Sign the following pledge below:
I pledge on my honor that I have not given or received any unauthorized assistance on this assignment/examination.

Note that the first 10 questions are true-false. Mark A for true, B for false. Questions 11 through 20 are multiple choice–mark the correct answer on your bubble sheet. Answers to the last five questions should be written directly on the exam, and should be written neatly and correctly.

True-false questions

1. Let $a$ and $b$ be integers. If $a|b$, then $a^2|b^2$.
2. There is a real number solution for the equation $x^8 + x^4 + 1 = 0$.
3. If $n \equiv 2 \text{ (mod 5)}$, then $n^2 \equiv 4 \text{ (mod 5)}$.
4. For every two sets $A$ and $B$, $(A \cap B) \cup (A - B) = A$.
5. For $x \in \mathbb{R}$ if $x^4 + 1 \leq x^5 + x$, then $x > 0$.
6. If $A$ is a subset of the rational numbers and has a least element, then $A$ is well-ordered.
7. Let $S_1, S_2, \ldots$ be statements. Assume that $S_1$ is true, and that if $S_k$ is true for some $k \in \mathbb{N}$, then $S_{k+1}$ is true. Then $S_n$ is true for all $n \in \mathbb{N}$.
8. Let $A$ be a set. If $A - B = A$ for every set $B$, then $A = \emptyset$.
9. Let $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $b \neq 0$. If $a|c$ and $b|c$, then $ab|c$.
10. If $A, B,$ and $C$ are sets such that $A \times C = B \times C$, then $A = B$.

Multiple Choice Questions

11. Let $R$ be a relation defined on the set $\mathbb{Z}$ by $a \sim b$ if and only if $a^2 + b^2$ is even. Then $R$ is (remember to choose the most correct answer)
   (a) Reflexive
   (b) Symmetric
   (c) Transitive
   (d) Reflexive and symmetric
   (e) Symmetric and transitive
   (f) Reflexive and transitive
   (g) An equivalence relation

12. How many different relations are there on $A \times A$ where $A = \{1, 2\}$?
   (a) 1  (b) 2  (c) 4  (d) 8  (e) 16  (f) 32  (g) 64
13. How many equivalence relations are there on \( A = \{1, 2\} \)?

(a) 2  (b) 3  (c) 4  (d) 8  (e) 10  (f) 14  (g) 16

14. Negate the statement

\[ \forall x > 0, \exists y \geq 0, x^2 > y^2. \]

(a) \( \exists x > 0, \exists y \geq 0, x^2 > y^2. \)
(b) \( \exists x > 0, \forall y \geq 0, x^2 > y^2. \)
(c) \( \exists x < 0, \forall y < 0, x^2 \leq y^2. \)
(d) \( \forall x > 0, \exists y \geq 0, x^2 \leq y^2. \)
(e) \( \forall x > 0, \forall y \geq 0, x^2 \leq y^2. \)
(f) \( \exists x > 0, \forall y \geq 0, x^2 \leq y^2. \)

15. Evaluate the proposed proof of the following result. Choose the most complete correct answer.

**Theorem:** A relation \( R \) is defined on \( \mathbb{Z} \) by \((a, b) \in R\) if and only if \(3|(a + 2b)\). Then \( R \) is reflexive and symmetric.

**Proof.** Assume that \((a, a) \in R\). Then \(3|(a + 2a)\). Since \(a + 2a = 3a\) and \(a \in \mathbb{Z}\), it follows that \(3|3a\). Therefore, \( R \) is reflexive.

Next we show that \( R \) is symmetric. Assume that \((a, b) \in R\). Then \(3|(a + 2b)\). So \(a + 2b = 3x\), where \(x \in \mathbb{Z}\). Hence, \(a = 3x - 2b\). Therefore,

\[ b + 2a = b + 2(3x - 2b) = b + 6x - 4b = 6x - 3b = 3(2x - b). \]

Since \(2x - b\) is an integer we know that \(3|(b + 2a)\). So \( R \) is symmetric. \( \square \)

(a) The theorem and the proof are correct.
(b) The proof is correct but the theorem is incorrect.
(c) The theorem is correct, but the proof is incorrect.
(d) The theorem and proof are incorrect.

16. Which of the following does not prove \( P \Rightarrow Q \).

(a) Assume \( \sim P \) and \( Q \) and prove a contradiction.
(b) Assume \( \sim Q \) and prove \( \sim P \).
(c) Assume \( P \) and \( \sim Q \) and prove a contradiction.
(d) Assume \( P \) and prove \( Q \).
17. Let $R$ be a relation defined on the set $\mathbb{Z}$ by $a \sim b$ if and only if $3a - 7b$ is even. Then $R$ is (remember to choose the most correct answer)

(a) Reflexive
(b) Symmetric
(c) Transitive
(d) Reflexive and symmetric
(e) Symmetric and transitive
(f) Reflexive and transitive
(g) An equivalence relation

18. Rewrite the following statement using quantifiers. There exists a positive integer $m$ such that for all positive integers $n$ we have $|\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{2}$

(a) $\exists m \in \mathbb{Z}, \exists n \in \mathbb{N}, |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{2}$
(b) $\exists m \in \mathbb{N}, \exists n \in \mathbb{Z}, |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{2}$
(c) $\exists m \in \mathbb{Z}, \forall n \in \mathbb{N}, |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{2}$
(d) $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}, |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{2}$
(e) $\forall m \in \mathbb{N}, \exists n \in \mathbb{N}, |\frac{1}{m} - \frac{1}{n}| \leq \frac{1}{2}$

19. Let $R$ be a relation defined on the set $\mathbb{Z}$ by $a \sim b$ if and only if $|a - b| \leq 2$. Then $R$ is (remember to choose the most correct answer)

(a) Reflexive
(b) Symmetric
(c) Transitive
(d) Reflexive and symmetric
(e) Symmetric and transitive
(f) Reflexive and transitive
(g) An equivalence relation

20. Evaluate the proposed proof of the following result. Choose the most complete correct answer.

**Theorem**: Every symmetric and transitive relation on a nonempty set is an equivalence relation.

**Proof.** Let $R$ be a symmetric and transitive relation defined on a nonempty set $A$. We need only show that $R$ is reflexive. Let $x \in A$. We show that $(x, x) \in R$. Let $y \in A$ such that $(x, y) \in R$. Since $R$ is symmetric we know that $(y, x) \in R$. Now we have $(x, y) \in R$ and $(y, x) \in R$. Since $R$ is transitive we have $(x, x) \in R$. Thus, $R$ is reflexive. $\square$

(a) The theorem and the proof are correct.
(b) The proof is correct but the theorem is incorrect.
(c) The theorem is correct, but the proof is incorrect.
(d) The theorem and proof are incorrect.
21. Prove that

\[ \sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n - 1}{2^n}. \]
22. Define the terms in boldface by completing the sentence.

(a) An integer \(a\) divides \(b\) if

(b) A nonempty set \(A\) is well-ordered if

(c) A relation \(R\) is irreflexive if

(d) A relation \(R\) is symmetric if

(e) Let \(R\) be a relation from \(A\) to \(B\). The domain of \(R\) is
23. Prove or disprove the following:

(a) If $x, y \in \mathbb{Z}$ are of the same parity, then $xy$ and $(x + y)^2$ are of the same parity.

(b) Let $A$ be a set. If $A \cup B \neq \emptyset$ for all sets $B$, then $A \neq \emptyset$. 

24. Prove that each natural number is 1 or divisible by a prime.
25. Prove that there exists no positive integer $x$ such that $2x < x^2 < 3x$. 