Math 316 Hwk 1

Problem 1. Assume that \( \{ v_1, v_2, \ldots, v_n \} \) spans the vector space \( V \), and let \( v \) be any other vector in \( V \). Show that \( \{ v, v_1, v_2, \ldots, v_n \} \) is linearly dependent.

Problem 2. Let \( x_1, x_2, \ldots, x_k \) be linearly independent vectors in \( \mathbb{R}^n \), and let \( A \) be a nonsingular \( n \times n \) matrix. Define \( y_i = Ax_i \) for \( i = 1, \ldots, k \). Show that \( y_1, y_2, \ldots, y_k \) are linearly independent.

Problem 3. Let \( X \) be a subspace of \( W \) and \( L : V \rightarrow W \) be a linear transformation. The preimage of \( X \), denoted \( L^{-1}(X) \), is defined by

\[
L^{-1}(X) = \{ v \in V \mid L(v) \in X \}.
\]

Prove that \( L^{-1}(X) \) is a subspace of \( V \).

Problem 4. Prove that the \( \ell^p \) norms satisfy the following inequalities:

(a). \( \| x \|_2 \leq \| x \|_1 \leq \sqrt{n} \| x \|_2 \).

(b). \( \| x \|_\infty \leq \| x \|_2 \leq \sqrt{n} \| x \|_\infty \).

Hint: Use the Cauchy-Schwarz inequality.

Problem 5. Let \( d(x, y) \) be a metric on a vector space \( V \). Show that

\[
\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}
\]

is also a metric.

Problem 6. Let \( V, W, X \) be vector spaces. Assume that \( L : V \rightarrow W \) and \( M : W \rightarrow X \) are linear transformations. Prove that \( M \circ L : V \rightarrow X \) is a linear transformation.

Problem 7. A set \( C \subset \mathbb{R}^n \) is convex if for each \( x, y \in C \), we have that \( \lambda x + (1 - \lambda)y \in C \), whenever \( 0 \leq \lambda \leq 1 \).

(a). Give the geometric interpretation of a convex set.

(b). Provide an example of a set that is convex and one that isn’t.
Problem 8. The convex hull of $S \subset \mathbb{R}^n$, denoted $\text{co}(S)$, is the set of all convex combinations of elements of $S$, that is, the set of all linear combinations

$$a_1x_1 + \cdots + a_nx_n$$

such that $a_1 + \cdots + a_n = 1$, each $a_j \geq 0$, and each $x_j \in S$, $j = 1, \ldots, n$, $n \in \mathbb{N}$. Prove that a convex set $C$ contains every convex combination of its elements, or in other words $\text{co}(C) \subset C$.

Problem 9. Let $\{C_\alpha\}_{\alpha \in J}$ be a collection of convex sets for some indexing set $J$. Prove that $\bigcap_{\alpha \in J} C_\alpha$ is convex.

Problem 10. Let $S \subset \mathbb{R}^n$. Prove that $\text{co}(S)$ is equal to the intersection of all convex sets containing $S$. 