{Any even number } > 2 \text{ can be written as the sum of two primes.}
\]

would require a proof (if true), or a counterexample (if false).
Types of mathematical statements

Single implication:

If $P$ holds, then $Q$ holds (written $P \implies Q$).

Note: $P \implies Q$ is not the same as $Q \implies P$ (one may be true, while the other may be false).

Example: If $n \in \mathbb{N}$ is divisible by 6, then $n$ is divisible by 3.

on the other hand

If $n$ is divisible by 3, then $n$ may or may not be divisible by 6.
Proof types - direct proof

Start with the assumptions, and try to reach the desired conclusion directly.

Examples:

**Theorem**
If \( \{v_1, \ldots, v_p\} \) is a set of \( p \) vectors in \( \mathbb{R}^n \), and \( n < p \), then the set \( \{v_1, \ldots, v_p\} \) is linearly dependent.

**Theorem**
If a set of vectors \( \{v_1, \ldots, v_p\} \) contains the zero vector \( \mathbf{0} \), then \( \{v_1, \ldots, v_p\} \) is linearly dependent.
The statement \((P \implies Q)\) is equivalent to \((\neg Q \implies \neg P)\).

**Examples:** If \(n \in \mathbb{N}\) is divisible by 6, then \(n\) is divisible by 3.

is the same as

If \(n\) is not divisible by 3, then \(n\) is not divisible by 6.

**Proof by contrapositive:** Prove \((\neg Q \implies \neg P)\) instead of \((P \implies Q)\).

Start by assuming \((\neg Q)\), and try to prove \((\neg P)\).

**Theorem**
If the product \(r \cdot s \geq 100\), then either \(r \geq 10\) or \(s \geq 10\).
Proof by contradiction: Start by assuming both $P$ and (not $Q$) and try to derive a contradiction.

Since having both $P$ and (not $Q$) leads to a contradiction, whenever $P$ is true $Q$ must also be true.

Theorem
The sum of a rational number and an irrational number is irrational.
Types of mathematical statements

If and only if:

$P$ if and only if $Q$ (written $P \iff Q$) is the same as

both $(P \implies Q)$ and $(Q \implies P)$.

To prove $P \iff Q$, we must prove both $P \implies Q$ and $Q \implies P$.

Theorem

The set $\{v_1, \ldots, v_p\}$ is linearly dependent if and only if some $v_j$ can be written as a linear combination of the other vectors.
Suppose we want to prove that something is true for all natural numbers $n \geq 1$.

**First step:** Prove that it is true for $n = 1$.

**Inductive step:** Assume that the statement is true for some $n \geq 1$. Prove that it must also be true for $n + 1$.

Then the statement must be true for all $n \geq 1$.

**Theorem**
For any $n \in \mathbb{N}$, we have $1 + 2 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2}$.