GEOMETRICALLY AND DIAGRAMMATICALLY MAXIMAL KNOTS

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Abstract. The ratio of volume to crossing number of a hyperbolic knot is known to be bounded above by the volume of a regular ideal octahedron, and a similar bound is conjectured for the knot determinant per crossing. We investigate a natural question motivated by these bounds: For which knots are these ratios nearly maximal? We show that many families of alternating knots and links simultaneously maximize both ratios. One such family is weaving knots, which are alternating knots with the same projection as a torus knot, and which were conjectured by Lin to be among the maximum volume knots for fixed crossing number. For weaving knots, we provide the first asymptotically correct volume bounds.

1. Introduction

Despite many new developments in the fields of hyperbolic geometry, quantum topology, 3–manifolds, and knot theory, there remain notable gaps in our understanding about how the invariants of knots and links that come from these different areas of mathematics are related to each other. In particular, significant recent work has focused on understanding how the hyperbolic volume of knots and links is related to diagrammatic knot invariants (see, e.g., [5, 14]). In this paper, we investigate such relationships between the volume, determinant, and crossing number for sequences of hyperbolic knots and links.

For any diagram of a hyperbolic link \( K \), an upper bound for the hyperbolic volume \( \text{vol}(K) \) was given by D. Thurston by decomposing \( S^3 - K \) into octahedra, placing one octahedron at each crossing, and pulling remaining vertices to \( \pm \infty \). Any hyperbolic octahedron has volume bounded above by the volume of the regular ideal octahedron, \( v_8 \approx 3.66386 \). So if \( c(K) \) is the crossing number of \( K \), then

\[
\frac{\text{vol}(K)}{c(K)} \leq v_8.
\]

This result motivates several natural questions about the quantity \( \frac{\text{vol}(K)}{c(K)} \), which we call the volume density of \( K \). How sharp is the bound of equation (1)? For which links is the volume density very near \( v_8 \)? In this paper, we address these questions from several different directions, and present several conjectures motivated by our work.

We also investigate another notion of density for a knot or link. For any non-split link \( K \), we say that \( 2\pi \log \det(K)/c(K) \) is its determinant density. The following conjectured upper bound for the determinant density is equivalent to a conjecture of Kenyon for planar graphs (Conjecture 2.3 below).

**Conjecture 1.1.** If \( K \) is any knot or link,

\[
\frac{2\pi \log \det(K)}{c(K)} \leq v_8.
\]

We study volume and determinant density by considering sequences of knots and links.
Definition 1.2. A sequence of links $K_n$ with $c(K_n) \to \infty$ is geometrically maximal if
\[
\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_8.
\]

Similarly, a sequence of knots or links $K_n$ with $c(K_n) \to \infty$ is diagrammatically maximal if
\[
\lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8.
\]

In this paper, we find many families of geometrically and diagrammatically maximal knots and links. Our examples are related to the infinite weave $W$, which we define to be the infinite alternating link with the square lattice projection, as in Figure 1. We will see in Section 3 that there is a complete hyperbolic structure on $\mathbb{R}^3 - W$ obtained by tessellating the manifold by regular ideal octahedra such that the volume density of $W$ is exactly $v_8$. Therefore, a natural place to look for geometrically maximal knots is among those with geometry approaching $\mathbb{R}^3 - W$. We will see that links whose diagrams converge to the diagram of $W$ in an appropriate sense are both geometrically and diagrammatically maximal. To state our results, we need to define the convergence of diagrams.

Definition 1.3. Let $G$ be any possibly infinite graph. For any finite subgraph $H$, the set $\partial H$ is the set of vertices of $H$ that share an edge with a vertex not in $H$. We let $|\cdot|$ denote the number of vertices in a graph. An exhaustive nested sequence of connected subgraphs, $\{H_n \subset G : H_n \subset H_{n+1}, \cup_n H_n = G\}$, is a Følner sequence for $G$ if
\[
\lim_{n \to \infty} \frac{|\partial H_n|}{|H_n|} = 0.
\]

The graph $G$ is amenable if a Følner sequence for $G$ exists. In particular, the infinite square lattice $G(W)$ is amenable.

For any link diagram $K$, let $G(K)$ denote the projection graph of the diagram. We will need a particular diagrammatic condition called a cycle of tangles, which is defined carefully in Definition 7.4 below. For an example, see Figure 2(a). We now show two strikingly similar ways to obtain geometrically and diagrammatically maximal links.
Figure 2. (a) A Celtic knot diagram that has a cycle of tangles. (b) A Celtic knot diagram with no cycle of tangles, which could be in a sequence that satisfies conditions of Theorem 1.4.

**Theorem 1.4.** Let $K_n$ be any sequence of hyperbolic alternating links that contain no cycle of tangles, such that

1. there are subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(W)$, and
2. $\lim_{n \to \infty} |G_n|/c(K_n) = 1$.

Then $K_n$ is geometrically maximal: $\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_8$.

**Theorem 1.5.** Let $K_n$ be any sequence of alternating links such that

1. there are subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(W)$, and
2. $\lim_{n \to \infty} |G_n|/c(K_n) = 1$.

Then $K_n$ is diagrammatically maximal: $\lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8$.

Ideas for the proof of Theorem 1.4 are due to Agol, who showed that many sequences of links are geometrically maximal using volume bounds from [3]. Agol’s results are unpublished, although when announced (for example, mentioned in [17]) they were claimed to apply to closures of finite pieces of $W$. In fact, Theorem 1.4 applies to more general links, including those obtained by introducing crossings before taking the closure of a finite piece of $W$, for example as in Figure 2(b). We include a complete proof of Theorem 1.4 in this paper.

Notice that any sequence of links satisfying the hypotheses of Theorem 1.4 also satisfies the hypotheses of Theorem 1.5. This motivates the following questions.

**Question 1.6.** Is any diagrammatically maximal sequence of knots geometrically maximal, and vice versa?

i.e. $\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_8 \iff \lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8$?

Both our diagrammatic and geometric arguments below rely on special properties of alternating links. With present tools, we cannot say much about links that are mostly alternating.

**Question 1.7.** Let $K_n$ be any sequence of links such that

1. there are subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(W)$,
2. $K_n$ restricted to $G_n$ is alternating, and
3. $\lim_{n \to \infty} |G_n|/c(K_n) = 1$.

Is $K_n$ geometrically and diagrammatically maximal?
1.1. Weaving knots. In this paper, we also provide an explicit example of a family of knots and links satisfying the conditions of Theorems 1.4 and 1.5.

A weaving knot $W(p,q)$ is the alternating knot or link with the same projection as the standard $p$–braid $(\sigma_1 \ldots \sigma_{p-1})^q$ projection of the torus knot or link $T(p,q)$. Thus, $c(W(p,q)) = q(p - 1)$. For example, $W(5,4)$ is the closure of the 5–braid in Figure 3.

X.-S. Lin suggested in the early 2000s that weaving knots would be among knots with largest volume for fixed crossing number. In fact, $W(5,4)$ has the second largest volume among all knots with $c(K) \leq 16$, which can be verified using Knotscape [20]. (Good guess among 379,799 alternating knots with 16 crossings!)

Theorems 1.4 and 1.5 imply that any sequence of knots $W(p,q)$, with $p,q \rightarrow \infty$, is both geometrically and diagrammatically maximal. However, for these knots, we can prove more.

For both volume density and determinant density, determining explicit asymptotically correct bounds for the density in finite cases seems harder and requires different methods than proving the asymptotic density itself. For example, Conjecture 1.1 on determinant density for any explicit example of a knot seems to require more than what is involved in the proof of Theorem 1.5, which is about asymptotic determinant density for a sequence of knots. Similarly, Conjecture 1.14 below illustrates the difficulty inherent in determining volume bounds in finite explicit examples of knots.

However, in the case that the finite knot is a weaving knot $W(p,q)$, we provide asymptotically sharp, explicit bounds on volumes in terms of $p$ and $q$ alone. Theorem 1.8 below leads to explicit bounds on volume density for weaving knots for given $p$ and $q$, and is another main result of this paper.

**Theorem 1.8.** If $p \geq 3$ and $q \geq 7$, then

$$v_8 (p - 2) q \left(1 - \frac{(2\pi)^2}{q^2}\right)^{3/2} \leq \text{vol}(W(p,q)) \leq (v_8 (p - 3) + 4v_3) q.$$

Here $v_3 \approx 1.01494$ is the volume of the regular ideal tetrahedron, and $v_8$ is the same as above. Since $c(W(p,q)) = q(p - 1)$, these bounds provide another proof that weaving knots are geometrically maximal.

The methods involved in proving Theorem 1.8 are completely different than those used in the proof of Theorem 1.4. While the proof of Theorem 1.4 uses volume bounds via guts of 3–manifolds cut along essential surfaces as in [3], the proof of Theorem 1.8 involves explicit angle structures and the convexity of volume, as in [32].

Moreover, applying these asymptotically sharp volume bounds for the links $W(p,q)$, we prove that their geometric structures converge in a well–behaved manner.

**Theorem 1.9.** As $p,q \rightarrow \infty$, $S^3 - W(p,q)$ approaches $\mathbb{R}^3 - \mathcal{W}$ as a geometric limit.

Proving that a class of knots or links approaches $\mathbb{R}^3 - \mathcal{W}$ as a geometric limit seems to be difficult in general. It is unknown, for example, whether all the links of Theorem 1.4 approach $\mathbb{R}^3 - \mathcal{W}$ as a geometric limit, and the proof of that theorem does not give this information. However, Theorem 1.9 provides the result for $W(p,q)$. 
1.2. Previous volume bounds. To put Theorem 1.8 in context, we recall the best volume bounds known at present for alternating links. Following the notation in [22], let $D$ be a prime alternating twist–reduced diagram of hyperbolic link $L$. Let $t(D)$ denote the twist number of $D$, which is an invariant of $L$. Combining results in [22, 3], we obtain the following bound for $\text{vol}(L)$:

$$\frac{v_8}{2}(t(D) - 2) \leq \text{vol}(L) < 10v_3(t(D) - 1). \quad (2)$$

Again, $v_3$ is the volume of the regular ideal tetrahedron, and $v_8$ is the volume of a regular ideal octahedron. The upper bound was shown in [22, Appendix] to be asymptotically correct. However, the only sequences of links that approach this upper bound have a large number of bigons. If $D$ has no bigons then $t(D) = c(D)$ since no two crossings are twist–equivalent. Note $10v_3$ is much larger than $v_8$, so for links without bigons the upper bound in (1) is better.

The upper bound in (1) was subsequently improved in [1] for any hyperbolic link $K$ with $c(K) \geq 5$. In this case,

$$\text{vol}(K) \leq v_8(c(K) - 5) + 4v_3. \quad (3)$$

Combining the lower bound in (2) and the upper bound in (3), we get the best current volume bounds for a knot or link $K$ with a prime alternating twist–reduced diagram with no bigons and $c(K) \geq 5$ crossings:

$$\frac{v_8}{2}(c(K) - 2) \leq \text{vol}(K) \leq v_8(c(K) - 5) + 4v_3. \quad (4)$$

Note that both upper and lower bounds of Theorem 1.8 are stronger for $W(p,q)$ than those provided by (4).

1.3. Knot determinant and hyperbolic volume. There is strong experimental evidence in support of a conjectured relationship between the hyperbolic volume and the determinant of a knot, which was first observed in Dunfield’s prescient online post [10]. A quick experimental snapshot can be obtained from Knotscape [20], which provides this data for all knots with at most 16 crossings. The top nine knots in the Knotscape census sorted by maximum volume and by maximum determinant agree, but only set-wise! More data and a broader context is provided by Friedl and Jackson [12], and Stoimenow [34]. In particular, Stoimenow [34] proved there are constants $C_1, C_2 > 0$, such that for any hyperbolic alternating link $K$,

$$2 \cdot 1.0355^{\text{vol}(K)} \leq \text{det}(K) \leq \left(\frac{C_1 c(K)}{\text{vol}(K)}\right)^{C_2 \text{vol}(K)}.$$

Experimentally, we discovered the following surprisingly simple relationship between the two quantities that arise in the volume and determinant densities. We have verified the following conjecture for all alternating knots up to 16 crossings, and weaving knots and links for $3 \leq p \leq 50$ and $2 \leq q \leq 50$.

**Conjecture 1.10.** For any alternating hyperbolic link $K$,

$$\text{vol}(K) < 2\pi \log \text{det}(K).$$

Conjectures 1.10 and 1.1 would imply one direction of Question 1.6, that any geometrically maximal sequence of knots is diagrammatically maximal. In contrast, we can obtain $K_n$ by twisting on two strands, such that $\text{vol}(K_n)$ is bounded but $\text{det}(K_n) \to \infty$.

Our main results imply that the constant $2\pi$ in Conjecture 1.10 is sharp:
Corollary 1.11. If $\alpha < 2\pi$ then there exists a knot $K$ such that $\alpha \log \det(K) < \text{vol}(K)$.

Proof. Let $K_n$ be a sequence of knots that is both geometrically and diagrammatically maximal. Then $\lim_{n \to \infty} \alpha \log \det(K_n) = \alpha v_8 / 2\pi < v_8$ and $\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_8$. Hence, for $n$ sufficiently large, $\alpha \log \det(K_n) < \text{vol}(K_n)$. □

Remark 1.12. For weaving knots and links, experimental observations using SnapPy [9] show that for $3 \leq p \leq 50$ and $2 \leq q \leq 50$,

$\text{vol}(W(p,q)) < (v_8(p-3) + 4v_3)q < 2\pi \log \det(W(p,q)) < v_8(p-1)q$ if $2p < q^2 + q + 4$

$\text{vol}(W(p,q)) < 2\pi \log \det(W(p,q)) < (v_8(p-3) + 4v_3)q < v_8(p-1)q$ if $2p \geq q^2 + q + 4$

Our focus on geometrically and diagrammatically maximal knots and links naturally emphasizes the importance of alternating links. Every non-alternating link can be viewed as a modification of a diagram of an alternating link with the same projection, by changing crossings. This modification affects the determinant as follows.

Proposition 1.13. Let $K$ be a reduced alternating link diagram, and $K'$ be obtained by changing any proper subset of crossings of $K$. Then

$$\det(K') < \det(K).$$

What happens to volume under this modification? Motivated by Proposition 1.13, the first two authors previously conjectured that alternating diagrams also maximize hyperbolic volume in a given projection. They have verified part (a) of the following conjecture for all alternating knots up to 18 crossings ($\approx 10.7$ million knots).

Conjecture 1.14. (a) Let $K$ be an alternating hyperbolic knot, and $K'$ be obtained by changing any crossing of $K$. Then

$$\text{vol}(K') < \text{vol}(K).$$

(b) The same result holds if $K'$ is obtained by changing any proper subset of crossings of $K$.

Note that by Thurston’s Dehn surgery theorem, the volume converges from below when twisting two strands of a knot, so $\text{vol}(K) - \text{vol}(K')$ can be an arbitrarily small positive number. For weaving knots, Conjecture 1.14 is especially interesting because every knot can be obtained by changing some crossings of $W(p,q)$ for some $p,q$. This follows from the proof for torus knots in [24].

A natural extension of Conjecture 1.10 to any hyperbolic knot is to replace the determinant with the rank of the reduced Khovanov homology $\bar{H}^{*,*}(K)$. Let $K$ be an alternating hyperbolic knot, and $K'$ be obtained by changing any proper subset of crossing of $K$. It follows from results in [6] that

$$\det(K') \leq \text{rank}(\bar{H}^{*,*}(K')) \leq \det(K).$$

Conjectures 1.10 and 1.14 would imply that $\text{vol}(K') < 2\pi \log \det(K)$, but using data from KhoHo [33] we have verified the following stronger conjecture for all non-alternating knots with up to 15 crossings.

Conjecture 1.15. For any hyperbolic knot $K$,

$$\text{vol}(K) < 2\pi \log \text{rank}(\bar{H}^{*,*}(K)).$$

Note that Conjecture 1.10 is a special case of Conjecture 1.15.
1.4. Spectra for volume and determinant density. We describe a more general context for Theorems 1.4 and 1.5.

**Definition 1.16.** Let $C_{\text{vol}} = \{ \text{vol}(K)/c(K) \}$ and $C_{\text{det}} = \{ 2\pi \log \det(K)/c(K) \}$ be the sets of respective densities for all hyperbolic links $K$. We define $\text{Spec}_{\text{vol}} = C_{\text{vol}}'$ and $\text{Spec}_{\text{det}} = C_{\text{det}}'$ as their derived sets (set of all limit points).

The upper bound in (4) shows that the volume density of any link is strictly less than $v_8$. Together with Conjecture 1.1, this implies:

$$\text{Spec}_{\text{vol}}, \text{Spec}_{\text{det}} \subset [0, v_8]$$

For infinite sequences of alternating links without bigons, equation (4) implies that $\text{Spec}_{\text{vol}}$ restricted to such links lies in $[v_8/2, v_8]$.

Twisting on two strands of an alternating link gives 0 as a limit point of both densities. Thus, we obtain the following corollary of Theorems 1.4 and 1.5:

**Corollary 1.17.** $\{0, v_8\} \subset \text{Spec}_{\text{vol}} \cap \text{Spec}_{\text{det}}$.

Although $v_8$ does not occur as a volume density of any finite link, $v_8$ is the volume density of $W$ (see Remark 3.2). Corollary 5.7 also shows that $2v_3 \in \text{Spec}_{\text{vol}}$. It is an interesting problem to understand the sets $\text{Spec}_{\text{vol}}, \text{Spec}_{\text{det}},$ and $\text{Spec}_{\text{vol}} \cap \text{Spec}_{\text{det}}$, and to explicitly describe and relate their elements.

1.5. A maximal volume conjecture. The Volume Conjecture (see, e.g. [5] and references therein) is an important mathematical program to bridge the gap between quantum and geometric topology. One interesting consequence of our discussion above is a maximal volume conjecture for a sequence of links that is geometrically and diagrammatically maximal.

The Volume Conjecture involves the Kashaev invariant

$$\langle K \rangle_N := \frac{J_N(K; \exp(2\pi i/N))}{J_N(\emptyset; \exp(2\pi i/N))},$$

and is the following limit:

$$\lim_{N \to \infty} 2\pi \log |\langle K \rangle_N|^\frac{1}{N} = \text{vol}(K).$$

For any knot $K$, Garoufalidis and Le [17] proved

$$\limsup_{N \to \infty} \frac{2\pi \log |\langle K \rangle_N|^\frac{1}{N}}{c(K)} \leq v_8.$$

Now, since the limits in Theorems 1.4 and 1.5 are both equal to $v_8$, we can make the maximal volume conjecture as follows.

**Conjecture 1.18 (Maximal volume conjecture).** For any sequence of links $K_n$ that is both geometrically and diagrammatically maximal, there exists an increasing integer-valued function $N = N(n)$ such that

$$\lim_{n \to \infty} \frac{2\pi \log |\langle K_n \rangle_N|^\frac{1}{N}}{c(K_n)} = v_8 = \lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)}.$$

To prove Conjecture 1.18 it suffices to prove

$$\lim_{n \to \infty} \frac{2\pi \log |\langle K_n \rangle_N|^\frac{1}{N}}{c(K_n)} = \lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_8,$$

which relates only diagrammatic invariants.
1.6. Organization. In Section 2, we give the proof of Theorem 1.5. Because it appears throughout the paper, we discuss the geometry of the infinite weave $W$ in Section 3. In Sections 4 and 5, we prove upper and lower volume bounds for $W(p,q)$ using explicit triangulations and angle structures. In Section 6, we prove Theorem 1.9. In Section 7, we begin the proof of Theorem 1.4 by proving that these links have volumes bounded below by the volumes of certain right-angled hyperbolic polyhedra. In Section 8, we complete the proof of Theorem 1.4 essentially using the rigidity of circle patterns associated to the right-angled polyhedra. We will assume throughout that our links are non-split.

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2. Diagrammatically maximal knots and spanning trees

In this section, we first give the proof of Theorem 1.5, then discuss conjectures related to Conjecture 1.1.

For any connected link diagram $K$, we can associate a connected graph $G_K$, called the Tait graph of $K$, by checkerboard coloring complementary regions of $K$, assigning a vertex to every shaded region, an edge to every crossing and a $\pm$ sign to every edge as follows:

\[
\begin{align*}
\begin{array}{c}
|\
\end{array}
\end{align*}

Thus, $e(G_K) = c(K)$, and the signs on the edges are all equal if and only if $K$ is alternating. So any alternating knot or link $K$ is determined up to mirror image by its unsigned Tait graph $G_K$.

Let $\tau(G_K)$ denote the number of spanning trees of $G_K$. For any alternating link, $\tau(G_K) = \det(K)$, the determinant of $K$. More generally, for links including non-alternating links, we have the following.

**Lemma 2.1** ([7]). For any spanning tree $T$ of $G_K$, let $\sigma(T)$ be the number of positive edges in $T$. Let $s_\sigma(K) = \# \{\text{spanning trees } T \text{ of } G_K \mid \sigma(T) = \sigma\}$. Then

\[
\det(K) = \left| \sum_\sigma (-1)^\sigma s_\sigma(K) \right|.
\]

With this notation, we can prove Proposition 1.13 from the introduction.

**Proposition 1.13.** Let $K$ be a reduced alternating link diagram, and $K'$ be obtained by changing any proper subset of crossings of $K$. Then

\[
\det(K') < \det(K).
\]

**Proof.** First, suppose only one crossing of $K$ is switched, and let $e$ be the corresponding edge of $G_K$, which is the only negative edge in $G_{K'}$. Since $K$ has no nugatory crossings, $e$ is neither a bridge nor a loop. Hence, there exist spanning trees $T_1$ and $T_2$ such that $e \in T_1$ and $e \notin T_2$. The result now follows by Lemma 2.1.
When a proper subset of crossings of $K$ is switched, by Lemma 2.1 it suffices to show that if $s_\sigma(K) \neq 0$ then $s_{\sigma+1}(K) \neq 0$ or $s_{\sigma-1}(K) \neq 0$. Since there are no bridges or loops, every pair of edges is contained in a cycle. So for any spanning tree $T_1$ with $\sigma(T_1) < e(T_1)$, we can find a pair of edges $e_1$ and $e_2$ with opposite signs, such that $e_1 \in \text{cyc}(T_1, e_2)$, where recall $\text{cyc}(T_1, e_2)$ is the set of edges in the unique cycle of $T_1 \cup e_2$. It follows that $T_2 = (T_1 - e_1) \cup e_2$ satisfies $\sigma(T_2) = \sigma(T_1) \pm 1$. □

We now show how Theorem 1.5 follows from previously known results about the asymptotic enumeration of spanning trees of finite planar graphs.

**Theorem 2.2.** Let $H_n$ be any Følner sequence for the square lattice, and let $K_n$ be any sequence of alternating links with corresponding Tait graphs $G_n \subset H_n$, such that

$$\lim_{n \to \infty} \frac{\# \{ x \in V(G_n) : \deg(x) = 4 \}}{|H_n|} = 1,$$

where $V(G_n)$ is the set of vertices of $G_n$. Then

$$\lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{e(K_n)} = v_8.$$

**Proof.** Burton and Pemantle (1993), Shrock and Wu (2000), and others (see [23] and references therein) computed the spanning tree entropy of graphs $H_n$ that approach $G(W)$:

$$\lim_{n \to \infty} \frac{\log \tau(H_n)}{e(H_n)} = 4C/\pi.$$

where $C \approx 0.9160$ is Catalan’s constant. The spanning tree entropy of $G_n$ is the same as for graphs $H_n$ that approach $G(W)$ by [23, Corollary 3.13]. Since $4C = v_8$ and by the two–to–one correspondence for edges to vertices of the square lattice, the result follows. □

Note that the subgraphs $G_n$ in Theorem 2.2 have small boundary (made precise in [23]) but they need not be nested, and need not exhaust the infinite square lattice $G(W)$. Because the Tait graph $G_W$ is isomorphic to $G(W)$, these results about the spanning tree entropy of Tait graphs $G_K$ are the same as for projection graphs $G(K)$ used in Theorem 1.5. Thus, Theorem 2.2 implies that theorem. This concludes the proof of Theorem 1.5.

### 2.1. Determinant density

We now return our attention to Conjecture 1.1 from the introduction. That conjecture is equivalent to the following conjecture due to Kenyon.

**Conjecture 2.3** (Kenyon [21]). *If $G$ is any finite planar graph,*

$$\frac{\log \tau(G)}{e(G)} \leq 2C/\pi \approx 0.58312$$

*where $C \approx 0.9160$ is Catalan’s constant.*

The equivalence can be seen as follows. Since $4C = v_8$ and $\tau(G_K) = \det(K)$, Conjecture 2.3 would immediately imply that Conjecture 1.1 holds for all alternating links $K$. If $K$ is not alternating, then there exists an alternating link with the same crossing number and strictly greater determinant by Proposition 1.13. Therefore, Conjecture 2.3 would still imply Conjecture 1.1 in the non-alternating case.

On the other hand, any finite planar graph is realized as the Tait graph of an alternating link, with edges corresponding to crossings. Hence Conjecture 1.1 implies Conjecture 2.3.

The equivalent conjectures 1.1 and 2.3 would imply a third conjecture, related again to the Jones polynomial.
Let $V_K(t) = \sum_i a_i t^i$ denote the Jones polynomial, with $d = \text{span}(V_K(t))$, which is the difference between the highest and lowest degrees of terms in $V_K(t)$. Let $\mu(K)$ denote the average of the absolute values of coefficients of $V_K(t)$, i.e.

$$\mu(K) = \frac{1}{d+1} \sum |a_i|.$$

**Conjecture 2.4.** If $K$ is any knot or link,

$$\frac{2\pi \log \mu(K)}{c(K)} \leq v_8.$$

Our evidence for Conjecture 2.4 is the following. It follows from the spanning tree expansion for $V_K(t)$ in [36] that if $K$ is an alternating link,

$$\mu(K) = \frac{\text{det}(K)}{c(K) + 1}.$$

Thus Conjecture 1.1 would immediately imply that Conjecture 2.4 holds for all alternating links $K$. By the spanning tree expansion for $V_K(t)$, $\sum |a_i| \leq \tau(G_K)$, with equality if and only if $K$ is alternating. Hence, if $K$ is not alternating, then there exists an alternating link with the same crossing number and strictly greater coefficient sum $\sum |a_i|$. Therefore, Conjecture 1.1 would still imply Conjecture 2.4 in the non-alternating case.

Moreover, for sequences of alternating diagrammatically maximal knots, we have:

**Proposition 2.5.** If $K_n$ is any sequence of alternating diagrammatically maximal links,

$$\lim_{n \to \infty} \frac{2\pi \log \mu(K_n)}{c(K_n)} = v_8.$$

**Proof.** Since $\mu(K) = \frac{\text{det}(K)}{c(K) + 1}$, we have

$$\log \mu(K) = \log \frac{\text{det}(K)}{c(K) + 1} = \frac{\log \text{det}(K) - \log(c(K) + 1)}{c(K)}.$$ $$\Box$$

We conjecture that the alternating condition in Proposition 2.5 can be dropped.

Note that for links $K_n$ that satisfy Theorem 1.4, their asymptotic volume density equals their asymptotic determinant density, so in this case,

$$\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = \lim_{n \to \infty} \frac{2\pi \log \text{det}(K_n)}{c(K_n)} = \lim_{n \to \infty} \frac{2\pi \log \mu(K_n)}{c(K_n)} = v_8.$$

### 3. Geometry of the infinite weave

In this section, we discuss the geometry and topology of the infinite weave $\mathcal{W}$ and its complement $\mathbb{R}^3 - \mathcal{W}$. Recall that $\mathcal{W}$ is the infinite alternating link whose diagram projects to the square lattice, as in Figure 1.

**Theorem 3.1.** $\mathbb{R}^3 - \mathcal{W}$ has a complete hyperbolic structure with a fundamental domain tessellated by regular ideal octahedra, one for each square of the infinite square lattice.

**Proof.** First, we view $\mathbb{R}^3$ as $\mathbb{R}^2 \times (-1,1)$, with the plane of projection for $\mathcal{W}$ the plane $\mathbb{R}^2 \times \{0\}$. Thus, $\mathcal{W}$ lies in a small neighborhood of $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^2 \times (-1,1)$. We can arrange the diagram so that $\mathbb{R}^3 - \mathcal{W}$ is biperiodic and equivariant under a $\mathbb{Z} \times \mathbb{Z}$ action given by translations along the $x$ and $y$-axes, translating by two squares in each direction to match the alternating property of the diagram. Notice that the quotient gives an alternating link in the thickened torus, with fundamental region as in Figure 4(a). A thickened torus, in turn, is homeomorphic to the complement of the Hopf link in $S^3$. Thus the quotient of $\mathbb{R}^3 - \mathcal{W}$ under the $\mathbb{Z} \times \mathbb{Z}$ action is the complement of a link $L$ in $S^3$, as in Figure 4(b).
The link complement $S^3 - L$ can be easily shown to be obtained by gluing four regular ideal octahedra, for example by computer using Snap [8] (which uses exact arithmetic). Below, we present an explicit geometric way to obtain this decomposition.

Consider the two surfaces of $S^3 - L$ on the projection plane of the thickened torus, i.e. the image of $\mathbb{R}^2 \times \{0\}$. These can be checkerboard colored on $T^2 \times \{0\}$. These intersect in four crossing arcs, running between crossings of the single square shown in the fundamental domain of Figure 4. Generalizing the usual polyhedral decomposition of alternating links, due to Menasco [27] (see also [22]), cut along these checkerboard surfaces. When we cut, the manifold falls into two pieces $X_1$ and $X_2$, each homeomorphic to $T^2 \times I$, with one boundary component $T^2 \times \{1\}$, say, coming from a Hopf link component in $S^3$, and the other now given faces, ideal edges, and ideal vertices from the checkerboard surfaces, as follows.

1. For each piece $X_1$ and $X_2$, there are four faces total, two red and two blue, all quadrilaterals coming from the checkerboard surfaces.
2. There are four equivalence class of edges, each corresponding to a crossing arc.
3. Ideal vertices come from remnants of the link in $T^2 \times \{0\}$: either overcrossings in the piece above the projection plane, or undercrossings in the piece below.

The faces (red and blue), edges (dark blue), and ideal vertices (white) for $X_1$ above the projection plane are shown in Figure 4(c).

Now, for each ideal vertex on $T^2 \times \{0\}$ of $X_i$, $i = 1, 2$, add an edge running vertically from that vertex to the boundary component $T^2 \times \{1\}$. Add triangular faces where two of these new edges together bound an ideal triangle with one of the ideal edges on $T^2 \times \{1\}$. These new edges and triangular faces cut each $X_i$ into four square pyramids. Since $X_1$ and $X_2$ are glued across the squares at the base of these pyramids, this gives a decomposition of $S^3 - L$ into four ideal octahedra, one for each square region in $T^2 \times \{0\}$.

Give each octahedron the hyperbolic structure of a hyperbolic regular ideal octahedron. Note that each edge meets exactly four octahedra, and so the monodromy map about each edge is the identity. Moreover, each cusp is tiled by Euclidean squares, and inherits a Euclidean structure in a horospherical cross-section. Thus by the Poincaré polyhedron theorem (see, e.g., [11]), this gives a complete hyperbolic structure on $S^3 - L$.

Thus the universal cover of $S^3 - L$ is $\mathbb{H}^3$, tiled by regular ideal octahedra, with a square through the center of each octahedron projecting to a square from the checkerboard decomposition. Taking the cover of $S^3 - L$ corresponding to the $\mathbb{Z} \times \mathbb{Z}$ subgroup associated with

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{(a) The $\mathbb{Z} \times \mathbb{Z}$ quotient of $\mathbb{R}^3 - \mathcal{W}$ is a link complement in a thickened torus (b) This is also the complement of link $L$ in $S^3$ (c) Cutting along checkerboard surfaces, viewed from above the projection plane.}
\end{figure}
the Hopf link, we obtain a complete hyperbolic structure on $\mathbb{R}^3 - \mathcal{W}$, with a fundamental domain tessellated by regular ideal octahedra, with one octahedron for each square of the square lattice, as claimed.

Figure 5. (a) Circle pattern for hyperbolic planes of the top piece $X_1$ of $\mathbb{R}^3 - \mathcal{W}$. (b) Hyperbolic planes bounding one top square pyramid.

Figure 6. A regular ideal hyperbolic octahedron is obtained by gluing the two square pyramids. Hyperbolic planes that form the bottom square pyramid and the associated circle pattern are shown.

The proof of Theorem 3.1 also provides the face pairings for the regular ideal octahedra that tessellate the fundamental domain for $\mathbb{R}^3 - \mathcal{W}$. We also discuss the associated circle patterns on $T^2 \times \{0\}$, which in the end play an important role in the proof of Theorem 1.4 in Section 8.

A regular ideal octahedron is obtained by gluing two square pyramids, which we will call the top and bottom pyramids. In Figure 5(b), the apex of the top square pyramid is at infinity, the triangular faces are shown in the vertical planes, and the square face is on the hemisphere.

From the proof above, $\mathbb{R}^3 - \mathcal{W}$ is cut into $\tilde{X}_1$ and $\tilde{X}_2$, such that $\tilde{X}_1$ is obtained by gluing top pyramids along triangular faces, and $\tilde{X}_2$ by gluing bottom pyramids along triangular faces. The circle pattern in Figure 5(a) shows how the square pyramids in $\tilde{X}_1$ are viewed from infinity on the $xy$-plane.

Similarly, in Figure 6, we show the hyperbolic planes that form the bottom square pyramid, and the associated circle pattern. In this figure, the apex of the bottom square pyramid is in the center, the triangular faces are on hemispherical planes, and the square face is on the upper hemisphere. The circle pattern shows how the bottom square pyramids on the $xy$-plane are viewed from infinity on the $xy$-plane.
A fundamental domain $P_W$ for $R^3-W$ in $H^3$ is explicitly obtained by attaching each top pyramid of $\tilde{X}_1$ to a bottom pyramid of $\tilde{X}_2$ along their common square face. Hence, $P_W$ is tessellated by regular ideal octahedra. By the proof above, an appropriate $\pi/2$ rotation is needed when gluing the square faces, which determines how adjacent triangular faces are glued to obtain $P_W$. Figure 7 shows the face pairings for the triangular faces of the bottom square pyramids, and the associated circle pattern. The face pairings are equivariant under the translations $(x, y) \mapsto (x \pm 1, y \pm 1)$. That is, when a pair of faces is identified, then the corresponding pair of faces under this translation is also identified.

Remark 3.2. Because every regular ideal octahedron corresponds to a square face that meets four crossings, and any crossing meets four square faces that correspond to four ideal octahedra, it follows that the volume density of the infinite link $W$ is exactly $v_8$.

4. Triangulation of weaving knots

We now return to the weaving knots $W(p,q)$, which we consider as a closed $p$–braid, and let $B$ denote the braid axis. In this section, we describe a decomposition of $S^3 - (W(p,q) \cup B)$ into ideal tetrahedra and octahedra. This leads to our upper bound on volume. We will use the decomposition in Section 5 to prove the lower bound as well.

Let $p \geq 3$. Consider $W(p,q)$ as a closed $p$–braid, as above. Then the complement of $W(p,q)$ in $S^3$ with the braid axis also removed is a $q$–fold cover of the complement of $W(p,1)$ and its braid axis.

Lemma 4.1. Let $B$ denote the braid axis of $W(p,1)$. Then $S^3 - (W(p,1) \cup B)$ admits an ideal polyhedral decomposition $P$ with four ideal tetrahedra and $p-3$ ideal octahedra.
Figure 8. Polygonal decomposition of cusp corresponding to braid axis. A fundamental region consists of four triangles and $2(p-3)$ quads. The example shown here is $p=5$.

Figure 9. Left: dividing projection plane into triangles and quadrilaterals. Right: Coning to ideal octahedra and tetrahedra.

Moreover, a meridian for the braid axis runs over exactly one side of one of the ideal tetrahedra. The polyhedra give a polygonal decomposition of the boundary of a horoball neighborhood of the braid axis, with a fundamental region consisting of four triangles and $2(p-3)$ quadrilaterals, as shown in Figure 8.

Proof. Obtain an ideal polyhedral decomposition as follows. First, for every crossing of the standard diagram of $W(p,1)$, take an ideal edge running from the knot strand at the top of the crossing to the knot strand at the bottom. This subdivides the projection plane into two triangles and $p-3$ quadrilaterals. This is shown on the left of Figure 9 when $p=5$. In the figure, note that the four dotted red edges shown at each crossing are homotopic to the crossing arc.

Now for each quadrilateral on the projection plane, add four ideal edges above the projection plane and four below, as follows. Those edges above the projection plane run vertically from the strand of $W(p,1)$ corresponding to an ideal vertex of the quadrilateral to the braid axis $B$. Those edges below the projection plane also run from strands of $W(p,1)$ corresponding to ideal vertices of the quadrilateral, only now they run below the projection plane to $B$. These edges bound eight triangles, as follows. Four of the triangles lie above the projection plane, with two sides running from a strand of $W(p,1)$ to $B$ and the third on the quadrilateral. The other four lie below, again each with two edges running from strands of $W(p,1)$ to $B$ and one edge on the quadrilateral. The eight triangles together bound a single octahedron. This is shown in Figure 9 (right). Note there are $p-3$ such octahedra coming from the $p-3$ quadrilaterals on the projection plane.

As for the tetrahedra, these come from the triangular regions on the projection plane. As above, draw three ideal edges above the projection plane and three below. Each ideal edge runs from a strand of $W(p,1)$ corresponding to an ideal vertex of the triangle. For each ideal vertex, one edge runs above the projection plane to $B$ and the other runs below to
Again we form six ideal triangles per triangular region on the projection plane. This triangular region along with the ideal triangles above the projection plane bounds one of the four tetrahedra. The triangular region along with ideal triangles below the projection plane bounds another. There are two more coming from the ideal triangles above and below the projection plane for the other region.

These glue to give the polyhedral decomposition as claimed. Note that tetrahedra are glued in pairs across the projection plane. Note also that one triangular face of a tetrahedron above the projection plane, with two edges running to $B$, is identified to one triangular face of a tetrahedron below, again with two edges running to $B$. The identification swings the triangle through the projection plane. In fact, note that there is a single triangle in each region of the projection plane meeting $B$: two edges of the triangle are identified to the single edge running from the nearest strand of $W(p,1)$ to $B$, and the third edge runs between the nearest crossing of $W(p,1)$. This triangle belongs to tetrahedra above and below the projection plane. All other triangular and quadrilateral faces are identified by obvious homotopies of the edges and faces.

Finally, to see that the cusp cross section of $B$ meets the polyhedra as claimed, we need to step through the gluings of the portions of polyhedra meeting $B$. As noted above, where $B$ meets the projection plane, there is a single triangular face of two tetrahedra. Notice that two sides of the triangle, namely the ideal edges running from $W(p,1)$ to $B$, are actually homotopic in $S^3 - (W(p,1) \cup B)$. Hence that triangle wraps around an entire meridian of $B$. Thus a meridian of $B$ runs over exactly one side of one of the ideal tetrahedra. Now, two tetrahedra, one from above the plane of projection, and one from below, are glued along that side. The other two sides of the tetrahedron above the projection plane are glued to two distinct sides of the octahedron directly adjacent, above the projection plane. The remaining two sides of this octahedron above the projection plane are glued to two distinct sides of the next adjacent octahedron, above the projection plane, and so on, until we meet the tetrahedron above the projection plane on the opposite end of $W(p,1)$, which is glued below the projection plane. Now following the same arguments, we see the triangles and quadrilaterals repeated below the projection plane, until we meet up with the original tetrahedron. Hence the cusp shape is as shown in Figure 8. □

**Corollary 4.2.** For $p \geq 3$, the volume of $W(p,q)$ is at most $(4v_3 + (p - 3)v_8)q$.

**Proof.** The maximal volume of a hyperbolic ideal tetrahedron is $v_3$, the volume of a regular ideal tetrahedron. The maximal volume of a hyperbolic ideal octahedron is at most $v_8$, the volume of a regular ideal octahedron. The result now follows immediately from the first part of Lemma 4.1, and the fact that volume decreases under Dehn filling [38]. □

### 4.1. Weaving knots with three strands

The case when $p = 3$ is particularly nice geometrically, and so we treat it separately in this section.

**Theorem 4.3.** If $p = 3$ then the upper bound in Corollary 4.2 is achieved exactly by the volume of $S^3 - (W(3,q) \cup B)$, where $B$ denotes the braid axis. Thus,

$$\text{vol}(W(3,q) \cup B) = 4q v_3.$$ 

**Proof.** Since the complement of $W(3,q) \cup B$ in $S^3$ is a $q$–fold cover of the complement of $W(3,1) \cup B$, it is enough to prove the statement for $q = 1$.

We proceed as in the proof of Lemma 4.1. If $p = 3$, then the projection plane of $W(3,1)$ is divided into two triangles; see Figure 10. This gives four tetrahedra, two each on the top
and bottom. The edges and faces on the top tetrahedra are glued to those of the bottom tetrahedra across the projection plane for the same reason as in the proof Lemma 4.1.

![Figure 10. The tetrahedral decomposition of $S^3 - (W(3,1) \cup B)$.

Thus the tetrahedra are glued as shown in Figure 10. The top figures indicate the top tetrahedra and the bottom figures indicate the bottom tetrahedra. The crossing edges are labelled by numbers and the edges from the knot to the braid axis are labelled by letters. The two triangles in the projection plane are labelled $S$ and $T$. Edges and faces are glued as shown.

In this case, all edges of $\mathcal{P}$ are 6–valent. We set all tetrahedra to be regular ideal tetrahedra, and obtain a solution to the gluing equations of $\mathcal{P}$. Since all links of tetrahedra are equilateral triangles, they are all similar, and all edges of any triangle are scaled by the same factor under dilations. Hence, the holonomy for every loop in the cusp has to expand and contract by the same factor (i.e. it is scaled by unity), and so it is a Euclidean isometry. This implies that the regular ideal tetrahedra are also a solution to the completeness equations. Thus this is a geometric triangulation giving the complete structure with volume $4v_3$. □

**Remark 4.4.** Since the volumes of $S^3 - (W(3, q) \cup B)$ are multiples of $v_3$, we investigated its commensurability with the figure–8 knot complement and checked (using SnapPy [9]) that $S^3 - (W(3,2) \cup B)$ is a 4–fold cover of $S^3 - 4_1$. Note that $W(3,2)$ is the figure–8 knot, thus the figure–8 knot complement is covered by its braid complement with the axis removed! Some other interesting links also appear in this commensurability class as illustrated in Figure 11.

5. **Angle structures and lower volume bounds**

In this section we find lower bounds on volumes of weaving knots. To do so, we use angle structures on the manifolds $S^3 - (W(p, q) \cup B)$.

**Definition 5.1.** Given an ideal triangulation $\{\Delta_i\}$ of a 3–manifold, an angle structure is a choice of three angles $(x_i, y_i, z_i) \in (0, \pi)^3$ for each tetrahedron $\Delta_i$, assigned to three edges of $\Delta_i$ meeting in a vertex, such that

1. $x_i + y_i + z_i = \pi$;
2. the edge opposite that assigned $x_i$ in $\Delta_i$ is also assigned angle $x_i$, and similarly for $y_i$ and $z_i$; and
Figure 11. The complement of the figure–8 knot and its braid axis, $S^3 - (W(3, 2) \cup B)$, is a 4–fold cover of the figure–8 knot complement, $S^3 - W(3, 2)$.

(3) angles about any edge add to $2\pi$.

For any tetrahedron $\Delta_i$ and angle assignment $(x_i, y_i, z_i)$ satisfying (1) and (2) above, there exists a unique hyperbolic ideal tetrahedron with the same dihedral angles. The volume of this hyperbolic ideal tetrahedron can be computed from $(x_i, y_i, z_i)$. We do not need the exact formula for our purposes. However, given an angle structure on a triangulation $\{\Delta_i\}$, we can compute the corresponding volume by summing all volumes of ideal tetrahedra with that angle assignment.

**Lemma 5.2.** For $p > 3$, the manifold $S^3 - (W(p, 1) \cup B)$ admits an angle structure with volume $v_8(p - 2)$.

**Proof.** For the ideal polyhedral decomposition of $S^3 - (W(p, 1) \cup B)$ in Lemma 4.1, assign to each edge in an octahedron the angle $\pi/2$. As for the four tetrahedra, assign angles $\pi/4$, $\pi/4$, and $\pi/2$ to each, such that pairs of the tetrahedra glue into squares in the cusp neighborhood of the braid axis. See Figure 8.

We need to show the angle sum around each edge is $2\pi$. Consider first ideal edges with one endpoint on $W(p, 1)$ and one on the braid axis. These correspond to vertices of the polygonal decomposition of the braid axis illustrated in Figure 8. Note that many of these edges meet exactly four ideal octahedra, hence the angle sum around them is $2\pi$. Any such edge that meets a tetrahedron either meets three other ideal octahedra and the angle in the tetrahedron is $\pi/2$, so the total angle sum is $2\pi$, or it is identified to four edges of tetrahedra with angle $\pi/4$, and two octahedra. Hence the angle sum around it is $2\pi$.

Finally consider the angle sum around edges which run from $W(p, 1)$ to $W(p, 1)$. These arise from crossings in the diagram of $W(p, 1)$. The first two crossings on the left side and the last two crossings on the right side give rise to ideal edges bordering (some) tetrahedra. The others (for $p > 4$) border only octahedra, and exactly four such octahedra, hence the angle sum for those is $2\pi$. So we need only consider the edges arising from two crossings
Figure 12. Each edge 1 and 2 is a part of two tetrahedra arising from the triangle, and an octahedron arising from the square as shown in Figure 13. The braid axis is shown in the center.

Figure 13. Edges are glued as shown in figure. From left to right, shown are tetrahedra $T_1$, $T'_1$, and adjacent octahedron.

on the far left and two crossings on the far right. We consider those on the far left; the argument for the far right is identical.

Label the edge at the first crossing on the left 1, and label that of the second 2. See Figure 12. The two tetrahedra arising on the far left have edges glued as shown on the left of Figure 13, and the adjacent octahedron has edges glued as on the right of that figure. We label the tetrahedra $T_1$ and $T'_1$.

Note that the edge labeled 1 in the figure is glued four times in tetrahedra, twice in $T_1$ and twice in $T'_1$, and once in an octahedron. However, note that in the tetrahedra it is assigned different angle measurements. In particular, in $T_1$, the edge labeled 1, which is opposite the edge labeled $B$, must have angle $\pi/2$, because that is the angle on the edge labeled $B$. The other edge of $T_1$ labeled 1 must have angle $\pi/4$. Similarly for $T'_1$. Thus the angle sum around the edge labeled 1 is $2\pi$.

In both $T_1$ and $T'_1$, the edge labeled 2 has angle $\pi/4$. Since the edge labeled 2 is also glued to two edges in one octahedron, and one edge in another, the total angle sum around that edge is also $2\pi$. Hence this gives an angle sum as claimed.

Take our ideal polyhedral decomposition of $S^3 - (W(p, 1) \cup B)$ and turn it into an ideal triangulation by stellating the octahedra, splitting them into four ideal tetrahedra. More precisely, this is done by adding an ideal edge running from the ideal vertex on the braid axis above the plane of projection, through the plane of projection to the ideal vertex on
the braid axis below the plane of projection. Using this ideal edge, the octahedron is split into four tetrahedra. Assign angle structures to these four tetrahedra in the obvious way, namely, on each tetrahedron the ideal edge through the plane of projection is given angle $\pi/2$, and the other two edges meeting that edge in an ideal vertex are labeled $\pi/4$. By the above work, this gives an angle structure.

The volume estimate comes from the fact that a regular ideal octahedron has volume $v_8$. Moreover, four ideal tetrahedra, each with angles $\pi/2, \pi/4, \pi/4$, can be glued to give an ideal octahedron, hence each such tetrahedron has volume $v_8/4$. We have $p-3$ octahedra and four such tetrahedra, and hence the corresponding volume is $(p-3)v_8$.

\[ \text{Lemma 5.3.} \text{ Let } P \text{ be an ideal polyhedron obtained by coning to } \pm \infty \text{ from any ideal quadrilateral in the projection plane. Then for any angle structure on } P, \text{ the volume of that angle structure } \text{vol}(P) \text{ satisfies } \text{vol}(P) \leq v_8, \text{ the volume of the regular ideal octahedron.} \]

\[ \text{Proof.} \text{ Suppose the volume for some angle structure is strictly greater than } v_8. \text{ The dihedral angles on the exterior of } P \text{ give a dihedral angle assignment } \Delta \text{ to } P. \text{ By Theorem 6.13 of that paper, there is a unique complete structure with angle assignment } \Delta, \text{ and the proof of Theorem 6.16 of [32] shows that the complete structure occurs exactly when the volume is maximized over } A(P, \Delta). \text{ Hence the volume of our angle structure is at most the volume of the complete hyperbolic structure on } P \text{ with angle assignment } \Delta. \]

On the other hand, for complete hyperbolic structures on $P$, it is known that the volume is maximized in the regular case, and thus the volume is strictly less than the volume of a regular ideal octahedron. The proof of this fact is given, for example, in Theorem 10.4.8 and the proof of Theorem 10.4.7 in [31]. This is a contradiction. \qed

Now consider the space $A(P)$ of angle structures on the ideal triangulation $P$ for $S^3 - (W(p, 1) \cup B)$.

\[ \text{Lemma 5.4.} \text{ The critical point for } \text{vol}: A(P) \to \mathbb{R} \text{ is in the interior of the space } A \text{ of angle structures on } P. \]

\[ \text{Proof.} \text{ It is known that the volume function is concave down on the space of angle structures [32, 13]. We will show that the volume function takes values strictly smaller on the boundary of } A(P) \text{ than at any point in the interior. Therefore, it will follow that the maximum occurs in the interior of } A(P). \]

Suppose we have a point $X$ on the boundary of $A(P)$ that maximizes volume. Because the point is on the boundary, there must be at least one triangle $\Delta$ with angles $(x_0, y_0, z_0)$ where one of $x_0$, $y_0$, and $z_0$ equals zero. A proposition of Guérin [18, Proposition 7.1], implies that if one of $x_0$, $y_0$, $z_0$ is zero, then another is $\pi$ and the third is also zero. (The proposition is stated for once–punctured torus bundles in [18], but only relies on the formula for volume of a single ideal tetrahedron, [18, Proposition 6.1].)

A tetrahedron with angles $0, 0, \pi$ is a flattened tetrahedron, and contributes nothing to volume. We consider which tetrahedra might be flattened.

Let $P_0$ be the original polyhedral decomposition described in the proof of Lemma 4.1. Suppose first that we have flattened one of the four tetrahedra which came from tetrahedra in $P_0$. Then the maximal volume we can obtain from these four tetrahedra is at most $3v_3$, which is strictly less than $v_8$, which is the volume we obtain from these four tetrahedra from the angle structure of Lemma 5.2. Thus, by Lemma 5.3, the maximal volume we can obtain from any such angle structure is $3v_3 + (p-3)v_8 < (p-2)v_8$. Since the volume on the right
is realized by an angle structure in the interior by Lemma 5.2, the maximum of the volume cannot occur at such a point of the boundary.

Now suppose one of the four tetrahedra coming from an octahedron is flattened. Then the remaining three tetrahedra can have volume at most $3v_3 < v_8$. Thus the volume of such a structure can be at most

$$4v_3 + 3v_3 + (p-4)v_8,$$

the first term comes from the maximum volume of the four tetrahedra in $P_0$, the second from the maximum volume of the stellated octahedron with one flat tetrahedron, and the last term from the maximal volume of the remaining ideal octahedra. Because $7v_3 < 2v_8$, the volume of this structure is still strictly less than that of Lemma 5.2.

Therefore, there does not exist $X$ on the boundary of the space of angle structures that maximizes volume.

\[\square\]

Theorem 5.5. If $p > 3$, then

$$v_8(p-2)q \leq \text{vol}(W(p,q) \cup B) \leq (v_8(p-3) + 4v_3)q.$$ 

If $p = 3$, $\text{vol}(W(3,q) \cup B) = 4qv_3$.

Proof. Theorem 4.3 provides the $p = 3$ case.

For $p > 3$, Casson and Rivin showed that if the critical point for the volume is in the interior of the space of angle structures, then the maximal volume angle structure is realized by the actual hyperbolic structure [32]. By Lemma 5.4, the critical point for volume is in the interior of the space of angle structures. By Lemma 5.2, the volume of one particular angle structure is $v_8(p-2)q$. So the maximal volume must be at least this. The upper bound is from Corollary 4.2.

\[\square\]

Lemma 5.6. The length of a meridian of the braid axis is at least $q$.

Proof. A meridian of the braid axis of $W(p,q)$ is a $q$–fold cover of a meridian of $W(p,1)$. The meridian of $W(p,1)$ must have length at least one [38, 2]. Hence the meridian of $W(p,q)$ has length at least $q$.

\[\square\]

We can now prove our main result on volumes of weaving knots:

Proof of Theorem 1.8. The upper bound is from Corollary 4.2.

As for the lower bound, the manifold $S^3 - W(p,q)$ is obtained by Dehn filling the meridian on the braid axis of $S^3 - (W(p,q) \cup B)$. When $q > 6$, Lemma 5.6 implies that the meridian of the braid axis has length greater than $2\pi$, and so [15, Theorem 1.1] will apply. Combining this with Theorem 5.5 implies

$$\left(1 - \left(\frac{2\pi}{q}\right)^2\right)^{3/2}((p-2)qv_8) \leq \text{vol}(S^3 - W(p,q)).$$

For $p = 3$ note that $v_8 < 4v_3$, so this gives the desired lower bound for all $p \geq 3$.

\[\square\]

Corollary 5.7. The links $K_n = W(3,n)$ satisfy

$$\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = 2v_3.$$ 

Proof. By Theorem 4.3 and the same argument as above, for $q > 6$ we have

$$\left(1 - \left(\frac{2\pi}{q}\right)^2\right)^{3/2}(4qv_3) \leq \text{vol}(S^3 - W(3,q)) \leq 4qv_3.$$ 

\[\square\]
6. Geometric convergence of weaving knots

In the section, we will prove Theorem 1.9, which states that as \( p,q \to \infty \), the manifold \( S^3 - W(p,q) \) approaches \( \mathbb{R}^3 - W \) as a geometric limit. This result connects the two different polyhedral decompositions above: The polyhedra described in Section 3 are the geometric limit of the polyhedra described in Section 4. From the proof below, we will see that these polyhedra converge as follows. If we cut the torus in Figure 9 in half along the horizontal plane shown, each half is tessellated mostly by square pyramids, as well as some tetrahedra. As \( p,q \to \infty \), the tetrahedra are pushed off to infinity, and the square pyramids converge to the square pyramids that are shown in Figure 5. Gluing the two halves of the torus along the square faces of the square pyramids, in the limit we obtain the tessellation by regular ideal octahedra.

First, we review the definition of a geometric limit.

**Definition 6.1.** For compact metric spaces \( X \) and \( Y \), define their bilipschitz distance to be

\[
\inf \{|\log \text{lip}(f)| + |\log \text{lip}(f^{-1})|\}
\]

where the infimum is taken over all bilipschitz mappings \( f \) from \( X \) to \( Y \), and \( \text{lip}(f) \) denotes the lipschitz constant. The lipschitz constant is defined to be infinite if there is no bilipschitz map between \( X \) and \( Y \).

**Definition 6.2.** A sequence \( \{(X_n, x_n)\} \) of locally compact complete length metric spaces with distinguished basepoints is said to converge in the pointed bilipschitz topology to \((Y, y)\) if for any \( R > 0 \), the closed balls \( B_R(x_n) \) of radius \( R \) about \( x_n \) in \( X_n \) converge to the closed ball \( B_R(y) \) about \( y \) in \( Y \) in the bilipschitz topology.

**Definition 6.3.** For \( X, Y \) locally compact complete metric spaces, we say that \( Y \) is a geometric limit of \( X_n \) if there exist basepoints \( y \in Y \) and \( x_n \in X_n \) such that \((X_n, x_n)\) converges in the pointed bilipschitz topology to \((Y, y)\).

In order to prove Theorem 1.9, we will consider \( M_{p,q} := S^3 - (W(p,q) \cup B) \). Since \( S^3 - W(p,q) \) is obtained by Dehn filling \( M_{p,q} \) along a slope of length at least \( q \) by Lemma 5.6, Thurston’s Dehn filling theorem implies that \( M_{p,q} \) is a geometric limit of \( S^3 - W(p,q) \). Thus it will suffice to show \( \mathbb{R}^3 - W \) is a geometric limit of \( M_{p,q} \). To show this, we need to find basepoints \( x_{p,q} \) for each \( M_{p,q} \) so that closed balls \( B_R(x_{p,q}) \) converge to a closed ball in \( \mathbb{R}^3 - W \).

We do this by considering structures on ideal polyhedra.

Let \( P_{p,q} \) denote the ideal polyhedra in the decomposition of \( M_{p,q} \) from the proofs of Lemma 4.1 and Lemma 5.2. We decomposed \( P_{p,q} \) into ideal tetrahedra and ideal octahedra, such that octahedra corresponding to non-peripheral squares of \( G(W(p,q)) \) satisfy the same local gluing condition on the faces as that for the fundamental domain \( P_{W} \) for \( \mathbb{R}^3 - W \) as illustrated in Figure 7. In particular, the faces of each octahedron are glued to faces of adjacent octahedra, with the gluings of the triangular faces of the top and bottom square pyramids locally the same as those for \( P_{W} \).

We find a sequence of consecutive octahedra in \( M_{p,1} = S^3 - (W(p,1) \cup B) \) with volume approaching \( v_8 \), and then use the \( q \)-fold cover \( M_{p,q} \to M_{p,1} \) to find a grid of octahedra in \( M_{p,q} \) all of which have volume nearly \( v_8 \).

**Lemma 6.4.** There exist \( k \to \infty \), \( \epsilon(k) \to 0 \), and \( n(k) \to \infty \) such that for \( p \geq n(k) \) there exist at least \( k \) consecutive ideal octahedra in \( P_{p,1} \) each of which has volume greater than \((v_8 - \epsilon(k))\).
Proof. Let $\epsilon(k) = \frac{1}{k}$ and $n(k) = k^3$. Suppose there are no $k$ consecutive octahedra each of whose volume is greater than $v_8 - \epsilon(k)$. This implies that there exist at least $n(k)/k = k^2$ octahedra each of which has volume at most $v_8 - \epsilon(k)$. Hence for $k > 12$ and $p > n(k)$,

$$\text{vol}(W(p,1) \cup B) \leq 4v_3 + (p - k^2)v_8 + k^2(v_8 - 1/k)$$

$$= 4v_3 + pv_8 - k$$

$$= (p - 2)v_8 + 4v_3 + 2v_8 - k$$

$$< (p - 2)v_8.$$  

This contradicts Theorem 5.5, which says that $(p - 2)v_8 < \text{vol}(W(p,1) \cup B)$. \hfill \Box

Corollary 6.5. For any $\epsilon > 0$ and any $k > 0$ there exists $N$ such that if $p,q > N$ then $\mathcal{P}_{p,q}$ contains a $k \times k$ grid of adjacent ideal octahedra, each of which has volume greater than $(v_8 - \epsilon)$.

Proof. Apply Lemma 6.4, taking $k$ sufficiently large so that $\epsilon(k) < \epsilon$. Then for any $N > n(k)$, if $p > N$ we obtain at least $k$ consecutive ideal octahedra with volume as desired. Now let $q > N$, so $q > k$. Use the $q$–fold cover $M_{p,q} \to M_{p,1}$. We obtain a $k \times q$ grid of octahedra, all of which have volume greater than $(v_8 - \epsilon(k))$, as shown in Figure 14. \hfill \Box

![Figure 14. Grid of octahedra with volumes near $v_8$ in $\mathcal{P}_{p,q}$, and base point.](image)

We are now ready to complete the proof of Theorem 1.9.

Proof of Theorem 1.9. Given $R > 0$, we will show that closed balls based in $M_{p,q}$ converge to a closed ball based in $\mathbb{R}^3 - W$. Since $M_{p,q} := S^3 - (W(p,q) \cup B)$ is a limit of $S^3 - W(p,q)$, this implies the result.

Take a basepoint $y \in \mathbb{R}^3 - W$ to lie in the interior of any octahedron, on one of the squares projecting to a checkerboard surface, say at the point where the diagonals of that square intersect. Consider the ball $B_R(y)$ of radius $R$ about the basepoint $y$. This will intersect some number of regular ideal octahedra. Notice that the octahedra are glued on all faces to adjacent octahedra, by the gluing pattern we obtained in Section 3, particularly Figure 7. Consider all octahedra in $\mathbb{R}^3 - W$ that meet the ball $B_R(y)$. Call these octahedra $\text{Oct}(R)$.

In $S^3 - (W(p,q) \cup B)$, for an octahedron of Lemma 4.1 coming from a square in the interior of the diagram of $W(p,q)$, the gluing pattern on each of its faces agrees with the gluing of an octahedron in $\mathbb{R}^3 - W$. Thus for $p, q$ large enough, we may find a collection of adjacent octahedra $\text{Oct}_{p,q}$ in $S^3 - W(p,q)$ with the same combinatorial gluing pattern as $\text{Oct}(R)$. Since all the octahedra are glued along faces to adjacent octahedra, Corollary 6.5 implies that if we choose $p, q$ large enough, then each ideal octahedron in $\text{Oct}_{p,q}$ has volume within $\epsilon$ of $v_8$.

It is known that the volume of a hyperbolic ideal octahedron is uniquely maximized by the volume of a regular ideal octahedron (see, e.g. [31, Theorem 10.4.7]). Thus as $\epsilon \to 0$, each ideal octahedron of $\text{Oct}_{p,q}$ must be converging to a regular ideal octahedron. So $\text{Oct}_{p,q}$
converges as a polyhedron to Oct(R). But then it follows that for suitable basepoints \(x_{p,q}\) in \(\mathcal{P}_{p,q}\), the balls \(B_R(x_{p,q})\) in \(\mathcal{P}_{p,q} \subset M_{p,q}\) converge to \(B_R(y)\) in the pointed bilipschitz topology. □

7. Guts and lower volume bounds

In this section, we begin the proof of Theorem 1.4 by showing that knots satisfying hypotheses of that theorem have volume bounded below by the volume of a certain right–angled polyhedron. The main result of this section is Theorem 7.13 below.

The techniques we use to bound volume from below involve guts of embedded essential surfaces, which we define below. Since we will be dealing with orientable as well as nonorientable surfaces, we say that any surface is essential if and only if the boundary of a regular neighborhood of the surface is an essential (orientable) surface, i.e. it is incompressible and boundary incompressible.

If \(N\) is a 3–manifold admitting an embedded essential surface \(\Sigma\), then \(N\setminus\Sigma\) denotes the manifold with boundary obtained by removing a regular open neighborhood of \(\Sigma\) from \(N\). Let \(\Sigma\) denote the boundary of \(N\setminus\Sigma\), which is homeomorphic to the unit normal bundle of \(\Sigma\). Note if \(N\) is an open manifold, i.e. \(N\) has nonempty topological frontier consisting of rank–2 cusps, then \(\Sigma\) will be a strict subset of the topological frontier of \(N\setminus\Sigma\), which consists of \(\Sigma\) and a collection of tori and annuli coming from cusps of \(N\).

**Definition 7.1.** The parabolic locus of \(N\setminus\Sigma\) consists of tori and annuli on the topological frontier of \(N\setminus\Sigma\) which come from cusps of \(N\).

We let \(D(N\setminus\Sigma)\) denote the double of the manifold \(N\setminus\Sigma\), doubled along the boundary \(\Sigma\). The manifold \(D(N\setminus\Sigma)\) admits a JSJ–decomposition. That is, it can be decomposed along essential annuli and tori into Seifert fibered and hyperbolic pieces. This gives an annulus decomposition of \(N\setminus\Sigma\): a collection of annuli in \(N\setminus\Sigma\), disjoint from the parabolic locus, that cut \(N\setminus\Sigma\) into \(I\)–bundles, Seifert fibered solid tori, and guts. Let \(\text{guts}(N\setminus\Sigma)\) denote the guts, which is the portion that admits a hyperbolic metric with geodesic boundary. Let \(D(\text{guts}(N\setminus\Sigma))\) denote the complete hyperbolic 3-manifold obtained by doubling the \(\text{guts}(N\setminus\Sigma)\) along the part of boundary contained in \(\Sigma\) (i.e. disjoint from the parabolic locus of \(N\setminus\Sigma\)).

**Theorem 7.2** (Agol–Storm–Thurston [3]). Let \(N\) be a finite volume hyperbolic manifold, and \(\Sigma\) an embedded \(\pi_1\)-injective surface in \(N\). Then

\[
\chi_-(\text{guts}(N\setminus\Sigma))\cdot\text{vol}(N) \geq \frac{1}{2} v_3||D(N\setminus\Sigma)|| = \frac{1}{2} \text{vol}(D(\text{guts}(N\setminus\Sigma)))).
\]

Here the value \(||\cdot||\) denotes the Gromov norm of the manifold.

We will prove Theorem 1.4 in a sequence of lemmas that concern the geometry and topology of alternating links, and particularly ideal checkerboard polyhedra that make up the complements of these alternating links. These ideal polyhedra were described by Menasco [27] (see also [22]). We review them briefly.

**Definition 7.3.** Let \(K\) be a hyperbolic alternating link with an alternating diagram (also denoted \(K\)) that is checkerboard colored. Let \(B\) (blue) and \(R\) (red) denote the checkerboard surfaces of \(K\). If we cut \(S^3-K\) along both \(B\) and \(R\), the manifold decomposes into two identical ideal polyhedra, denoted by \(P_1\) and \(P_2\). We call these the checkerboard ideal polyhedra of \(K\). They have the following properties.
(1) For each $P_i$, the ideal vertices and edges form a 4–valent graph on $\partial P_i$, and that graph is isomorphic to the projection graph of $K$ on the projection plane.

(2) The faces of $P_i$ are colored blue and red corresponding to the checkerboard coloring of $K$.

(3) To obtain $S^3 - K$ from $P_1$ and $P_2$, glue each red face of $P_1$ to the same red face of $P_2$, and glue each blue face of $P_1$ to the same blue face of $P_2$.

The gluing maps in item (3) are not the identity maps, but rather involve a single clockwise or counterclockwise “twist” (see [27] for details). In this paper, we won’t need the precise gluing maps, just which faces are attached.

The checkerboard surfaces $B$ and $R$ are well known to be essential in the alternating link complement [26]. We will cut along these surfaces, and investigate the manifolds $(S^3 - K)\setminus B$ and $(S^3 - K)\setminus R$. Note that because $K \subset B$, there is a homeomorphism $(S^3 - K)\setminus B \cong S^3\setminus B$, with parabolic locus mapping to identical parabolic locus. We will simplify notation by writing $S^3\setminus B$.

We will need to work with diagrams without Conway spheres. Menasco [25] and Thistlethwaite [37, 35] showed that for a prime alternating link diagram, Conway spheres can appear in very limited ways. Following Thistlethwaite’s notation, a Conway sphere is called visible if it is parallel to one dividing the diagram into two tangles, as in Figure 15. In [37, Proposition 5.1], it was shown that if there is a visible Conway sphere, then it is always visible in any prime alternating diagram of the same link.

**Definition 7.4.** We call a 2–tangle knotty if it is nontrivial, and not a (portion of a) single twist region; i.e. not a rational tangle of type $n$ or $1/n$ for $n \in \mathbb{Z}$. We will say that $K_n$ contains a cycle of tangles if $K_n$ contains a visible Conway sphere with a knotty tangle on each side.

For any link that contains a cycle of tangles, one of its two Tait graphs has a 2–vertex cut set coming from the regions on either side of a tangle. On the other hand, the Tait graphs of $W$ are both the square lattice, which is 4–connected. So $W$ has no cycle of tangles.

**Lemma 7.5.** Let $K$ be a link diagram that is prime, alternating, and twist-reduced with no cycle of tangles, with red and blue checkerboard surfaces. Obtain a new link diagram $K_R$ (resp. $K_B$) by removing red (resp. blue) bigons from the diagram of $K$ and replacing adjacent red (resp. blue) bigons in a twist region with a single crossing in the same direction. Then the resulting diagram $K_R$ (resp. $K_B$) is prime, alternating, and twist-reduced with no cycle of tangles.

**Proof.** Because twist regions bounding red bigons are replaced by a single crossing in the same direction, the diagram of $K_R$ remains alternating. If it is not prime, there would be
a simple closed curve $\gamma$ meeting the diagram transversely twice in two edges, with crossings on either side. Because the closed curve $\gamma$ does not meet crossings, we may re-insert the red bigons into a small neighborhood of the crossings of $K_R$ without meeting $\gamma$. Then $\gamma$ gives a simple closed curve in the diagram of $K$ meeting the diagram twice with crossings on either side, contradicting the fact that $K$ is prime.

Next suppose the diagram of $K_R$ contains a cycle of tangles. The corresponding visible Conway sphere must avoid crossings of $K_R$, so we may re-insert red bigons into a neighborhood of crossings of $K_R$ without meeting $\gamma$. Then $\gamma$ gives a simple closed curve in the diagram of $K$ meeting the diagram twice with crossings on either side, contradicting the fact that $K$ is prime.

Finally suppose that $K_R$ is not twist-reduced. Then there exists a simple closed curve $\gamma'$ meeting the diagram in exactly two crossings $x$ and $y$, with $\gamma'$ running through opposite sides of the two crossings, such that neither side of $\gamma'$ bounds a string of bigons in a twist region. Perturb $\gamma'$ slightly so that it contains $x$ on one side, and $y$ on the other. Then $\gamma'$ defines two nontrivial tangles, one on either side of $\gamma'$, neither of which can be knotty. This contradicts the above paragraph.

\[ \square \]

**Corollary 7.6.** For $K$ as in Lemma 7.5, let $K_{BR}$ be obtained by replacing any twist region in the diagram of $K$ by a single crossing (removing both red and blue bigons). Then the diagram of $K_{BR}$ will be prime, alternating, and twist-reduced with no cycle of tangles.

**Proof.** Replace $K$ in the statement of Lemma 7.5 with $K_R$ and apply the lemma to the blue bigons of $K_R$. □

Given a twist region in the diagram of a knot or link, recall that a crossing circle at that twist region is a simple closed curve in the diagram, bounding a disk in $S^3$ that is punctured exactly twice by the diagram, by strands of the link running through that twist region.

**Lemma 7.7.** Let $K$ be a hyperbolic link with a prime, alternating, twist-reduced diagram (also called $K$) with no cycle of tangles. Let $B$ denote the blue checkerboard surface of $K$. Let $K_R$ be the link with diagram obtained from that of $K$ by replacing adjacent red bigons by a single crossing, and let $B_R$ be the blue checkerboard surface for $K_R$. Then there exists a collection of twist regions bounding blue bigons in $K_R$, and Seifert fibered solid tori $E$, with the core of each solid torus in $E$ isotopic to a crossing circle encircling one of these twist regions, such that

\[
guts(S^3 \setminus B) = guts(S^3 \setminus B_R) = (S^3 \setminus B_R) - E,
\]

and

\[
\text{vol}(S^3 - K) \geq \frac{1}{2} v_3 \| D(S^3 \setminus B_R)\| = \frac{1}{2} \text{vol}(D((S^3 \setminus B_R) - E)).
\]

**Proof.** By Theorem 7.2,

\[
\text{vol}(S^3 - K) \geq \frac{1}{2} v_3 \| D((S^3 \setminus B)\| = \frac{1}{2} \text{vol}(D(guts(S^3 \setminus B))),
\]

so the claim about volumes follows from the claim about guts.

Lackenby notes that for a prime, twist–reduced alternating diagram $K$, $\text{guts}((S^3 - K) \setminus B)$ is equal to $\text{guts}((S^3 - K_R) \setminus B_R)$ [22, Section 5]. By [22, Theorem 13], $\chi(\text{guts}(S^3 \setminus B_R)) = \ldots$
Figure 16. Left: a cycle of three fused units. Right: All but one of the tangles are trivial.

\( \chi(S^3 \setminus B_R) \), where \( \chi(\cdot) \) denotes Euler characteristic. In fact, in the proof of that theorem, Lackenby shows that a bounding annulus of the characteristic submanifold is either boundary parallel, or separates off a Seifert fibered solid torus. We review the important features of that proof to determine the form of the Seifert fibered solid tori in the collection \( E \) required by this lemma.

In the case that there is a Seifert fibered solid torus, its boundary is made up of at least one annulus on \( B_R \) and at least one essential annulus \( A \) in \( S^3 \setminus B_R \). The essential annulus \( A \) is either parabolically compressible or parabolically incompressible, as defined in [22] (see also [14, Definition 4.5]). If it is parabolically compressible, then it decomposes into essential product disks. Lackenby proves in [22, Theorem 14] that there are no essential product disks. But if \( A \) is parabolically incompressible, then following the proof of [22, Theorem 14] carefully, we see that such a Seifert fibered solid torus determines a cycle of fused units, as shown in Figure 16 (left), which is a reproduction of Figure 14 of [22], with three fused units shown. More generally, there must be at least two fused units; otherwise, we have a Möbius band and not an essential annulus by the proof of [16, Lemma 4.1].

The ellipses in dotted lines in Figure 16 represent the boundaries of normal squares that form the essential annulus. Each of these encircles a fused unit, which is made up of two crossings and a (possibly trivial) tangle, represented by a circle in the figure. The Seifert fibered solid torus is made up of two copies of such a figure, one in each polyhedron, and consists of the region exterior to the ellipses. That is, it meets the blue surface in strips between dotted ellipses, and meets the red surface in a disk in the center of the diagram, and one outside the diagram. This gives a ball, with fibering of an \( I \)-bundle, with each interval of \( I \) parallel to the blue strips and with its endpoints on the red disks. The two balls are attached by gluing red faces, giving a Seifert fibered solid torus whose core runs through the center of the two red disks.

Now, we want to show that such a Seifert fibered solid torus only arises in a twist region of blue bigons. Consider a single cycle of fused units. Note that if any one of the tangles in that fused unit is non-trivial, then the boundary of the fused unit is a visible Conway sphere bounding at least one knotty tangle. If more than one of the fused units in the cycle have this property, then by grouping other tangles in the cycle of fused units into these non-trivial tangles, we find that our diagram contains a cycle of tangles, contrary to assumption.

So at most one of the fused units in the cycle can have a non-trivial tangle. If both tangles in a fused unit are non-trivial, then by joining one non-trivial tangle to all other tangles in the cycle of fused units, we obtain again a cycle of tangles, contrary to assumption.
So at most one of the tangles in the cycle of fused units is non-trivial. If all the tangles are trivial, then the diagram is that of a \((2,q)\)–torus link, contradicting the fact that it is hyperbolic. Hence exactly one of the tangles is non-trivial. This is shown in Figure 16 (right). Notice in this case, the cycle of fused units is simply a twist region of the diagram bounding blue bigons. Notice also that the Seifert fibered solid torus has the form claimed in the statement of the lemma.

Then \(guts(S^3 \setminus B_R) = (S^3 \setminus B_R) - E\). \(\Box\)

To simplify notation, let \(M_B = D(S^3 \setminus B)\) and let \(M_R = D(S^3 \setminus R)\).

**Lemma 7.8.** Let \(K\) be a link with a prime, alternating diagram with checkerboard surfaces \(B\) and \(R\). Then the manifold \(M_B\) contains an embedded essential surface \(DR\) obtained by doubling \(R\).

*Proof.* Note that \(M_B\) has an ideal polyhedral decomposition coming from the checkerboard ideal polyhedra of Definition 7.3. That is, \(S^3 - K\) is obtained from two polyhedra \(P_1\) and \(P_2\) with red and blue faces glued. The manifold \(S^3 \setminus B\) is obtained by cutting along blue faces, or removing the gluing maps on blue faces of \(P_i\).

Then the double, \(M_B\), is obtained by taking two copies, \(P^1_i\) and \(P^2_i\), of \(P_i\), gluing red faces of \(P^1_i\) to red faces of \(P^2_j\) by a twist, and gluing blue faces of \(P^1_i\) to those of \(P^2_j\) by the identity. Note that \(DR\) consists of all red faces of the four polyhedra.

Now, suppose that \(DR\) is not essential. Suppose first that the boundary of a regular neighborhood of \(DR\), call it \(\overline{DR}\), is compressible. Let \(F\) be a compressing disk for \(\overline{DR}\). Then \(\partial F\) lies on \(\overline{DR}\), but the interior of \(F\) is disjoint from a regular neighborhood of \(DR\). Make \(F\) transverse to the faces of the \(P^j_i\). Note now that \(F\) must intersect \(B\), else \(F\) lies completely in one of the \(P^j_i\), hence \(F\) can be mapped into \(S^3 - K\) to give a compression disk for \(R\) in \(S^3 - K\). This is impossible since \(R\) is incompressible in \(S^3 - K\).

Now consider the intersections of \(B\) with \(F\). We may assume there are no simple closed curves of intersection, or an innermost such curve would bound a compressing disk for \(B\), which we can isotope off using the fact that \(B\) is essential. Thus \(B \cap F\) consists of arcs running from \(\partial F\) to \(\partial F'\).

An outermost arc of \(B \cap F\) cuts off a subdisk \(F'\) of \(F\) whose boundary consists of an arc on \(\partial F \subset \overline{DR}\) and an arc on \(B\). The boundary \(\partial F'\) gives a closed curve on the checkerboard colored polyhedron which meets exactly two edges and two faces. Using the correspondence between the boundary of the polyhedron and the diagram of the link, item (1) of Definition 7.3, it follows that \(\partial F'\) gives a closed curve on the diagram of \(K\) that intersects the diagram exactly twice. Because the diagram of \(K\) is prime, there can be crossings on only one side of \(F'\). Thus the arc of \(\partial F'\) on \(B\) must have its endpoints on the same ideal edge of the polyhedron, and we may isotope it off, reducing the number of intersections \(|F \cap B|\). Continuing in this manner, we reduce to the case \(F \cap B = \emptyset\), which is a contradiction.

The proof that \(\overline{DR}\) is boundary incompressible follows a similar idea. Suppose as above that \(F\) is a boundary compressing disk for \(\overline{DR}\). Then \(\partial F\) consists of an arc \(\alpha\) on a neighborhood \(N(K)\) of \(K\), and an arc \(\beta\) on \(\overline{DR}\). As before, \(F\) must intersect \(B\) or it gives a boundary compression disk for \(R\) in \(S^3 - K\). As before, \(F\) cannot intersect \(B\) in closed curves, and primality of the diagram of \(K\) again implies \(F\) cannot intersect \(B\) in arcs that cut off subdisks of \(F\) with boundary disjoint from \(N(K)\). Hence all arcs of intersection \(F \cap B\) have one endpoint on \(\alpha = \partial F \cap N(K)\) and one endpoint on \(\beta = \partial F \cap \overline{DR}\). Again there must be an outermost such arc, cutting off a disk \(F' \subset F\) embedded in a single ideal polyhedron with \(\partial F'\)
consisting of three arcs, one on \( R \), one on \( B \), and one on an ideal vertex of the polyhedron (coming from \( N(K) \)). But then \( \partial F' \) must run through a vertex and an adjacent edge, hence it can be isotoped off, reducing the number of intersections of \( F \) and \( B \). Repeating a finite number of times, again \( B \cap F = \emptyset \), which is a contradiction. \( \square \)

**Lemma 7.9.** Let \( K_R \) be a link with a prime, twist-reduced diagram with no red bigons and no cycle of tangles, with checkerboard surfaces \( B_R \) and \( R_R \), and let \( E \) be the Seifert fibered solid tori from Lemma 7.7. Denote the double of the red surface in \( D(S^3 \setminus B_R) - E \) by \( DR_E \). Then \( DR_E \) is essential in \( D(S^3 \setminus B_R) - E \).

**Proof.** Recall that to prove this, we need to show that the boundary of a regular neighborhood of \( DR_E \), call it \( \widetilde{DR_E} \), is incompressible and boundary incompressible in \( D(S^3 \setminus B_R) - E \).

By the previous lemma, we know \( \widetilde{DR_R} \subset D(S^3 \setminus B_R) \) is incompressible. We will use the fact that \( D((S^3 \setminus B_R) - E) \) is an embedded submanifold of \( D(S^3 \setminus B_R) \), with \( DR_E \subset DR_R \), and that the cores of the Seifert fibered solid tori in \( E \) are all isotopic to crossing circles for the diagram, encircling blue bigons, by Lemma 7.7. Such a crossing circle intersects the polyhedra in the decomposition of \( D(S^3 \setminus B_R) \) in arcs with endpoints on distinct red faces.

Now, suppose there exists a compressing disk \( \Phi \) for \( \widetilde{DR_E} \). Then \( \Phi \subset D((S^3 \setminus B_R) - E) \subset D(S^3 \setminus B_R) \), which has a decomposition into ideal polyhedra coming from \( S^3 - K \), and so we may isotop \( \Phi \), keeping it disjoint from \( E \), so that it meets faces and edges of the polyhedra transversely. The boundary of \( \Phi \) lies entirely on \( \widetilde{DR_E} \), which we may isotope to lie entirely on the red faces of the polyhedra. As in the proof of the previous lemma, we consider how \( \Phi \) intersects blue faces.

Suppose first that \( \Phi \) does not meet any blue faces. Then \( \partial \Phi \) must lie in a single red face. But then it bounds a disk in that red face. If it does not bound a disk in \( \widetilde{DR_E} \), then that disk must meet \( E \). Then the disk meets the core of a component of \( E \), which is a portion of a crossing circle in the polyhedron. However, since \( \Phi \cap E \) is empty, the entire portion of the crossing circle in the polyhedron must lie within \( \partial \Phi \), and thus have both its endpoints in the polyhedron within the disk bounded by \( \partial \Phi \). This is a contradiction: any crossing circle meets two distinct red faces.

So suppose \( \Phi \) meets a blue face. Then just as above, an outermost arc of intersection on \( \Phi \) defines a curve on the diagram of \( S^3 - K \) meeting the knot exactly twice. It must bound no crossings on one side, by primality of \( K \). Then the curve bounds a disk on the polyhedron, and a disk on \( \Phi \), so either we may use these disks to isotope away the intersection with the blue face, or the disk in the polyhedron meets \( E \). But again, this implies that a portion of crossing circle lies between \( \Phi \) and this disk on the polyhedron. Again, since the crossing circle has endpoints in distinct red faces, this is impossible without intersecting \( \Phi \). This proves that \( \widetilde{DR_E} \) is incompressible.

The proof of boundary incompressibility is very similar. If there is a boundary compressing disk \( \Phi \), then \( \partial \Phi \) consists of an arc \( \beta \) on the boundary of \( D((S^3 \setminus B_R) - E) \) and an arc \( \alpha \) on red faces of the polyhedra. A similar argument to that above implies that \( \alpha \) cannot lie in a single red face: since \( \Phi \) does not meet crossing circles at the cores of \( E \), \( \partial \Phi \) would bound a disk on \( \widetilde{DR_E} \). So \( \alpha \) intersects a blue face of the polyhedron. Then consider an outermost blue arc of intersection. As above, it cannot cut off a disk with boundary consisting of exactly one red arc and one blue arc. So it cuts off a disk \( \Phi' \) on \( \Phi \) with boundary a red arc \( \alpha' \), an arc \( \beta' \) on the boundary of \( D((S^3 \setminus B_R) - E) \), and a blue arc \( \gamma' \). This bounds a disk on the boundary of the polyhedron. The interior of the disk cannot intersect \( E \), else \( \Phi' \) would
Lemma 7.10. Let \( K \) be a hyperbolic link with prime, alternating, twist-reduced diagram with no bigons and no cycle of tangles. Let \( B \) and \( R \) denote its checkerboard surfaces. Let \( M_B = D(S^3 \setminus B) \), and \( DR \) be the double of \( R \) in \( M_B \) as above. Then

\[
guts(M_B \setminus DR) = M_B \setminus DR.
\]

That is, in the annulus version of the JSJ decomposition of \( M_B \setminus DR \), there are no I–bundle or Seifert fibered solid torus components.

Proof. The manifold \( M_B \setminus DR \) is obtained by gluing four copies of the checkerboard ideal polyhedra of Definition 7.3, by gluing \( P^1_i \) to \( P^2_i \) by the identity on blue faces, \( i = 1, 2 \), and leaving red faces unglued.

If \( M_B \setminus DR \) does contain I–bundle or Seifert fibered solid torus components, then there must be an essential annulus \( A \) in \( M_B \setminus DR \) disjoint from the parabolic locus. Suppose \( A \) is such an annulus. Then \( A \) has boundary components on \( DR \) and interior disjoint from a neighborhood of \( R \). Put \( A \) into normal form with respect to the polyhedra \( P^j_i \) of \( M_B \setminus DR \). Because \( A \) is essential, it must intersect \( B \) in arcs running from one component of \( \partial A \) to the other, cutting \( A \) into an even number of squares \( S_1, S_2, \ldots, S_{2n} \) alternating between \( P^1_i \) and \( P^2_i \), for fixed \( i \).

The square \( S_2 \) is glued along some arc \( \alpha_1 \) in \( A \cap B \) to the square \( S_1 \), and \( S_2 \) is glued along another arc \( \alpha_2 \) in \( A \cap B \) to the square \( S_3 \). The squares \( S_1 \) and \( S_3 \) lie in the same polyhedron \( P^j_i \). Superimpose \( \partial S_2 \) on that polyhedron. Because \( S_1 \) and \( S_3 \) are glued by the identity on \( B \), the arcs of all three squares coming from one component of \( \partial A \) lie on the same red face of the polyhedron. Similarly for the other component of \( \partial A \). The same argument applies to any three consecutive squares, showing in general that one component of \( \partial A \) lies entirely in two identical red faces of \( P^1_i \) and \( P^2_i \), and these are glued by the identity on adjacent blue faces. The result is shown in Figure 17 (right).

By hypothesis, we have no cycle of tangles in our diagram. Thus one of the \( T_i \) in Figure 17 must be trivial or a part of a twist region. If trivial, then the square \( S_1 \) is not normal, which is a contradiction. But because \( K \) has a diagram with no bigons, \( T_i \) cannot contain bigons, so \( T_i \) must be a single crossing. Note neither \( T_{i-1} \) nor \( T_{i+1} \) can be a single crossing (it could be that \( T_{i-1} = T_{i+1} \)), else \( T_i \) and this tangle would form a bigon. Thus \( T_i \cup T_{i-1} \) is a tangle that is non-trivial, and does not bound a portion of a twist region. If \( T_{i+1} \) is a distinct tangle,
then we have a cycle of tangles (possibly after performing this same move elsewhere to move single crossings into larger tangles).

The only remaining possibility is that there are just two tangles, \( T_1 \) and \( T_2 \), and \( T_1 \) is a single crossing, and \( T_2 \) is a knotty tangle. But then \( S_1 \cup S_2 \) is a boundary parallel annulus in \( M_B \setminus DR \), parallel to the double of the ideal vertex corresponding to this single crossing. A boundary parallel annulus is not essential. □

Lemma 7.11. Let \( K_R \) be a hyperbolic link with prime, alternating, twist-reduced diagram with no red bigons and no cycle of tangles, with checkerboard surfaces \( B_R \) and \( R_R \), and let \( E \) be the Seifert fibered solid tori from Lemma 7.7. Finally, let \( K_{BR} \) be the new link obtained by removing all the blue bigons from the diagram of \( K_R \), and let \( B_{BR} \) and \( R_{BR} \) be its checkerboard surfaces. Then

\[
guts(D((S^3 \setminus B_R) - E) \setminus DR_R) = \text{guts}(D(S^3 \setminus B_{BR}) \setminus DR_{BR}) = D(S^3 \setminus B_{BR}) \setminus DR_{BR}.
\]

Thus

\[
\text{vol}(D((S^3 \setminus B_R) - E)) \geq \frac{1}{2} \text{vol}(D(S^3 \setminus B_{BR}) \setminus DR_{BR}).
\]

Proof. Because \( K_{BR} \) has no bigons, Lemma 7.10 implies the last equality:

\[
guts(D(S^3 \setminus B_{BR}) \setminus DR_{BR}) = D(S^3 \setminus B_{BR}) \setminus DR_{BR}.
\]

Hence it remains to show the first equality.

Recall that \( S^3 \setminus B_R \) is obtained by gluing two checkerboard ideal polyhedra along their red faces, leaving blue faces unglued. The Seifert fibered solid tori of \( E \) lie in those polyhedra in blue twist regions, meeting the bigons of the twist region and the adjacent red faces, as on the right of Figure 16.

When we double along blue, the Seifert fibered solid tori in \( E \) glue to give a Seifert fibered submanifold, with boundary in \( D(S^3 \setminus B_R) \) an essential torus obtained by gluing two annuli \( A \), where \( A \) is made up of squares bounding the fused units in Figure 16. Thus \( (S^3 \setminus B_R) - E \) consists of portions of the polyhedra that lie outside of \( E \). For each such solid torus, in each polyhedron these consist of regions bounding a single bigon, and one region bounding the fused unit with the non-trivial tangle.

Note that each region consisting of a square encircling a single bigon is fibered. The bigon itself is fibered, with fibers meeting each edge of the bigon in a single point and parallel to the ideal vertex, which is part of the parabolic locus. Similarly, the square encircling the bigon gives a fibered disk. Together, these two squares bound a fibered box, where fibers have one endpoint on one red face, one endpoint on the other, and are parallel to the fibers of the bigon.

To obtain \( D((S^3 \setminus B_R) - E) \setminus DR_R \), we take four polyhedra, and glue them in pairs by the identity along their blue faces. Notice that the fibered boxes at blue twist regions must belong to the characteristic \( I \)-bundle of the cut manifold, so these twist regions cannot be part of the guts of this manifold. Therefore the guts is obtained by considering only the quadrilateral bounding the non-trivial tangle. The corresponding polyhedron is equivalent to the polyhedron obtained by replacing the blue twist region with a single bigon.

But now consider any remaining blue bigons in the diagram, including this, and including blue bigons from twist regions that did not give rise to Seifert fibered solid tori in the previous step. As before, a neighborhood of any such bigon and the parabolic locus is an \( I \)-bundle, and is part of the characteristic \( I \)-bundle of the manifold. So if \( Y \) is a neighborhood of the union of the parabolic locus and all the bigons in the polyhedra, then the guts of
\[ D((S^3 \setminus B_R) \setminus E) \setminus DR_R := N_R \] is the guts of the closure of \( N_R - Y \), where the latter is given parabolic locus \( c(\partial Y - \partial N_R) \).

As in the beginning of Section 5 of [22], this can be identified explicitly. If we replace the bigons of each blue twist region by a single ideal vertex of the polyhedron, then the remaining portions of the polyhedra will be identical. But this is exactly the polyhedron of the link with no bigons \( K_{BR} \). The desired result follows.

\[ \square \]

**Lemma 7.12.** When \( K \) is a link with prime, alternating, twist-reduced diagram with no bigons and no cycle of tangles, the doubles of manifolds \( M_B \setminus DR \) and \( M_R \setminus DB \) admit isometric finite volume hyperbolic structures. In these structures, the surfaces coming from \( DR \) and \( DB \) are totally geodesic and meet at angle \( \pi/2 \). The manifolds are both obtained by gluing eight isometric copies of a right angled hyperbolic ideal polyhedron \( P \), and this polyhedron is equivalent to the checkerboard ideal polyhedron of \( K \).

**Proof.** By Lemma 7.10, \( M_B \setminus DR \) is all guts, with no \( I \)-bundle or Seifert fibered pieces. Thus when we double it, the resulting manifold admits a complete finite volume hyperbolic structure, in which \( DR \) is a totally geodesic surface. The same argument applies to the double of \( M_R \setminus DB \).

Now we show that these two doubles are homeomorphic. Both are homeomorphic to eight copies of the checkerboard polyhedron \( P \) coming from \( K \), glued by identity maps on faces, as follows. In particular, recall that \( M_B \setminus DR = D(S^3 \setminus B) \setminus DR \) is obtained by taking four copies of the polyhedron \( P \), denoted by \( P^1_i, P^2_i, P^1_j, \) and \( P^2_j \), with red faces unglued, and blue faces of \( P^1_i \) glued to those of \( P^2_j \) by the identity, for \( j = 1, 2 \). When we double across \( DR \), we obtain four more polyhedra, \( P^1_j, i, j = 1, 2 \), with blue faces of \( P^1_j \) glued to blue faces of \( P^2_j \) by the identity for each \( i = 1, 2 \), and red faces of \( P^1_j \) glued to red faces of \( P^2_j \) by the identity, for \( i, j = 1, 2 \). To form the double of \( M_R \setminus DB \), we repeat the process, only cut and glue along red first, then along blue. Again we obtain eight copies of the checkerboard polyhedron, denoted \( Q^1_i \) and \( Q^2_i \), for \( i, j = 1, 2 \), with \( Q^1_i \) glued to \( Q^2_i \) by the identity map on red faces, \( \bar{Q}^1_i \) glued to \( \bar{Q}^2_i \) by the identity on red faces, and \( \bar{Q}^1_i \) glued to \( \bar{Q}^2_i \) by the identity on blue faces. This gluing is summarized in the following diagram.

```
\[
\begin{array}{c}
P^1_i \xleftarrow{\text{blue}} \xrightarrow{\text{red}} P^2_i \\
\text{red} \downarrow \quad \downarrow \quad \text{blue} \\
P^1_i \xleftarrow{\text{blue}} \xrightarrow{\text{red}} P^2_i \\
\end{array}
\quad\quad\quad
\begin{array}{c}
Q^1_i \xleftarrow{\text{red}} \xrightarrow{\text{blue}} Q^2_i \\
\text{red} \downarrow \quad \downarrow \quad \text{blue} \\
\bar{Q}^1_i \xleftarrow{\text{red}} \xrightarrow{\text{blue}} \bar{Q}^2_i \\
\end{array}
\]
```

Now build a homeomorphism by mapping by the identity between \( P \)'s and \( Q \)'s, rotating the diagram on the left to match that on the right. That is, map \( P^1_i \) to \( Q^2_i \), map \( P^1_i \) to \( Q^1_i \), map \( P^2_i \) to \( Q^2_i \), and map \( P^2_i \) to \( Q^1_i \), for \( i = 1, 2 \). These maps give the identity on the interiors of the polyhedra, the identity on interiors of faces, and extend to identity maps on edges between faces. Thus they give a homeomorphism of spaces.

Since \( D(M_B \setminus DB) \) and \( D(M_R \setminus DR) \) are finite volume hyperbolic manifolds, by Mostow–Prasad rigidity [28, 30], the doubles are actually isometric. Thus \( DB \) and \( DR \) are totally geodesic in each of them. Hence cutting along these totally geodesic surfaces yields eight copies of \( P \), each with totally geodesic red and blue faces.

Finally, since \( DR \) is geodesic when we double along \( DB \), it must intersect \( DB \) at right angles. Thus the surfaces meet everywhere at angle \( \pi/2 \), as claimed. \[ \square \]
Theorem 7.13. Suppose $K$ is a link with a prime, alternating, twist-reduced, diagram with no cycle of tangles. Let $K'$ be the link with both red and blue bigons (if any) removed. Let $P(K')$ denote the checkerboard polyhedron coming from $K'$, given an ideal hyperbolic structure with all right angles. Then

$$\text{vol}(S^3 - K) \geq 2 \text{vol}(P(K')).$$

Proof. Lemmas 7.7, 7.8, and 7.11 imply that when $K$ admits a prime, alternating diagram with no bigons and no cycle of tangles,

$$\text{vol}(S^3 - K) \geq \frac{1}{4} \text{vol}(D(M_B \backslash DR)).$$

Lemma 7.12 further implies that $\text{vol}(D(M_B \backslash DR)) = 8 \text{vol}(P(K))$. Hence $\text{vol}(S^3 - K) \geq 2 \text{vol}(P(K)).$

If $K$ contains bigons, let $K_B$ and $K_R$ denote the link with the blue and red bigons removed, respectively, and let $K_{BR} = K'$ denote the link with both blue and red bigons removed. Let $B_B$ and $R_B$ denote the checkerboard surfaces of $K_B$, let $B_R$ and $R_R$ denote the checkerboard surfaces of $K_R$, and let $B_{BR}$ and $R_{BR}$ denote the checkerboard surfaces of $K_{BR}$.

Lemma 7.7 implies that

$$\text{vol}(S^3 - K) \geq \frac{1}{2} \text{vol}(D((S^3 \backslash B_R) - E_R)),$$

where $E_R$ is a collection of Seifert fibered solid tori. By Lemma 7.11,

$$\text{vol}(D((S^3 \backslash B_R) - E_R)) \geq \frac{1}{2} \text{vol}(D(D(S^3 \backslash B_{BR}) \backslash DR_{BR})).$$

By Lemma 7.12, $\text{vol}(D(D(S^3 \backslash B_{BR}) \backslash DR_{BR})) = 8 \text{vol}(P(K_{BR})).$ Thus

$$\text{vol}(S^3 - K) \geq 2 \text{vol}(P(K_{BR})).$$

\[\Box\]

8. Geometrically maximal knots

In this section, we complete the proof of Theorem 1.4.

Since the number of crossings in a diagram of $K$ is equal to the number of ideal vertices of $P(K)$ by item (1) of Definition 7.3, our goal is to bound the ratio of the volume of $P(K)$ to the number of vertices of $P(K)$. We will do so using methods of Atkinson [4], which rely on fundamental results of He [19] on the rigidity of circle patterns. In particular, we employ the proof of Proposition 6.3 of [4], which obtains the volume per vertex bounds we desire, but for a different class of polyhedra. We first set up notation.

If we lift the ideal polyhedron $P := P(K)$ into $\mathbb{H}^3$, the geodesic faces lift to lie on geodesic planes. These correspond to Euclidean hemispheres, and each extends to give a circle on $\mathbb{C}$. For every such polyhedron, we obtain a finite collection of circles (or disks) on $\mathbb{C}$ meeting at right angles in pairs, and meeting at ideal vertices in sets of four. This defines a finite disk pattern $D$ on $\mathbb{C}$, with angle $\pi/2$ between disks. Let $G(D)$ be the graph with a vertex for each disk and an edge between two vertices when the corresponding disks overlap on $\mathbb{C}$. Edges of $G(D)$ are labeled by the angle at which the disks meet, which in our case is $\pi/2$ for each edge. Note all faces of $G(D)$ in our case are quadrilaterals, since all vertices of $P$ are 4–valent, hence the disk pattern $D$ is rigid [19].

Similarly, as described in Section 3, we can form an infinite polyhedron $P_W$ corresponding to a checkerboard polyhedron of the infinite weave $W$: view the diagram of $W$ as squares with vertices on the integer lattice, and for each square draw the Euclidean circle on $\mathbb{C}$ running
through its four vertices. Each such circle on $C$ corresponds to a hemisphere in $\mathbb{H}^3$. Let $P_W$ be the infinite polyhedron obtained from $\mathbb{H}^3$ by cutting out all half-spaces of $\mathbb{H}^3$ bounded by these hemispheres. (In Section 3, this polyhedron was called $\tilde{X}_1$.) By construction, faces meet in pairs at right angles, and at ideal vertices in fours. We obtain a corresponding rigid disk pattern $D_\infty$, as in Figure 5(a).

**Definition 8.1.** Let $D$ and $D'$ be disk patterns. Give $G(D)$ and $G(D')$ the path metric in which each edge has length 1. For disks $d$ in $D$ and $d'$ in $D'$, we say $(D,d)$ and $(D',d')$ agree to generation $n$ if the balls of radius $n$ about vertices corresponding to $d$ and $d'$ admit a graph isomorphism, with labels on edges preserved.

For any disk $d$, we let $S(d)$ be the geodesic hyperplane in $\mathbb{H}^3$ whose boundary agrees with that of $d$. That is, $S(d)$ is the Euclidean hemisphere in $\mathbb{H}^3$ with boundary on the boundary of $d$. For a disk pattern coming from a right angled ideal polyhedron, the planes $S(d)$ form the boundary faces of the polyhedron. In this case, the disk pattern $D$ is said to be simply connected and ideal.

If $d$ is a disk in a disk pattern $D$, with intersecting neighboring disks $d_1, \ldots, d_m$ in $D$, then $S(d) \cap S(d_i)$ is a geodesic $\gamma_i$ in $\mathbb{H}^3$. Assume that the boundary of $d$ is disjoint from the point at infinity. Then the geodesics $\gamma_i$ on $S(d)$ bound an ideal polygon in $\mathbb{H}^3$, and we may take the cone over this polygon to the point at infinity. Denote the ideal polyhedron obtained in this manner by $C(d)$ (see Figure 18).

**Figure 18.** (a) Disk pattern $(D,d)$ and graph $G(D)$. (b) Ideal polyhedron $C(d)$.

The following lemma restates Lemma 6.2 of [4].

**Lemma 8.2** (Atkinson [4]). There exists a bounded sequence $0 \leq \epsilon_\ell \leq b < \infty$ converging to zero such that if $D$ is a simply connected, ideal, rigid finite disk pattern containing a disk $d$ so that $(D_\infty,d_\infty)$ and $(D,d)$ agree to generation $\ell$ then
\[ |\text{vol}(C(d)) - \text{vol}(C(d_\infty))| \leq \epsilon_\ell. \]

We now generalize Proposition 6.3 of [4] to classes of polyhedra that include the checkerboard ideal polyhedra of interest in this paper. The proof of the following lemma is essentially contained in [4], but we present it here for completeness.

**Lemma 8.3.** Let $D_\infty$ denote the infinite disk pattern coming from $P_W$, as defined above, with fixed disk $d_\infty$. Let $P_n$ be a sequence of right angled hyperbolic polyhedra with corresponding disk patterns $D_n$. Suppose the following hold.

1. If $F^n_\ell$ is the set of disks $d$ in $D_n$ such that $(D_n,d)$ agrees to generation $\ell$ but not to generation $\ell + 1$ with $(D_\infty,d_\infty)$, then
\[ \lim_{n \to \infty} \frac{|\bigcup F^n_\ell|}{v(P_n)} = 1, \quad \text{and} \quad \lim_{n \to \infty} \frac{|F^n_\ell|}{v(P_n)} = 0. \]
(2) For every positive integer $k$, let $|f_k^n|$ denote the number of faces of $P_n$ with $k$ sides that are not contained in $\cup F^m_\ell$ and do not meet the point at infinity. Then

$$\lim_{n \to \infty} \frac{\sum k |f_k^n|}{v(P_n)} = 0.$$ 

Under these hypotheses,

$$\lim_{n \to \infty} \frac{\text{vol}(P_n)}{v(P_n)} = \frac{v_8}{2}.$$ 

Proof. First, let $f_k^n$ be a face with $k$ sides that is not contained in $\cup F^m_\ell$, and which does not meet the point at infinity. Then $\text{vol}(C(f_k^n))$ has volume at most $k$ times the maximum value of the Lobachevsky function $\Lambda$, which is $\Lambda(\pi/6)$ [38, Chapter 7]. Let $E^n$ denote the sum of the actual volumes of all the faces $f_k^n$, for every integer $k$. Then we have

$$E^n \leq \sum_k \sum_{f_k^n} k \Lambda(\pi/6) = \sum_k k |f_k^n| \Lambda(\pi/6).$$

For any face $f$ in $F^n_\ell$, let $\delta^n_\ell(f)$ be a positive number such that $\text{vol}(C(f)) = v_8/2 + \delta^n_\ell(f)$. Then

$$\text{vol}(P_n) = \sum_\ell \sum_{f \in F^n_\ell} \left( \frac{v_8}{2} \pm \delta^n_\ell(f) \right) + E^n.$$

Hence

$$\text{vol}(P_n) = \left| \bigcup_\ell F^n_\ell \right| \frac{v_8}{2} + \sum_\ell \sum_{f \in F^n_\ell} (\pm \delta^n_\ell(f)) + E^n.$$

We divide each term by $v(P_n)$ and take the limit. For the first term, we obtain

$$\lim_{n \to \infty} \frac{\left| \bigcup_\ell F^n_\ell \right| v_8}{2 v(P_n)} = \frac{v_8}{2}.$$

By Lemma 8.2, there are positive numbers $\epsilon_\ell$ such that $\delta^n_\ell(f) \leq \epsilon_\ell$, so the second term becomes

$$\lim_{n \to \infty} \frac{\sum_\ell \sum_{f \in F^n_\ell} (\pm \delta^n_\ell(f))}{v(P_n)} \leq \lim_{n \to \infty} \frac{\sum_\ell |F^n_\ell|^\epsilon_\ell}{v(P_n)}.$$

This can be seen to be zero, as follows. Fix any $\varepsilon > 0$. Because $\lim_{\ell \to \infty} \epsilon_\ell = 0$, there is $L$ sufficiently large that $\epsilon_\ell < \varepsilon/3$, for $\ell > L$. Then $\sum_{\ell=1}^L \epsilon_\ell$ is a finite number, say $M$.

By item (1) in the statement of this lemma, there exists $N$ such that if $n > N$ then $\max_{\ell \leq L} |F^n_\ell|/v(P_n) < \varepsilon/(3M \cdot L)$ and $|\bigcup_\ell F^n_\ell|/v(P_n) < (1 + \varepsilon)$. Then for $n > N$,

$$\sum_\ell \frac{|F^n_\ell|^\epsilon_\ell}{v(P_n)} = \sum_{\ell=1}^L \frac{|F^n_\ell|^\epsilon_\ell}{v(P_n)} + \sum_{\ell > L} \frac{|F^n_\ell|^\epsilon_\ell}{v(P_n)} < \frac{\varepsilon L}{3M \cdot L} M + (1 + \varepsilon) \frac{\varepsilon}{3} < \varepsilon.$$

Hence the limit of the second term is zero.

Finally, the third term gives us

$$\lim_{n \to \infty} \frac{E^n}{v(P_n)} \leq \sum_k k |f_k^n| \Lambda(\pi/6) = 0.$$

Therefore, $\lim_{n \to \infty} \text{vol}(P_n)/v(P_n) = v_8/2$. \qed

We can now prove Theorem 1.4, which we recall from the introduction.
Theorem 1.4. Let $K_n$ be any sequence of hyperbolic alternating links that contain no cycle of tangles, such that

1. there are subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(W)$, and
2. $\lim_{n \to \infty} |G_n|/c(K_n) = 1$.

Then $K_n$ is geometrically maximal: $\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_8$.

Proof. We may assume that $K_n$ are prime and twist-reduced diagrams. Any hyperbolic link is prime. If a sequence $K_n$ satisfies the other conditions above, then the twist-reduced diagrams $K_n$ also satisfy these conditions because the twist reduction does not change crossing number, and only changes the diagram in $G(K_n) - G_n$.

We first consider the case that $K_n$ contains no bigons. Because $K_n$ is prime, alternating, has no bigons and no cycle of tangles, Theorem 7.13 implies that $\text{vol}(S^3 - K_n) \geq 2 \text{vol}(P(K_n))$.

The crossing number of $K_n$ is equal to the number of vertices of $P(K_n)$, so dividing by crossing number gives

$$\frac{\text{vol}(S^3 - K_n)}{c(K_n)} \geq 2 \frac{\text{vol}(P(K_n))}{v(P(K_n))}.$$

Using the fact that $W$ is an amenable Cayley graph for $\mathbb{Z} \oplus \mathbb{Z}$, the Følner condition (1) above actually implies a stronger condition: There is a sequence of convex sets $S_n$ in $\mathbb{R}^2$ such that $S_n \subset S_{n+1}$ and each $S_n$ contains a ball of radius $r_n$, with $r_n \to \infty$ as $n \to \infty$, and $G_n = S_n \cap G$ is a Følner sequence for $G$. See, for example, [29, Lemma 5.3]. Hence, there exist vertex sets $A_n = \{x_i\} \subset G(K_n)$, with $A_n \subset A_{n+1}$, such that each $G_n$ can be expressed as a union of balls, $G_n = \bigcup_{x_i \in A_n} B(x_i; r_i)$. Therefore, there are sequences $p_n, q_n \to \infty$ monotonically, such that $|G_n| = p_n q_n + O(p_n + q_n)$ and $|\partial G_n| = O(p_n + q_n)$.

Now, the disk pattern graph $G(D_n)$ associated with $P(K_n)$ is the dual of $G(K_n)$. Since $G(W)$ is isomorphic to its own dual, to simplify notation we may assume $G_n \subset G(D_n)$. Pick a point in $G(D_n)$ which is outside $G_n$ to send to infinity. We want to apply Lemma 8.3.

The set $F^\ell_n$ of disks $d$ such that $(D_n, d)$ agrees to generation $\ell$ but not to generation $\ell + 1$ with $(D_\infty, d_\infty)$ consists of all disks of distance $\ell$ from the boundary of $G_n$ in $D_n$. First, consider a maximal $p_n \times q_n$ rectangular lattice $G_n$. The points of distance $\ell$ from the boundary of the lattice are contained in a rectangle of distance $\ell$ from the boundary of the lattice. There are at most $2(p_n - \ell) + 2(q_n - \ell)$ of these points, so $|F^\ell_n| \leq 2(p_n + q_n)$ for each $\ell$. The number of disks in the union $\bigcup_\ell F^\ell_n$ consists of the disks in the $p_n q_n$ lattice, which is $p_n q_n$ disks. More generally, since $|G_n| = p_n q_n + O(p_n + q_n)$ and $|\partial G_n| = O(p_n + q_n)$,

(a) $|F^\ell_n| = O(p_n + q_n)$ for each $\ell$, and
(b) $|\bigcup_\ell F^\ell_n| = p_n q_n + O(p_n + q_n)$.

Hence as $n$ approaches infinity, we obtain the limits required for item (1) of Lemma 8.3.

As for item (2), by counting vertices, $\sum k |f^\ell_k| \leq 4 |G(K_n) - G_n|$. The factor 4 appears because every vertex belongs to four faces, so it will be counted at most four times in the sum. Item (2) now follows from $|G(K_n)| = v(P(K_n)) = c(K_n)$ and $\lim_{n \to \infty} |G_n|/c(K_n) = 1$.

Thus by Lemma 8.3,

$$\lim_{n \to \infty} \frac{\text{vol}(S^3 - K_n)}{c(K_n)} \geq \lim_{n \to \infty} 2 \frac{\text{vol}(P(K_n))}{v(P(K_n))} = v_8.$$

By equation (1), this must be an equality.

Finally, if $K_n$ contains bigons, let $K'_n$ denote the link with both blue and red bigons removed. Theorem 7.13 implies that $\text{vol}(S^3 - K_n) \geq 2 \text{vol}(P(K'_n))$.
Now, since all bigons of $K_n$ must be in $G(K_n) - G_n$, the above proof implies
\[
\lim_{n \to \infty} \frac{\text{vol}(S^3 - K_n)}{c(K_n)} \geq \lim_{n \to \infty} 2 \frac{\text{vol}(P(K'_n))}{v(P(K'_n))} = v_8. \]
\[\square\]

References


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