VOLUME BOUNDS FOR WEAVING KNOTS

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Abstract. Weaving knots are alternating knots with the same projection as torus knots, and were conjectured by X.-S. Lin to be among the maximum volume knots for fixed crossing number. We provide the first asymptotically correct volume bounds for weaving knots, and we prove that the infinite weave is their geometric limit.

1. Introduction

The crossing number of a knot is one of the oldest knot invariants, and it has been used to study knots since the 19th century. Since the 1980s, hyperbolic volume has also been used to study and distinguish knots. We are interested in the relationship between volume and crossing number. On the one hand, it is very easy to construct sequences of knots with crossing number approaching infinity but bounded volume. For example, start with a reduced alternating diagram of an alternating knot, and add crossings by twisting two strands in a fixed twist region of the diagram. By work of Thurston [16], the volume of the knots is bounded, but the crossing number increases with the number of crossings in the twist region (see, e.g., [15]). On the other hand, since there are only a finite number of knots with bounded crossing number, among all such knots there must be one (or more) with maximal volume. It is an open problem to determine the maximal volume of all knots with bounded crossing number, and to find the knots that realize the maximal volume per crossing number.

In this paper, we study a class of knots that are candidates for those with the largest volume per crossing number: weaving knots. For these knots, we provide explicit, asymptotically sharp bounds on their volumes. We also prove that they converge geometrically to an infinite link complement which asymptotically maximizes volume per crossing number. Thus, while our methods cannot answer the question of which knots maximize volume per crossing number, they provide evidence that weaving knots are among those with largest volume.

A weaving knot $W(p,q)$ is the alternating knot or link with the same projection as the standard $p$–braid $(\sigma_1 \ldots \sigma_{p-1})^q$ projection of the torus knot or link $T(p,q)$. Thus, the crossing number $c(W(p,q)) = q(p-1)$. For example, $W(5,4)$ and $W(7,12)$ are shown in Figure 1. By our definition, weaving knots can include links with many components.

Xiao-Song Lin suggested in the early 2000s that weaving knots would be among the knots with largest volume for fixed crossing number. In fact, $W(5,4)$ has the second largest volume among all knots with $c(K) \leq 16$, which can be verified using Knotscape [8] or SnapPy [4]. (Good guess among 1,701,954 hyperbolic knots with at most 16 crossings!)

It is a consequence of our main results in [3] that weaving knots are geometrically maximal. That is, they satisfy:

\begin{equation}
\lim_{p,q \to \infty} \frac{\text{vol}(W(p,q))}{c(W(p,q))} = v_{\text{oct}},
\end{equation}

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where $v_{\text{oct}} \approx 3.66$ is the volume of a regular ideal octahedron, and $\text{vol}(\cdot)$ and $c(\cdot)$ denote volume and crossing number, respectively. Moreover, it is known that for any link the volume density $\text{vol}(K)/c(K)$ is always bounded above by $v_{\text{oct}}$.

What was not known is how to obtain sharp estimates on the volumes of $W(p,q)$ in terms of $p$ and $q$ alone, which is needed to bound volume for fixed crossing number. Lackenby gave bounds on volumes of alternating knots [9], improved by Agol, Storm and Thurston [2]. However, their bounds are not asymptotically correct for weaving knots; nor can they be used to establish the limit of equation (1). Our methods in [3] also fail to give bounds on volumes of knots for fixed crossing number, including $W(p,q)$. Thus, it seems that determining explicit, asymptotically correct bounds for the volume density $\text{vol}(K)/c(K)$ in finite cases is harder and requires different methods than proving the asymptotic volume density for sequences.

In this paper, for weaving knots $W(p,q)$ we provide asymptotically sharp, explicit bounds on volumes in terms of $p$ and $q$ alone.

Theorem 1.1. If $p \geq 3$ and $q \geq 7$, then
\[
v_{\text{oct}} (p - 2) q \left(1 - \frac{(2\pi)^2}{q^2}\right)^{3/2} \leq \text{vol}(W(p,q)) \leq (v_{\text{oct}} (p - 3) + 4v_{\text{tet}}) q.
\]

Here $v_{\text{tet}} \approx 1.01494$ is the volume of the regular ideal tetrahedron, and $v_{\text{oct}}$ is the same as above. Since $c(W(p,q)) = q (p - 1)$, these bounds provide another proof of equation (1).

The methods involved in proving Theorem 1.1 are completely different than those used in [3], which relied on volume bounds via guts of 3–manifolds cut along essential surfaces as in [2]. Instead, the proof of Theorem 1.1 involves explicit angle structures and the convexity of volume, as in [14].

Moreover, applying these asymptotically sharp volume bounds for the links $W(p,q)$, we prove that their geometric structures converge, as follows.

The infinite weave $W$ is defined to be the infinite alternating link with the square grid projection, as in Figure 2. In [3], we showed that there is a complete hyperbolic structure on $\mathbb{R}^3 - W$ obtained by tessellating the manifold by regular ideal octahedra such that the volume density of $W$ is exactly $v_{\text{oct}}$.

Theorem 1.2. As $p,q \to \infty$, $S^3 - W(p,q)$ approaches $\mathbb{R}^3 - W$ as a geometric limit.

Proving that a class of knots or links approaches $\mathbb{R}^3 - W$ as a geometric limit seems to be difficult. For example, in [3] we showed that many families of knots $K_n$ with diagrams approaching that of $W$, in an appropriate sense, satisfy $\text{vol}(K_n)/c(K_n) \to v_{\text{oct}}$. However, it is unknown whether their complements $S^3 - K_n$ approach $\mathbb{R}^3 - W$ as a geometric limit, and the proof in [3] does not give this information. Theorem 1.2 provides the result for $W(p,q)$.

It is an interesting fact that every knot can be obtained by changing some crossings of $W(p,q)$ for some $p,q$, which follows from the proof for torus knots in [11]. We conjecture that
Figure 2. The infinite alternating weave

Figure 3. Polygonal decomposition of cusp corresponding to braid axis. A fundamental region consists of four triangles and $2(p-3)$ quads. The example shown here is $p=5$.

the upper volume bound in Theorem 1.1 applies to any knot or link obtained by changing crossings of $W(p,q)$. This is a special case of the following conjecture, which appears in [3].

**Conjecture 1.3.** Let $K$ be an alternating hyperbolic knot, and $K'$ be obtained by changing any proper subset of crossings of $K$. Then $\text{vol}(K') < \text{vol}(K)$.

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2. **Triangulation of weaving knots**

Consider the weaving knot $W(p,q)$ as a closed $p$–braid. Let $B$ denote the braid axis. In this section, we describe a decomposition of $S^3 - (W(p,q) \cup B)$ into ideal tetrahedra and octahedra. This leads to our upper bound on volume, obtained in this section. In Section 3 we will use this decomposition to prove the lower bound as well.

Let $p \geq 3$. Note that the complement of $W(p,q)$ in $S^3$ with the braid axis also removed is a $q$–fold cover of the complement of $W(p,1)$ and its braid axis.

**Lemma 2.1.** Let $B$ denote the braid axis of $W(p,1)$. Then $S^3 - (W(p,1) \cup B)$ admits an ideal polyhedral decomposition $\mathcal{P}$ with four ideal tetrahedra and $p-3$ ideal octahedra.

Moreover, a meridian for the braid axis runs over exactly one side of one of the ideal tetrahedra. The polyhedra give a polygonal decomposition of the boundary of a horoball neighborhood of the braid axis, with a fundamental region consisting of four triangles and $2(p-3)$ quadrilaterals, as shown in Figure 3.

**Proof.** Obtain an ideal polyhedral decomposition as follows. First, for every crossing of the standard diagram of $W(p,1)$, take an ideal edge, the *crossing arc*, running from the knot strand at the top of the crossing to the knot strand at the bottom. This subdivides the
projection plane into two triangles and $p - 3$ quadrilaterals. This is shown on the left of Figure 4 when $p = 5$. In the figure, note that the four dotted red edges shown at each crossing are homotopic to the crossing arc.

Now for each quadrilateral on the projection plane, add four ideal edges above the projection plane and four below, as follows. Those edges above the projection plane run vertically from the strand of $W(p, 1)$ corresponding to an ideal vertex of the quadrilateral to the braid axis $B$. Those edges below the projection plane also run from strands of $W(p, 1)$ corresponding to ideal vertices of the quadrilateral, only now they run below the projection plane to $B$. These edges bound eight triangles, as follows. Four of the triangles lie above the projection plane, with two sides running from a strand of $W(p, 1)$ to $B$ and the third on the quadrilateral. The other four lie below, again each with two edges running from strands of $W(p, 1)$ to $B$ and one edge on the quadrilateral. The eight triangles together bound a single octahedron. This is shown in Figure 4 (right). Note there are $p - 3$ such octahedra coming from the $p - 3$ quadrilaterals on the projection plane.

As for the tetrahedra, these come from the triangular regions on the projection plane. As above, draw three ideal edges above the projection plane and three below. Each ideal edge runs from a strand of $W(p, 1)$ corresponding to an ideal vertex of the triangle. For each ideal vertex, one edge runs above the projection plane to $B$ and the other runs below to $B$. Again we form six ideal triangles per triangular region on the projection plane. This triangular region along with the ideal triangles above the projection plane bounds one of the four tetrahedra. The triangular region along with ideal triangles below the projection plane bounds another. There are two more coming from the ideal triangles above and below the projection plane for the other region.

These glue to give the polyhedral decomposition as claimed. Note that tetrahedra are glued in pairs across the projection plane. Note also that one triangular face of a tetrahedron above the projection plane, with two edges running to $B$, is identified to one triangular face of a tetrahedron below, again with two edges running to $B$. The identification swings the triangle through the projection plane. In fact, note that there is a single triangle in each region of the projection plane meeting $B$: two edges of the triangle are identified to the single edge running from the nearest strand of $W(p, 1)$ to $B$, and the third edge runs between the nearest crossing of $W(p, 1)$. This triangle belongs to tetrahedra above and below the projection plane. All other triangular and quadrilateral faces are identified by obvious homotopies of the edges and faces.

Finally, to see that the cusp cross section of $B$ meets the polyhedra as claimed, we need to step through the gluings of the portions of polyhedra meeting $B$. As noted above, where
$B$ meets the projection plane, there is a single triangular face of two tetrahedra. Notice that two sides of the triangle, namely the ideal edges running from $W(p,1)$ to $B$, are actually homotopic in $S^3 - (W(p,1) \cup B)$. Hence that triangle wraps around an entire meridian of $B$. Thus a meridian of $B$ runs over exactly one side of one of the ideal tetrahedra. Now, two tetrahedra, one from above the plane of projection, and one from below, are glued along that side. The other two sides of the tetrahedron above the projection plane are glued to two distinct sides of the octahedron directly adjacent, above the projection plane. The remaining two sides of this octahedron above the projection plane are glued to two distinct sides of the next adjacent octahedron, above the projection plane, and so on, until we meet the tetrahedron above the projection plane on the opposite end of $W(p,1)$, which is glued below the projection plane. Now following the same arguments, we see the triangles and quadrilaterals repeated below the projection plane, until we meet up with the original tetrahedron. Hence the cusp shape is as shown in Figure 3.

\[ \square \]

**Corollary 2.2.** For $p \geq 3$, the volume of $W(p,q)$ is at most $(4v_{\text{tet}} + (p - 3)v_{\text{oct}})q$.

**Proof.** The maximal volume of a hyperbolic ideal tetrahedron is $v_{\text{tet}}$, the volume of a regular ideal tetrahedron. The maximal volume of a hyperbolic ideal octahedron is at most $v_{\text{oct}}$, the volume of a regular ideal octahedron. The result now follows immediately from the first part of Lemma 2.1, and the fact that volume decreases under Dehn filling [16]. \[ \square \]

2.1. **Weaving knots with three strands.** The case when $p = 3$ is particularly nice geometrically, and so we treat it separately in this section.

**Theorem 2.3.** If $p = 3$ then the upper bound in Corollary 2.2 is achieved exactly by the volume of $S^3 - (W(3,q) \cup B)$, where $B$ denotes the braid axis. That is,

$$\text{vol}(W(3,q) \cup B) = 4q v_{\text{tet}}.$$  

**Proof.** Since the complement of $W(3,q) \cup B$ in $S^3$ is a $q$–fold cover of the complement of $W(3,1) \cup B$, it is enough to prove the statement for $q = 1$.

We proceed as in the proof of Lemma 2.1. If $p = 3$, then the projection plane of $W(3,1)$ is divided into two triangles; see Figure 5. This gives four tetrahedra, two each on the top and bottom. The edges and faces on the top tetrahedra are glued to those of the bottom tetrahedra across the projection plane for the same reason as in the proof Lemma 2.1. Thus the tetrahedra are glued as shown in Figure 5. The top figures indicate the top tetrahedra and the bottom figures indicate the bottom tetrahedra. The crossing edges are labelled by numbers and the edges from the knot to the braid axis are labelled by letters. The two triangles in the projection plane are labelled $S$ and $T$. Edges and faces are glued as shown.

In this case, all edges of $P$ are 6–valent. We set all tetrahedra to be regular ideal tetrahedra, and obtain a solution to the gluing equations of $P$. Since all links of tetrahedra are equilateral triangles, they are all similar, and all edges of any triangle are scaled by the same factor under dilations. Hence, the holonomy for every loop in the cusp has to expand and contract by the same factor (i.e. it is scaled by unity), and so it is a Euclidean isometry. This implies that the regular ideal tetrahedra are also a solution to the completeness equations. Thus this is a geometric triangulation giving the complete structure with volume $4v_{\text{tet}}$.

\[ \square \]

**Remark 2.4.** Since the volumes of $S^3 - (W(3,q) \cup B)$ are multiples of $v_{\text{tet}}$, we investigated their commensurability with the complement of the figure–8 knot, which is $W(3,2)$. Using SnapPy [4], we verified that $S^3 - (W(3,2) \cup B)$ is a 4–fold cover of $S^3 - W(3,2)$. Thus, the
Figure 5. The tetrahedral decomposition of $S^3 - (W(3, 1) \cup B)$. \\

Figure 6. The complement of the figure–8 knot and its braid axis, $S^3 - (W(3, 2) \cup B)$, is a 4–fold cover of the figure–8 knot complement, $S^3 - W(3, 2)$.

The figure–8 knot complement is covered by its braid complement with the axis removed! Some other interesting links also appear in this commensurability class, as illustrated in Figure 6.

3. Angle structures and lower volume bounds

In this section we find lower bounds on volumes of weaving knots. To do so, we use angle structures on the manifolds $S^3 - (W(p, q) \cup B)$.

**Definition 3.1.** Given an ideal triangulation $\{\Delta_i\}$ of a 3–manifold, an *angle structure* is a choice of three angles $(x_i, y_i, z_i) \in (0, \pi)^3$ for each tetrahedron $\Delta_i$, assigned to three edges of $\Delta_i$ meeting in a vertex, such that
(1) \(x_i + y_i + z_i = \pi\);
(2) the edge opposite the edge assigned angle \(x_i\) in \(\Delta_i\) is also assigned angle \(x_i\), and similarly for \(y_i\) and \(z_i\);
(3) angles about any edge add to \(2\pi\).

For any tetrahedron \(\Delta_i\) and angle assignment \((x_i, y_i, z_i)\) satisfying (1) and (2) above, there exists a unique hyperbolic ideal tetrahedron with the same dihedral angles. The volume of this hyperbolic ideal tetrahedron can be computed from \((x_i, y_i, z_i)\). We do not need the exact formula for our purposes. However, given an angle structure on a triangulation \(\{\Delta_i\}\), we can compute the corresponding volume by summing all volumes of ideal tetrahedra with that angle assignment.

**Lemma 3.2.** For \(p > 3\), the manifold \(S^3 - (W(p, 1) \cup B)\) admits an angle structure with volume \(v_{oct}(p - 2)\).

**Proof.** For the ideal polyhedral decomposition of \(S^3 - (W(p, 1) \cup B)\) in Lemma 2.1, assign to each edge in an octahedron the angle \(\pi/2\). As for the four tetrahedra, assign angles \(\pi/4\), \(\pi/4\), and \(\pi/2\) to each, such that pairs of the tetrahedra glue into squares in the cusp neighborhood of the braid axis. See Figure 3.

We need to show the angle sum around each edge is \(2\pi\). Consider first the ideal edges with one endpoint on \(W(p, 1)\) and one on the braid axis. These correspond to vertices of the polygonal decomposition of the braid axis illustrated in Figure 3. Note that many of these edges meet exactly four ideal octahedra, hence the angle sum around them is \(2\pi\). Any such edge that meets a tetrahedron either meets three other ideal octahedra and the angle in the tetrahedron is \(\pi/2\), so the total angle sum is \(2\pi\), or it is identified to four edges of tetrahedra with angle \(\pi/4\), and two octahedra. Hence the angle sum around it is \(2\pi\).

Finally consider the angle sum around edges which run from \(W(p, 1)\) to \(W(p, 1)\). These arise from crossings in the diagram of \(W(p, 1)\). The first two crossings on the left side and the last two crossings on the right side give rise to ideal edges bordering (some) tetrahedra. The others (for \(p > 4\)) border only octahedra, and exactly four such octahedra, hence the angle sum for those is \(2\pi\). So we need only consider the edges arising from two crossings on the far left and two crossings on the far right. We consider those on the far left; the argument for the far right is identical.

Label the edge at the first crossing on the left 1, and label that of the second 2. See Figure 7. The two tetrahedra arising on the far left have edges glued as shown on the left of Figure 8, and the adjacent octahedron has edges glued as on the right of that figure. We label the tetrahedra \(T_1\) and \(T_1'\).

Note that the edge labeled 1 in the figure is glued four times in tetrahedra, twice in \(T_1\) and twice in \(T_1'\), and once in an octahedron. However, note that in the tetrahedra it is assigned different angle measurements. In particular, in \(T_1\), the edge labeled 1, which is opposite the edge labeled \(B\), must have angle \(\pi/2\), because that is the angle on the edge labeled \(B\). The other edge of \(T_1\) labeled 1 must have angle \(\pi/4\). Similarly for \(T_1'\). Thus the angle sum around the edge labeled 1 is \(2\pi\).

In both \(T_1\) and \(T_1'\), the edge labeled 2 has angle \(\pi/4\). Since the edge labeled 2 is also glued to two edges in one octahedron, and one edge in another, the total angle sum around that edge is also \(2\pi\). Hence this gives an angle sum as claimed.

Take our ideal polyhedral decomposition of \(S^3 - (W(p, 1) \cup B)\) and turn it into an ideal triangulation by stellating the octahedra, splitting them into four ideal tetrahedra. More precisely, this is done by adding an ideal edge running from the ideal vertex on the braid axis above the plane of projection, through the plane of projection to the ideal vertex on
Figure 7. Each edge 1 and 2 is a part of two tetrahedra arising from the triangle, and an octahedron arising from the square as shown in Figure 8. The braid axis is shown in the center.

Figure 8. Edges are glued as shown in figure. From left to right, shown are tetrahedra $T_1$, $T'_1$, and adjacent octahedron.

The braid axis below the plane of projection. Using this ideal edge, the octahedron is split into four tetrahedra. Assign angle structures to these four tetrahedra in the obvious way, namely, on each tetrahedron the ideal edge through the plane of projection is given angle $\pi/2$, and the other two edges meeting that edge in an ideal vertex are labeled $\pi/4$. By the above work, this gives an angle structure.

The volume estimate comes from the fact that a regular ideal octahedron has volume $v_{oct}$. Moreover, four ideal tetrahedra, each with angles $\pi/2, \pi/4, \pi/4$, can be glued to give an ideal octahedron, hence each such tetrahedron has volume $v_{oct}/4$. We have $p - 3$ octahedra and four such tetrahedra, and hence the corresponding volume is $(p - 2) v_{oct}$.

Lemma 3.3. Let $P$ be an ideal polyhedron obtained by coning to $\pm\infty$ from any ideal quadrilateral in the projection plane. Then for any angle structure on $P$, the volume of that angle structure $\text{vol}(P)$ satisfies $\text{vol}(P) \leq v_{oct}$, the volume of the regular ideal octahedron.

Proof. Suppose the volume for some angle structure is strictly greater than $v_{oct}$. The dihedral angles on the exterior of $P$ give a dihedral angle assignment $\Delta$ to $P$, and so in the terminology of Rivin [14], $A(P, \Delta)$ is nonempty. By Theorem 6.13 of that paper, there is a unique complete structure with angle assignment $\Delta$, and the proof of Theorem 6.16 of [14] shows that the complete structure occurs exactly when the volume is maximized over $A(P, \Delta)$. Hence the volume of our angle structure is at most the volume of the complete hyperbolic structure on $P$ with angle assignment $\Delta$. 

□
On the other hand, for complete hyperbolic structures on $P$, it is known that the volume is maximized in the regular case, and thus the volume is strictly less than the volume of a regular ideal octahedron. The proof of this fact is given, for example, in Theorem 10.4.8 and the proof of Theorem 10.4.7 in [13]. This is a contradiction.

Now consider the space $A(P)$ of angle structures on the ideal triangulation $P$ for $S^3 - (\hat{W}(p,1) \cup B)$.

**Lemma 3.4.** The critical point for $\text{vol}: A(P) \to \mathbb{R}$ is in the interior of the space $A$ of angle structures on $P$.

**Proof.** It is known that the volume function is concave down on the space of angle structures [14, 5]. We will show that the volume function takes values strictly smaller on the boundary of $A(P)$ than at any point in the interior. Therefore, it will follow that the maximum occurs in the interior of $A(P)$.

Suppose we have a point $X$ on the boundary of $A(P)$ that maximizes volume. Because the point is on the boundary, there must be at least one triangle $\Delta$ with angles $(x_0, y_0, z_0)$ where one of $x_0, y_0,$ and $z_0$ equals zero. A proposition of Guéritaud, [7, Proposition 7.1], implies that if one of $x_0, y_0, z_0$ is zero, then another is $\pi$ and the third is also zero. (The proposition is stated for once-punctured torus bundles in [7], but only relies on the formula for volume of a single ideal tetrahedron, [7, Proposition 6.1].)

A tetrahedron with angles 0, 0, and $\pi$ is a flattened tetrahedron, and contributes nothing to volume. We consider which tetrahedra might be flattened.

Let $P_0$ be the original polyhedral decomposition described in the proof of Lemma 2.1. Suppose first that we have flattened one of the four tetrahedra which came from tetrahedra in $P_0$. Then the maximal volume we can obtain from these four tetrahedra is at most $3v_{\text{tet}}$, which is strictly less than $v_{\text{oct}}$, which is the volume we obtain from these four tetrahedra from the angle structure of Lemma 3.2. Thus, by Lemma 3.3, the maximal volume we can obtain from any such angle structure is $3v_{\text{tet}} + (p - 3)v_{\text{oct}} < (p - 2)v_{\text{oct}}$. Since the volume on the right is realized by an angle structure in the interior by Lemma 3.2, the maximum of the volume cannot occur at such a point of the boundary.

Now suppose one of the four tetrahedra coming from an octahedron is flattened. Then the remaining three tetrahedra can have volume at most $3v_{\text{tet}} < v_{\text{oct}}$. Thus the volume of such a structure can be at most $4v_{\text{tet}} + 3v_{\text{tet}} + (p - 4)v_{\text{oct}}$, where the first term comes from the maximum volume of the four tetrahedra in $P_0$, the second from the maximum volume of the stellated octahedron with one flat tetrahedron, and the last term from the maximal volume of the remaining ideal octahedra. Because $7v_{\text{tet}} < 2v_{\text{oct}}$, the volume of this structure is still strictly less than that of Lemma 3.2.

Therefore, there does not exist $X$ on the boundary of the space of angle structures that maximizes volume.

**Theorem 3.5.** If $p > 3$, then

$$v_{\text{oct}}(p - 2)q \leq \text{vol}(W(p, q) \cup B) \leq (v_{\text{oct}}(p - 3) + 4v_{\text{tet}})q.$$  

If $p = 3$, $\text{vol}(W(3, q) \cup B) = 4q v_{\text{tet}}$.

**Proof.** Theorem 2.3 provides the $p = 3$ case.

For $p > 3$, Casson and Rivin showed that if the critical point for the volume is in the interior of the space of angle structures, then the maximal volume angle structure is realized by the actual hyperbolic structure [14]. By Lemma 3.4, the critical point for volume is in the interior of the space of angle structures. By Lemma 3.2, the volume of one particular angle
structure is $v_{\text{oct}}(p - 2)q$. So the maximal volume must be at least this. The upper bound is from Corollary 2.2.

**Lemma 3.6.** The length of a meridian of the braid axis is at least $q$.

**Proof.** A meridian of the braid axis of $W(p, q)$ is a $q$-fold cover of a meridian of the braid axis of $W(p, 1)$. The meridian of the braid axis of $W(p, 1)$ must have length at least one [16, 1]. Hence the meridian of the braid axis of $W(p, q)$ has length at least $q$. □

We can now prove our main result on volumes of weaving knots:

**Proof of Theorem 1.1.** The upper bound is from Corollary 2.2.

As for the lower bound, the manifold $S^3 - W(p, q)$ is obtained by Dehn filling the meridian on the braid axis of $S^3 - (W(p, q) \cup B)$. When $q > 6$, Lemma 3.6 implies that the meridian of the braid axis has length greater than $2\pi$, and so [6, Theorem 1.1] will apply. Combining this with Theorem 3.5 implies

$$\left(1 - \left(\frac{2\pi}{q}\right)^2\right)^{3/2} (p - 2) q v_{\text{oct}} \leq \text{vol}(S^3 - W(p, q)).$$

For $p = 3$ note that $v_{\text{oct}} < 4v_{\text{tet}}$, so this gives the desired lower bound for all $p \geq 3$. □

**Corollary 3.7.** The links $K_n = W(3, n)$ satisfy

$$\lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = 2v_{\text{tet}}.$$

**Proof.** By Theorem 2.3 and the same argument as above, for $q > 6$ we have

$$\left(1 - \left(\frac{2\pi}{q}\right)^2\right)^{3/2} (4q v_{\text{tet}}) \leq \text{vol}(S^3 - W(3, q)) \leq 4q v_{\text{tet}}.$$

□

4. **Geometric convergence of weaving knots**

In this section, we will prove Theorem 1.2, which states that as $p, q \to \infty$, the manifold $S^3 - W(p, q)$ approaches $\mathbb{R}^3 - W$ as a geometric limit.

A regular ideal octahedron is obtained by gluing two square pyramids, which we will call the *top* and *bottom* pyramids. As discussed in more detail in [3, Section 3], $\mathbb{R}^3 - W$ is cut into $\tilde{X}_1$ and $\tilde{X}_2$, such that $\tilde{X}_1$ is obtained by gluing top pyramids along triangular faces, and $\tilde{X}_2$ by gluing bottom pyramids along triangular faces. The circle pattern in Figure 9(a) shows how the square pyramids in $\tilde{X}_1$ are viewed from infinity on the $xy$-plane.

A fundamental domain $\mathcal{P}_W$ for $R^3 - W$ in $\mathbb{H}^3$ is explicitly obtained by attaching each top pyramid of $\tilde{X}_1$ to a bottom pyramid of $\tilde{X}_2$ along their common square face. Hence, $\mathcal{P}_W$ is tessellated by regular ideal octahedra. An appropriate $\pi/2$ rotation is needed when gluing the square faces, which determines how adjacent triangular faces are glued to obtain $\mathcal{P}_W$. Figure 10 shows the face pairings for the triangular faces of the bottom square pyramids, and the associated circle pattern. The face pairings are equivariant under the translations $(x, y) \mapsto (x \pm 1, y \pm 1)$. That is, when a pair of faces is identified, then the corresponding pair of faces under this translation is also identified.

The proof below provides the geometric limit of the polyhedra described in Section 2. We will see that these polyhedra converge as follows. If we cut the torus in Figure 4 in half along the horizontal plane shown, each half is tessellated mostly by square pyramids, as well as some tetrahedra. As $p, q \to \infty$, the tetrahedra are pushed off to infinity, and the
square pyramids converge to the square pyramids that are shown in Figure 9. Gluing the two halves of the torus along the square faces of the square pyramids, in the limit we obtain the tessellation by regular ideal octahedra.

\[ \text{Figure 9. (a) Circle pattern for hyperbolic planes of the top polyhedron of } \mathbb{R}^3 - \mathcal{W}. \text{ (b) Hyperbolic planes bounding one top square pyramid.} \]

To make this precise, we review the definition of a geometric limit.

**Definition 4.1.** For compact metric spaces \( X \) and \( Y \), define their **bilipschitz distance** to be

\[
\inf \{ |\log \text{lip}(f)| + |\log \text{lip}(f^{-1})| \}
\]

where the infimum is taken over all bilipschitz mappings \( f \) from \( X \) to \( Y \), and \( \text{lip}(f) \) denotes the lipschitz constant. The lipschitz constant is defined to be infinite if there is no bilipschitz map between \( X \) and \( Y \).

\[ \text{Figure 10. Face pairings for a fundamental domain } \mathcal{P}_W \text{ of } \mathbb{R}^3 - \mathcal{W}. \]
Definition 4.2. A sequence \( \{(X_n, x_n)\} \) of locally compact complete length metric spaces with distinguished basepoints is said to converge in the pointed bilipschitz topology to \((Y, y)\) if for any \(R > 0\), the closed balls \(B_R(x_n)\) of radius \(R\) about \(x_n\) in \(X_n\) converge to the closed ball \(B_R(y)\) about \(y\) in \(Y\) in the bilipschitz topology.

Definition 4.3. For \(X, Y\) locally compact complete metric spaces, we say that \(Y\) is a geometric limit of \(X\) if there exist basepoints \(y \in Y\) and \(x_n \in X\) such that \((X_n, x_n)\) converges in the pointed bilipschitz topology to \((Y, y)\).

In order to prove Theorem 1.2, we will consider \(M_{p,q} := S^3 - (W(p,q) \cup B)\). Since \(S^3 - W(p,q)\) is obtained by Dehn filling \(M_{p,q}\) along a slope of length at least \(q\) by Lemma 3.6, Thurston’s Dehn filling theorem implies that \(M_{p,q}\) is a geometric limit of \(S^3 - W(p,q)\). Thus it will suffice to show \(\mathbb{R}^3 - W\) is a geometric limit of \(M_{p,q}\). To show this, we need to find basepoints \(x_{p,q}\) for each \(M_{p,q}\) so that closed balls \(B_R(x_{p,q})\) converge to a closed ball in \(\mathbb{R}^3 - W\). We do this by considering structures on ideal polyhedra.

Let \(\mathcal{P}_{p,q}\) denote the ideal polyhedra in the decomposition of \(M_{p,q}\) from the proofs of Lemma 2.1 and Lemma 3.2. We decomposed \(\mathcal{P}_{p,q}\) into ideal tetrahedra and ideal octahedra, such that octahedra corresponding to non-peripheral squares of the diagram projection graph of \(W(p,q)\) satisfy the same local gluing condition on the faces as that for the fundamental domain \(\mathcal{P}_W\) for \(\mathbb{R}^3 - W\) as illustrated in Figure 10. In particular, the faces of each octahedron are glued to faces of adjacent octahedra, with the gluings of the triangular faces of the top and bottom square pyramids locally the same as those for \(\mathcal{P}_W\).

We find a sequence of consecutive octahedra in \(M_{p,1} = S^3 - (W(p,1) \cup B)\) with volume approaching \(v_{\text{oct}}\), and then use the \(q\)-fold cover \(M_{p,q} \to M_{p,1}\) to find a grid of octahedra in \(M_{p,q}\) all of which have volume nearly \(v_{\text{oct}}\).

Lemma 4.4. There exist \(k \to \infty\), \(\epsilon(k) \to 0\), and \(n(k) \to \infty\) such that for \(p \geq n(k)\) there exist at least \(k\) consecutive ideal octahedra in \(\mathcal{P}_{p,1}\) each of which has volume greater than \((v_{\text{oct}} - \epsilon(k))\).

**Proof.** Let \(\epsilon(k) = \frac{1}{k}\) and \(n(k) = k^3\). Suppose there are no \(k\) consecutive octahedra each of whose volume is greater than \(v_{\text{oct}} - \epsilon(k)\). This implies that there exist at least \(n(k)/k = k^2\) octahedra each of which has volume at most \(v_{\text{oct}} - \epsilon(k)\). Hence for \(k > 12\) and \(p > n(k)\),
\[
\text{vol}(W(p,1) \cup B) \leq 4v_{\text{tet}} + (p - k^2)v_{\text{oct}} + k^2(v_{\text{oct}} - 1/k) = 4v_{\text{tet}} + pv_{\text{oct}} - k = (p - 2)v_{\text{oct}} + 4v_{\text{tet}} + 2v_{\text{oct}} - k < (p - 2)v_{\text{oct}}.
\]
This contradicts Theorem 3.5, which says that \((p - 2)v_{\text{oct}} < \text{vol}(W(p,1) \cup B)\). 

**Corollary 4.5.** For any \(\epsilon > 0\) and any \(k > 0\) there exists \(N\) such that if \(p, q > N\) then \(\mathcal{P}_{p,q}\) contains a \(k \times k\) grid of adjacent ideal octahedra, each of which has volume greater than \((v_{\text{oct}} - \epsilon)\).

**Proof.** Apply Lemma 4.4, taking \(k\) sufficiently large so that \(\epsilon(k) < \epsilon\). Then for any \(N > n(k)\), if \(p > N\) we obtain at least \(k\) consecutive ideal octahedra with volume as desired. Now let \(q > N\), so \(q > k\). Use the \(q\)-fold cover \(M_{p,q} \to M_{p,1}\). We obtain a \(k \times q\) grid of octahedra, all of which have volume greater than \((v_{\text{oct}} - \epsilon(k))\), as shown in Figure 11.

We are now ready to complete the proof of Theorem 1.2.
Proof of Theorem 1.2. Given $R > 0$, we will show that closed balls based in $M_{p,q}$ converge to a closed ball based in $\mathbb{R}^3 - W$. Since $M_{p,q} := S^3 - (W(p, q) \cup B)$ is a geometric limit of $S^3 - W(p, q)$, this implies the result.

Take a basepoint $y \in \mathbb{R}^3 - W$ to lie in the interior of any octahedron, on one of the squares projecting to a checkerboard surface, say at the point where the diagonals of that square intersect. Consider the ball $B_R(y)$ of radius $R$ about the basepoint $y$. This will intersect some number of regular ideal octahedra. Notice that the octahedra are glued on all faces to adjacent octahedra, by the gluing pattern we obtained in Figure 10. Consider all octahedra in $\mathbb{R}^3 - W$ that meet the ball $B_R(y)$. Call these octahedra $\text{Oct}(R)$.

In $S^3 - (W(p, q) \cup B)$, for an octahedron of Lemma 2.1 coming from a square in the interior of the diagram of $W(p, q)$, the gluing pattern on each of its faces agrees with the gluing of an octahedron in $\mathbb{R}^3 - W$. Thus for $p, q$ large enough, we may find a collection of adjacent octahedra $\text{Oct}_{p,q}$ in $S^3 - W(p, q)$ with the same combinatorial gluing pattern as $\text{Oct}(R)$. Since all the octahedra are glued along faces to adjacent octahedra, Corollary 4.5 implies that if we choose $p, q$ large enough, then each ideal octahedron in $\text{Oct}_{p,q}$ has volume within $\epsilon$ of $v_{\text{oct}}$.

It is known that the volume of a hyperbolic ideal octahedron is uniquely maximized by the volume of a regular ideal octahedron (see, e.g. [13, Theorem 10.4.7]). Thus as $\epsilon \to 0$, each ideal octahedron of $\text{Oct}_{p,q}$ must be converging to a regular ideal octahedron. So $\text{Oct}_{p,q}$ converges as a polyhedron to $\text{Oct}(R)$. But then it follows that for suitable basepoints $x_{p,q}$ in $\mathcal{P}_{p,q}$, the balls $B_R(x_{p,q})$ in $\mathcal{P}_{p,q} \subset M_{p,q}$ converge to $B_R(y)$ in the pointed bilipschitz topology. □

References


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