A Short Differential Equations Course

Kenneth Kuttler

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Contents

Chapter 1

Calculus Review

1.1 The Limit Of A Sequence

The definition of the limit of a sequence was define by Bolzano[1].

Definition 1.1.1 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to* a,

$$\lim_{n\to\infty} a_n = a \text{ or } a_n \to a$$

if and only if for every $\varepsilon > 0$ *there exists* n_ε *such that whenever* $n \geq n_\varepsilon$,

$$|a_n - a| < \varepsilon.$$

In words the definition says that given any measure of closeness, ε, the terms of the sequence are eventually all this close to a. Note the similarity with the concept of limit. Here, the word "eventually" refers to n being sufficiently large. The above definition is always the definition of what is meant by the limit of a sequence. If the a_n are complex numbers or later on, vectors the definition remains the same. If $a_n = x_n + iy_n$ and $a = x + iy, |a_n - a| = \sqrt{(x_n - x)^2 + (y_n - y)^2}$. Recall the way you measure distance between two complex numbers.

Theorem 1.1.2 *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} a_n = a_1$ *then* $a_1 = a$.

Proof: Suppose $a_1 \neq a$. Then let $0 < \varepsilon < |a_1 - a|/2$ in the definition of the limit. It follows there exists n_ε such that if $n \geq n_\varepsilon$, then $|a_n - a| < \varepsilon$ and $|a_n - a_1| < \varepsilon$. Therefore, for such n,

$$\begin{aligned} |a_1 - a| &\leq |a_1 - a_n| + |a_n - a| \\ &< \varepsilon + \varepsilon < |a_1 - a|/2 + |a_1 - a|/2 = |a_1 - a|, \end{aligned}$$

a contradiction.

Example 1.1.3 *Let* $a_n = \frac{1}{n^2+1}$.

[1] Bernhard Bolzano lived from 1781 to 1848. He was a Catholic priest and held a position in philosophy at the University of Prague. He had strong views about the absurdity of war, educational reform, and the need for individual concience. His convictions got him in trouble with Emporer Franz I of Austria and when he refused to recant, was forced out of the university. He understood the need for absolute rigor in mathematics. He also did work on physics.

Then it seems clear that

$$\lim_{n\to\infty} \frac{1}{n^2+1} = 0.$$

In fact, this is true from the definition. Let $\varepsilon > 0$ be given. Let $n_\varepsilon \geq \sqrt{\varepsilon^{-1}}$. Then if

$$n > n_\varepsilon \geq \sqrt{\varepsilon^{-1}},$$

it follows that $n^2 + 1 > \varepsilon^{-1}$ and so

$$0 < \frac{1}{n^2+1} = a_n < \varepsilon..$$

Thus $|a_n - 0| < \varepsilon$ whenever n is this large.

Note the definition was of no use in finding a candidate for the limit. This had to be produced based on other considerations. The definition is for verifying beyond any doubt that something is the limit. It is also what must be referred to in establishing theorems which are good for finding limits.

Example 1.1.4 *Let* $a_n = n^2$

Then in this case $\lim_{n\to\infty} a_n$ does not exist. Sometimes this situation is also referred to by saying $\lim_{n\to\infty} a_n = \infty$.

Example 1.1.5 *Let* $a_n = (-1)^n$.

In this case, $\lim_{n\to\infty} (-1)^n$ does not exist. This follows from the definition. Let $\varepsilon = 1/2$. If there exists a limit, l, then eventually, for all n large enough, $|a_n - l| < 1/2$. However, $|a_n - a_{n+1}| = 2$ and so,

$$2 = |a_n - a_{n+1}| \leq |a_n - l| + |l - a_{n+1}| < 1/2 + 1/2 = 1$$

which cannot hold. Therefore, there can be no limit for this sequence.

Theorem 1.1.6 *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are sequences and that*

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

Also suppose x *and* y *are real numbers. Then*

$$\lim_{n\to\infty} xa_n + yb_n = xa + yb \tag{1.1}$$

$$\lim_{n\to\infty} a_n b_n = ab \tag{1.2}$$

If $b \neq 0$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}. \tag{1.3}$$

Proof: The first of these claims is left for you to do. To do the second, let $\varepsilon > 0$ be given and choose n_1 such that if $n \geq n_1$ then

$$|a_n - a| < 1.$$

Then for such n, the triangle inequality implies

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &\leq |a_n| |b_n - b| + |b| |a_n - a| \\ &\leq (|a| + 1) |b_n - b| + |b| |a_n - a| . \end{aligned}$$

Now let n_2 be large enough that for $n \geq n_2$,

$$|b_n - b| < \frac{\varepsilon}{2(|a|+1)}, \text{ and } |a_n - a| < \frac{\varepsilon}{2(|b|+1)}.$$

Such a number exists because of the definition of limit. Therefore, let

$$n_\varepsilon > \max(n_1, n_2).$$

For $n \geq n_\varepsilon$,

$$\begin{aligned} |a_n b_n - ab| &\leq (|a|+1)|b_n - b| + |b||a_n - a| \\ &< (|a|+1)\frac{\varepsilon}{2(|a|+1)} + |b|\frac{\varepsilon}{2(|b|+1)} \leq \varepsilon. \end{aligned}$$

This proves 1.2. Next consider 1.3.

Let $\varepsilon > 0$ be given and let n_1 be so large that whenever $n \geq n_1$,

$$|b_n - b| < \frac{|b|}{2}.$$

Thus for such n,

$$\left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \left|\frac{a_n b - a b_n}{b b_n}\right| \leq \frac{2}{|b|^2}\left[|a_n b - ab| + |ab - a b_n|\right]$$

$$\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b|.$$

Now choose n_2 so large that if $n \geq n_2$, then

$$|a_n - a| < \frac{\varepsilon|b|}{4}, \text{ and } |b_n - b| < \frac{\varepsilon|b|^2}{4(|a|+1)}.$$

Letting $n_\varepsilon > \max(n_1, n_2)$, it follows that for $n \geq n_\varepsilon$,

$$\begin{aligned} \left|\frac{a_n}{b_n} - \frac{a}{b}\right| &\leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b_n - b| \\ &< \frac{2}{|b|}\frac{\varepsilon|b|}{4} + \frac{2|a|}{|b|^2}\frac{\varepsilon|b|^2}{4(|a|+1)} < \varepsilon. \end{aligned}$$

Another very useful theorem for finding limits is the squeezing theorem.

Theorem 1.1.7 *Suppose* $\lim_{n\to\infty} a_n = a = \lim_{n\to\infty} b_n$ *and* $a_n \leq c_n \leq b_n$ *for all* n *large enough. Then* $\lim_{n\to\infty} c_n = a$.

Proof: Let $\varepsilon > 0$ be given and let n_1 be large enough that if $n \geq n_1$,

$$|a_n - a| < \varepsilon/2 \text{ and } |b_n - a| < \varepsilon/2.$$

Then for such n,

$$|c_n - a| \leq |a_n - a| + |b_n - a| < \varepsilon.$$

The reason for this is that if $c_n \geq a$, then

$$|c_n - a| = c_n - a \leq b_n - a \leq |a_n - a| + |b_n - a|$$

because $b_n \geq c_n$. On the other hand, if $c_n \leq a$, then

$$|c_n - a| = a - c_n \leq a - a_n \leq |a - a_n| + |b - b_n|.$$

This proves the theorem.

As an example, consider the following.

Example 1.1.8 *Let*

$$c_n \equiv (-1)^n \frac{1}{n}$$

and let $b_n = \frac{1}{n}$, *and* $a_n = -\frac{1}{n}$. *Then you may easily show that*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0.$$

Since $a_n \leq c_n \leq b_n$, *it follows* $\lim_{n\to\infty} c_n = 0$ *also.*

Theorem 1.1.9 $\lim_{n\to\infty} r^n = 0$. *Whenever* $|r| < 1$.

Proof: If $0 < r < 1$ if follows $r^{-1} > 1$. Why? Letting $\alpha = \frac{1}{r} - 1$, it follows

$$r = \frac{1}{1+\alpha}.$$

Therefore, by the binomial theorem,

$$0 < r^n = \frac{1}{(1+\alpha)^n} \leq \frac{1}{1+\alpha n}.$$

Therefore, $\lim_{n\to\infty} r^n = 0$ if $0 < r < 1$. Now in general, if $|r| < 1$, $|r^n| = |r|^n \to 0$ by the first part. This proves the theorem.

Definition 1.1.10 *Let* $\{a_n\}$ *be a sequence and let* $n_1 < n_2 < n_3, \cdots$ *be any strictly increasing list of integers such that* n_1 *is at least as large as the first number in the domain of the function. Then if* $b_k \equiv a_{n_k}$, $\{b_k\}$ *is called a subsequence of* $\{a_n\}$.

An important theorem is the one which states that if a sequence converges, so does every subsequence.

Theorem 1.1.11 *Let* $\{x_n\}$ *be a sequence with* $\lim_{n\to\infty} x_n = x$ *and let* $\{x_{n_k}\}$ *be a subsequence. Then* $\lim_{k\to\infty} x_{n_k} = x$.

Proof: Let $\varepsilon > 0$ be given. Then there exists n_ε such that if $n > n_\varepsilon$, then $|x_n - x| < \varepsilon$. Suppose $k > n_\varepsilon$. Then $n_k \geq k > n_\varepsilon$ and so

$$|x_{n_k} - x| < \varepsilon$$

showing $\lim_{k\to\infty} x_{n_k} = x$ as claimed.

1.2 Continuity And The Limit Of A Sequence

There is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 1.2.1 *A function* $f : D(f) \to \mathbb{R}$ *is continuous at* $x \in D(f)$ *if and only if, whenever* $x_n \to x$ *with* $x_n \in D(f)$, *it follows* $f(x_n) \to f(x)$.

Proof: Suppose first that f is continuous at x and let $x_n \to x$. Let $\varepsilon > 0$ be given. By continuity, there exists $\delta > 0$ such that if $|y - x| < \delta$, then $|f(x) - f(y)| < \varepsilon$. However, there exists n_δ such that if $n \geq n_\delta$, then $|x_n - x| < \delta$ and so for all n this large,

$$|f(x) - f(x_n)| < \varepsilon$$

which shows $f(x_n) \to f(x)$.

Now suppose the condition about taking convergent sequences to convergent sequences holds at x. Suppose f fails to be continuous at x. Then there exists $\varepsilon > 0$ and $x_n \in D(f)$ such that $|x - x_n| < \frac{1}{n}$, yet

$$|f(x) - f(x_n)| \geq \varepsilon.$$

But this is clearly a contradiction because, although $x_n \to x$, $f(x_n)$ fails to converge to $f(x)$. It follows f must be continuous after all. This proves the theorem.

1.3 The Integral

The integral is needed in order to consider the basic questions of differential equations. Consider the following initial value problem, differential equation and initial condition.

$$A'(x) = e^{x^2}, \ A(0) = 0.$$

So what is the solution to this initial value problem and does it even have a solution? More generally, for which functions, f does there exist a solution to the initial value problem, $y'(x) = f(x), y(0) = y_0$? The solution to these sorts of questions depend on the integral. Since this is usually not done well in beginning calculus courses, I will give a presentation of the theory of the integral. I assume the reader is familiar with the usual techniques for finding antiderivatives and integrals such as partial fractions, integration by parts and integration by substitution. These topics are usually done very well in begining calculus courses.

1.4 Upper And Lower Sums

The Riemann integral pertains to bounded functions which are defined on a bounded interval. Let $[a, b]$ be a closed interval. A set of points in $[a, b]$, $\{x_0, \cdots, x_n\}$ is a partition if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

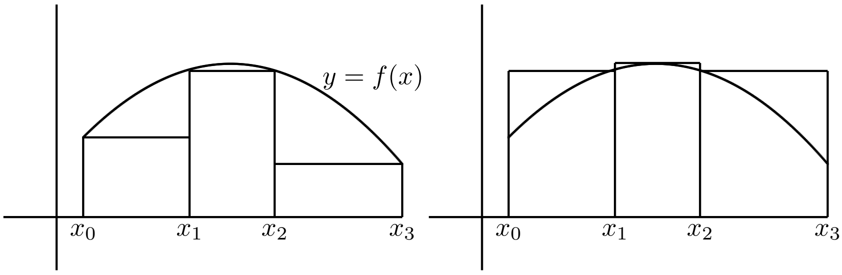
Such partitions are denoted by P or Q. For f a bounded function defined on $[a, b]$, let

$$M_i(f) \equiv \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$
$$m_i(f) \equiv \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

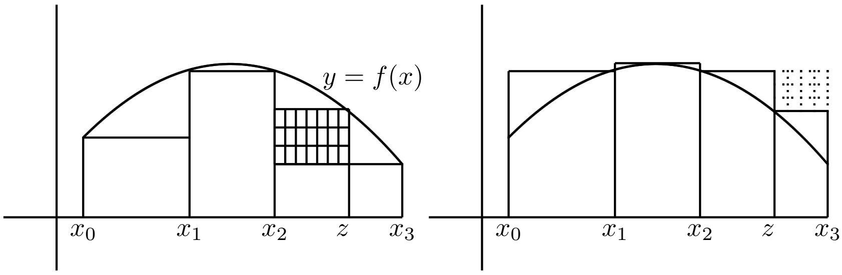
Also let $\Delta x_i \equiv x_i - x_{i-1}$. Then define upper and lower sums as

$$U(f, P) \equiv \sum_{i=1}^{n} M_i(f) \Delta x_i \text{ and } L(f, P) \equiv \sum_{i=1}^{n} m_i(f) \Delta x_i$$

respectively. The numbers, $M_i(f)$ and $m_i(f)$, are well defined real numbers because f is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S = \{f(x) : x \in [x_{i-1}, x_i]\}$ is bounded above and below. In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition.



What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled z has been added in.



Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 1.4.1 *If* $P \subseteq Q$ *then*

$$U(f,Q) \leq U(f,P), \text{ and } L(f,P) \leq L(f,Q).$$

Proof: This is verified by adding in one point at a time. Thus let $P = \{x_0, \cdots, x_n\}$ and let $Q = \{x_0, \cdots, x_k, y, x_{k+1}, \cdots, x_n\}$. Thus exactly one point, y, is added between x_k and x_{k+1}. Now the term in the upper sum which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f,P)$ is

$$\sup\{f(x) : x \in [x_k, x_{k+1}]\}(x_{k+1} - x_k) \tag{1.4}$$

and the term which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f,Q)$ is

$$\sup\{f(x) : x \in [x_k, y]\}(y - x_k) + \sup\{f(x) : x \in [y, x_{k+1}]\}(x_{k+1} - y) \tag{1.5}$$

$$\equiv M_1(y - x_k) + M_2(x_{k+1} - y) \tag{1.6}$$

All the other terms in the two sums coincide. Now $\sup\{f(x) : x \in [x_k, x_{k+1}]\} \geq \max(M_1, M_2)$ and so the expression in 1.5 is no larger than

$$\sup\{f(x) : x \in [x_k, x_{k+1}]\}(x_{k+1} - y) + \sup\{f(x) : x \in [x_k, x_{k+1}]\}(y - x_k)$$

$$= \sup\{f(x) : x \in [x_k, x_{k+1}]\}(x_{k+1} - x_k),$$

the term corresponding to the interval, $[x_k, x_{k+1}]$ and $U(f,P)$. This proves the first part of the lemma pertaining to upper sums because if $Q \supseteq P$, one can obtain Q from P by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part is similar and is left as an exercise.

Lemma 1.4.2 *If P and Q are two partitions, then*

$$L(f,P) \leq U(f,Q).$$

Proof: By Lemma 1.4.1,

$$L(f,P) \leq L(f,P\cup Q) \leq U(f,P\cup Q) \leq U(f,Q).$$

Definition 1.4.3

$$\overline{I} \equiv \inf\{U(f,Q) \text{ where } Q \text{ is a partition}\}$$

$$\underline{I} \equiv \sup\{L(f,P) \text{ where } P \text{ is a partition}\}.$$

Note that $\underline{I}$ and $\overline{I}$ are well defined real numbers.

Theorem 1.4.4 $\underline{I} \leq \overline{I}$.

Proof: From Lemma 1.4.2,

$$\underline{I} = \sup\{L(f,P) \text{ where } P \text{ is a partition}\} \leq U(f,Q)$$

because $U(f,Q)$ is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since Q is arbitrary,

$$\begin{aligned}\underline{I} &= \sup\{L(f,P) \text{ where } P \text{ is a partition}\} \\ &\leq \inf\{U(f,Q) \text{ where } Q \text{ is a partition}\} \equiv \overline{I}\end{aligned}$$

where the inequality holds because it was just shown that $\underline{I}$ is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 1.4.5 *A bounded function f is Riemann integrable, written as*

$$f \in R([a,b])$$

if

$$\underline{I} = \overline{I}$$

and in this case,

$$\int_a^b f(x)\, dx \equiv \underline{I} = \overline{I}.$$

Thus, in words, the Riemann integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall Proposition **??**. It is stated here for ease of reference.

Proposition 1.4.6 *Let S be a nonempty set and suppose $\sup(S)$ exists. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ exists, then for every $\delta > 0$,

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

This proposition implies the following theorem which is used to determine the question of Riemann integrability.

Theorem 1.4.7 *A bounded function f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a partition P such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{1.7}$$

Proof: First assume f is Riemann integrable. Then let P and Q be two partitions such that

$$U(f,Q) < \overline{I} + \varepsilon/2, \; L(f,P) > \underline{I} - \varepsilon/2.$$

Then since $\underline{I} = \overline{I}$,

$$U(f, Q \cup P) - L(f, P \cup Q) \leq U(f,Q) - L(f,P) < \overline{I} + \varepsilon/2 - (\underline{I} - \varepsilon/2) = \varepsilon.$$

Now suppose that for all $\varepsilon > 0$ there exists a partition such that 1.7 holds. Then for given ε and partition P corresponding to ε

$$\overline{I} - \underline{I} \leq U(f,P) - L(f,P) \leq \varepsilon.$$

Since ε is arbitrary, this shows $\underline{I} = \overline{I}$ and this proves the theorem.

The condition described in the theorem is called the Riemann criterion .

Not all bounded functions are Riemann integrable. For example, let

$$f(x) \equiv \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{1.8}$$

Then if $[a,b] = [0,1]$ all upper sums for f equal 1 while all lower sums for f equal 0. Therefore the Riemann criterion is violated for $\varepsilon = 1/2$.

1.5 Exercises

1. Prove the second half of Lemma 1.4.1 about lower sums.

2. Verify that for f given in 1.8, the lower sums on the interval $[0,1]$ are all equal to zero while the upper sums are all equal to one.

3. Let $f(x) = 1 + x^2$ for $x \in [-1,3]$ and let $P = \left\{-1, -\frac{1}{3}, 0, \frac{1}{2}, 1, 2\right\}$. Find $U(f,P)$ and $L(f,P)$.

4. Show that if $f \in R([a,b])$, there exists a partition, $\{x_0, \cdots, x_n\}$ such that for any $z_k \in [x_k, x_{k+1}]$,

$$\left| \int_a^b f(x)\, dx - \sum_{k=1}^{n} f(z_k)(x_k - x_{k-1}) \right| < \varepsilon$$

   This sum, $\sum_{k=1}^{n} f(z_k)(x_k - x_{k-1})$, is called a Riemann sum and this exercise shows that the integral can always be approximated by a Riemann sum.

5. Let $P = \left\{1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2\right\}$. Find upper and lower sums for the function, $f(x) = \frac{1}{x}$ using this partition. What does this tell you about $\ln(2)$?

6. If $f \in R([a,b])$ and f is changed at finitely many points, show the new function is also in $R([a,b])$ and has the same integral as the unchanged function.

7. Consider the function, $y = x^2$ for $x \in [0,1]$. Show this function is Riemann integrable and find the integral using the definition and the formula

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}(n+1)^3 - \frac{1}{2}(n+1)^2 + \frac{1}{6}(n+1)$$

which you should verify by using math induction. This is not a practical way to find integrals in general.

8. Define a "left sum" as

$$\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

and a "right sum",

$$\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$

Also suppose that all partitions have the property that $x_k - x_{k-1}$ equals a constant, $(b-a)/n$ so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as n gets larger and larger. Show that for f given in 1.8, $\int_0^x f(t)\, dt = 1$ if x is rational and $\int_0^x f(t)\, dt = 0$ if x is irrational. It turns out that the correct answer should always equal zero for that function, regardless of whether x is rational. This is shown in more advanced courses when the Lebesgue integral is studied. This illustrates why the method of defining the integral in terms of left and right sums is nonsense.

1.6 Functions Of Riemann Integrable Functions

It is often necessary to consider functions of Riemann integrable functions and a natural question is whether these are Riemann integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

Theorem 1.6.1 *Let f, g be bounded functions and let $f([a,b]) \subseteq [c_1, d_1]$ and $g([a,b]) \subseteq [c_2, d_2]$. Let $H : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R}$ satisfy,*

$$|H(a_1, b_1) - H(a_2, b_2)| \leq K[|a_1 - a_2| + |b_1 - b_2|]$$

for some constant K. Then if $f, g \in R([a,b])$ it follows that $H \circ (f,g) \in R([a,b])$.

Proof: In the following claim, $M_i(h)$ and $m_i(h)$ have the meanings assigned above with respect to some partition of $[a,b]$ for the function, h.

Claim: The following inequality holds.

$$|M_i(H \circ (f,g)) - m_i(H \circ (f,g))| \leq$$

$$K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|].$$

Proof of the claim: By the above proposition, there exist $x_1, x_2 \in [x_{i-1}, x_i]$ be such that

$$H(f(x_1), g(x_1)) + \eta > M_i(H \circ (f,g)),$$

and

$$H(f(x_2), g(x_2)) - \eta < m_i(H \circ (f,g)).$$

Then

$$|M_i (H \circ (f, g)) - m_i (H \circ (f, g))|$$

$$< 2\eta + |H (f (x_1), g (x_1)) - H (f (x_2), g (x_2))|$$
$$< 2\eta + K [|f (x_1) - f (x_2)| + |g (x_1) - g (x_2)|]$$
$$\leq 2\eta + K [|M_i (f) - m_i (f)| + |M_i (g) - m_i (g)|] .$$

Since $\eta > 0$ is arbitrary, this proves the claim.

Now continuing with the proof of the theorem, let P be such that

$$\sum_{i=1}^{n} (M_i (f) - m_i (f)) \Delta x_i < \frac{\varepsilon}{2K}, \sum_{i=1}^{n} (M_i (g) - m_i (g)) \Delta x_i < \frac{\varepsilon}{2K}.$$

Then from the claim,

$$\sum_{i=1}^{n} (M_i (H \circ (f, g)) - m_i (H \circ (f, g))) \Delta x_i$$

$$< \sum_{i=1}^{n} K [|M_i (f) - m_i (f)| + |M_i (g) - m_i (g)|] \Delta x_i < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows $H \circ (f, g)$ satisfies the Riemann criterion and hence $H \circ (f, g)$ is Riemann integrable as claimed. This proves the theorem.

This theorem implies that if f, g are Riemann integrable, then so is $af + bg, |f|, f^2$, along with infinitely many other such continuous combinations of Riemann integrable functions. For example, to see that $|f|$ is Riemann integrable, let $H (a, b) = |a|$. Clearly this function satisfies the conditions of the above theorem and so $|f| = H (f, f) \in R ([a, b])$ as claimed. The following theorem gives an example of many functions which are Riemann integrable.

Theorem 1.6.2 *Let $f : [a, b] \to \mathbb{R}$ be either increasing or decreasing on $[a, b]$. Then $f \in R ([a, b])$.*

Proof: Let $\varepsilon > 0$ be given and let

$$x_i = a + i \left(\frac{b - a}{n}\right), \ i = 0, \cdots, n.$$

Then since f is increasing,

$$U (f, P) - L (f, P) = \sum_{i=1}^{n} (f (x_i) - f (x_{i-1})) \left(\frac{b - a}{n}\right)$$
$$= (f (b) - f (a)) \left(\frac{b - a}{n}\right) < \varepsilon$$

whenever n is large enough. Thus the Riemann criterion is satisfied and so the function is Riemann integrable. The proof for decreasing f is similar.

Corollary 1.6.3 *Let $[a, b]$ be a bounded closed interval and let $\phi : [a, b] \to \mathbb{R}$ be Lipschitz continuous. Then $\phi \in R ([a, b])$. Recall that a function, ϕ, is Lipschitz continuous if there is a constant, K, such that for all x, y,*

$$|\phi (x) - \phi (y)| < K |x - y| .$$

Proof: Let $f (x) = x$. Then by Theorem 1.6.2, f is Riemann integrable. Let $H (a, b) \equiv \phi (a)$. Then by Theorem 1.6.1 $H \circ (f, f) = \phi \circ f = \phi$ is also Riemann integrable. This proves the corollary.

1.7 The Integral Of A Continuous Function

There is a theorem about the integral of a continuous function which requires the notion of uniform continuity. This is discussed in this section. Consider the function $f(x) = \frac{1}{x}$ for $x \in (0,1)$. This is a continuous function because, by Theorem **??**, it is continuous at every point of $(0,1)$. However, for a given $\varepsilon > 0$, the δ needed in the ε, δ definition of continuity becomes very small as x gets close to 0. The notion of uniform continuity involves being able to choose a single δ which works on the whole domain of f. Here is the definition. For the sake of contrast, I will first review the definition of what it means to be continuous at a point. Then when this is done, the definition of uniform continuity is given.

Definition 1.7.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then f is continuous at $x \in D$ if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|y - x| < \delta$ and $y \in D$, then*

$$|f(x) - f(y)| < \varepsilon.$$

*Note the δ could depend on x. The function is said to be continuous on D if it is continuous at every point of D. Furthermore, this is a pointwise property valid at this or that point of D. f is uniformly continuous if for every $\varepsilon > 0$, there exists a δ **depending only on** ε such that if x, y are any two points of D with $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.*

It is an amazing fact that under certain conditions continuity implies uniform continuity. First here is an important lemma called the nested interval lemma.

Lemma 1.7.2 *Let $\{[a_k, b_k]\}_{k=1}^{\infty}$ be a sequence of closed and bounded intervals. Suppose also that for all k,*

$$[a_k, b_k] \supseteq [a_{k+1}, b_{k+1}]$$

That is, the intervals are nested. Then there exists a point x which is contained in all the intervals.

Proof: First note that for k, l given, every a_k is less than any b_l. Here is why. The sequence $\{a_k\}$ is increasing and the sequence $\{b_k\}$ is decreasing. Thus if $k \leq l$.

$$a_k \leq a_l \leq b_l$$

while if $k > l$,

$$a_k \leq a_{k+1} \leq b_{k+1} \leq b_l$$

Therefore, $\{a_k\}_{k=1}^{\infty}$ is an increasing sequence which is bounded above by b_l for each l. Therefore, letting x be the least upper bound of this sequence, it follows that for all l,

$$a_l \leq x \leq b_l$$

which says x is contained in all the intervals.

Definition 1.7.3 *A set, $K \subseteq \mathbb{R}$ is sequentially compact if whenever $\{a_n\} \subseteq K$ is a sequence, there exists a subsequence, $\{a_{n_k}\}$ such that this subsequence converges to a point of K.*

The following theorem is part of the Heine Borel theorem.

Theorem 1.7.4 *Every closed interval, $[a, b]$ is sequentially compact.*

Proof: Let $\{x_n\} \subseteq [a,b] \equiv I_0$. Consider the two intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ each of which has length $(b-a)/2$. At least one of these intervals contains x_n for infinitely many values of n. Call this interval I_1. Now do for I_1 what was done for I_0. Split it in half and let I_2 be the interval which contains x_n for infinitely many values of n. Continue this way obtaining a sequence of nested intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots$ where the length of I_n is $(b-a)/2^n$. Now pick n_1 such that $x_{n_1} \in I_1$, n_2 such that $n_2 > n_1$ and $x_{n_2} \in I_2, n_3$ such that $n_3 > n_2$ and $x_{n_3} \in I_3$, etc. (This can be done because in each case the intervals contained x_n for infinitely many values of n.) By the nested interval lemma there exists a point, c contained in all these intervals. Furthermore,

$$|x_{n_k} - c| < (b-a)\, 2^{-k}$$

and so $\lim_{k\to\infty} x_{n_k} = c \in [a,b]$. This proves the theorem.

Theorem 1.7.5 *Let $f : K \to \mathbb{R}$ be continuous where K is a sequentially compact set in. Then f is uniformly continuous on K.*

Proof: If this is not true, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a pair of points, x_δ and y_δ such that even though $|x_\delta - y_\delta| < \delta$, $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. Taking a succession of values for δ equal to $1, 1/2, 1/3, \cdots$, and letting the exceptional pair of points for $\delta = 1/n$ be denoted by x_n and y_n,

$$|x_n - y_n| < \frac{1}{n}, |f(x_n) - f(y_n)| \geq \varepsilon.$$

Now since K is sequentially compact, there exists a subsequence, $\{x_{n_k}\}$ such that $x_{n_k} \to z \in K$. Now $n_k \geq k$ and so

$$|x_{n_k} - y_{n_k}| < \frac{1}{k}.$$

Consequently, $y_{n_k} \to z$ also. (x_{n_k} is like a person walking toward a certain point and y_{n_k} is like a dog on a leash which is constantly getting shorter. Obviously y_{n_k} must also move toward the point also. You should give a precise proof of what is needed here.) By continuity of f and Theorem 1.2.1,

$$0 = |f(z) - f(z)| = \lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon,$$

an obvious contradiction. Therefore, the theorem must be true.

The following corollary follows from this theorem and Theorem 1.7.4.

Corollary 1.7.6 *Suppose I is a closed interval, $I = [a,b]$ and $f : I \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

The next theorem is the one about being able to take the integral of a continuous function.

Theorem 1.7.7 *Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then $f \in R([a,b])$.*

Proof: By Corollary 1.7.6, f is uniformly continuous on $[a,b]$. Therefore, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ such that if $|x_i - x_{i-1}| < \delta$, then $M_i - m_i < \frac{\varepsilon}{b-a}$. (Recall $M_i \equiv \sup\{|f(x)| : x \in [x_{i-1}, x_i]\}$Let

$$P \equiv \{x_0, \cdots, x_n\}$$

be a partition with $|x_i - x_{i-1}| < \delta$. Then

$$U(f,P) - L(f,P) < \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

By the Riemann criterion, $f \in R([a,b])$. This proves the theorem.

1.8 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.

Lemma 1.8.1 *Let S be a nonempty set which is bounded above and below. Then if $-S \equiv \{-x : x \in S\}$,*

$$\sup(-S) = -\inf(S) \tag{1.9}$$

and

$$\inf(-S) = -\sup(S). \tag{1.10}$$

Proof: Consider 1.9. Let $x \in S$. Then $-x \leq \sup(-S)$ and so $x \geq -\sup(-S)$. If follows that $-\sup(-S)$ is a lower bound for S and therefore, $-\sup(-S) \leq \inf(S)$. This implies $\sup(-S) \geq -\inf(S)$. Now let $-x \in -S$. Then $x \in S$ and so $x \geq \inf(S)$ which implies $-x \leq -\inf(S)$. Therefore, $-\inf(S)$ is an upper bound for $-S$ and so $-\inf(S) \geq \sup(-S)$. This shows 1.9. Formula 1.10 is similar and is left as an exercise.

In particular, the above lemma implies that for $M_i(f)$ and $m_i(f)$ defined above $M_i(-f) = -m_i(f)$, and $m_i(-f) = -M_i(f)$.

Lemma 1.8.2 *If $f \in R([a,b])$ then $-f \in R([a,b])$ and*

$$-\int_a^b f(x)\, dx = \int_a^b -f(x)\, dx.$$

Proof: The first part of the conclusion of this lemma follows from Theorem 1.6.2 since the function $\phi(y) \equiv -y$ is Lipschitz continuous. Now choose P such that

$$\int_a^b -f(x)\, dx - L(-f, P) < \varepsilon.$$

Then since $m_i(-f) = -M_i(f)$,

$$\varepsilon > \int_a^b -f(x)\, dx - \sum_{i=1}^n m_i(-f)\,\Delta x_i = \int_a^b -f(x)\, dx + \sum_{i=1}^n M_i(f)\,\Delta x_i$$

which implies

$$\varepsilon > \int_a^b -f(x)\, dx + \sum_{i=1}^n M_i(f)\,\Delta x_i \geq \int_a^b -f(x)\, dx + \int_a^b f(x)\, dx.$$

Thus, since ε is arbitrary,

$$\int_a^b -f(x)\, dx \leq -\int_a^b f(x)\, dx$$

whenever $f \in R([a,b])$. It follows

$$\int_a^b -f(x)\, dx \leq -\int_a^b f(x)\, dx = -\int_a^b -(-f(x))\, dx \leq \int_a^b -f(x)\, dx$$

and this proves the lemma.

Theorem 1.8.3 *The integral is linear,*

$$\int_a^b (\alpha f + \beta g)(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

whenever $f, g \in R([a,b])$ and $\alpha, \beta \in \mathbb{R}$.

Proof: First note that by Theorem 1.6.1, $\alpha f+\beta g\in R\left([a,b]\right)$. To begin with, consider the claim that if $f,g\in R\left([a,b]\right)$ then

$$\int_a^b (f+g)(x)\ dx=\int_a^b f(x)\ dx+\int_a^b g(x)\ dx. \tag{1.11}$$

Let P_1,Q_1 be such that

$$U(f,Q_1)-L(f,Q_1)<\varepsilon/2,\ U(g,P_1)-L(g,P_1)<\varepsilon/2.$$

Then letting $P\equiv P_1\cup Q_1$, Lemma 1.4.1 implies

$$U(f,P)-L(f,P)<\varepsilon/2,\ \text{and } U(g,P)-U(g,P)<\varepsilon/2.$$

Next note that

$$m_i(f+g)\geq m_i(f)+m_i(g),\ M_i(f+g)\leq M_i(f)+M_i(g).$$

Therefore,

$$L(g+f,P)\geq L(f,P)+L(g,P),\ U(g+f,P)\leq U(f,P)+U(g,P).$$

For this partition,

$$\int_a^b (f+g)(x)\ dx\in\left[L(f+g,P),U(f+g,P)\right]$$
$$\subseteq\left[L(f,P)+L(g,P),U(f,P)+U(g,P)\right]$$

and

$$\int_a^b f(x)\ dx+\int_a^b g(x)\ dx\in\left[L(f,P)+L(g,P),U(f,P)+U(g,P)\right].$$

Therefore,

$$\left|\int_a^b (f+g)(x)\ dx-\left(\int_a^b f(x)\ dx+\int_a^b g(x)\ dx\right)\right|\leq$$
$$U(f,P)+U(g,P)-(L(f,P)+L(g,P))<\varepsilon/2+\varepsilon/2=\varepsilon.$$

This proves 1.11 since ε is arbitrary.

It remains to show that

$$\alpha\int_a^b f(x)\ dx=\int_a^b \alpha f(x)\ dx.$$

Suppose first that $\alpha\geq 0$. Then

$$\int_a^b \alpha f(x)\ dx\equiv\sup\{L(\alpha f,P):P\text{ is a partition}\}=$$

$$\alpha\sup\{L(f,P):P\text{ is a partition}\}\equiv\alpha\int_a^b f(x)\ dx.$$

If $\alpha<0$, then this and Lemma 1.8.2 imply

$$\int_a^b \alpha f(x)\ dx=\int_a^b (-\alpha)(-f(x))\ dx$$

$$=(-\alpha)\int_a^b (-f(x))\ dx=\alpha\int_a^b f(x)\ dx.$$

This proves the theorem.

Theorem 1.8.4 *If $f \in R([a,b])$ and $f \in R([b,c])$, then $f \in R([a,c])$ and*

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx. \tag{1.12}$$

Proof: Let P_1 be a partition of $[a,b]$ and P_2 be a partition of $[b,c]$ such that

$$U(f,P_i) - L(f,P_i) < \varepsilon/2,\ i = 1,2.$$

Let $P \equiv P_1 \cup P_2$. Then P is a partition of $[a,c]$ and

$$\begin{aligned} &U(f,P) - L(f,P) \\ &= U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \tag{1.13}$$

Thus, $f \in R([a,c])$ by the Riemann criterion and also for this partition,

$$\begin{aligned} \int_a^b f(x)\, dx + \int_b^c f(x)\, dx &\in [L(f,P_1) + L(f,P_2), U(f,P_1) + U(f,P_2)] \\ &= [L(f,P), U(f,P)] \end{aligned}$$

and

$$\int_a^c f(x)\, dx \in [L(f,P), U(f,P)].$$

Hence by 1.13,

$$\left| \int_a^c f(x)\, dx - \left(\int_a^b f(x)\, dx + \int_b^c f(x)\, dx \right) \right| < U(f,P) - L(f,P) < \varepsilon$$

which shows that since ε is arbitrary, 1.12 holds. This proves the theorem.

Corollary 1.8.5 *Let $[a,b]$ be a closed and bounded interval and suppose that*

$$a = y_1 < y_2 \cdots < y_l = b$$

and that f is a bounded function defined on $[a,b]$ which has the property that f is either increasing on $[y_j, y_{j+1}]$ or decreasing on $[y_j, y_{j+1}]$ for $j = 1, \cdots, l-1$. Then $f \in R([a,b])$.

Proof: This follows from Theorem 1.8.4 and Theorem 1.6.2.
The symbol, $\int_a^b f(x)\, dx$ when $a > b$ has not yet been defined.

Definition 1.8.6 *Let $[a,b]$ be an interval and let $f \in R([a,b])$. Then*

$$\int_b^a f(x)\, dx \equiv -\int_a^b f(x)\, dx.$$

Note that with this definition,

$$\int_a^a f(x)\, dx = -\int_a^a f(x)\, dx$$

and so

$$\int_a^a f(x)\, dx = 0.$$

Theorem 1.8.7 *Assuming all the integrals make sense,*

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx.$$

Proof: This follows from Theorem 1.8.4 and Definition 1.8.6. For example, assume

$$c \in (a,b).$$

Then from Theorem 1.8.4,

$$\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$$

and so by Definition 1.8.6,

$$\begin{aligned}\int_a^c f(x)\, dx &= \int_a^b f(x)\, dx - \int_c^b f(x)\, dx \\ &= \int_a^b f(x)\, dx + \int_b^c f(x)\, dx.\end{aligned}$$

The other cases are similar.

The following properties of the integral have either been established or they follow quickly from what has been shown so far.

$$\text{If } f \in R([a,b]) \text{ then if } c \in [a,b],\ f \in R([a,c]), \tag{1.14}$$

$$\int_a^b \alpha\, dx = \alpha(b-a), \tag{1.15}$$

$$\int_a^b (\alpha f + \beta g)(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx, \tag{1.16}$$

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx, \tag{1.17}$$

$$\int_a^b f(x)\, dx \geq 0 \text{ if } f(x) \geq 0 \text{ and } a < b, \tag{1.18}$$

$$\left| \int_a^b f(x)\, dx \right| \leq \left| \int_a^b |f(x)|\, dx \right|. \tag{1.19}$$

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

$$|f(x)| - f(x) \geq 0,\ |f(x)| + f(x) \geq 0.$$

Therefore, by 1.18 and 1.16, if $a < b$,

$$\int_a^b |f(x)|\, dx \geq \int_a^b f(x)\, dx$$

and

$$\int_a^b |f(x)|\, dx \geq -\int_a^b f(x)\, dx.$$

Therefore,

$$\int_a^b |f(x)|\, dx \geq \left| \int_a^b f(x)\, dx \right|.$$

If $b < a$ then the above inequality holds with a and b switched. This implies 1.19.

1.9 Fundamental Theorem Of Calculus

With these properties, it is easy to prove the fundamental theorem of calculus[2]. Let $f \in R([a,b])$. Then by 1.14 $f \in R([a,x])$ for each $x \in [a,b]$. The first version of the fundamental theorem of calculus is a statement about the derivative of the function

$$x \to \int_a^x f(t)\, dt.$$

Theorem 1.9.1 *Let $f \in R([a,b])$ and let*

$$F(x) \equiv \int_a^x f(t)\, dt.$$

Then if f is continuous at $x \in (a,b)$,

$$F'(x) = f(x).$$

Proof: Let $x \in (a,b)$ be a point of continuity of f and let h be small enough that $x+h \in [a,b]$. Then by using 1.17,

$$h^{-1}(F(x+h) - F(x)) = h^{-1} \int_x^{x+h} f(t)\, dt.$$

Also, using 1.15,

$$f(x) = h^{-1} \int_x^{x+h} f(x)\, dt.$$

Therefore, by 1.19,

$$\left|h^{-1}(F(x+h) - F(x)) - f(x)\right| = \left|h^{-1} \int_x^{x+h} (f(t) - f(x))\, dt\right|$$

$$\leq \left|h^{-1} \int_x^{x+h} |f(t) - f(x)|\, dt\right|.$$

Let $\varepsilon > 0$ and let $\delta > 0$ be small enough that if $|t - x| < \delta$, then

$$|f(t) - f(x)| < \varepsilon.$$

Therefore, if $|h| < \delta$, the above inequality and 1.15 shows that

$$\left|h^{-1}(F(x+h) - F(x)) - f(x)\right| \leq |h|^{-1} \varepsilon |h| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\lim_{h \to 0} h^{-1}(F(x+h) - F(x)) = f(x)$$

and this proves the theorem.

Note this gives existence for the initial value problem,

$$F'(x) = f(x),\ F(a) = 0$$

whenever f is Riemann integrable and continuous.[3]

The next theorem is also called the fundamental theorem of calculus.

[2]This theorem is why Newton and Liebnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton's predecessor, Isaac Barrow had made some progress in this direction.

[3]Of course it was proved that if f is continuous on a closed interval, $[a,b]$, then $f \in R([a,b])$ but this is a hard theorem using the difficult result about uniform continuity.

Theorem 1.9.2 *Let $f \in R([a,b])$ and suppose there exists an antiderivative for f, G, such that*

$$G'(x) = f(x)$$

for every point of (a,b) and G is continuous on $[a,b]$. Then

$$\int_a^b f(x)\, dx = G(b) - G(a). \tag{1.20}$$

Proof: Let $P = \{x_0, \cdots, x_n\}$ be a partition satisfying

$$U(f,P) - L(f,P) < \varepsilon.$$

Then

$$\begin{aligned} G(b) - G(a) &= G(x_n) - G(x_0) \\ &= \sum_{i=1}^{n} G(x_i) - G(x_{i-1}). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} G(b) - G(a) &= \sum_{i=1}^{n} G'(z_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(z_i)\Delta x_i \end{aligned}$$

where z_i is some point in $[x_{i-1}, x_i]$. It follows, since the above sum lies between the upper and lower sums, that

$$G(b) - G(a) \in [L(f,P), U(f,P)],$$

and also

$$\int_a^b f(x)\, dx \in [L(f,P), U(f,P)].$$

Therefore,

$$\left| G(b) - G(a) - \int_a^b f(x)\, dx \right| < U(f,P) - L(f,P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, 1.20 holds. This proves the theorem.

The following notation is often used in this context. Suppose F is an antiderivative of f as just described with F continuous on $[a,b]$ and $F' = f$ on (a,b). Then

$$\int_a^b f(x)\, dx = F(b) - F(a) \equiv F(x)|_a^b.$$

The next theorem is a significant existence theorem which tells you that solutions of the initial value problem exist.

Theorem 1.9.3 *Suppose f is a continuous function defined on an interval, (a,b), $c \in (a,b)$, and $y_0 \in \mathbb{R}$. Then there exists a unique solution to the initial value problem,*

$$F'(x) = f(x),\ F(c) = y_0.$$

This solution is given by

$$F(x) = y_0 + \int_c^x f(t)\, dt. \tag{1.21}$$

Proof: From Theorem 1.7.7, it follows the integral in 1.21 is well defined. Now by the fundamental theorem of calculus, $F'(x) = f(x)$. Therefore, F solves the given differential equation. Also, $F(c) = y_0 + \int_c^c f(t)\,dt = y_0$ so the initial condition is also satisfied. This establishes the existence part of the theorem.

Suppose F and G both solve the initial value problem. Then

$$F'(x) - G'(x) = f(x) - f(x) = 0$$

and so $F(x) - G(x) = C$ for some constant C by an application of the mean value theorem. However, $F(c) - G(c) = y_0 - y_0 = 0$ and so the constant C can only equal 0. This proves the uniqueness part of the theorem.

1.10 Limits Of A Vector Valued Function Of One Variable

The above discussion considered expressions like

$$\frac{\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)}{h}$$

and determined what they get close to as h gets small. In other words it is desired to consider

$$\lim_{h \to 0} \frac{\mathbf{f}(t_0 + h) - \mathbf{f}(t_0)}{h}$$

Specializing to functions of one variable, one can give a meaning to

$$\lim_{s \to t+} \mathbf{f}(s), \lim_{s \to t-} \mathbf{f}(s), \lim_{s \to +\infty} \mathbf{f}(s),$$

and

$$\lim_{s \to -\infty} \mathbf{f}(s).$$

Definition 1.10.1 *In the case where $D(\mathbf{f})$ is only assumed to satisfy $D(\mathbf{f}) \supseteq (t, t+r)$,*

$$\lim_{s \to t+} \mathbf{f}(s) = \mathbf{L}$$

if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < s - t < \delta,$$

then

$$|\mathbf{f}(s) - \mathbf{L}| < \varepsilon.$$

In the case where $D(\mathbf{f})$ is only assumed to satisfy $D(\mathbf{f}) \supseteq (t - r, t)$,

$$\lim_{s \to t-} \mathbf{f}(s) = \mathbf{L}$$

if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < t - s < \delta,$$

then

$$|\mathbf{f}(s) - \mathbf{L}| < \varepsilon.$$

One can also consider limits as a variable "approaches" infinity. Of course nothing is "close" to infinity and so this requires a slightly different definition.

$$\lim_{t\to\infty}\mathbf{f}(t)=\mathbf{L}$$

if for every $\varepsilon>0$ *there exists* l *such that whenever* $t>l$,

$$|\mathbf{f}(t)-\mathbf{L}|<\varepsilon \tag{1.22}$$

and

$$\lim_{t\to-\infty}\mathbf{f}(t)=\mathbf{L}$$

if for every $\varepsilon>0$ *there exists* l *such that whenever* $t<l$, *1.22 holds.*

Note that in all of this the definitions are identical to the case of scalar valued functions. The only difference is that here $|\cdot|$ refers to the norm or length in $\mathbb{R}^p$ where maybe $p>1$. Here is an important observation which further reduces to the case of scalar valued functions.

Proposition 1.10.2 *Suppose* $\mathbf{f}(s)=(f_1(s),\cdots,f_p(s))^T$ *and* $\mathbf{L}\equiv(L_1,\cdots,L_p)^T$. *Then*

$$\lim_{s\to t}\mathbf{f}(s)=\mathbf{L}$$

if and only if for every k,

$$\lim_{s\to t}f_k(s)=L_k$$

The proof comes directly from the definitions and is left to you. If you like, you can take this as a definition and no harm will be done.

Example 1.10.3 *Let* $\mathbf{f}(t)=\left(\cos t,\sin t,t^2+1,\ln(t)\right)$. *Find* $\lim_{t\to\pi/2}\mathbf{f}(t)$.

From the above, this equals

$$\begin{aligned}&\left(\lim_{t\to\pi/2}\cos t,\ \lim_{t\to\pi/2}\sin t,\ \lim_{t\to\pi/2}\left(t^2+1\right),\ \lim_{t\to\pi/2}\ln(t)\right)\\ =&\left(0,1,\ln\left(\frac{\pi^2}{4}+1\right),\ln\left(\frac{\pi}{2}\right)\right).\end{aligned}$$

Example 1.10.4 *Let* $\mathbf{f}(t)=\left(\frac{\sin t}{t},t^2,t+1\right)$. *Find* $\lim_{t\to 0}\mathbf{f}(t)$.

Recall that $\lim_{t\to 0}\frac{\sin t}{t}=1$. Then from Theorem **??** on Page **??**, $\lim_{t\to 0}\mathbf{f}(t)=(1,0,1)$.

1.11 The Derivative And Integral

The following definition is on the derivative and integral of a vector valued function of one variable.

Definition 1.11.1 *The derivative of a function,* $\mathbf{f}'(t)$, *is defined as the following limit whenever the limit exists. If the limit does not exist, then neither does* $\mathbf{f}'(t)$.

$$\lim_{h\to 0}\frac{\mathbf{f}(t+h)-\mathbf{f}(x)}{h}\equiv\mathbf{f}'(t)$$

The function of h on the left is called the difference quotient just as it was for a scalar valued function. If $\mathbf{f}(t) = (f_1(t), \cdots, f_p(t))$ and $\int_a^b f_i(t)\, dt$ exists for each $i = 1, \cdots, p$, then $\int_a^b \mathbf{f}(t)\, dt$ is defined as the vector,

$$\left(\int_a^b f_1(t)\, dt, \cdots, \int_a^b f_p(t)\, dt \right).$$

This is what is meant by saying $\mathbf{f} \in R([a,b])$.

This is exactly like the definition for a scalar valued function. As before,

$$\mathbf{f}'(x) = \lim_{y \to x} \frac{\mathbf{f}(y) - \mathbf{f}(x)}{y - x}.$$

As in the case of a scalar valued function, differentiability implies continuity but not the other way around.

Theorem 1.11.2 *If $\mathbf{f}'(t)$ exists, then $\mathbf{f}$ is continuous at t.*

Proof: Suppose $\varepsilon > 0$ is given and choose $\delta_1 > 0$ such that if $|h| < \delta_1$,

$$\left| \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h} - \mathbf{f}'(t) \right| < 1.$$

then for such h, the triangle inequality implies

$$|\mathbf{f}(t+h) - \mathbf{f}(t)| < |h| + |\mathbf{f}'(t)|\, |h|.$$

Now letting $\delta < \min\left(\delta_1, \frac{\varepsilon}{1+|\mathbf{f}'(x)|}\right)$ it follows if $|h| < \delta$, then

$$|\mathbf{f}(t+h) - \mathbf{f}(t)| < \varepsilon.$$

Letting $y = h + t$, this shows that if $|y - t| < \delta$,

$$|\mathbf{f}(y) - \mathbf{f}(t)| < \varepsilon$$

which proves $\mathbf{f}$ is continuous at t. This proves the theorem.

As in the scalar case, there is a fundamental theorem of calculus.

Theorem 1.11.3 *If $\mathbf{f} \in R([a,b])$ and if $\mathbf{f}$ is continuous at $t \in (a,b)$, then*

$$\frac{d}{dt}\left(\int_a^t \mathbf{f}(s)\, ds \right) = \mathbf{f}(t).$$

Proof: Say $\mathbf{f}(t) = (f_1(t), \cdots, f_p(t))$. Then it follows

$$\frac{1}{h}\int_a^{t+h} \mathbf{f}(s)\, ds - \frac{1}{h}\int_a^t \mathbf{f}(s)\, ds = \left(\frac{1}{h}\int_t^{t+h} f_1(s)\, ds, \cdots, \frac{1}{h}\int_t^{t+h} f_p(s)\, ds \right)$$

and $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f_i(s)\, ds = f_i(t)$ for each $i = 1, \cdots, p$ from the fundamental theorem of calculus for scalar valued functions. Therefore,

$$\lim_{h \to 0} \frac{1}{h}\int_a^{t+h} \mathbf{f}(s)\, ds - \frac{1}{h}\int_a^t \mathbf{f}(s)\, ds = (f_1(t), \cdots, f_p(t)) = \mathbf{f}(t)$$

and this proves the claim.

Example 1.11.4 *Let* $\mathbf{f}(x) = \mathbf{c}$ *where* $\mathbf{c}$ *is a constant. Find* $\mathbf{f}'(x)$.

The difference quotient,

$$\frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h} = \frac{\mathbf{c} - \mathbf{c}}{h} = \mathbf{0}$$

Therefore,

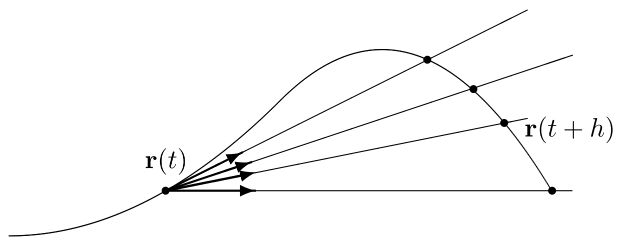
$$\lim_{h \to 0} \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h} = \lim_{h \to 0} \mathbf{0} = \mathbf{0}$$

Example 1.11.5 *Let* $\mathbf{f}(t) = (at, bt)$ *where* a, b *are constants. Find* $\mathbf{f}'(t)$.

From the above discussion this derivative is just the vector valued functions whose components consist of the derivatives of the components of $\mathbf{f}$. Thus $\mathbf{f}'(t) = (a, b)$.

1.11.1 Geometric And Physical Significance Of The Derivative

Suppose $\mathbf{r}$ is a vector valued function of a parameter, t not necessarily time and consider the following picture of the points traced out by $\mathbf{r}$.



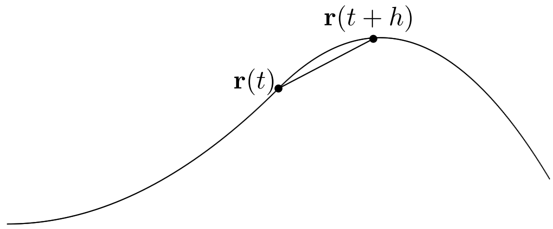
In this picture there are unit vectors in the direction of the vector from $\mathbf{r}(t)$ to $\mathbf{r}(t+h)$. You can see that it is reasonable to suppose these unit vectors, if they converge, converge to a unit vector, $\mathbf{T}$ which is tangent to the curve at the point $\mathbf{r}(t)$. Now each of these unit vectors is of the form

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{|\mathbf{r}(t+h) - \mathbf{r}(t)|} \equiv \mathbf{T}_h.$$

Thus $\mathbf{T}_h \to \mathbf{T}$, a unit tangent vector to the curve at the point $\mathbf{r}(t)$. Therefore,

$$\begin{aligned}
\mathbf{r}'(t) &\equiv \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h \to 0} \frac{|\mathbf{r}(t+h) - \mathbf{r}(t)|}{h} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{|\mathbf{r}(t+h) - \mathbf{r}(t)|} \\
&= \lim_{h \to 0} \frac{|\mathbf{r}(t+h) - \mathbf{r}(t)|}{h} \mathbf{T}_h = |\mathbf{r}'(t)| \mathbf{T}.
\end{aligned}$$

In the case that t is time, the expression $|\mathbf{r}(t+h) - \mathbf{r}(t)|$ is a good approximation for the distance traveled by the object on the time interval $[t, t+h]$. The real distance would be the length of the curve joining the two points but if h is very small, this is essentially equal to $|\mathbf{r}(t+h) - \mathbf{r}(t)|$ as suggested by the picture below.



Therefore,

$$\frac{|\mathbf{r}(t+h)-\mathbf{r}(t)|}{h}$$

gives for small h, the approximate distance travelled on the time interval, $[t, t+h]$ divided by the length of time, h. Therefore, this expression is really the average speed of the object on this small time interval and so the limit as $h \to 0$, deserves to be called the instantaneous speed of the object. Thus $|\mathbf{r}'(t)| \mathbf{T}$ represents the speed times a unit direction vector, $\mathbf{T}$ which defines the direction in which the object is moving. Thus $\mathbf{r}'(t)$ is the velocity of the object. This is the physical significance of the derivative when t is time.

How do you go about computing $\mathbf{r}'(t)$? Letting $\mathbf{r}(t) = (r_1(t), \cdots, r_q(t))$, the expression

$$\frac{\mathbf{r}(t_0+h)-\mathbf{r}(t_0)}{h} \tag{1.23}$$

is equal to

$$\left(\frac{r_1(t_0+h)-r_1(t_0)}{h}, \cdots, \frac{r_q(t_0+h)-r_q(t_0)}{h}\right).$$

Then as h converges to 0, 1.23 converges to

$$\mathbf{v} \equiv (v_1, \cdots, v_q)$$

where $v_k = r_k'(t)$. This by Theorem **??** on Page **??**, which says that the term in 1.23 gets close to a vector, $\mathbf{v}$ if and only if all the coordinate functions of the term in 1.23 get close to the corresponding coordinate functions of $\mathbf{v}$.

In the case where t is time, this simply says the velocity vector equals the vector whose components are the derivatives of the components of the displacement vector, $\mathbf{r}(t)$.

In any case, the vector, $\mathbf{T}$ determines a direction vector which is tangent to the curve at the point, $\mathbf{r}(t)$ and so it is possible to find parametric equations for the line tangent to the curve at various points.

Example 1.11.6 *Let* $\mathbf{r}(t) = \left(\sin t, t^2, t+1\right)$ *for* $t \in [0,5]$. *Find a tangent line to the curve parameterized by* $\mathbf{r}$ *at the point* $\mathbf{r}(2)$.

From the above discussion, a direction vector has the same direction as $\mathbf{r}'(2)$. Therefore, it suffices to simply use $\mathbf{r}'(2)$ as a direction vector for the line. $\mathbf{r}'(2) = (\cos 2, 4, 1)$. Therefore, a parametric equation for the tangent line is

$$(\sin 2, 4, 3) + t(\cos 2, 4, 1) = (x, y, z).$$

Example 1.11.7 *Let* $\mathbf{r}(t) = \left(\sin t, t^2, t+1\right)$ *for* $t \in [0,5]$. *Find the velocity vector when* $t = 1$.

From the above discussion, this is simply $\mathbf{r}'(1) = (\cos 1, 2, 1)$.

1.11.2 Differentiation Rules

There are rules which relate the derivative to the various operations done with vectors such as the dot product, the cross product, and vector addition and scalar multiplication.

Theorem 1.11.8 *Let $a, b \in \mathbb{R}$ and suppose $\mathbf{f}'(t)$ and $\mathbf{g}'(t)$ exist. Then the following formulas are obtained.*

$$(a\mathbf{f} + b\mathbf{g})'(t) = a\mathbf{f}'(t) + b\mathbf{g}'(t). \tag{1.24}$$

$$(\mathbf{f} \cdot \mathbf{g})'(t) = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t) \tag{1.25}$$

If $\mathbf{f}, \mathbf{g}$ have values in $\mathbb{R}^3$, then

$$(\mathbf{f} \times \mathbf{g})'(t) = \mathbf{f}(t) \times \mathbf{g}'(t) + \mathbf{f}'(t) \times \mathbf{g}(t) \tag{1.26}$$

The formulas, 1.25, and 1.26 are referred to as the product rule.

Proof: The first formula is left for you to prove. Consider the second, 1.25.

$$\lim_{h \to 0} \frac{\mathbf{f} \cdot \mathbf{g}(t+h) - \mathbf{fg}(t)}{h}$$

$$\begin{aligned}
&= \lim_{h \to 0} \frac{\mathbf{f}(t+h) \cdot \mathbf{g}(t+h) - \mathbf{f}(t+h) \cdot \mathbf{g}(t)}{h} + \frac{\mathbf{f}(t+h) \cdot \mathbf{g}(t) - \mathbf{f}(t) \cdot \mathbf{g}(t)}{h} \\
&= \lim_{h \to 0} \left(\mathbf{f}(t+h) \cdot \frac{(\mathbf{g}(t+h) - \mathbf{g}(t))}{h} + \frac{(\mathbf{f}(t+h) - \mathbf{f}(t))}{h} \cdot \mathbf{g}(t) \right) \\
&= \lim_{h \to 0} \sum_{k=1}^{n} f_k(t+h) \frac{(g_k(t+h) - g_k(t))}{h} + \sum_{k=1}^{n} \frac{(f_k(t+h) - f_k(t))}{h} g_k(t) \\
&= \sum_{k=1}^{n} f_k(t) g_k'(t) + \sum_{k=1}^{n} f_k'(t) g_k(t) \\
&= \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t).
\end{aligned}$$

Formula 1.26 is left as an exercise which follows from the product rule and the definition of the cross product in terms of components given on Page **??**.

Example 1.11.9 *Let*

$$\mathbf{r}(t) = \left(t^2, \sin t, \cos t\right)$$

and let $\mathbf{p}(t) = (t, \ln(t+1), 2t)$. Find $(\mathbf{r}(t) \times \mathbf{p}(t))'$.

From 1.26 this equals$(2t, \cos t, -\sin t) \times (t, \ln(t+1), 2t) + \left(t^2, \sin t, \cos t\right) \times \left(1, \frac{1}{t+1}, 2\right)$.

Example 1.11.10 *Let $\mathbf{r}(t) = \left(t^2, \sin t, \cos t\right)$ Find $\int_0^\pi \mathbf{r}(t)\, dt$.*

This equals $\left(\int_0^\pi t^2\, dt, \int_0^\pi \sin t\; dt, \int_0^\pi \cos t\, dt\right) = \left(\frac{1}{3}\pi^3, 2, 0\right)$.

Example 1.11.11 *An object has position $\mathbf{r}(t) = \left(t^3, \frac{t}{1+1}, \sqrt{t^2+2}\right)$ kilometers where t is given in hours. Find the velocity of the object in kilometers per hour when $t = 1$.*

Recall the velocity at time t was $\mathbf{r}'(t)$. Therefore, find $\mathbf{r}'(t)$ and plug in $t = 1$ to find the velocity.

$$\mathbf{r}'(t) = \left(3t^2, \frac{1(1+t)-t}{(1+t)^2}, \frac{1}{2}\left(t^2+2\right)^{-1/2} 2t\right)$$
$$= \left(3t^2, \frac{1}{(1+t)^2}, \frac{1}{\sqrt{(t^2+2)}}t\right)$$

When $t = 1$, the velocity is

$$\mathbf{r}'(1) = \left(3, \frac{1}{4}, \frac{1}{\sqrt{3}}\right) \text{ kilometers per hour.}$$

Obviously, this can be continued. That is, you can consider the possibility of taking the derivative of the derivative and then the derivative of that and so forth. The main thing to consider about this is the notation and it is exactly like it was in the case of a scalar valued function presented earlier. Thus $\mathbf{r}''(t)$ denotes the second derivative.

When you are given a vector valued function of one variable, sometimes it is possible to give a simple description of the curve which results. Usually it is not possible to do this!

Example 1.11.12 *Describe the curve which results from the vector valued function,* $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ *where* $t \in \mathbb{R}$.

The first two components indicate that for $\mathbf{r}(t) = (x(t), y(t), z(t))$, the pair, $(x(t), y(t))$ traces out a circle. While it is doing so, $z(t)$ is moving at a steady rate in the positive direction. Therefore, the curve which results is a cork skrew shaped thing called a helix.

As an application of the theorems for differentiating curves, here is an interesting application. It is also a situation where the curve can be identified as something familiar.

Example 1.11.13 *Sound waves have the angle of incidence equal to the angle of reflection. Suppose you are in a large room and you make a sound. The sound waves spread out and you would expect your sound to be inaudible very far away. But what if the room were shaped so that the sound is reflected off the wall toward a single point, possibly far away from you? Then you might have the interesting phenomenon of someone far away hearing what you said quite clearly. How should the room be designed?*

Suppose you are located at the point $\mathbf{P}_0$ and the point where your sound is to be reflected is $\mathbf{P}_1$. Consider a plane which contains the two points and let $\mathbf{r}(t)$ denote a parameterization of the intersection of this plane with the walls of the room. Then the condition that the angle of reflection equals the angle of incidence reduces to saying the angle between $\mathbf{P}_0 - \mathbf{r}(t)$ and $-\mathbf{r}'(t)$ equals the angle between $\mathbf{P}_1 - \mathbf{r}(t)$ and $\mathbf{r}'(t)$. Draw a picture to see this. Therefore,

$$\frac{(\mathbf{P}_0 - \mathbf{r}(t)) \cdot (-\mathbf{r}'(t))}{|\mathbf{P}_0 - \mathbf{r}(t)| |\mathbf{r}'(t)|} = \frac{(\mathbf{P}_1 - \mathbf{r}(t)) \cdot (\mathbf{r}'(t))}{|\mathbf{P}_1 - \mathbf{r}(t)| |\mathbf{r}'(t)|}.$$

This reduces to

$$\frac{(\mathbf{r}(t) - \mathbf{P}_0) \cdot (-\mathbf{r}'(t))}{|\mathbf{r}(t) - \mathbf{P}_0|} = \frac{(\mathbf{r}(t) - \mathbf{P}_1) \cdot (\mathbf{r}'(t))}{|\mathbf{r}(t) - \mathbf{P}_1|} \tag{1.27}$$

Now

$$\frac{(\mathbf{r}(t) - \mathbf{P}_1) \cdot (\mathbf{r}'(t))}{|\mathbf{r}(t) - \mathbf{P}_1|} = \frac{d}{dt} |\mathbf{r}(t) - \mathbf{P}_1|$$

and a similar formula holds for $\mathbf{P}_1$ replaced with $\mathbf{P}_0$. This is because

$$|\mathbf{r}(t) - \mathbf{P}_1| = \sqrt{(\mathbf{r}(t) - \mathbf{P}_1) \cdot (\mathbf{r}(t) - \mathbf{P}_1)}$$

and so using the chain rule and product rule,

$$\begin{aligned} \frac{d}{dt} |\mathbf{r}(t) - \mathbf{P}_1| &= \frac{1}{2} ((\mathbf{r}(t) - \mathbf{P}_1) \cdot (\mathbf{r}(t) - \mathbf{P}_1))^{-1/2} 2 ((\mathbf{r}(t) - \mathbf{P}_1) \cdot \mathbf{r}'(t)) \\ &= \frac{(\mathbf{r}(t) - \mathbf{P}_1) \cdot (\mathbf{r}'(t))}{|\mathbf{r}(t) - \mathbf{P}_1|}. \end{aligned}$$

Therefore, from 1.27,

$$\frac{d}{dt} (|\mathbf{r}(t) - \mathbf{P}_1|) + \frac{d}{dt} (|\mathbf{r}(t) - \mathbf{P}_0|) = 0$$

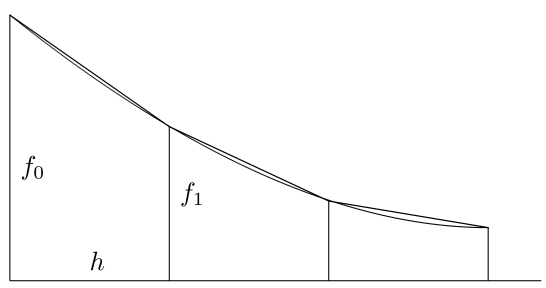
showing that $|\mathbf{r}(t) - \mathbf{P}_1| + |\mathbf{r}(t) - \mathbf{P}_0| = C$ for some constant, C.This implies the curve of intersection of the plane with the room is an ellipse having $\mathbf{P}_0$ and $\mathbf{P}_1$ as the foci.

1.11.3 Leibniz's Notation

Leibniz's notation also generalizes routinely. For example, $\frac{d\mathbf{y}}{dt} = \mathbf{y}'(t)$ with other similar notations holding.

1.12 Exercises

1. Let $F(x) = \int_{x^2}^{x^3} \frac{t^5+7}{t^7+87t^6+1} \, dt$. Find $F'(x)$.
2. Let $F(x) = \int_2^x \frac{1}{1+t^4} \, dt$. Sketch a graph of F and explain why it looks the way it does.
3. There is a general procedure for estimating the integral of a function, f on an interval, $[a, b]$. Form a uniform partition, $P = \{x_0, x_1, \cdots, x_n\}$ where for each j, $x_j - x_{j-1} = h$. Let $f_i = f(x_i)$ and assuming $f \geq 0$ on the interval $[x_{i-1}, x_i]$, approximate the area above this interval and under the curve with the area of a trapezoid having vertical sides, f_{i-1}, and f_i as shown in the following picture.



Thus $\frac{1}{2}\left(\frac{f_i+f_{i-1}}{2}\right)$ approximates the area under the curve. Show that adding these up yields

$$\frac{h}{2} [f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n]$$

as an approximation to $\int_a^b f(x)\,dx$. This is known as the trapezoid rule. Verify that if $f(x) = mx + b$, the trapezoid rule gives the exact answer for the integral. Would this be true of upper and lower sums for such a function? Can you show that in the case of the function, $f(t) = 1/t$ the trapezoid rule will always yield an answer which is too large for $\int_1^2 \frac{1}{t}dt$?

4. Let there be three equally spaced points, $x_{i-1}, x_{i-1} + h \equiv x_i$, and $x_i + 2h \equiv x_{i+1}$. Suppose also a function, f, has the value f_{i-1} at x, f_i at $x + h$, and f_{i+1} at $x + 2h$. Then consider

$$g_i(x) \equiv \frac{f_{i-1}}{2h^2}(x - x_i)(x - x_{i+1}) - \frac{f_i}{h^2}(x - x_{i-1})(x - x_{i+1}) + \frac{f_{i+1}}{2h^2}(x - x_{i-1})(x - x_i).$$

Check that this is a second degree polynomial which equals the values f_{i-1}, f_i, and f_{i+1} at the points x_{i-1}, x_i, and x_{i+1} respectively. The function, g_i is an approximation to the function, f on the interval $[x_{i-1}, x_{i+1}]$. Also,

$$\int_{x_{i-1}}^{x_{i+1}} g_i(x)\,dx$$

is an approximation to $\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx$. Show $\int_{x_{i-1}}^{x_{i+1}} g_i(x)\,dx$ equals

$$\frac{hf_{i-1}}{3} + \frac{hf_i 4}{3} + \frac{hf_{i+1}}{3}.$$

Now suppose n is even and $\{x_0, x_1, \cdots, x_n\}$ is a partition of the interval, $[a, b]$ and the values of a function, f defined on this interval are $f_i = f(x_i)$. Adding these approximations for the integral of f on the succession of intervals,

$$[x_0, x_2], [x_2, x_4], \cdots, [x_{n-2}, x_n],$$

show that an approximation to $\int_a^b f(x)\,dx$ is

$$\frac{h}{3}[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_2 + \cdots + 4f_{n-1} + f_n].$$

This is called Simpson's rule. Use Simpson's rule to compute an approximation to $\int_1^2 \frac{1}{t}dt$ letting $n = 4$. Compare with the answer from a calculator or computer.

5. Let a and b be positive numbers and consider the function,

$$F(x) = \int_0^{ax} \frac{1}{a^2 + t^2}\,dt + \int_b^{a/x} \frac{1}{a^2 + t^2}\,dt.$$

Show that F is a constant.

6. Solve the following initial value problem from ordinary differential equations which is to find a function y such that

$$y'(x) = \frac{x^7 + 1}{x^6 + 97x^5 + 7},\; y(10) = 5.$$

7. If $F, G \in \int f(x)\,dx$ for all $x \in \mathbb{R}$, show $F(x) = G(x) + C$ for some constant, C. Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion $\int_a^b f(t)\,dt = G(b) - G(a)$ where $G'(x) = f(x)$.

8. Suppose f is continuous on $[a,b]$. Show there exists $c \in (a,b)$ such that
$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ dx.$$
Hint: You might consider the function $F(x) \equiv \int_a^x f(t)\ dt$ and use the mean value theorem for derivatives and the fundamental theorem of calculus.

9. Suppose f and g are continuous functions on $[a,b]$ and that $g(x) \neq 0$ on (a,b). Show there exists $c \in (a,b)$ such that
$$f(c) \int_a^b g(x)\ dx = \int_a^b f(x) g(x)\ dx.$$
Hint: Define $F(x) \equiv \int_a^x f(t) g(t)\ dt$ and let $G(x) \equiv \int_a^x g(t)\ dt$. Then use the Cauchy mean value theorem from calculus on these two functions.

10. Consider the function
$$f(x) \equiv \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$
Is f Riemann integrable? Explain why or why not.

11. Prove the second part of Theorem 1.6.2 about decreasing functions.

12. Suppose it is desired to find a function, $L : (0,\infty) \to \mathbb{R}$ which satisfies
$$L(xy) = Lx + Ly,\ L(1) = 0. \tag{1.28}$$
Show the only differentiable solutions to this equation are functions of the form $L_k(x) = \int_1^x \frac{k}{t}\, dt$. **Hint:** Fix $x > 0$ and differentiate both sides of the above equation with respect to y. Then let $y = 1$.

13. Recall that $\ln e = 1$. In fact, this was how e was defined. Show that
$$\lim_{y \to 0+} (1 + yx)^{1/y} = e^x.$$
Hint: Consider $\ln(1+yx)^{1/y} = \frac{1}{y} \ln(1+yx) = \frac{1}{y} \int_1^{1+yx} \frac{1}{t}\, dt$, use upper and lower sums and then the squeezing theorem to verify $\ln(1+yx)^{1/y} \to x$. Recall that $x \to e^x$ is continuous.

Chapter 2

First Order Scalar Equations

2.1 Solving And Classifying First Order Differential Equations

2.1.1 Classifying Equations

To begin with here are some definitions which explain terminology related to differential equations. The differential equation

$$F'(x) = f(x)$$

for the unknown function F and the given function f is the simplest example of a differential equation and this is well solved by consideration of the integral. A solution is

$$F(x) = \int_a^x f(t)\,dt$$

for suitable a. However, there are many more interesting differential equations which cannot be solved so easily.

Definition 2.1.1 ***Differential equations*** *are equations involving an unknown function and some of its derivatives. A differential equation is first order if the highest order derivative of the unknown function found in the equation is 1. Thus a first order equation is one which is of the form*

$$f(t, y, y') = 0.$$

A second order differential equation is of the form

$$f(t, y, y', y'') = 0$$

with a similar definition holding for higher order equations.

First order differential equations are classified as being either **linear** or **nonlinear**.

Definition 2.1.2 *A first order linear differential equation is one which can be written in the form*

$$y' + p(t)\,y = q(t)$$

If it can't be written in this form, it is called a nonlinear equation. A second order equation is called linear if it can be written in the form

$$y'' + a(t)\,y' + b(t)\,y = c(t)\,.$$

Example 2.1.3 $y' + t^2 y = \sin(t)$ *is first order linear while* $y' + y^2 = \sin(xy)$ *is nonlinear.*

Example 2.1.4 *Verify* $y = \tan(t)$ *is a solution of* $y' = 1 + y^2$.

This is easy. $y'(t) = \sec^2(t) = 1 + \tan^2(t) = 1 + y^2(t)$.

Of course it is trivial to verify something solves a differential equation. You just differentiate and plug in. A more interesting problem is in coming up with the solution to a differential equation in the first place.

2.1.2 Solving First Order Linear Equations

The homogeneous first order constant coefficient linear differential equation is a differential equation of the form

$$y' + ay = 0. \tag{2.1}$$

It is arguably the most important differential equation in existence. Generalizations of it include the entire subject of linear differential equations and even many of the most important partial differential equations occurring in applications.

Here is how to find the solutions to this equation. Multiply both sides of the equation by e^{at}. Then use the product and chain rules to verify that

$$e^{at}(y' + ay) = \frac{d}{dt}\left(e^{at}y\right) = 0.$$

Therefore, since the derivative of the function $t \to e^{at}y(t)$ equals zero, it follows this function must equal some constant, C. Consequently, $ye^{at} = C$ and so $y(t) = Ce^{-at}$. This shows that if there is a solution of the equation, $y' + ay = 0$, then it must be of the form Ce^{-at} for some constant, C. You should verify that every function of the form, $y(t) = Ce^{-at}$ is a solution of the above differential equation, showing this yields all solutions. This proves the following theorem.

Theorem 2.1.5 *The solutions to the equation,* $y' + ay = 0$ *consist of all functions of the form,* Ce^{-at} *where* C *is some constant.*

Example 2.1.6 *Radioactive substances decay in the following way. The rate of decay is proportional to the amount present. In other words, letting* $A(t)$ *denote the amount of the radioactive substance at time* t, $A(t)$ *satisfies the following initial value problem.*

$$A'(t) = -k^2 A(t), \ A(0) = A_0$$

where A_0 *is the initial amount of the substance. What is the solution to the initial value problem?*

Write the differential equation as $A'(t) + k^2 A(t) = 0$. From Theorem 2.1.5 the solution is

$$A(t) = Ce^{-k^2 t}$$

and it only remains to find C. Letting $t = 0$, it follows $A_0 = A(0) = C$. Thus $A(t) = A_0 \exp\left(-k^2 t\right)$.

Now consider a slightly harder equation.

$$y' + a(t)y = b(t).$$

In the easier case, you multiplied both sides by e^{at}. In this case, you multiply both sides by $e^{A(t)}$ where $A'(t) = a(t)$. In other words, you find an antiderivative of $a(t)$ and multiply both sides of the equation by e raised to that function. Thus

$$e^{A(t)}(y' + a(t)y) = e^{A(t)}b(t).$$

Now you notice that this becomes

$$\frac{d}{dt}\left(e^{A(t)}y\right) = e^{A(t)}b(t). \tag{2.2}$$

This follows from the chain rule.

$$\frac{d}{dt}\left(e^{A(t)}y\right) = A'(t)e^{A(t)}y + e^{A(t)}y' = e^{A(t)}(y' + a(t)y).$$

Then from 2.2,

$$e^{A(t)}y \in \int e^{A(t)}b(t)\,dt.$$

Therefore, to find the solution, you find a function in $\int e^{A(t)}b(t)\,dt$, say $F(t)$, and

$$e^{A(t)}y = F(t) + C$$

for some constant, C, so the solution is given by $y = e^{-A(t)}F(t) + e^{-A(t)}C$. This proves the following theorem.

Theorem 2.1.7 *The solutions to the equation, $y' + a(t)y = b(t)$ consist of all functions of the form*

$$y = e^{-A(t)}F(t) + e^{-A(t)}C$$

where $F(t) \in \int e^{A(t)}b(t)\,dt$ and C is a constant.

Example 2.1.8 *Find the solution to the initial value problem $y' + 2ty = \sin(t)e^{-t^2}$, $y(0) = 3$.*

Multiply both sides by e^{t^2} because $t^2 \in \int tdt$. Then $\frac{d}{dt}\left(e^{t^2}y\right) = \sin(t)$ and so $e^{t^2}y = -\cos(t) + C$. Hence the solution is of the form $y(t) = -\cos(t)e^{-t^2} + Ce^{-t^2}$. It only remains to choose C in such a way that the initial condition is satisfied. From the initial condition, $3 = y(0) = -1 + C$ and so $C = 4$. Therefore, the solution is $y = -\cos(t)e^{-t^2} + 4e^{-t^2}$. Now at this point, you should check and see if it works. It needs to solve both the initial condition and the differential equation.

Finally, here is a uniqueness theorem.

Theorem 2.1.9 *If $a(t)$ is a continuous function, there is at most one solution to the initial value problem, $y' + a(t)y = b(t)$, $y(r) = y_0$.*

Proof: If there were two solutions, y_1, and y_2, then letting $w = y_1 - y_2$, it follows $w' + a(t)w = 0$ and $w(r) = 0$. Then multiplying both sides of the differential equation by $e^{A(t)}$ where $A'(t) = a(t)$, it follows

$$\left(e^{A(t)}w\right)' = 0$$

and so $e^{A(t)}w(t) = C$ for some constant, C. However, $w(r) = 0$ and so this constant can only be 0. Hence $w = 0$ and so $y_1 = y_2$.

2.1.3 Bernouli Equations

Some kinds of nonlinear equations can be changed to get a linear equation. An equation of the form

$$y' + a(t)\,y = b(t)\,y^{\alpha}$$

is called a Bernouli equation. The trick is to define a new variable, $z = y^{1-\alpha}$. Then $y^{\alpha}z = y$ and so

$$z' = (1-\alpha)\,y^{-\alpha}y'$$

which implies

$$\frac{1}{(1-\alpha)}y^{\alpha}z' = y'.$$

Then

$$\frac{1}{(1-\alpha)}y^{\alpha}z' + a(t)\,y^{\alpha}z = b(t)\,y^{\alpha}$$

and so

$$z' + (1-\alpha)\,a(t)\,z = (1-\alpha)\,b(t)\,.$$

Now this is a linear equation for z. Solve it and then use the transformation to find y.

Example 2.1.10 *Solve $y' + y = ty^3$.*

You let $z = y^{-2}$ and make the above substitution. Thus

$$z' - 2z = (-2)\,t$$

Then

$$\frac{d}{dt}\left(e^{-2t}z\right) = -2te^{-2t}$$

and so

$$e^{-2t}z = te^{-2t} + \frac{1}{2}e^{-2t} + C$$

and so

$$y^{-2} = z = t + \frac{1}{2} + Ce^{2t}$$

and so

$$y^2 = \frac{1}{t + \frac{1}{2} + Ce^{2t}}.$$

2.1.4 Separable Differential Equations

Definition 2.1.11 *Separable differential equations are those which can be written in the form*

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

The reason these are called separable is that if you formally cross multiply,

$$g(y)\,dy = f(x)\,dx$$

and the variables are "separated". The x variables are on one side and the y variables are on the other.

Proposition 2.1.12 *If $G'(y) = g(y)$ and $F'(x) = f(x)$, then if the equation, $F(x) - G(y) = c$ specifies y as a differentiable function of x, then $x \to y(x)$ solves the separable differential equation*

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}. \tag{2.3}$$

Proof: Differentiate both sides of $F(x) - G(y) = c$ with respect to x. Using the chain rule,

$$F'(x) - G'(y)\frac{dy}{dx} = 0.$$

Therefore, since $F'(x) = f(x)$ and $G'(y) = g(y)$, $f(x) = g(y)\frac{dy}{dx}$ which is equivalent to 2.3.

Example 2.1.13 *Find the solution to the initial value problem,*

$$y' = \frac{x}{y^2},\ y(0) = 1.$$

This is a separable equation and in fact, $y^2 dy = xdx$ so the solution to the differential equation is of the form

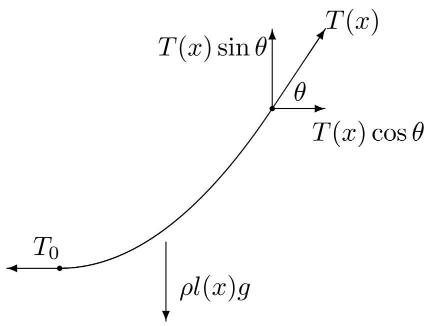
$$\frac{y^3}{3} - \frac{x^2}{2} = C \tag{2.4}$$

and it only remains to find the constant, C. To do this, you use the initial condition. Letting $x = 0$, it follows $\frac{1}{3} = C$ and so

$$\frac{y^3}{3} - \frac{x^2}{2} = \frac{1}{3}$$

Example 2.1.14 *What is the equation of a hanging chain?*

Consider the following picture of a portion of this chain.



In this picture, ρ denotes the density of the chain which is assumed to be constant and g is the acceleration due to gravity. $T(x)$ and T_0 represent the magnitude of the tension in the chain at t and at 0 respectively, as shown. Let the bottom of the chain be at the origin as shown. If this chain does not move, then all these forces acting on it must balance. In particular,

$$T(x)\sin\theta = l(x)\rho g,\ T(x)\cos\theta = T_0.$$

Therefore, dividing these yields

$$\frac{\sin\theta}{\cos\theta} = l(x)\overbrace{\rho g/T_0}^{\equiv c}.$$

Now letting $y(x)$ denote the y coordinate of the hanging chain corresponding to x,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = y'(x).$$

Therefore, this yields

$$y'(x) = cl(x).$$

Now differentiating both sides of the differential equation,

$$y''(x) = cl'(x) = c\sqrt{1 + y'(x)^2}$$

and so

$$\frac{y''(x)}{\sqrt{1 + y'(x)^2}} = c.$$

Let $z(x) = y'(x)$ so the above differential equation becomes

$$\frac{z'(x)}{\sqrt{1+z^2}} = c.$$

Therefore, $\int \frac{z'(x)}{\sqrt{1+z^2}}\,dx = cx + d$. Change the variable in the antiderivative letting $u = z(x)$ and this yields

$$\begin{aligned}\int \frac{z'(x)}{\sqrt{1+z^2}}\,dx &= \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1}(u) + C \\ &= \sinh^{-1}(z(x)) + C.\end{aligned}$$

Therefore, combining the constants of integration,

$$\sinh^{-1}(y'(x)) = cx + d$$

and so

$$y'(x) = \sinh(cx + d).$$

Therefore,

$$y(x) = \frac{1}{c}\cosh(cx + d) + k$$

where d and k are some constants and $c = \rho g/T_0$. Curves of this sort are called catenaries. Note these curves result from an assumption the only forces acting on the chain are as shown.

2.1.5 Homogeneous Equations

Sometimes equations can be made separable by changing the variables appropriately. This occurs in the case of the so called homogeneous equations, those of the form

$$y' = f\left(\frac{y}{x}\right).$$

When this sort of equation occurs, there is an easy trick which will allow you to consider a separable equation.

You define a new variable,

$$u \equiv \frac{y}{x}.$$

Thus $y = ux$ and so

$$y' = u'x + u = f(u).$$

Thus

$$\frac{du}{dx}x = f(u) - u$$

and so

$$\frac{du}{f(u) - u} = \frac{dx}{x}.$$

The variables have now been separated and you go to work on it in the usual way.

Example 2.1.15 *Find the solutions of the equation*

$$y' = \frac{y^2 + xy}{x^2}.$$

First note this is of the form

$$y' = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right).$$

Let $u = \frac{y}{x}$ so $y = xu$. Then

$$u'x + u = u^2 + u$$

and so, separating the variables yields

$$\frac{du}{u^2} = \frac{dx}{x}$$

Hence

$$-\frac{1}{u} = \ln|x| + C$$

and so

$$\frac{y}{x} = u = \frac{1}{K - \ln|x|}$$

where $K = -C$. Hence

$$y(x) = \frac{x}{K - \ln|x|}$$

2.2 Linear And Nonlinear Differential Equations

Recall initial value problems for linear differential equations are those of the form

$$y' + p(t)y = q(t), \; y(t_0) = y_0 \tag{2.5}$$

where $p(t)$ and $q(t)$ are continuous functions of t. Then if $t_0 \in [a,b]$, an interval, there exists a unique solution to the initial value problem given above which is defined for all $t \in [a,b]$. The following theorem which is really something of a review gives a proof.

Theorem 2.2.1 *Let $[a,b]$ be an interval containing t_0 and let $p(t)$ and $q(t)$ be continuous functions defined on $[a,b]$. Then there exists a unique solution to 2.5 valid for all $t \in [a,b]$.*

Proof: Let $P'(t) = p(t), P(t_0) = 0$. For example, let $P(t) \equiv \int_{t_0}^{t} p(s)\, ds$. Then multiply both sides of the differential equation by $\exp(P(t))$. This yields

$$(y(t)\exp(P(t)))' = q(t)\exp(P(t))$$

and so, integrating both sides from t_0 to t,

$$y(t)\exp(P(t)) - y_0 = \int_{t_0}^{t} q(s)\exp(P(s))\, ds$$

and so

$$y(t) = \exp(-P(t))\, y_0 + \exp(-P(t)) \int_{t_0}^{t} q(s)\exp(P(s))\, ds$$

which shows that if there is a solution to 2.5, then the above formula gives that solution. Thus there is at most one solution. Also you see the above formula makes perfect sense on the whole interval. Since the steps are reversible, this shows $y(t)$ given in the above formula is a solution. You should provide the details. Use the fundamental theorem of calculus. This proves the theorem.

It is not so simple for a nonlinear initial value problem of the form

$$y' = f(t, y),\ y(t_0) = y_0.$$

Theorem 2.2.2 *Let f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle, $a < t < b, c < y < d$ containing the point (t_0, y_0). Then there exists a unique local solution to the initial value problem*

$$y' = f(t, y),\ y(t_0) = y_0.$$

This means there exists an interval, I such that $t_0 \in I \subseteq (a, b)$ and a unique function, y defined on this interval which solves the above initial value problem on that interval.

Example 2.2.3 *Solve $y' = 1 + y^2, y(0) = 0$.*

This satisfies the conditions of Theorem 2.2.2. Therefore, there is a unique solution to the above initial value problem defined on some interval containing 0. However, in this case, we can solve the initial value problem and determine exactly what happens. The equation is separable.

$$\frac{dy}{1 + y^2} = dt$$

and so $\arctan(y) = t + C$. Then from the initial condition, $C = 0$. Therefore, the solution to the equation is $y = \tan(t)$. Of course this function is defined on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It is impossible to extend it further because it has an assymptote at the two ends of this interval.

The Theorem 2.2.2 does not say that the local solution can never be extended beyond some small interval. Sometimes it can. It depends very much on the nonlinear equation. For example, the initial value problem

$$y' = 1 + y^2 - \varepsilon y^3,\ y(0) = y_0$$

turns out to have a solution on the whole real line. Here ε is a small positive number. You might think about why this is so. It is related to the fact that in this new equation, the extra term prevents y' from becoming unbounded.

If you assume less on f in the above theorem, you sometimes can get existence but not uniqueness for the initial value problem.

Example 2.2.4 *Find the solutions to the initial value problem*

$$y' = y^{1/3},\ y(0) = 0.$$

The equation is separable so

$$\frac{dy}{y^{1/3}} = dt$$

and so the solutions are of the form

$$\frac{3}{2} y^{2/3} = t + C.$$

Letting $C = 0$ from the initial condition, one solution is

$$y = \left(\frac{2}{3} t \right)^{3/2}$$

for $t > 0$. However, you can also see that $y = 0$ is also a solution. Thus uniqueness is violated. Note there are two solutions to the initial value problem and both exist and solve the initial value problem on all of $\mathbb{R}$.

What is the main difference between linear and nonlinear equations? Linear initial value problems have an interval of existence which is as long as desired. Nonlinear initial value problems sometimes don't. Solutions to linear initial value problems are unique. This is not always true for nonlinear equations although if in the nonlinear equation, f and $\partial f / \partial y$ are both continuous, then you at least get uniqueness as well as existence on some possibly small interval.

Chapter 3

Higher Order Differential Equations

3.1 Linear Equations

3.1.1 Real Solutions To The Characteristic Equation

These differential equations are of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = f$$

where f is some function. When $f = 0$ the equation is called **homogeneous.** To find solutions to this equation you look for a solution in the form

$$y = e^{rt}$$

and then try to choose r in such a way that it works. Here is a simple example.

Example 3.1.1 *Find solutions to the homogeneous equation*

$$y'' - 3y' + 2y = 0.$$

Following the above suggestion, you look for $y = e^{rt}$. Then plugging this in to the equation yields

$$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = e^{rt}\left(r^2 - 3r + 2\right) = 0$$

Now it is clear this happens exactly when $r = 2, 1$. Therefore, both $y = e^{2t}$ and $y = e^t$ solve the equation.

How would this work in general? You have

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

and you look for a solution in the form $y = e^{rt}$. Plugging this in to the equation yields

$$e^{rt}\left(r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0\right) = 0$$

so you need to choose r such that the following **characteristic equation** is satisfied

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

Then when this is done,

$$y = e^{rt}$$

will be a solution to the equation. At least this is the case if r is real. The question of what to do when r is not real will be dealt with a little later. The problem is that we do not know at this time the proper definition of $e^{(a+ib)t}$.

Example 3.1.2 *Find solutions to the equation*

$$y''' - 6y'' + 11y' - 6y = 0$$

First you write the characteristic equation

$$r^3 - 6r^2 + 11r - 6 = 0$$

and find the solutions to the equation, $r = 1, 2, 3$ in this case. Then some solutions to the differential equation are $y = e^t, y = e^{2t}$, and $y = e^{3t}$.

What happens in the case of a repeated zero to the characteristic equation? Here is an example.

Example 3.1.3 *Find solutions to the equation*

$$y''' - y'' - y' + y = 0$$

In this case the characteristic equation is

$$r^3 - r^2 - r + 1 = 0$$

and when the polynomial is factored this yields

$$(r + 1)(r - 1)^2 = 0$$

Therefore, $y = e^{-t}$ and $y = e^t$ are both solutions to the equation. Now in this case $y = te^t$ is also a solution. This is because the solutions to the characteristic equation are $1, 1, -1$ where 1 is listed twice because of the $(r - 1)^2$ in the factored characteristic polynomial. Corresponding to the first occurance of 1 you get e^t and corresponding to the second occurance you get te^t.

If the factored characteristic polynomial were of the form

$$(r + 1)^2 (r - 1)^3,$$

you would write

$$e^{-t}, te^{-t}, e^t, te^t, t^2 e^t$$

This is described in the following procedure

Procedure 3.1.4 *To find solutions to the homogeneous equation*

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

You find solutions r to the characteristic equation

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

and then $y = e^{rt}$ will be a solution to the differential equation. For every λ a repeated zero of order k, meaning $(r - \lambda)^k$ occurs in the factored characteristic polynomial, you also obtain as solutions the following functions.

$$te^{\lambda t}, t^2 e^{\lambda t}, \cdots, t^{r-1} e^{rt}$$

Why do we care about these other functions? This involves the notion of general solution. You notice the above procedure always delivers exactly n solutions to the differential equation. It turns out that to obtain all possible solutions you need all n.

3.1.2 Superposition And General Solutions

This is concerned with differential equations of the form

$$Ly \equiv y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \cdots + a_1(t) y'(t) + a_0(t) y(t) = 0 \tag{3.1}$$

in which the functions $a_k(t)$ are continuous. The fundamental thing to observe about L is that it is linear.

Definition 3.1.5 *Suppose L satisfies the following condition in which a and b are numbers and y_1, y_2 are functions.*

$$L(ay_1 + by_2) = aLy_1 + bLy_2$$

Then L is called a linear operator.

Proposition 3.1.6 *Let L be given in 3.1. Then L is linear.*

Proof: To save space, note that L can be written in summation notation as

$$Ly(t) = y^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t) y^{(k)}(t)$$

Then letting a, b be numbers and y_1, y_2 functions,

$$L(ay_1 + by_2)(t) \equiv (ay_1 + by_2)^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)(ay_1 + by_2)^{(k)}(t)$$

Now remember from calculus that the derivative of a sum is the sum of the derivatives and also the derivative of a constant times a function is the constant times the derivative of the function. Therefore,

$$L(ay_1 + by_2)(t) = ay_1^{(n)}(t) + by_2^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\left(ay_1^{(k)}(t) + by_2^{(k)}(t)\right)$$

and this equals

$$ay_1^{(n)}(t) + a\sum_{k=0}^{n-1} a_k(t) y_1^{(k)}(t) + by^{(n)}(t) + b\sum_{k=0}^{n-1} a_k(t) y_2^{(k)}(t)$$

which equals

$$aLy_1(t) + bLy_2(t)$$

which shows L is linear. This proves the proposition.

Corollary 3.1.7 *If L is linear, then for a_k scalars and y_k functions,*

$$L\left(\sum_{k=1}^{m} a_k y_k\right) = \sum_{k=1}^{m} a_k L y_k.$$

Proof: The statement that L is linear applies to $m = 2$. Suppose you have three functions.

$$\begin{aligned} L(ay_1 + by_2 + cy_3) &= L((ay_1 + by_2) + cy_3) \\ &= L(ay_1 + by_2) + cLy_3 \\ &= aLy_1 + bLy_2 + cLy_3 \end{aligned}$$

Thus the conclusion holds for $m = 3$. Following the same pattern just illustrated, you see it holds for $m = 4, 5, \cdots$ also.

More precisely, assuming the conclusion holds for m,

$$\begin{aligned} L\left(\sum_{k=1}^{m+1} a_k y_k\right) &= L\left(\sum_{k=1}^{m} a_k y_k + a_{m+1} y_{m+1}\right) \\ &= L\left(\sum_{k=1}^{m} a_k y_k\right) + a_{m+1} L y_{m+1} \end{aligned}$$

and assuming the conclusion holds for m, the above equals

$$\sum_{k=1}^{m} a_k L y_k + a_{m+1} L y_{m+1} = \sum_{k=1}^{m+1} a_k L y_k$$

This proves the corollary.

The principle of superposition applies to any linear operator in any context. Here the operator is the one defined above but the same result applies to any other example of a linear operator. The following is called the principle of superposition. It says that if you have some solutions to $Ly = 0$ you can multiply them by constants and add up the products and you will still have a solution.

Theorem 3.1.8 *Let L be a linear operator and suppose $Ly_k = 0$ for $k = 1, 2, \cdots, m$. Then if $a_1, \cdots, a_m$ are scalars,*

$$L\left(\sum_{k=1}^{m} a_k y_k\right) = 0$$

Proof: This follows because L is linear.

$$L\left(\sum_{k=1}^{m} a_k y_k\right) = \sum_{k=1}^{m} a_k L y_k = \sum_{k=1}^{m} a_k 0 = 0.$$

This proves the principle of superposition.

Example 3.1.9 *Find lots of solutions to the equation*

$$y'' - 2y' + y = 0$$

Recall how you do this. You write down the characteristic equation

$$r^2 - 2r + 1 = 0$$

finding the solutions are $r = 1, 1$, there being a repeated zero. Then you know both e^t and te^t are solutions. It follows from the principle of superposition that any function of the form

$$C_1 e^t + C_2 t e^t$$

is a solution.

Consider the above example. You can pick the C_1 and C_2 any way you want so you have indeed found lots of solutions. What is the obvious question to ask at this point? In case you are not sure, here it is:

You have lots of solutions but do you have them all?

The answer to this question comes from linear algebra and a fundamental existence and uniqueness theorem. First, here is the fundamental existence and uniqueness theorem.

Theorem 3.1.10 *Let L be given in 3.1 and let $y_0, y_1, \cdots, y_{n-1}$ be given numbers and $(a,b), \infty \leq a < b \leq \infty$, be an interval on which each $a_k(t)$ in the definition of L is continuous. Also let $f(t)$ be a function which is continuous on (a,b). Then if $c \in (a,b)$, there exists a unique solution y to the initial value problem*

$$Ly = f,\ y(c) = y_0, y'(c) = y_1, \cdots, y^{(n-1)}(c) = y_{n-1}.$$

I will present a proof of a generalization of this important result later. For now, just use it. The following is the definition of something called the Wronskian. It is this which determines whether you have all the solutions.

Definition 3.1.11 *Let $y_1, \cdots, y_n$ be functions which have $n-1$ derivatives. Then $W(y_1(t), \cdots, y_n(t))$ is defined by the following determinant.*

$$\det \begin{pmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{pmatrix}$$

This determinant is called the Wronskian.

Example 3.1.12 *Find the Wronskian of the functions t^2, t^3, t^4.*

By the above definition

$$W\left(t^2, t^3, t^4\right) \equiv \det \begin{pmatrix} t^2 & t^3 & t^4 \\ 2t & 3t^2 & 4t^3 \\ 2 & 6t & 12t^2 \end{pmatrix} = 2t^6$$

Now the way to tell whether you have all possible solutions is contained in the following fundamental theorem. Sometimes this theorem is referred to as the Wronskian alternative.

Theorem 3.1.13 *Let L be given in 3.1 and suppose each $a_k(t)$ in the definition of L is continuous on (a,b) some interval such that $\infty \leq a < b \leq \infty$. Suppose for $i = 1, 2, \cdots, n$,*

$$Ly_i = 0.$$

Then for any choice of scalars $C_1, \cdots, C_n$,

$$\sum_{i=1}^{n} C_i y_i$$

is a solution of the equation $Ly = 0$. All possible solutions of this equation are obtained in this form if and only if for some $c \in (a,b)$,

$$W(y_1(c), y_2(c), \cdots, y_n(c)) \neq 0.$$

*Furthermore, $W(y_1(t), y_2(t), \cdots, y_n(t))$ is either **always** equal to 0 for all $t \in (a,b)$ or **never** equal to 0 on (a,b) for any $t \in (a,b)$. In the case that all possible solutions are obtained as the above sum, we say $\sum_{i=1}^{n} C_i y_i$ is the **general solution.***

Proof: Suppose for some $c \in (a,b)$,

$$W(y_1(c), y_2(c), \cdots, y_n(c)) \neq 0.$$

Suppose $Lz = 0$. Then consider the numbers

$$z(c), z'(c), \cdots, z^{(n-1)}(c)$$

Since $W(y_1(c), y_2(c), \cdots, y_n(c)) \neq 0$, it follows the matrix

$$\begin{pmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c) & \cdots & y_n^{(n-1)}(c) \end{pmatrix}$$

has an inverse. Therefore, there exists unique scalars $C_1, C_2, \cdots, C_n$ such that

$$\begin{pmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c) & \cdots & y_n^{(n-1)}(c) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} z(c) \\ z'(c) \\ \vdots \\ z^{(n-1)}(c) \end{pmatrix}$$

Now consider the function

$$y(t) = \sum_{k=1}^{n} C_k y_k(t)$$

By the principle of superposition, $Ly = 0$ and in addition, from the above it follows

$$y^{(k)}(c) = z^{(k)}(c)$$

for $k = 0, 1, \cdots, n-1$. By the uniqueness part of Theorem 3.1.13 it follows $y(t) = z(t)$. Therefore, since $z(t)$ was arbitrary, this has shown all solutions are obtained by varying the constants in the sum

$$\sum_{k=1}^{n} C_k y_k(t).$$

This shows that if $W(y_1(c), y_2(c), \cdots, y_n(c)) \neq 0$ for some $c \in (a,b)$ then the general solution is obtained.

Suppose now that for some $c \in (a,b)$, $W(y_1(c), y_2(c), \cdots, y_n(c)) = 0$. I will show that in this case the general solution is **not obtained.** Since this Wronskian is equal to 0, the matrix

$$\begin{pmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c) & \cdots & y_n^{(n-1)}(c) \end{pmatrix}$$

is not invertible. Therefore, there exists $\mathbf{z} = \begin{pmatrix} z_0 & z_1 & \cdots & z_{n-1} \end{pmatrix}^T$ such that there is no solution

$$\begin{pmatrix} C_1 & C_2 & \cdots & C_n \end{pmatrix}^T$$

to the system of equations

$$\begin{pmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c) & \cdots & y_n^{(n-1)}(c) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix}$$

From the existence part of Theorem 3.1.13, there exists z such that $Lz = 0$ and it satisfies the initial conditions

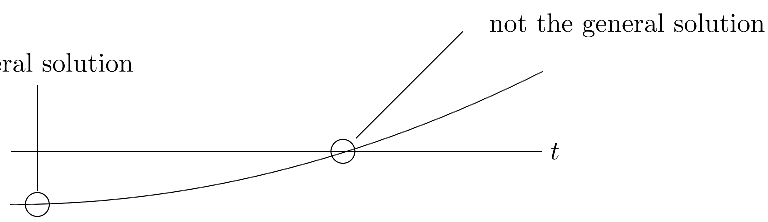
$$z(c) = z_0, z'(c) = z_1, \cdots, z^{(n-1)}(c) = z_{n-1}.$$

Therefore, there is no way to write $z(t)$ in the form

$$\sum_{k=1}^{n} C_k y_k(t) \tag{3.2}$$

because if you could do so, you could plug in $t = c$ and get a solution to the above system of equations which is given to not have a solution. Therefore, when $W(y_1(c), y_2(c), \cdots, y_n(c)) = 0$, you do not get all the solutions to $Ly = 0$ by looking at sums of the form in 3.2 for various choices of $C_1, C_2, \cdots, C_n$.

Why does the Wronskian either vanish for all $t \in (a, b)$ or for no $t \in (a, b)$? This is because 3.2 either yields all possible solutions or it does not. If it does, then the Wronskian cannot equal zero at any $c \in (a, b)$. If it does not yield all possible solutions, then at any point $c \in (a, b)$ the Wronskian cannot be nonzero there. Hence it must be zero there. The following picture illustrates this alternative. The curved line represents the graph of the Wronskian.



This proves the theorem.

Example 3.1.14 *Find all solutions of the equation*

$$y^{(4)} - 2y^{(3)} + 2y' - y = 0.$$

In this case the characteristic equation is

$$r^4 - 2r^3 + 2r - 1 = 0$$

The polynomial factors.

$$r^4 - 2r^3 + 2r - 1 = (r-1)^3 (r+1)$$

Therefore, you can find 4 solutions $e^t, te^t, t^2e^t, e^{-t}$. The general solution will be

$$C_1 e^t + C_2 te^t + C_3 t^2 e^t + C_4 e^{-t} \tag{3.3}$$

if and only if the Wronskian of these functions is non zero at some point. First obtain the Wronskian. This is

$$\det \begin{pmatrix} e^t & te^t & t^2e^t & e^{-t} \\ e^t & e^t + te^t & 2te^t + t^2e^t & -e^{-t} \\ e^t & 2e^t + te^t & 2e^t + 4te^t + t^2e^t & e^{-t} \\ e^t & 3e^t + te^t & 6e^t + 6te^t + t^2e^t & -e^{-t} \end{pmatrix}$$

Now you only have to check at one point. I pick $t = 0$. Then it reduces to

$$\det \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 6 & -1 \end{pmatrix} = -16 \neq 0$$

Therefore, 3.3 is the general solution.

Example 3.1.15 *Does there exist any interval* (a, b) *containing* 0 *and continuous functions* $a_2(t), a_1(t), a_0(t)$ *such that the functions* t^2, t^3, t^4 *are solutions of the equation*

$$y''' + a_2(t) y'' + a_1(t) y' + a_0 y = 0?$$

The answer is **NO.** This follows from Example 3.1.12 in which it is shown these functions have Wronskian equal to $2t^6$ which equals zero at 0 but is nonzero elsewhere. Therefore, if such functions and an interval containing 0 did exist, it would violate the conclusion of Theorem 3.1.13.

Is there an easy way to tell if the Wronskian is nonzero? Yes if you are looking at two solutions of a second order equation.

Proposition 3.1.16 *Let* $a_1(t), a_0(t)$ *be continuous on an interval* (a, b) *and suppose* $y_1(t), y_2(t)$ *are two functions which solve*

$$y'' + a_1(t) y' + a_0 y = 0.$$

Then the general solution is

$$C_1 y_1(t) + C_2 y_2(t)$$

if and only if the ratio $y_2(t)/y_1(t)$ *is not constant.*

Proof: First suppose the ratio is not constant. Then the derivative of the quotient is nonzero. Hence by the quotient rule,

$$0 \neq \frac{y_2'(t) y_1(t) - y_1'(t) y_2(t)}{y_1(t)^2} = \frac{W(y_1(t), y_2(t))}{y_1(t)^2}$$

and this shows that at some point $W(y_1(t), y_2(t)) \neq 0$. Therefore, the general solution is obtained.

Conversely, if the ratio is constant, then by the quotient rule as above, the Wronskian equals 0. This proves the proposition.

Example 3.1.17 *Show the general solution of the equation*

$$y'' + 2y' + 2y = 0$$

is

$$C_1 e^{-t} \cos(t) + C_2 e^{-t} \sin(t).$$

You can check and see both of these functions are solutions of the differential equation. Furthermore, their ratio is not a constant. Therefore, by Proposition 3.1.16 the above is the general solution.

3.2 The Case Of Complex Zeros

Consider the following equation

$$y'' - 2y' + 2y = 0$$

The characteristic equation for this ordinary differential equation is

$$r^2 - 2r + 2 = 0$$

and the solutions to this equation are

$$r = 1 \pm i$$

where i is the imaginary number which when squared gives -1. You would want the solutions to be

$$e^{\alpha t}$$

where α is one of the complex numbers $1 + i$ or $1 - i$. However, the above function has not been defined but in a sense this is really fortunate. It is fortunate because, since it is presently meaningless, we are free to give it a meaning. We do this in such a way that the function is useful for finding solutions to equations.

Why is it a good idea to look for solutions to a linear constant coefficient equation in the form $e^{\alpha t}$? It is because when you differentiate $e^{\alpha t}$ you get $\alpha e^{\alpha t}$ and so every time you differentiate it, it brings down another factor of α. Finally you obtain a polynomial in α times $e^{\alpha t}$ and then you simply cancel the $e^{\alpha t}$ and find α. This is the case if α is real. Thus we want to define $e^{(a+ib)t}$ in such a way that its derivative is $(a + ib)\, e^{(a+ib)t}$. Also, to conform to the case where α is real, we require $e^{(a+ib)0} = 1$. Thus it is desired to find a function $y(t)$ which satisfies the following two properties.

$$y(0) = 1,\ y'(t) = (a + ib)\, y(t)\,. \tag{3.4}$$

Proposition 3.2.1 *Let $y(t) = e^{at}(\cos(bt) + i\sin(bt))$. Then $y(t)$ is a solution to 3.4 and furthermore, this is the only function which satisfies the conditions of 3.4.*

Proof: It is easy to see that if $y(t)$ is as given above, then it satisfies the desired conditions. First

$$y(0) = e^0(\cos(0) + i\sin(0)) = 1.$$

Next

$$\begin{aligned} y'(t) &= ae^{at}(\cos(bt) + i\sin(bt)) + e^{at}(-b\sin(bt) + ib\cos(bt)) \\ &= ae^{at}\cos bt - e^{at}b\sin bt + i\left(ae^{at}\sin bt + e^{at}b\cos bt\right) \end{aligned}$$

On the other hand,

$$(a + ib)\left(e^{at}(\cos(bt) + i\sin(bt))\right)$$
$$= ae^{at}\cos bt - e^{at}b\sin bt + i\left(ae^{at}\sin bt + e^{at}b\cos bt\right)$$

which is the same thing. Remember $i^2 = -1$.

It remains to verify this is the only function which satisfies 3.4. Suppose $y_1(t)$ is another function which works. Then letting $z(t) \equiv y(t) - y_1(t)$, it follows

$$z'(t) = (a + ib)\, z(t)\,,\ z(0) = 0.$$

Now $z(t)$ has a real part and an imaginary part, $z(t) = u(t) + iv(t)$. Then $\overline{z}(t) \equiv u(t) - iv(t)$ and

$$\overline{z}'(t) = (a - ib)\, \overline{z}(t)\,,\ \overline{z}(0) = 0$$

Then $|z(t)|^2 = z(t)\overline{z}(t)$ and by the product rule,

$$\begin{aligned}\frac{d}{dt}|z(t)|^2 &= z'(t)\overline{z}(t) + z(t)\overline{z}'(t) \\ &= (a+ib)\,z(t)\overline{z}(t) + (a-ib)\,z(t)\overline{z}(t) \\ &= (a+ib)|z(t)|^2 + (a-ib)|z(t)|^2 \\ &= 2a|z(t)|^2,\ |z(0)|^2 = 0.\end{aligned}$$

Therefore, since this is a first order system for $|z(t)|^2$, it follows you know how to solve this and the solution is

$$|z(t)|^2 = 0e^{2at} = 0.$$

Thus $z(t) = 0$ and so $y(t) = y_1(t)$ and this proves the uniqueness assertion of the proposition.

Note that the function $e^{(a+ib)t}$ is never equal to 0. This is because its absolute value is e^{at}(why?).

Chapter 4

Theory Of Ordinary Differential Equations*

4.1 Picard Iteration

We suppose that $\mathbf{f} : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following two conditions.

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{x}_1)| \leq K |\mathbf{x} - \mathbf{x}_1|, \tag{4.1}$$

$$\mathbf{f} \text{ is continuous.} \tag{4.2}$$

The first of these conditions is known as a Lipschitz condition.

Lemma 4.1.1 *Suppose $\mathbf{x} : [a, b] \to \mathbb{R}^n$ is a continuous function and $c \in [a, b]$. Then $\mathbf{x}$ is a solution to the initial value problem,*

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \ \mathbf{x}(c) = \mathbf{x}_0 \tag{4.3}$$

if and only if $\mathbf{x}$ is a solution to the integral equation,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_c^t \mathbf{f}(s, \mathbf{x}(s))\, ds. \tag{4.4}$$

Proof: If $\mathbf{x}$ solves 4.4, then since $\mathbf{f}$ is continuous, we may apply the fundamental theorem of calculus to differentiate both sides and obtain $\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}(t))$. Also, letting $t = c$ on both sides, gives $\mathbf{x}(c) = \mathbf{x}_0$. Conversely, if $\mathbf{x}$ is a solution of the initial value problem, we may integrate both sides from c to t to see that $\mathbf{x}$ solves 4.4. This proves the lemma.

It follows from this lemma that we may study the initial value problem, 4.3 by considering the integral equation 4.4. The most famous technique for studying this integral equation is the method of Picard iteration . In this method, we start with an initial function, $\mathbf{x}_0(t) \equiv \mathbf{x}_0$ and then iterate as follows.

$$\mathbf{x}_1(t) \equiv \mathbf{x}_0 + \int_c^t \mathbf{f}(s, \mathbf{x}_0(s))\, ds$$

$$= \mathbf{x}_0 + \int_c^t \mathbf{f}(s, \mathbf{x}_0)\, ds,$$

$$\mathbf{x}_2(t) \equiv \mathbf{x}_0 + \int_c^t \mathbf{f}(s, \mathbf{x}_1(s))\, ds,$$

and if $\mathbf{x}_{k-1}(s)$ has been determined,

$$\mathbf{x}_k(t) \equiv \mathbf{x}_0 + \int_c^t \mathbf{f}(s, \mathbf{x}_{k-1}(s))\, ds.$$

Now we will consider some simple estimates which come from these definitions. Let

$$M_{\mathbf{f}} \equiv \max\{|\mathbf{f}(s, \mathbf{x}_0)| : s \in [a, b]\}.$$

Then for any $t \in [a, b]$,

$$|\mathbf{x}_1(t) - \mathbf{x}_0| \leq \left| \int_c^t |\mathbf{f}(s, \mathbf{x}_0)|\, ds \right| \leq M_{\mathbf{f}} |t - c|. \tag{4.5}$$

Now using this estimate and the Lipschitz condition for $\mathbf{f}$,

$$|\mathbf{x}_2(t) - \mathbf{x}_1(t)| \leq \left| \int_c^t |\mathbf{f}(s, \mathbf{x}_1(s)) - \mathbf{f}(s, \mathbf{x}_0)|\, ds \right|$$

$$\leq \left| \int_c^t K |\mathbf{x}_1(s) - \mathbf{x}_0|\, ds \right| \leq$$

$$K M_{\mathbf{f}} \left| \int_c^t |s - c|\, ds \right| \leq K M_{\mathbf{f}} \frac{|t - c|^2}{2}. \tag{4.6}$$

Continuing in this way we will establish the following lemma.

Lemma 4.1.2 *Let $k \geq 2$. Then for any $t \in [a, b]$,*

$$|\mathbf{x}_k(t) - \mathbf{x}_{k-1}(t)| \leq M_{\mathbf{f}} K^{k-1} \frac{|t - c|^k}{k!}. \tag{4.7}$$

Proof: We have verified this estimate in the case when $k = 2$. Assume it is true for k. Then we use the Lipshitz condition on $\mathbf{f}$ to write

$$|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq$$

$$\left| \int_c^t |\mathbf{f}(s, \mathbf{x}_k(s)) - \mathbf{f}(s, \mathbf{x}_{k-1}(s))|\, ds \right|$$

$$\leq \left| \int_c^t K |\mathbf{x}_k(s) - \mathbf{x}_{k-1}(s)|\, ds \right|$$

which, by the induction hypothesis, is dominated by

$$\leq \left| \int_c^t K M_{\mathbf{f}} K^{k-1} \frac{|s - c|^k}{k!} ds \right|$$

$$\leq M_{\mathbf{f}} K^k \frac{|t - c|^{k+1}}{(k+1)!}.$$

This proves the lemma.

Lemma 4.1.3 *For each $t \in [a, b]$, $\{\mathbf{x}_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence.*

Proof: Pick such a $t \in [a,b]$. Then if $k > l$, we use the triangle inequality and the estimate of the above lemma to write

$$|\mathbf{x}_k(t) - \mathbf{x}_l(t)| \leq \sum_{r=l}^{k-1} |\mathbf{x}_{r+1}(t) - \mathbf{x}_r(t)| \leq$$

$$M_{\mathbf{f}} \sum_{r=l}^{\infty} K^r \frac{|t-c|^{r+1}}{(r+1)!} \leq M_{\mathbf{f}}(b-a) \sum_{r=l}^{\infty} \frac{(K(b-a))^r}{(r+1)!} \tag{4.8}$$

a quantity which converges to zero as $l \to \infty$ due to the convergence of the series, $\sum_{r=0}^{\infty} \frac{(K(b-a))^r}{(r+1)!}$, a fact which is easily seen by an application of the ratio test. This shows that for every $\varepsilon > 0$, there exists L such that if $k, l > L$, then $|\mathbf{x}_k(t) - \mathbf{x}_l(t)| < \varepsilon$. In other words, $\{\mathbf{x}_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence. This proves the lemma.

Since $\{\mathbf{x}_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence, we denote by $\mathbf{x}(t)$ the point to which it converges. Letting $k \to \infty$ in 4.8, we see that

$$|\mathbf{x}(t) - \mathbf{x}_l(t)| \leq M_{\mathbf{f}} \sum_{r=l}^{\infty} \frac{(K(b-a))^r}{(r+1)!} < \varepsilon \tag{4.9}$$

whenever l is sufficiently large. Since the right side of the inequality does not depend on t, it follows that for all $\varepsilon > 0$, there exists L such that if $l \geq L$, then for all $t \in [a,b]$, $|\mathbf{x}(t) - \mathbf{x}_l(t)| < \varepsilon$.

Lemma 4.1.4 *The function $t \to \mathbf{x}(t)$ is continuous and*

$$\lim_{l\to\infty} \left(\sup\{|\mathbf{x}(t) - \mathbf{x}_l(t)| : t \in [a,b]\}\right) = 0. \tag{4.10}$$

Proof: Formula 4.10 was established above in 4.9.

Let $\varepsilon > 0$ be given and let $t \in [a,b]$. Then letting l be large enough that

$$\lim_{l\to\infty} \left(\sup\{|\mathbf{x}(t) - \mathbf{x}_l(t)| : t \in [a,b]\}\right) < \varepsilon/3$$

$$|\mathbf{x}(s) - \mathbf{x}(t)| \leq$$

$$|\mathbf{x}(s) - \mathbf{x}_l(s)| + |\mathbf{x}_l(s) - \mathbf{x}_l(t)| + |\mathbf{x}_l(t) - \mathbf{x}(t)|$$

$$\leq \frac{2\varepsilon}{3} + |\mathbf{x}_l(s) - \mathbf{x}_l(t)|.$$

By the continuity of $\mathbf{x}_l$, we see the last term is dominated by $\frac{\varepsilon}{3}$ whenever $|s-t|$ is small enough. This verifies continuity of $t \to \mathbf{x}(t)$ and proves the lemma.

Letting l be large enough that

$$\lim_{l\to\infty} \left(\sup\{|\mathbf{x}(t) - \mathbf{x}_l(t)| : t \in [a,b]\}\right) < \frac{\varepsilon}{K(b-a)},$$

$$\left| \int_c^t \mathbf{f}(s, \mathbf{x}_l(s))\, ds - \int_c^t \mathbf{f}(s, \mathbf{x}(s))\, ds \right|$$

$$\leq \left| \int_c^t |\mathbf{f}(s, \mathbf{x}_l(s)) - \mathbf{f}(s, \mathbf{x}(s))|\, ds \right|$$

$$\leq \left| \int_c^t K |\mathbf{x}_l(s) - \mathbf{x}(s)|\, ds \right|$$

$$< K(b-a)\frac{\varepsilon}{K(b-a)} = \varepsilon.$$

Since ε is arbitrary, this verifies that

$$\lim_{l\to\infty}\int_c^t \mathbf{f}(s,\mathbf{x}_l(s))\,ds = \int_c^t \mathbf{f}(s,\mathbf{x}(s))\,ds.$$

It follows that

$$\begin{aligned}\mathbf{x}(t) &= \lim_{k\to\infty}\mathbf{x}_k(t) \\ &= \lim_{k\to\infty}\left(\mathbf{x}_0 + \int_c^t \mathbf{f}(s,\mathbf{x}_{k-1}(s))\,ds\right) \\ &= \mathbf{x}_0 + \int_c^t \mathbf{f}(s,\mathbf{x}(s))\,ds\end{aligned}$$

and so by Lemma 4.1.1, $\mathbf{x}$ is a solution to the initial value problem, 4.3. This proves the existence part of the following theorem.

Theorem 4.1.5 *Let $\mathbf{f}$ satisfy 4.1 and 4.2. Then there exists a unique solution to the initial value problem, 4.3.*

Proof: It only remains to verify the uniqueness assertion. Therefore, assume $\mathbf{x}$ and $\mathbf{x}_1$ are two solutions to the initial value problem. Then by Lemma 4.1.1 it follows that

$$\mathbf{x}(t) - \mathbf{x}_1(t) = \int_c^t (\mathbf{f}(s,\mathbf{x}(s)) - \mathbf{f}(s,\mathbf{x}_1(s)))\,ds.$$

Suppose first that $t < c$. Then this equation implies that for all such t,

$$\begin{aligned}|\mathbf{x}(t) - \mathbf{x}_1(t)| &\le \int_t^c |\mathbf{f}(s,\mathbf{x}(s)) - \mathbf{f}(s,\mathbf{x}_1(s))| \\ &\le \int_t^c K|\mathbf{x}(s) - \mathbf{x}_1(s)|\,ds.\end{aligned}$$

Letting $g(t) = \int_t^c |\mathbf{x}(s) - \mathbf{x}_1(s)|\,ds$, it follows that $-g'(t) = |\mathbf{x}(t) - \mathbf{x}_1(t)|$ and so

$$-g'(t) \le Kg(t)$$

which implies $0 \le g'(t) + Kg(t)$ and consequently,

$$0 \le \left(e^{Kt}g(t)\right)'$$

which means, integration from t to c gives

$$0 \le e^{Kc}g(c) - e^{Kt}g(t).$$

But $g(c) = 0$ and so $0 \ge g(t) \ge 0$. Hence $g(t) = 0$ for all $t \le c$ and so $\mathbf{x}(t) = \mathbf{x}_1(t)$ for these values of t.

Now suppose that $t \ge c$. Then

$$\begin{aligned}|\mathbf{x}(t) - \mathbf{x}_1(t)| &\le \int_c^t |\mathbf{f}(s,\mathbf{x}(s)) - \mathbf{f}(s,\mathbf{x}_1(s))| \\ &\le K\int_c^t |\mathbf{x}(s) - \mathbf{x}_1(s)|\,ds.\end{aligned}$$

Letting $h(t) = \int_c^t |\mathbf{x}(s) - \mathbf{x}_1(s)|\, ds$, it follows that

$$h'(t) \leq Kh(t)$$

and so

$$\left(e^{-Kt}h(t)\right)' \leq 0$$

which means $e^{-Kt}h(t) - e^{-Kc}h(c) \leq 0$. But $h(c) = 0$ and so this requires

$$h(t) \equiv \int_c^t |\mathbf{x}(s) - \mathbf{x}_1(s)|\, ds \leq 0$$

and so $\mathbf{x}(t) = \mathbf{x}_1(t)$ whenever $t \geq c$. Therefore, $\mathbf{x}_1 = \mathbf{x}$. This proves uniqueness and establishes the theorem.

4.2 Linear Systems

As an example of the above theorem, consider for $t \in [a,b]$ the system

$$\mathbf{x}' = A(t)\mathbf{x}(t) + \mathbf{g}(t),\ \mathbf{x}(c) = \mathbf{x}_0 \tag{4.11}$$

where $A(t)$ is an $n \times n$ matrix whose entries are continuous functions of $t, (a_{ij}(t))$ and $\mathbf{g}(t)$ is a vector whose components are continuous functions of t satisfies the conditions of Theorem 4.1.5 with $\mathbf{f}(t,\mathbf{x}) = A(t)\mathbf{x} + \mathbf{g}(t)$. To see this, let $\mathbf{x} = (x_1, \cdots, x_n)^T$ and $\mathbf{x}_1 = (x_{11}, \cdots x_{1n})^T$. Then letting $M = \max\{|a_{ij}(t)| : t \in [a,b], i,j \leq n\}$,

$$|\mathbf{f}(t,\mathbf{x}) - \mathbf{f}(t,\mathbf{x}_1)| = |A(t)(\mathbf{x} - \mathbf{x}_1)|$$

$$\begin{aligned}
&= \left|\left(\sum_{i=1}^n \left|\sum_{j=1}^n a_{ij}(t)(x_j - x_{1j})\right|^2\right)^{1/2}\right| \\
&\leq M\left|\left(\sum_{i=1}^n \left(\sum_{j=1}^n |x_j - x_{1j}|\right)^2\right)^{1/2}\right| \\
&\leq M\left|\left(\sum_{i=1}^n n\sum_{j=1}^n |x_j - x_{1j}|^2\right)^{1/2}\right| \\
&= Mn\left(\sum_{j=1}^n |x_j - x_{1j}|^2\right)^{1/2} = Mn|\mathbf{x} - \mathbf{x}_1|.
\end{aligned}$$

Therefore, we can let $K = Mn$. This proves

Theorem 4.2.1 *Let $A(t)$ be a continuous $n \times n$ matrix and let $\mathbf{g}(t)$ be a continuous vector for $t \in [a,b]$ and let $c \in [a,b]$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Then there exists a unique solution to 4.11 valid for $t \in [a,b]$.*

This includes more examples of linear equations than we will encounter in the entire course.

4.3 Local Solutions

Lemma 4.3.1 *Let $D(\mathbf{x}_0, r) \equiv \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and suppose U is an open set containing $D(\mathbf{x}_0, r)$ such that $\mathbf{f} : U \to \mathbb{R}^n$ is $C^1(U)$. (Recall this means all partial derivatives of $\mathbf{f}$ exist and are continuous.) Then for $K = Mn$, where M denotes the maximum of $\left|\frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{z})\right|$ for $\mathbf{z} \in D(\mathbf{x}_0, r)$, it follows that for all $\mathbf{x}, \mathbf{y} \in D(\mathbf{x}_0, r)$,*

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|.$$

Proof: Let $\mathbf{x}, \mathbf{y} \in D(\mathbf{x}_0, r)$ and consider the line segment joining these two points, $\mathbf{x}+t(\mathbf{y} - \mathbf{x})$ for $t \in [0, 1]$. If we let $\mathbf{h}(t) = \mathbf{f}(\mathbf{x}+t(\mathbf{y} - \mathbf{x}))$ for $t \in [0, 1]$, then

$$\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) = \mathbf{h}(1) - \mathbf{h}(0) = \int_0^1 \mathbf{h}'(t)\, dt.$$

Also, by the chain rule,

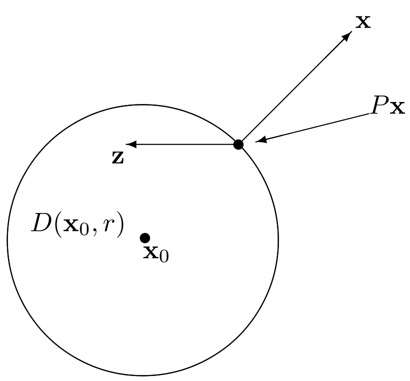
$$\mathbf{h}'(t) = \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}+t(\mathbf{y} - \mathbf{x}))(y_i - x_i).$$

Therefore, we must have

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| =$$

$$\begin{aligned} & \left|\int_0^1 \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}+t(\mathbf{y} - \mathbf{x}))(y_i - x_i)\, dt\right| \\ \leq \ & \int_0^1 \sum_{i=1}^{n} \left|\frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}+t(\mathbf{y} - \mathbf{x}))\right| |y_i - x_i|\, dt \\ \leq \ & M \sum_{i=1}^{n} |y_i - x_i| \leq Mn |\mathbf{x} - \mathbf{y}|. \end{aligned}$$

Now consider the map, P which maps all of $\mathbb{R}^n$ to $D(\mathbf{x}_0, r)$ given as follows. For $\mathbf{x} \in D(\mathbf{x}_0, r)$, we have $P\mathbf{x} = \mathbf{x}$. For $\mathbf{x} \notin D(\mathbf{x}_0, r)$ we have $P\mathbf{x}$ will be the closest point in $D(\mathbf{x}_0, r)$ to $\mathbf{x}$. Such a closest point exists because $D(\mathbf{x}_0, r)$ is a closed and bounded set. Taking $f(\mathbf{y}) \equiv |\mathbf{y} - \mathbf{x}|$, it follows f is a continuous function defined on $D(\mathbf{x}_0, r)$ which must achieve its minimum value by the extreme value theorem from calculus.



Lemma 4.3.2 *For any pair of points,* $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ *we have* $|P\mathbf{x}-P\mathbf{y}|\leq|\mathbf{x}-\mathbf{y}|$.

Proof: From the above picture it follows that for $\mathbf{z}\in D(\mathbf{x}_0,r)$ arbitrary, the angle between the vectors $\mathbf{x}-P\mathbf{x}$ and $\mathbf{z}-P\mathbf{x}$ is always greater than $\pi/2$ radians. Therefore, the cosine of this angle is always negative. It follows that

$$(\mathbf{y}-P\mathbf{y})\cdot(P\mathbf{x}-P\mathbf{y})\leq 0$$

and

$$(\mathbf{x}-P\mathbf{x})\cdot(P\mathbf{y}-P\mathbf{x})\leq 0.$$

Thus $(\mathbf{x}-P\mathbf{x})\cdot(P\mathbf{x}-P\mathbf{y})\geq 0$ and so if we subtract, we find

$$(\mathbf{x}-P\mathbf{x}-(\mathbf{y}-P\mathbf{y}))\cdot(P\mathbf{x}-P\mathbf{y})\geq 0$$

which implies

$$\begin{aligned}(\mathbf{x}-\mathbf{y})\cdot(P\mathbf{x}-P\mathbf{y}) &\geq (P\mathbf{x}-P\mathbf{y})\cdot(P\mathbf{x}-P\mathbf{y})\\ &= |P\mathbf{x}-P\mathbf{y}|^2.\end{aligned}$$

Now apply the Cauchy Schwarz inequality to the left side of the above inequality to obtain

$$|\mathbf{x}-\mathbf{y}|\,|P\mathbf{x}-P\mathbf{y}|\geq|P\mathbf{x}-P\mathbf{y}|^2$$

which yields the claim of the lemma.

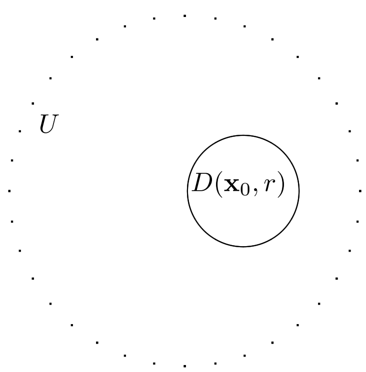
With this here is the local existence and uniqueness theorem.

Theorem 4.3.3 *Let* $[a,b]$ *be a closed interval and let* U *be an open subset of* $\mathbb{R}^n$. *Let* $\mathbf{f}:[a,b]\times U\to\mathbb{R}^n$ *be continuous and suppose that for each* $t\in[a,b]$, *the map* $\mathbf{x}\to\frac{\partial\mathbf{f}}{\partial x_i}(t,\mathbf{x})$ *is continuous. Also let* $\mathbf{x}_0\in U$ *and* $c\in[a,b]$. *Then there exists an interval,* $I\subseteq[a,b]$ *such that* $c\in I$ *and there exists a unique solution to the initial value problem,*

$$\mathbf{x}'=\mathbf{f}(t,\mathbf{x}),\ \mathbf{x}(c)=\mathbf{x}_0 \tag{4.12}$$

valid for $t\in I$.

Proof: Consider the following picture.



The large dotted circle represents U and the little solid circle represents $D(\mathbf{x}_0,r)$ as indicated. Here we have chosen r so small that $D(\mathbf{x}_0,r)$ is contained in U as shown. Now let P denote the projection map defined above. Consider the initial value problem

$$\mathbf{x}'=\mathbf{f}(t,P\mathbf{x}),\ \mathbf{x}(c)=\mathbf{x}_0. \tag{4.13}$$

From Lemma 4.3.1 and the continuity of $\mathbf{x} \to \frac{\partial \mathbf{f}}{\partial x_i}(t, \mathbf{x})$, we know there exists a constant, K such that if $\mathbf{x}, \mathbf{y} \in D(\mathbf{x}_0, r)$, then $|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$ for all $t \in [a, b]$. Therefore, by Lemma 4.3.2

$$|\mathbf{f}(t, P\mathbf{x}) - \mathbf{f}(t, P\mathbf{y})| \leq K |P\mathbf{x} - P\mathbf{y}| \leq K |\mathbf{x} - \mathbf{y}|.$$

It follows from Theorem 4.1.5 that 4.13 has a unique solution valid for $t \in [a, b]$. Since $\mathbf{x}$ is continuous, it follows that there exists an interval, I containing c such that for $t \in I$, we have $\mathbf{x}(t) \in D(\mathbf{x}_0, r)$. Therefore, for these values of t, we have $\mathbf{f}(t, P\mathbf{x}) = \mathbf{f}(t, \mathbf{x})$ and so there is a unique solution to 4.12 on I. This proves the theorem.

Now suppose $\mathbf{f}$ has the property that for every $R > 0$ there exists a constant, K_R such that for all $\mathbf{x}, \mathbf{x}_1 \in \overline{B(0, R)}$,

$$|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{x}_1)| \leq K_R |\mathbf{x} - \mathbf{x}_1|. \tag{4.14}$$

Corollary 4.3.4 *Let $\mathbf{f}$ satisfy 4.14 and suppose also that $(t, \mathbf{x}) \to \mathbf{f}(t, \mathbf{x})$ is continuous. Suppose now that $\mathbf{x}_0$ is given and there exists an estimate of the form $|\mathbf{x}(t)| < R$ for all $t \in [0, T)$ where $T \leq \infty$ on the local solution to*

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \ \mathbf{x}(0) = \mathbf{x}_0. \tag{4.15}$$

Then there exists a unique solution to the initial value problem, 4.15 valid on $[0, T)$.

Proof: Replace $\mathbf{f}(t, \mathbf{x})$ with $\mathbf{f}(t, P\mathbf{x})$ where P is the projection onto $\overline{B(0, R)}$. Then by Theorem 4.1.5 there exists a unique solution to the system

$$\mathbf{x}' = \mathbf{f}(t, P\mathbf{x}), \ \mathbf{x}(0) = \mathbf{x}_0$$

valid on $[0, T_1]$ for every $T_1 < T$. Therefore, the above system has a unique solution on $[0, T)$ and from the estimate, $P\mathbf{x} = \mathbf{x}$. This proves the corollary.

Chapter 5

First Order Linear Systems

5.1 The Cayley Hamilton Theorem

Definition 5.1.1 *Let A be an $n \times n$ matrix. The characteristic polynomial is defined as*

$$p_A(t) \equiv \det(tI - A)$$

and the solutions to $p_A(t) = 0$ are called eigenvalues. For A a matrix and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, denote by $p(A)$ the matrix defined by

$$p(A) \equiv A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

The explanation for the last term is that A^0 is interpreted as I, the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $P_A(t) = 0$. It is one of the most important theorems in linear algebra. The following lemma will help with its proof.

Lemma 5.1.2 *Suppose for all $|\lambda|$ large enough,*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = 0,$$

where the A_i are $n \times n$ matrices. Then each $A_i = 0$.

Proof: Multiply by λ^{-m} to obtain

$$A_0\lambda^{-m} + A_1\lambda^{-m+1} + \cdots + A_{m-1}\lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_m = 0$. With this, multiply by λ to obtain

$$A_0\lambda^{-m+1} + A_1\lambda^{-m+2} + \cdots + A_{m-1} = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by λ and letting $\lambda \to \infty$ to obtain that all the $A_i = 0$. This proves the lemma.

With the lemma, here is a simple corollary.

Corollary 5.1.3 *Let A_i and B_i be $n \times n$ matrices and suppose*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = B_0 + B_1\lambda + \cdots + B_m\lambda^m$$

for all $|\lambda|$ large enough. Then $A_i = B_i$ for all i. Consequently if λ is replaced by any $n \times n$ matrix, the two sides will be equal. That is, for C any $n \times n$ matrix,

$$A_0 + A_1C + \cdots + A_mC^m = B_0 + B_1C + \cdots + B_mC^m.$$

Proof: Subtract and use the result of the lemma.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

Theorem 5.1.4 *Let A be an $n \times n$ matrix and let $p(\lambda) \equiv \det(\lambda I - A)$ be the characteristic polynomial. Then $p(A) = 0$.*

Proof: Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I - A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then λ cannot be in the finite list of eigenvalues of A and so for such λ, $(\lambda I - A)^{-1}$ exists.) Therefore, by Theorem **??**

$$C(\lambda) = p(\lambda)(\lambda I - A)^{-1}.$$

Note that each entry in $C(\lambda)$ is a polynomial in λ having degree no more than $n-1$. Therefore, collecting the terms,

$$C(\lambda) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}$$

for C_j some $n \times n$ matrix. It follows that for all $|\lambda|$ large enough,

$$(A - \lambda I)\left(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}\right) = p(\lambda) I$$

and so Corollary 5.1.3 may be used. It follows the matrix coefficients corresponding to equal powers of λ are equal on both sides of this equation. Therefore, if λ is replaced with A, the two sides will be equal. Thus

$$0 = (A - A)\left(C_0 + C_1 A + \cdots + C_{n-1}A^{n-1}\right) = p(A) I = p(A).$$

This proves the Cayley Hamilton theorem.

5.2 First Order Linear Systems

Here is a discussion of linear systems of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$$

where A is an $n \times n$ matrix and $\mathbf{f}$ is a vector valued function having all entries continuous. Of course it is a very special case of the general considerations above but I will give a self contained presentation.

Definition 5.2.1 *Suppose $t \to M(t)$ is a matrix valued function of t. Thus $M(t) = (m_{ij}(t))$. Then define*

$$M'(t) \equiv \left(m_{ij}'(t)\right).$$

In words, the derivative of $M(t)$ is the matrix whose entries consist of the derivatives of the entries of $M(t)$. Integrals of matrices are defined the same way. Thus

$$\int_a^b M(t)\, di \equiv \left(\int_a^b m_{ij}(t)\, dt\right).$$

In words, the integral of $M(t)$ is the matrix obtained by replacing each entry of $M(t)$ by the integral of that entry.

With this definition, it is easy to prove the following theorem.

Theorem 5.2.2 *Suppose $M(t)$ and $N(t)$ are matrices for which $M(t)N(t)$ makes sense. Then if $M'(t)$ and $N'(t)$ both exist, it follows that*

$$(M(t)N(t))' = M'(t)N(t) + M(t)N'(t).$$

Proof:

$$\begin{aligned}
\left((M(t)N(t))'\right)_{ij} &\equiv \left((M(t)N(t))_{ij}\right)' \\
&= \left(\sum_k M(t)_{ik} N(t)_{kj}\right)' \\
&= \sum_k (M(t)_{ik})' N(t)_{kj} + M(t)_{ik}\left(N(t)_{kj}\right)' \\
&\equiv \sum_k \left(M(t)'\right)_{ik} N(t)_{kj} + M(t)_{ik}\left(N(t)'\right)_{kj} \\
&\equiv \left(M'(t)N(t) + M(t)N'(t)\right)_{ij}
\end{aligned}$$

and this proves the theorem.

In the study of differential equations, one of the most important theorems is Gronwall's inequality which is next.

Theorem 5.2.3 *Suppose $u(t) \geq 0$ and for all $t \in [0,T]$,*

$$u(t) \leq u_0 + \int_0^t Ku(s)\,ds. \tag{5.1}$$

where K is some constant. Then

$$u(t) \leq u_0 e^{Kt}. \tag{5.2}$$

Proof: Let $w(t) = \int_0^t u(s)\,ds$. Then using the fundamental theorem of calculus, 5.1 $w(t)$ satisfies the following.

$$u(t) - Kw(t) = w'(t) - Kw(t) \leq u_0,\ w(0) = 0. \tag{5.3}$$

Multiply both sides of this inequality by e^{-Kt} and using the product rule and the chain rule,

$$e^{-Kt}(w'(t) - Kw(t)) = \frac{d}{dt}\left(e^{-Kt}w(t)\right) \leq u_0 e^{-Kt}.$$

Integrating this from 0 to t,

$$e^{-Kt}w(t) \leq u_0 \int_0^t e^{-Ks}ds = u_0\left(-\frac{e^{-tK}-1}{K}\right).$$

Now multiply through by e^{Kt} to obtain

$$w(t) \leq u_0\left(-\frac{e^{-tK}-1}{K}\right)e^{Kt} = -\frac{u_0}{K} + \frac{u_0}{K}e^{tK}.$$

Therefore, 5.3 implies

$$u(t) \leq u_0 + K\left(-\frac{u_0}{K} + \frac{u_0}{K}e^{tK}\right) = u_0 e^{Kt}.$$

This proves the theorem.

With Gronwall's inequality, here is a theorem on uniqueness of solutions to the initial value problem,

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t),\ \mathbf{x}(a) = \mathbf{x}_a, \tag{5.4}$$

in which A is an $n \times n$ matrix and $\mathbf{f}$ is a continuous function having values in $\mathbb{C}^n$.

Theorem 5.2.4 *Suppose* $\mathbf{x}$ *and* $\mathbf{y}$ *satisfy 5.4. Then* $\mathbf{x}(t)=\mathbf{y}(t)$ *for all* t.

Proof: Let $\mathbf{z}(t)=\mathbf{x}(t+a)-\mathbf{y}(t+a)$. Then for $t\geq 0$,

$$\mathbf{z}'=A\mathbf{z},\ \mathbf{z}(0)=\mathbf{0}. \tag{5.5}$$

Note that for $K=\max\{|a_{ij}|\}$, where $A=(a_{ij})$,

$$|(A\mathbf{z},\mathbf{z})|=\left|\sum_{ij}a_{ij}z_j\overline{z_i}\right|\leq K\sum_{ij}|z_i||z_j|$$

$$\leq K\sum_{ij}\left(\frac{|z_i|^2}{2}+\frac{|z_j|^2}{2}\right)=nK|\mathbf{z}|^2.$$

(For x and y real numbers, $xy\leq\frac{x^2}{2}+\frac{y^2}{2}$ because this is equivalent to saying $(x-y)^2\geq 0$.) Similarly,

$$|(\mathbf{z},A\mathbf{z})|\leq nK|\mathbf{z}|^2$$

Thus,

$$|(\mathbf{z},A\mathbf{z})|,|(A\mathbf{z},\mathbf{z})|\leq nK|\mathbf{z}|^2. \tag{5.6}$$

Now multiplying 5.5 by $\mathbf{z}$ and observing that

$$\frac{d}{dt}\left(|\mathbf{z}|^2\right)=(\mathbf{z}',\mathbf{z})+(\mathbf{z},\mathbf{z}')=(A\mathbf{z},\mathbf{z})+(\mathbf{z},A\mathbf{z}),$$

it follows from 5.6 and the observation that $\mathbf{z}(0)=0$,

$$|\mathbf{z}(t)|^2\leq\int_0^t 2nK|\mathbf{z}(s)|^2\,ds$$

and so by Gronwall's inequality, $|\mathbf{z}(t)|^2=0$ for all $t\geq 0$. Thus,

$$\mathbf{x}(t)=\mathbf{y}(t)$$

for all $t\geq a$.

Now let $\mathbf{w}(t)=\mathbf{x}(a-t)-\mathbf{y}(a-t)$ for $t\geq 0$. Then $\mathbf{w}'(t)=(-A)\mathbf{w}(t)$ and you can repeat the argument which was just given to conclude that $\mathbf{x}(t)=\mathbf{y}(t)$ for all $t\leq a$. This proves the theorem.

Definition 5.2.5 *Let* A *be an* $n\times n$ *matrix. We say* $\Phi(t)$ *is a fundamental matrix for* A *if*

$$\Phi'(t)=A\Phi(t),\ \Phi(0)=I, \tag{5.7}$$

and $\Phi(t)^{-1}$ *exists for all* $t\in\mathbb{R}$.

Why should anyone care about a fundamental matrix? The reason is that such a matrix valued function makes possible a convenient description of the solution of the initial value problem,

$$\mathbf{x}'=A\mathbf{x}+\mathbf{f}(t),\ \mathbf{x}(0)=\mathbf{x}_0, \tag{5.8}$$

on the interval, $[0,T]$. First consider the special case where $n=1$. This is the first order linear differential equation,

$$r'=\lambda r+g,\ r(0)=r_0, \tag{5.9}$$

where g is a continuous scalar valued function. First consider the case where $g=0$.

Lemma 5.2.6 *There exists a unique solution to the initial value problem,*

$$r' = \lambda r,\ r(0) = 1, \tag{5.10}$$

and the solution for $\lambda = a + ib$ is given by

$$r(t) = e^{at}(\cos bt + i \sin bt). \tag{5.11}$$

This solution to the initial value problem is denoted as $e^{\lambda t}$. (If λ is real, $e^{\lambda t}$ as defined here reduces to the usual exponential function so there is no contradiction between this and earlier notation seen in Calculus.)

Proof: From the uniqueness theorem presented above, Theorem 5.2.4, applied to the case where $n = 1$, there can be no more than one solution to the initial value problem, 5.10. Therefore, it only remains to verify 5.11 is a solution to 5.10. However, this is an easy calculus exercise. This proves the Lemma.

Note the differential equation in 5.10 says

$$\frac{d}{dt}\left(e^{\lambda t}\right) = \lambda e^{\lambda t}. \tag{5.12}$$

With this lemma, it becomes possible to easily solve the case in which $g \neq 0$.

Theorem 5.2.7 *There exists a unique solution to 5.9 and this solution is given by the formula,*

$$r(t) = e^{\lambda t} r_0 + e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds. \tag{5.13}$$

Proof: By the uniqueness theorem, Theorem 5.2.4, there is no more than one solution. It only remains to verify that 5.13 is a solution. But $r(0) = e^{\lambda 0} r_0 + \int_0^0 e^{-\lambda s} g(s)\, ds = r_0$ and so the initial condition is satisfied. Next differentiate this expression to verify the differential equation is also satisfied. Using 5.12, the product rule and the fundamental theorem of calculus,

$$\begin{aligned} r'(t) &= \lambda e^{\lambda t} r_0 + \lambda e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds + e^{\lambda t} e^{-\lambda t} g(t) \\ &= \lambda r(t) + g(t). \end{aligned}$$

This proves the Theorem.

Now consider the question of finding a fundamental matrix for A. When this is done, it will be easy to give a formula for the general solution to 5.8 known as the variation of constants formula, arguably the most important result in differential equations.

The next theorem gives a formula for the fundamental matrix 5.7. It is known as Putzer's method [**?**].

Theorem 5.2.8 *Let A be an $n \times n$ matrix whose eigenvalues are $\{\lambda_1, \cdots, \lambda_n\}$ listed according to multiplicity. Define*

$$P_k(A) \equiv \prod_{m=1}^{k} (A - \lambda_m I),\ P_0(A) \equiv I,$$

and let the scalar valued functions, $r_k(t)$ be defined as the solutions to the following initial value problem

$$\begin{pmatrix} r_0'(t) \\ r_1'(t) \\ r_2'(t) \\ \vdots \\ r_n'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_1 r_1(t) + r_0(t) \\ \lambda_2 r_2(t) + r_1(t) \\ \vdots \\ \lambda_n r_n(t) + r_{n-1}(t) \end{pmatrix}, \quad \begin{pmatrix} r_0(0) \\ r_1(0) \\ r_2(0) \\ \vdots \\ r_n(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Note the system amounts to a list of single first order linear differential equations. Now define

$$\Phi(t) \equiv \sum_{k=0}^{n-1} r_{k+1}(t) P_k(A).$$

Then

$$\Phi'(t) = A\Phi(t), \ \Phi(0) = I. \tag{5.14}$$

Furthermore, if $\Phi(t)$ is a solution to 5.14 for all t, then it follows $\Phi(t)^{-1}$ exists for all t and $\Phi(t)$ is the unique fundamental matrix for A.

Proof: The first part of this follows from a computation. First note that by the Cayley Hamilton theorem, $P_n(A) = 0$. Now for the computation:

$$\Phi'(t) = \sum_{k=0}^{n-1} r_{k+1}'(t) P_k(A) = \sum_{k=0}^{n-1} (\lambda_{k+1} r_{k+1}(t) + r_k(t)) P_k(A) =$$

$$\sum_{k=0}^{n-1} \lambda_{k+1} r_{k+1}(t) P_k(A) + \sum_{k=0}^{n-1} r_k(t) P_k(A) = \sum_{k=0}^{n-1} (\lambda_{k+1} I - A) r_{k+1}(t) P_k(A) +$$

$$\sum_{k=0}^{n-1} r_k(t) P_k(A) + \sum_{k=0}^{n-1} A r_{k+1}(t) P_k(A)$$

$$= -\sum_{k=0}^{n-1} r_{k+1}(t) P_{k+1}(A) + \sum_{k=0}^{n-1} r_k(t) P_k(A) + A \sum_{k=0}^{n-1} r_{k+1}(t) P_k(A). \tag{5.15}$$

Now using $r_0(t) = 0$, the first term equals

$$\begin{aligned} -\sum_{k=1}^{n} r_k(t) P_k(A) &= -\sum_{k=1}^{n-1} r_k(t) P_k(A) \\ &= -\sum_{k=0}^{n-1} r_k(t) P_k(A) \end{aligned}$$

and so 5.15 reduces to

$$A \sum_{k=0}^{n-1} r_{k+1}(t) P_k(A) = A\Phi(t).$$

This shows $\Phi'(t) = A\Phi(t)$. That $\Phi(0) = 0$ follows from

$$\Phi(0) = \sum_{k=0}^{n-1} r_{k+1}(0) P_k(A) = r_1(0) P_0 = I.$$

It remains to verify that if 5.14 holds, then $\Phi(t)^{-1}$ exists for all t. To do so, consider $\mathbf{v} \neq \mathbf{0}$ and suppose for some t_0, $\Phi(t_0)\mathbf{v} = \mathbf{0}$. Let $\mathbf{x}(t) \equiv \Phi(t_0 + t)\mathbf{v}$. Then

$$\mathbf{x}'(t) = A\Phi(t_0 + t)\mathbf{v} = A\mathbf{x}(t), \ \mathbf{x}(0) = \Phi(t_0)\mathbf{v} = \mathbf{0}.$$

But also $\mathbf{z}(t) \equiv \mathbf{0}$ also satisfies

$$\mathbf{z}'(t) = A\mathbf{z}(t), \ \mathbf{z}(0) = \mathbf{0},$$

and so by the theorem on uniqueness, it must be the case that $\mathbf{z}(t) = \mathbf{x}(t)$ for all t, showing that $\Phi(t + t_0)\mathbf{v} = \mathbf{0}$ for all t, and in particular for $t = -t_0$. Therefore,

$$\Phi(-t_0 + t_0)\mathbf{v} = I\mathbf{v} = \mathbf{0}$$

and so $\mathbf{v} = \mathbf{0}$, a contradiction. It follows that $\Phi(t)$ must be one to one for all t and so, $\Phi(t)^{-1}$ exists for all t.

It only remains to verify that the solution to 5.14 is unique. Suppose Ψ is another fundamental matrix solving 5.14. Then letting $\mathbf{v}$ be an arbitrary vector,

$$\mathbf{z}(t) \equiv \Phi(t)\mathbf{v}, \ \mathbf{y}(t) \equiv \Psi(t)\mathbf{v}$$

both solve the initial value problem,

$$\mathbf{x}' = A\mathbf{x}, \ \mathbf{x}(0) = \mathbf{v},$$

and so by the uniqueness theorem, $\mathbf{z}(t) = \mathbf{y}(t)$ for all t showing that $\Phi(t)\mathbf{v} = \Psi(t)\mathbf{v}$ for all t. Since $\mathbf{v}$ is arbitrary, this shows that $\Phi(t) = \Psi(t)$ for every t. This proves the theorem.

It is useful to consider the differential equations for the r_k for $k \geq 1$. As noted above, $r_0(t) = 0$ and $r_1(t) = e^{\lambda_1 t}$.

$$r_{k+1}' = \lambda_{k+1} r_{k+1} + r_k, \ r_{k+1}(0) = 0.$$

Thus

$$r_{k+1}(t) = \int_0^t e^{\lambda_{k+1}(t-s)} r_k(s)\, ds.$$

Therefore,

$$r_2(t) = \int_0^t e^{\lambda_2(t-s)} e^{\lambda_1 s} ds = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{-\lambda_2 + \lambda_1}$$

assuming $\lambda_1 \neq \lambda_2$.

Sometimes people define a fundamental matrix to be a matrix, $\Phi(t)$ such that $\Phi'(t) = A\Phi(t)$ and $\det(\Phi(t)) \neq 0$ for all t. Thus this avoids the initial condition, $\Phi(0) = I$. The next proposition has to do with this situation.

Proposition 5.2.9 *Suppose A is an $n \times n$ matrix and suppose $\Phi(t)$ is an $n \times n$ matrix for each $t \in \mathbb{R}$ with the property that*

$$\Phi'(t) = A\Phi(t). \tag{5.16}$$

Then either $\Phi(t)^{-1}$ exists for all $t \in \mathbb{R}$ or $\Phi(t)^{-1}$ fails to exist for all $t \in \mathbb{R}$.

Proof: Suppose $\Phi(0)^{-1}$ exists and 5.16 holds. Let $\Psi(t) \equiv \Phi(t)\Phi(0)^{-1}$. Then $\Psi(0) = I$ and

$$\Psi'(t) = \Phi'(t)\Phi(0)^{-1} = A\Phi(t)\Phi(0)^{-1} = A\Psi(t)$$

so by Theorem 5.2.8, $\Psi(t)^{-1}$ exists for all t. Therefore, $\Phi(t)^{-1}$ also exists for all t.

Next suppose $\Phi(0)^{-1}$ does not exist. I need to show $\Phi(t)^{-1}$ does not exist for any t. Suppose then that $\Phi(t_0)^{-1}$ does exist. Then let$\Psi(t) \equiv \Phi(t_0+t)\Phi(t_0)^{-1}$. Then $\Psi(0) = I$ and $\Psi' = A\Psi$ so by Theorem 5.2.8 it follows $\Psi(t)^{-1}$ exists for all t and so for all $t, \Phi(t+t_0)^{-1}$ must also exist, even for $t = -t_0$ which implies $\Phi(0)^{-1}$ exists after all. This proves the proposition.

The conclusion of this proposition is usually referred to as the Wronskian alternative and another way to say it is that if 5.16 holds, then either $\det(\Phi(t)) = 0$ for all t or $\det(\Phi(t))$ is never equal to 0. The Wronskian is the usual name of the function, $t \to \det(\Phi(t))$.

The following theorem gives the variation of constants formula,.

Theorem 5.2.10 *Let $\mathbf{f}$ be continuous on $[0,T]$ and let A be an $n\times n$ matrix and $\mathbf{x}_0$ a vector in $\mathbb{C}^n$. Then there exists a unique solution to 5.8, $\mathbf{x}$, given by the variation of constants formula,*

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \Phi(t)\int_0^t \Phi(s)^{-1}\mathbf{f}(s)\,ds \tag{5.17}$$

for $\Phi(t)$ the fundamental matrix for A. Also, $\Phi(t)^{-1} = \Phi(-t)$ and $\Phi(t+s) = \Phi(t)\Phi(s)$ for all t,s and the above variation of constants formula can also be written as

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(t-s)\mathbf{f}(s)\,ds \tag{5.18}$$

$$= \Phi(t)\mathbf{x}_0 + \int_0^t \Phi(s)\mathbf{f}(t-s)\,ds \tag{5.19}$$

Proof: From the uniqueness theorem there is at most one solution to 5.8. Therefore, if 5.17 solves 5.8, the theorem is proved. The verification that the given formula works is identical with the verification that the scalar formula given in Theorem 5.2.7 solves the initial value problem given there. $\Phi(s)^{-1}$ is continuous because of the formula for the inverse of a matrix in terms of the transpose of the cofactor matrix. Therefore, the integrand in 5.17 is continuous and the fundamental theorem of calculus applies. To verify the formula for the inverse, fix s and consider $\mathbf{x}(t) = \Phi(s+t)\mathbf{v}$, and $\mathbf{y}(t) = \Phi(t)\Phi(s)\mathbf{v}$. Then

$$\mathbf{x}'(t) = A\Phi(t+s)\mathbf{v} = A\mathbf{x}(t),\ \mathbf{x}(0) = \Phi(s)\mathbf{v}$$

$$\mathbf{y}'(t) = A\Phi(t)\Phi(s)\mathbf{v} = A\mathbf{y}(t),\ \mathbf{y}(0) = \Phi(s)\mathbf{v}.$$

By the uniqueness theorem, $\mathbf{x}(t) = \mathbf{y}(t)$ for all t. Since s and $\mathbf{v}$ are arbitrary, this shows $\Phi(t+s) = \Phi(t)\Phi(s)$ for all t,s. Letting $s = -t$ and using $\Phi(0) = I$ verifies $\Phi(t)^{-1} = \Phi(-t)$.

Next, note that this also implies $\Phi(t-s)\Phi(s) = \Phi(t)$ and so $\Phi(t-s) = \Phi(t)\Phi(s)^{-1}$. Therefore, this yields 5.18 and then 5.19follows from changing the variable. This proves the theorem.

If $\Phi' = A\Phi$ and $\Phi(t)^{-1}$ exists for all t, you should verify that the solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f},\ \mathbf{x}(t_0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = \Phi(t-t_0)\mathbf{x}_0 + \int_{t_0}^t \Phi(t-s)\mathbf{f}(s)\,ds.$$

Theorem 5.2.10 is general enough to include all constant coefficient linear differential equations or any order. Thus it includes as a special case the main topics of an entire elementary differential equations class. This is illustrated in the following example. One can reduce an arbitrary linear differential equation to a first order system and then apply the above theory to solve the problem. The next example is a differential equation of damped vibration.

Example 5.2.11 *The differential equation is* $y'' + 2y' + 2y = \cos t$ *and initial conditions,* $y(0) = 1$ *and* $y'(0) = 0$.

To solve this equation, let $x_1 = y$ and $x_2 = x_1' = y'$. Then, writing this in terms of these new variables, yields the following system.

$$\begin{gathered} x_2' + 2x_2 + 2x_1 = \cos t \\ x_1' = x_2 \end{gathered}$$

This system can be written in the above form as

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' &= \begin{pmatrix} x_2 \\ -2x_2 - 2x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}. \end{aligned}$$

and the initial condition is of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Now $P_0(A) \equiv I$. The eigenvalues are $-1+i, -1-i$ and so

$$\begin{aligned} P_1(A) &= \left(\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} - (-1+i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}. \end{aligned}$$

Recall $r_0(t) \equiv 0$ and $r_1(t) = e^{(-1+i)t}$. Then

$$r_2' = (-1-i) r_2 + e^{(-1+i)t}, \; r_2(0) = 0$$

and so

$$r_2(t) = \frac{e^{(-1+i)t} - e^{(-1-i)t}}{2i} = e^{-t} \sin(t)$$

Putzer's method yields the fundamental matrix as

$$\begin{aligned} \Phi(t) &= e^{(-1+i)t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e^{-t} \sin(t) \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \\ &= \begin{pmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t} \sin t \\ -2e^{-t} \sin t & e^{-t}(\cos(t) - \sin(t)) \end{pmatrix} \end{aligned}$$

From variation of constants formula the desired solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (t) = \begin{pmatrix} e^{-t}(\cos(t) + \sin(t)) & e^{-t} \sin t \\ -2e^{-t} \sin t & e^{-t}(\cos(t) - \sin(t)) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$+\int_0^t \begin{pmatrix} e^{-s}(\cos(s)+\sin(s)) & e^{-s}\sin s \\ -2e^{-s}\sin s & e^{-s}(\cos(s)-\sin(s)) \end{pmatrix} \begin{pmatrix} 0 \\ \cos(t-s) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}(\cos(t)+\sin(t)) \\ -2e^{-t}\sin t \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-s}\sin(s)\cos(t-s) \\ e^{-s}(\cos s - \sin s)\cos(t-s) \end{pmatrix} ds$$

$$= \begin{pmatrix} e^{-t}(\cos(t)+\sin(t)) \\ -2e^{-t}\sin t \end{pmatrix} + \begin{pmatrix} -\frac{1}{5}(\cos t)e^{-t} - \frac{3}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t \\ -\frac{2}{5}(\cos t)e^{-t} + \frac{4}{5}e^{-t}\sin t + \frac{2}{5}\cos t - \frac{1}{5}\sin t \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5}(\cos t)e^{-t} + \frac{2}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t \\ -\frac{6}{5}e^{-t}\sin t - \frac{2}{5}(\cos t)e^{-t} + \frac{2}{5}\cos t - \frac{1}{5}\sin t \end{pmatrix}$$

Thus $y(t) = x_1(t) = \frac{4}{5}(\cos t)e^{-t} + \frac{2}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t$.

Example 5.2.12 *Find the solution to the initial value problem $y''' - y'' - y' + y = e^t$ along with the initial condition $y(0) = 1, y'(0) = 0, y''(0) = 0$.*

As before, you take $x_0 = y, x_1 = y' = x_0', x_2 = y'' = x_1'$. Then the above initial value problem can be written in the form

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix},$$

$$\begin{pmatrix} x_0(0) \\ x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Rather than do the long process described above, it would be easier to work with the Jordan form of the matrix. This is what is really happening in most elementary differential equations classes when they look for generalized eigenvectors and such things.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

and so the above system reduces to

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

and the system for the y variables,

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}$$

Thus

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -y_0 + e^t \\ y_1 + y_2 - e^t \\ y_2 - e^t \end{pmatrix},$$

$$\begin{pmatrix} y_0(0) \\ y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Thus $y_2' = y_2 - e^{-t}, y_2(0) = 1$ and so $y_2(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$. Then

$$\begin{aligned} y_1' &= y_1 + \frac{1}{2}e^t + \frac{1}{2}e^{-t} - e^{-t} \\ &= y_1 + \frac{1}{2}e^t - \frac{1}{2}e^{-t} \end{aligned}$$

along with the initial condition $y_1(0) = 0$. Thus

$$y_1(t) = \frac{1}{4}e^{-t} + \frac{1}{2}te^t - \frac{1}{4}e^t.$$

Then it only remains to find $y_0(t)$. From the top line,

$$y_0' = -y_0 + e^t, y_0(0) = 1.$$

and so

$$y_0(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}$$

Then the solution in terms of the x variables,

$$\begin{aligned} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{2} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} \\ \frac{1}{4}e^{-t} + \frac{1}{2}te^t - \frac{1}{4}e^t \\ \frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{8}e^t + \frac{3}{8}e^{-t} - \frac{1}{4}te^t \\ -\frac{1}{8}e^{-t} - \frac{1}{4}te^t + \frac{1}{8}e^t \\ -\frac{1}{8}e^{-t} - \frac{1}{4}te^t + \frac{1}{8}e^t \end{pmatrix}. \end{aligned}$$

The solution to the original third order equation is then

$$y = x_0 = \frac{5}{8}e^t + \frac{3}{8}e^{-t} - \frac{1}{4}te^t.$$

In a similar way other scalar differential equations with initial conditions can be reduced to systems of the form just discussed.

What if you could only find the eigenvalues approximately? Then Putzer's method applied to the approximate eigenvalues would end up giving a fairly good solution because there are no eigenvectors mentioned. You are not left stranded if you can't find the eigenvalues exactly, as you are if you use the methods usually taught.

Is the above approach computationally more involved than what is usually taught? Of course it is but surely it is even easier to use a computer algebra system to get the answer. Why should we pretend the students are computer algebra systems? Wouldn't it be better to give them an approach they could completely understand without the Jordan form and save the routine solutions to trivial problems for a real computer algebra system?

5.3 Geometric Theory Of Autonomous Systems

Here a sufficient condition is given for stability of a first order system. First of all, here is a fundamental estimate for the entries of a fundamental matrix.

Lemma 5.3.1 *Let the functions, r_k be given in the statement of Theorem 5.2.8 and suppose that A is an $n \times n$ matrix whose eigenvalues are $\{\lambda_1, \cdots, \lambda_n\}$. Suppose that these eigenvalues are ordered such that*

$$\operatorname{Re}(\lambda_1) \leq \operatorname{Re}(\lambda_2) \leq \cdots \leq \operatorname{Re}(\lambda_n) < 0.$$

Then if $0 > -\delta > \operatorname{Re}(\lambda_n)$ *is given, there exists a constant,* C *such that for each* $k = 0, 1, \cdots, n$,

$$|r_k(t)| \leq Ce^{-\delta t} \tag{5.20}$$

for all $t > 0$.

Proof: This is obvious for $r_0(t)$ because it is identically equal to 0. From the definition of the r_k,

$$r_1' = \lambda_1 r_1, r_1(0) = 1$$

and so

$$r_1(t) = e^{\lambda_1 t}$$

which implies

$$|r_1(t)| \leq e^{\operatorname{Re}(\lambda_1)t}.$$

Suppose for some $m \geq 1$ there exists a constant, C_m such that

$$|r_k(t)| \leq C_m t^m e^{\operatorname{Re}(\lambda_m)t}$$

for all $k \leq m$ for all $t > 0$. Then

$$r_{m+1}'(t) = \lambda_{m+1} r_{m+1}(t) + r_m(t),\ r_{m+1}(0) = 0$$

and so

$$r_{m+1}(t) = e^{\lambda_{m+1}t} \int_0^t e^{-\lambda_{m+1}s} r_m(s)\, ds.$$

Then by the induction hypothesis,

$$\begin{aligned} |r_{m+1}(t)| &\leq e^{\operatorname{Re}(\lambda_{m+1})t} \int_0^t \left| e^{-\lambda_{m+1}s} \right| C_m s^m e^{\operatorname{Re}(\lambda_m)s} ds \\ &\leq e^{\operatorname{Re}(\lambda_{m+1})t} \int_0^t s^m C_m e^{-\operatorname{Re}(\lambda_{m+1})s} e^{\operatorname{Re}(\lambda_m)s} ds \\ &\leq e^{\operatorname{Re}(\lambda_{m+1})t} \int_0^t s^m C_m ds = \frac{C_m}{m+1} t^{m+1} e^{\operatorname{Re}(\lambda_{m+1})t} \end{aligned}$$

It follows by induction there exists a constant, C such that for all $k \leq n$,

$$|r_k(t)| \leq Ct^n e^{\operatorname{Re}(\lambda_n)t}$$

and this obviously implies the conclusion of the lemma.

The proof of the above lemma yields the following corollary.

Corollary 5.3.2 *Let the functions,* r_k *be given in the statement of Theorem 5.2.8 and suppose that* A *is an* $n \times n$ *matrix whose eigenvalues are* $\{\lambda_1, \cdots, \lambda_n\}$. *Suppose that these eigenvalues are ordered such that*

$$\operatorname{Re}(\lambda_1) \leq \operatorname{Re}(\lambda_2) \leq \cdots \leq \operatorname{Re}(\lambda_n).$$

Then there exists a constant C *such that for all* $k \leq m$

$$|r_k(t)| \leq Ct^m e^{\operatorname{Re}(\lambda_m)t}.$$

With the lemma, the following sloppy estimate is available for a fundamental matrix.

Theorem 5.3.3 *Let A be an $n \times n$ matrix and let $\Phi(t)$ be the fundamental matrix for A. That is,*

$$\Phi'(t) = A\Phi(t), \ \Phi(0) = I.$$

Suppose also the eigenvalues of A are $\{\lambda_1, \cdots, \lambda_n\}$ where these eigenvalues are ordered such that

$$\operatorname{Re}(\lambda_1) \leq \operatorname{Re}(\lambda_2) \leq \cdots \leq \operatorname{Re}(\lambda_n) < 0.$$

Then if $0 > -\delta > \operatorname{Re}(\lambda_n)$, is given, there exists a constant, C such that

$$\left|\Phi(t)_{ij}\right| \leq Ce^{-\delta t}$$

for all $t > 0$. Also

$$|\Phi(t)\mathbf{x}| \leq Cn^{3/2}e^{-\delta t}|\mathbf{x}|. \tag{5.21}$$

Proof: Let

$$M \equiv \max\left\{P_k(A)_{ij} \text{ for all } i,j,k\right\}.$$

Then from Putzer's formula for $\Phi(t)$ and Lemma 5.3.1, there exists a constant, C such that

$$\left|\Phi(t)_{ij}\right| \leq \sum_{k=0}^{n-1} Ce^{-\delta t}M.$$

Let the new C be given by nCM. This proves the theorem.

Next,

$$\begin{aligned} |\Phi(t)\mathbf{x}|^2 &\equiv \sum_{i=1}^n \left(\sum_{j=1}^n \Phi_{ij}(t)x_j\right)^2 \\ &\leq \sum_{i=1}^n \left(\sum_{j=1}^n |\Phi_{ij}(t)||x_j|\right)^2 \\ &\leq \sum_{i=1}^n \left(\sum_{j=1}^n Ce^{-\delta t}|\mathbf{x}|\right)^2 \\ &= C^2e^{-2\delta t}\sum_{i=1}^n (n|\mathbf{x}|)^2 = C^2e^{-2\delta t}n^3|\mathbf{x}|^2 \end{aligned}$$

This proves 5.21 and completes the proof.

Definition 5.3.4 *Let $\mathbf{f} : U \to \mathbb{R}^n$ where U is an open subset of $\mathbb{R}^n$ such that $\mathbf{a} \in U$ and $\mathbf{f}(\mathbf{a}) = \mathbf{0}$. A point, $\mathbf{a}$ where $\mathbf{f}(\mathbf{a}) = \mathbf{0}$ is called an equilibrium point. Then $\mathbf{a}$ is assymptotically stable if for any $\varepsilon > 0$ there exists $r > 0$ such that whenever $|\mathbf{x}_0 - \mathbf{a}| < r$ and $\mathbf{x}(t)$ the solution to the initial value problem,*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(0) = \mathbf{x}_0,$$

it follows

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{a}, \ |\mathbf{x}(t) - \mathbf{a}| < \varepsilon$$

A differential equation of the form $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ is called autonomous as opposed to a nonautonomous equation of the form $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$. The equilibrium point $\mathbf{a}$ is stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{x}_0 - \mathbf{a}| < \delta$, then if $\mathbf{x}$ is the solution of

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \ \mathbf{x}(0) = \mathbf{x}_0, \tag{5.22}$$

then $|\mathbf{x}(t) - \mathbf{a}| < \varepsilon$ for all $t > 0$.

Obviously assymptotical stability implies stability.

An ordinary differential equation is called almost linear if it is of the form

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x})$$

where A is an $n \times n$ matrix and

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\mathbf{g}(\mathbf{x})}{|\mathbf{x}|} = \mathbf{0}.$$

Now the stability of an equilibrium point of an autonomous system,

$$\mathbf{x}' = \mathbf{f}(\mathbf{x})$$

can always be reduced to the consideration of the stability of $\mathbf{0}$ for an almost linear system. Here is why. If you are considering the equilibrium point, $\mathbf{a}$ for $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, you could define a new variable, $\mathbf{y}$ by

$$\mathbf{a} + \mathbf{y} = \mathbf{x}.$$

Then assymptotic stability would involve $|\mathbf{y}(t)| < \varepsilon$ and $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ while stability would only require $|\mathbf{y}(t)| < \varepsilon$. Then since $\mathbf{a}$ is an equilibrium point, $\mathbf{y}$ solves the following initial value problem.

$$\mathbf{y}' = \mathbf{f}(\mathbf{a} + \mathbf{y}) - \mathbf{f}(\mathbf{a}),\ \mathbf{y}(0) = \mathbf{y}_0,$$

where $\mathbf{y}_0 = \mathbf{x}_0 - \mathbf{a}$.

Let $A = D\mathbf{f}(\mathbf{a})$. Then from the definition of the derivative of a function,

$$\mathbf{y}' = A\mathbf{y} + \mathbf{g}(\mathbf{y}),\ \mathbf{y}(0) = \mathbf{y}_0 \tag{5.23}$$

where

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\mathbf{g}(\mathbf{y})}{|\mathbf{y}|} = \mathbf{0}.$$

Thus there is never any loss of generality in considering only the equilibrium point $\mathbf{0}$ for an almost linear system.[1] Therefore, from now on I will only consider the case of almost linear systems and the equilibrium point $\mathbf{0}$.

Theorem 5.3.5 *Consider the almost linear system of equations,*

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(\mathbf{x})$$

where

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\mathbf{g}(\mathbf{x})}{|\mathbf{x}|} = \mathbf{0}$$

and $\mathbf{g}$ *is a* C^1 *function. Suppose that for all* λ *an eigenvalue of* A, $\operatorname{Re}\lambda < 0$. *Then* $\mathbf{0}$ *is assymptotically stable.*

Proof: By Theorem 5.3.3 there exist constants $\delta > 0$ and K such that for $\Phi(t)$ the fundamental matrix for A,

$$|\Phi(t)\mathbf{x}| \leq Ke^{-\delta t}|\mathbf{x}|.$$

Let $\varepsilon > 0$ be given and let r be small enough that $Kr < \varepsilon$ and for $|\mathbf{x}| < (K+1)r, |\mathbf{g}(\mathbf{x})| < \eta|\mathbf{x}|$ where η is so small that $K\eta < \delta$, and let $|\mathbf{y}_0| < r$. Then by the variation of constants formula, the solution to **??**, at least for small t satisfies

$$\mathbf{y}(t) = \Phi(t)\mathbf{y}_0 + \int_0^t \Phi(t-s)\mathbf{g}(\mathbf{y}(s))\,ds.$$

[1] This is no longer true when you study partial differential equatioins as ordinary differential equationis in infinite dimensional spaces.

The following estimate holds.

$$\begin{aligned} |\mathbf{y}(t)| &\leq Ke^{-\delta t}|\mathbf{y}_0| + \int_0^t Ke^{-\delta(t-s)}\eta|\mathbf{y}(s)|\,ds \\ &< Ke^{-\delta t}r + \int_0^t Ke^{-\delta(t-s)}\eta|\mathbf{y}(s)|\,ds. \end{aligned}$$

Therefore,

$$e^{\delta t}|\mathbf{y}(t)| < Kr + \int_0^t K\eta e^{\delta s}|\mathbf{y}(s)|\,ds.$$

By Gronwall's inequality,

$$e^{\delta t}|\mathbf{y}(t)| < Kre^{K\eta t}$$

and so

$$|\mathbf{y}(t)| < Kre^{(K\eta-\delta)t} < \varepsilon e^{(K\eta-\delta)t}$$

Therefore, $|\mathbf{y}(t)| < Kr < \varepsilon$ for all t and so from Corollary 4.3.4, the solution to **??** exists for all $t \geq 0$ and since $K\eta - \delta < 0$,

$$\lim_{t\to\infty}|\mathbf{y}(t)| = 0.$$

This proves the theorem.

It can be proved that if the matrix, A has eigenvalues such that the real parts are either positive or negative and it also has some whose real parts are positive, then 0 is not stable for the almost linear system. In fact there exists a set containing 0 such that if the initial condition is not on that set, then the solution to the differential equation fails to stay in some open ball containing 0 but if the initial condition is on this set, then the solution does converge to 0. However, this requires more work to establish. When one of the eigenvalues has real part equal to 0 the situation is much more problematic and requires further analysis since many different things can happen in this case.

Chapter 6

Power Series Methods

6.1 Second Order Linear Equations

Second order linear equations are those which can be written in the following form.

$$y'' + p(x) y' + q(x) y = 0 \tag{6.1}$$

These are the equations for which power series methods are used the most. The equation can be considered as a first order system as follows.

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -p(x) & -q(x) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

and so if p, q are continuous on some interval, (c, d) containing a, there exists a unique solution to the initial value problem

$$\begin{aligned} \begin{pmatrix} y \\ z \end{pmatrix}' &= \begin{pmatrix} 0 & 1 \\ -p(x) & -q(x) \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \\ \begin{pmatrix} y(a) \\ z(a) \end{pmatrix} &= \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}. \end{aligned} \tag{6.2}$$

In terms of the original equation and the variable y, this is the following initial value problem.

$$\begin{aligned} y'' + p(x) y' + q(x) y &= 0 \\ y(a) &= y_0, \\ y'(a) &= y_1. \end{aligned} \tag{6.3}$$

Suppose y_1 and y_2 are two solutions to 6.3. Then

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}, \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

are two solutions to the differential equation 6.2 and so the matrix,

$$\Phi(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

either is invertible for all x or is not invertible for any x.

Now suppose y is a solution to the equation of 6.3 and suppose $\Phi(x)$ is invertible for all x. Let C_1, C_2 be the unique solution to the system of equations,

$$C_1 \begin{pmatrix} y_1(a) \\ y_1'(a) \end{pmatrix} + C_2 \begin{pmatrix} y_2(a) \\ y_2'(a) \end{pmatrix} = \begin{pmatrix} y(a) \\ y'(a) \end{pmatrix}.$$

Such a unique solution exists because by assumption,

$$\det(\Phi(a)) \neq 0.$$

Then if $z(x) = C_1 y_1(x) + C_2 y_2(x)$, it follows both y and z are solutions to the system

$$\begin{aligned} w'' + p(x) w' + q(x) w &= 0 \\ w(a) &= y(a) \\ w'(a) &= y'(a) \end{aligned}$$

and by uniqueness of this initial value problem, it follows $z = y$. Thus whenever the Wronskian

$$\det \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \neq 0$$

for some x, it follows every solution to the differential equation, 6.3 can be obtained in the form

$$C_1 y_1 + C_2 y_2$$

for some choice of C_1 and C_2. In this context, it is customary to refer to the Wronskian as

$$W(y_1, y_2) \equiv \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

It is convenient to observe that

$$W(y_1, y_2) = (y_1)^2 \left(\frac{y_2}{y_1}\right)' = y_2' y_1 - y_2 y_1' \tag{6.4}$$

and so, in checking whether all solutions to the equation of 6.3 can be obtained in the form

$$C_1 y_1 + C_2 y_2 \tag{6.5}$$

for suitable constants, C_1, C_2, all that is required is to observe whether the ratio, y_2/y_1 is a constant. If it is not a constant, then 6.4 implies the Wronskian must be non zero and so the general solution to 6.3 is indeed 6.5 which means that all solutions to 6.3 are obtained in the form 6.5 for suitable constants.

One of the problems often encountered is that of finding the general solution to 6.3 given one solution to 6.3. There is a very nice way to do this using Abel's formula and this is discussed next.

Suppose y_2 and y_1 are two solutions to 6.3. Then both of the following equations must hold.

$$y_2 y_1'' + p(x) y_2 y_1' + q(x) y_2 y_1 = 0,$$

and

$$y_1 y_2'' + p(x) y_1 y_2' + q(x) y_1 y_2 = 0.$$

Subtracting the first from the second yields

$$y_1 y_2'' - y_2 y_1'' + p(x)(y_1 y_2' - y_2 y_1') = 0.$$

Now the term, $y_1 y_2'' - y_2 y_1''$ equals $(y_1 y_2' - y_2 y_1')' = W(y_1, y_2)'$ and so this reduces to

$$W' + p(x) W = 0,$$

a first order linear equation for the Wronskian, W. Letting $P'(x) = p(x)$, the solution to this equation is

$$W(y_1, y_2)(x) = Ce^{-P(x)}.$$

Note this shows the Wronskian either vanishes identically ($C = 0$) or not at all ($C \neq 0$), as claimed earlier. This formula, called Abel's formula, can be used to find the general solution.

Theorem 6.1.1 *If y_1 solves the equation 6.3, then the general solution is given by 6.5 where y_2 is a solution to the differential equation,*

$$y_2' y_1 - y_2 y_1' = e^{-P(x)}. \tag{6.6}$$

Proof: We know from the theory of differential equations, in particular the fundamental existence and uniqueness theorem, that there does exist $\widetilde{y}_2$ such that 6.5 with $\widetilde{y}_2$ in place of y_2 is a general solution. Just choose initial conditions for y_2,

$$y_2(a), y_2'(a)$$

such that

$$\det \begin{pmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{pmatrix} \neq 0$$

and it will follow from the Wronskian alternative that

$$W(y_1, y_2) \neq 0.$$

By Abel's formula

$$\widetilde{y}_2' y_1 - \widetilde{y}_2 y_1' = Ce^{-P(x)}$$

where $C \neq 0$. Now define $y_2 \equiv C^{-1} \widetilde{y}_2$ and it follows 6.6 holds. This proves the theorem.

This theorem says that in order to find the general solution given one solution, y_1, it suffices to find a solution y_2 to the differential equation 6.6 and then the general solution will be given by 6.5.

6.2 Differential Equations Near An Ordinary Point

The problem is to find the solution to something like this.

$$y'' + p(x) y' + q(x) y = 0 \tag{6.7}$$
$$y(a) = y_0, y'(a) = y_1 \tag{6.8}$$

given $p(x)$ and $q(x)$ are analytic near the point a. The same techniques work for linear equations of any order however.

Definition 6.2.1 *A function f is analytic near a if for all $|x - a|$ small enough,*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - a)^k$$

That is, f is correctly given by a power series for x near a. The radius of convergence of a power series $\sum_{k=0}^{\infty} a_k (x - a)^k$ is a nonnegative number r such that if $|x - a| < r$, then the series converges.

Then there is a nice theorem which follows.

Theorem 6.2.2 *In 6.7 suppose p and q are analytic near a and that r is the minimum of the radii of convergence of these functions. Then the initial value problem 6.7 - 6.8 has a unique solution given by a power series*

$$y(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

which is valid at least for all $|x-a| < r$.

Example 6.2.3 *Where does there exist a power series solution to the initial value problem*

$$y'' + \frac{1}{x^2+1} y = 0, y(0) = 1, y'(0) = 0?$$

The answer is there exists a power series solution for all x satisfying

$$|x| < 1$$

This is because the function

$$\frac{1}{1+x^2}$$

has a power series,

$$\sum_{k=0}^{\infty} (-1)^k x^{2k}$$

which converges for all $|x| < 1$. An easy way to see this is to note that the function is undefined when $x = i$ and that i is the closest point to 0 such that the function is undefined. The distance between 0 and i is 1 and so the radius of convergence will be 1. You can also use the root test or ratio test to verify this.

Now here is a variation.

Example 6.2.4 *On what interval does there exist a power series solution to the initial value problem*

$$y'' + \frac{1}{x^2+1} y = 0, y(3) = 1, y'(3) = 0?$$

In this case, you observe that $1/\left(1+x^2\right)$ is zero at i or $-i$ and the distance between either of these points and 3 is $\sqrt{10}$. Therefore, the initial value problem will have a solution on the interval

$$|x-3| < \sqrt{10}$$

The radius of convergence is $\sqrt{10}$.

These examples illustrate why power series methods are really stupid. The initial value problem has a solution for all $x \in \mathbb{R}$. However, if you insist on using power series methods, you can only get the solution on a small interval because of something happening in the complex plane which you couldn't care less about.

The following example shows how you can find the first several terms of a power series solution.

Example 6.2.5 *Find several terms of the power series solution to*

$$y'' + \frac{1}{x^2+1} y = 0, y(3) = 1, y'(3) = 0?$$

The way this is usually done is to plug in a power series and compute the coefficients. I will illustrate another simple way to do the same thing which is adequate for finding several terms of the power series solution.

$$y(x) = \sum_{k=0}^{\infty} a_k (x-3)^k$$

From calculus you know

$$a_k = \frac{y^{(k)}(3)}{k!}$$

so all we have to do is find the derivatives of y. We can do this from the equation itself along with the initial conditions. Thus

$$\begin{aligned} a_0 &= y(3) = 1 \\ a_1 &= y'(3) = 0 \end{aligned}$$

How do you find $y''(3)$? Use the equation.

$$y''(3) = -\frac{1}{1+3^2} y(3) = -\frac{1}{10}$$

and so

$$a_2 = \left(-\frac{1}{10}\right)/2 = \frac{-1}{20}$$

Now lets find $y'''(3)$. From the equation,

$$y'''(x) + y'(x)\left(\frac{1}{1+x^2}\right) + y(x)\left(-\frac{2}{(1+x^2)^2}x\right) = 0$$

Plug in $x = 3$. This gives

$$\begin{aligned} y'''(3) + y'(3)\left(\frac{1}{10}\right) + y(3)\left(-\frac{2}{100}3\right) &= 0 \\ y'''(3) + 0\left(\frac{1}{10}\right) + 1\left(-\frac{2}{100}3\right) &= 0 \end{aligned}$$

and so

$$y'''(3) = \frac{6}{100} = \frac{3}{50}$$

Therefore since $3! = 6$,

$$a_3 = \left(\frac{3}{50}\right)/6 = \frac{1}{100}$$

The first four terms of the power series solution for this equation are

$$y(x) = 1 - \frac{1}{20}(x-3)^2 + \frac{1}{100}(x-3)^3 + \cdots$$

You could keep on finding terms for this series but you know by the above theorem that the convergence takes place if $|x-3| < \sqrt{10}$. Note I didn't find a general formula for the power series. It was too much trouble. There is one and it is easy to find as many terms as needed. Furthermore, in applications this is often all that is needed anyway because after all, the series only represents the solution to the differential equation for x pretty close to 3.

Example 6.2.6 *Do the above example another way by plugging in the power series to the equation.*

Now lets do it another way by plugging in a power series. If you do this, you need to use the power series for the function $1/\left(1+x^2\right)$ expanded about 3. After doing some fussy work you find

$$\begin{aligned} 1/\left(1+x^2\right) &= \frac{1}{10}-\frac{3}{50}(x-3)+\frac{13}{500}(x-3)^2 \\ &\quad -\frac{6}{625}(x-3)^3+\frac{79}{25\,000}(x-3)^4+O\left((x-3)^5\right) \end{aligned}$$

I will just find the first several terms by matching coefficients.

$$\begin{aligned} y &= \sum_{k=0}^{\infty} a_k (x-3)^k, y' = \sum_{k=1}^{\infty} a_k k (x-3)^{k-1}, \\ y'' &= \sum_{k=2}^{\infty} a_k k (k-1)(x-3)^{k-2} \end{aligned}$$

Plugging in the power series, we get

$$\sum_{k=2}^{\infty} a_k k(k-1)(x-3)^{k-2} + \left(\frac{1}{10}-\frac{3}{50}(x-3)+\frac{13}{500}(x-3)^2-\frac{6}{625}(x-3)^3\right.$$
$$\left.+\frac{79}{25\,000}(x-3)^4\right)\sum_{k=0}^{\infty} a_k (x-3)^k = 0$$

Now we match the terms. First, the constant tems. Recall $a_1 = 0$ and $a_0 = 1$.

$$2a_2 + \frac{1}{10}a_0 = 0$$

so $a_2 = -1/20$. Next consider the first order terms.

$$a_3 6 + \frac{1}{10}a_1 + \left(\frac{-3}{50}\right)a_0 = 0$$

and so

$$a_3 = \frac{3}{6\times 50} = \frac{1}{100}$$

Next you could consider the second order terms but I am sick of doing this. Therefore, we get up to first order terms the following expression for the power series.

$$y = 1 - \frac{1}{20}(x-3)^2 + \frac{1}{100}(x-3)^3$$

Note that in this simple problem it would be better to consider the equation in the form

$$\left(1+x^2\right)y'' + y = 0,\ y(0) = 1, y'(0) = 0.$$

To solve it in this form you write $1+x^2$ as a power series

$$1+x^2 = 10 + 6(x-3) + (x-3)^2$$

Then you plug in the series for y and y'' as before.

$$\left(10+6(x-3)+(x-3)^2\right)\sum_{k=2}^{\infty}a_k k(k-1)(x-3)^{k-2}+\sum_{k=0}^{\infty}a_k(x-3)^k=0$$

and then match powers of $(x-3)$. This problem is actually easy enough you could get a nice recurrance relation.

The reason the power series method works for solving ordinary differential equations is dependent on the fact that a power series can be differentiated term by term on its radius of convergence. This is a theorem in calculus which is seldom proved these days.

Example 6.2.7 *The Legendre equation is*

$$\left(1-x^2\right)y''-2xy'+\alpha(\alpha+1)y=0$$

Describe the radius of convergence of solutions for which the initial data is given at $x=0$*. Repeat the problem if the initial data is given at* $x=5$*.*

You divide by $\left(1-x^2\right)$ and note that the nearest "singular" point of

$$2x/\left(1-x^2\right)$$

occurs when $x=1$ or -1 and so the power series solutions to this equation with initial data given at $x=0$ converge on $(-1,1)$. In the case where the initial data is given at 5, the series solution will converge for $|x-5|<4$. In other words, on the interval $(1,9)$. Actually, in this example, you sometimes get polynomials for the solution and so the power series converges on all of $\mathbb{R}$.

Example 6.2.8 *Determine a lower bound for the radius of convergence of series solutions of the differential equation*

$$\left(1+x^2\right)y''+2xy'+4x^2y=0$$

about the point 0 and -7 and give the intervals on which the solutions will be sure to converge.

Divide by $1+x^2$. Then you have two functions

$$\frac{2x}{1+x^2},\frac{4x^2}{1+x^2}$$

and the power series centered at 0 for these functions converge converge if $|x|<1$ so when the initial data is given at 0 the power series converge on $(-1,1)$. Now consider the case where the initial condition is given at -7. In this case you need

$$|x-(-7)|<\sqrt{7^2+1}=\sqrt{50}$$

and so the series will converge on the interval

$$\left(-7-\sqrt{50},-7+\sqrt{50}\right)$$

Perhaps this is not the first thing you would think of.

Example 6.2.9 *Where will power series solutions to the equation*

$$y''+\sin(x)y'+\left(1+x^2\right)y=0$$

converge?

These will converge for all values of x because the power series for $\sin(x)$ converges for all x and so does the power series for $\left(1+x^2\right)$ and this is true no matter where the initial data is given.

Could you find the first several terms of a power series solution for the initial value problem

$$\begin{gathered} y''+\sin(x)\,y'+\left(1+x^2\right)y=0 \\ y(0)=1 \\ y'(0)=1 \end{gathered}$$

Yes you can and a little computing using the above technique will show that this solution is

$$y(x)=1+x-\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{24}x^4+O\left(x^5\right)$$

The expression $O\left(x^5\right)$ means that it is a sum of terms each of which have x^m in them where $m\geq 5$.

6.3 The Euler Equations

The next step is to progress from ordinary points to regular singular points. The simplest equation to illustrate the concept of a regular singular point is the so called Euler equation, sometimes called a Cauchy Euler equation.

Definition 6.3.1 *A differential equation is called an Euler equation if it can be written in the form*

$$x^2y''+axy'+by=0.$$

Solving a Cauchy Euler equation is really easy. You look for a solution in terms $y=x^r$ and try to choose r in such a way that it solves the equation. Plugging this in to the above equation,

$$x^2r(r-1)x^{r-2}+xarx^{r-1}+bx^r=0$$

This reduces to

$$x^r(r(r-1)+ar+b)=0$$

and so you have to solve the equation

$$r(r-1)+ar+b=0$$

to find the values of r. If these values of r are different, say $r_1\neq r_2$ then the general solution must be

$$C_1x^{r_1}+C_2x^{r_2}$$

because the Wronskian of the two functions will be nonzero. I know this because the ratio of the two functions is not a constant so the derivative of their ratio is not 0. However, by the quotient rule, the numerator is ± 1 times the Wronskian.

Example 6.3.2 *Find the general solution to $x^2y''-2xy'+2y=0$.*

You plug in x^r and look for r. Then as above this yields

$$r(r-1)-2r+2=r^2-3r+2=0$$

and so the two values of r are $1,2$. Therefore, the general solution to this equation is

$$C_1x+C_2x^2.$$

Of course there are three cases for solutions to the so called indicial equation

$$r(r-1)+ar+b=0$$

Either the zeros are distinct and real, distinct and complex or repeated. Consider the case where they are distinct and complex next.

Example 6.3.3 *Find the general solution to* $x^2y''+3xy'+2y=0$.

This time you have

$$r^2+2r+2=0$$

and the solutions are $r=-1\pm i$. How do we interpret

$$x^{-1+i}, x^{-1-i}?$$

It is real easy.

$$x^{-1+i}=e^{\ln(x)(-1+i)}=e^{-\ln(x)+i\ln(x)}$$

and by Euler's formula this equals

$$\begin{aligned} x^{-1+i} &= e^{\ln\left(x^{-1}\right)}\left(\cos\left(\ln\left(x\right)\right)+i\sin\left(\ln\left(x\right)\right)\right) \\ &= \frac{1}{x}\left(\cos\left(\ln\left(x\right)\right)+i\sin\left(\ln\left(x\right)\right)\right) \end{aligned}$$

Corresponding to x^{-1-i} we get something similar.

$$x^{-1-i}=\frac{1}{x}\left(\left(\cos\left(\ln\left(x\right)\right)-i\sin\left(\ln\left(x\right)\right)\right)\right)$$

Adding these together and dividing by 2 to get the real part, the principle of superposition implies

$$\frac{1}{x}\cos\left(\ln\left(x\right)\right)$$

is a solution. Then subtacting them and dividing by $2i$ you get

$$\frac{1}{x}\sin\left(\ln\left(x\right)\right)$$

is a solution. Hence anything of the form

$$C_1\frac{1}{x}\cos\left(\ln\left(x\right)\right)+C_2\frac{1}{x}\sin\left(\ln\left(x\right)\right)$$

is a solution. Is this the general solution? Of course. This follows because the ratio of the two functions is not constant and this implies their Wronskian is nonzero.

In the general case, suppose the solutions of the indicial equation

$$r(r-1)+ar+b=0$$

are $\alpha\pm i\beta$. Then the general solution for $x>0$ is

$$C_1x^{\alpha}\cos\left(\beta\ln\left(x\right)\right)+C_2x^{\alpha}\sin\left(\beta\ln\left(x\right)\right)$$

Finally consider the case where the zeros of the indicial equation are real and repeated. Note I have included all cases because since the coefficients of this equation are real, the zeros come in conjugate pairs if they are not real. Suppose then that x^r is a solution of

$$x^2y''+axy'+by=0$$

Then if $z(x)$ is another solution which is not a multiple of x^r, you would have the following by Theorem 6.1.1.

$$\begin{vmatrix} x^r & z \\ rx^{r-1} & z' \end{vmatrix} = e^{-a\ln(x)} = x^{-a}$$

and so

$$z'x^r - zrx^{r-1} = x^{-a}$$

and so

$$z' - \frac{1}{x}rz = x^{-a-r}$$

Then doing the usual thing for first order linear equations,

$$\frac{d}{dx}\left(x^{-r}z\right) = x^{-a-r}x^{-r} = x^{-a-2r}$$

and so

$$\left(x^{-r}z\right) = \frac{x^{-a-2r+1}}{-a-2r+1}$$

and so z is a multiple of x^{r_2} for some $r_2 \neq r$, which is assumed not to happen, unless $a + 2r = 1$ which must therefore, be the case. In this case, you get

$$x^{-r}z = \ln(x) + C$$

and so another solution is

$$z = x^r \ln(x)$$

Example 6.3.4 *Find the general solution of the equation*

$$x^2y'' + 3xy' + y = 0.$$

In this case the indicial equation is

$$r(r-1) + 3r + 1 = r^2 + 2r + 1 = 0$$

and there is a repeated zero, $r = -1$. Therefore, the general solution is

$$y = C_1x^{-1} + C_2\ln(x)x^{-1}.$$

This is pretty easy isn't it?

How would things be different if the equation was of the form

$$(x-a)^2y'' + a(x-a)y' + by = 0?$$

The answer is that is wouldn't be any different. You could just define a new independent variable $t \equiv (x-a)$ and then the equation in terms of t becomes

$$t^2z'' + atz + bz = 0$$

where $z(t) \equiv y(x) = y(t+a)$. You can always reduce these sorts of equations to the case where the singular point is at 0. However, you might not want to do this. If not, you look for a solution in the form $y = (x-a)^r$, plug in and determine the correct value of r. In the case of real and distinct zeros you get

$$y = C_1(x-a)^{r_1} + C_2(x-a)^{r_2}$$

In the case where $r = \alpha \pm i\beta$ you get

$$y = C_1(x-a)^\alpha\cos(\beta\ln(x-a)) + C_2(x-a)^\alpha\sin(\beta\ln(x-a))$$

for the general solution for $x > a$

In the case where r is a repeated zero, you get

$$y = C_1(x-a)^r + C_2\ln(x-a)(x-a)^r.$$

6.4 Some Simple Observations On Power Series

This section is a review of a few facts about power series which should have been learned in calculus.

Theorem 6.4.1 *Suppose $f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$ for x near a and suppose $a_0 \neq 0$. Then*

$$f(x)^{-1} = \frac{1}{a_0} + h(x)$$

where $h(x) = \sum_{n=1}^{\infty} b_n (x-a)^n$ so $h(a) = 0$.

Proof: From theorems in mathematics (complex variable), we know $f(x)^{-1}$ has a power series representation near a. Therefore, the result follows from the observation that for $g(x) \equiv f(x)^{-1}$, $g(0) = \frac{1}{a_0}$.

Theorem 6.4.2 *Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ for x near 0. Then $f(x) g(x)$ also has a power series near 0 and in fact,*

$$f(x) g(x) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k \right) x^n.$$

Proof:

$$f(x) g(x) = \sum_{n=0}^{\infty} a_n x^n \sum_{k=0}^{\infty} b_k x^k$$

$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} a_n x^n \right) b_k x^k = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_{n-k} x^{n-k} \right) b_k x^k.$$

From mathematics, we know the power series of a function converges absolutely on its interval of convergence. Therefore from another very significant theorem in mathematics, we can interchange the order of summation in the last sum to write

$$f(x) g(x) = \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} a_{n-k} x^{n-k} \right) b_k x^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n-k} x^{n-k} b_k x^k$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_{n-k} b_k \right) x^n$$

and this proves all the banal aspects of the theorem.

6.5 Regular Singular Points

First of all, here is the definition of what a regular singular point is.

Definition 6.5.1 *A differential equation has a regular singular point at 0 if the equation can be written in the form*

$$x^2 y'' + x b(x) y' + c(x) y = 0 \tag{6.9}$$

where

$$b(x) = \sum_{n=0}^{\infty} b_n x^n, \ \sum_{n=0}^{\infty} c_n x^n = c(x)$$

for all x near 0. More generally, a differential equation

$$P(x) y'' + Q(x) y' + R(x) y = 0 \tag{6.10}$$

where P, Q, R are analytic near a has a regular singular point at a if it can be written in the form

$$(x-a)^2 y'' + (x-a) b(x) y' + c(x) y = 0 \tag{6.11}$$

where

$$b(x) = \sum_{n=0}^{\infty} b_n (x-a)^n, \ \sum_{n=0}^{\infty} c_n (x-a)^n = c(x)$$

for all $|x-a|$ small enough. The equation 6.10 has a singular point at a if $P(a) = 0$.

The following table emphasizes the similarities between the Euler equations and the regular singular point equations. I have featured the point 0. If you are interested in another point a, you just replace x with $x - a$ everywhere it occurs.

	Euler Equation	Regular Singular Point
Form of equation	$x^2 y'' + x b_0 y' + c_0 y = 0$	$x^2 y'' + x(b_0 + b_1 x + \cdots) y' + (c_0 + c_1 x + \cdots) y = 0$
Indicial Equation	$r(r-1) + b_0 r + c_0 = 0$	$r(r-1) + b_0 r + c_0 = 0$
One solution	$y = x^r$	$y = x^r \sum_{k=0}^{\infty} a_k x^k, \ a_0 = 1.$

Recognizing regular singular points

How do you know a singular differential equation can be written a certain way? In particular, how can you recognize a regular singular point when you see one? Suppose

$$P(x) y'' + Q(x) y' + R(x) y = 0$$

where all of P, Q, R are analytic functions near a. How can you tell if it has a regular singular point at a? Here is how. It has a regular singular point at a if

$$\lim_{x \to a} (x-a) \frac{Q(x)}{P(x)} \text{ exists}$$

$$\lim_{x \to a} (x-a)^2 \frac{R(x)}{P(x)} \text{ exists}$$

If these conditions hold, then by theorems in complex analysis it will be the case that

$$(x-a) \frac{Q(x)}{P(x)} = \sum_{n=0}^{\infty} b_n (x-a)^n,$$

and

$$(x-a)^2 \frac{R(x)}{P(x)} = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for x near a. Indeed, equations of this form reduce to the form in 6.11 upon dividing by $P(x)$ and multiplying by $(x-a)^2$.

Example 6.5.2 *Find the regular singular points of the equation and find the singular points.*

$$x^3 (x-2)^2 (x-1)^2 y'' + (x-2) \sin(x) y' + (1+x) y = 0$$

The singular points are $0, 2, 1$. Lets consider 0 first.

$$\lim_{x \to 0} x \frac{(x-2)\sin(x)}{x^3 (x-2)^2 (x-1)^2}$$

does not exist. Therefore, 0 is not a regular singular point. I don't have to check any further. Now consider the singular point 2.

$$\lim_{x \to 2} (x-2) \frac{(x-2)\sin(x)}{x^3 (x-2)^2 (x-1)^2} = \frac{1}{8}\sin 2$$

and

$$\lim_{x \to 2} (x-2)^2 \frac{1+x}{x^3 (x-2)^2 (x-1)^2} = \frac{3}{8}$$

and so yes, 2 is a regular singular point. Now consider 1.

$$\lim_{x \to 1} (x-1) \frac{(x-2)\sin(x)}{x^3 (x-2)^2 (x-1)^2}$$

does not exist so 1 is not a regular singular point. Thus the above equation has only one regular singular point and this is where $x = 2$.

Example 6.5.3 *Find the regular singular points of*

$$x\sin(x)\,y'' + 3\tan(x)\,y' + 2y = 0$$

The singular points are $0, n\pi$ where n is an integer. Lets consider a point at $n\pi$ where $n \neq 0$. To be specific, lets let $n = 3$

$$\lim_{x \to 3\pi} (x - 3\pi) \frac{3\tan(x)}{x\sin(x)} = 0$$

Similarly the limit exists for other values of n. Now consider

$$\lim_{x \to 3\pi} (x - 3\pi)^2 \frac{2}{x\sin(x)} = 0$$

Similarly the limit exists for other values of n. What about 0?

$$\lim_{x \to 0} x \frac{3\tan(x)}{x\sin(x)} = 3$$

and

$$\lim_{x \to 0} x^2 \frac{2}{x\sin(x)} = 2$$

so it appears all these singular points are regular singular points.

Example 6.5.4 *Find the regular singular points of*

$$x^2\sin(x)\,y'' + 3\tan(x)\,y' + 2y = 0$$

Lets look at $x = 0$ first. The equation has the same singular points.

$$\lim_{x \to 0} x \frac{3\tan(x)}{x^2\sin(x)} = \text{undefined}$$

so 0 is not a regular singular point.

$$\lim_{x \to 3\pi} (x - 3\pi) \frac{3 \tan(x)}{x^2 \sin(x)} = 0$$

and the situation is similar for other singular points $n\pi$. Also

$$\lim_{x \to 3\pi} (x - 3\pi)^2 \frac{2}{x^2 \sin(x)} = 0$$

with similar result for arbitrary $n\pi$ where $n \neq 0$. Thus in this case 0 is not a regular singular point but $n\pi$ is a regular singular point for all integers $n \neq 0$.

In general, if you have an equation which has a regular singular point at a so that the equation can be massaged to give something of the form

$$(x - a)^2 y'' + (x - a) b(x) y' + c(x) y = 0$$

you could always define a new variable $t \equiv (x - a)$ and letting $z(t) = y(x)$, you could rewrite the equation in terms of t in the form

$$t^2 z'' + tb(a + t) z' + c(a + t) z = 0$$

and thereby reduce to the case where the regular singular point is at 0. Thus there is no loss of generality in concentrating on the case where the regular singular point is at 0. In addition, the most important examples are like this. Therefore, from now on, I will consider this case. This just means you have all the series in terms of powers of x rather than the more general powers of $x - a$.

6.6 Finding The Solution

Suppose you have reduced the equation to

$$x^2 y'' + xp(x) y' + q(x) y = 0 \tag{6.12}$$

where each of p, q is analytic near 0. Then letting

$$p(x) = b_0 + b_1 x + \cdots$$

$$q(x) = c_0 + c_1 x + \cdots$$

you see that for small x the equation should be approximately equal to

$$x^2 y'' + xb_0 y' + c_0 y = 0$$

which is an Euler equation. This would have a solution in the form x^r where

$$r(r - 1) + b_0 r + c_0 = 0,$$

the indicial equation for the Euler equation, and so it is not unreasonable to look for a solution to the equation in 6.12 which is of the form

$$x^r \sum_{k=0}^{\infty} a_k x^k, \; a_0 \neq 0.$$

You perturb the coefficients of the Euler equation to get 6.12 and so it is not unreasonable to think you should look for a solution to 6.12 of the above form.

Example 6.6.1 *Find the general solution to the equation*

$$x^2y'' + x\left(1 + x^2\right)y' - 2y = 0.$$

The associated Euler equation is of the form

$$x^2y'' + xy' - 2y = 0$$

and so the indicial equation is

$$r(r-1) + r - 2 = 0 \tag{6.13}$$

so $r = \sqrt{2}, r = -\sqrt{2}$. Then you would look for a solution in the form

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}$$

where $r = \pm\sqrt{2}$. Plug in to the equation.

$$x^2 \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2} + x\left(1+x^2\right) \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

$$-2\sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

This simplifies to

$$\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} a_k (k+r) x^{k+r} + \sum_{k=0}^{\infty} a_k (k+r) x^{k+r+2} \tag{6.14}$$

$$-2\sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

The lowest order term is the x^r term and it yields

$$a_0(r)(r-1) + a_0(r) - 2a_0 = 0$$

but this is just $a_0(r(r-1) + r - 2) = 0$. Since r is one of the zeros of 6.13, there is no restriction on the choice of a_0. In fact, as discussed below, this lack of a requirement on a_0 is equivalent to finding the right value of r. Next consider the x^{r+1} terms. There are no such terms in the third of the above sums just as there were no x^r terms in this sum. Then

$$a_1((1+r)(r) + (1+r) - 2) = 0$$

Now if r solves 6.13 then $1 + r$ does not do so because the two solutions to this equation do not differ by an integer. Therefore, the above equation requires $a_1 = 0$. At this point we can give a recurrence relation for the other a_k. To do this, change the variable of summation in the third sum of 6.14 to obtain

$$\sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} a_k (k+r) x^{k+r} + \sum_{k=2}^{\infty} a_{k-2} (k-2+r) x^{k+r}$$

$$-2\sum_{k=0}^{\infty} a_k x^{k+r} = 0$$

Thus for $k \geq 2$,

$$a_k\left[(k+r)(k+r-1)+(k+r)-2\right]+a_{k-2}(k-2+r)=0$$

Hence for $k \geq 2$,

$$\begin{aligned} a_k &= \frac{-a_{k-2}(k-2+r)}{[(k+r)(k+r-1)+(k+r)-2]} \\ &= \frac{-a_{k-2}(k-2+r)}{[(k+r)(k+r-1)+(k+r)-2]} \end{aligned}$$

and we take $a_0 \neq 0$ while $a_1 = 0$. Now lets find the first several terms of two independent solutions, one for $r = \sqrt{2}$ and the other for $r = -\sqrt{2}$. Let $a_0 = 1$ for simplicity. Then the above recurrence relation shows that since $a_1 = 0$ all the odd terms equal 0. Also

$$a_2 = \frac{-r}{[(2+r)(2+r-1)+(2+r)-2]} = -\frac{r}{[(2+r)(1+r)+r]}$$

while

$$a_4 = \frac{-\left(-\frac{r}{[(2+r)(1+r)+r]}\right)(4-2+r)}{[(4+r)(4+r-1)+(4+r)-2]} = \frac{r}{[2+4r+r^2]}\frac{2+r}{[14+8r+r^2]}$$

Continuing this way, you can get as many terms as you want. Now lets put in the two values of r to obtain the beginning of the two solutions. First let $r = \sqrt{2}$

$$y_1(x) = x^{\sqrt{2}}\left(1+\left(-\frac{\sqrt{2}}{\left[(2+\sqrt{2})\left(\sqrt{2}+1\right)+\sqrt{2}\right]}\right)x^2 + \right.$$
$$\left. + \left(\frac{\sqrt{2}}{[4+4\sqrt{2}]}\frac{2+\sqrt{2}}{[16+8\sqrt{2}]}\right)x^4 \cdots\right)$$

the solution which corresponds to $r = -\sqrt{2}$ is

$$y_2(x) = x^{-\sqrt{2}}\left(1+\left(\frac{\sqrt{2}}{\left[(2-\sqrt{2})(1-\sqrt{2})-\sqrt{2}\right]}\right)x^2 + \right.$$
$$\left. \sqrt{2}\frac{-2+\sqrt{2}}{[4-4\sqrt{2}][16-8\sqrt{2}]}x^4+\cdots\right)$$

Then the general solution is

$$C_1y_1 + C_2y_2$$

and this is valid for $x > 0$.

Generalities

For an equation having a regular singular point at 0, one looks for solutions in the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n} \tag{6.15}$$

where r is a constant which is to be determined in such a way that $a_0 \neq 0$. It turns out that such equations **always** have such solutions although solutions of this sort are not always enough to obtain the general solution to the equation. The constant r is called the exponent of the singularity because the solution is of the form

$$x^r a_0 + \text{ higher order terms.}$$

Thus the behavior of the solution to the equation given above is like x^r for x near the singularity, 0.

If you require that 6.15 solves 6.12 and plug in, you obtain using Theorem 6.4.2

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{n+r} +$$

$$+\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k (k+r) b_{n-k} \right) x^{n+r}$$

$$+\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} c_{n-k} a_k \right) x^{n+r} = 0. \tag{6.16}$$

Since $a_0 \neq 0$,

$$p(r) \equiv r(r-1) + b_0 r + c_0 = 0 \tag{6.17}$$

and this is called the indicial equation. (Note it is the indicial equation for the Euler equation which comes from deleting all the nonconstant terms in the power series for $p(x)$ and $q(x)$.) Also the following equation must hold for $n = 1, \cdots$.

$$p(n+r) a_n = -\sum_{k=0}^{n-1} a_k (k+r) b_{n-k}$$

$$-\sum_{k=0}^{n-1} c_{n-k} a_k \equiv f_n(a_i, b_i, c_i) \tag{6.18}$$

These equations are all obtained by setting the coefficient of x^{n+r} equal to 0.

There are various cases depending on the nature of the solutions to this indicial equation. I will always assume the zeros are real, but will consider the case when the zeros are distinct and do not differ by an integer and the case when the zeros differ by a non negative integer.

It turns out that the nature of the problem changes according to which of these cases holds. You can see why this is the case by looking at the equations 6.17 and 6.18. If r_1, r_2 solve 6.17 and $r_1 - r_2 \neq$ an integer, then with r in equation 6.18 replaced by either r_1 or r_2, for $n = 1, \cdots$, $p(n+r) \neq 0$ and so there is a unique solution to 6.18 for each $n \geq 1$ once $a_0 \neq 0$ has been chosen. Therefore, in this case that $r_1 - r_2 \neq$ an integer, equation 6.9 has a general solution in the form

$$C_1 \sum_{n=0}^{\infty} a_n x^{n+r_1} + C_2 \sum_{n=0}^{\infty} b_n x^{n+r_2}, a_0, b_0 \neq 0.$$

It is obvious this is the general solution because the ratio of the two solutions is non constant.

On the other hand, if $r_1 - r_2 =$ an integer, then there exists a unique solution to 6.18 for each $n \geq 1$ if r is replaced by the larger of the two zeros, r_1. Therefore, in this case there is always a solution of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1},\ a_0 = 1, \tag{6.19}$$

but you might very well hit a snag when you attempt to find a solution of this form with r_1 replaced with the smaller of the two zeros r_2 due to the possibility that for some $m \geq 1$, $p(m + r_2) = p(r_1) = 0$ without the right side of 6.18 vanishing. In the case when both zeros are equal, there is only one solution of the form in 6.19 since there is always a unique solution to 6.18 for $n \geq 1$. Therefore, in the case when $r_1 - r_2 =$ a non negative integer either 0 or some positive integer, you must consider other solutions. I will use Abel's formula to find the second solution. The equation solved by these two solutions is

$$x^2 y'' + xp(x) y' + q(x) y = 0$$

and dividing by x^2 to place in the right form for using Abel's formula,

$$y'' + \frac{1}{x} p(x) y' + \frac{1}{x^2} q(x) y = 0$$

Thus letting y_1 be the solution of the form in 6.19, and y_2 another solution,

$$y_2' y_1 - y_2 y_1' = e^{-P(x)} \tag{6.20}$$

where $P(x) \in \int x^{-1} p(x)\, dx$. Thus

$$\begin{aligned} P(x) &\in \int \left(\frac{b_0}{x} + b_1 + b_2 x + \cdots \right) dx \\ &= b_0 \ln x + b_1 x + b_2 x^2/2 + \cdots \end{aligned}$$

and so

$$-P(x) = \ln x^{-b_0} + k(x)$$

for $k(x)$ some analytic function, $k(0) = 0$. Therefore,

$$e^{-P(x)} = e^{\ln\left(x^{-b_0}\right) + k(x)} = x^{-b_0} g(x)$$

for $g(x)$ some analytic function, $g(0) = 1$. Therefore, 6.20 is of the form

$$y_2' y_1 - y_2 y_1' = x^{-b_0} g(x),\ g(0) = 1. \tag{6.21}$$

Now consider the zeros to the indicial equation,

$$r(r-1) + b_0 r + c_0 = r^2 - r + b_0 r + c_0 = 0.$$

It is given that $r_1 = r_2 + m$ where m is a non negative integer. Thus the left side of the above equals

$$(r - r_2)(r - r_2 - m) = r^2 - 2rr_2 - rm + r_2^2 + r_2 m$$

and so

$$-2r_2 - m = b_0 - 1$$

which implies

$$r_2 = \frac{1 - b_0}{2} - \frac{m}{2}$$

and hence

$$r_1 = r_2 + m = \frac{1 - b_0}{2} + \frac{m}{2}$$

Therefore,

$$y_1(x) = x^{\frac{1 - b_0 + m}{2}} \sum_{n=0}^{\infty} a_n x^n,\ a_0 = 1 \tag{6.22}$$

The left side of 6.21 equals

$$y_1^2\left(\frac{y_2}{y_1}\right)'$$

and so this equation reduces to

$$y_1^2\left(\frac{y_2}{y_1}\right)' = x^{-b_0}g(x)\,,\ g(0)=1.$$

Now from Theorem 6.4.1 and looking at 6.22 $y_1(x)^{-2}$ is of the form

$$\frac{1}{x^{1-b_0+m}\left(\sum_{n=0}^{\infty}a_n x^n\right)^2} = x^{b_0-1-m}(1+h(x))$$

where $h(x)$ is analytic, $h(0)=0$.

$$\begin{aligned}\left(\frac{y_2}{y_1}\right)' &= x^{-b_0}x^{b_0-1-m}(1+h(x))\,g(x)\\ &= x^{-1-m}\left(1+\sum_{n=1}^{\infty}A_n x^n\right)\\ &= x^{-1-m}+\sum_{n=1}^{\infty}A_n x^{n-1-m}. \qquad (6.23)\end{aligned}$$

Now suppose that $m>0$. Then,

$$\frac{y_2}{y_1} = \frac{-x^{-m}}{m}+\sum_{n=1}^{m-1}A_n\frac{x^{n-m}}{n-m}$$

$$+A_m\ln(x)+\sum_{n=m+1}^{\infty}A_n\frac{x^{n-m}}{n-m}.$$

It follows

$$y_2 = A_m\ln(x)\,y_1+$$

$$x^{-m}\left(\frac{-1}{m}+\sum_{n=1}^{\infty}B_n x^n\right)\overbrace{x^{r_1}\sum_{n=0}^{\infty}a_n x^n}^{y_1}.$$

Where $B_n=\frac{A_n}{n-m}$ for $n\neq m$. Therefore, y_2 has the following form.

$$y_2 = A_m\ln(x)\,y_1 + x^{r_2}\sum_{n=0}^{\infty}C_n x^n.$$

If $m=0$ so there is a repeated zero to the indicial equation then 6.23 implies

$$\frac{y_2}{y_1} = \ln x+\sum_{n=1}^{\infty}\frac{A_n}{n}x^n + A_0$$

where A_0 is a constant of integration. Thus, the second solution is of the form

$$y_2 = \ln(x)\,y_1 + x^{r_2}\sum_{n=0}^{\infty}C_n x^n.$$

The following theorem summarizes the above discussion.

Theorem 6.6.2 *Let 6.9 be an equation with a regular singular point and let r_1 and r_2 be real solutions of the indicial equation, 6.17 with $r_1 \geq r_2$. Then if $r_1 - r_2$ is not equal to an integer, the general solution 6.9 may be written in the form :*

$$C_1 \sum_{n=0}^{\infty} a_n x^{n+r_1} + C_2 \sum_{n=0}^{\infty} b_n x^{n+r_2}$$

where we can have $a_0 = 1$ and $b_0 = 1$. If $r_1 = r_2 = r$ then the general solution of 6.9 may be obtained in the form

$$C_1 \overbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}^{y_1} + C_2 \left(\ln(x) \overbrace{\sum_{n=0}^{\infty} a_n x^{n+r}}^{y_1} + \sum_{n=0}^{\infty} C_n x^{n+r} \right)$$

where we may take $a_0 = 1$. If $r_1 - r_2 = m$, a positive integer, then the general solution to 6.9 may be written as

$$C_1 \overbrace{\left(\sum_{n=0}^{\infty} a_n x^{n+r_1} \right)}^{y_1} +$$

$$C_2 \left(k \ln(x) \overbrace{\left(\sum_{n=0}^{\infty} a_n x^{n+r_1} \right)}^{y_1} + x^{r_2} \sum_{n=0}^{\infty} C_n x^n \right),$$

where k may or may not equal zero and we may take $a_0 = 1$.

This theorem indicates what one should look for in the various cases.

Bibliography

[1] Coddington and Levinson, *Theory of Ordinary Differential Equations* McGraw Hill 1955.