Chapter 11

Infinite Series

11.1 Approximation By Taylor Polynomials

By now, you have noticed there are two sorts of functions, those which come from a formula like $f(x) = x^2 + 2$ which are easy to evaluate by following a simple procedure, and those which come as short words; things like $\ln(x)$ or $\sin(x)$. This latter type of function is not so easy to evaluate. For example, what is $\sin 2$? Can you get it by doing a simple sequence of operations like you can with $f(x) = x^2+2$? How can you find $\sin 2$? It turns out there are many ways to do so. In this section, the method of Taylor polynomials is discussed. The following theorem is called Taylor's theorem. Before presenting it, recall the meaning of $n!$ for n a positive integer. Define $0! \equiv 1 = 1!$ and $(n+1)! \equiv (n+1)\,n!$ so that $n! = n(n-1)\cdots 1$. In particular, $2! = 2, 3! = 3\times 2! = 6, 4! = 4\times 3! = 24$, etc. A version of the following theorem is due to Lagrange, about 1790.

Theorem 11.1. *Suppose f has $n+1$ derivatives on an interval (a,b) and let $c \in (a,b)$. Then if $x \in (a,b)$, there exists ξ between c and x such that*

$$f(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}.$$

Proof: It can be assumed $x \neq c$ because if $x = c$ there is nothing to show. Then there exists K such that

$$f(x) - \left(f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + K(x-c)^{n+1} \right) = 0 \tag{11.1}$$

In fact,

$$K = \frac{-f(x) + \left(f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \right)}{(x-c)^{n+1}}.$$

Now define $F(t)$ for t in the closed interval determined by x and c by

$$F(t) \equiv f(x) - \left(f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + K(x-t)^{n+1} \right).$$

The c in (11.1) got replaced by t.

Therefore, $F(c) = 0$ by the way K was chosen and also $F(x) = 0$. By the mean value theorem or Rolle's theorem, there exists ξ between x and c such that $F'(\xi) = 0$. Therefore,

$$\begin{aligned} 0 &= f'(\xi) + \sum_{k=1}^{n} \frac{f^{(k+1)}(\xi)}{k!}(x-\xi)^k - \sum_{k=1}^{n} \frac{f^{(k)}(\xi)}{(k-1)!}(x-\xi)^{k-1} - K(n+1)(x-\xi)^n \\ &= f'(\xi) + \sum_{k=1}^{n} \frac{f^{(k+1)}(\xi)}{k!}(x-\xi)^k - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(\xi)}{k!}(x-\xi)^{k} - K(n+1)(x-\xi)^n \\ &= f'(\xi) + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n - f'(\xi) - K(n+1)(x-\xi)^n \\ &= \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n - K(n+1)(x-\xi)^n \end{aligned}$$

Then therefore,

$$K = \frac{f^{(n+1)}(\xi)}{(n+1)!} \quad \blacksquare$$

The term $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$, is called the remainder, and this particular form of the remainder is called the Lagrange form of the remainder.

Example 11.1. Approximate $\sin x$ for x in some open interval containing 0.

Use Taylor's formula just presented and let $c = 0$. Then for $f(x) = \sin x$,

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,$$

etc. Therefore, $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1$, etc. Thus the Taylor polynomial for $\sin x$ is of the form

$$x - \frac{x^3}{3!} + \cdots \pm \frac{x^{2n+1}}{(2n+1)!} = \sum_{k=1}^{n+1} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$$

while the remainder is of the form

$$\frac{f^{(2n+2)}(\xi)\, x^{2n+2}}{(2n+2)!}$$

for some ξ between 0 and x. For $n = 2$ in the above, the resulting polynomial is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

and the error between this polynomial and $\sin x$ must be measured by the remainder term. Therefore,

$$\left| \sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \right) \right| \leq \left| \frac{f^{(6)}(\xi)\, x^6}{6!} \right| \leq \frac{x^6}{6!}.$$