Math 316 Hwk 3

Problem 1. If $U \subset \mathbb{R}^n$ is open and $K \subset U$ is compact, prove that there is a compact set $D$ such that $K \subset D^\circ$ and $D \subset U$.

Problem 2. Recall that a homeomorphism is bijective, continuous, and has a continuous inverse. Prove that there is no homeomorphism from $\mathbb{R}^2$ onto $\mathbb{R}$. HINT: Remove a point from each and consider a connectedness argument.

Problem 3. If $f, g$ are continuous on the Euclidian spaces $\mathbb{R}^m$ into $\mathbb{R}$, prove that $f + g$ and $f \cdot g$ are continuous.

Problem 4. A hyperplane is a set of the form \( \{ x \in \mathbb{R}^m | a^T x = b \} \), where $a \in \mathbb{R}^m$, $a \neq 0$, and $b \in \mathbb{R}$. Prove that a hyperplane is convex.

Problem 5. What is the distance between two parallel hyperplanes. Use the Euclidian metric in $\mathbb{R}^n$.

Problem 6. A halfspace is a set of the form \( \{ x \in \mathbb{R}^m | a^T x \leq b \} \), where $a \in \mathbb{R}^m$, $a \neq 0$, and $b \in \mathbb{R}$. Prove that a halfspace is convex.

Problem 7. Under what conditions does one halfspace contain another? Use the Euclidian metric in $\mathbb{R}^n$.

Problem 8. We know that $\mathbb{R}$ is complete. Use this to show that $\mathbb{R}^n$ is complete in the Euclidian norm.

Problem 9. Prove carefully the following: If a Cauchy sequence $\{x_k\}_{k=1}^\infty$ has a cluster point $x \in X$, then it converges to $x$.

Problem 10. If $\{x_k\}_{k=1}^\infty$ and $\{y_k\}_{k=1}^\infty$ are Cauchy sequences in $(X, d)$, then $d(x_n, y_n)$ converges. Note that $X$ may not be complete.