Contents

1. Review Of Advanced Calculus 23

1. Set Theory 25
   1.1 Basic Definitions ............................................... 25
   1.2 The Schroder Bernstein Theorem ............................. 28
   1.3 Equivalence Relations ...................................... 31
   1.4 Partially Ordered Sets ................................... 32

2. Continuous Functions Of One Variable 33
   2.1 Exercises .................................................. 34
   2.2 Theorems About Continuous Functions .................... 35

3. The Riemann Stieltjes Integral 41
   3.1 Upper And Lower Riemann Stieltjes Sums ................. 41
   3.2 Exercises .................................................. 45
   3.3 Functions Of Riemann Integrable Functions .............. 46
   3.4 Properties Of The Integral ................................ 49
   3.5 Fundamental Theorem Of Calculus ......................... 53
   3.6 Exercises .................................................. 57

4. Some Important Linear Algebra 59
   4.1 Algebra In \(\mathbb{F}^n\) ........................................ 61
   4.2 Exercises .................................................. 62
   4.3 The Inner Product And Distance In \(\mathbb{C}^n\) ............ 63
   4.4 Subspaces Spanned And Bases ............................... 65
   4.5 An Application To Matrices ................................. 70
   4.6 The Mathematical Theory Of Determinants ................ 71
      4.6.1 The Function \(\text{sgn}\) .................................. 71
   4.7 The Determinant ........................................... 74
      4.7.1 The Definition ....................................... 74
      4.7.2 Permuting Rows Or Columns ............................ 74
      4.7.3 A Symmetric Definition .............................. 76
      4.7.4 The Alternating Property Of The Determinant ...... 76
      4.7.5 Linear Combinations And Determinants .............. 77
      4.7.6 The Determinant Of A Product ....................... 78
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.10</td>
<td>Exercises</td>
<td>175</td>
</tr>
<tr>
<td>7</td>
<td>Weierstrass Approximation Theorem</td>
<td>181</td>
</tr>
<tr>
<td>7.1</td>
<td>The Bernstein Polynomials</td>
<td>181</td>
</tr>
<tr>
<td>7.2</td>
<td>Stone Weierstrass Theorem</td>
<td>185</td>
</tr>
<tr>
<td>7.2.1</td>
<td>The Case Of Compact Sets</td>
<td>185</td>
</tr>
<tr>
<td>7.2.2</td>
<td>The Case Of Locally Compact Sets</td>
<td>188</td>
</tr>
<tr>
<td>7.2.3</td>
<td>The Case Of Complex Valued Functions</td>
<td>189</td>
</tr>
<tr>
<td>7.3</td>
<td>The Holder Spaces</td>
<td>190</td>
</tr>
<tr>
<td>7.4</td>
<td>Exercises</td>
<td>192</td>
</tr>
<tr>
<td>8</td>
<td>Brouwer Fixed Point Theorem</td>
<td>195</td>
</tr>
<tr>
<td>8.1</td>
<td>Simplices And Triangulations</td>
<td>195</td>
</tr>
<tr>
<td>8.2</td>
<td>Labeling Vertices</td>
<td>196</td>
</tr>
<tr>
<td>8.3</td>
<td>The Brouwer Fixed Point Theorem</td>
<td>199</td>
</tr>
<tr>
<td>8.4</td>
<td>Invariance Of Domain</td>
<td>201</td>
</tr>
<tr>
<td>II</td>
<td>Real And Abstract Analysis</td>
<td>205</td>
</tr>
<tr>
<td>9</td>
<td>Abstract Measure And Integration</td>
<td>207</td>
</tr>
<tr>
<td>9.1</td>
<td>(\sigma) Algebras</td>
<td>207</td>
</tr>
<tr>
<td>9.2</td>
<td>Exercises</td>
<td>218</td>
</tr>
<tr>
<td>9.3</td>
<td>The Abstract Lebesgue Integral</td>
<td>220</td>
</tr>
<tr>
<td>9.3.1</td>
<td>Preliminary Observations</td>
<td>220</td>
</tr>
<tr>
<td>9.3.2</td>
<td>The Lebesgue Integral Nonnegative Functions</td>
<td>222</td>
</tr>
<tr>
<td>9.3.3</td>
<td>The Lebesgue Integral For Nonnegative Simple Functions</td>
<td>224</td>
</tr>
<tr>
<td>9.3.4</td>
<td>Simple Functions And Measurable Functions</td>
<td>227</td>
</tr>
<tr>
<td>9.3.5</td>
<td>The Monotone Convergence Theorem</td>
<td>231</td>
</tr>
<tr>
<td>9.3.6</td>
<td>Other Definitions</td>
<td>232</td>
</tr>
<tr>
<td>9.3.7</td>
<td>Fatou's Lemma</td>
<td>233</td>
</tr>
<tr>
<td>9.3.8</td>
<td>The Righteous Algebraic Desires Of The Lebesgue Integral</td>
<td>235</td>
</tr>
<tr>
<td>9.4</td>
<td>The Space (L^p)</td>
<td>236</td>
</tr>
<tr>
<td>9.5</td>
<td>Vital Convergence Theorem</td>
<td>243</td>
</tr>
<tr>
<td>9.6</td>
<td>Exercises</td>
<td>245</td>
</tr>
<tr>
<td>10</td>
<td>The Construction Of Measures</td>
<td>249</td>
</tr>
<tr>
<td>10.1</td>
<td>Outer Measures</td>
<td>249</td>
</tr>
<tr>
<td>10.2</td>
<td>Urysohn's lemma</td>
<td>262</td>
</tr>
<tr>
<td>10.3</td>
<td>Positive Linear Functionals</td>
<td>269</td>
</tr>
<tr>
<td>10.4</td>
<td>One Dimensional Lebesgue Measure</td>
<td>275</td>
</tr>
<tr>
<td>10.5</td>
<td>One Dimensional Lebesgue Stieltjes Measure</td>
<td>276</td>
</tr>
<tr>
<td>10.6</td>
<td>The Distribution Function</td>
<td>278</td>
</tr>
<tr>
<td>10.7</td>
<td>Good Lambda Inequality</td>
<td>281</td>
</tr>
<tr>
<td>10.8</td>
<td>The Ergodic Theorem</td>
<td>282</td>
</tr>
<tr>
<td>10.9</td>
<td>Product Measures</td>
<td>288</td>
</tr>
</tbody>
</table>
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.10</td>
<td>Alternative Treatment Of Product Measure</td>
<td>300</td>
</tr>
<tr>
<td>10.10.1</td>
<td>Monotone Classes And Algebras</td>
<td>300</td>
</tr>
<tr>
<td>10.10.2</td>
<td>Product Measure</td>
<td>303</td>
</tr>
<tr>
<td>10.11</td>
<td>Completion Of Measures</td>
<td>308</td>
</tr>
<tr>
<td>10.12</td>
<td>Another Version Of Product Measures</td>
<td></td>
</tr>
<tr>
<td>10.12.1</td>
<td>General Theory</td>
<td>312</td>
</tr>
<tr>
<td>10.12.2</td>
<td>Completion Of Product Measure Spaces</td>
<td>316</td>
</tr>
<tr>
<td>10.13</td>
<td>Disturbing Examples</td>
<td>318</td>
</tr>
<tr>
<td>10.14</td>
<td>Exercises</td>
<td>320</td>
</tr>
<tr>
<td>11</td>
<td>Lebesgue Measure</td>
<td>323</td>
</tr>
<tr>
<td>11.1</td>
<td>Basic Properties</td>
<td>323</td>
</tr>
<tr>
<td>11.2</td>
<td>The Vitali Covering Theorem</td>
<td>327</td>
</tr>
<tr>
<td>11.3</td>
<td>The Vitali Covering Theorem (Elementary Version)</td>
<td>329</td>
</tr>
<tr>
<td>11.4</td>
<td>Vitali Coverings</td>
<td>332</td>
</tr>
<tr>
<td>11.5</td>
<td>Change Of Variables For Linear Maps</td>
<td>336</td>
</tr>
<tr>
<td>11.6</td>
<td>Change Of Variables For $C^1$ Functions</td>
<td>339</td>
</tr>
<tr>
<td>11.7</td>
<td>Mappings Which Are Not One To One</td>
<td>344</td>
</tr>
<tr>
<td>11.8</td>
<td>Lebesgue Measure And Iterated Integrals</td>
<td>345</td>
</tr>
<tr>
<td>11.9</td>
<td>Spherical Coordinates In $p$ Dimensions</td>
<td>347</td>
</tr>
<tr>
<td>11.10</td>
<td>The Brouwer Fixed Point Theorem</td>
<td>351</td>
</tr>
<tr>
<td>11.11</td>
<td>The Brouwer Fixed Point Theorem Another Proof</td>
<td>354</td>
</tr>
<tr>
<td>11.12</td>
<td>Invariance Of Domain</td>
<td>357</td>
</tr>
<tr>
<td>12</td>
<td>Some Extension Theorems</td>
<td>363</td>
</tr>
<tr>
<td>12.1</td>
<td>Algebras</td>
<td>363</td>
</tr>
<tr>
<td>12.2</td>
<td>Caratheodory Extension Theorem</td>
<td>365</td>
</tr>
<tr>
<td>12.3</td>
<td>The Tychonoff Theorem</td>
<td>368</td>
</tr>
<tr>
<td>12.4</td>
<td>Kolmogorov Extension Theorem</td>
<td>370</td>
</tr>
<tr>
<td>12.5</td>
<td>Exercises</td>
<td>377</td>
</tr>
<tr>
<td>13</td>
<td>The $L^p$ Spaces</td>
<td>379</td>
</tr>
<tr>
<td>13.1</td>
<td>Basic Inequalities And Properties</td>
<td>379</td>
</tr>
<tr>
<td>13.2</td>
<td>Density Considerations</td>
<td>387</td>
</tr>
<tr>
<td>13.3</td>
<td>Separability</td>
<td>389</td>
</tr>
<tr>
<td>13.4</td>
<td>Continuity Of Translation</td>
<td>391</td>
</tr>
<tr>
<td>13.5</td>
<td>Mollifiers And Density Of Smooth Functions</td>
<td>392</td>
</tr>
<tr>
<td>13.6</td>
<td>Exercises</td>
<td>397</td>
</tr>
<tr>
<td>14</td>
<td>Stone’s Theorem And Partitions Of Unity</td>
<td>403</td>
</tr>
<tr>
<td>14.1</td>
<td>Partitions Of Unity And Stone’s Theorem</td>
<td>407</td>
</tr>
<tr>
<td>14.2</td>
<td>An Extension Theorem, Retracts</td>
<td>409</td>
</tr>
<tr>
<td>14.3</td>
<td>Something Which Is Not A Retract</td>
<td>412</td>
</tr>
<tr>
<td>14.4</td>
<td>Exercises</td>
<td>416</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>17.9</td>
<td>Compact Operators In Banach Space</td>
<td>548</td>
</tr>
<tr>
<td>17.10</td>
<td>The Fredholm Alternative</td>
<td>549</td>
</tr>
<tr>
<td>17.11</td>
<td>Square Roots</td>
<td>551</td>
</tr>
<tr>
<td>17.12</td>
<td>General Theory Of Continuous Semigroups</td>
<td>556</td>
</tr>
<tr>
<td></td>
<td>17.12.1 An Evolution Equation</td>
<td>566</td>
</tr>
<tr>
<td></td>
<td>17.12.2 Adjoints, Hilbert Space</td>
<td>569</td>
</tr>
<tr>
<td></td>
<td>17.12.3 Adjoints, Reflexive Banach Space</td>
<td>573</td>
</tr>
<tr>
<td>18</td>
<td>Representation Theorems</td>
<td>579</td>
</tr>
<tr>
<td></td>
<td>18.1 Radon Nikodym Theorem</td>
<td>579</td>
</tr>
<tr>
<td></td>
<td>18.2 Vector Measures</td>
<td>585</td>
</tr>
<tr>
<td></td>
<td>18.3 Representation Theorems For The Dual Space Of $L^p$</td>
<td>593</td>
</tr>
<tr>
<td></td>
<td>18.4 The Dual Space Of $L^\infty(\Omega)$</td>
<td>601</td>
</tr>
<tr>
<td></td>
<td>18.5 Non-$\sigma$ Finite Case</td>
<td>605</td>
</tr>
<tr>
<td></td>
<td>18.6 The Dual Space Of $C_0(\mathbb{X})$</td>
<td>610</td>
</tr>
<tr>
<td></td>
<td>18.7 The Dual Space Of $C_0(\mathbb{X})$, Another Approach</td>
<td>615</td>
</tr>
<tr>
<td></td>
<td>18.8 More Attractive Formulations</td>
<td>616</td>
</tr>
<tr>
<td></td>
<td>18.9 Sequential Compactness In $L^1$</td>
<td>618</td>
</tr>
<tr>
<td></td>
<td>18.10 Exercises</td>
<td>624</td>
</tr>
<tr>
<td>19</td>
<td>The Bochner Integral</td>
<td>627</td>
</tr>
<tr>
<td></td>
<td>19.1 Strong And Weak Measurability</td>
<td>627</td>
</tr>
<tr>
<td></td>
<td>19.2 The Bochner Integral</td>
<td>635</td>
</tr>
<tr>
<td></td>
<td>19.2.1 Definition And Basic Properties</td>
<td>635</td>
</tr>
<tr>
<td></td>
<td>19.2.2 Taking A Closed Operator Out Of The Integral</td>
<td>640</td>
</tr>
<tr>
<td></td>
<td>19.3 Operator Valued Functions</td>
<td>645</td>
</tr>
<tr>
<td></td>
<td>19.3.1 Review Of Hilbert Schmidt Theorem</td>
<td>647</td>
</tr>
<tr>
<td></td>
<td>19.3.2 Measurable Compact Operators</td>
<td>651</td>
</tr>
<tr>
<td></td>
<td>19.4 Fubini's Theorem For Bochner Integrals</td>
<td>652</td>
</tr>
<tr>
<td></td>
<td>19.5 The Spaces $L^p(\Omega;\mathbb{X})$</td>
<td>655</td>
</tr>
<tr>
<td></td>
<td>19.6 Measurable Representatives</td>
<td>662</td>
</tr>
<tr>
<td></td>
<td>19.7 Vector Measures</td>
<td>664</td>
</tr>
<tr>
<td></td>
<td>19.8 The Riesz Representation Theorem</td>
<td>669</td>
</tr>
<tr>
<td></td>
<td>19.9 Pointwise Behavior Of Weakly Convergent Sequences</td>
<td>678</td>
</tr>
<tr>
<td></td>
<td>19.10 Exercises</td>
<td>679</td>
</tr>
<tr>
<td>20</td>
<td>The Derivative</td>
<td>681</td>
</tr>
<tr>
<td></td>
<td>20.1 Limits Of A Function</td>
<td>681</td>
</tr>
<tr>
<td></td>
<td>20.2 Basic Definitions</td>
<td>685</td>
</tr>
<tr>
<td></td>
<td>20.3 The Chain Rule</td>
<td>687</td>
</tr>
<tr>
<td></td>
<td>20.4 The Derivative Of A Compact Mapping</td>
<td>687</td>
</tr>
<tr>
<td></td>
<td>20.5 The Matrix Of The Derivative</td>
<td>688</td>
</tr>
<tr>
<td></td>
<td>20.6 A Mean Value Inequality</td>
<td>690</td>
</tr>
<tr>
<td></td>
<td>20.7 Higher Order Derivatives</td>
<td>696</td>
</tr>
<tr>
<td></td>
<td>20.8 The Derivative And The Cartesian Product</td>
<td>697</td>
</tr>
</tbody>
</table>
## CONTENTS

20.9 Mixed Partial Derivatives .................................................. 701
20.10 Implicit Function Theorem .................................................. 703
20.11 More Derivatives .............................................................. 710
20.12 Lyapunov Schmidt Procedure .............................................. 711
20.13 Analytic Functions ........................................................... 715
20.14 Ordinary Differential Equations .......................................... 721
20.15 Exercises ................................................................. 724

21 Degree Theory, An Introduction ............................................ 729
   21.1 Sard's Lemma ............................................................... 730
   21.2 Preliminary Results ...................................................... 732
   21.3 Definitions And Elementary Properties .............................. 737
      21.3.1 The Degree For $C^\infty(\Omega; \mathbb{R}^n)$ .................. 738
      21.3.2 Definition Of The Degree For Continuous Functions ....... 741
   21.4 Borsuk's Theorem ....................................................... 745
   21.5 Applications ............................................................ 748
   21.6 The Product Formula ................................................... 752
   21.7 A Function With Values In Smaller Dimensions .................... 763
   21.8 The Leray Schauder Degree ........................................... 768
   21.9 Exercises ............................................................... 778

22 Critical Points .............................................................. 785
   22.1 Mountain Pass Theorem In Hilbert Space ............................ 785
      22.1.1 A Locally Lipschitz Selection, Pseudogradients ........... 789
      22.1.2 Mountain Pass Theorem In Banach Space ................... 796

23 Nonlinear Operators ........................................................ 803
   23.1 Some Nonlinear Single Valued Operators ............................ 803
   23.2 Duality Maps ............................................................ 813
   23.3 Penalization And Projection Operators ............................... 821
   23.4 Set-Valued Maps, Pseudomonotone Operators ....................... 824
   23.5 Sum Of Pseudomonotone Operators ................................... 836
   23.6 Generalized Gradients .................................................. 846
   23.7 Maximal Monotone Operators ......................................... 849
      23.7.1 The min max Theorem ............................................ 850
      23.7.2 Equivalent Conditions For Maximal Monotone .......... 855
      23.7.3 Surjectivity Theorems .......................................... 864
      23.7.4 Approximation Theorems ....................................... 876
      23.7.5 Sum Of Maximal Monotone Operators ...................... 884
      23.7.6 Convex Functions, An Example ............................. 889
   23.8 Perturbation Theorems ................................................ 903
## 24 Integrals And Derivatives

24.1 The Fundamental Theorem Of Calculus ........................................ 917
24.2 Absolutely Continuous Functions ........................................ 923
24.3 Weak Derivatives .................................................................. 928
24.4 Lipschitz Functions ................................................................ 932
24.5 Rademacher's Theorem First Version ...................................... 935
24.6 Rademacher's Theorem .......................................................... 940
   24.6.1 Morrey's Inequality ..................................................... 941
   24.6.2 Rademacher's Theorem ............................................... 943
24.7 Differentiation Of Measures With Respect To Lebesgue Measure .... 948
24.8 Exercises ............................................................................ 954

## 25 Orlitz Spaces

25.1 Basic Theory .................................................................... 959
25.2 Dual Spaces In Orlitz Space .................................................. 974

## 26 Hausdorff Measure

26.1 Definition Of Hausdorff Measures ........................................... 979
   26.1.1 Properties Of Hausdorff Measure ................................... 980
26.2 \( H^n \) And \( m_n \) .............................................................. 983
26.3 Technical Considerations ....................................................... 985
   26.3.1 Steiner Symmetrization ................................................ 987
   26.3.2 The Isodiametric Inequality ......................................... 989
26.4 The Proper Value Of \( \beta (n) \) ................................................ 989
   26.4.1 A Formula For \( \alpha (n) \) ............................................... 990
   26.4.2 Hausdorff Measure And Linear Transformations .......... 993
26.5 The Area Formula ................................................................ 995
   26.5.1 Preliminary Results ..................................................... 995
   26.5.2 The Area Formula ....................................................... 1004
   26.5.3 Mappings That Are Not One To One ............................ 1009
26.6 The Divergence Theorem ...................................................... 1012
26.7 Integration And The Degree .................................................. 1026
26.8 The Case Of \( W^{1,1} \) ............................................................ 1030

## 27 Integration Of Differential Forms

27.1 Manifolds ........................................................................... 1041
27.2 The Binet Cauchy Formula .................................................... 1044
27.3 Integration Of Differential Forms On Manifolds ...................... 1045
   27.3.1 The Derivative Of A Differential Form ......................... 1049
27.4 Stokes' Theorem And The Orientation Of \( \partial \Omega \) .................... 1049
27.5 Green's Theorem .................................................................. 1054
   27.5.1 An Oriented Manifold .................................................. 1055
   27.5.2 Green's Theorem ......................................................... 1056
27.6 The Divergence Theorem ...................................................... 1057
## 28 Differentiation Of Radon Measures

28.1 Besicovitch Covering Theorem ........................................... 1061
28.2 Fundamental Theorem Of Calculus For Radon Measures .......... 1066
28.3 Slicing Measures .......................................................... 1069
28.4 Vital Coverings ............................................................. 1076
28.5 Differentiation Of Radon Measures ..................................... 1078
28.6 The Radon Nikodym Theorem For Radon Measures ................. 1081

## 29 Fourier Transforms

29.1 An Algebra Of Special Functions ................................. 1085
29.2 Fourier Transforms Of Functions In $G$ .......................... 1086
29.3 Fourier Transforms Of Just About Anything ...................... 1088
29.3.1 Fourier Transforms Of $G^*$ ....................................... 1088
29.3.2 Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$ .......... 1092
29.3.3 Fourier Transforms Of Functions In $L^2(\mathbb{R}^n)$ .......... 1095
29.3.4 The Schwartz Class .................................................. 1099
29.4 Convolutions .............................................................. 1101
29.4 Exercises ................................................................. 1103

## 30 Fourier Analysis In $\mathbb{R}^n$: An Introduction

30.1 The Marcinkiewicz Interpolation Theorem ......................... 1109
30.2 The Calderon Zygmund Decomposition ............................. 1112
30.3 Mihlin’s Theorem ....................................................... 1114
30.4 Singular Integrals ........................................................ 1127
30.5 Helmholtz Decompositions ............................................. 1137

## 31 Gelfand Triples And Related Stuff

31.1 An Unnatural Example .................................................. 1147
31.2 Standard Techniques In Evolution Equations ....................... 1152
31.3 An Important Formula .................................................. 1163
31.4 The Implicit Case ........................................................ 1170
31.5 The Implicit Case, $B = B(t)$ ....................................... 1189
31.6 Another Approach ....................................................... 1202
31.7 Some Imbedding Theorems ............................................. 1209
31.8 Some Evolution Inclusions ............................................. 1218

## 32 Maximal Monotone Operators, Hilbert Space

32.1 Basic Theory .............................................................. 1223
32.2 Evolution Inclusions ..................................................... 1230
32.3 Subgradients ............................................................... 1235
32.3.1 General Results ..................................................... 1235
32.3.2 Hilbert Space ......................................................... 1249
32.4 A Perturbation Theorem ................................................. 1255
32.5 An Evolution Inclusion ................................................ 1256
32.6 A More Complicated Perturbation Theorem ....................... 1259
32.7 An Evolution Inclusion ................................................ 1261
CONTENTS

III Sobolev Spaces 1269

33 Weak Derivatives 1271

33.1 Weak ∗ Convergence .................................................. 1271
33.2 Test Functions And Weak Derivatives .............................. 1272
33.3 Weak Derivatives In $L^p_{loc}$ ....................................... 1276
33.4 Morrey’s Inequality ..................................................... 1279
33.5 Rademacher’s Theorem .................................................. 1282
33.6 Change Of Variables Formula Lipschitz Maps .................... 1285

34 The Area And Coarea Formulas 1295

34.1 The Area Formula Again .............................................. 1295
34.2 Mappings That Are Not One To One ................................ 1298
34.3 The Coarea Formula ..................................................... 1302
34.4 A Nonlinear Fubini’s Theorem ...................................... 1313

35 Integration On Manifolds 1315

35.1 Partitions Of Unity ...................................................... 1315
35.2 Integration On Manifolds .............................................. 1319
35.3 Comparison With $H^n$ .................................................. 1325

36 Basic Theory Of Sobolev Spaces 1327

36.1 Embedding Theorems For $W^{m,p}(\mathbb{R}^n)$ ..................... 1336
36.2 An Extension Theorem .................................................. 1350
36.3 General Embedding Theorems ...................................... 1357
36.4 More Extension Theorems ............................................. 1360

37 Sobolev Spaces Based On $L^p$ 1367

37.1 Fourier Transform Techniques ...................................... 1367
37.2 Fractional Order Spaces .............................................. 1372
37.3 An Intrinsic Norm ....................................................... 1374
37.4 Embedding Theorems .................................................... 1381
37.5 The Trace On The Boundary Of A Half Space .................... 1383
37.6 Sobolev Spaces On Manifolds ....................................... 1390

37.6.1 General Theory ....................................................... 1390
37.6.2 The Trace On The Boundary ...................................... 1395

38 Weak Solutions 1401

38.1 The Lax Milgram Theorem ............................................ 1401
38.2 An Application Of The Mountain Pass Theorem .................. 1406

39 Korn’s Inequality 1413

39.1 A Fundamental Inequality ............................................ 1413
39.2 Korn’s Inequality ....................................................... 1419
CONTENTS

10 Elliptic Regularity And Nirenberg Differences 1421
  10.1 The Case Of A Half Space 1421
  10.2 The Case Of Bounded Open Sets 1431

11 Interpolation In Banach Space 1441
  11.1 Some Standard Techniques In Evolution Equations 1441
    11.1.1 Weak Vector Valued Derivatives 1441
  11.2 An Important Formula 1452
  11.3 The Implicit Case 1459
  11.4 Some Implicit Inclusions 1470
  11.5 Some Imbedding Theorems 1473
  11.6 The $A$ Method 1478
  11.7 The $J$ Method 1484
  11.8 Duality And Interpolation 1490

12 Trace Spaces 1501
  12.1 Definition And Basic Theory Of Trace Spaces 1501
  12.2 Trace And Interpolation Spaces 1508

13 Traces Of Sobolev Spaces And Fractional Order Spaces 1515
  13.1 Traces Of Sobolev Spaces On The Boundary Of A Half Space 1515
  13.2 A Right Inverse For The Trace For A Half Space 1518
  13.3 Intrinsic Norms 1520
  13.4 Fractional Order Sobolev Spaces 1541

14 Sobolev Spaces On Manifolds 1547
  14.1 Basic Definitions 1547
  14.2 The Trace On The Boundary Of An Open Set 1549

IV Multifunctions 1553

15 The Yankov von Neumann Aumann theorem 1555

16 Multifunctions And Their Measurability 1567
  16.1 The General Case 1567
    16.1.1 A Special Case Which Is Easier 1572
    16.1.2 Other Measurability Considerations 1572
  16.2 Existence Of Measurable Fixed Points 1576
    16.2.1 Simplices And Labeling 1576
    16.2.2 Labeling Vertices 1577
    16.2.3 Measurability Of Brouwer Fixed Points 1580
    16.2.4 Measurability Of Schauder Fixed Points 1585
  16.3 A Set Valued Browder Lemma With Measurability 1589
  16.4 A Measurable Kakutani Theorem 1596
  16.5 Some Variational Inequalities 1598
46.6 An Example .................................................. 1603
46.7 Limit Conditions For Nemytskii Operators ................. 1608

V Complex Analysis .............................................. 1627

17 The Complex Numbers ........................................ 1629
  17.1 The Extended Complex Plane ............................ 1631
  17.2 Exercises ................................................. 1632

18 Riemann Stieltjes Integrals ................................ 1635
  18.1 Exercises ................................................. 1645

19 Fundamentals Of Complex Analysis .......................... 1647
  19.1 Analytic Functions ....................................... 1647
    19.1.1 Cauchy Riemann Equations ......................... 1649
    19.1.2 An Important Example .............................. 1651
  19.2 Exercises ................................................. 1652
  19.3 Cauchy’s Formula For A Disk ........................... 1653
  19.4 Exercises ................................................. 1661
  19.5 Zeros Of An Analytic Function .......................... 1664
  19.6 Liouville’s Theorem ...................................... 1666
  19.7 The General Cauchy Integral Formula ................... 1667
    19.7.1 The Cauchy Goursat Theorem ....................... 1667
    19.7.2 A Redundant Assumption ........................... 1670
    19.7.3 Classification Of Isolated Singularities .......... 1671
    19.7.4 The Cauchy Integral Formula ....................... 1674
    19.7.5 An Example Of A Cycle ............................. 1681
  19.8 Exercises ................................................. 1684

50 The Open Mapping Theorem .................................. 1687
  50.1 A Local Representation .................................. 1687
  50.2 Branches Of The Logarithm ................................ 1689
  50.3 Maximum Modulus Theorem ................................ 1691
  50.4 Extensions Of Maximum Modulus Theorem ................. 1693
    50.4.1 Phragmen Lindelof Theorem ......................... 1693
    50.4.2 Hadamard Three Circles Theorem ................... 1695
    50.4.3 Schwarz’s Lemma .................................... 1696
    50.4.4 One To One Analytic Maps On The Unit Ball ....... 1697
  50.5 Exercises ................................................. 1698
  50.6 Counting Zeros .......................................... 1700
  50.7 An Application To Linear Algebra ....................... 1704
  50.8 Exercises ................................................. 1708
CONTENTS

51 Residues 1711

51.1 Rouche’s Theorem And The Argument Principle ✎ 1715

51.1.1 Argument Principle ✎ 1715

51.1.2 Rouche’s Theorem ✎ 1717

51.1.3 A Different Formulation ✎ 1718

51.2 Singularities And The Laurent Series ✎ 1720

51.2.1 What Is An Annulus? ✎ 1720

51.2.2 The Laurent Series ✎ 1722

51.2.3 Contour Integrals And Evaluation Of Integrals ✎ 1726

51.3 Exercises ✎ 1735

52 Some Important Functional Analysis Applications 1739

52.1 The Spectral Radius Of A Bounded Linear Transformation ✎ 1739

52.2 Analytic Semigroups ✎ 1742

52.2.1 Sectorial Operators And Analytic Semigroups ✎ 1742

52.2.2 The Numerical Range ✎ 1753

52.2.3 An Interesting Example ✎ 1755

52.2.4 Fractional Powers Of Sectorial Operators ✎ 1758

52.2.5 A Scale Of Banach Spaces ✎ 1773

53 Complex Mappings 1777

53.1 Conformal Maps ✎ 1777

53.2 Fractional Linear Transformations ✎ 1778

53.2.1 Circles And Lines ✎ 1778

53.2.2 Three Points To Three Points ✎ 1780

53.3 Riemann Mapping Theorem ✎ 1782

53.3.1 Montel’s Theorem ✎ 1782

53.3.2 Regions With Square Root Property ✎ 1785

53.4 Analytic Continuation ✎ 1788

53.4.1 Regular And Singular Points ✎ 1788

53.4.2 Continuation Along A Curve ✎ 1790

53.5 The Picard Theorems ✎ 1792

53.5.1 Two Competing Lemmas ✎ 1793

53.5.2 The Little Picard Theorem ✎ 1797

53.5.3 Schottky’s Theorem ✎ 1797

53.5.4 A Brief Review ✎ 1802

53.5.5 Montel’s Theorem ✎ 1803

53.5.6 The Great Big Picard Theorem ✎ 1805

53.6 Exercises ✎ 1806

54 Approximation By Rational Functions 1809

54.1 Runge’s Theorem ✎ 1809

54.1.1 Approximation With Rational Functions ✎ 1809

54.1.2 Moving The Poles And Keeping The Approximation ✎ 1811

54.1.3 Merten’s Theorem ✎ 1811
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>57.12 Characteristic Functions For Measures</td>
<td>1942</td>
</tr>
<tr>
<td>57.13 Characteristic Functions And Independence In Banach Space</td>
<td>1945</td>
</tr>
<tr>
<td>57.14 Convolution And Sums</td>
<td>1947</td>
</tr>
<tr>
<td>57.15 The Convergence Of Sums Of Symmetric Random Variables</td>
<td>1953</td>
</tr>
<tr>
<td>57.16 The Multivariate Normal Distribution</td>
<td>1958</td>
</tr>
<tr>
<td>57.17 Use Of Characteristic Functions To Find Moments</td>
<td>1965</td>
</tr>
<tr>
<td>57.18 The Central Limit Theorem</td>
<td>1966</td>
</tr>
<tr>
<td>57.19 Characteristic Functions Of Probability Measures, Prokhorov Theorem</td>
<td>1974</td>
</tr>
<tr>
<td>57.20 Generalized Multivariate Normal</td>
<td>1981</td>
</tr>
<tr>
<td>57.21 Positive Definite Functions, Bochner’s Theorem</td>
<td>1986</td>
</tr>
<tr>
<td>58 Conditional Expectation And Martingales</td>
<td>1993</td>
</tr>
<tr>
<td>58.1 Conditional Expectation</td>
<td>1993</td>
</tr>
<tr>
<td>58.2 Discrete Martingales</td>
<td>1997</td>
</tr>
<tr>
<td>58.2.1 Upcrossings</td>
<td>1999</td>
</tr>
<tr>
<td>58.2.2 The Submartingale Convergence Theorem</td>
<td>2000</td>
</tr>
<tr>
<td>58.2.3 Doob Submartingale Estimate</td>
<td>2001</td>
</tr>
<tr>
<td>58.3 Optional Sampling And Stopping Times</td>
<td>2002</td>
</tr>
<tr>
<td>58.3.1 Stopping Times And Their Properties</td>
<td>2002</td>
</tr>
<tr>
<td>58.4 Optional Stopping Times And Martingales</td>
<td>2006</td>
</tr>
<tr>
<td>58.4.1 Stopping Times And Their Properties</td>
<td>2006</td>
</tr>
<tr>
<td>58.5 Submartingale Convergence Theorem</td>
<td>2013</td>
</tr>
<tr>
<td>58.5.1 Upcrossings</td>
<td>2013</td>
</tr>
<tr>
<td>58.5.2 Maximal Inequalities</td>
<td>2015</td>
</tr>
<tr>
<td>58.5.3 The Upcrossing Estimate</td>
<td>2018</td>
</tr>
<tr>
<td>58.6 The Submartingale Convergence Theorem</td>
<td>2021</td>
</tr>
<tr>
<td>58.7 A Reverse Submartingale Convergence Theorem</td>
<td>2025</td>
</tr>
<tr>
<td>58.8 Strong Law Of Large Numbers</td>
<td>2028</td>
</tr>
<tr>
<td>59 Probability In Infinite Dimensions</td>
<td>2031</td>
</tr>
<tr>
<td>59.1 Conditional Expectation In Banach Spaces</td>
<td>2031</td>
</tr>
<tr>
<td>59.2 Probability Measures And Tightness</td>
<td>2034</td>
</tr>
<tr>
<td>59.3 Tight Measures</td>
<td>2037</td>
</tr>
<tr>
<td>59.4 A Major Existence And Convergence Theorem</td>
<td>2042</td>
</tr>
<tr>
<td>59.5 Bochner’s Theorem In Infinite Dimensions</td>
<td>2049</td>
</tr>
<tr>
<td>59.6 The Multivariate Normal Distribution</td>
<td>2054</td>
</tr>
<tr>
<td>59.7 Gaussian Measures</td>
<td>2059</td>
</tr>
<tr>
<td>59.7.1 Definitions And Basic Properties</td>
<td>2059</td>
</tr>
<tr>
<td>59.7.2 Fernique’s Theorem</td>
<td>2062</td>
</tr>
<tr>
<td>59.8 Gaussian Measures For A Separable Hilbert Space</td>
<td>2068</td>
</tr>
<tr>
<td>59.9 Abstract Wiener Spaces</td>
<td>2078</td>
</tr>
<tr>
<td>59.10 White Noise</td>
<td>2092</td>
</tr>
<tr>
<td>59.11 Existence Of Abstract Wiener Spaces</td>
<td>2092</td>
</tr>
</tbody>
</table>
18

CONTENTS

60 Stochastic Processes 2097
   60.1 Fundamental Definitions And Properties 2097
   60.2 Kolmogorov Censtov Continuity Theorem 2100
   60.3 Filtrations 2109
   60.4 Martingales 2118
   60.5 Some Maximal Estimates 2119
   60.6 Optional Sampling Theorems
      60.6.1 Stopping Times And Their Properties 2122
      60.6.2 Doob Optional Sampling Theorem 2126
   60.7 Doob Optional Sampling Continuous Case
      60.7.1 Stopping Times 2129
      60.7.2 The Optional Sampling Theorem Continuous Case 2134
   60.8 Right Continuity Of Submartingales 2140
   60.9 Some Maximal Inequalities 2147
   60.10 Continuous Submartingale Convergence Theorem 2151
   60.11 Hitting This Before That 2157
   60.12 The Space \( M^p_T (E) \) 2160

61 The Quadratic Variation Of A Martingale 2165
   61.1 How To Recognize A Martingale 2165
   61.2 The Quadratic Variation 2170
   61.3 The Covariation 2178
   61.4 The Burkholder Davis Gundy Inequality 2181
   61.5 The Quadratic Variation And Stochastic Integration 2187
   61.6 Another Limit For Quadratic Variation 2195
   61.7 Doob Meyer Decomposition 2201
   61.8 Levy's Theorem 2221

62 Wiener Processes 2231
   62.1 Real Wiener Processes 2231
   62.2 Nowhere Differentiability Of Wiener Processes 2236
   62.3 Wiener Processes In Separable Banach Space 2238
   62.4 An Example Of Martingales, Independent Increments 2242
   62.5 Hilbert Space Valued Wiener Processes 2247
   62.6 Wiener Processes, Another Approach
      62.6.1 Lots Of Independent Normally Distributed Random Variables 2261
      62.6.2 The Wiener Processes 2270
      62.6.3 Q Wiener Processes In Hilbert Space 2271
      62.6.4 Levy's Theorem In Hilbert Space 2278

63 Stochastic Integration 2283
   63.1 Integrals Of Elementary Processes 2283
   63.2 Different Definition Of Elementary Functions 2289
   63.3 Approximating With Elementary Functions 2290
   63.4 Some Hilbert Space Theory 2294
CONTENTS

73.5 The General Integral ................................. 2299
73.6 The Case That Q Is Trace Class ..................... 2307
73.7 A Short Comment On Measurability .......... 2308
73.8 Localization For Elementary Functions .......... 2309
73.9 Localization In General ............................. 2311
73.10 The Stochastic Integral As A Local Martingale 2313
73.11 The Quadratic Variation Of The Stochastic Integral 2316
73.12 The Holder Continuity Of The Integral .... 2319
73.13 Taking Out A Linear Transformation .......... 2320
73.14 A Technical Integration By Parts Result .... 2322

74 The Integral $\int_0^t (Y, dM)_H$ .......................... 2331

75 The Easy Ito Formula .................................. 2349
75.1 The Situation ........................................ 2349
75.2 Assumptions And A Lemma ......................... 2350
75.3 A Special Case ..................................... 2351
75.4 The Case Of Elementary Functions .......... 2355
75.5 The Integrable Case ................................ 2356
75.6 The General Stochastically Integrable Case .... 2358
75.7 Remembering The Formula ......................... 2359
75.8 An Interesting Formula ............................ 2360
75.9 Some Representation Theorems ................. 2360

76 A Different Kind Of Stochastic Integration .... 2375
76.1 Hermite Polynomials ................................. 2377
76.2 A Remarkable Theorem Involving The Hermite Polynomials 2382
76.3 A Multiple Integral ................................. 2389
76.4 The Skorokhod Integral ............................. 2404
76.4.1 The Derivative ................................ 2404
76.4.2 The Integral .................................. 2406
76.4.3 The Ito And Skorokhod Integrals .......... 2412

77 Gelfand Triples ....................................... 2417
77.1 An Unnatural Example .............................. 2419
77.2 Standard Techniques In Evolution Equations 2424
77.3 An Important Formula .............................. 2434
77.4 The Implicit Case .................................. 2439
77.5 Some Imbedding Theorems ......................... 2450

78 Measurability Without Uniqueness ............... 2457
78.1 Multifunctions And Their Measurability .. 2457
78.2 A Measurable Selection ............................ 2459
78.3 Measurability In Finite Dimensional Problems 2465
78.4 The Navier–Stokes Equations .................... 2470
78.5 A Friction contact problem ....................... 2479
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>68.5.1</td>
<td>The Abstract Problem</td>
<td>2481</td>
</tr>
<tr>
<td>68.5.2</td>
<td>An Approximate Problem</td>
<td>2483</td>
</tr>
<tr>
<td>68.5.3</td>
<td>Discontinuous coefficient of friction</td>
<td>2488</td>
</tr>
<tr>
<td>69</td>
<td>Stochastic O.D.E. One Space</td>
<td>2493</td>
</tr>
<tr>
<td>69.1</td>
<td>Adapted Solutions With Uniqueness</td>
<td>2493</td>
</tr>
<tr>
<td>69.2</td>
<td>Including Stochastic Integrals</td>
<td>2494</td>
</tr>
<tr>
<td>69.3</td>
<td>Stochastic Differential Equations In A Hilbert Space</td>
<td>2499</td>
</tr>
<tr>
<td>69.3.1</td>
<td>The Lipschitz Case</td>
<td>2500</td>
</tr>
<tr>
<td>69.3.2</td>
<td>The Locally Lipschitz Case</td>
<td>2502</td>
</tr>
<tr>
<td>70</td>
<td>The Hard Ito Formula</td>
<td>2509</td>
</tr>
<tr>
<td>70.1</td>
<td>Predictable And Stochastic Continuity</td>
<td>2509</td>
</tr>
<tr>
<td>70.2</td>
<td>Approximating With Step Functions</td>
<td>2511</td>
</tr>
<tr>
<td>70.3</td>
<td>The Situation</td>
<td>2513</td>
</tr>
<tr>
<td>70.4</td>
<td>The Main Estimate</td>
<td>2515</td>
</tr>
<tr>
<td>70.5</td>
<td>Converging In Probability</td>
<td>2522</td>
</tr>
<tr>
<td>70.6</td>
<td>The Ito Formula</td>
<td>2523</td>
</tr>
<tr>
<td>71</td>
<td>The Hard Ito Formula, Implicit Case</td>
<td>2531</td>
</tr>
<tr>
<td>71.1</td>
<td>Approximating With Step Functions</td>
<td>2531</td>
</tr>
<tr>
<td>71.2</td>
<td>The Situation</td>
<td>2533</td>
</tr>
<tr>
<td>71.3</td>
<td>Preliminary Results</td>
<td>2534</td>
</tr>
<tr>
<td>71.4</td>
<td>The Main Estimate</td>
<td>2540</td>
</tr>
<tr>
<td>71.5</td>
<td>A Simplification Of The Formula</td>
<td>2549</td>
</tr>
<tr>
<td>71.6</td>
<td>Convergence</td>
<td>2551</td>
</tr>
<tr>
<td>71.7</td>
<td>The Ito Formula</td>
<td>2557</td>
</tr>
<tr>
<td>72</td>
<td>A More Attractive Version</td>
<td>2569</td>
</tr>
<tr>
<td>72.1</td>
<td>The Situation</td>
<td>2570</td>
</tr>
<tr>
<td>72.2</td>
<td>Preliminary Results</td>
<td>2571</td>
</tr>
<tr>
<td>72.3</td>
<td>The Main Estimate</td>
<td>2574</td>
</tr>
<tr>
<td>72.4</td>
<td>A Simplification Of The Formula</td>
<td>2583</td>
</tr>
<tr>
<td>72.5</td>
<td>Convergence</td>
<td>2584</td>
</tr>
<tr>
<td>72.6</td>
<td>The Ito Formula</td>
<td>2587</td>
</tr>
<tr>
<td>73</td>
<td>Some Nonlinear Operators</td>
<td>2599</td>
</tr>
<tr>
<td>73.1</td>
<td>An Assortment Of Nonlinear Operators</td>
<td>2599</td>
</tr>
<tr>
<td>73.2</td>
<td>Duality Maps</td>
<td>2605</td>
</tr>
<tr>
<td>74</td>
<td>Implicit Stochastic Equations</td>
<td>2611</td>
</tr>
<tr>
<td>74.1</td>
<td>Introduction</td>
<td>2611</td>
</tr>
<tr>
<td>74.2</td>
<td>Preliminary Results</td>
<td>2611</td>
</tr>
<tr>
<td>74.3</td>
<td>The Existence Of Approximate Solutions</td>
<td>2616</td>
</tr>
<tr>
<td>74.4</td>
<td>The General Case</td>
<td>2629</td>
</tr>
<tr>
<td>74.5</td>
<td>Replacing $\Phi$ With $\sigma(u)$</td>
<td>2647</td>
</tr>
</tbody>
</table>
CONTENTS

74.6 Examples ................................................. 2654
74.7 Other Examples, Inclusions .......................... 2660

75 Stochastic Inclusions ................................. 2673
  75.1 The General Context ................................. 2673
  75.2 Some Fundamental Theorems ......................... 2674
  75.3 Preliminary Results ................................ 2690
  75.4 Measurable Approximate Solutions .................. 2693
  75.5 The Main Result ..................................... 2700
  75.6 Variational Inequalities ............................ 2714
  75.7 Progressively Measurable Solutions ............... 2719
  75.8 Adding A Quasi-bounded Operator .................. 2721

76 A Different Approach .................................. 2735
  76.1 Summary Of The Problem .............................. 2735
    76.1.1 General Assumptions On $A$ ...................... 2736
    76.1.2 Preliminary Results ............................. 2738
  76.2 Measurable Solutions To Evolution Inclusions .... 2745
  76.3 Relaxed Coercivy Condition ......................... 2751
  76.4 Progressively Measurable Solutions .............. 2759

77 Including Stochastic Integrals ....................... 2763
  77.1 Replacing $\Phi$ With $\sigma(u)$ ................... 2779

A The Hausdorff Maximal Theorem ......................... 2783
  A.1 The Hamel Basis .................................... 2787
  A.2 Exercises ........................................... 2788

Copyright © 2004,
Part I

Review Of Advanced Calculus
Chapter 1

Set Theory

1.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as 1, 2, −4, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, \{ \} specify a set by listing the things which are in the set between the parentheses. For example the set of integers between −1 and 2, including these numbers could be denoted as \{−1, 0, 1, 2\}. The notation signifying $x$ is an element of a set $S$, is written as $x \in S$. Thus, $1 \in \{-1, 0, 1, 2\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.

2. To every set $A$, and to every condition $S(x)$ there corresponds a set, $B$, whose elements are exactly those elements $x$ of $A$ for which $S(x)$ holds.

3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

4. The Cartesian product of a nonempty family of nonempty sets is nonempty.

5. If $A$ is a set there exists a set, $\mathcal{P}(A)$ such that $\mathcal{P}(A)$ is the set of all subsets of $A$. This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1, 2, 3\} = \{2, 3, 1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some “condition” which can be used as a test to
CHAPTER 1. SET THEORY

determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

\[ \{ x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z} \} . \]

In this notation, the colon is read as “such that” and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way to determine to everyone’s satisfaction whether a given judge is an activist. Also, just because something is grammatically correct does not mean it makes any sense. For example consider the following nonsense.

\[ S = \{ x \in \text{set of dogs} : \text{it is colder in the mountains than in the winter} \} . \]

So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say “set of sets” or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If \( S \) is such a set whose elements are sets,

\[ \bigcup \{ A : A \in S \} \text{ or } \bigcup S \]

signify this union.

Something is in the Cartesian product of a set or “family” of sets if it consists of a single thing taken from each set in the family. Thus \( (1, 2, 3) \in \{1, 4, 2\} \times \{1, 2, 7\} \times \{4, 3, 7, 9\} \) because it consists of exactly one element from each of the sets which are separated by \( \times \). Also, this is the notation for the Cartesian product of finitely many sets. If \( S \) is a set whose elements are sets,

\[ \prod_{A \in S} A \]

signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of \( S \). You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

\( A \) is a subset of \( B \), written \( A \subseteq B \), if every element of \( A \) is also an element of \( B \). This can also be written as \( B \supseteq A \). \( A \) is a proper subset of \( B \), written \( A \subset B \) or \( B \supset A \) if \( A \) is a subset of \( B \) but \( A \) is not equal to \( B \), \( A \neq B \). \( A \cap B \) denotes the intersection of the two sets, \( A \) and \( B \) and it means the set of elements of \( A \) which
are also elements of $B$. The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, $A$ and $B$ and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set ) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of $A$, denoted as $A^C$ (or more precisely as $X \setminus A$) is a set obtained from using the axiom of specification to write

$$A^C \equiv \{x \in X : x \notin A\}$$

The symbol $\notin$ means: “is not an element of”. Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as “all” or “there exists” are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7. Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical even though such usage may be grammatically correct. Quantifiers are used often enough that there are symbols for them. The symbol $\forall$ is read as “for all” or “for every” and the symbol $\exists$ is read as “there exists”. Thus $\forall \exists$ could mean for every upside down $A$ there exists a backwards $E$.

DeMorgan’s laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, $U$. Then

$$\bigcup\{A^C : A \in \mathcal{S}\} = (\bigcap\{A : A \in \mathcal{S}\})^C$$

and

$$\bigcap\{A^C : A \in \mathcal{S}\} = (\bigcup\{A : A \in \mathcal{S}\})^C.$$ 

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathcal{S}$ be a set of sets then

$$B \cup \bigcup\{A : A \in \mathcal{S}\} = \bigcup\{B \cup A : A \in \mathcal{S}\}.$$ 

and: Let $\mathcal{S}$ be a set of sets show

$$B \cap \bigcup\{A : A \in \mathcal{S}\} = \bigcup\{B \cap A : A \in \mathcal{S}\}.$$ 

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it $S$. Then you could consider $A$ the set of all elements of $S$ which are not elements of themselves, this from the axiom of specification. If $A$ is an element of itself, then it fails to qualify for inclusion in $A$. Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in $A$ so it is an element of itself and so this can’t be true either. Thus
the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, “Nothing contains everything”. Always beware of statements involving quantifiers wherever they occur, even this one.

1.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

**Definition 1.2.1** Let $X$ and $Y$ be sets.

$$X \times Y \equiv \{(x, y) : x \in X \text{ and } y \in Y\}$$

A relation is defined to be a subset of $X \times Y$. A function, $f$, also called a mapping, is a relation which has the property that if $(x, y)$ and $(x, y_1)$ are both elements of the $f$, then $y = y_1$. The domain of $f$ is defined as

$$D(f) \equiv \{x : (x, y) \in f\}$$

written as $f : D(f) \rightarrow Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, $x$ and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, $(x, y)$ which is an element of the function or mapping has an input, $x$ and a unique output, $y$, denoted as $f(x)$ while the name of the function is $f$. “mapping” is often a noun meaning function. However, it also is a verb as in “$f$ is mapping $A$ to $B$”. That which a function is thought of as doing is also referred to using the word “maps” as in: $f$ maps $X$ to $Y$. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

**Theorem 1.2.2** Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be two functions. Then there exist sets $A, B, C, D$, such that

$$A \cup B = X, \quad C \cup D = Y, \quad A \cap B = \emptyset, \quad C \cap D = \emptyset,$$

$$f(A) = C, \quad g(D) = B.$$
1.2. THE SCHRODER BERNSTEIN THEOREM

The following picture illustrates the conclusion of this theorem.

\[
\begin{array}{c}
X \\
\begin{array}{c}
A \\
B = g(D)
\end{array} \\
\end{array}
\begin{array}{c}
\begin{array}{c}
f \\
g
\end{array} \\
Y \end{array}
\begin{array}{c}
C = f(A) \\
D \\
\end{array}
\end{array}
\]

**Proof:** Consider the empty set, \(\emptyset \subseteq X\). If \(y \in Y \setminus f(\emptyset)\), then \(g(y) \notin \emptyset\) because \(\emptyset\) has no elements. Also, if \(A, B, C,\) and \(D\) are as described above, \(A\) also would have this same property that the empty set has. However, \(A\) is probably larger. Therefore, say \(A_0 \subseteq X\) satisfies \(\mathcal{P}\) if whenever \(y \in Y \setminus f(A_0)\), \(g(y) \notin A_0\).

\[A \equiv \{A_0 \subseteq X : A_0 \text{ satisfies } \mathcal{P}\}\]

Let \(A = \bigcup \mathcal{A}\). If \(y \in Y \setminus f(A)\), then for each \(A_0 \in \mathcal{A}\), \(y \in Y \setminus f(A_0)\) and so \(g(y) \notin A_0\). Since \(g(y) \notin A_0\) for all \(A_0 \in \mathcal{A}\), it follows \(g(y) \notin A\). Hence \(A\) satisfies \(\mathcal{P}\) and is the largest subset of \(X\) which does so. Now define \(C \equiv f(A), D \equiv Y \setminus C, B \equiv X \setminus A\).

It only remains to verify that \(g(D) = B\).

Suppose \(x \in B = X \setminus A\). Then \(A \cup \{x\}\) does not satisfy \(\mathcal{P}\) and so there exists \(y \in Y \setminus f(A \cup \{x\}) \subseteq D\) such that \(g(y) \in A \cup \{x\}\). But \(y \notin f(A)\) and so since \(A\) satisfies \(\mathcal{P}\), it follows \(g(y) \notin A\). Hence \(g(y) = x\) and so \(x \in g(D)\) and this proves the theorem.

**Theorem 1.2.3 (Schroder Bernstein)** If \(f : X \to Y\) and \(g : Y \to X\) are one to one, then there exists \(h : X \to Y\) which is one to one and onto.

**Proof:** Let \(A, B, C, D\) be the sets of Theorem 1.2.2 and define

\[h(x) \equiv \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in B \end{cases}\]

Then \(h\) is the desired one to one and onto mapping.

Recall that the Cartesian product may be considered as the collection of choice functions.

**Definition 1.2.4** Let \(I\) be a set and let \(X_i\) be a set for each \(i \in I\). \(f\) is a choice function written as

\[f \in \prod_{i \in I} X_i\]

if \(f(i) \in X_i\) for each \(i \in I\).
CHAPTER 1. SET THEORY

The axiom of choice says that if \( X_i \neq \emptyset \) for each \( i \in I \), for \( I \) a set, then

\[
\prod_{i \in I} X_i \neq \emptyset.
\]

Sometimes the two functions, \( f \) and \( g \) are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

**Corollary 1.2.5** If \( f : X \rightarrow Y \) is onto and \( g : Y \rightarrow X \) is onto, then there exists \( h : X \rightarrow Y \) which is one to one and onto.

**Proof:** For each \( y \in Y \), \( f^{-1}(y) = \{ x \in X : f(x) = y \} \neq \emptyset \). Therefore, by the axiom of choice, there exists \( f^{-1}_0 \in \prod_{y \in Y} f^{-1}(y) \) which is the same as saying that for each \( y \in Y \), \( f^{-1}_0(y) \in f^{-1}(y) \). Similarly, there exists \( g^{-1}_0(x) \in g^{-1}(x) \) for all \( x \in X \). Then \( f^{-1}_0 \) is one to one because if \( f^{-1}_0(y_1) = f^{-1}_0(y_2) \), then

\[
y_1 = f(f^{-1}_0(y_1)) = f(f^{-1}_0(y_2)) = y_2.
\]

Similarly \( g^{-1}_0 \) is one to one. Therefore, by the Schroder Bernstein theorem, there exists \( h : X \rightarrow Y \) which is one to one and onto.

**Definition 1.2.6** A set \( S \), is finite if there exists a natural number \( n \) and a map \( \theta \) which maps \( \{1, \ldots, n\} \) one to one and onto \( S \). \( S \) is infinite if it is not finite. A set \( S \), is called countable if there exists a map \( \theta \) mapping \( \mathbb{N} \) one to one and onto \( S \). (When \( \theta \) maps a set \( A \) to a set \( B \), this will be written as \( \theta : A \rightarrow B \) in the future.) Here \( \mathbb{N} = \{1, 2, \ldots \} \) the natural numbers. \( S \) is at most countable if there exists a map \( \theta : \mathbb{N} \rightarrow S \) which is onto.

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

**Theorem 1.2.7** If \( X \) and \( Y \) are both at most countable, then \( X \times Y \) is also at most countable. If either \( X \) or \( Y \) is countable, then \( X \times Y \) is also countable.

**Proof:** It is given that there exists a mapping \( \eta : \mathbb{N} \rightarrow X \) which is onto. Define \( \eta(i) = x_i \) and consider \( X \) as the set \( \{x_1, x_2, x_3, \ldots \} \). Similarly, consider \( Y \) as the set \( \{y_1, y_2, y_3, \ldots \} \). It follows the elements of \( X \times Y \) are included in the following rectangular array.

\[
\begin{array}{cccc}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \leftarrow \text{Those which have } x_1 \text{ in first slot.} \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \leftarrow \text{Those which have } x_2 \text{ in first slot.} \\
(x_3, y_1) & (x_3, y_2) & (x_3, y_3) & \cdots \leftarrow \text{Those which have } x_3 \text{ in first slot.} \\
\vdots & \vdots & \vdots & \vdots
\end{array}
\]
Follow a path through this array as follows.

\[(x_1, y_1) \rightarrow (x_1, y_2) \rightarrow (x_1, y_3) \rightarrow \]
\[(x_2, y_1) \rightarrow (x_2, y_2) \rightarrow (x_2, y_3) \rightarrow \]
\[(x_3, y_1) \rightarrow (x_3, y_2) \rightarrow (x_3, y_3) \rightarrow \]

Thus the first element of \(X \times Y\) is \((x_1, y_1)\), the second element of \(X \times Y\) is \((x_1, y_2)\), the third element of \(X \times Y\) is \((x_2, y_1)\) etc. This assigns a number from \(\mathbb{N}\) to each element of \(X \times Y\). Thus \(X \times Y\) is at most countable.

It remains to show the last claim. Suppose without loss of generality that \(X\) is countable. Then there exists \(\alpha : \mathbb{N} \rightarrow X\) which is one to one and onto. Let \(\beta : X \times Y \rightarrow \mathbb{N}\) be defined by \(\beta ((x, y)) \equiv \alpha^{-1} (x)\). Thus \(\beta\) is onto \(\mathbb{N}\). By the first part there exists a function from \(\mathbb{N}\) onto \(X \times Y\). Therefore, by Corollary 1.2.5, there exists a one to one and onto mapping from \(X \times Y\) to \(\mathbb{N}\). This proves the theorem.

**Theorem 1.2.8** If \(X\) and \(Y\) are at most countable, then \(X \cup Y\) is at most countable. If either \(X\) or \(Y\) are countable, then \(X \cup Y\) is countable.

**Proof:** As in the preceding theorem,

\[X = \{x_1, x_2, x_3, \ldots\}\]

and

\[Y = \{y_1, y_2, y_3, \ldots\}\]

Consider the following array consisting of \(X \cup Y\) and path through it.

\[x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \]
\[y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow \]

Thus the first element of \(X \cup Y\) is \(x_1\), the second is \(x_2\) the third is \(y_1\) the fourth is \(y_2\) etc.

Consider the second claim. By the first part, there is a map from \(\mathbb{N}\) onto \(X \times Y\). Suppose without loss of generality that \(X\) is countable and \(\alpha : \mathbb{N} \rightarrow X\) is one to one and onto. Then define \(\beta (y) \equiv 1\), for all \(y \in Y\),and \(\beta (x) \equiv \alpha^{-1} (x)\). Thus, \(\beta\) maps \(X \times Y\) onto \(\mathbb{N}\) and this shows there exist two onto maps, one mapping \(X \times Y\) onto \(\mathbb{N}\) and the other mapping \(\mathbb{N}\) onto \(X \cup Y\). Then Corollary 1.2.5 yields the conclusion. This proves the theorem.

1.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same
person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

**Definition 1.3.1** Let $S$ be a set. $\sim$ is an equivalence relation on $S$ if it satisfies the following axioms.

1. $x \sim x$ for all $x \in S$. (Reflexive)
2. If $x \sim y$ then $y \sim x$. (Symmetric)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)

**Definition 1.3.2** $[x]$ denotes the set of all elements of $S$ which are equivalent to $x$ and $[x]$ is called the equivalence class determined by $x$ or just the equivalence class of $x$.

With the above definition one can prove the following simple theorem.

**Theorem 1.3.3** Let $\sim$ be an equivalence class defined on a set, $S$ and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.

### 1.4 Partially Ordered Sets

**Definition 1.4.1** Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that

- $x \leq x$ for all $x \in \mathcal{F}$.
- If $x \leq y$ and $y \leq z$ then $x \leq z$.

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

**Theorem 1.4.2** (Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.
Chapter 2

Continuous Functions Of One Variable

There is a theorem about the integral of a continuous function which requires the notion of uniform continuity. This is discussed in this section. Consider the function \( f(x) = \frac{1}{x} \) for \( x \in (0, 1) \). This is a continuous function because it is continuous at every point of \((0, 1)\). However, for a given \( \varepsilon > 0 \), the \( \delta \) needed in the \( \varepsilon, \delta \) definition of continuity becomes very small as \( x \) gets close to 0. The notion of uniform continuity involves being able to choose a single \( \delta \) which works on the whole domain of \( f \). Here is the definition.

**Definition 2.0.3** Let \( f: D \subseteq \mathbb{R} \to \mathbb{R} \) be a function. Then \( f \) is uniformly continuous if for every \( \varepsilon > 0 \), there exists a \( \delta \) depending only on \( \varepsilon \) such that if \( |x - y| < \delta \) then \( |f(x) - f(y)| < \varepsilon \).

It is an amazing fact that under certain conditions continuity implies uniform continuity.

**Definition 2.0.4** A set, \( K \subseteq \mathbb{R} \) is sequentially compact if whenever \( \{a_n\} \subseteq K \) is a sequence, there exists a subsequence, \( \{a_{n_k}\} \) such that this subsequence converges to a point of \( K \).

The following theorem is part of the Heine Borel theorem.

**Theorem 2.0.5** Every closed interval, \([a, b]\) is sequentially compact.

**Proof:** Let \( \{x_n\} \subseteq [a, b] \equiv I_0 \). Consider the two intervals \([a, \frac{a+b}{2}]\) and \([\frac{a+b}{2}, b]\) each of which has length \((b-a)/2\). At least one of these intervals contains \( x_n \) for infinitely many values of \( n \). Call this interval \( I_1 \). Now do for \( I_1 \) what was done for \( I_0 \). Split it in half and let \( I_2 \) be the interval which contains \( x_n \) for infinitely many values of \( n \). Continue this way obtaining a sequence of nested intervals \( I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots \) where the length of \( I_n \) is \((b-a)/2^n\). Now pick \( n_1 \) such that \( x_{n_1} \in I_1 \), \( n_2 \) such that
$n_2 > n_1$ and $x_{n_2} \in I_2, n_3$ such that $n_3 > n_2$ and $x_{n_3} \in I_3$, etc. (This can be done because in each case the intervals contained $x_n$ for infinitely many values of $n$.) By the nested interval lemma there exists a point, $c$ contained in all these intervals. Furthermore, 

$$|x_{nk} - c| < (b - a)2^{-k}$$

and so \( \lim_{k \to \infty} x_{nk} = c \in [a,b] \). This proves the theorem.

**Theorem 2.0.6** Let $f : K \to \mathbb{R}$ be continuous where $K$ is a sequentially compact set in $\mathbb{R}$. Then $f$ is uniformly continuous on $K$.

**Proof:** If this is not true, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a pair of points, $x_\delta$ and $y_\delta$ such that even though $|x_\delta - y_\delta| < \delta$, $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. Taking a succession of values for $\delta$ equal to $1, 1/2, 1/3, \ldots$, and letting the exceptional pair of points for $\delta = 1/n$ be denoted by $x_n$ and $y_n$,

$$|x_n - y_n| < \frac{1}{n}, |f(x_n) - f(y_n)| \geq \varepsilon.$$

Now since $K$ is sequentially compact, there exists a subsequence, $\{x_{nk}\}$ such that $x_{nk} \to z \in K$. Now $n_k \geq k$ and so

$$|x_{nk} - y_{nk}| < \frac{1}{k}.$$ 

Consequently, $y_{nk} \to z$ also. ( $x_{nk}$ is like a person walking toward a certain point and $y_{nk}$ is like a dog on a leash which is constantly getting shorter. Obviously $y_{nk}$ must also move toward the point also. You should give a precise proof of what is needed here.) By continuity of $f$

$$0 = |f(z) - f(z)| = \lim_{k \to \infty} |f(x_{nk}) - f(y_{nk})| \geq \varepsilon,$$

an obvious contradiction. Therefore, the theorem must be true.

The following corollary follows from this theorem and Theorem 2.0.5.

**Corollary 2.0.7** Suppose $I$ is a closed interval, $I = [a,b]$ and $f : I \to \mathbb{R}$ is continuous. Then $f$ is uniformly continuous.

### 2.1 Exercises

1. A function, $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, $K$ such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in D$. Show every Lipschitz function is uniformly continuous.

2. If $|x_n - y_n| \to 0$ and $x_n \to z$, show that $y_n \to z$ also.
3. Consider $f : (1, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Show $f$ is uniformly continuous even though the set on which $f$ is defined is not sequentially compact.

4. If $f$ is uniformly continuous, does it follow that $|f|$ is also uniformly continuous? If $|f|$ is uniformly continuous does it follow that $f$ is uniformly continuous? Answer the same questions with “uniformly continuous” replaced with “continuous”. Explain why.

2.2 Theorems About Continuous Functions

In this section, proofs of some theorems which have not been proved yet are given.

**Theorem 2.2.1** The following assertions are valid

1. The function, $af + bg$ is continuous at $x$ when $f$, $g$ are continuous at $x \in D(f) \cap D(g)$ and $a, b \in \mathbb{R}$.

2. If and $f$ and $g$ are each real valued functions continuous at $x$, then $fg$ is continuous at $x$. If, in addition to this, $g(x) \neq 0$, then $f/g$ is continuous at $x$.

3. If $f$ is continuous at $x$, $f(x) \in D(g) \subseteq \mathbb{R}$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$.

4. The function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x|$ is continuous.

**Proof:** First consider 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|x - y| < \delta_1$, it follows $|f(x) - f(y)| < \frac{\varepsilon}{2(a|a| + |b| + 1)}$ and there exists $\delta_2 > 0$ such that whenever $|x - y| < \delta_2$, it follows that $|g(x) - g(y)| < \frac{\varepsilon}{2(a|a| + |b| + 1)}$. Then let $0 < \delta = \min(\delta_1, \delta_2)$. If $|x - y| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$|af(x) + bf(x) - (ag(y) + bg(y))|$$

$$\leq |a||f(x) - f(y)| + |b||g(x) - g(y)|$$

$$< |a| \left( \frac{\varepsilon}{2(|a| + |b| + 1)} \right) + |b| \left( \frac{\varepsilon}{2(|a| + |b| + 1)} \right) < \varepsilon.$$

Now consider 2.) There exists $\delta_1 > 0$ such that if $|y - x| < \delta_1$, then $|f(x) - f(y)| < 1$.

Therefore, for such $y$,

$$|f(y)| < 1 + |f(x)|.$$

It follows that for such $y$,

$$|fg(x) - fg(y)| \leq |f(x)g(x) - g(x)f(y)| + |g(x)f(y) - f(y)g(y)|$$
|g(x)| \cdot |f(x) - f(y)| + |f(y)| \cdot |g(x) - g(y)| \\
&\leq (1 + |g(x)|) \cdot |g(x) - g(y)| + |f(x) - f(y)| \\

Now let \( \varepsilon > 0 \) be given. There exists \( \delta_2 \) such that if \( |x - y| < \delta_2 \), then 
\[
|g(x) - g(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)},
\]
and there exists \( \delta_3 \) such that if \( |x - y| < \delta_3 \), then 
\[
|f(x) - f(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)}.
\]
Now let \( 0 < \delta \leq \min(\delta_1, \delta_2, \delta_3) \). Then if \( |x - y| < \delta \), all the above hold at once and so
\[
|f g (x) - f g (y)| \leq (1 + |g(x)| + |f(y)|) \cdot |g(x) - g(y)| + |f(x) - f(y)| \\
< \left(1 + |g(x)| + |f(y)|\right) \left(\frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} + \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)}\right) = \varepsilon.
\]

This proves the first part of 2.) To obtain the second part, let \( \delta_1 \) be as described above and let \( \delta_0 > 0 \) be such that for \( |x - y| < \delta_0 \),
\[
|g(x) - g(y)| < \frac{|g(x)|}{2}
\]
and so by the triangle inequality,
\[
-\frac{|g(x)|}{2} \leq |g(y)| - |g(x)| \leq \frac{|g(x)|}{2}
\]
which implies \( |g(y)| \geq \frac{|g(x)|}{2} \), and \( |g(y)| < 3 |g(x)|/2 \).

Then if \( |x - y| < \min(\delta_0, \delta_1) \),
\[
\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} = \frac{f(x) g(y) - f(y) g(x)}{g(x) g(y)} \\
\leq \frac{|f(x) g(y) - f(y) g(x)|}{\left(\frac{|g(x)|}{2}\right)^2} \\
= \frac{2 |f(x) g(y) - f(y) g(x)|}{|g(x)|^2}
\]
\[
\leq \frac{2}{|g(x)|^2} \left[|f(x) g(y) - f(y) g(x)| + |f(x)| \cdot |g(y) - g(x)|\right] \\
\leq \frac{2}{|g(x)|^2} \left[3 |g(x)| \cdot |f(x) - f(y)| + (1 + |f(x)|) |g(y) - g(x)|\right] \\
\leq \frac{2}{|g(x)|^2} (1 + 2 |f(x)| + 2 |g(x)|) \left[|f(x) - f(y)| + |g(y) - g(x)|\right] \\
= M \left[|f(x) - f(y)| + |g(y) - g(x)|\right]
2.2. THEOREMS ABOUT CONTINUOUS FUNCTIONS

where $M$ is defined by

\[ M \equiv \frac{2}{|g(x)|^2} (1 + 2|f(x)| + 2|g(x)|) \]

Now let $\delta_2$ be such that if $|x-y| < \delta_2$, then

\[ |f(x) - f(y)| < \frac{\varepsilon}{2} M^{-1} \]

and let $\delta_3$ be such that if $|x-y| < \delta_3$, then

\[ |g(y) - g(x)| < \frac{\varepsilon}{2} M^{-1}. \]

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|x-y| < \delta$, everything holds and

\[ \left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| \leq M \left[ |f(x) - f(y)| + |g(y) - g(x)| \right] \]

\[ < M \left[ \frac{\varepsilon}{2} M^{-1} + \frac{\varepsilon}{2} M^{-1} \right] = \varepsilon. \]

This completes the proof of the second part of 2.)

Note that in these proofs no effort is made to find some sort of “best” $\delta$. The problem is one which has a yes or a no answer. Either is it or it is not continuous.

Now consider 3.). If $f$ is continuous at $x$, $f(x) \in D(f) \subseteq \mathbb{R}$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$ such that if $|y-f(x)| < \eta$ and $y \in D(f)$, it follows that $|g(y) - g(f(x))| < \varepsilon$. From continuity of $f$ at $x$, there exists $\delta > 0$ such that if $|x-z| < \delta$ and $z \in D(f)$, then $|f(z) - f(x)| < \eta$. Then if $|x-z| < \delta$ and $z \in D(g \circ f) \subseteq D(f)$, all the above hold and so

\[ |g(f(z)) - g(f(x))| < \varepsilon. \]

This proves part 3.)

To verify part 4.), let $\varepsilon > 0$ be given and let $\delta = \varepsilon$. Then if $|x-y| < \delta$, the triangle inequality implies

\[ |f(x) - f(y)| = |x| - |y| \leq |x-y| < \delta = \varepsilon. \]

This proves part 4.) and completes the proof of the theorem.

Next here is a proof of the intermediate value theorem.

**Theorem 2.2.2** Suppose $f : [a, b] \to \mathbb{R}$ is continuous and suppose $f(a) < c < f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = c$.

**Proof:** Let $d = \frac{a+b}{2}$ and consider the intervals $[a, d]$ and $[d, b]$. If $f(d) \geq c$, then on $[a, d]$, the function is $\leq c$ at one end point and $\geq c$ at the other. On the other hand, if $f(d) \leq c$, then on $[d, b]$ $f \geq 0$ at one end point and $\leq 0$ at the
other. Pick the interval on which \( f \) has values which are at least as large as \( c \) and values no larger than \( c \). Now consider that interval, divide it in half as was done for the original interval and argue that on one of these smaller intervals, the function has values at least as large as \( c \) and values no larger than \( c \). Continue in this way. Next apply the nested interval lemma to get \( x \) in all these intervals. In the \( n^{th} \) interval, let \( x_n, y_n \) be elements of this interval such that \( f(x_n) \leq c, f(y_n) \geq c \). Now \( |x_n - x| \leq (b - a)2^{-n} \) and \( |y_n - x| \leq (b - a)2^{-n} \) and so \( x_n \to x \) and \( y_n \to x \). Therefore,
\[
 f(x) - c = \lim_{n \to \infty} (f(x_n) - c) \leq 0
\]
while
\[
 f(x) - c = \lim_{n \to \infty} (f(y_n) - c) \geq 0.
\]
Consequently \( f(x) = c \) and this proves the theorem.

**Lemma 2.2.3** Let \( \phi : [a, b] \to \mathbb{R} \) be a continuous function and suppose \( \phi \) is \( 1 - 1 \) on \( (a, b) \). Then \( \phi \) is either strictly increasing or strictly decreasing on \( [a, b] \).

**Proof:** First it is shown that \( \phi \) is either strictly increasing or strictly decreasing on \( (a, b) \).

If \( \phi \) is not strictly decreasing on \( (a, b) \), then there exists \( x_1 < y_1, x_1, y_1 \in (a, b) \) such that
\[
(\phi(y_1) - \phi(x_1))(y_1 - x_1) > 0.
\]
If for some other pair of points, \( x_2 < y_2 \) with \( x_2, y_2 \in (a, b) \), the above inequality does not hold, then since \( \phi \) is \( 1 - 1 \),
\[
(\phi(y_2) - \phi(x_2))(y_2 - x_2) < 0.
\]
Let \( x_t \equiv tx_1 + (1 - t)x_2 \) and \( y_t \equiv ty_1 + (1 - t)y_2 \). Then \( x_t < y_t \) for all \( t \in [0, 1] \) because
\[
tx_1 \leq ty_1 \text{ and } (1 - t)x_2 \leq (1 - t)y_2
\]
with strict inequality holding for at least one of these inequalities since not both \( t \) and \( (1 - t) \) can equal zero. Now define
\[
h(t) \equiv (\phi(y_t) - \phi(x_t))(y_t - x_t).
\]
Since \( h \) is continuous and \( h(0) < 0 \), while \( h(1) > 0 \), there exists \( t \in (0, 1) \) such that \( h(t) = 0 \). Therefore, both \( x_t \) and \( y_t \) are points of \( (a, b) \) and \( \phi(y_t) - \phi(x_t) = 0 \) contradicting the assumption that \( \phi \) is one to one. It follows \( \phi \) is either strictly increasing or strictly decreasing on \( (a, b) \).

This property of being either strictly increasing or strictly decreasing on \( (a, b) \) carries over to \( [a, b] \) by the continuity of \( \phi \). Suppose \( \phi \) is strictly increasing on \( (a, b) \), a similar argument holding for \( \phi \) strictly decreasing on \( (a, b) \). If \( x > a \), then pick \( y \in (a, x) \) and from the above, \( \phi(y) < \phi(x) \). Now by continuity of \( \phi \) at \( a \),
\[
\phi(a) = \lim_{x \to a^+} \phi(z) \leq \phi(y) < \phi(x).
\]
Therefore, \( \phi(a) < \phi(x) \) whenever \( x \in (a, b) \). Similarly \( \phi(b) > \phi(x) \) for all \( x \in (a, b) \). This proves the lemma.
Corollary 2.2.4 Let \( f : (a, b) \to \mathbb{R} \) be one to one and continuous. Then \( f (a, b) \) is an open interval, \((c, d)\) and \( f^{-1} : (c, d) \to (a, b) \) is continuous.

Proof: Since \( f \) is either strictly increasing or strictly decreasing, it follows that \( f (a, b) \) is an open interval, \((c, d)\). Assume \( f \) is decreasing. Now let \( x \in (a, b) \). Why is \( f^{-1} \) is continuous at \( f (x) \)? Since \( f \) is decreasing, if \( f (x) < f (y) \), then \( y = f^{-1} (f (y)) < x = f^{-1} (f (x)) \) and so \( f^{-1} \) is also decreasing. Let \( \varepsilon > 0 \) be given. Let \( \varepsilon > \eta > 0 \) and \( (x - \eta, x + \eta) \subseteq (a, b) \). Then \( f (x) \in (f (x + \eta), f (x - \eta)) \). Let \( \delta = \min (f (x) - f (x + \eta), f (x - \eta) - f (x)) \). Then if
\[
|f (z) - f (x)| < \delta,
\]
it follows
\[
z = f^{-1} (f (z)) \in (x - \eta, x + \eta) \subseteq (x - \varepsilon, x + \varepsilon)
\]
so
\[
|f^{-1} (f (z)) - x| = |f^{-1} (f (z)) - f^{-1} (f (x))| < \varepsilon.
\]
This proves the theorem in the case where \( f \) is strictly decreasing. The case where \( f \) is increasing is similar.
Chapter 3

The Riemann Stieltjes
Integral

The integral originated in attempts to find areas of various shapes and the ideas involved in finding integrals are much older than the ideas related to finding derivatives. In fact, Archimedes was finding areas of various curved shapes about 250 B.C. using the main ideas of the integral. What is presented here is a generalization of these ideas. The main interest is in the Riemann integral but if it is easy to generalize to the so called Stieltjes integral in which the length of an interval, \([x, y]\) is replaced with an expression of the form \(F(y) - F(x)\) where \(F\) is an increasing function, then the generalization is given. However, there is much more that can be written about Stieltjes integrals than what is presented here. A good source for this is the book by Apostol, \([3]\).

3.1 Upper And Lower Riemann Stieltjes Sums

The Riemann integral pertains to bounded functions which are defined on a bounded interval. Let \([a, b]\) be a closed interval. A set of points in \([a, b]\), \(\{x_0, \cdots, x_n\}\) is a partition if

\[
  a = x_0 < x_1 < \cdots < x_n = b.
\]

Such partitions are denoted by \(P\) or \(Q\). For \(f\) a bounded function defined on \([a, b]\), let

\[
  M_i(f) \equiv \sup \{ f(x) : x \in [x_{i-1}, x_i] \},
\]

\[
  m_i(f) \equiv \inf \{ f(x) : x \in [x_{i-1}, x_i] \}.
\]

1Archimedes 287-212 B.C. found areas of curved regions by stuffing them with simple shapes which he knew the area of and taking a limit. He also made fundamental contributions to physics. The story is told about how he determined that a gold smith had cheated the king by giving him a crown which was not solid gold as had been claimed. He did this by finding the amount of water displaced by the crown and comparing with the amount of water it should have displaced if it had been solid gold.
Definition 3.1.1 Let $F$ be an increasing function defined on $[a, b]$ and let $\Delta F_i \equiv F(x_i) - F(x_{i-1})$. Then define upper and lower sums as

$$U(f, P) \equiv \sum_{i=1}^{n} M_i(f) \Delta F_i \quad \text{and} \quad L(f, P) \equiv \sum_{i=1}^{n} m_i(f) \Delta F_i$$

respectively. The numbers, $M_i(f)$ and $m_i(f)$, are well defined real numbers because $f$ is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S = \{f(x) : x \in [x_{i-1}, x_i]\}$ is bounded above and below.

In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition. In these pictures the function, $F$ is given by $F(x) = x$ and these are the ordinary upper and lower sums from calculus.

What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled $z$ has been added in.

Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 3.1.2 If $P \subseteq Q$ then

$$U(f, Q) \leq U(f, P), \quad \text{and} \quad L(f, P) \leq L(f, Q).$$
3.1. UPPER AND LOWER RIEMANN STIELTJES SUMS

**Proof:** This is verified by adding in one point at a time. Thus let

\[ P = \{x_0, \ldots, x_n\} \]

and let

\[ Q = \{x_0, \ldots, x_k, y, x_{k+1}, \ldots, x_n\}. \]

Thus exactly one point, \( y \), is added between \( x_k \) and \( x_{k+1} \). Now the term in the upper sum which corresponds to the interval \([x_k, x_{k+1}]\) in \( U(f, P) \) is

\[
\sup \{ f(x) : x \in [x_k, x_{k+1}] \} (F(x_{k+1}) - F(x_k))
\]

and the term which corresponds to the interval \([x_k, x_{k+1}]\) in \( U(f, Q) \) is

\[
\sup \{ f(x) : x \in [x_k, y] \} (F(y) - F(x_k)) + \sup \{ f(x) : x \in [y, x_{k+1}] \} (F(x_{k+1}) - F(y))
\]

All the other terms in the two sums coincide. Now \( \sup \{ f(x) : x \in [x_k, x_{k+1}] \} \geq \max(M_1, M_2) \) and so the expression in (3.1.2) is no larger than

\[
\sup \{ f(x) : x \in [x_k, x_{k+1}] \} (F(x_{k+1}) - F(x_k))
\]

the term corresponding to the interval, \([x_k, x_{k+1}]\) and \( U(f, P) \). This proves the first part of the lemma pertaining to upper sums because if \( Q \supseteq P \), one can obtain \( Q \) from \( P \) by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part about lower sums is similar and is left as an exercise.

**Lemma 3.1.3** If \( P \) and \( Q \) are two partitions, then

\[ L(f, P) \leq U(f, Q). \]

**Proof:** By Lemma 3.1.2,

\[ L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q). \]

**Definition 3.1.4**

\[ T = \inf \{ U(f, Q) \text{ where } Q \text{ is a partition} \} \]

\[ L = \sup \{ L(f, P) \text{ where } P \text{ is a partition} \}. \]

Note that \( L \) and \( T \) are well defined real numbers.
CHAPTER 3. THE RIEMANN STIELTJES INTEGRAL

Theorem 3.1.5 \( I \leq \overline{T} \).

Proof: From Lemma 3.1.3,
\[
I = \sup \{L(f,P) \mid P \text{ is a partition}\} \leq U(f,Q)
\]
because \( U(f,Q) \) is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since \( Q \) is arbitrary,
\[
I = \sup \{L(f,P) \mid P \text{ is a partition}\} 
\leq \inf \{U(f,Q) \mid Q \text{ is a partition}\} \equiv \overline{T}
\]
where the inequality holds because it was just shown that \( I \) is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 3.1.6 A bounded function \( f \) is Riemann Stieltjes integrable, written as \( f \in R([a,b]) \) if
\[
I = \overline{T}
\]
and in this case,
\[
\int_a^b f(x) \, dF \equiv I = \overline{T}.
\]
When \( F(x) = x \), the integral is called the Riemann integral and is written as
\[
\int_a^b f(x) \, dx.
\]
Thus, in words, the Riemann integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall the following Proposition which comes from the definitions.

Proposition 3.1.7 Let \( S \) be a nonempty set and suppose \( \sup(S) \) exists. Then for every \( \delta > 0 \),
\[
S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.
\]
If \( \inf(S) \) exists, then for every \( \delta > 0 \),
\[
S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.
\]

This proposition implies the following theorem which is used to determine the question of Riemann Stieltjes integrability.

Theorem 3.1.8 A bounded function \( f \) is Riemann integrable if and only if for all \( \varepsilon > 0 \), there exists a partition \( P \) such that
\[
U(f,P) - L(f,P) < \varepsilon. \quad (3.1.3)
\]
Proof: First assume \( f \) is Riemann integrable. Then let \( P \) and \( Q \) be two partitions such that
\[
U(f,Q) < \bar{T} + \varepsilon/2, \quad L(f,P) > \bar{L} - \varepsilon/2.
\]
Then since \( \bar{I} = \bar{T} \),
\[
U(f,Q \cup P) - L(f,P \cup Q) \leq U(f,Q) - L(f,P) < \bar{T} + \varepsilon/2 - (\bar{L} - \varepsilon/2) = \varepsilon.
\]
Now suppose that for all \( \varepsilon > 0 \) there exists a partition such that \( 3.1.3 \) holds. Then for given \( \varepsilon \) and partition \( P \) corresponding to \( \varepsilon \)
\[
I - I \leq U(f,P) - L(f,P) \leq \varepsilon.
\]
Since \( \varepsilon \) is arbitrary, this shows \( I = I \) and this proves the theorem.

The condition described in the theorem is called the Riemann criterion. Not all bounded functions are Riemann integrable. For example, let \( F(x) = x \) and \( f(x) \equiv \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \) (3.1.4)
Then if \([a,b] = [0,1]\) all upper sums for \( f \) equal 1 while all lower sums for \( f \) equal 0. Therefore the Riemann criterion is violated for \( \varepsilon = 1/2 \).

3.2 Exercises

1. Prove the second half of Lemma 3.1.2 about lower sums.

2. Verify that for \( f \) given in 3.1.3, the lower sums on the interval \([0,1]\) are all equal to zero while the upper sums are all equal to one.

3. Let \( f(x) = 1 + x^2 \) for \( x \in [-1,3] \) and let \( P = \{-1,-1/3,0,1/2,1,2\} \). Find \( U(f,P) \) and \( L(f,P) \) for \( F(x) = x \) and for \( F(x) = x^3 \).

4. Show that if \( f \in R([a,b]) \) for \( F(x) = x \), there exists a partition, \( \{x_0,\cdots,x_n\} \) such that for any \( z_k \in [x_k,x_{k+1}] \),
\[
\left| \int_a^b f(x) \, dx - \sum_{k=1}^n f(z_k) (x_k - x_{k-1}) \right| < \varepsilon
\]
This sum, \( \sum_{k=1}^n f(z_k) (x_k - x_{k-1}) \), is called a Riemann sum and this exercise shows that the Riemann integral can always be approximated by a Riemann sum. For the general Riemann Stieltjes case, does anything change?

5. Let \( P = \{1,1^{1/3},1^{1/2},1^{3/3},2\} \) and \( F(x) = x \). Find upper and lower sums for the function, \( f(x) = \frac{1}{x} \) using this partition. What does this tell you about \( \ln(2) \)?

6. If \( f \in R([a,b]) \) with \( F(x) = x \) and \( f \) is changed at finitely many points, show the new function is also in \( R([a,b]) \). Is this still true for the general case where \( F \) is only assumed to be an increasing function? Explain.
7. In the case where \( F(x) = x \), define a “left sum” as
\[
\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})
\]
and a “right sum”,
\[
\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).
\]
Also suppose that all partitions have the property that \( x_k - x_{k-1} \) equals a constant, \((b - a)/n\) so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as \( n \) gets larger and larger. Show that for \( f \) given in 3.1.4,
\[
\int_{0}^{x} f(t) \, dt = 1 \text{ if } x \text{ is rational and } \int_{0}^{x} f(t) \, dt = 0 \text{ if } x \text{ is irrational.}
\]
It turns out that the correct answer should always equal zero for that function, regardless of whether \( x \) is rational. This is shown when the Lebesgue integral is studied. This illustrates why this method of defining the integral in terms of left and right sums is total nonsense. Show that even though this is the case, it makes no difference if \( f \) is continuous.

3.3 Functions Of Riemann Integrable Functions

It is often necessary to consider functions of Riemann integrable functions and a natural question is whether these are Riemann integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

**Theorem 3.3.1** Let \( f, g \) be bounded functions and let
\[
f ([a, b]) \subseteq [c_1, d_1], \; g ([a, b]) \subseteq [c_2, d_2].
\]
Let \( H : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R} \) satisfy,
\[
|H(a_1, b_1) - H(a_2, b_2)| \leq K |a_1 - a_2| + |b_1 - b_2|
\]
for some constant \( K \). Then if \( f, g \in R([a, b]) \) it follows that \( H \circ (f, g) \in R([a, b]) \).

**Proof:** In the following claim, \( M_i (h) \) and \( m_i (h) \) have the meanings assigned above with respect to some partition of \([a, b]\) for the function, \( h \).

**Claim:** The following inequality holds.
\[
|M_i (H \circ (f, g)) - m_i (H \circ (f, g))| \leq K \left( |M_i (f) - m_i (f)| + |M_i (g) - m_i (g)| \right).
\]

**Proof of the claim:** By the above proposition, there exist \( x_1, x_2 \in [x_{i-1}, x_i] \) be such that
\[
H (f(x_1), g(x_1)) + \eta > M_i (H \circ (f, g))\]
and
\[ H( f(x_2), g(x_2)) - \eta < m_i(H \circ (f, g)). \]
Then
\[ |M_i(H \circ (f, g)) - m_i(H \circ (f, g))| \]
\[ < 2\eta + |H(f(x_1), g(x_1)) - H(f(x_2), g(x_2))| \]
\[ < 2\eta + K[|f(x_1) - f(x_2)| + |g(x_1) - g(x_2)|] \]
\[ \leq 2\eta + K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|]. \]

Since \( \eta > 0 \) is arbitrary, this proves the claim.

Now continuing with the proof of the theorem, let \( P \) be such that
\[ \sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta F_i < \frac{\varepsilon}{2K}, \sum_{i=1}^{n} (M_i(g) - m_i(g)) \Delta F_i < \frac{\varepsilon}{2K}. \]
Then from the claim,
\[ \sum_{i=1}^{n} (M_i(H \circ (f, g)) - m_i(H \circ (f, g))) \Delta F_i \]
\[ < \sum_{i=1}^{n} K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|] \Delta F_i < \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, this shows \( H \circ (f, g) \) satisfies the Riemann criterion and hence \( H \circ (f, g) \) is Riemann integrable as claimed. This proves the theorem.

This theorem implies that if \( f, g \) are Riemann Stieltjes integrable, then so is \( af + bg, |f|, f^2 \), along with infinitely many other such continuous combinations of Riemann Stieltjes integrable functions. For example, to see that \( |f| \) is Riemann integrable, let \( H(a, b) = |a| \). Clearly this function satisfies the conditions of the above theorem and so \( |f| = H(f, f) \in R([a, b]) \) as claimed. The following theorem gives an example of many functions which are Riemann integrable.

**Theorem 3.3.2** Let \( f : [a, b] \to \mathbb{R} \) be either increasing or decreasing on \([a, b]\) and suppose \( F \) is continuous. Then \( f \in R([a, b]) \).

**Proof:** Let \( \varepsilon > 0 \) be given and let
\[ x_i = a + i \left( \frac{b - a}{n} \right), \quad i = 0, \ldots, n. \]
Since \( F \) is continuous, it follows from Corollary 2.0.7 on Page 41 that it is uniformly continuous. Therefore, if \( n \) is large enough, then for all \( i \),
\[ F(x_i) - F(x_{i-1}) < \frac{\varepsilon}{f(b) - f(a) + 1} \]
Then since \( f \) is increasing,
\[
U(f,P) - L(f,P) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) (F(x_i) - F(x_{i-1})) \leq \varepsilon \frac{f(b) - f(a)}{f(b) - f(a) + 1} (f(b) - f(a)) < \varepsilon.
\]
Thus the Riemann criterion is satisfied and so the function is Riemann Stieltjes integrable. The proof for decreasing \( f \) is similar.

**Corollary 3.3.3** Let \([a,b]\) be a bounded closed interval and let \( \phi : [a,b] \to \mathbb{R} \) be Lipschitz continuous and suppose \( F \) is continuous. Then \( \phi \in R([a,b]) \). Recall that a function, \( \phi \), is Lipschitz continuous if there is a constant, \( K \), such that for all \( x,y \),
\[
|\phi(x) - \phi(y)| < K|x-y|.
\]

**Proof:** Let \( f(x) = x \). Then by Theorem 3.3.3, \( f \) is Riemann Stieltjes integrable. Let \( H(a,b) \equiv \phi(a) \). Then by Theorem 3.3.4 \( H \circ (f,f) = \phi \circ f = \phi \) is also Riemann Stieltjes integrable. This proves the corollary. In fact, it is enough to assume \( \phi \) is continuous, although this is harder. This is the content of the next theorem which is where the difficult theorems about continuity and uniform continuity are used. This is the main result on the existence of the Riemann Stieltjes integral for this book.

**Theorem 3.3.4** Suppose \( f : [a,b] \to \mathbb{R} \) is continuous and \( F \) is just an increasing function defined on \([a,b]\). Then \( f \in R([a,b]) \).

**Proof:** By Corollary 2.0.7 on Page 34, \( f \) is uniformly continuous on \([a,b]\). Therefore, if \( \varepsilon > 0 \) is given, there exists an \( \delta > 0 \) such that if \( |x_i - x_{i-1}| < \delta \), then \( M_i - m_i < \frac{\varepsilon}{F(b) - F(a) + 1} \). Let
\[
P \equiv \{x_0, \ldots, x_n\}
\]
be a partition with \( |x_i - x_{i-1}| < \delta \). Then
\[
U(f,P) - L(f,P) < \sum_{i=1}^{n} (M_i - m_i) (F(x_i) - F(x_{i-1})) < \frac{\varepsilon}{F(b) - F(a) + 1} (F(b) - F(a)) < \varepsilon.
\]
By the Riemann criterion, \( f \in R([a,b]) \). This proves the theorem.
3.4 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.

**Lemma 3.4.1** Let $S$ be a nonempty set which is bounded above and below. Then if $-S \equiv \{-x : x \in S\}$,

$$\sup(-S) = -\inf(S) \quad (3.4.5)$$

and

$$\inf(-S) = -\sup(S). \quad (3.4.6)$$

**Proof:** Consider (3.4.5). Let $x \in S$. Then $-x \leq \sup(-S)$ and so $x \geq -\sup(-S)$. This implies $-\sup(-S) \geq -\inf(S)$. Now let $-x \in -S$. Then $x \in S$ and so $x \geq \inf(S)$ which implies $-x \leq -\inf(S)$. Therefore, $-\inf(S)$ is a lower bound for $-S$ and so $-\inf(S) \geq \sup(-S)$. This shows (3.4.5). Formula (3.4.6) is similar and is left as an exercise.

In particular, the above lemma implies that for $M_i(f)$ and $m_i(f)$ defined above

$$M_i(-f) = -m_i(f), \quad m_i(-f) = -M_i(f).$$

**Lemma 3.4.2** If $f \in R([a,b])$ then $-f \in R([a,b])$ and

$$-\int_a^b f(x) \, dF = \int_a^b -f(x) \, dF.$$

**Proof:** The first part of the conclusion of this lemma follows from Theorem 3.3.2 since the function $\phi(y) \equiv -y$ is Lipschitz continuous. Now choose $P$ such that

$$\int_a^b -f(x) \, dF - L(-f, P) < \varepsilon.$$

Then since $m_i(-f) = -M_i(f)$,

$$\varepsilon > \int_a^b -f(x) \, dF - \sum_{i=1}^n m_i(-f) \Delta F_i = \int_a^b -f(x) \, dF + \sum_{i=1}^n M_i(f) \Delta F_i$$

which implies

$$\varepsilon > \int_a^b -f(x) \, dF + \sum_{i=1}^n M_i(f) \Delta F_i \geq \int_a^b -f(x) \, dF + \int_a^b f(x) \, dF.$$

Thus, since $\varepsilon$ is arbitrary,

$$\int_a^b -f(x) \, dF \leq -\int_a^b f(x) \, dF$$

whenever $f \in R([a,b])$. It follows

$$\int_a^b -f(x) \, dF \leq -\int_a^b f(x) \, dF = -\int_a^b (-f(x)) \, dF \leq \int_a^b -f(x) \, dF$$

and this proves the lemma.
Theorem 3.4.3  The integral is linear,
\[ \int_a^b (\alpha f + \beta g) (x) \, dF = \alpha \int_a^b f (x) \, dF + \beta \int_a^b g (x) \, dF. \]
whenever \( f, g \in R([a,b]) \) and \( \alpha, \beta \in \mathbb{R} \).

**Proof:** First note that by Theorem 3.3.1, \( \alpha f + \beta g \in R([a,b]) \). To begin with, consider the claim that if \( f, g \in R([a,b]) \) then
\[ \int_a^b (f + g) (x) \, dF = \int_a^b f (x) \, dF + \int_a^b g (x) \, dF. \tag{3.4.7} \]
Let \( P_1, Q_1 \) be such that
\[ U (f, Q_1) - L (f, Q_1) < \varepsilon/2, \quad U (g, P_1) - L (g, P_1) < \varepsilon/2. \]
Then letting \( P \equiv P_1 \cup Q_1 \), Lemma 3.1.2 implies
\[ U (f, P) - L (f, P) < \varepsilon/2, \quad \text{and} \quad U (g, P) - U (g, P) < \varepsilon/2. \]
Next note that
\[ m_i (f + g) \geq m_i (f) + m_i (g), \quad M_i (f + g) \leq M_i (f) + M_i (g). \]
Therefore,
\[ L (g + f, P) \geq L (f, P) + L (g, P), \quad U (g + f, P) \leq U (f, P) + U (g, P). \]
For this partition,
\[ \int_a^b (f + g) (x) \, dF \in [L (f + g, P), U (f + g, P)] \]
\[ \subseteq [L (f, P) + L (g, P), U (f, P) + U (g, P)] \]
and
\[ \int_a^b f (x) \, dF + \int_a^b g (x) \, dF \in [L (f, P) + L (g, P), U (f, P) + U (g, P)]. \]
Therefore,
\[ \left| \int_a^b (f + g) (x) \, dF - \left( \int_a^b f (x) \, dF + \int_a^b g (x) \, dF \right) \right| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \]
This proves (3.4.7) since \( \varepsilon \) is arbitrary.
It remains to show that
\[ \alpha \int_a^b f(x) \, dF = \int_a^b \alpha f(x) \, dF. \]

Suppose first that \( \alpha \geq 0 \). Then
\[ \int_a^b \alpha f(x) \, dF \equiv \sup \{ L(\alpha f, P) : P \text{ is a partition} \} = \alpha \sup \{ L(f, P) : P \text{ is a partition} \} \equiv \alpha \int_a^b f(x) \, dF. \]

If \( \alpha < 0 \), then this and Lemma 3.4.2 imply
\[ \int_a^b \alpha f(x) \, dF = \int_a^b (-\alpha)(-f(x)) \, dF = (-\alpha) \int_a^b (-f(x)) \, dF = \alpha \int_a^b f(x) \, dF. \]

This proves the theorem.

In the next theorem, suppose \( F \) is defined on \([a, b] \cup [b, c]\).

**Theorem 3.4.4** If \( f \in R([a, b]) \) and \( f \in R([b, c]) \), then \( f \in R([a, c]) \) and
\[ \int_a^c f(x) \, dF = \int_a^b f(x) \, dF + \int_b^c f(x) \, dF. \] \hspace{1cm} (3.4.8)

**Proof:** Let \( P_1 \) be a partition of \([a, b]\) and \( P_2 \) be a partition of \([b, c]\) such that
\[ U(f, P_i) - L(f, P_i) < \varepsilon/2, \ i = 1, 2. \]

Let \( P \equiv P_1 \cup P_2 \). Then \( P \) is a partition of \([a, c]\) and
\[ U(f, P) - L(f, P) = U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \] \hspace{1cm} (3.4.9)

Thus, \( f \in R([a, c]) \) by the Riemann criterion and also for this partition,
\[ \int_a^b f(x) \, dF + \int_b^c f(x) \, dF \in [L(f, P_1) + L(f, P_2), U(f, P_1) + U(f, P_2)] \]
\[ = [L(f, P), U(f, P)] \]
and
\[ \int_a^c f(x) \, dF \in [L(f, P), U(f, P)]. \]

Hence by (3.4.9),
\[ \left| \int_a^c f(x) \, dF - \left( \int_a^b f(x) \, dF + \int_b^c f(x) \, dF \right) \right| < U(f, P) - L(f, P) < \varepsilon \]
which shows that since \( \varepsilon \) is arbitrary, (3.4.8) holds. This proves the theorem.
Corollary 3.4.5  Let $F$ be continuous and let $[a, b]$ be a closed and bounded interval and suppose that
\[ a = y_1 < y_2 < \cdots < y_l = b \]
and that $f$ is a bounded function defined on $[a, b]$ which has the property that $f$ is either increasing on $[y_j, y_{j+1}]$ or decreasing on $[y_j, y_{j+1}]$ for $j = 1, \ldots, l - 1$. Then $f \in R([a, b])$.

Proof: This follows from Theorem 3.4.4 and Theorem 3.3.2.

The symbol, $\int_a^b f(x) \, dF$ when $a > b$ has not yet been defined.

Definition 3.4.6  Let $[a, b]$ be an interval and let $f \in R([a, b])$. Then
\[ \int_b^a f(x) \, dF \equiv - \int_a^b f(x) \, dF. \]

Note that with this definition,
\[ \int_a^a f(x) \, dF = - \int_a^a f(x) \, dF \]
and so
\[ \int_a^a f(x) \, dF = 0. \]

Theorem 3.4.7  Assuming all the integrals make sense,
\[ \int_a^b f(x) \, dF + \int_b^c f(x) \, dF = \int_a^c f(x) \, dF. \]

Proof: This follows from Theorem 3.4.4 and Definition 3.4.6. For example, assume
\[ c \in (a, b). \]

Then from Theorem 3.4.4,
\[ \int_a^c f(x) \, dF + \int_c^b f(x) \, dF = \int_a^b f(x) \, dF \]
and so by Definition 3.4.6
\[ \int_a^c f(x) \, dF = \int_a^b f(x) \, dF - \int_c^b f(x) \, dF \]
\[ = \int_a^b f(x) \, dF + \int_b^c f(x) \, dF. \]

The other cases are similar.
3.5. FUNDAMENTAL THEOREM OF CALCULUS

The following properties of the integral have either been established or they follow quickly from what has been shown so far.

If \( f \in R([a,b]) \) then if \( c \in [a,b] \), \( f \in R([a,c]) \),

\[
\int_a^b \alpha \, dF = \alpha \left( F(b) - F(a) \right),
\]

(3.4.10)

\[
\int_a^b (\alpha f + \beta g) \, dF = \alpha \int_a^b f \, dF + \beta \int_a^b g \, dF,
\]

(3.4.11)

\[
\int_a^b \alpha f \, dF + \int_b^c f(x) \, dF = \int_a^c f(x) \, dF,
\]

(3.4.12)

\[
\int_a^b f(x) \, dF \geq 0 \text{ if } f(x) \geq 0 \text{ and } a < b,
\]

(3.4.13)

\[
\left| \int_a^b f(x) \, dF \right| \leq \left| \int_a^b |f(x)| \, dF \right|.
\]

(3.4.14)

\[
\int_a^b \alpha f \, dF \geq \left| \int_a^b \alpha f \, dF \right|.
\]

(3.4.15)

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

\[
|f(x)| - f(x) \geq 0, \quad |f(x)| + f(x) \geq 0.
\]

Therefore, by (3.4.14) and (3.4.12), if \( a < b \),

\[
\int_a^b |f(x)| \, dF \geq \int_a^b f(x) \, dF
\]

and

\[
\int_a^b |f(x)| \, dF \geq -\int_a^b f(x) \, dF.
\]

Therefore,

\[
\int_a^b |f(x)| \, dF \geq \left| \int_a^b f(x) \, dF \right|.
\]

If \( b < a \) then the above inequality holds with \( a \) and \( b \) switched. This implies (3.4.15).

3.5 Fundamental Theorem Of Calculus

In this section \( F(x) = x \) so things are specialized to the ordinary Riemann integral. With these properties, it is easy to prove the fundamental theorem of calculus.

\[\text{This theorem is why Newton and Liebnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton’s predecessor, Isaac Barrow had made some progress in this direction.}\]
Let \( f \in R([a,b]) \). Then by \( f \in R([a,x]) \) for each \( x \in [a,b] \). The first version of the fundamental theorem of calculus is a statement about the derivative of the function \( x \to \int_a^x f(t) \, dt \).

**Theorem 3.5.1** Let \( f \in R([a,b]) \) and let \( F(x) \equiv \int_a^x f(t) \, dt \).

Then if \( f \) is continuous at \( x \in (a,b) \),

\[
F'(x) = f(x).
\]

**Proof:** Let \( x \in (a,b) \) be a point of continuity of \( f \) and let \( h \) be small enough that \( x + h \in [a,b] \). Then by using \( 3.4.13 \),

\[
h^{-1} (F(x+h) - F(x)) = h^{-1} \int_x^{x+h} f(t) \, dt.
\]

Also, using \( 3.4.11 \),

\[
f(x) = h^{-1} \int_x^{x+h} f(t) \, dt.
\]

Therefore, by \( 3.4.15 \),

\[
|h^{-1} (F(x+h) - F(x)) - f(x)| = \left| h^{-1} \int_x^{x+h} (f(t) - f(x)) \, dt \right|
\]

\[
\leq \left| h^{-1} \int_x^{x+h} |f(t) - f(x)| \, dt \right|.
\]

Let \( \varepsilon > 0 \) and let \( \delta > 0 \) be small enough that if \( |t-x| < \delta \), then

\[
|f(t) - f(x)| < \varepsilon.
\]

Therefore, if \( |h| < \delta \), the above inequality and \( 3.4.11 \) shows that

\[
|h^{-1} (F(x+h) - F(x)) - f(x)| \leq |h|^{-1} \varepsilon |h| = \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this shows

\[
\lim_{h \to 0} h^{-1} (F(x+h) - F(x)) = f(x)
\]

and this proves the theorem.

Note this gives existence for the initial value problem,

\[
F'(x) = f(x), \quad F(a) = 0
\]
whenever \( f \) is Riemann integrable and continuous.

The next theorem is also called the fundamental theorem of calculus.

**Theorem 3.5.2** Let \( f \in R([a, b]) \) and suppose there exists an antiderivative for \( f, G, \) such that
\[
G'(x) = f(x)
\]
for every point of \((a, b)\) and \( G \) is continuous on \([a, b]\). Then
\[
\int_a^b f(x) \, dx = G(b) - G(a).
\]  
(3.5.16)

**Proof:** Let \( P = \{x_0, \ldots, x_n\} \) be a partition satisfying
\[
U(f, P) - L(f, P) < \varepsilon.
\]
Then
\[
G(b) - G(a) = G(x_n) - G(x_0) = \sum_{i=1}^n G(x_i) - G(x_{i-1}).
\]
By the mean value theorem,
\[
G(b) - G(a) = \sum_{i=1}^n G'(z_i) (x_i - x_{i-1}) = \sum_{i=1}^n f(z_i) \Delta x_i
\]
where \( z_i \) is some point in \([x_{i-1}, x_i]\). It follows, since the above sum lies between the upper and lower sums, that
\[
G(b) - G(a) \in [L(f, P), U(f, P)],
\]
and also
\[
\int_a^b f(x) \, dx \in [L(f, P), U(f, P)].
\]
Therefore,
\[
\left| G(b) - G(a) - \int_a^b f(x) \, dx \right| < U(f, P) - L(f, P) < \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, (3.5.16) holds. This proves the theorem.

\(^3\)Of course it was proved that if \( f \) is continuous on a closed interval, \([a, b]\), then \( f \in R([a, b]) \) but this is a hard theorem using the difficult result about uniform continuity.
The following notation is often used in this context. Suppose $F$ is an antiderivative of $f$ as just described with $F$ continuous on $[a, b]$ and $F' = f$ on $(a, b)$. Then

$$\int_a^b f(x) \, dx = F(b) - F(a) \equiv F(x)|_a^b.$$ 

**Definition 3.5.3** Let $f$ be a bounded function defined on a closed interval $[a, b]$ and let $P \equiv \{x_0, \cdots, x_n\}$ be a partition of the interval. Suppose $z_i \in [x_{i-1}, x_i]$ is chosen. Then the sum

$$\sum_{i=1}^n f(z_i) (x_i - x_{i-1})$$

is known as a Riemann sum. Also,

$$||P|| \equiv \max \{|x_i - x_{i-1}| : i = 1, \cdots, n\}.$$ 

**Proposition 3.5.4** Suppose $f \in R([a,b])$. Then there exists a partition, $P \equiv \{x_0, \cdots, x_n\}$ with the property that for any choice of $z_k \in [x_{k-1}, x_k]$,

$$\left| \int_a^b f(x) \, dx - \sum_{k=1}^n f(z_k) (x_k - x_{k-1}) \right| < \varepsilon.$$ 

**Proof:** Choose $P$ such that $U(f, P) - L(f, P) < \varepsilon$ and then both $\int_a^b f(x) \, dx$ and $\sum_{k=1}^n f(z_k) (x_k - x_{k-1})$ are contained in $[L(f, P), U(f, P)]$ and so the claimed inequality must hold. This proves the proposition.

It is significant because it gives a way of approximating the integral.

The definition of Riemann integrability given in this chapter is also called Darboux integrability and the integral defined as the unique number which lies between all upper sums and all lower sums which is given in this chapter is called the Darboux integral. The definition of the Riemann integral in terms of Riemann sums is given next.

**Definition 3.5.5** A bounded function, $f$ defined on $[a,b]$ is said to be Riemann integrable if there exists a number, $I$ with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$P \equiv \{x_0, x_1, \cdots, x_n\}$$

is any partition having $||P|| < \delta$, and $z_i \in [x_{i-1}, x_i]$,

$$\left| I - \sum_{i=1}^n f(z_i) (x_i - x_{i-1}) \right| < \varepsilon.$$ 

The number $\int_a^b f(x) \, dx$ is defined as $I$.

Thus, there are two definitions of the Riemann integral. It turns out they are equivalent which is the following theorem of of Darboux.
Theorem 3.5.6  A bounded function defined on \([a, b]\) is Riemann integrable in the sense of Definition 3.5.5 if and only if it is integrable in the sense of Darboux. Furthermore the two integrals coincide.

The proof of this theorem is left for the exercises in Problems 10 - 12. It isn’t essential that you understand this theorem so if it does not interest you, leave it out. Note that it implies that given a Riemann integrable function \(f\) in either sense, it can be approximated by Riemann sums whenever \(||P||\) is sufficiently small. Both versions of the integral are obsolete but entirely adequate for most applications and as a point of departure for a more up to date and satisfactory integral. The reason for using the Darboux approach to the integral is that all the existence theorems are easier to prove in this context.

3.6 Exercises

1. Let \(F(x) = \int_a^x \frac{t^2 + 7}{t^7 + 87t^3 + 1} \, dt\). Find \(F'(x)\).

2. Let \(F(x) = \int_2^x \frac{1}{1 + 7t} \, dt\). Sketch a graph of \(F\) and explain why it looks the way it does.

3. Let \(a\) and \(b\) be positive numbers and consider the function,

\[
F(x) = \int_0^a \frac{1}{a^2 + t^2} \, dt + \int_b^a \frac{1}{a^2 + t^2} \, dt.
\]

Show that \(F\) is a constant.

4. Solve the following initial value problem from ordinary differential equations which is to find a function \(y\) such that

\[
y'(x) = \frac{x^7 + 1}{x^6 + 97x^5 + 7}, \quad y(10) = 5.
\]

5. If \(F, G \in \int f(x) \, dx\) for all \(x \in \mathbb{R}\), show \(F(x) = G(x) + C\) for some constant, \(C\). Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion \(\int_a^b f(t) \, dt = G(b) - G(a)\) where \(G'(x) = f(x)\).

6. Suppose \(f\) is Riemann integrable on \([a, b]\) and continuous. (In fact continuous implies Riemann integrable.) Show there exists \(c \in (a, b)\) such that

\[
f(c) = \frac{1}{b - a} \int_a^b f(x) \, dx.
\]

**Hint:** You might consider the function \(F(x) = \int_a^x f(t) \, dt\) and use the mean value theorem for derivatives and the fundamental theorem of calculus.
7. Suppose \( f \) and \( g \) are continuous functions on \([a, b]\) and that \( g(x) \neq 0 \) on \((a, b)\).
   Show there exists \( c \in (a, b) \) such that
   \[
   f(c) \int_a^b g(x) \, dx = \int_a^b f(x) g(x) \, dx.
   \]
   \[\text{Hint:} \] Define \( F(x) = \int_a^x f(t) g(t) \, dt \) and let \( G(x) = \int_a^x g(t) \, dt \). Then use the Cauchy mean value theorem on these two functions.

8. Consider the function
   \[
   f(x) = \begin{cases} 
   \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
   0 & \text{if } x = 0 
   \end{cases}
   \]
   Is \( f \) Riemann integrable? Explain why or why not.

9. Prove the second part of Theorem 3.3.2 about decreasing functions.

10. Suppose \( f \) is a bounded function defined on \([a, b]\) and \(|f(x)| < M\) for all \( x \in [a, b] \). Now let \( Q \) be a partition having \( n \) points, \( \{x_0^*, \cdots, x_n^*\} \) and let \( P \) be any other partition. Show that
    \[
    |U(f, P) - L(f, P)| \leq 2Mn \|P\| + |U(f, Q) - L(f, Q)|.
    \]
    \[\text{Hint:} \] Write the sum for \( U(f, P) - L(f, P) \) and split this sum into two sums, the sum of terms for which \([x_{i-1}, x_i]\) contains at least one point of \( Q \), and terms for which \([x_{i-1}, x_i]\) does not contain any points of \( Q \). In the latter case, \([x_{i-1}, x_i]\) must be contained in some interval, \([x_{k-1}^*, x_k^*]\). Therefore, the sum of these terms should be no larger than \(|U(f, Q) - L(f, Q)|\).

11. ↑ If \( \varepsilon > 0 \) is given and \( f \) is a Darboux integrable function defined on \([a, b]\), show there exists \( \delta > 0 \) such that whenever \( \|P\| < \delta \), then
    \[
    |U(f, P) - L(f, P)| < \varepsilon.
    \]

12. ↑ Prove Theorem 3.5.6.
Chapter 4

Some Important Linear Algebra

This chapter contains some important linear algebra as distinguished from that which is normally presented in undergraduate courses consisting mainly of uninteresting things you can do with row operations.

The notation, $\mathbb{C}^n$ refers to the collection of ordered lists of $n$ complex numbers. Since every real number is also a complex number, this simply generalizes the usual notion of $\mathbb{R}^n$, the collection of all ordered lists of $n$ real numbers. In order to avoid worrying about whether it is real or complex numbers which are being referred to, the symbol $\mathbb{F}$ will be used. If it is not clear, always pick $\mathbb{C}$.

**Definition 4.0.1** Define

$$\mathbb{F}^n \equiv \{(x_1, \cdots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \cdots, n\}.$$ 

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \text{ if and only if for all } j = 1, \cdots, n, x_j = y_j.$$ 

When $$(x_1, \cdots, x_n) \in \mathbb{F}^n,$$ 

it is conventional to denote $(x_1, \cdots, x_n)$ by the single bold face letter, $\mathbf{x}$. The numbers, $x_j$ are called the coordinates. The set

$$\{(0, \cdots, 0, t, 0, \cdots, 0) : t \in \mathbb{F}\}$$

for $t$ in the $i^{\text{th}}$ slot is called the $i^{\text{th}}$ coordinate axis. The point $\mathbf{0} = (0, \cdots, 0)$ is called the origin.

Thus $(1, 2, 4i) \in \mathbb{F}^3$ and $(2, 1, 4i) \in \mathbb{F}^3$ but $(1, 2, 4i) \neq (2, 1, 4i)$ because, even though the same numbers are involved, they don’t match up. In particular, the first entries are not equal.

The geometric significance of $\mathbb{R}^n$ for $n \leq 3$ has been encountered already in calculus or in precalculus. Here is a short review. First consider the case when
n = 1. Then from the definition, $\mathbb{R}^1 = \mathbb{R}$. Recall that $\mathbb{R}$ is identified with the points of a line. Look at the number line again. Observe that this amounts to identifying a point on this line with a real number. In other words a real number determines where you are on this line. Now suppose $n = 2$ and consider two lines which intersect each other at right angles as shown in the following picture.

\[ (-8, 3), \quad (2, 6) \]

Notice how you can identify a point shown in the plane with the ordered pair, $(2, 6)$. You go to the right a distance of 2 and then up a distance of 6. Similarly, you can identify another point in the plane with the ordered pair $(-8, 3)$. Go to the left a distance of 8 and then up a distance of 3. The reason you go to the left is that there is a $-$ sign on the eight. From this reasoning, every ordered pair determines a unique point in the plane. Conversely, taking a point in the plane, you could draw two lines through the point, one vertical and the other horizontal and determine unique points, $x_1$ on the horizontal line in the above picture and $x_2$ on the vertical line in the above picture, such that the point of interest is identified with the ordered pair, $(x_1, x_2)$. In short, points in the plane can be identified with ordered pairs similar to the way that points on the real line are identified with real numbers. Now suppose $n = 3$. As just explained, the first two coordinates determine a point in a plane. Letting the third component determine how far up or down you go, depending on whether this number is positive or negative, this determines a point in space. Thus, $(1, 4, -5)$ would mean to determine the point in the plane that goes with $(1, 4)$ and then to go below this plane a distance of 5 to obtain a unique point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

You can’t stop here and say that you are only interested in $n \leq 3$. What if you were interested in the motion of two objects? You would need three coordinates to describe where the first object is and you would need another three coordinates to describe where the other object is located. Therefore, you would need to be considering $\mathbb{R}^6$. If the two objects moved around, you would need a time coordinate as well. As another example, consider a hot object which is cooling and suppose you want the temperature of this object. How many coordinates would be needed? You would need one for the temperature, three for the position of the point in the object and one more for the time. Thus you would need to be considering $\mathbb{R}^5$. Many other examples can be given. Sometimes $n$ is very large. This is often the case in applications to business when they are trying to maximize profit subject
to constraints. It also occurs in numerical analysis when people try to solve hard
problems on a computer.

There are other ways to identify points in space with three numbers but the one
presented is the most basic. In this case, the coordinates are known as Cartesian
coordinates after Descartes\footnote{René Descartes 1596-1650 is often credited
with inventing analytic geometry although it seems the ideas were actually
known much earlier. He was interested in many different subjects, physiology,
chemistry, and physics being some of them. He also wrote a large book in
which he tried to explain the book of Genesis scientifically. Descartes ended
up dying in Sweden.} who invented this idea in the first half of the seventeenth century. I will often not bother to draw a distinction between the point in $n$
dimensional space and its Cartesian coordinates.

The geometric significance of $\mathbb{C}^n$ for $n > 1$ is not available because each copy of
$\mathbb{C}$ corresponds to the plane or $\mathbb{R}^2$.

\section{Algebra In $\mathbb{F}^n$}

There are two algebraic operations done with elements of $\mathbb{F}^n$. One is addition and
the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^n$ the scalars
are complex numbers while in the case of $\mathbb{R}^n$ the only allowed scalars are real
numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

**Definition 4.1.1** If $\mathbf{x} \in \mathbb{F}^n$ and $a \in \mathbb{F}$, also called a scalar, then $a\mathbf{x} \in \mathbb{F}^n$ is defined
by

$$a\mathbf{x} = a(x_1, \cdots, x_n) \equiv (ax_1, \cdots, ax_n). \tag{4.1.1}$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ then $\mathbf{x} + \mathbf{y} \in \mathbb{F}^n$ and is defined
by

$$\mathbf{x} + \mathbf{y} = (x_1, \cdots, x_n) + (y_1, \cdots, y_n) \equiv (x_1 + y_1, \cdots, x_n + y_n) \tag{4.1.2}$$

With this definition, the algebraic properties satisfy the conclusions of the fol-
lowing theorem.

**Theorem 4.1.2** For $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$ and $\alpha, \beta$ scalars, (real numbers), the following hold.

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \tag{4.1.3}$$

the commutative law of addition,

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}), \tag{4.1.4}$$

the associative law for addition,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \tag{4.1.5}$$

the existence of an additive identity,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}, \tag{4.1.6}$$
the existence of an additive inverse, Also

\[ \alpha (v + w) = \alpha v + \alpha w, \quad (4.1.7) \]

\[ (\alpha + \beta) v = \alpha v + \beta v, \quad (4.1.8) \]

\[ \alpha (\beta v) = \alpha \beta (v), \quad (4.1.9) \]

\[ 1v = v. \quad (4.1.10) \]

In the above \(0 = (0, \cdots, 0)\).

You should verify these properties all hold. For example, consider

\[\begin{align*}
\alpha (v + w) &= \alpha (v_1 + w_1, \cdots, v_n + w_n) \\
&= (\alpha (v_1 + w_1), \cdots, \alpha (v_n + w_n)) \\
&= (\alpha v_1 + \alpha w_1, \cdots, \alpha v_n + \alpha w_n) \\
&= (\alpha v_1, \cdots, \alpha v_n) + (\alpha w_1, \cdots, \alpha w_n) \\
&= \alpha v + \alpha w.
\end{align*}\]

As usual subtraction is defined as \(x - y \equiv x + (-y)\).

### 4.2 Exercises

1. Verify all the properties.

2. Compute \(5(1, 2 + 3i, 3, -2) + 6(2 - i, 1, -2, 7)\).

3. Draw a picture of the points in \(\mathbb{R}^2\) which are determined by the following ordered pairs.

   (a) \((1, 2)\)
   (b) \((-2, -2)\)
   (c) \((-2, 3)\)
   (d) \((2, -5)\)

4. Does it make sense to write \((1, 2) + (2, 3, 1)\)? Explain.

5. Draw a picture of the points in \(\mathbb{R}^3\) which are determined by the following ordered triples.

   (a) \((1, 2, 0)\)
   (b) \((-2, -2, 1)\)
   (c) \((-2, 3, -2)\)
4.3 The Inner Product And Distance In $\mathbb{C}^n$

It is necessary to give a generalization of the dot product for vectors in $\mathbb{C}^n$. This is often called the inner product. It reduces to the definition of the dot product in the case the components of the vector are real.

**Definition 4.3.1** Let $x, y \in \mathbb{C}^n$. Thus $x = (x_1, \cdots, x_n)$ where each $x_k \in \mathbb{C}$ and a similar formula holding for $y$. Then the inner product of these two vectors is defined to be

$$x \cdot y \equiv \sum_j x_j \overline{y}_j \equiv x_1 \overline{y}_1 + \cdots + x_n \overline{y}_n.$$  

The inner product is often denoted as $(x, y)$ or $\langle x, y \rangle$.

Notice how you put the conjugate on the entries of the vector $y$. It makes no difference if the vectors happen to be real vectors but with complex vectors you must do it this way. The reason for this is that when you take the inner product of a vector with itself, you want to get the square of the length of the vector, a positive number. Placing the conjugate on the components of $y$ in the above definition assures this will take place. Thus

$$x \cdot x = \sum_j x_j \overline{x}_j = \sum_j |x_j|^2 \geq 0.$$  

If you didn’t place a conjugate as in the above definition, things wouldn’t work out correctly. For example,

$$(1 + i)^2 + 2 = 4 + 2i$$  

and this is not a positive number.

The following properties of the inner product follow immediately from the definition and you should verify each of them.

**Properties of the inner product:**

1. $u \cdot v = \overline{v} \cdot \overline{u}$.
2. If $a, b$ are numbers and $u, v, z$ are vectors then $(au + bv) \cdot z = a(u \cdot z) + b(v \cdot z)$.
3. $u \cdot u \geq 0$ and it equals 0 if and only if $u = 0$.

Note this implies $(x \cdot \alpha y) = \overline{\alpha} (x \cdot y)$ because

$$(x \cdot \alpha y) = (\overline{\alpha} y \cdot x) = \overline{\alpha} (y \cdot x) = \overline{\alpha} (x \cdot y)$$  

The norm is defined in the usual way.

**Definition 4.3.2** For $x \in \mathbb{C}^n$,

$$|x| \equiv \left( \sum_{k=1}^n |x_k|^2 \right)^{1/2} = (x \cdot x)^{1/2}$$  

= (x \cdot x)^{1/2}
Here is a fundamental inequality called the **Cauchy Schwarz inequality** which is stated here in $\mathbb{C}^n$. First here is a simple lemma.

**Lemma 4.3.3** If $z \in \mathbb{C}$ there exists $\theta \in \mathbb{C}$ such that $\theta z = |z|$ and $|\theta| = 1$.

**Proof:** Let $\theta = 1$ if $z = 0$ and otherwise, let $\theta = \frac{\overline{z}}{|z|}$. Recall that for $z = x + iy, \overline{z} = x - iy$ and $zz = |z|^2$.

I will give a proof of this important inequality which depends only on the above list of properties of the inner product. It will be slightly different than the earlier proof.

**Theorem 4.3.4** (Cauchy Schwarz) The following inequality holds for $x$ and $y \in \mathbb{C}^n$.

$$|(x \cdot y)| \leq (x \cdot x)^{1/2} (y \cdot y)^{1/2} \quad (4.3.11)$$

Equality holds in this inequality if and only if one vector is a multiple of the other.

**Proof:** Let $\theta \in \mathbb{C}$ such that $|\theta| = 1$ and

$$\theta (x \cdot y) = |(x \cdot y)|$$

Consider $p(t) \equiv (x + \theta t y, x + t \overline{y})$ where $t \in \mathbb{R}$. Then from the above list of properties of the dot product,

$$0 \leq p(t) = (x \cdot x) + t \theta (x \cdot y) + t \overline{y}(x \cdot y) + t^2 (y \cdot y)$$

$$= (x \cdot x) + t \theta (x \cdot y) + t \overline{y}(x \cdot y) + t^2 (y \cdot y)$$

$$= (x \cdot x) + 2t \text{Re}(\theta (x \cdot y)) + t^2 (y \cdot y)$$

$$= (x \cdot x) + 2t |(x \cdot y)| + t^2 (y \cdot y) \quad (4.3.12)$$

and this must hold for all $t \in \mathbb{R}$. Therefore, if $(y \cdot y) = 0$ it must be the case that $|(x \cdot y)| = 0$ also since otherwise the above inequality would be violated. Therefore, in this case,

$$|(x \cdot y)| \leq (x \cdot x)^{1/2} (y \cdot y)^{1/2}.$$ 

On the other hand, if $(y \cdot y) \neq 0$, then $p(t) \geq 0$ for all $t$ means the graph of $y = p(t)$ is a parabola which opens up and it either has exactly one real zero in the case its vertex touches the $t$ axis or it has no real zeros.

From the quadratic formula this happens exactly when

$$4 |(x \cdot y)|^2 - 4 (x \cdot x) (y \cdot y) \leq 0$$

which is equivalent to $(4.3.11)$.
It is clear from a computation that if one vector is a scalar multiple of the other that equality holds in (4.3.11). Conversely, suppose equality does hold. Then this is equivalent to saying $4|\langle x \cdot y \rangle|^2 - 4\langle x \cdot x \rangle \langle y \cdot y \rangle = 0$ and so from the quadratic formula, there exists one real zero to $p(t) = 0$. Call it $t_0$. Then

$$p(t_0) = \langle (x + \theta t_0 y) \cdot (x + t_0 \bar{y}) \rangle = |x + \theta t_0 y|^2 = 0$$

and so $x = -\bar{y}t_0$. □

Note that I only used part of the above properties of the inner product. It was not necessary to use the one which says that if $\langle x \cdot x \rangle = 0$ then $x = 0$.

By analogy to the case of $\mathbb{R}^n$, length or magnitude of vectors in $\mathbb{C}^n$ can be defined.

**Definition 4.3.5** Let $z \in \mathbb{C}^n$. Then $|z| \equiv (\langle z \cdot z \rangle)^{1/2}$.

The conclusions of the following theorem are also called the axioms for a norm.

**Theorem 4.3.6** For length defined in Definition 4.3.5, the following hold.

$$|z| \geq 0 \text{ and } |z| = 0 \text{ if and only if } z = 0 \quad (4.3.13)$$

If $\alpha$ is a scalar, $|\alpha z| = |\alpha||z|$ \hspace{1cm} (4.3.14)

$$|z + w| \leq |z| + |w|. \hspace{1cm} (4.3.15)$$

**Proof:** The first two claims are left as exercises. To establish the third, you use the same argument which was used in $\mathbb{R}^n$.

$$|z + w|^2 = (z + w, z + w) = z \cdot z + w \cdot w + w \cdot z + z \cdot w$$

$$= |z|^2 + |w|^2 + 2 \text{Re} w \cdot z$$

$$\leq |z|^2 + |w|^2 + 2 |w| |z|$$

$$\leq |z|^2 + |w|^2 + 2 |w||z| = (|z| + |w|)^2. \Box$$

This dot product is often denoted as $(x, y)$. It is also called an inner product. An inner product space is a vector space which has an inner product. More is given on this later. If an inner product space is complete, meaning that Cauchy sequences converge, then it is called a Hilbert space. More is given on this later.

### 4.4 Subspaces Spans And Bases

**Definition 4.4.1** Let $\{x_1, \cdots, x_p\}$ be vectors in $\mathbb{F}^n$. A linear combination is any expression of the form

$$\sum_{i=1}^{p} c_i x_i$$
where the $c_i$ are scalars. The set of all linear combinations of these vectors is called $\text{span}(x_1, \cdots, x_n)$. If $V \subseteq \mathbb{F}^n$, then $V$ is called a subspace if whenever $\alpha, \beta$ are scalars and $u$ and $v$ are vectors of $V$, it follows $\alpha u + \beta v \in V$. That is, it is “closed under the algebraic operations of vector addition and scalar multiplication”.

A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus $\{x_1, \cdots, x_n\}$ is called linearly independent if whenever

$$
\sum_{k=1}^{p} c_k x_k = 0
$$

it follows that all the scalars, $c_k$ equal zero. A set of vectors, $\{x_1, \cdots, x_p\}$, is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars, $c_i, i = 1, \cdots, n$, not all zero such that $\sum_{k=1}^{p} c_k x_k = 0$.

**Lemma 4.4.2** A set of vectors $\{x_1, \cdots, x_p\}$ is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.

**Proof:** Suppose first that $\{x_1, \cdots, x_p\}$ is linearly independent. If

$$
x_k = \sum_{j \neq k} c_j x_j,
$$

then

$$
0 = 1x_k + \sum_{j \neq k} (-c_j) x_j,
$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\{x_1, \cdots, x_p\}$ linearly independent? If it is not there exist scalars, $c_i$, not all zero such that

$$
\sum_{i=1}^{p} c_i x_i = 0.
$$

Say $c_k \neq 0$. Then you can solve for $x_k$ as

$$
x_k = \sum_{j \neq k} (-c_j) / c_k x_j
$$

contrary to assumption. This proves the lemma.

The following is called the exchange theorem.

**Theorem 4.4.3** *(Exchange Theorem)* Let $\{x_1, \cdots, x_r\}$ be a linearly independent set of vectors such that each $x_i$ is in $\text{span}(y_1, \cdots, y_s)$. Then $r \leq s$. 

4.4. SUBSPACES SPANS AND BASES

Proof: Define \( \text{span}\{y_1, \cdots, y_s\} = V \), it follows there exist scalars, \( c_1, \cdots, c_s \) such that

\[
x_1 = \sum_{i=1}^{s} c_i y_i.
\]

(4.4.16)

Not all of these scalars can equal zero because if this were the case, it would follow that \( x_1 = 0 \) and so \( \{x_1, \cdots, x_r\} \) would not be linearly independent. Indeed, if \( x_1 = 0 \), \( 1x_1 + \sum_{i=2}^{r} 0x_i = x_1 = 0 \) and so there would exist a nontrivial linear combination of the vectors \( \{x_1, \cdots, x_r\} \) which equals zero.

Say \( c_k \neq 0 \). Then solve (4.4.16) for \( y_k \) and obtain

\[
y_k \in \text{span} \left( x_1, y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_s \right).
\]

Define \( \{z_1, \cdots, z_{s-1}\} \) by

\[
\{z_1, \cdots, z_{s-1}\} = \{y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_s\}
\]

Therefore, \( \text{span}\{x_1, z_1, \cdots, z_{s-1}\} = V \) because if \( v \in V \), there exist constants \( c_1, \cdots, c_s \) such that

\[
v = \sum_{i=1}^{s} c_i z_i + c_s y_k.
\]

Now replace the \( y_k \) in the above with a linear combination of the vectors,

\[
\{x_1, z_1, \cdots, z_{s-1}\}
\]

to obtain

\[
v \in \text{span}\{x_1, z_1, \cdots, z_{s-1}\}.
\]

The vector \( y_k \), in the list \( \{y_1, \cdots, y_s\}\), has now been replaced with the vector \( x_1 \) and the resulting modified list of vectors has the same span as the original list of vectors, \( \{y_1, \cdots, y_s\} \).

Now suppose that \( r > s \) and that

\[
\text{span}\{x_1, \cdots, x_l, z_1, \cdots, z_p\} = V
\]

where the vectors, \( z_1, \cdots, z_p \), are each taken from the set, \( \{y_1, \cdots, y_s\} \) and \( l + p = s \). This has now been done for \( l = 1 \) above. Then since \( r > s \), it follows that \( l \leq s < r \) and so \( l + 1 \leq r \). Therefore, \( x_{l+1} \) is a vector not in the list, \( \{x_1, \cdots, x_l\} \) and since \( \text{span}\{x_1, \cdots, x_l, z_1, \cdots, z_p\} = V \), there exist scalars, \( c_i \) and \( d_j \) such that

\[
x_{l+1} = \sum_{i=1}^{l} c_i x_i + \sum_{j=1}^{p} d_j z_j.
\]

(4.4.17)

Now not all the \( d_j \) can equal zero because if this were so, it would follow that \( \{x_1, \cdots, x_r\} \) would be a linearly dependent set because one of the vectors would
equal a linear combination of the others. Therefore, (4.4.17) can be solved for one of the $z_i$, say $z_k$, in terms of $x_{l+1}$ and the other $z_i$ and just as in the above argument, replace that $z_i$ with $x_{l+1}$ to obtain

$$\text{span} \left( x_1, \cdots x_{l+1}, z_1, \cdots z_{k-1}, z_{k+1}, \cdots, z_p \right) = V.$$  

Continue this way, eventually obtaining

$$\text{span} (x_1, \cdots, x_s) = V.$$  

But then $x_r \in \text{span} (x_1, \cdots, x_s)$ contrary to the assumption that $\{x_1, \cdots, x_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

**Definition 4.4.4** A finite set of vectors, $\{x_1, \cdots, x_r\}$ is a basis for $\mathbb{F}^n$ if

$$\text{span} (x_1, \cdots, x_r) = \mathbb{F}^n$$

and $\{x_1, \cdots, x_r\}$ is linearly independent.

**Corollary 4.4.5** Let $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_s\}$ be two bases\footnote{This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in “The sixth shiek’s sixth sheep is sick”. This is the reason that bases is used instead of basiss.} of $\mathbb{F}^n$. Then $r = s = n$.

**Proof:** From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$e_i = (0, \cdots, 0, 1, 0 \cdots, 0)$$

for $i = 1, 2, \cdots, n$ are a basis for $\mathbb{F}^n$. This proves the corollary.

**Lemma 4.4.6** Let $\{v_1, \cdots, v_r\}$ be a set of vectors. Then $V \equiv \text{span} (v_1, \cdots, v_r)$ is a subspace.

**Proof:** Suppose $\alpha, \beta$ are two scalars and let $\sum_{k=1}^{r} c_k v_k$ and $\sum_{k=1}^{r} d_k v_k$ are two elements of $V$. What about

$$\alpha \sum_{k=1}^{r} c_k v_k + \beta \sum_{k=1}^{r} d_k v_k?$$

Is it also in $V$?

$$\alpha \sum_{k=1}^{r} c_k v_k + \beta \sum_{k=1}^{r} d_k v_k = \sum_{k=1}^{r} (\alpha c_k + \beta d_k) v_k \in V$$

so the answer is yes. This proves the lemma.
4.4. SUBSPACES SPANS AND BASES

**Definition 4.4.7** A finite set of vectors, \( \{x_1, \ldots, x_r\} \) is a basis for a subspace, \( V \) of \( \mathbb{F}^n \) if \( \text{span} (x_1, \ldots, x_r) = V \) and \( \{x_1, \ldots, x_r\} \) is linearly independent.

**Corollary 4.4.8** Let \( \{x_1, \ldots, x_r\} \) and \( \{y_1, \ldots, y_s\} \) be two bases for \( V \). Then \( r = s \).

**Proof:** From the exchange theorem, \( r \leq s \) and \( s \leq r \). Therefore, this proves the corollary.

**Definition 4.4.9** Let \( V \) be a subspace of \( \mathbb{F}^n \). Then \( \dim (V) \) read as the dimension of \( V \) is the number of vectors in a basis.

Of course you should wonder right now whether an arbitrary subspace even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

**Lemma 4.4.10** Suppose \( v \notin \text{span} (u_1, \ldots, u_k) \) and \( \{u_1, \ldots, u_k\} \) is linearly independent. Then \( \{u_1, \ldots, u_k, v\} \) is also linearly independent.

**Proof:** Suppose \( \sum_{i=1}^k c_i u_i + dv = 0 \). It is required to verify that each \( c_i = 0 \) and that \( d = 0 \). But if \( d \neq 0 \), then you can solve for \( v \) as a linear combination of the vectors, \( \{u_1, \ldots, u_k\} \),

\[
v = -\sum_{i=1}^k \left( \frac{c_i}{d} \right) u_i
\]

contrary to assumption. Therefore, \( d = 0 \). But then \( \sum_{i=1}^k c_i u_i = 0 \) and the linear independence of \( \{u_1, \ldots, u_k\} \) implies each \( c_i = 0 \) also. This proves the lemma.

**Theorem 4.4.11** Let \( V \) be a nonzero subspace of \( \mathbb{F}^n \). Then \( V \) has a basis.

**Proof:** Let \( v_1 \in V \) where \( v_1 \neq 0 \). If \( \text{span} \{v_1\} = V \), stop. \( \{v_1\} \) is a basis for \( V \). Otherwise, there exists \( v_2 \in V \) which is not in \( \text{span} \{v_1\} \). By Lemma 4.4.10, \( \{v_1, v_2\} \) is a linearly independent set of vectors. If \( \text{span} \{v_1, v_2\} = V \), stop. \( \{v_1, v_2\} \) is a basis for \( V \). If \( \text{span} \{v_1, v_2\} \neq V \), then there exists \( v_3 \notin \text{span} \{v_1, v_2\} \) and \( \{v_1, v_2, v_3\} \) is a larger linearly independent set of vectors. Continuing this way, the process must stop before \( n + 1 \) steps because if not, it would be possible to obtain \( n + 1 \) linearly independent vectors contrary to the exchange theorem. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

**Corollary 4.4.12** Let \( V \) be a subspace of \( \mathbb{F}^n \) and let \( \{v_1, \ldots, v_r\} \) be a linearly independent set of vectors in \( V \). Then either it is a basis for \( V \) or there exist vectors, \( v_{r+1}, \ldots, v_s \) such that \( \{v_1, \ldots, v_r, v_{r+1}, \ldots, v_s\} \) is a basis for \( V \).

**Proof:** This follows immediately from the proof of Theorem 4.4.11. You do exactly the same argument except you start with \( \{v_1, \ldots, v_r\} \) rather than \( \{v_1\} \).

It is also true that any spanning set of vectors can be restricted to obtain a basis.
**Theorem 4.4.13** Let $V$ be a subspace of $\mathbb{F}^n$ and suppose $\text{span} (u_1, \ldots, u_p) = V$ where the $u_i$ are nonzero vectors. Then there exist vectors, $\{v_1, \ldots, v_r\}$ such that $\{v_1, \ldots, v_r\} \subseteq \{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_r\}$ is a basis for $V$.

**Proof:** Let $r$ be the smallest positive integer with the property that for some set, $\{v_1, \ldots, v_r\} \subseteq \{u_1, \ldots, u_p\}$, $\text{span} (v_1, \ldots, v_r) = V$.

Then $r \leq p$ and it must be the case that $\{v_1, \ldots, v_r\}$ is linearly independent because if it were not so, one of the vectors, say $v_k$ would be a linear combination of the others. But then you could delete this vector from $\{v_1, \ldots, v_r\}$ and the resulting list of $r-1$ vectors would still span $V$ contrary to the definition of $r$. This proves the theorem.

### 4.5 An Application To Matrices

The following is a theorem of major significance.

**Theorem 4.5.1** Suppose $A$ is an $n \times n$ matrix. Then $A$ is one to one if and only if $A$ is onto. Also, if $B$ is an $n \times n$ matrix and $AB = I$, then it follows $BA = I$.

**Proof:** First suppose $A$ is one to one. Consider the vectors, $\{Ae_1, \ldots, Ae_n\}$ where $e_k$ is the column vector which is all zeros except for a 1 in the $k^{th}$ position. This set of vectors is linearly independent because if

$$\sum_{k=1}^{n} c_k Ae_k = 0,$$

then since $A$ is linear,

$$A \left( \sum_{k=1}^{n} c_k e_k \right) = 0$$

and since $A$ is one to one, it follows

$$\sum_{k=1}^{n} c_k e_k = 0$$

which implies each $c_k = 0$. Therefore, $\{Ae_1, \ldots, Ae_n\}$ must be a basis for $\mathbb{F}^n$ because if not there would exist a vector, $y \notin \text{span} (Ae_1, \ldots, Ae_n)$ and then by Lemma 4.4.10, $\{Ae_1, \ldots, Ae_n, y\}$ would be an independent set of vectors having $n+1$ vectors in it, contrary to the exchange theorem. It follows that for $y \in \mathbb{F}^n$ there exist constants, $c_i$ such that

$$y = \sum_{k=1}^{n} c_k Ae_k = A \left( \sum_{k=1}^{n} c_k e_k \right)$$
4.6. THE MATHEMATICAL THEORY OF DETERMINANTS

showing that, since \( y \) was arbitrary, \( A \) is onto.

Next suppose \( A \) is onto. This means the span of the columns of \( A \) equals \( \mathbb{F}^n \). If these columns are not linearly independent, then by Lemma 4.4.2 on Page 66, one of the columns is a linear combination of the others and so the span of the columns of \( A \) equals the span of the \( n-1 \) other columns. This violates the exchange theorem because \( \{e_1, \cdots, e_n\} \) would be a linearly independent set of vectors contained in the span of only \( n-1 \) vectors. Therefore, the columns of \( A \) must be independent and this equivalent to saying that \( Ax = 0 \) if and only if \( x = 0 \). This implies \( A \) is one to one because if \( Ax = Ay \), then \( A(x-y) = 0 \) and so \( x - y = 0 \).

Now suppose \( AB = I \). Why is \( BA = I \)? Since \( AB = I \) it follows \( B \) is one to one since otherwise, there would exist, \( x \neq 0 \) such that \( Bx = 0 \) and then \( ABx = A0 = 0 \neq Ix \). Therefore, from what was just shown, \( B \) is also onto. In addition to this, \( A \) must be one to one because if \( Ay = 0 \), then \( y = Bx \) for some \( x \) and then \( x = ABx = Ay = 0 \) showing \( y = 0 \). Now from what is given to be so, it follows \( (AB)A = A \) and so using the associative law for matrix multiplication,

\[
A(BA) - A = (BA - I) = 0.
\]

But this means \( (BA - I)x = 0 \) for all \( x \) since otherwise, \( A \) would not be one to one. Hence \( BA = I \) as claimed. This proves the theorem.

This theorem shows that if an \( n \times n \) matrix, \( B \) acts like an inverse when multiplied on one side of \( A \) it follows that \( B = A^{-1} \) and it will act like an inverse on both sides of \( A \).

The conclusion of this theorem pertains to square matrices only. For example, let

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}
\]

Then

\[
BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

but

\[
AB = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.
\]

4.6 The Mathematical Theory Of Determinants

4.6.1 The Function \( \text{sgn} \)

The following Lemma will be essential in the definition of the determinant.

**Lemma 4.6.1** There exists a function, \( \text{sgn}_n \), which maps each ordered list of numbers from \( \{1, \cdots, n\} \) to one of the three numbers, 0, 1, or \(-1\) which also has the following properties.

\[
\text{sgn}_n (1, \cdots, n) = 1
\]
\[ \text{sgn}_n (i_1, \cdots, p, \cdots, q, \cdots, i_n) = -\text{sgn}_n (i_1, \cdots, q, \cdots, p, \cdots, i_n) \quad (4.6.20) \]

In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by \(-1\). Also, in the case where \(n > 1\) and \(\{i_1, \cdots, i_n\} = \{1, \cdots, n\}\) so that every number from \(\{1, \cdots, n\}\) appears in the ordered list, \((i_1, \cdots, i_n)\),

\[ \text{sgn}_n (i_1, \cdots, i_{\theta - 1}, n, i_{\theta + 1}, \cdots, i_n) \equiv (-1)^{n-\theta} \text{sgn}_{n-1} (i_1, \cdots, i_{\theta - 1}, i_{\theta + 1}, \cdots, i_n) \quad (4.6.21) \]

where \(n = i_\theta\) in the ordered list, \((i_1, \cdots, i_n)\).

**Proof:** Define \(\text{sign} (x) = 1\) if \(x > 0\), \(-1\) if \(x < 0\) and \(0\) if \(x = 0\). If \(n = 1\), there is only one list and it is just the number 1. Thus one can define \(\text{sgn}_1 (1) \equiv 1\). For the general case where \(n > 1\), simply define

\[ \text{sgn}_n (i_1, \cdots, i_n) \equiv \text{sign} \left( \prod_{r<s} (i_s - i_r) \right) \]

This delivers either \(-1, 1, \) or 0 by definition. What about the other claims? Suppose you switch \(i_p\) with \(i_q\) where \(p < q\) so two numbers in the ordered list \((i_1, \cdots, i_n)\) are switched. Denote the new ordered list of numbers as \((j_1, \cdots, j_n)\). Thus \(j_p = i_q\) and \(j_q = i_p\) and if \(r \notin \{p, q\}\), \(j_r = i_r\). See the following illustration

\[
\begin{array}{cccccccc}
  i_1 & i_2 & \cdots & i_p & \cdots & i_q & \cdots & i_n \\
  1 & 2 & \cdots & p & \cdots & q & \cdots & n \\
  \hline \\
  i_1 & i_2 & \cdots & i_q & \cdots & i_p & \cdots & i_n \\
  1 & 2 & \cdots & q & \cdots & p & \cdots & n \\
  \hline \\
  j_1 & j_2 & \cdots & j_p & \cdots & j_q & \cdots & j_n \\
  1 & 2 & \cdots & p & \cdots & q & \cdots & n \\
\end{array}
\]

Then

\[ \text{sgn}_n (j_1, \cdots, j_n) \equiv \text{sign} \left( \prod_{r<s} (j_s - j_r) \right) \]

\[ = \text{sign} \left( \frac{\text{both } p, q }{p < q} \prod_{p < q} (i_j - i_q) \frac{\text{one of } p, q }{p < q} \prod_{p < q} (i_p - i_j) \frac{\text{neither } p \text{ nor } q }{r < s, r, s \notin \{p, q\}} \prod_{r < s, r, s \notin \{p, q\}} (i_s - i_r) \right) \]

The last product consists of the product of terms which were in the un-switched product \(\prod_{r<s} (i_s - i_r)\) so produces no change in sign, while the two products in the middle both introduce \(q - p - 1\) minus signs. Thus their product produces no
change in sign. The first factor is of opposite sign to the \( i_q - i_p \) which occurred in 
\( \text{sgn}_n (i_1, \ldots, i_n) \). Therefore, this switch introduced a minus sign and

\[
\text{sgn}_n (j_1, \ldots, j_n) = - \text{sgn}_n (i_1, \ldots, i_n)
\]

Now consider the last claim. In computing \( \text{sgn}_n (i_1, \ldots, i_{\theta-1}, n, i_{\theta+1}, \ldots, i_n) \)
there will be the product of \( n - \theta \) negative terms

\[
(i_{\theta+1} - n) \cdots (i_n - n)
\]

and the other terms in the product for computing \( \text{sgn}_n (i_1, \ldots, i_{\theta-1}, n, i_{\theta+1}, \ldots, i_n) \)
are those which are required to compute \( \text{sgn}_{n-1} (i_1, \ldots, i_{\theta-1}, i_{\theta+1}, \ldots, i_n) \) multiplied by terms of the form \((n - i_j)\) which are nonnegative. It follows that

\[
\text{sgn}_n (i_1, \ldots, i_{\theta-1}, n, i_{\theta+1}, \ldots, i_n) = (-1)^{n-\theta} \text{sgn}_{n-1} (i_1, \ldots, i_{\theta-1}, i_{\theta+1}, \ldots, i_n)
\]

It is obvious that if there are repeats in the list the function gives 0.

**Lemma 4.6.2** Every ordered list of distinct numbers from \( \{1, 2, \ldots, n\} \) can be obtained from every other such ordered list by a finite number of switches. Also, \( \text{sgn}_n \) is unique.

**Proof:** This is obvious if \( n = 1 \) or 2. Suppose then that it is true for sets of \( n - 1 \) elements. Take two ordered lists of numbers, \( P_1, P_2 \). Make one switch in both to place \( n \) at the end. Call the result \( P^n_1 \) and \( P^n_2 \). Then using induction, there are finitely many switches in \( P^n_1 \) so that it will coincide with \( P^n_2 \). Now switch the \( n \) in what results to where it was in \( P_2 \).

To see \( \text{sgn}_n \) is unique, if there exist two functions, \( f \) and \( g \) both satisfying 4.6.19
and 4.6.20, you could start with \( f (1, \ldots, n) = g (1, \ldots, n) = 1 \) and applying the same sequence of switches, eventually arrive at \( f (i_1, \ldots, i_n) = g (i_1, \ldots, i_n) \). If any numbers are repeated, then 4.6.20 gives both functions are equal to zero for that ordered list.

**Definition 4.6.3** When you have an ordered list of distinct numbers from \( \{1, 2, \ldots, n\} \), say

\[
(i_1, \ldots, i_n),
\]

this ordered list is called a permutation. The symbol for all such permutations is \( S_n \). The number \( \text{sgn}_n (i_1, \ldots, i_n) \) is called the sign of the permutation.

A permutation can also be considered as a function from the set

\[
\{1, 2, \ldots, n\}
\]
as follows. Let \( f (k) = i_k \). Permutations are of fundamental importance in certain areas of math. For example, it was by considering permutations that Galois was able to give a criterion for solution of polynomial equations by radicals, but this is a different direction than what is being attempted here.

In what follows \( \text{sgn} \) will often be used rather than \( \text{sgn}_n \) because the context supplies the appropriate \( n \).
4.7 The Determinant

Definition 4.7.1 Let \( f \) be a function which has the set of ordered lists of numbers from \( \{1, \cdots, n\} \) as its domain. Define

\[
\sum_{(k_1, \cdots, k_n)} f(k_1 \cdots k_n)
\]

to be the sum of all the \( f(k_1 \cdots k_n) \) for all possible choices of ordered lists \( (k_1, \cdots, k_n) \) of numbers of \( \{1, \cdots, n\} \). For example,

\[
\sum_{(k_1, k_2)} f(k_1, k_2) = f(1, 2) + f(2, 1) + f(1, 1) + f(2, 2).
\]

4.7.1 The Definition

Definition 4.7.2 Let \((a_{ij}) = A\) denote an \( n \times n \) matrix. The determinant of \( A \), denoted by \( \det (A) \) is defined by

\[
\det (A) \equiv \sum_{(k_1, \cdots, k_n)} \text{sgn}(k_1, \cdots, k_n) a_{1k_1} \cdots a_{nk_n}
\]

where the sum is taken over all ordered lists of numbers from \( \{1, \cdots, n\} \). Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, \( \text{sgn}(k_1, \cdots, k_n) = 0 \) and so that term contributes 0 to the sum.

4.7.2 Permuting Rows Or Columns

Let \( A \) be an \( n \times n \) matrix, \( A = (a_{ij}) \) and let \((r_1, \cdots, r_n)\) denote an ordered list of \( n \) numbers from \( \{1, \cdots, n\} \). Let \( A(r_1, \cdots, r_n) \) denote the matrix whose \( k^{th} \) row is the \( r_k \) row of the matrix \( A \). Thus

\[
\det (A(r_1, \cdots, r_n)) = \sum_{(k_1, \cdots, k_n)} \text{sgn}(k_1, \cdots, k_n) a_{r_1k_1} \cdots a_{r_nk_n}
\]  

(4.7.22)

and

\[
A(1, \cdots, n) = A.
\]

Proposition 4.7.3 Let

\[
(r_1, \cdots, r_n)
\]

be an ordered list of numbers from \( \{1, \cdots, n\} \). Then

\[
\text{sgn}(r_1, \cdots, r_n) \det (A)
\]

\[
= \sum_{(k_1, \cdots, k_n)} \text{sgn}(k_1, \cdots, k_n) a_{r_1k_1} \cdots a_{r_nk_n}
\]

(4.7.23)

\[
= \det (A(r_1, \cdots, r_n)).
\]

(4.7.24)
4.7. THE DETERMINANT

Proof: Let \((1, \cdots, n) = (1, \cdots, r, \cdots, s, \cdots, n)\) so \(r < s\).

\[
\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = \sum_{(k_1, \cdots, k_n)} \text{sgn}(k_1, \cdots, k_r, \cdots, k_s, \cdots, k_n) a_{1k_1} \cdots a_{rk_r} \cdots a_{sk_s} \cdots a_{nk_n},
\]

and renaming the variables, calling \(k_s, k_r\) and \(k_r, k_s\), this equals

\[
= \sum_{(k_1, \cdots, k_n)} \text{sgn}(k_1, \cdots, k_s, \cdots, k_r, \cdots, k_n) a_{1k_1} \cdots a_{rk_r} \cdots a_{sk_s} \cdots a_{nk_n}
\]

\[
= -\det(A(1, \cdots, s, \cdots, r, \cdots, n)). \tag{4.7.25}
\]

Consequently,

\[
\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = -\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = -\det(A)
\]

Now letting \(A(1, \cdots, s, \cdots, r, \cdots, n)\) play the role of \(A\), and continuing in this way, switching pairs of numbers,

\[
\det(A(r_1, \cdots, r_n)) = (-1)^p \det(A)
\]

where it took \(p\) switches to obtain \((r_1, \cdots, r_n)\) from \((1, \cdots, n)\). By Lemma 4.6.1, this implies

\[
\det(A(r_1, \cdots, r_n)) = (-1)^p \det(A) = \text{sgn}(r_1, \cdots, r_n) \det(A)
\]

and proves the proposition in the case when there are no repeated numbers in the ordered list, \((r_1, \cdots, r_n)\). However, if there is a repeat, say the \(r^{th}\) row equals the \(s^{th}\) row, then the reasoning of 4.7.25-4.7.26 shows that \(\det(A(r_1, \cdots, r_n)) = 0\) and also \(\text{sgn}(r_1, \cdots, r_n) = 0\) so the formula holds in this case also. 

Observation 4.7.4 There are \(n!\) ordered lists of distinct numbers from \(\{1, \cdots, n\}\).

To see this, consider \(n\) slots placed in order. There are \(n\) choices for the first slot. For each of these choices, there are \(n - 1\) choices for the second. Thus there are \(n(n - 1)\) ways to fill the first two slots. Then for each of these ways there are \(n - 2\) choices left for the third slot. Continuing this way, there are \(n!\) ordered lists of distinct numbers from \(\{1, \cdots, n\}\) as stated in the observation.
4.7.3 A Symmetric Definition

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\det(A) = \det(A^T)$.

**Corollary 4.7.5** The following formula for $\det(A)$ is valid.

$$\det(A) = \frac{1}{n!} \sum_{(r_1, \cdots, r_n)} \sum_{(k_1, \cdots, k_n)} \text{sgn}(r_1, \cdots, r_n) \text{sgn}(k_1, \cdots, k_n) a_{r_1k_1} \cdots a_{r_nk_n}. \quad (4.7.27)$$

And also $\det(A^T) = \det(A)$ where $A^T$ is the transpose of $A$. (Recall that for $A^T = (a^T_{ij}), a^T_{ij} = a_{ji}$.)

**Proof:** From Proposition 4.7.3, if the $r_i$ are distinct, $\det(A) = \sum_{(k_1, \cdots, k_n)} \text{sgn}(r_1, \cdots, r_n) \text{sgn}(k_1, \cdots, k_n) a_{r_1k_1} \cdots a_{r_nk_n}$. Summing over all ordered lists, $(r_1, \cdots, r_n)$ where the $r_i$ are distinct. (If the $r_i$ are not distinct, $\text{sgn}(r_1, \cdots, r_n) = 0$ and so there is no contribution to the sum.)

$$n! \det(A) = \sum_{(r_1, \cdots, r_n)} \sum_{(k_1, \cdots, k_n)} \text{sgn}(r_1, \cdots, r_n) \text{sgn}(k_1, \cdots, k_n) a_{r_1k_1} \cdots a_{r_nk_n}.$$  

This proves the corollary since the formula gives the same number for $A$ as it does for $A^T$. ■

4.7.4 The Alternating Property Of The Determinant

**Corollary 4.7.6** If two rows or two columns in an $n \times n$ matrix $A$, are switched, the determinant of the resulting matrix equals $(-1)$ times the determinant of the original matrix. If $A$ is an $n \times n$ matrix in which two rows are equal or two columns are equal then $\det(A) = 0$. Suppose the $i^{th}$ row of $A$ equals $(xa_1 + yb_1, \cdots, xa_n + yb_n)$. Then

$$\det(A) = x \det(A_1) + y \det(A_2)$$

where the $i^{th}$ row of $A_1$ is $(a_1, \cdots, a_n)$ and the $i^{th}$ row of $A_2$ is $(b_1, \cdots, b_n)$, all other rows of $A_1$ and $A_2$ coinciding with those of $A$. In other words, $\det$ is a linear function of each row $A$. The same is true with the word “row” replaced with the word “column”.

**Proof:** By Proposition 4.7.3, when two rows are switched, the determinant of the resulting matrix is $(-1)$ times the determinant of the original matrix. By Corollary 4.7.3, the same holds for columns because the columns of the matrix equal the rows
4.7. THE DETERMINANT

of the transposed matrix. Thus if $A_1$ is the matrix obtained from $A$ by switching two columns,

$$\det (A) = \det (A^T) = - \det (A_1^T) = - \det (A_1).$$

If $A$ has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\det (A) = - \det (A)$ and so $\det (A) = 0$.

It remains to verify the last assertion.

$$\det (A) \equiv \sum_{(k_1, \cdots, k_n)} \text{sgn} (k_1, \cdots, k_n) a_{1k_1} \cdots (xa_{k_1} + yb_{k_1}) \cdots a_{nk_n}$$

$$= x \sum_{(k_1, \cdots, k_n)} \text{sgn} (k_1, \cdots, k_n) a_{1k_1} \cdots a_{k_1} \cdots a_{nk_n}$$

$$+ y \sum_{(k_1, \cdots, k_n)} \text{sgn} (k_1, \cdots, k_n) a_{1k_1} \cdots b_{k_1} \cdots a_{nk_n}$$

$$\equiv x \det (A_1) + y \det (A_2).$$

The same is true of columns because $\det (A^T) = \det (A)$ and the rows of $A^T$ are the columns of $A$.

4.7.5 Linear Combinations And Determinants

Linear combinations have been discussed already. However, here is a review and some new terminology.

**Definition 4.7.7** A vector $w$, is a linear combination of the vectors $\{v_1, \cdots, v_r\}$ if there exists scalars, $c_1, \cdots, c_r$ such that $w = \sum_{k=1}^{r} c_k v_k$. This is the same as saying $w \in \text{span} (v_1, \cdots, v_r)$.

The following corollary is also of great use.

**Corollary 4.7.8** Suppose $A$ is an $n \times n$ matrix and some column (row) is a linear combination of $r$ other columns (rows). Then $\det (A) = 0$.

**Proof:** Let $A = (\begin{array}{c} a_1 \cdots a_n \end{array})$ be the columns of $A$ and suppose the condition that one column is a linear combination of $r$ of the others is satisfied. Then by using Corollary 4.7.6 the determinant of $A$ is zero if and only if the determinant of the matrix $B$, which has this special column placed in the last position, equals zero. Thus $a_n = \sum_{k=1}^{r} c_k a_k$ and so

$$\det (B) = \det (\begin{array}{c} a_1 \cdots a_r \cdots a_{n-1} \sum_{k=1}^{r} c_k a_k \end{array}).$$

By Corollary 4.7.6

$$\det (B) = \sum_{k=1}^{r} c_k \det (\begin{array}{c} a_1 \cdots a_r \cdots a_{n-1} a_k \end{array}) = 0.$$

because there are two equal columns. The case for rows follows from the fact that $\det (A) = \det (A^T)$. ■
4.7.6 The Determinant Of A Product

Recall the following definition of matrix multiplication.

**Definition 4.7.9** If $A$ and $B$ are $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, $AB = (c_{ij})$ where
\[
c_{ij} \equiv \sum_{k=1}^{n} a_{ik}b_{kj}.
\]

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

**Theorem 4.7.10** Let $A$ and $B$ be $n \times n$ matrices. Then
\[
\det(AB) = \det(A) \det(B).
\]

**Proof:** Let $c_{ij}$ be the $ij^{th}$ entry of $AB$. Then by Proposition 4.7.3,
\[
det(AB) = \sum_{(k_1, \ldots, k_n)} \prod_{r=1}^{n} a_{1r}b_{r_1k_1}\cdots a_{nr}b_{r_nk_n} = \sum_{(r_1, \ldots, r_n)} \prod_{r=1}^{n} a_{1r_1}b_{r_1k_1}\cdots a_{nr}b_{r_nk_n} \det(B) = det(A) \det(B).
\]

4.7.7 Cofactor Expansions

**Lemma 4.7.11** Suppose a matrix is of the form
\[
M = \begin{pmatrix} A & * \\ 0 & a \end{pmatrix}
\]

or
\[
M = \begin{pmatrix} A & 0 \\ * & a \end{pmatrix}
\]

where $a$ is a number and $A$ is an $(n-1) \times (n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the $0$ denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then $\det(M) = a \det(A)$. 
4.7. THE DETERMINANT

Proof: Denote \( M \) by \((m_{ij})\). Thus in the first case, \( m_{nn} = a \) and \( m_{ni} = 0 \) if \( i \neq n \) while in the second case, \( m_{nn} = a \) and \( m_{in} = 0 \) if \( i \neq n \). From the definition of the determinant,

\[
\det (M) = \sum_{(k_1, \ldots, k_n)} \text{sgn}_n (k_1, \ldots, k_n) m_{1k_1} \cdots m_{nk_n}
\]

Letting \( \theta \) denote the position of \( n \) in the ordered list, \((k_1, \ldots, k_n)\) then using Lemma \[\ref{lemma:4.6.1}\], \( \det (M) \) equals

\[
\sum_{(k_1, \ldots, k_n)} (-1)^{n-\theta} \text{sgn}_{n-1} (k_1, \ldots, k_{\theta-1}, k_{\theta+1}, \ldots, k_n) m_{1k_1} \cdots m_{nk_n}
\]

Now suppose \[\ref{4.7.28}\]. Then if \( n \neq n \), the term involving \( m_{nk_n} \) in the above expression equals zero. Therefore, the only terms which survive are those for which \( \theta = n \) or in other words, those for which \( k_n = n \). Therefore, the above expression reduces to

\[
a \sum_{(k_1, \ldots, k_{n-1})} \text{sgn}_{n-1} (k_1, \ldots, k_{n-1}) m_{1k_1} \cdots m_{(n-1)k_{n-1}} = a \det (A).
\]

To get the assertion in the situation of \[\ref{4.7.28}\] use Corollary \[\ref{corollary:4.7.5}\] and \[\ref{4.7.29}\] to write

\[
\det (M) = \det (M^T) = \det \left( \begin{pmatrix} A^T & \mathbf{0} \\ \mathbf{*} & a \end{pmatrix} \right) = a \det (A^T) = a \det (A).
\]

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 4.7.12 Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. Then a new matrix called the cofactor matrix, \( \text{cof} (A) \) is defined by \( \text{cof} (A) = (c_{ij}) \) where to obtain \( c_{ij} \) delete the \( i \)th row and the \( j \)th column of \( A \), take the determinant of the \( (n-1) \times (n-1) \) matrix which results, (This is called the \( ij \)th minor of \( A \).) and then multiply this number by \((-1)^{i+j}\). To make the formulas easier to remember, \( \text{cof} (A)_{ij} \) will denote the \( ij \)th entry of the cofactor matrix.

The following is the main result. Earlier this was given as a definition and the outrageous totally unjustified assertion was made that the same number would be obtained by expanding the determinant along any row or column. The following theorem proves this assertion.

Theorem 4.7.13 Let \( A \) be an \( n \times n \) matrix where \( n \geq 2 \). Then

\[
\det (A) = \sum_{j=1}^{n} a_{ij} \text{cof} (A)_{ij} = \sum_{i=1}^{n} a_{ij} \text{cof} (A)_{ij}.
\]

(4.7.30)

The first formula consists of expanding the determinant along the \( i \)th row and the second expands the determinant along the \( j \)th column.
**Proof:** Let \((a_{i_1}, \cdots, a_{i_n})\) be the \(i^{th}\) row of \(A\). Let \(B_j\) be the matrix obtained from \(A\) by leaving every row the same except the \(i^{th}\) row which in \(B_j\) equals 

\[(0, \cdots, 0, a_{ij}, 0, \cdots, 0)\].

Then by Corollary \ref{c4.7.6},

\[
\det(A) = \sum_{j=1}^{n} \det(B_j)
\]

Denote by \(A_{ij}\) the \((n-1) \times (n-1)\) matrix obtained by deleting the \(i^{th}\) row and the \(j^{th}\) column of \(A\). Thus \(\text{cof}(A)_{ij} = (-1)^{i+j} \det(A^{ij})\). At this point, recall that from Proposition \ref{p4.7.3}, when two rows or two columns in a matrix \(M\), are switched, this results in multiplying the determinant of the old matrix by \(-1\) to get the determinant of the new matrix. Therefore, by Lemma \ref{l4.7.11},

\[
\det(B_j) = (-1)^{n-j}(-1)^{n-i} \det\left( \begin{pmatrix} A^{ij} & * \\ 0 & a_{ij} \end{pmatrix} \right)
\]

\[
= (-1)^{i+j} \det\left( \begin{pmatrix} A^{ij} & * \\ 0 & a_{ij} \end{pmatrix} \right) = a_{ij} \text{cof}(A)_{ij}.
\]

Therefore,

\[
\det(A) = \sum_{j=1}^{n} a_{ij} \text{cof}(A)_{ij}
\]

which is the formula for expanding \(\det(A)\) along the \(i^{th}\) row. Also,

\[
\det(A) = \det(A^T) = \sum_{j=1}^{n} a_{ij}^T \text{cof}(A^T)_{ij}
\]

\[
= \sum_{j=1}^{n} a_{ji} \text{cof}(A)_{ji}
\]

which is the formula for expanding \(\det(A)\) along the \(i^{th}\) column. \(\blacksquare\)

4.7.8 Formula For The Inverse

Note that this gives an easy way to write a formula for the inverse of an \(n \times n\) matrix.

**Theorem 4.7.14** \(A^{-1}\) exists if and only if \(\det(A) \neq 0\). If \(\det(A) \neq 0\), then \(A^{-1} = (a^{-1}_{ij})\) where

\[
a^{-1}_{ij} = \det(A)^{-1} \text{cof}(A)_{ji}
\]

for \(\text{cof}(A)_{ij}\) the \(ij^{th}\) cofactor of \(A\).
Proof: By Theorem 4.7.13 and letting $A = A$, if $\det(A) \neq 0$,
\[
\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1.
\]
Now consider
\[
\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}
\]
when $k \neq r$. Replace the $k^{th}$ column with the $r^{th}$ column to obtain a matrix $B_{k}$ whose determinant equals zero by Corollary 4.7.6. However, expanding this matrix along the $k^{th}$ column yields
\[
0 = \det(B_{k}) \det(A)^{-1} = \sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}
\]
Summarizing,
\[
\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.
\]
Using the other formula in Theorem 4.7.13 and similar reasoning,
\[
\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}
\]
This proves that if $\det(A) \neq 0$, then $A^{-1}$ exists with $A^{-1} = (a_{ij}^{-1})$, where
\[
a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.
\]
Now suppose $A^{-1}$ exists. Then by Theorem 4.7.10,
\[
1 = \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1})
\]
so $\det(A) \neq 0$. ■

The next corollary points out that if an $n \times n$ matrix $A$ has a right or a left inverse, then it has an inverse.

Corollary 4.7.15 Let $A$ be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix $B$ such that $BA = I$. Then $A^{-1}$ exists and $A^{-1} = B$. Also, if there exists $C$ an $n \times n$ matrix such that $AC = I$, then $A^{-1}$ exists and $A^{-1} = C$.

Proof: Since $BA = I$, Theorem 4.7.11 implies
\[
\det B \det A = 1
\]
and so $\det A \neq 0$. Therefore from Theorem 4.7.10, $A^{-1}$ exists. Therefore,
\[
A^{-1} = (BA) A^{-1} = B (AA^{-1}) = BI = B.
\]
The case where $CA = I$ is handled similarly. ■

The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 4.7.14 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix $A$. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, $A^{-1}$ is equal to one over the determinant of $A$ times the adjugate matrix of $A$.

4.7.9 Cramer’s Rule

In case you are solving a system of equations, $Ax = y$ for $x$, it follows that if $A^{-1}$ exists,

$$x = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}y$$

thus solving the system. Now in the case that $A^{-1}$ exists, there is a formula for $A^{-1}$ given above. Using this formula,

$$x_i = \sum_{j=1}^{n} a_{ij}^{-1} y_j = \sum_{j=1}^{n} \frac{1}{\det(A)} \text{cof}(A)_{ji} y_j.$$ 

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{pmatrix} \ast & \cdots & y_1 & \cdots & \ast \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \ast & \cdots & y_n & \cdots & \ast \end{pmatrix},$$

where here the $i^{th}$ column of $A$ is replaced with the column vector $(y_1, \ldots, y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer’s rule.

4.7.10 Upper Triangular Matrices

Definition 4.7.16 A matrix $M$, is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form $M_{ii}$ as shown.

$$\begin{pmatrix} \ast & \ast & \cdots & \ast \\ 0 & \ast & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ast \\ 0 & \cdots & 0 & \ast \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 4.7.14.
Corollary 4.7.17 Let $M$ be an upper (lower) triangular matrix. Then $\text{det}(M)$ is obtained by taking the product of the entries on the main diagonal.

4.8 The Cayley Hamilton Theorem*

**Definition 4.8.1** Let $A$ be an $n \times n$ matrix. The characteristic polynomial is defined as

$$q_A(t) \equiv \text{det}(tI - A)$$

and the solutions to $q_A(t) = 0$ are called eigenvalues. For $A$ a matrix and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, denote by $p(A)$ the matrix defined by

$$p(A) \equiv A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

The explanation for the last term is that $A^0$ is interpreted as $I$, the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $q_A(t) = 0$. It is one of the most important theorems in linear algebra. The proof in this section is not the most general proof, but works well when the field of scalars is $\mathbb{R}$ or $\mathbb{C}$. The following lemma will help with its proof.

**Lemma 4.8.2** Suppose for all $|\lambda|$ large enough,

$$A_0 + A_1 \lambda + \cdots + A_m \lambda^m = 0,$$

where the $A_i$ are $n \times n$ matrices. Then each $A_i = 0$.

**Proof:** Multiply by $\lambda^{-m}$ to obtain

$$A_0 \lambda^{-m} + A_1 \lambda^{-m+1} + \cdots + A_{m-1} \lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_m = 0$. With this, multiply by $\lambda$ to obtain

$$A_0 \lambda^{-m+1} + A_1 \lambda^{-m+2} + \cdots + A_{m-1} = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by $\lambda$ and letting $\lambda \to \infty$ to obtain that all the $A_i = 0$. ■

With the lemma, here is a simple corollary.

**Corollary 4.8.3** Let $A_i$ and $B_i$ be $n \times n$ matrices and suppose

$$A_0 + A_1 \lambda + \cdots + A_m \lambda^m = B_0 + B_1 \lambda + \cdots + B_m \lambda^m$$

for all $|\lambda|$ large enough. Then $A_i = B_i$ for all $i$. If $A_i = B_i$ for each $A_i, B_i$ then one can substitute an $n \times n$ matrix $M$ for $\lambda$ and the identity will continue to hold.

---

*A special case was first proved by Hamilton in 1853. The general case was announced by Cayley some time later and a proof was given by Frobenius in 1878.*
Proof: Subtract and use the result of the lemma. The last claim is obvious by matching terms.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

**Theorem 4.8.4** Let $A$ be an $n \times n$ matrix and let $q(\lambda) \equiv \det (\lambda I - A)$ be the characteristic polynomial. Then $q(A) = 0$.

**Proof:** Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I - A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then $\lambda$ cannot be in the finite list of eigenvalues of $A$ and so for such $\lambda$, $(\lambda I - A)^{-1}$ exists.) Therefore, by Theorem 4.8.3

$$C(\lambda) = q(\lambda) (\lambda I - A)^{-1}.$$  

Say

$$q(\lambda) = a_0 + a_1 \lambda + \cdots + \lambda^n$$

Note that each entry in $C(\lambda)$ is a polynomial in $\lambda$ having degree no more than $n - 1$. For example, you might have something like

$$C(\lambda) = \begin{pmatrix}
9 & 3 & 0 \\
-6 & 0 & 0 \\
-1 & -1 & 2
\end{pmatrix} + \lambda \begin{pmatrix}
-6 & -1 & 0 \\
2 & -3 & 0 \\
1 & 1 & -3
\end{pmatrix} + \lambda^2 \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

Therefore, collecting the terms in the general case,

$$C(\lambda) = C_0 + C_1 \lambda + \cdots + C_{n-1} \lambda^{n-1}$$

for $C_j$ some $n \times n$ matrix. Then

$$C(\lambda) (\lambda I - A) = (C_0 + C_1 \lambda + \cdots + C_{n-1} \lambda^{n-1}) (\lambda I - A) = q(\lambda) I$$

Then multiplying out the middle term, it follows that for all $|\lambda|$ sufficiently large,

$$a_0 I + a_1 I \lambda + \cdots + I \lambda^n = C_0 \lambda + C_1 \lambda^2 + \cdots + C_{n-1} \lambda^n$$

$$- [C_0 A + C_1 A \lambda + \cdots + C_{n-1} A \lambda^{n-1}]$$

$$= -C_0 A + (C_0 - C_1 A) \lambda + (C_1 - C_2 A) \lambda^2 + \cdots + (C_{n-2} - C_{n-1} A) \lambda^{n-1} + C_{n-1} \lambda^n$$

Then, using Corollary 4.8.3, one can replace $\lambda$ on both sides with $A$. Then the right side is seen to equal 0. Hence the left side, $q(A) I$ is also equal to 0.
4.9  An Identity Of Cauchy

There is a very interesting identity for determinants due to Cauchy.

**Theorem 4.9.1** The following identity holds.

\[
\prod_{i,j} (a_i + b_j) \begin{vmatrix}
\frac{1}{a_1 + b_1} & \cdots & \frac{1}{a_1 + b_n} \\
\vdots & \ddots & \vdots \\
\frac{1}{a_n + b_1} & \cdots & \frac{1}{a_n + b_n}
\end{vmatrix} = \prod_{j<i} (a_i - a_j) (b_i - b_j). \tag{4.9.31}
\]

**Proof:** What is the exponent of \(a_2\) on the right? It occurs in \((a_2 - a_1)\) and in \((a_m - a_2)\) for \(m > 2\). Therefore, there are exactly \(n - 1\) factors which contain \(a_2\). Therefore, \(a_2\) has an exponent of \(n - 1\).

Similarly, each \(a_k\) is raised to the \(n - 1\) power and the same holds for the \(b_k\) as well. Therefore, the right side of (4.9.31) is of the form

\[ca_1^{n-1}a_2^{n-1} \cdots a_n^{n-1}b_1^{n-1} \cdots b_n^{n-1}\]

where \(c\) is some constant. Now consider the left side of (4.9.31).

This is of the form

\[
\frac{1}{n!} \prod_{i,j} (a_i + b_j) \sum_{i_1 \cdots i_n, j_1 \cdots j_n} \operatorname{sgn} (i_1 \cdots i_n) \operatorname{sgn} (j_1 \cdots j_n) \cdot \frac{1}{a_{i_1} + b_{j_1}} \frac{1}{a_{i_2} + b_{j_2}} \cdots \frac{1}{a_{i_n} + b_{j_n}}.
\]

For a given \(i_1 \cdots i_n, j_1 \cdots j_n\), let

\[S(i_1 \cdots i_n, j_1 \cdots j_n) \equiv \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}.
\]

This equals

\[
\frac{1}{n!} \sum_{i_1 \cdots i_n, j_1 \cdots j_n} \operatorname{sgn} (i_1 \cdots i_n) \operatorname{sgn} (j_1 \cdots j_n) \prod_{(i,j) \notin \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}} (a_i + b_j)
\]

where you can assume the \(i_k\) are all distinct and the \(j_k\) are also all distinct because otherwise \(\operatorname{sgn}\) will produce a 0. Therefore, in

\[
\prod_{(i,j) \notin \{(i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)\}} (a_i + b_j),
\]

there are exactly \(n - 1\) factors which contain \(a_k\) for each \(k\) and similarly, there are exactly \(n - 1\) factors which contain \(b_k\) for each \(k\). Therefore, the left side of (4.9.31) is of the form

\[da_1^{n-1}a_2^{n-1} \cdots a_n^{n-1}b_1^{n-1} \cdots b_n^{n-1}\]
and it remains to verify that \( c = d \). Using the properties of determinants, the left side of (4.9.31) is of the form

\[
\prod_{i \neq j} (a_i + b_j) \begin{vmatrix}
1 & a_1 + b_1 & \cdots & a_1 + b_n \\
\frac{a_2 + b_2}{a_2 + b_1} & 1 & \cdots & \frac{a_1 + b_2}{a_2 + b_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{a_n + b_n}{a_n + b_1} & \frac{a_2 + b_n}{a_n + b_2} & \cdots & 1
\end{vmatrix}
\]

Let \( a_k \to -b_k \). Then this converges to \( \prod_{i \neq j} (-b_i + b_j) \). The right side of (4.9.31) converges to

\[
\prod_{j < i} (-b_i + b_j) (b_i - b_j) = \prod_{i \neq j} (-b_i + b_j).
\]

Therefore, \( d = c \) and this proves the identity.

### 4.10 Block Multiplication Of Matrices

Consider the following problem

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
E & F \\
G & H
\end{pmatrix}
\]

You know how to do this. You get

\[
\begin{pmatrix}
AE + BG & AF + BH \\
CE + DG & CF + DH
\end{pmatrix}
\]

Now what if instead of numbers, the entries, \( A, B, C, D, E, F, G \) are matrices of a size such that the multiplications and additions needed in the above formula all make sense. Would the formula be true in this case? I will show below that this is true.

Suppose \( A \) is a matrix of the form

\[
A = \begin{pmatrix}
A_{11} & \cdots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{r1} & \cdots & A_{rm}
\end{pmatrix}
\]  

(4.10.32)

where \( A_{ij} \) is a \( s_i \times p_j \) matrix where \( s_i \) is constant for \( j = 1, \cdots, m \) for each \( i = 1, \cdots, r \). Such a matrix is called a block matrix, also a partitioned matrix. How do you get the block \( A_{ij} \)? Here is how for \( A \) an \( m \times n \) matrix:

\[
\begin{pmatrix}
s_i \times m \\
0
\end{pmatrix}
\begin{pmatrix}
I_{s_i \times s_i} & 0 \\
0 & I_{p_j \times p_j}
\end{pmatrix}
\]

(4.10.33)
In the block column matrix on the right, you need to have \( c_j - 1 \) rows of zeros above the small \( p_j \times p_j \) identity matrix where the columns of \( A \) involved in \( A_{ij} \) are \( c_j, \ldots, c_j + p_j \) and in the block row matrix on the left, you need to have \( r_i - 1 \) columns of zeros to the left of the \( s_i \times s_i \) identity matrix where the rows of \( A \) involved in \( A_{ij} \) are \( r_i, \ldots, r_i + s_i \). An important observation to make is that the matrix on the right specifies columns to use in the block and the one on the left specifies the rows used. There is no overlap between the blocks of \( A \). Thus the identity \( n \times n \) identity matrix corresponding to multiplication on the right of \( A \) is of the form

\[
\begin{pmatrix}
I_{p_1 \times p_1} & 0 \\
& \ddots & \vdots \\
& 0 & I_{p_m \times p_m}
\end{pmatrix}
\]

these little identity matrices don’t overlap. A similar conclusion follows from consideration of the matrices \( I_{s_i \times s_i} \).

Next consider the question of multiplication of two block matrices. Let \( B \) be a block matrix of the form

\[
\begin{pmatrix}
B_{11} & \cdots & B_{1p} \\
\vdots & \ddots & \vdots \\
B_{r1} & \cdots & B_{rp}
\end{pmatrix}
\]  

(4.10.34)

and \( A \) is a block matrix of the form

\[
\begin{pmatrix}
A_{11} & \cdots & A_{1m} \\
\vdots & \ddots & \vdots \\
A_{p1} & \cdots & A_{pm}
\end{pmatrix}
\]  

(4.10.35)

and that for all \( i, j \), it makes sense to multiply \( B_{is}A_{sj} \) for all \( s \in \{1, \ldots, p\} \). (That is the two matrices, \( B_{is} \) and \( A_{sj} \) are conformable.) and that for fixed \( ij \), it follows \( B_{is}A_{sj} \) is the same size for each \( s \) so that it makes sense to write \( \sum_s B_{is}A_{sj} \).

The following theorem says essentially that when you take the product of two matrices, you can do it two ways. One way is to simply multiply them forming \( BA \). The other way is to partition both matrices, formally multiply the blocks to get another block matrix and this one will be \( BA \) partitioned. Before presenting this theorem, here is a simple lemma which is really a special case of the theorem.

**Lemma 4.10.1** Consider the following product.

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

where the first is \( n \times r \) and the second is \( r \times n \). The small identity matrix \( I \) is an \( r \times r \) matrix and there are \( l \) zero rows above \( I \) and \( l \) zero columns to the left of \( I \) in the right matrix. Then the product of these matrices is a block matrix of the form

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
Proof: From the definition of the way you multiply matrices, the product is
\[
\left( \begin{array}{ccc}
0 & \cdots & 0 \\
I & \cdots & I \\
0 & \cdots & 0
\end{array} \right) e_1 \cdots \left( \begin{array}{ccc}
0 & \cdots & 0 \\
I & \cdots & I \\
0 & \cdots & 0
\end{array} \right) e_r = \sum_{s=1}^{r} e_s
\]
which yields the claimed result. In the formula $e_j$ refers to the column vector of length $r$ which has a 1 in the $j$th position. This proves the lemma.

**Theorem 4.10.2** Let $B$ be a $q \times p$ block matrix as in 4.10.34 and let $A$ be a $p \times n$ block matrix as in 4.10.35 such that $B_{is}$ is conformable with $A_{sj}$ and each product, $B_{is}A_{sj}$ for $s = 1, \cdots, p$ is of the same size so they can be added. Then $BA$ can be obtained as a block matrix such that the $ij$th block is of the form
\[
\sum_s B_{is}A_{sj}. \quad (4.10.36)
\]

**Proof:** From 4.10.33

\[
B_{is}A_{sj} = \left( \begin{array}{ccc}
0 & I_{r_i \times r_i} & 0 \\
0 & I_{p_s \times p_s} & 0 \\
0 & 0 & 0
\end{array} \right) B \left( \begin{array}{ccc}
0 & I_{p_s \times p_s} & 0 \\
0 & I_{q_j \times q_j} & 0 \\
0 & 0 & 0
\end{array} \right) A
\]
where here it is assumed $B_{is}$ is $r_i \times p_s$ and $A_{sj}$ is $p_s \times q_j$. The product involves the $s$th block in the $i$th row of blocks for $B$ and the $s$th block in the $j$th column of $A$. Thus there are the same number of rows above the $I_{p_s \times p_s}$ as there are columns to the left of $I_{p_s \times p_s}$ in those two inside matrices. Then from Lemma 4.10.1
\[
\left( \begin{array}{ccc}
I_{p_s \times p_s} & 0 & I_{p_s \times p_s} \\
0 & I_{p_s \times p_s} & 0 \\
0 & 0 & 0
\end{array} \right) = \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right)
\]
Since the blocks of small identity matrices do not overlap,
\[
\sum_s \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & I_{p_s \times p_s} & 0 \\
0 & 0 & 0
\end{array} \right) = \left( \begin{array}{ccc}
I_{p_1 \times p_1} & 0 & 0 \\
0 & \cdots & 0 \\
0 & 0 & I_{p_n \times p_n}
\end{array} \right) = I
\]
and so
\[
\sum_s B_{is}A_{sj} = \sum_s (0 \ I_{r_i \times r_i} \ 0 ) B \left( \begin{array}{ccc}
0 & I_{p_s \times p_s} & 0 \\
0 & I_{p_s \times p_s} & 0 \\
0 & 0 & 0
\end{array} \right) A \left( \begin{array}{ccc}
0 & I_{q_j \times q_j} \\
0 & 0 & 0
\end{array} \right)
= (0 \ I_{r_i \times r_i} \ 0 ) BIA \left( \begin{array}{ccc}
0 & I_{q_j \times q_j} \\
0 & 0 & 0
\end{array} \right) = (0 \ I_{r_i \times r_i} \ 0 ) BA \left( \begin{array}{ccc}
0 & I_{q_j \times q_j} \\
0 & 0 & 0
\end{array} \right)
\]
Hence the $ij^{th}$ block of $BA$ equals the formal multiplication according to matrix multiplication,

$$
\sum_s B_{is}A_{sj}.
$$

This proves the theorem.

**Example 4.10.3** Let an $n \times n$ matrix have the form

$$A = \begin{pmatrix}
a & b \\
c & P
\end{pmatrix}
$$

where $P$ is $n-1 \times n-1$. Multiply it by

$$B = \begin{pmatrix}
p & q \\
r & Q
\end{pmatrix}
$$

where $B$ is also an $n \times n$ matrix and $Q$ is $n-1 \times n-1$.

You use block multiplication

$$
\begin{pmatrix}
a & b \\
c & P
\end{pmatrix}
\begin{pmatrix}
p & q \\
r & Q
\end{pmatrix} = \begin{pmatrix}
ap + br & aq + bQ \\
pr + cpQ & cq + PQ
\end{pmatrix}
$$

Note that this all makes sense. For example, $b = 1 \times n - 1$ and $r = n - 1 \times 1$ so $br$ is a $1 \times 1$. Similar considerations apply to the other blocks.

Here is an interesting and significant application of block multiplication. In this theorem, $p_M(t)$ denotes the characteristic polynomial, $\det(tI - M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, $M$.

**Theorem 4.10.4** Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times m$ matrix for $m \leq n$. Then

$$p_{BA}(t) = t^{n-m}p_{AB}(t),$$

so the eigenvalues of $BA$ and $AB$ are the same including multiplicities except that $BA$ has $n - m$ extra zero eigenvalues.

**Proof:** Use block multiplication to write

$$
\begin{pmatrix}
AB & 0 \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix} = 
\begin{pmatrix}
AB & ABA \\
B & BA
\end{pmatrix}
$$

$$
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
B & BA
\end{pmatrix} = 
\begin{pmatrix}
AB & ABA \\
B & BA
\end{pmatrix}.
$$

Therefore,

$$
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix}^{-1}
\begin{pmatrix}
AB & 0 \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 \\
B & BA
\end{pmatrix}.$$
Since the two matrices above are similar it follows that $egin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $egin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials. Therefore, noting that $BA$ is an $n \times n$ matrix and $AB$ is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m} \det(tI - AB) = t^{n-m} p_{AB}(t)$. This proves the theorem.

### 4.11 Exercises

1. Show that matrix multiplication is associative. That is, $(AB)C = A(BC)$.

2. Show the inverse of a matrix, if it exists, is unique. Thus if $AB = BA = I$, then $B = A^{-1}$.

3. In the proof of Theorem 4.7.14 it was claimed that $\det(I) = 1$. Here $I = (\delta_{ij})$. Prove this assertion. Also prove Corollary 4.7.17.

4. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{F}^n$ and let $M(v_1, \ldots, v_n)$ denote the matrix whose $i^{th}$ column equals $v_i$. Define

$$d(v_1, \ldots, v_n) \equiv \det(M(v_1, \ldots, v_n)).$$

Prove that $d$ is linear in each variable, (multilinear), that

$$d(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -d(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n), \quad (4.11.37)$$

and

$$d(e_1, \ldots, e_n) = 1 \quad (4.11.38)$$

where here $e_j$ is the vector in $\mathbb{F}^n$ which has a zero in every position except the $j^{th}$ position in which it has a one.

5. Suppose $f : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ satisfies 4.11.37 and 4.11.38 and is linear in each variable. Show that $f = d$.

6. Show that if you replace a row (column) of an $n \times n$ matrix $A$ with itself added to some multiple of another row (column) then the new matrix has the same determinant as the original one.

7. If $A = (a_{ij})$, show $\det(A) = \sum_{(k_1, \ldots, k_n)} \text{sgn}(k_1, \ldots, k_n) a_{k_11} \cdots a_{k_n n}$.

8. Use the result of Problem 4 to evaluate by hand the determinant

$$\det \begin{pmatrix} 1 & 2 & 3 & 2 \\ -6 & 3 & 2 & 3 \\ 5 & 2 & 2 & 3 \\ 3 & 4 & 6 & 4 \end{pmatrix}.$$
9. Find the inverse if it exists of the matrix,
\[
\begin{pmatrix}
e^t & \cos t & \sin t \\
e^t & -\sin t & \cos t \\
e^t & -\cos t & -\sin t
\end{pmatrix}
\]

10. Let \(Ly = y^{(n)} + a_{n-1} (x) y^{(n-1)} + \cdots + a_1 (x) y' + a_0 (x) y\) where the \(a_i\) are given continuous functions defined on a closed interval, \((a, b)\) and \(y\) is some function which has \(n\) derivatives so it makes sense to write \(Ly\). Suppose \(Ly_k = 0\) for \(k = 1, 2, \cdots, n\). The Wronskian of these functions, \(y_i\) is defined as
\[
W(y_1, \cdots, y_n)(x) \equiv \det
\begin{pmatrix}
y_1 (x) & \cdots & y_n (x) \\
y_1' (x) & \cdots & y_n' (x) \\
\vdots & \ddots & \vdots \\
y_1^{(n-1)} (x) & \cdots & y_n^{(n-1)} (x)
\end{pmatrix}
\]
Show that for \(W(x) = W(y_1, \cdots, y_n)(x)\) to save space,
\[
W'(x) = \det
\begin{pmatrix}
y_1 (x) & \cdots & y_n (x) \\
y_1' (x) & \cdots & y_n' (x) \\
\vdots & \ddots & \vdots \\
y_1^{(n)} (x) & \cdots & y_n^{(n)} (x)
\end{pmatrix}
\]
Now use the differential equation, \(Ly = 0\) which is satisfied by each of these functions, \(y_i\) and properties of determinants presented above to verify that \(W' + a_{n-1} (x) W = 0\). Give an explicit solution of this linear differential equation, Abel's formula, and use your answer to verify that the Wronskian of these solutions to the equation, \(Ly = 0\) either vanishes identically on \((a, b)\) or never.

11. Two \(n \times n\) matrices, \(A\) and \(B\), are similar if \(B = S^{-1}AS\) for some invertible \(n \times n\) matrix, \(S\). Show that if two matrices are similar, they have the same characteristic polynomials.

12. Suppose the characteristic polynomial of an \(n \times n\) matrix, \(A\) is of the form
\[
t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0
\]
and that \(a_0 \neq 0\). Find a formula \(A^{-1}\) in terms of powers of the matrix, \(A\). Show that \(A^{-1}\) exists if and only if \(a_0 \neq 0\).

13. In constitutive modeling of the stress and strain tensors, one sometimes considers sums of the form \(\sum_{k=0}^{\infty} a_k A^k\) where \(A\) is a \(3 \times 3\) matrix. Show using the Cayley Hamilton theorem that if such a thing makes any sense, you can always obtain it as a finite sum having no more than \(n\) terms.
4.12 Shur’s Theorem

Every matrix is related to an upper triangular matrix in a particularly significant way. This is Shur’s theorem and it is the most important theorem in the spectral theory of matrices.

Lemma 4.12.1 Let \( \{x_1, \cdots, x_n\} \) be a basis for \( F^n \). Then there exists an orthonormal basis for \( F^n \),
\[ \{u_1, \cdots, u_n\} \]
which has the property that for each \( k \leq n \),
\[ \text{span} (x_1, \cdots, x_k) = \text{span} (u_1, \cdots, u_k). \]

Proof: Let \( \{x_1, \cdots, x_n\} \) be a basis for \( F^n \). Let \( u_1 \equiv x_1/|x_1| \). Thus for \( k = 1 \), span \( (u_1) = \text{span} (x_1) \) and \( \{u_1\} \) is an orthonormal set. Now suppose for some \( k < n \), \( u_1, \cdots, u_k \) have been chosen such that \( (u_j \cdot u_l) = \delta_{jl} \) and span \( (x_1, \cdots, x_k) = \text{span} (u_1, \cdots, u_k) \). Then define
\[ u_{k+1} \equiv \frac{x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j}{\|x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j\|}, \tag{4.12.39} \]
where the denominator is not equal to zero because the \( x_j \) form a basis and so \( x_{k+1} \not\in \text{span} (x_1, \cdots, x_k) = \text{span} (u_1, \cdots, u_k) \). Thus by induction,
\[ u_{k+1} \in \text{span} (u_1, \cdots, u_k, x_{k+1}) = \text{span} (x_1, \cdots, x_k, x_{k+1}). \]
Also, \( x_{k+1} \in \text{span} (u_1, \cdots, u_k, u_{k+1}) \) which is seen easily by solving \( 4.12.39 \) for \( x_{k+1} \) and it follows
\[ \text{span} (x_1, \cdots, x_k, x_{k+1}) = \text{span} (u_1, \cdots, u_k, u_{k+1}). \]
If \( l \leq k \),
\[ (u_{k+1} \cdot u_l) = C \left( (x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) (u_j \cdot u_l) \right) \]
\[ = C \left( (x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) \delta_{lj} \right) \]
\[ = C ((x_{k+1} \cdot u_l) - (x_{k+1} \cdot u_l)) = 0. \]
The vectors, \( \{u_j\}_{j=1}^{n} \), generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process. Recall the following definition.
Definition 4.12.2 An $n \times n$ matrix, $U$, is unitary if $UU^* = I = U^*U$ where $U^*$ is defined to be the transpose of the conjugate of $U$.

Theorem 4.12.3 Let $A$ be an $n \times n$ matrix. Then there exists a unitary matrix, $U$ such that

$$U^*AU = T,$$

(4.12.40)

where $T$ is an upper triangular matrix having the eigenvalues of $A$ on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: Let $v_1$ be a unit eigenvector for $A$. Then there exists $\lambda_1$ such that

$$Av_1 = \lambda_1 v_1, \quad |v_1| = 1.$$ 

Extend \{v_1\} to a basis and then use Lemma 4.12.1 to obtain \{v_1, \ldots, v_n\}, an orthonormal basis in $F^n$. Let $U_0$ be a matrix whose $i^{th}$ column is $v_i$. Then from the above, it follows $U_0$ is unitary. Then $U_0^*AU_0$ is of the form

$$\begin{pmatrix}
\lambda_1 & * & \cdots & * \\
0 & & & \\
\vdots & & & A_1 \\
0 & & & 
\end{pmatrix}$$

where $A_1$ is an $(n-1) \times (n-1)$ matrix. Repeat the process for the matrix, $A_1$ above. There exists a unitary matrix $U_1$ such that $U_1^*A_1 U_1$ is of the form

$$\begin{pmatrix}
\lambda_2 & * & \cdots & * \\
0 & & & \\
\vdots & & & A_2 \\
0 & & & 
\end{pmatrix}.$$ 

Now let $U_1$ be the $n \times n$ matrix of the form

$$\begin{pmatrix}
1 & 0 \\
0 & U_1 \end{pmatrix}.$$ 

This is also a unitary matrix because by block multiplication,

$$\begin{pmatrix}
1 & 0 \\
0 & U_1 \end{pmatrix}^* \begin{pmatrix}
1 & 0 \\
0 & U_1 \end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & U_1^* \end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & U_1 \end{pmatrix}$$

$$= \begin{pmatrix}
1 & 0 \\
0 & U_1^*U_1 \end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & I \end{pmatrix}.$$ 

Then using block multiplication, $U_1^*U_0^*AU_0U_1$ is of the form

$$\begin{pmatrix}
\lambda_1 & * & \cdots & * \\
0 & \lambda_2 & * & \cdots & * \\
0 & 0 & \cdots & \\
\vdots & \vdots & & A_2 \\
0 & 0 & & 
\end{pmatrix}.$$
where $A_2$ is an $n - 2 \times n - 2$ matrix. Continuing in this way, there exists a unitary matrix, $U$ given as the product of the $U_i$ in the above construction such that

$$U^*AU = T$$

where $T$ is some upper triangular matrix. Since the matrix is upper triangular, the characteristic equation is $\prod_{i=1}^{n} (\lambda - \lambda_i)$ where the $\lambda_i$ are the diagonal entries of $T$. Therefore, the $\lambda_i$ are the eigenvalues.

What if $A$ is a real matrix and you only want to consider real unitary matrices?

**Theorem 4.12.4** Let $A$ be a real $n \times n$ matrix. Then there exists a real unitary matrix, $Q$ and a matrix $T$ of the form

$$T = \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_r \end{pmatrix} \quad (4.12.41)$$

where $P_i$ equals either a real $1 \times 1$ matrix or $P_i$ equals a real $2 \times 2$ matrix having two complex eigenvalues of $A$ such that $Q^*AQ = T$. The matrix, $T$ is called the real Schur form of the matrix $A$.

**Proof:** Suppose

$$Av_1 = \lambda_1 v_1, \ |v_1| = 1$$

where $\lambda_1$ is real. Then let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of vectors in $\mathbb{R}^n$. Let $Q_0$ be a matrix whose $i^{th}$ column is $v_i$. Then $Q_0^*AQ_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where $A_1$ is a real $n - 1 \times n - 1$ matrix. This is just like the proof of Theorem 4.12.3 up to this point.

Now in case $\lambda_1 = \alpha + i\beta$, it follows since $A$ is real that $v_1 = z_1 + iw_1$ and that $v_1 = z_1 - iw_1$ is an eigenvector for the eigenvalue, $\alpha - i\beta$. Here $z_1$ and $w_1$ are real vectors. It is clear that $\{z_1, w_1\}$ is an independent set of vectors in $\mathbb{R}^n$. Indeed, $\{v_1, v_1\}$ is an independent set and it follows span $(v_1, v_1) = \text{span} (z_1, w_1)$.

Now using the Gram Schmidt theorem in $\mathbb{R}^n$, there exists $\{u_1, u_2\}$, an orthonormal set of real vectors such that span $(u_1, u_2) = \text{span} (v_1, v_1)$. Now let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis in $\mathbb{R}^n$ and let $Q_0$ be a unitary matrix whose $i^{th}$ column is $u_i$. Then $Au_j$ are both in span $(u_1, u_2)$ for $j = 1, 2$ and so $u_i^TAu_j = 0$ whenever $k \geq 3$. It follows that $Q_0^*AQ_0$ is of the form

$$\begin{pmatrix} * & * & \cdots & * \\ * & * & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$
where $A_1$ is now an $n - 2 \times n - 2$ matrix. In this case, find $\tilde{Q}_1$ an $n - 2 \times n - 2$ matrix to put $A_1$ in an appropriate form as above and come up with $A_2$ either an $n - 4 \times n - 4$ matrix or an $n - 3 \times n - 3$ matrix. Then the only other difference is to let

$$ Q_1 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \ddots & \tilde{Q}_1 \\
0 & 0 & \cdots & \cdots & 1
\end{pmatrix} $$

thus putting a $2 \times 2$ identity matrix in the upper left corner rather than a one. Repeating this process with the above modification for the case of a complex eigenvalue leads eventually to $Q$ where $Q$ is the product of real unitary matrices $Q_i$ above. Finally,

$$ \lambda I - T = \begin{pmatrix}
\lambda I_1 - P_1 & \cdots & * \\
& \ddots & \vdots \\
0 & \cdots & \lambda I_r - P_r
\end{pmatrix} $$

where $I_k$ is the $2 \times 2$ identity matrix in the case that $P_k$ is $2 \times 2$ and is the number 1 in the case where $P_k$ is a $1 \times 1$ matrix. Now, it follows that $\det(\lambda I - T) = \prod_{k=1}^r \det(\lambda I_k - P_k)$. Therefore, $\lambda$ is an eigenvalue of $T$ if and only if it is an eigenvalue of some $P_k$. This proves the theorem since the eigenvalues of $T$ are the same as those of $A$ because they have the same characteristic polynomial due to the similarity of $A$ and $T$.

**Definition 4.12.5** When a linear transformation, $A$, mapping a linear space, $V$ to $V$ has a basis of eigenvectors, the linear transformation is called non defective. Otherwise it is called defective. An $n \times n$ matrix, $A$, is called normal if $AA^* = A^*A$. An important class of normal matrices is that of the Hermitian or self adjoint matrices. An $n \times n$ matrix, $A$ is self adjoint or Hermitian if $A = A^*$.

The next lemma is the basis for concluding that every normal matrix is unitarily similar to a diagonal matrix.

**Lemma 4.12.6** If $T$ is upper triangular and normal, then $T$ is a diagonal matrix.

**Proof:** Since $T$ is normal, $T^*T = TT^*$. Writing this in terms of components and using the description of the adjoint as the transpose of the conjugate, yields the following for the $ik^{th}$ entry of $T^*T = TT^*$.

$$ \sum_j t_{ij}^* t_{jk} = \sum_j t_{ij} \overline{t_{kj}} = \sum_j t_{ij}^* t_{jk} = \sum_j t_{ji}^* t_{jk}. $$

Now use the fact that $T$ is upper triangular and let $i = k = 1$ to obtain the following from the above.

$$ \sum_j |t_{1j}|^2 = \sum_j |t_{j1}|^2 = |t_{11}|^2 $$
You see, $t_{j1} = 0$ unless $j = 1$ due to the assumption that $T$ is upper triangular. This shows $T$ is of the form

$$
\begin{pmatrix}
* & 0 & \cdots & 0 \\
0 & * & \cdots & * \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & *
\end{pmatrix}.
$$

Now do the same thing only this time take $i = k = 2$ and use the result just established. Thus, from the above,

$$
\sum_j |t_{2j}|^2 = \sum_j |t_{j2}|^2 = |t_{22}|^2,
$$

showing that $t_{2j} = 0$ if $j > 2$ which means $T$ has the form

$$
\begin{pmatrix}
* & 0 & 0 & \cdots & 0 \\
0 & * & 0 & \cdots & 0 \\
0 & 0 & * & \cdots & * \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & *
\end{pmatrix}.
$$

Next let $i = k = 3$ and obtain that $T$ looks like a diagonal matrix in so far as the first 3 rows and columns are concerned. Continuing in this way it follows $T$ is a diagonal matrix.

**Theorem 4.12.7** Let $A$ be a normal matrix. Then there exists a unitary matrix, $U$ such that $U^*AU$ is a diagonal matrix.

**Proof:** From Theorem 4.12.3 there exists a unitary matrix, $U$ such that $U^*AU$ equals an upper triangular matrix. The theorem is now proved if it is shown that the property of being normal is preserved under unitary similarity transformations. That is, verify that if $A$ is normal and if $B = U^*AU$, then $B$ is also normal. But this is easy.

$$
B^*B = U^*A^*U^*AU = U^*A^*AU \\
= U^*AA^*U = U^*AUU^*A^*U = BB^*.
$$

Therefore, $U^*AU$ is a normal and upper triangular matrix and by Lemma 4.12.6 it must be a diagonal matrix. This proves the theorem.

**Corollary 4.12.8** If $A$ is Hermitian, then all the eigenvalues of $A$ are real and there exists an orthonormal basis of eigenvectors.

**Proof:** Since $A$ is normal, there exists unitary, $U$ such that $U^*AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of $A$. Therefore, $D^* = U^*A^*U = U^*AU = D$ showing $D$ is real.
Finally, let
\[ U = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \]
where the \( u_i \) denote the columns of \( U \) and
\[ D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \]
The equation, \( U^*AU = D \) implies
\[ AU = \begin{pmatrix} Au_1 & Au_2 & \cdots & Au_n \end{pmatrix} = UD = \begin{pmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{pmatrix} \]
where the entries denote the columns of \( AU \) and \( UD \) respectively. Therefore, \( Au_i = \lambda_i u_i \) and since the matrix is unitary, the \( ij^{th} \) entry of \( U^*U \) equals \( \delta_{ij} \) and so
\[ \delta_{ij} = u_i^T u_j = u_i^T u_i u_j = u_i \cdot u_j. \]
This proves the corollary because it shows the vectors \( \{u_i\} \) form an orthonormal basis.

**Corollary 4.12.9** If \( A \) is a real symmetric matrix, then \( A \) is Hermitian and there exists a real unitary matrix, \( U \) such that \( U^T AU = D \) where \( D \) is a diagonal matrix.

**Proof:** This follows from Theorem 4.12.4 and Corollary 4.12.8.

### 4.13 The Right Polar Decomposition

The right polar decomposition involves writing a matrix as a product of two other matrices, one which preserves distances and the other which stretches and distorts. First here are some lemmas.

**Lemma 4.13.1** Let \( A \) be a Hermitian matrix such that all its eigenvalues are non-negative. Then there exists a Hermitian matrix, \( A^{1/2} \) such that \( A^{1/2} \) has all non-negative eigenvalues and \( (A^{1/2})^2 = A \).

**Proof:** Since \( A \) is Hermitian, there exists a diagonal matrix \( D \) having all real nonnegative entries and a unitary matrix \( U \) such that \( A = U^*DU \). Then denote by \( D^{1/2} \) the matrix which is obtained by replacing each diagonal entry of \( D \) with its square root. Thus \( D^{1/2}D^{1/2} = D \). Then define
\[ A^{1/2} = U^*D^{1/2}U. \]
Then
\[ (A^{1/2})^2 = U^*D^{1/2}UU^*D^{1/2}U = U^*DU = A. \]
Since $D^{1/2}$ is real,

$$\left(U^* D^{1/2} U\right)^* = U^* \left(D^{1/2} \right)^* (U^*)^* = U^* D^{1/2} U$$

so $A^{1/2}$ is Hermitian. This proves the lemma.

There is also a useful observation about orthonormal sets of vectors which is stated in the next lemma.

**Lemma 4.13.2** Suppose $\{x_1, x_2, \cdots, x_r\}$ is an orthonormal set of vectors. Then if $c_1, \cdots, c_r$ are scalars,

$$\sum_{k=1}^{r} c_k x_k = \sum_{k=1}^{r} |c_k|^2 .$$

**Proof:** This follows from the definition. From the properties of the dot product and using the fact that the given set of vectors is orthonormal,

$$\sum_{k=1}^{r} c_k x_k = \left( \sum_{k=1}^{r} c_k x_k, \sum_{j=1}^{r} c_j x_j \right) = \sum_{k,j} c_k \overline{c_j} (x_k, x_j) = \sum_{k=1}^{r} |c_k|^2 .$$

This proves the lemma.

Next it is helpful to recall the Gram Schmidt algorithm and observe a certain property stated in the next lemma.

**Lemma 4.13.3** Suppose $\{w_1, \cdots, w_r, v_{r+1}, \cdots, v_p\}$ is a linearly independent set of vectors such that $\{w_1, \cdots, w_r\}$ is an orthonormal set of vectors. Then when the Gram Schmidt process is applied to the vectors in the given order, it will not change any of the $w_1, \cdots, w_r$.

**Proof:** Let $\{u_1, \cdots, u_p\}$ be the orthonormal set delivered by the Gram Schmidt process. Then $u_1 = w_1$ because by definition, $u_1 \equiv w_1 / |w_1| = w_1$. Now suppose $u_j = w_j$ for all $j \leq k \leq r$. Then if $k < r$, consider the definition of $u_{k+1}$.

$$u_{k+1} = \frac{w_{k+1} - \sum_{j=1}^{k+1} (w_{k+1}, u_j) u_j}{|w_{k+1} - \sum_{j=1}^{k+1} (w_{k+1}, u_j) u_j|}$$

By induction, $u_j = w_j$ and so this reduces to $w_{k+1} / |w_{k+1}| = w_{k+1}$. This proves the lemma.

This lemma immediately implies the following lemma.

**Lemma 4.13.4** Let $V$ be a subspace of dimension $p$ and let $\{w_1, \cdots, w_r\}$ be an orthonormal set of vectors in $V$. Then this orthonormal set of vectors may be extended to an orthonormal basis for $V$,

$$\{w_1, \cdots, w_r, y_{r+1}, \cdots, y_p\}$$
4.13. THE RIGHT POLAR DECOMPOSITION

Proof: First extend the given linearly independent set \( \{w_1, \cdots, w_r\} \) to a basis for \( V \) and then apply the Gram Schmidt theorem to the resulting basis. Since \( \{w_1, \cdots, w_r\} \) is orthonormal it follows from Lemma 4.13.3 the result is of the desired form, an orthonormal basis extending \( \{w_1, \cdots, w_r\} \). This proves the lemma.

Here is another lemma about preserving distance.

Lemma 4.13.5 Suppose \( R \) is an \( m \times n \) matrix with \( m > n \) and \( R \) preserves distances. Then \( \Re (R^* R - I) = 0 \).

Proof: Since \( R \) preserves distances, \( |Rx| = |x| \) for every \( x \). Therefore from the axioms of the dot product,

\[
|\mathbf{x}|^2 + |\mathbf{y}|^2 + (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{x}) = (\mathbf{R}(\mathbf{x} + \mathbf{y}), (\mathbf{x} + \mathbf{y})) = (\mathbf{Rx}, \mathbf{Rx}) + (\mathbf{Ry}, \mathbf{Ry}) + (\mathbf{Rx}, \mathbf{Ry}) + (\mathbf{Ry}, \mathbf{Rx})
\]

and so for all \( \mathbf{x}, \mathbf{y}, \)

\[
(R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y}) + (\mathbf{y}, R^* \mathbf{Rx} - \mathbf{x}) = 0
\]

Hence for all \( \mathbf{x}, \mathbf{y}, \)

\[
\Re (R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y}) = 0
\]

Now for a \( \mathbf{x}, \mathbf{y} \) given, choose \( \alpha \in \mathbb{C} \) such that

\[
\alpha (R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y}) = |(R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y})|
\]

Then

\[
0 = \Re (R^* \mathbf{Rx} - \mathbf{x}, \overline{\alpha \mathbf{y}}) = \Re \alpha (R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y}) = |(R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y})|
\]

Thus \( |(R^* \mathbf{Rx} - \mathbf{x}, \mathbf{y})| = 0 \) for all \( \mathbf{x}, \mathbf{y} \) because the given \( \mathbf{x}, \mathbf{y} \) were arbitrary. Let \( \mathbf{y} = R^* \mathbf{Rx} - \mathbf{x} \) to conclude that for all \( \mathbf{x}, \)

\[
R^* \mathbf{Rx} - \mathbf{x} = 0
\]

which says \( R^* R = I \) since \( \mathbf{x} \) is arbitrary. This proves the lemma.

With this preparation, here is the big theorem about the right polar decomposition.

Theorem 4.13.6 Let \( F \) be an \( m \times n \) matrix where \( m \geq n \). Then there exists a Hermitian \( n \times n \) matrix, \( U \) which has all nonnegative eigenvalues and an \( m \times n \) matrix, \( R \) which preserves distances and satisfies \( R^* R = I \) such that

\[
F = RU.
\]
**Proof:** Consider $F^*F$. This is a Hermitian matrix because

$$(F^*F)^* = F^*(F^*)^* = F^*F$$

Also the eigenvalues of the $n \times n$ matrix $F^*F$ are all nonnegative. This is because if $x$ is an eigenvalue,

$$\lambda(x, x) = (F^*Fx, x) = (Fx, Fx) \geq 0.$$ 

Therefore, by Lemma 4.13.1, there exists an $n \times n$ Hermitian matrix, $U$ having all nonnegative eigenvalues such that

$$U^2 = F^*F.$$ 

Consider the subspace $U(F^n)$. Let $\{Ux_1, \ldots, Ux_r\}$ be an orthonormal basis for $U(F^n) \subseteq F^n$. Note that $U(F^n)$ might not be all of $F^n$. Using Lemma 4.13.4, extend to an orthonormal basis for all of $F^n$,

$$\{Ux_1, \ldots, Ux_r, y_{r+1}, \ldots, y_n\}.$$ 

Next observe that $\{Fx_1, \ldots, Fx_r\}$ is also an orthonormal set of vectors in $F^m$. This is because

$$(Fx_k, Fx_j) = (F^*Fx_k, x_j) = (U^2x_k, x_j) = (Ux_k, U^*x_j) = (Ux_k, Ux_j) = \delta_{jk}.$$ 

Therefore, from Lemma 4.13.4 again, this orthonormal set of vectors can be extended to an orthonormal basis for $F^m$,

$$\{Fx_1, \ldots, Fx_r, z_{r+1}, \ldots, z_m\}.$$ 

Thus there are at least as many $z_k$ as there are $y_j$. Now for $x \in F^n$, since

$$\{Ux_1, \ldots, Ux_r, y_{r+1}, \ldots, y_n\}$$ 

is an orthonormal basis for $F^n$, there exist unique scalars,

$$c_1, \ldots, c_r, d_{r+1}, \ldots, d_n$$ 

such that

$$x = \sum_{k=1}^r c_k Ux_k + \sum_{j=r+1}^n d_j y_j.$$ 

Define

$$Rx \equiv \sum_{k=1}^r c_k Fx_k + \sum_{j=r+1}^n d_j z_j \quad (4.13.42)$$ 

Then also there exist scalars $b_k$ such that

$$Ux = \sum_{k=1}^r b_k Ux_k.$$
and so from applied to \( Ux \) in place of \( x \)

\[
RUx = \sum_{k=1}^{r} b_k Fx_k = F \left( \sum_{k=1}^{r} b_k x_k \right)
\]

Is \( F(\sum_{k=1}^{r} b_k x_k) = F(x) \)?

\[
\left( F \left( \sum_{k=1}^{r} b_k x_k \right) - F(x) , F \left( \sum_{k=1}^{r} b_k x_k \right) - F(x) \right)
\]

\[
= \left( (F^*F) \left( \sum_{k=1}^{r} b_k x_k - x \right) , \left( \sum_{k=1}^{r} b_k x_k - x \right) \right)
\]

\[
= \left( U^2 \left( \sum_{k=1}^{r} b_k x_k - x \right) , \left( \sum_{k=1}^{r} b_k x_k - x \right) \right)
\]

\[
= \left( U \left( \sum_{k=1}^{r} b_k x_k - x \right) , U \left( \sum_{k=1}^{r} b_k x_k - x \right) \right)
\]

\[
= \left( \sum_{k=1}^{r} b_k Ux_k - Ux , \sum_{k=1}^{r} b_k Ux_k - Ux \right) = 0
\]

Therefore, \( F(\sum_{k=1}^{r} b_k x_k) = F(x) \) and this shows

\[
RUx = Fx.
\]

From and Lemma \( R \) preserves distances. Therefore, by Lemma \( R^*R = I \). This proves the theorem.
Chapter 5

Multi-variable Calculus

5.1 Continuous Functions

In what follows, $F$ will denote either $\mathbb{R}$ or $\mathbb{C}$. It turns out it is more efficient to not make a distinction. However, the main interest is in $\mathbb{R}$ so if you like, you can think $\mathbb{R}$ whenever you see $F$.

5.2 Open And Closed Sets

Eventually, one must consider functions which are defined on subsets of $F^n$ and their properties. The next definition will end up being quite important. It describe a type of subset of $F^n$ with the property that if $x$ is in this set, then so is $y$ whenever $y$ is close enough to $x$. In all of this, for $x$ a vector, $|x|$ is given by $(x, x)^{1/2}$ where this denotes the square root of the inner product of the vector with itself as described earlier. Then the distance between the vectors $x$ and $y$ is defined as $|x - y|$.

**Definition 5.2.1** Let $U \subseteq F^n$. $U$ is an open set if whenever $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. More generally, if $U$ is any subset of $F^n$, $x \in U$ is an interior point of $U$ if there exists $r > 0$ such that $x \in B(x, r) \subseteq U$. In other words $U$ is an open set exactly when every point of $U$ is an interior point of $U$.

If there is something called an open set, surely there should be something called a closed set and here is the definition of one.

**Definition 5.2.2** A subset, $C$, of $F^n$ is called a closed set if $F^n \setminus C$ is an open set. They symbol, $F^n \setminus C$ denotes everything in $F^n$ which is not in $C$. It is also called the complement of $C$. The symbol, $S^C$ is a short way of writing $F^n \setminus S$.

To illustrate this definition, consider the following picture.
You see in this picture how the edges are dotted. This is because an open set, can not include the edges or the set would fail to be open. For example, consider what would happen if you picked a point out on the edge of $U$ in the above picture. Every open ball centered at that point would have in it some points which are outside $U$. Therefore, such a point would violate the above definition. You also see the edges of $B(x, r)$ dotted suggesting that $B(x, r)$ ought to be an open set. This is intuitively clear but does require a proof. This will be done in the next theorem and will give examples of open sets. Also, you can see that if $x$ is close to the edge of $U$, you might have to take $r$ to be very small.

It is roughly the case that open sets don’t have their skins while closed sets do. Here is a picture of a closed set, $C$.

Note that $x \notin C$ and since $\mathbb{F}^n \setminus C$ is open, there exists a ball, $B(x, r)$ contained entirely in $\mathbb{F}^n \setminus C$. If you look at $\mathbb{F}^n \setminus C$, what would be its skin? It can’t be in $\mathbb{F}^n \setminus C$ and so it must be in $C$. This is a rough heuristic explanation of what is going on with these definitions. Also note that $\mathbb{F}^n$ and $\emptyset$ are both open and closed. Here is why. If $x \in \emptyset$, then there must be a ball centered at $x$ which is also contained in $\emptyset$. This must be considered to be true because there is nothing in $\emptyset$ so there can be no example to show it false. Therefore, from the definition, it follows $\emptyset$ is open.

To a mathematician, the statement: Whenever a pig is born with wings it can fly must be taken as true. We do not consider biological or aerodynamic considerations in such statements. There is no such thing as a winged pig and therefore, all winged pigs must be superb flyers since there can be no example of one which is not. On the other hand we would also consider the statement: Whenever a pig is born with wings it can’t possibly fly, as equally true. The point is, you can say
5.2. OPEN AND CLOSED SETS

It is also closed because if $x \notin \emptyset$, then $B(x, 1)$ is also contained in $\mathbb{F}^n \setminus \emptyset = \mathbb{F}^n$. Therefore, $\emptyset$ is both open and closed. From this, it follows $\mathbb{F}^n$ is also both open and closed.

**Theorem 5.2.3** Let $x \in \mathbb{F}^n$ and let $r \geq 0$. Then $B(x, r)$ is an open set. Also, 

$$D(x, r) \equiv \{ y \in \mathbb{F}^n : |y - x| \leq r \}$$

is a closed set.

**Proof:** Suppose $y \in B(x, r)$. It is necessary to show there exists $r_1 > 0$ such that $B(y, r_1) \subseteq B(x, r)$. Define $r_1 \equiv r - |x - y|$. Then if $|z - y| < r_1$, it follows from the above triangle inequality that 

$$|z - x| = |z - y + y + x| \leq |z - y| + |y - x| < r_1 + |y - x| = r - |x - y| + |y - x| = r.$$ 

Note that if $r = 0$ then $B(x, r) = \emptyset$, the empty set. This is because if $y \in \mathbb{F}^n$, $|x - y| \geq 0$ and so $y \notin B(x, 0)$. Since $\emptyset$ has no points in it, it must be open because every point in it, (There are none.) satisfies the desired property of being an interior point.

Now suppose $y \notin D(x, r)$. Then $|x - y| > r$ and defining $\delta \equiv |x - y| - r$, it follows that if $z \in B(y, \delta)$, then by the triangle inequality,

$$|x - z| \geq |x - y| - |y - z| > |x - y| - \delta = |x - y| - (|x - y| - r) = r$$

and this shows that $B(y, \delta) \subseteq \mathbb{F}^n \setminus D(x, r)$. Since $y$ was an arbitrary point in $\mathbb{F}^n \setminus D(x, r)$, it follows $\mathbb{F}^n \setminus D(x, r)$ is an open set which shows from the definition that $D(x, r)$ is a closed set as claimed.

A picture which is descriptive of the conclusion of the above theorem which also implies the manner of proof is the following.

\[ \text{Diagram} \]

---

anything you want about the elements of the empty set and no one can gainsay your statement. Therefore, such statements are considered as true by default. You may say this is a very strange way of thinking about truth and ultimately this is because mathematics is not about truth. It is more about consistency and logic.
5.3 Continuous Functions

With the above definition of the norm in $\mathbb{F}^p$, it becomes possible to define continuity.

**Definition 5.3.1** A function $f : D(f) \subseteq \mathbb{F}^p \to \mathbb{F}^q$ is continuous at $x \in D(f)$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $y \in D(f)$ and

$$|y - x| < \delta$$

it follows that

$$|f(x) - f(y)| < \varepsilon.$$

$f$ is continuous if it is continuous at every point of $D(f)$.

Note the total similarity to the scalar valued case.

5.3.1 Sufficient Conditions For Continuity

The next theorem is a fundamental result which will allow us to worry less about the $\varepsilon \delta$ definition of continuity.

**Theorem 5.3.2** The following assertions are valid.

1. The function, $af + bg$ is continuous at $x$ whenever $f, g$ are continuous at $x$ $\in D(f) \cap D(g)$ and $a, b \in \mathbb{F}$.

2. If $f$ is continuous at $x$, $f(x) \in D(g) \subseteq \mathbb{F}^p$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$.

3. If $f = (f_1, \ldots, f_q) : D(f) \to \mathbb{F}^q$, then $f$ is continuous if and only if each $f_k$ is a continuous $\mathbb{F}$ valued function.

4. The function $f : \mathbb{F}^p \to \mathbb{F}$, given by $f(x) = |x|$ is continuous.

The proof of this theorem is in the last section of this chapter. Its conclusions are not surprising. For example the first claim says that $(af + bg)(y)$ is close to $(af + bg)(x)$ when $y$ is close to $x$ provided the same can be said about $f$ and $g$. For the second claim, if $y$ is close to $x$, $f(x)$ is close to $f(y)$ and so by continuity of $g$ at $f(x)$, $g(f(y))$ is close to $g(f(x))$. To see the third claim is likely, note that closeness in $\mathbb{F}^p$ is the same as closeness in each coordinate. The fourth claim is immediate from the triangle inequality.

For functions defined on $\mathbb{F}^n$, there is a notion of polynomial just as there is for functions defined on $\mathbb{R}$.

**Definition 5.3.3** Let $\alpha$ be an $n$ dimensional multi-index. This means

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$
where each $\alpha_i$ is a natural number or zero. Also, let

$$|\alpha| \equiv \sum_{i=1}^{n} |\alpha_i|$$

The symbol, $x^\alpha$, means

$$x^\alpha \equiv x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$ 

An $n$ dimensional polynomial of degree $m$ is a function of the form

$$p(x) = \sum_{|\alpha| \leq m} d_\alpha x^\alpha.$$ 

where the $d_\alpha$ are complex or real numbers.

The above theorem implies that polynomials are all continuous.

### 5.4 Exercises

1. Let $f(t) = (t, \sin t)$. Show $f$ is continuous at every point $t$.

2. Suppose $|f(x) - f(y)| \leq K|x - y|$ where $K$ is a constant. Show that $f$ is everywhere continuous. Functions satisfying such an inequality are called Lipschitz functions.

3. Suppose $|f(x) - f(y)| \leq K|x - y|^\alpha$ where $K$ is a constant and $\alpha \in (0, 1)$. Show that $f$ is everywhere continuous.

4. Suppose $f : \mathbb{F}^3 \to \mathbb{F}$ is given by $f(x) = 3x_1x_2 + 2x_1^2$. Use Theorem \[\text{[Theorem]}\] to verify that $f$ is continuous. \textbf{Hint:} You should first verify that the function, $\pi_k : \mathbb{F}^3 \to \mathbb{F}$ given by $\pi_k(x) = x_k$ is a continuous function.

5. Generalize the previous problem to the case where $f : \mathbb{F}^q \to \mathbb{F}$ is a polynomial.

6. State and prove a theorem which involves quotients of functions encountered in the previous problem.

### 5.5 Limits Of A Function

As in the case of scalar valued functions of one variable, a concept closely related to continuity is that of the limit of a function. The notion of limit of a function makes sense at points, $x$, which are limit points of $D(f)$ and this concept is defined next.

**Definition 5.5.1** Let $A \subseteq \mathbb{F}^m$ be a set. A point, $x$, is a limit point of $A$ if $B(x, r)$ contains infinitely many points of $A$ for every $r > 0$. 
Definition 5.5.2  Let $f : D(f) \subseteq \mathbb{F}^p \rightarrow \mathbb{F}^q$ be a function and let $x$ be a limit point of $D(f)$. Then
\[
\lim_{y \rightarrow x} f(y) = L
\]
if and only if the following condition holds. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if
\[
0 < |y - x| < \delta, \text{ and } y \in D(f)
\]
then,
\[
|L - f(y)| < \varepsilon.
\]

Theorem 5.5.3  If $\lim_{y \rightarrow x} f(y) = L$ and $\lim_{y \rightarrow x} f(y) = L_1$, then $L = L_1$.

Proof: Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $0 < |y - x| < \delta$ and $y \in D(f)$, then
\[
|f(y) - L| < \varepsilon, \text{ and } |f(y) - L_1| < \varepsilon.
\]
Pick such a $y$. There exists one because $x$ is a limit point of $D(f)$. Then
\[
|L - L_1| \leq |L - f(y)| + |f(y) - L_1| < \varepsilon + \varepsilon = 2\varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary, this shows $L = L_1$.

As in the case of functions of one variable, one can define what it means for
\[
\lim_{y \rightarrow x} f(x) = \pm \infty.
\]

Definition 5.5.4  If $f(x) \in \mathbb{F}$, $\lim_{y \rightarrow x} f(x) = \infty$ if for every number $l$, there exists $\delta > 0$ such that whenever $|y - x| < \delta$ and $y \in D(f)$, then $f(x) > l$.

The following theorem is just like the one variable version presented earlier.

Theorem 5.5.5  Suppose $\lim_{y \rightarrow x} f(y) = L$ and $\lim_{y \rightarrow x} g(y) = K$ where $K, L \in \mathbb{F}^q$. Then if $a, b \in \mathbb{F}$,
\[
\lim_{y \rightarrow x} (af(y) + bg(y)) = aL + bK, \quad (5.5.1)
\]
\[
\lim_{y \rightarrow x} f \cdot g(y) =LK \quad (5.5.2)
\]
and if $g$ is scalar valued with $\lim_{y \rightarrow x} g(y) = K \neq 0$,
\[
\lim_{y \rightarrow x} f(y) g(y) = LK. \quad (5.5.3)
\]

Also, if $h$ is a continuous function defined near $L$, then
\[
\lim_{y \rightarrow x} h \circ f(y) = h(L). \quad (5.5.4)
\]

Suppose $\lim_{y \rightarrow x} f(y) = L$. If $|f(y) - b| \leq r$ for all $y$ sufficiently close to $x$, then $|L - b| \leq r$ also.
5.5. LIMITS OF A FUNCTION

Proof: The proof of 5.5.1 is left for you. It is like a corresponding theorem for continuous functions. Now 5.5.2 is to be verified. Let \( \varepsilon > 0 \) be given. Then by the triangle inequality,

\[
|f \cdot g(y) - L \cdot K| \leq |f(g(y) - f(y)) \cdot K| + |f(y) \cdot K - L \cdot K|
\]

\[
\leq |f(y)||g(y) - K| + |K||f(y) - L|.
\]

There exists \( \delta_1 \) such that if \( 0 < |y - x| < \delta_1 \) and \( y \in D(f) \), then

\[
|f(y) - L| < 1,
\]

and so for such \( y \), the triangle inequality implies, \(|f(y)| < 1 + |L|\). Therefore, for \( 0 < |y - x| < \delta_1 \),

\[
|f \cdot g(y) - L \cdot K| \leq (1 + |K| + |L|)(|g(y) - K| + |f(y) - L|).
\]

Now let \( 0 < \delta_2 \) be such that if \( y \in D(f) \) and \( 0 < |x - y| < \delta_2 \),

\[
|f(y) - L| < \frac{\varepsilon}{2(1 + |K| + |L|)}, \quad |g(y) - K| < \frac{\varepsilon}{2(1 + |K| + |L|)}.
\]

Then letting \( 0 < \delta = \min(\delta_1, \delta_2) \), it follows from 5.5.3 that

\[
|f \cdot g(y) - L \cdot K| < \varepsilon
\]

and this proves 5.5.2.

The proof of 5.5.3 is left to you.

Consider 5.5.4. Since \( h \) is continuous near \( L \), it follows that for \( \varepsilon > 0 \) given, there exists \( \eta > 0 \) such that if \( |y - L| < \eta \), then

\[
|h(y) - h(L)| < \varepsilon
\]

Now since \( \lim_{y \to x} f(y) = L \), there exists \( \delta > 0 \) such that if \( 0 < |y - x| < \delta \), then

\[
|f(y) - L| < \eta.
\]

Therefore, if \( 0 < |y - x| < \delta \),

\[
|h(f(y)) - h(L)| < \varepsilon.
\]

It only remains to verify the last assertion. Assume \( |f(y) - b| \leq r \) for all \( y \) close enough to \( x \). It is required to show that \( |L - b| \leq r \). If this is not true, then \( |L - b| > r \). Consider \( B(L, |L - b| - r) \). Since \( L \) is the limit of \( f \), it follows \( f(y) \in B(L, |L - b| - r) \) whenever \( y \in D(f) \) is close enough to \( x \). Thus, by the triangle inequality,

\[
|f(y) - L| < |L - b| - r
\]

and so

\[
r < |L - b| - |f(y) - L| \leq ||b - L| - |f(y) - L||
\]

\[
\leq |b - f(y)|,
\]

a contradiction to the assumption that \( |b - f(y)| \leq r \).
Theorem 5.5.6 For \( f : D(f) \to \mathbb{R}^q \) and \( x \in D(f) \) a limit point of \( D(f) \), \( f \) is continuous at \( x \) if and only if
\[
\lim_{y \to x} f(y) = f(x).
\]

Proof: First suppose \( f \) is continuous at \( x \) a limit point of \( D(f) \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |y - x| < \delta \) and \( y \in D(f) \), then \( |f(x) - f(y)| < \varepsilon \). In particular, this holds if \( 0 < |x - y| < \delta \) and this is just the definition of the limit. Hence \( f(x) = \lim_{y \to x} f(y) \).

Next suppose \( x \) is a limit point of \( D(f) \) and \( \lim_{y \to x} f(y) = f(x) \). This means that if \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for \( 0 < |x - y| < \delta \) and \( y \in D(f) \), it follows \( |f(y) - f(x)| < \varepsilon \). However, if \( y = x \), then \( |f(y) - f(x)| = |f(x) - f(x)| = 0 \) and so whenever \( y \in D(f) \) and \( |x - y| < \delta \), it follows \( |f(x) - f(y)| < \varepsilon \), showing \( f \) is continuous at \( x \).

The following theorem is important.

Theorem 5.5.7 Suppose \( f : D(f) \to \mathbb{R}^q \). Then for \( x \) a limit point of \( D(f) \),
\[
\lim_{y \to x} f(y) = L
\]
if and only if
\[
\lim_{y \to x} f_k(y) = L_k
\]
where \( f(y) \equiv (f_1(y), \ldots, f_p(y)) \) and \( L \equiv (L_1, \ldots, L_p) \).

Proof: Suppose \( \ref{5.5.6} \). Then letting \( \varepsilon > 0 \) be given there exists \( \delta > 0 \) such that if \( 0 < |y - x| < \delta \), it follows
\[
|f_k(y) - L_k| \leq |f(y) - L| < \varepsilon
\]
which verifies \( \ref{5.5.7} \).

Now suppose \( \ref{5.5.7} \) holds. Then letting \( \varepsilon > 0 \) be given, there exists \( \delta_k \) such that if \( 0 < |y - x| < \delta_k \), then
\[
|f_k(y) - L_k| < \frac{\varepsilon}{\sqrt{p}}.
\]
Let \( 0 < \delta < \min(\delta_1, \ldots, \delta_p) \). Then if \( 0 < |y - x| < \delta \), it follows
\[
|f(y) - L| = \left( \sum_{k=1}^p |f_k(y) - L_k|^2 \right)^{1/2} < \left( \sum_{k=1}^p \frac{\varepsilon^2}{p} \right)^{1/2} = \varepsilon.
\]
This proves the theorem.

This theorem shows it suffices to consider the components of a vector valued function when computing the limit.
Example 5.5.8 Find \( \lim_{(x,y)\to(3,1)} \left( \frac{x^2-9}{x-3}, y \right) \).

It is clear that \( \lim_{(x,y)\to(3,1)} \frac{x^2-9}{x-3} = 6 \) and \( \lim_{(x,y)\to(3,1)} y = 1 \). Therefore, this limit equals \((6, 1)\).

Example 5.5.9 Find \( \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \).

First of all observe the domain of the function is \( \mathbb{F}^2 \setminus \{(0,0)\} \), every point in \( \mathbb{F}^2 \) except the origin. Therefore, \((0,0)\) is a limit point of the domain of the function so it might make sense to take a limit. However, just as in the case of a function of one variable, the limit may not exist. In fact, this is the case here. To see this, take points on the line \( y = 0 \). At these points, the value of the function equals 0. Now consider points on the line \( y = x \) where the value of the function equals \( \frac{1}{2} \). Since arbitrarily close to \((0,0)\) there are points where the function equals \( \frac{1}{2} \) and points where the function has the value 0, it follows there can be no limit. Just take \( \varepsilon = 1/10 \) for example. You can’t be within \( 1/10 \) of \( \frac{1}{2} \) and also within \( 1/10 \) of 0 at the same time.

Note it is necessary to rely on the definition of the limit much more than in the case of a function of one variable and it is the case there are no easy ways to do limit problems for functions of more than one variable. It is what it is and you will not deal with these concepts without agony.

5.6 Exercises

1. Find the following limits if possible
   (a) \( \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \)
   (b) \( \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{(x^2+y^2)} \)
   (c) \( \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{(x^2+y^2)^2} \) \quad \text{Hint: Consider along } y = 0 \text{ and along } x = y^2.
   (d) \( \lim_{(x,y)\to(0,0)} x \sin \left( \frac{1}{x^2+y^2} \right) \)
   (e) The limit as \( (x,y) \to (1,2) \) of the expression
      \[ -2yx^2 + 8yx + 34y + 3y^3 - 18y^2 + 6x^2 - 13x - 20 - xy^2 - x^3 \]
      \[ -y^2 + 4y - 5 - x^2 + 2x \]
      \[ = \frac{-2yx^2 + 8yx + 34y + 3y^3 - 18y^2 + 6x^2 - 13x - 20 - xy^2 - x^3}{-y^2 + 4y - 5 - x^2 + 2x} \]
      \[ = \frac{-2(1)(1) + 8(1)(2) + 34(2) + 3(2)^3 - 18(2)^2 + 6(1)^2 - 13(1) - 20 - (1)(2)^2 - (1)^3}{-2^2 + 4(2) - 5 - 1^2 + 2(1)} \]
      \[ = \frac{-20 + 32 + 68 + 24 - 72 + 6 - 13 - 20 - 4 - 1}{-4 + 8 - 5 - 1 + 2} \]
      \[ = \frac{30}{2} = 15 \]
      \quad \text{Hint: It might help to write this in terms of the variables } (s,t) = (x-1, y-2).

2. In the definition of limit, why must \( x \) be a limit point of \( D(f) \)? \textbf{Hint:} If \( x \) were not a limit point of \( D(f) \), show there exists \( \delta > 0 \) such that \( B(x, \delta) \) contains no points of \( D(f) \) other than possibly \( x \) itself. Argue that 33.3 is a limit and that so is 22 and 7 and 11. In other words the concept is totally worthless.
5.7 The Limit Of A Sequence

As in the case of real numbers, one can consider the limit of a sequence of points in $\mathbb{F}^p$.

**Definition 5.7.1** A sequence $\{a_n\}_{n=1}^{\infty}$ converges to $a$, and write

$$\lim_{n \to \infty} a_n = a \quad \text{or} \quad a_n \to a$$

if and only if for every $\varepsilon > 0$ there exists $n_\varepsilon$ such that whenever $n \geq n_\varepsilon$,

$$|a_n - a| < \varepsilon.$$

In words the definition says that given any measure of closeness, $\varepsilon$, the terms of the sequence are eventually all this close to $a$. There is absolutely no difference between this and the definition for sequences of numbers other than here bold face is used to indicate $a_n$ and $a$ are points in $\mathbb{F}^p$.

**Theorem 5.7.2** If $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} a_n = a_1$ then $a_1 = a$.

**Proof:** Suppose $a_1 \neq a$. Then let $0 < \varepsilon < |a_1 - a|/2$ in the definition of the limit. It follows there exists $n_\varepsilon$ such that if $n \geq n_\varepsilon$, then $|a_n - a| < \varepsilon$ and $|a_n - a_1| < \varepsilon$. Therefore, for such $n$,

$$|a_1 - a| \leq |a_1 - a_n| + |a_n - a| < \varepsilon + \varepsilon < |a_1 - a|/2 + |a_1 - a|/2 = |a_1 - a|,$$

a contradiction.

As in the case of a vector valued function, it suffices to consider the components. This is the content of the next theorem.

**Theorem 5.7.3** Let $a_n = (a_1^n, \ldots, a_p^n) \in \mathbb{F}^p$. Then $\lim_{n \to \infty} a_n = a \equiv (a_1, \ldots, a_p)$ if and only if for each $k = 1, \ldots, p$,

$$\lim_{n \to \infty} a_k^n = a_k. \quad (5.7.8)$$

**Proof:** First suppose $\lim_{n \to \infty} a_n = a$. Then given $\varepsilon > 0$ there exists $n_\varepsilon$ such that if $n > n_\varepsilon$, then

$$|a^n_k - a_k| \leq |a_n - a| < \varepsilon$$

which establishes $5.7.8$.

Now suppose $5.7.8$ holds for each $k$. Then letting $\varepsilon > 0$ be given there exist $n_k$ such that if $n > n_k$,

$$|a^n_k - a_k| < \varepsilon/\sqrt{p}.$$

Therefore, letting $n_\varepsilon > \max(n_1, \ldots, n_p)$, it follows that for $n > n_\varepsilon$,

$$|a_n - a| = \left( \sum_{k=1}^{n} |a_k^n - a_k|^2 \right)^{1/2} < \left( \sum_{k=1}^{n} \frac{\varepsilon^2}{p} \right)^{1/2} = \varepsilon,$$

showing that $\lim_{n \to \infty} a_n = a$. This proves the theorem.
5.7. THE LIMIT OF A SEQUENCE

Example 5.7.4 Let \(a_n = \left( \frac{1}{n+1}, \frac{1}{n} \sin (n), \frac{n^2 + 3}{3n^2 + 5n} \right).\)

It suffices to consider the limits of the components according to the following theorem. Thus the limit is \((0, 0, 1/3)\).

Theorem 5.7.5 Suppose \(\{a_n\}\) and \(\{b_n\}\) are sequences and that
\[
\lim_{n \to \infty} a_n = a \quad \text{and} \quad \lim_{n \to \infty} b_n = b.
\]
Also suppose \(x\) and \(y\) are numbers in \(F\). Then
\[
\lim_{n \to \infty} xa_n + yb_n = xa + yb \tag{5.7.9}
\]
\[
\lim_{n \to \infty} a_n \cdot b_n = a \cdot b \tag{5.7.10}
\]
If \(b_n \in F\), then
\[a_n b_n \to ab.\]

Proof: The first of these claims is left for you to do. To do the second, let \(\varepsilon > 0\) be given and choose \(n_1\) such that if \(n \geq n_1\) then
\[|a_n - a| < \frac{\varepsilon}{2(|a| + 1)}.
\]
Then for such \(n\), the triangle inequality and Cauchy Schwarz inequality imply
\[
|a_n \cdot b_n - a \cdot b| \leq |a_n \cdot b_n - a_n \cdot b| + |a_n \cdot b - a \cdot b| \\
\leq |a_n| |b_n - b| + |b| |a_n - a| \\
\leq \left( |a| + 1 \right) |b_n - b| + |b| |a_n - a|.
\]
Now let \(n_2\) be large enough that for \(n \geq n_2\),
\[|b_n - b| < \frac{\varepsilon}{2(|a| + 1)}, \quad \text{and} \quad |a_n - a| < \frac{\varepsilon}{2(|b| + 1)}.
\]
Such a number exists because of the definition of limit. Therefore, let
\[n_\varepsilon > \max(n_1, n_2).
\]
For \(n \geq n_\varepsilon\),
\[
|a_n \cdot b_n - a \cdot b| \leq (|a| + 1) |b_n - b| + |b| |a_n - a| \\
\leq \left( |a| + 1 \right) \frac{\varepsilon}{2(|a| + 1)} + |b| \frac{\varepsilon}{2(|b| + 1)} \leq \varepsilon.
\]
This proves 5.7.9. The proof of 5.7.10 is entirely similar and is left for you.
5.7.1 Sequences And Completeness

Recall the definition of a Cauchy sequence.

Definition 5.7.6 \( \{a_n\} \) is a Cauchy sequence if for all \( \varepsilon > 0 \), there exists \( n_\varepsilon \) such that whenever \( n, m \geq n_\varepsilon \),

\[
|a_n - a_m| < \varepsilon.
\]

A sequence is Cauchy means the terms are “bunching up to each other” as \( m, n \) get large.

Theorem 5.7.7 Let \( \{a_n\}_{n=1}^{\infty} \) be a Cauchy sequence in \( \mathbb{F}^p \). Then there exists a unique \( a \in \mathbb{F}^p \) such that \( a_n \to a \).

Proof: Let \( a_n = (a_n^1, \ldots, a_n^p) \). Then

\[
|a_n^k - a_m^k| \leq |a_n - a_m|
\]

which shows for each \( k = 1, \ldots, p \), it follows \( \{a_k\}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{F} \). This requires that both the real and imaginary parts of \( a_n^k \) are Cauchy sequences in \( \mathbb{R} \) which means the real and imaginary parts converge in \( \mathbb{R} \). This shows \( \{a_n\}_{n=1}^{\infty} \) must converge to some \( a_k \). That is \( \lim_{n \to \infty} a_n^k = a_k \). Letting \( a = (a_1, \ldots, a_p) \), it follows from Theorem 5.7.3 that

\[
\lim_{n \to \infty} a_n = a.
\]

This proves the theorem.

Theorem 5.7.8 The set of terms in a Cauchy sequence in \( \mathbb{F}^p \) is bounded in the sense that for all \( n \), \( |a_n| < M \) for some \( M < \infty \).

Proof: Let \( \varepsilon = 1 \) in the definition of a Cauchy sequence and let \( n > n_1 \). Then from the definition,

\[
|a_n - a_{n_1}| < 1.
\]

It follows that for all \( n > n_1 \),

\[
|a_n| < 1 + |a_{n_1}|.
\]

Therefore, for all \( n \),

\[
|a_n| \leq 1 + |a_{n_1}| + \sum_{k=1}^{n_1} |a_k|.
\]

This proves the theorem.

Theorem 5.7.9 If a sequence \( \{a_n\} \) in \( \mathbb{F}^p \) converges, then the sequence is a Cauchy sequence.
5.8. PROPERTIES OF CONTINUOUS FUNCTIONS

Proof: Let \( \varepsilon > 0 \) be given and suppose \( a_n \to a \). Then from the definition of convergence, there exists \( n_\varepsilon \) such that if \( n > n_\varepsilon \), it follows that

\[
|a_n - a| < \frac{\varepsilon}{2}
\]

Therefore, if \( m, n \geq n_\varepsilon + 1 \), it follows that

\[
|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

showing that, since \( \varepsilon > 0 \) is arbitrary, \( \{a_n\} \) is a Cauchy sequence.

5.7.2 Continuity And The Limit Of A Sequence

Just as in the case of a function of one variable, there is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 5.7.10 A function \( f : D(f) \to \mathbb{R}^q \) is continuous at \( x \in D(f) \) if and only if, whenever \( x_n \to x \) with \( x_n \in D(f) \), it follows \( f(x_n) \to f(x) \).

Proof: Suppose first that \( f \) is continuous at \( x \) and let \( x_n \to x \). Let \( \varepsilon > 0 \) be given. By continuity, there exists \( \delta > 0 \) such that if \( |y - x| < \delta \), then \( |f(y) - f(x)| < \varepsilon \). However, there exists \( n_\delta \) such that if \( n \geq n_\delta \), then \( |x_n - x| < \delta \) and so for all \( n \) this large,

\[
|f(x) - f(x_n)| < \varepsilon
\]

which shows \( f(x_n) \to f(x) \).

Now suppose the condition about taking convergent sequences to convergent sequences holds at \( x \). Suppose \( f \) fails to be continuous at \( x \). Then there exists \( \varepsilon > 0 \) and \( x_n \in D(f) \) such that \( |x - x_n| < \frac{1}{n} \), yet

\[
|f(x) - f(x_n)| \geq \varepsilon.
\]

But this is clearly a contradiction because, although \( x_n \to x \), \( f(x_n) \) fails to converge to \( f(x) \). It follows \( f \) must be continuous after all. This proves the theorem.

5.8 Properties Of Continuous Functions

Functions of \( p \) variables have many of the same properties as functions of one variable. First there is a version of the extreme value theorem generalizing the one dimensional case.

Theorem 5.8.1 Let \( C \) be closed and bounded and let \( f : C \to \mathbb{R} \) be continuous. Then \( f \) achieves its maximum and its minimum on \( C \). This means there exist, \( x_1, x_2 \in C \) such that for all \( x \in C \),

\[
f(x_1) \leq f(x) \leq f(x_2).
\]
There is also the long technical theorem about sums and products of continuous functions. These theorems are proved in the next section.

**Theorem 5.8.2** The following assertions are valid

1. The function, \( af + bg \) is continuous at \( x \) when \( f, g \) are continuous at \( x \in D(f) \cap D(g) \) and \( a, b \in \mathbb{F} \).

2. If and \( f \) and \( g \) are each \( \mathbb{F} \) valued functions continuous at \( x \), then \( fg \) is continuous at \( x \). If, in addition to this, \( g(x) \neq 0 \), then \( f/g \) is continuous at \( x \).

3. If \( f \) is continuous at \( x \), \( f(x) \in D(g) \subseteq \mathbb{F}^p \), and \( g \) is continuous at \( f(x) \), then \( g \circ f \) is continuous at \( x \).

4. If \( f = (f_1, \cdots, f_q) : D(f) \to \mathbb{F}^q \), then \( f \) is continuous if and only if each \( f_k \) is a continuous \( \mathbb{F} \) valued function.

5. The function \( f : \mathbb{F}^p \to \mathbb{F} \), given by \( f(x) = |x| \) is continuous.

### 5.9 Exercises

1. \( f : D \subseteq \mathbb{F}^p \to \mathbb{F}^q \) is Lipschitz continuous or just Lipschitz for short if there exists a constant, \( K \) such that
   \[
   |f(x) - f(y)| \leq K|x - y|
   \]
   for all \( x, y \in D \). Show every Lipschitz function is uniformly continuous which means that given \( \varepsilon > 0 \) there exists \( \delta > 0 \) independent of \( x \) such that if \( |x - y| < \delta \), then \( |f(x) - f(y)| < \varepsilon \).

2. If \( f \) is uniformly continuous, does it follow that \( |f| \) is also uniformly continuous? If \( |f| \) is uniformly continuous does it follow that \( f \) is uniformly continuous? Answer the same questions with “uniformly continuous” replaced with “continuous”. Explain why.

### 5.10 Proofs Of Theorems

This section contains the proofs of the theorems which were just stated without proof.

**Theorem 5.10.1** The following assertions are valid

1. The function, \( af + bg \) is continuous at \( x \) when \( f, g \) are continuous at \( x \in D(f) \cap D(g) \) and \( a, b \in \mathbb{F} \).
5.10. PROOFS OF THEOREMS

2. If and $f$ and $g$ are each $\mathbb{F}$ valued functions continuous at $x$, then $fg$ is continuous at $x$. If, in addition to this, $g(x) \neq 0$, then $f/g$ is continuous at $x$.

3. If $f$ is continuous at $x$, $f(x) \in D(g) \subseteq \mathbb{F}$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$.

4. If $f = (f_1, \cdots, f_q) : D(f) \rightarrow \mathbb{F}^q$, then $f$ is continuous if and only if each $f_k$ is a continuous $\mathbb{F}$ valued function.

5. The function $f : \mathbb{F}^p \rightarrow \mathbb{F}$, given by $f(x) = |x|$ is continuous.

**Proof:** Begin with 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|x - y| < \delta_1$, it follows $|f(x) - f(y)| < \frac{\varepsilon}{2(|a| + |b| + 1)}$ and there exists $\delta_2 > 0$ such that whenever $|x - y| < \delta_2$, it follows $|g(x) - g(y)| < \frac{\varepsilon}{2(|a| + |b| + 1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|x - y| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$|af(x) + bf(y) - (ag(y) + bg(y))|$$

$$\leq |a||f(x) - f(y)| + |b||g(x) - g(y)|$$

$$< |a| \left( \frac{\varepsilon}{2(|a| + |b| + 1)} \right) + |b| \left( \frac{\varepsilon}{2(|a| + |b| + 1)} \right) < \varepsilon.$$

Now begin on 2.) There exists $\delta_1 > 0$ such that if $|y - x| < \delta_1$, then $|f(x) - f(y)| < 1$.

Therefore, for such $y$,

$$|f(y)| < 1 + |f(x)|.$$ 

It follows that for such $y$,

$$|fg(x) - fg(y)| \leq |f(x)g(x) - f(x)g(y)| + |g(x)f(y) - f(y)g(y)|$$

$$\leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)|$$

$$\leq (1 + |g(x)| + |f(y)|)||g(x) - g(y)| + |f(x) - f(y)||.$$ 

Now let $\varepsilon > 0$ be given. There exists $\delta_2$ such that if $|x - y| < \delta_2$, then

$$|g(x) - g(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)},$$

and there exists $\delta_3$ such that if $|x - y| < \delta_3$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)}.$$
Now let $0 < \delta \leq \min (\delta_1, \delta_2, \delta_3)$. Then if $|x - y| < \delta$, all the above hold at once and

$$|fg(x) - fg(y)| \leq (1 + |g(x)| + |f(y)|) (|g(x) - g(y)| + |f(x) - f(y)|)$$

$$< (1 + |g(x)| + |f(y)|) \left( \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} + \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} \right) = \varepsilon.$$  

This proves the first part of 2.) To obtain the second part, let $\delta_1$ be as described above and let $\delta_0 > 0$ be such that for $|x - y| < \delta_0$,

$$|g(x) - g(y)| < |g(x)|/2$$

and so by the triangle inequality,

$$-|g(x)|/2 \leq |g(y)| - |g(x)| \leq |g(x)|/2$$

which implies $|g(y)| \geq |g(x)|/2$, and $|g(y)| < 3|g(x)|/2$.

Then if $|x - y| < \min (\delta_0, \delta_1)$,

$$\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \leq \frac{|f(x)g(y) - f(y)g(x)|}{|g(x)g(y)|} \leq \frac{2|f(x)g(y) - f(y)g(x)|}{|g(x)|^2}$$

$$\leq \frac{2}{|g(x)|^2} \left[ |f(x)g(y) - f(y)g(x)| + |f(y)g(x) - f(x)g(y)| \right]$$

$$\leq \frac{2}{|g(x)|^2} \left[ |g(y)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \right]$$

$$\leq \frac{2}{|g(x)|^2} \left[ \frac{3}{2} |g(x)||f(x) - f(y)| + (1 + |f(x)|)|g(y) - g(x)| \right]$$

$$\leq \frac{2}{|g(x)|^2} (1 + 2|f(x)| + 2|g(x)|)|f(x) - f(y)| + |g(y) - g(x)|$$

$$= M |f(x) - f(y)| + |g(y) - g(x)|$$

where

$$M \equiv \frac{2}{|g(x)|^2} (1 + 2|f(x)| + 2|g(x)|)$$

Now let $\delta_2$ be such that if $|x - y| < \delta_2$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2} M^{-1}$$
and let $\delta_3$ be such that if $|x - y| < \delta_3$, then

$$|g(y) - g(x)| < \frac{\varepsilon}{2}M^{-1}.$$  

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|x - y| < \delta$, everything holds and

$$\left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| \leq M \left[ |f(x) - f(y)| + |g(y) - g(x)| \right]$$

$$< M \left[ \frac{\varepsilon}{2}M^{-1} + \frac{\varepsilon}{2}M^{-1} \right] = \varepsilon.$$

This completes the proof of the second part of 2.) Note that in these proofs no effort is made to find some sort of “best” $\delta$. The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now begin on 3.). If $f$ is continuous at $x$, $f(x) \in D(g) \subseteq \mathbb{F}^p$, and $g$ is continuous at $f(x)$, then $g \circ f$ is continuous at $x$. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$ such that if $|y - f(x)| < \eta$ and $y \in D(g)$, it follows that $|g(y) - g(f(x))| < \varepsilon$. It follows from continuity of $f$ at $x$ that there exists $\delta > 0$ such that if $|x - z| < \delta$ and $z \in D(f)$, then $|f(z) - f(x)| < \eta$. Then if $|x - z| < \delta$ and $z \in D(g \circ f) \subseteq D(f)$, all the above hold and so

$$|g(f(z)) - g(f(x))| < \varepsilon.$$  

This proves part 3.)

Part 4.) says: If $f = (f_1, \ldots, f_q) : D(f) \to \mathbb{F}^q$, then $f$ is continuous if and only if each $f_k$ is a continuous $\mathbb{F}$ valued function. Then

$$\left| \frac{f_k(x) - f_k(y)}{f(x) - f(y)} \right| \leq \frac{\varepsilon}{|f(x) - f(y)|} \leq \frac{\varepsilon}{\varepsilon} = 1.$$  

Suppose first that $f$ is continuous at $x$. Then there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. The first part of the above inequality then shows that for each $k = 1, \ldots, q$, $|f_k(x) - f_k(y)| < \varepsilon$. This shows the only if part. Now suppose each function, $f_k$ is continuous. Then if $\varepsilon > 0$ is given, there exists $\delta_k > 0$ such that whenever $|x - y| < \delta_k$

$$|f_k(x) - f_k(y)| < \varepsilon/q.$$  

Now let $0 < \delta \leq \min(\delta_1, \ldots, \delta_q)$. For $|x - y| < \delta$, the above inequality holds for all $k$ and so the last part of 5.10.11 implies

$$|f(x) - f(y)| \leq \sum_{i=1}^{q} |f_i(x) - f_i(y)|$$

$$< \sum_{i=1}^{q} \frac{\varepsilon}{q} = \varepsilon.$$
This proves part 4.)

To verify part 5.), let \( \varepsilon > 0 \) be given and let \( \delta = \varepsilon \). Then if \( |x - y| < \delta \), the triangle inequality implies
\[
|f(x) - f(y)| = |x| - |y| 
\leq |x - y| < \delta = \varepsilon.
\]
This proves part 5.) and completes the proof of the theorem.

Here is a multidimensional version of the nested interval lemma.

The following definition is similar to that given earlier. It defines what is meant by a sequentially compact set in \( \mathbb{F}^p \).

**Definition 5.10.2** A set, \( K \subseteq \mathbb{F}^p \) is sequentially compact if and only if whenever \( \{x_n\}_{n=1}^\infty \) is a sequence of points in \( K \), there exists a point, \( x \in K \) and a subsequence, \( \{x_{n_k}\}_{k=1}^\infty \) such that \( x_{n_k} \to x \).

It turns out the sequentially compact sets in \( \mathbb{F}^p \) are exactly those which are closed and bounded. Only half of this result will be needed in this book and this is proved next. First note that \( C \) can be considered as \( \mathbb{R}^2 \). Therefore, \( C^p \) may be considered as \( \mathbb{R}^{2p} \).

**Theorem 5.10.3** Let \( C \subseteq \mathbb{F}^p \) be closed and bounded. Then \( C \) is sequentially compact.

**Proof:** Let \( \{a_n\} \subseteq C \). Then let \( a_n = (a_{n1}, \cdots, a_{np}) \). It follows the real and imaginary parts of the terms of the sequence, \( \{a_j\}_{j=1}^\infty \) are each contained in some sufficiently large closed bounded interval. By Theorem 2.0.5 on Page 33, there is a subsequence of the sequence of real parts of \( \{a_j\}_{j=1}^\infty \) which converges. Also there is a further subsequence of the imaginary parts of \( \{a_{i}^n\}_{j=1}^\infty \) which converges. Thus there is a subsequence, \( n_k \) with the property that \( a_{n_k} \) converges to a point, \( a_j \in \mathbb{F} \). Taking further subsequences, one obtains the existence of a subsequence, still called \( n_k \) such that for each \( r = 1, \cdots, p \), \( a_{n_k}^r \) converges to a point, \( a_r \in \mathbb{F} \) as \( k \to \infty \). Therefore, letting \( a = (a_1, \cdots, a_p) \), \( \lim_{k \to \infty} a_{n_k} = a \). Since \( C \) is closed, it follows \( a \in C \). This proves the theorem.

Here is a proof of the extreme value theorem.

**Theorem 5.10.4** Let \( C \) be closed and bounded and let \( f : C \to \mathbb{R} \) be continuous. Then \( f \) achieves its maximum and its minimum on \( C \). This means there exist, \( x_1, x_2 \in C \) such that for all \( x \in C \),
\[
f(x_1) \leq f(x) \leq f(x_2).
\]

**Proof:** Let \( M = \sup \{f(x) : x \in C\} \). Recall this means \( +\infty \) if \( f \) is not bounded above and it equals the least upper bound of these values of \( f \) if \( f \) is bounded above. Then there exists a sequence, \( \{x_n\} \) such that \( f(x_n) \to M \). Since \( C \) is sequentially compact, there exists a subsequence, \( x_{n_k} \), and a point, \( x \in C \) such that \( x_{n_k} \to x \).
But then since \( f \) is continuous at \( x \), it follows from Theorem 5.7.10 on Page 115 that \( f(x) = \lim_{k \to \infty} f(x_{n_k}) = M \). This proves \( f \) achieves its maximum and also shows its maximum is less than \( \infty \). Let \( x_2 = x \). The case of a minimum is handled similarly.

Recall that a function is uniformly continuous if the following definition holds.

**Definition 5.10.5** Let \( f : D(f) \to \mathbb{F}^q \). Then \( f \) is uniformly continuous if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( |x - y| < \delta \), it follows \( |f(x) - f(y)| < \varepsilon \).

**Theorem 5.10.6** Let \( f : C \to \mathbb{F}^q \) be continuous where \( C \) is a closed and bounded set in \( \mathbb{F}^p \). Then \( f \) is uniformly continuous on \( C \).

**Proof:** If this is not so, there exists \( \varepsilon > 0 \) and pairs of points, \( x_n \) and \( y_n \) satisfying \( |x_n - y_n| < 1/n \) but \( |f(x_n) - f(y_n)| \geq \varepsilon \). Since \( C \) is sequentially compact, there exists \( x \in C \) and a subsequence, \( \{x_{n_k}\} \) satisfying \( x_{n_k} \to x \). But \( |x_{n_k} - y_{n_k}| < 1/k \) and so \( y_{n_k} \to x \) also. Therefore, from Theorem 5.7.10 on Page 115,

\[
\varepsilon \leq \lim_{k \to \infty} |f(x_{n_k}) - f(y_{n_k})| = |f(x) - f(x)| = 0,
\]

a contradiction. This proves the theorem.

### 5.11 The Space \( \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \)

**Definition 5.11.1** The symbol, \( \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \) will denote the set of linear transformations mapping \( \mathbb{F}^n \) to \( \mathbb{F}^m \). Thus \( L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \) means that for \( \alpha, \beta \) scalars and \( x, y \) vectors in \( \mathbb{F}^n \),

\[
L(\alpha x + \beta y) = \alpha L(x) + \beta L(y).
\]

It is convenient to give a norm for the elements of \( \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \). This will allow the consideration of questions such as whether a function having values in this space of linear transformations is continuous.

#### 5.11.1 The Operator Norm

How do you measure the distance between linear transformations defined on \( \mathbb{F}^n \)? It turns out there are many ways to do this but I will give the most common one here.

**Definition 5.11.2** \( \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \) denotes the space of linear transformations mapping \( \mathbb{F}^n \) to \( \mathbb{F}^m \). For \( A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \), the **operator norm** is defined by

\[
||A|| \equiv \max \{|Ax|_{\mathbb{F}^m} : |x|_{\mathbb{F}^n} \leq 1\} < \infty.
\]

**Theorem 5.11.3** Denote by \(|\cdot|\) the norm on either \( \mathbb{F}^n \) or \( \mathbb{F}^m \). Then \( \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \) with this operator norm is a complete normed linear space of dimension \( nm \) with

\[
||Ax|| \leq ||A|| \cdot |x|.
\]

Here **Completeness** means that every Cauchy sequence converges.
Proof: It is necessary to show the norm defined on $L(F^n, F^m)$ really is a norm. This means it is necessary to verify 

$$||A|| \geq 0 \text{ and equals zero if and only if } A = 0.$$ 

For $\alpha$ a scalar, 

$$||\alpha A|| = |\alpha|||A||,$$

and for $A, B \in L(F^n, F^m)$, 

$$||A + B|| \leq ||A|| + ||B||$$

The first two properties are obvious but you should verify them. It remains to verify the norm is well defined and also to verify the triangle inequality above. First if $|x| \leq 1$, and $(A_{ij})$ is the matrix of the linear transformation with respect to the usual basis vectors, then

$$||A|| = \max \left\{ \left( \sum_i |(A_{ij})|^2 \right)^{1/2} : |x| \leq 1 \right\}$$

$$= \max \left\{ \left( \sum_i \sum_j A_{ij}x_j^2 \right)^{1/2} : |x| \leq 1 \right\}$$

which is a finite number by the extreme value theorem.

It is clear that a basis for $L(F^n, F^m)$ consists of linear transformations whose matrices are of the form $E_{ij}$ where $E_{ij}$ consists of the $m \times n$ matrix having all zeros except for a 1 in the $ij^{th}$ position. In effect, this considers $L(F^n, F^m)$ as $F^{nm}$. Think of the $m \times n$ matrix as a long vector folded up.

If $x \neq 0$,

$$|Ax| \frac{1}{|x|} = \left| A \frac{x}{|x|} \right| \leq ||A|| \quad (5.11.12)$$

It only remains to verify completeness. Suppose then that $\{A_k\}$ is a Cauchy sequence in $L(F^n, F^m)$. Then from (5.11.12) $\{A_kx\}$ is a Cauchy sequence for each $x \in F^n$. This follows because

$$|A_kx - A_lx| \leq ||A_k - A_l|| |x|$$

which converges to 0 as $k, l \to \infty$. Therefore, by completeness of $F^m$, there exists $Ax$, the name of the thing to which the sequence, $\{A_kx\}$ converges such that

$$\lim_{k \to \infty} A_kx = Ax.$$
Then $A$ is linear because

$$A(ax + by) = \lim_{k \to \infty} A_k(ax + by)$$
$$= \lim_{k \to \infty} (aA_kx + bA_ky)$$
$$= a \lim_{k \to \infty} A_kx + b \lim_{k \to \infty} A_ky$$
$$= aA x + bA y.$$ 

By the first part of this argument, $||A|| < \infty$ and so $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. This proves the theorem.

**Proposition 5.11.4** Let $A(x) \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ for each $x \in U \subseteq \mathbb{F}^p$. Then letting $(A_{ij}(x))$ denote the matrix of $A(x)$ with respect to the standard basis, it follows $A_{ij}$ is continuous at $x$ for each $i, j$ if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $||A(x) - A(y)|| < \varepsilon$. That is, $A$ is a continuous function having values in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ at $x$.

**Proof:** Suppose first the second condition holds. Then from the material on linear transformations,

$$|A_{ij}(x) - A_{ij}(y)| = |e_i \cdot (A(x) - A(y)) e_j|$$
$$\leq |e_i| ||(A(x) - A(y)) e_j||$$
$$\leq ||A(x) - A(y)||.$$

Therefore, the second condition implies the first.

Now suppose the first condition holds. That is each $A_{ij}$ is continuous at $x$. Let $|v| \leq 1$.

$$|(A(x) - A(y))(v)| = \left( \sum_i \left| \sum_j (A_{ij}(x) - A_{ij}(y)) v_j \right|^2 \right)^{1/2} \tag{5.11.13}$$

$$\leq \left( \sum_i \left( \sum_j |A_{ij}(x) - A_{ij}(y)||v_j|^2 \right)^2 \right)^{1/2}.$$ 

By continuity of each $A_{ij}$, there exists a $\delta > 0$ such that for each $i, j$

$$|A_{ij}(x) - A_{ij}(y)| < \frac{\varepsilon}{n\sqrt{m}}.$$
whenever $|x - y| < \delta$. Then from $\text{5.11.13}$, if $|x - y| < \delta$, 

$$
|(A(x) - A(y))(v)| < \left( \sum_i \left( \sum_j \frac{\varepsilon}{n\sqrt{m}} |v| \right)^2 \right)^{1/2} \leq \left( \sum_i \left( \sum_j \frac{\varepsilon}{n\sqrt{m}} \right)^2 \right)^{1/2} = \varepsilon
$$

This proves the proposition.

### 5.12 The Frechet Derivative

Let $U$ be an open set in $\mathbb{F}^n$, and let $f : U \to \mathbb{F}^m$ be a function.

**Definition 5.12.1** A function $g$ is $o(v)$ if

$$
\lim_{|v| \to 0} \frac{g(v)}{|v|} = 0 \quad (5.12.14)
$$

A function $f : U \to \mathbb{F}^m$ is differentiable at $x \in U$ if there exists a linear transformation $L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ such that

$$
f(x + v) = f(x) + Lv + o(v)
$$

This linear transformation $L$ is the definition of $Df(x)$. This derivative is often called the Frechet derivative.

Usually no harm is occasioned by thinking of this linear transformation as its matrix taken with respect to the usual basis vectors.

The definition $5.12.14$ means that the error,

$$
f(x + v) - f(x) - Lv
$$

converges to $0$ faster than $|v|$. Thus the above definition is equivalent to saying

$$
\lim_{|v| \to 0} \frac{|f(x + v) - f(x) - Lv|}{|v|} = 0 \quad (5.12.15)
$$

or equivalently,

$$
\lim_{y \to x} \frac{|f(y) - f(x) - Df(x)(y - x)|}{|y - x|} = 0. \quad (5.12.16)
$$

Now it is clear this is just a generalization of the notion of the derivative of a function of one variable because in this more specialized situation,

$$
\lim_{|v| \to 0} \frac{|f(x + v) - f(x) - f'(x)v|}{|v|} = 0,
$$
due to the definition which says

$$f'(x) = \lim_{v \to 0} \frac{f(x + v) - f(x)}{v}.$$  

For functions of $n$ variables, you can't define the derivative as the limit of a difference quotient like you can for a function of one variable because you can't divide by a vector. That is why there is a need for a more general definition.

The term $o(v)$ is notation that is descriptive of the behavior in $\mathbb{R}^n$ and it is only this behavior that is of interest. Thus, if $t$ and $k$ are constants,

$$o(v) = o(v) + o(v), \quad o(tv) = o(v), \quad ko(v) = o(v)$$

and other similar observations hold. The sloppiness built in to this notation is useful because it ignores details which are not important. It may help to think of $o(v)$ as an adjective describing what is left over after approximating $f(x + v)$ by $f(x) + Df(x)v$.

**Theorem 5.12.2** The derivative is well defined.

**Proof:** First note that for a fixed vector, $v$, $o(tv) = o(t)$. Now suppose both $L_1$ and $L_2$ work in the above definition. Then let $v$ be any vector and let $t$ be a real scalar which is chosen small enough that $tv + x \in U$. Then

$$f(x + tv) = f(x) + L_1tv + o(tv), \quad f(x + tv) = f(x) + L_2tv + o(tv).$$

Therefore, subtracting these two yields $(L_2 - L_1)(tv) = o(tv) = o(t)$. Therefore, dividing by $t$ yields $(L_2 - L_1)(v) = \frac{o(t)}{t}$. Now let $t \to 0$ to conclude that $(L_2 - L_1)(v) = 0$. Since this is true for all $v$, it follows $L_2 = L_1$. This proves the theorem.

**Lemma 5.12.3** Let $f$ be differentiable at $x$. Then $f$ is continuous at $x$ and in fact, there exists $K > 0$ such that whenever $|v|$ is small enough,

$$|f(x + v) - f(x)| \leq K|v|$$

**Proof:** From the definition of the derivative, $f(x + v) - f(x) = Df(x)v + o(v)$. Let $|v|$ be small enough that $\frac{o(v)}{|v|} < 1$ so that $|o(v)| \leq |v|$. Then for such $v$,

$$|f(x + v) - f(x)| \leq |Df(x)v| + |v| \leq (|Df(x)| + 1)|v|$$

This proves the lemma with $K = |Df(x)| + 1$.

**Theorem 5.12.4** (The chain rule) Let $U$ and $V$ be open sets, $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$. Suppose $f : U \to V$ is differentiable at $x \in U$ and suppose $g : V \to \mathbb{R}^q$ is differentiable at $f(x) \in V$. Then $g \circ f$ is differentiable at $x$ and

$$D(g \circ f)(x) = Dg(f(x)) Df(x).$$
CHAPTER 5. MULTI-VARIABLE CALCULUS

**Proof:** This follows from a computation. Let \( B (x, r) \subseteq U \) and let \( r \) also be small enough that for \( |v| \leq r \), it follows that \( f (x + v) \in V \). Such an \( r \) exists because \( f \) is continuous at \( x \). For \( |v| < r \), the definition of differentiability of \( g \) and \( f \) implies

\[
g (f (x + v)) - g (f (x)) =
\]

\[
D g (f (x)) (f (x + v) - f (x)) + o (f (x + v) - f (x))
\]

\[
= D g (f (x)) [D f (x) v + o (v)] + o (f (x + v) - f (x))
\]

\[
= D g (f (x)) D f (x) v + o (v) + o (f (x + v) - f (x)). \tag{5.12.17}
\]

It remains to show \( o (f (x + v) - f (x)) = o (v) \).

By Lemma \( \text{Lemma 12.1.} \) with \( K \) given there, letting \( \varepsilon > 0 \), it follows that for \( |v| \) small enough,

\[
|o (f (x + v) - f (x))| \leq (\varepsilon / K) |f (x + v) - f (x)| \leq (\varepsilon / K) K |v| = \varepsilon |v|.
\]

Since \( \varepsilon > 0 \) is arbitrary, this shows \( o (f (x + v) - f (x)) = o (v) \) because whenever \( |v| \) is small enough,

\[
\frac{|o (f (x + v) - f (x))|}{|v|} \leq \varepsilon.
\]

By \( \text{Lemma 12.1.} \) this shows

\[
g (f (x + v)) - g (f (x)) = D g (f (x)) D f (x) v + o (v)
\]

which proves the theorem.

The derivative is a linear transformation. What is the matrix of this linear transformation taken with respect to the usual basis vectors? Let \( e_i \) denote the vector of \( \mathbb{F}^n \) which has a one in the \( i^{th} \) entry and zeroes elsewhere. Then the matrix of the linear transformation is the matrix whose \( i^{th} \) column is \( D f (x) e_i \). What is this? Let \( t \in \mathbb{R} \) such that \( |t| \) is sufficiently small.

\[
f (x + te_i) - f (x) = D f (x) te_i + o (te_i)
\]

\[
= D f (x) te_i + o (t).
\]

Then dividing by \( t \) and taking a limit,

\[
D f (x) e_i = \lim_{t \to 0} \frac{f (x + te_i) - f (x)}{t} = \frac{\partial f}{\partial x_i} (x).
\]

Thus the matrix of \( D f (x) \) with respect to the usual basis vectors is the matrix of the form

\[
\begin{pmatrix}
    f_{1,x_1} (x) & f_{1,x_2} (x) & \cdots & f_{1,x_n} (x) \\
    \vdots & \vdots & \ddots & \vdots \\
    f_{m,x_1} (x) & f_{m,x_2} (x) & \cdots & f_{m,x_n} (x)
\end{pmatrix}.
\]

As mentioned before, there is no harm in referring to this matrix as \( D f (x) \) but it may also be referred to as \( J f (x) \).

This is summarized in the following theorem.
Theorem 5.12.5 Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) and suppose \( f \) is differentiable at \( x \). Then all the partial derivatives \( \frac{\partial f_i}{\partial x_j}(x) \) exist and if \( Jf(x) \) is the matrix of the linear transformation with respect to the standard basis vectors, then the \( ij \)th entry is given by \( f_{i,j} \) or \( \frac{\partial f_i}{\partial x_j}(x) \).

What if all the partial derivatives of \( f \) exist? Does it follow that \( f \) is differentiable? Consider the following function.

\[
f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
\]

Then from the definition of partial derivatives,

\[
\lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

and

\[
\lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

However \( f \) is not even continuous at \((0, 0)\) which may be seen by considering the behavior of the function along the line \( y = x \) and along the line \( x = 0 \). By Lemma 5.12.2, this implies \( f \) is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

5.13 \( C^1 \) Functions

However, there are theorems which can be used to get differentiability of a function based on existence of the partial derivatives.

**Definition 5.13.1** When all the partial derivatives exist and are continuous the function is called a \( C^1 \) function.

Because of Proposition 5.11.4 on Page 123 and Theorem 5.12.5 which identifies the entries of \( Jf \) with the partial derivatives, the following definition is equivalent to the above.

**Definition 5.13.2** Let \( U \subseteq \mathbb{R}^n \) be an open set. Then \( f : U \to \mathbb{R}^m \) is \( C^1 (U) \) if \( f \) is differentiable and the mapping \( x \to Df(x) \),

is continuous as a function from \( U \) to \( \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \).

The following is an important abstract generalization of the familiar concept of partial derivative.
Definition 5.13.3 Let \( g : U \subseteq \mathbb{F}^n \times \mathbb{F}^m \rightarrow \mathbb{F}^q \), where \( U \) is an open set in \( \mathbb{F}^n \times \mathbb{F}^m \). Denote an element of \( \mathbb{F}^n \times \mathbb{F}^m \) by \((x, y)\) where \( x \in \mathbb{F}^n \) and \( y \in \mathbb{F}^m \). Then the map \( x \rightarrow g(x, y) \) is a function from the open set in \( \mathbb{F}^n \),
\[
\{ x : (x, y) \in U \}
\]to \( \mathbb{F}^q \). When this map is differentiable, its derivative is denoted by
\[
D_1g(x, y), \text{ or sometimes by } D_xg(x, y).
\]
Thus,
\[
g(x + v, y) - g(x, y) = D_1g(x, y)v + o(v).
\]
A similar definition holds for the symbol \( D_yg \) or \( D_2g \). The special case seen in beginning calculus courses is where \( g : U \rightarrow \mathbb{F}^q \) and
\[
g_{x_i}(x) \equiv \frac{\partial g(x)}{\partial x_i} \equiv \lim_{h \to 0} \frac{g(x + he_i) - g(x)}{h}.
\]

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem. You might call it the mean value inequality.

Theorem 5.13.4 Suppose \( U \) is an open subset of \( \mathbb{F}^n \) and \( f : U \rightarrow \mathbb{F}^m \) has the property that \( D_f(x) \) exists for all \( x \) in \( U \) and that, \( x + t(\ y - x) \in U \) for all \( t \in [0, 1] \). (The line segment joining the two points lies in \( U \).) Suppose also that for all points on this line segment,
\[
||Df(x + t(\ y - x))|| \leq M.
\]
Then
\[
|f(y) - f(x)| \leq M |y - x|.
\]

Proof: Let
\[
S \equiv \{ t \in [0, 1] : \text{ for all } s \in [0, t], |f(x + s(\ y - x)) - f(x)| \leq (M + \varepsilon) s |y - x| \}.
\]
Then \( 0 \in S \) and by continuity of \( f \), it follows that if \( t = \sup S \), then \( t \in S \) and if \( t < 1 \),
\[
|f(x + t(\ y - x)) - f(x)| = (M + \varepsilon) t |y - x|.
\]
If \( t < 1 \), then there exists a sequence of positive numbers, \( \{h_k\}_{k=1}^\infty \) converging to 0 such that
\[
|f(x + (t + h_k)(\ y - x)) - f(x)| > (M + \varepsilon)(t + h_k)|y - x|
\]
which implies that
\[
|f(x + (t + h_k)(\ y - x)) - f(x + t(\ y - x))| + |f(x + t(\ y - x)) - f(x)| > (M + \varepsilon)(t + h_k)|y - x|.
\]
5.13. \( C^1 \) FUNCTIONS

By \( \text{Theorem 5.13.5} \) this inequality implies

\[
|f(x + (t + h_k)(y-x)) - f(x + t(y-x))| > (M + \varepsilon)h_k|y-x|
\]

which yields upon dividing by \( h_k \) and taking the limit as \( h_k \to 0 \),

\[
|DF(x + t(y-x))(y-x)| \geq (M + \varepsilon)|y-x|.
\]

Now by the definition of the norm of a linear operator,

\[
M|y-x| \geq |DF(x + t(y-x))||y-x|
\]

\[
\geq |DF(x + t(y-x))(y-x)| \geq (M + \varepsilon)|y-x|,
\]

a contradiction. Therefore, \( t = 1 \) and so

\[
|f(x + (y-x)) - f(x)| \leq (M + \varepsilon)|y-x|.
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves the theorem.

The next theorem proves that if the partial derivatives exist and are continuous, then the function is differentiable.

**Theorem 5.13.5** Let \( g: U \subseteq \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q \). Then \( g \) is \( C^1(U) \) if and only if \( D_1g \) and \( D_2g \) both exist and are continuous on \( U \). In this case,

\[
Dg(x,y)(u,v) = D_1g(x,y)u + D_2g(x,y)v.
\]

**Proof:** Suppose first that \( g \in C^1(U) \). Then if \((x,y) \in U\),

\[
g(x + u, y) - g(x, y) = Dg(x,y)(u,0) + o(u).
\]

Therefore, \( D_1g(x,y)u = Dg(x,y)(u,0) \). Then

\[
|(D_1g(x,y) - D_1g(x',y'))(u)| =
\]

\[
|(Dg(x,y) - Dg(x',y'))(u,0)| \leq
\]

\[
|Dg(x,y) - Dg(x',y')||u,0||.
\]

Therefore,

\[
|D_1g(x,y) - D_1g(x',y')| \leq |Dg(x,y) - Dg(x',y')|.
\]

A similar argument applies for \( D_2g \) and this proves the continuity of the function, \((x,y) \to D_1g(x,y)\) for \( i = 1, 2 \). The formula follows from

\[
Dg(x,y)(u,v) = Dg(x,y)(u,0) + Dg(x,y)(0,v)
\]

\[
\equiv D_1g(x,y)u + D_2g(x,y)v.
\]

Now suppose \( D_1g(x,y) \) and \( D_2g(x,y) \) exist and are continuous.

\[
g(x + u, y + v) - g(x, y) = g(x + u, y + v) - g(x, y + v)
\]

\[
- g(x, y) + g(x, y + v) - g(x, y).
\]

\[
= D_1g(x,y)u + D_2g(x,y)v.
\]
5.13.4 Let \( h(x,u) \equiv g(x+u,y+v) - g(x+u,y) \). Then the expression in \([\ ]\) is of the form,
\[
h(x,u) = -g(x,u,y) + g(x+y,v) - g(x+y,v)\]
and so, by continuity of \((x,y) \rightarrow D_1g(x,y)\),
\[
\|D_2h(x,u)\| < \varepsilon
\]
whenever \(\|(u,v)\|\) is small enough. By Theorem 5.13.2 on Page 128, there exists \(\delta > 0\) such that if \(\|D_2h(x,u)\| < \delta\), the norm of the last term in \(5.13.14\) satisfies the inequality,
\[
\|D_2h(x,u)\| < \varepsilon
\]
whenever \(\|D_2h(x,u)\| < \varepsilon\) is small enough. By Theorem 5.13.2 on Page 128, there exists \(\delta > 0\) such that if \(\|D_2h(x,u)\| < \delta\), the norm of the last term in \(5.13.14\) satisfies the inequality,
\[
\|D_2h(x,u)\| < \varepsilon
\]
whenever \(\|D_2h(x,u)\| < \varepsilon\) is small enough. By Theorem 5.13.2 on Page 128, there exists \(\delta > 0\) such that if \(\|D_2h(x,u)\| < \delta\), the norm of the last term in \(5.13.14\) satisfies the inequality,
\[
\|D_2h(x,u)\| < \varepsilon
\]
whenever \(\|D_2h(x,u)\| < \varepsilon\) is small enough. By Theorem 5.13.2 on Page 128, there exists \(\delta > 0\) such that if \(\|D_2h(x,u)\| < \delta\), the norm of the last term in \(5.13.14\) satisfies the inequality,
\[
\|D_2h(x,u)\| < \varepsilon
\]
whenever \(\|D_2h(x,u)\| < \varepsilon\) is small enough. By Theorem 5.13.2 on Page 128, there exists \(\delta > 0\) such that if \(\|D_2h(x,u)\| < \delta\), the norm of the last term in \(5.13.14\) satisfies the inequality,
\[
\|D_2h(x,u)\| < \varepsilon
\]
Here is a generalization of Theorem 5.13.21.

**Theorem 5.13.7** Let \( g, U, \prod_{i=1}^{n} \mathbb{R}^{r_i} \), be given as in Definition 5.13.20. Then \( g \) is \( C^1(U) \) if and only if \( D_i g \) exists and is continuous on \( U \) for each \( i \). In this case,

\[
D g (x) (v) = \sum_{k} D_k g (x) v_k
\]

(5.13.21)

where \( v = (v_1, \cdots, v_n) \).

**Proof:** Suppose then that \( D_i g \) exists and is continuous for each \( i \). Note that \( \sum_{j=1}^{k} \theta_j v_j = (v_1, \cdots, v_k, 0, \cdots, 0) \). Thus \( \sum_{j=1}^{n} \theta_j v_j = v \) and define \( \sum_{j=1}^{0} \theta_j v_j = 0 \). Therefore,

\[
g (x + v) - g (x) = \sum_{k=1}^{n} \left[ g \left( x + \sum_{j=1}^{k} \theta_j v_j \right) - g \left( x + \sum_{j=1}^{k-1} \theta_j v_j \right) \right]
\]

(5.13.22)

Consider the terms in this sum.

\[
g \left( x + \sum_{j=1}^{k} \theta_j v_j \right) - g \left( x + \sum_{j=1}^{k-1} \theta_j v_j \right) = g (x + \theta_k v_k) - g (x) + \tag{5.13.23}
\]

\[
\left( g \left( x + \sum_{j=1}^{k} \theta_j v_j \right) - g (x + \theta_k v_k) \right) - \left( g \left( x + \sum_{j=1}^{k-1} \theta_j v_j \right) - g (x) \right)
\]

(5.13.24)

and the expression in (5.13.24) is of the form \( h (v_k) - h (0) \) where for small \( w \in \mathbb{R}^r \),

\[
h (w) \equiv g \left( x + \sum_{j=1}^{k-1} \theta_j v_j + \theta_k w \right) - g \left( x + \theta_k w \right).
\]

Therefore,

\[
D h (w) = D_k g \left( x + \sum_{j=1}^{k-1} \theta_j v_j + \theta_k w \right) - D_k g (x + \theta_k w)
\]

and by continuity, \(|D h (w)|| < \varepsilon\) provided \(|v|\) is small enough. Therefore, by Theorem 5.13.21 whenever \(|v|\) is small enough, \(|h (v_k) - h (0)| \leq \varepsilon |v_k| \leq \varepsilon |v|\) which shows that since \( \varepsilon \) is arbitrary, the expression in (5.13.24) is \( o (v) \). Now in

\[
g (x + \theta_k v_k) - g (x) = D_k g (x) v_k + o (v_k) = D_k g (x) v_k + o (v).
\]

Therefore, referring to (5.13.24),

\[
g (x + v) - g (x) = \sum_{k=1}^{n} D_k g (x) v_k + o (v)
\]
which shows \( Dg \) exists and equals the formula given in \( 5.13.21 \).

Next suppose \( g \) is \( C^1 \). I need to verify that \( D_k g(x) \) exists and is continuous. Let \( v \in \mathbb{F}^k \) sufficiently small. Then

\[
g(x + \theta_k v) - g(x) = Dg(x) \theta_k v + o(\theta_k v)
\]

since \( |\theta_k v| = |v| \). Then \( D_k g(x) \) exists and equals

\[
Dg(x) \circ \theta_k
\]

Since \( x \to Dg(x) \) is continuous and \( \theta_k : \mathbb{F}^k \to \prod_{i=1}^n \mathbb{F}^i \) is also continuous, this proves the theorem.

The way this is usually used is in the following corollary, a case of Theorem \( 5.13.7 \) obtained by letting \( F_r j = F_i \) in the above theorem.

**Corollary 5.13.8** Let \( U \) be an open subset of \( \mathbb{F}^n \) and let \( f : U \to \mathbb{F}^m \) be \( C^1 \) in the sense that all the partial derivatives of \( f \) exist and are continuous. Then \( f \) is differentiable and

\[
f(x + v) = f(x) + \sum_{k=1}^n \frac{\partial f}{\partial x_k}(x) v_k + o(v).
\]

### 5.14 \( C^k \) Functions

Recall the notation for partial derivatives in the following definition.

**Definition 5.14.1** Let \( g : U \to \mathbb{F}^n \). Then

\[
g_{x_k}(x) \equiv \frac{\partial g}{\partial x_k}(x) \equiv \lim_{h \to 0} \frac{g(x + he_k) - g(x)}{h}
\]

Higher order partial derivatives are defined in the usual way.

\[
g_{x_k x_l}(x) \equiv \frac{\partial^2 g}{\partial x_l \partial x_k}(x)
\]

and so forth.

To deal with higher order partial derivatives in a systematic way, here is a useful definition.

**Definition 5.14.2** \( \alpha = (\alpha_1, \cdots, \alpha_n) \) for \( \alpha_1 \cdots \alpha_n \) positive integers is called a multi-index. For \( \alpha \) a multi-index, \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and if \( x \in \mathbb{F}^n \),

\[
x = (x_1, \cdots, x_n),
\]

and \( f \) a function, define

\[
x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},
\]

\[
D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.
\]
The following is the definition of what is meant by a \( C^k \) function.

**Definition 5.14.3** Let \( U \) be an open subset of \( \mathbb{F}^n \) and let \( f : U \to \mathbb{F}^m \). Then for \( k \) a nonnegative integer, \( f \) is \( C^k \) if for every \( |\alpha| \leq k \), \( D^\alpha f \) exists and is continuous.

### 5.15 Mixed Partial Derivatives

Under certain conditions the mixed partial derivatives will always be equal. This astonishing fact is due to Euler in 1734.

**Theorem 5.15.1** Suppose \( f : U \subseteq \mathbb{F}^2 \to \mathbb{R} \) where \( U \) is an open set on which \( f_x, f_y, f_{xy}, \) and \( f_{yx} \) exist. Then if \( f_{xy} \) and \( f_{yx} \) are continuous at the point \((x,y) \in U\), it follows

\[
f_{xy}(x,y) = f_{yx}(x,y).
\]

**Proof:** Since \( U \) is open, there exists \( r > 0 \) such that \( B((x,y), r) \subseteq U \). Now let \( |t|, |s| < r/2, t, s \) real numbers and consider

\[
\Delta(s,t) = f_x(x+t, y+s) - f_x(x+t, y) - (f(x, y+s) - f(x, y)). \tag{5.15.25}
\]

Note that \((x+t, y+s) \in U\) because

\[
| (x+t, y+s) - (x,y) | = |(t,s)| = (t^2 + s^2)^{1/2} \leq \left( \frac{r^2}{4} + \frac{r^2}{4} \right)^{1/2} = \frac{r}{\sqrt{2}} < r.
\]

As implied above, \( h(t) = f(x+t, y+s) - f(x+t, y) \). Therefore, by the mean value theorem from calculus and the (one variable) chain rule,

\[
\Delta(s,t) = \frac{1}{st} (h(t) - h(0)) = \frac{1}{st} h'(\alpha t) t
\]

for some \( \alpha \in (0, 1) \). Applying the mean value theorem again,

\[
\Delta(s,t) = f_{xy}(x+\alpha t, y+\beta s)
\]

where \( \alpha, \beta \in (0, 1) \).

If the terms \( f(x+t, y) \) and \( f(x, y+s) \) are interchanged in (5.15.25), \( \Delta(s,t) \) is unchanged and the above argument shows there exist \( \gamma, \delta \in (0, 1) \) such that

\[
\Delta(s,t) = f_{yx}(x+\gamma t, y+\delta s).
\]

Letting \((s,t) \to (0,0)\) and using the continuity of \( f_{xy} \) and \( f_{yx} \) at \((x,y)\),

\[
\lim_{(s,t) \to (0,0)} \Delta(s,t) = f_{xy}(x,y) = f_{yx}(x,y).
\]
This proves the theorem.

The following is obtained from the above by simply fixing all the variables except for the two of interest.

**Corollary 5.15.2** Suppose $U$ is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ has the property that for two indices, $k, l$, $f_{x_k}$, $f_{x_l}$, and $f_{x_kx_l}$ exist on $U$ and $f_{x_kx_l}$ and $f_{x_lx_k}$ are both continuous at $x \in U$. Then $f_{x_kx_l}(x) = f_{x_lx_k}(x)$.

By considering the real and imaginary parts of $f$ in the case where $f$ has values in $\mathbb{F}$ you obtain the following corollary.

**Corollary 5.15.3** Suppose $U$ is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{F}$ has the property that for two indices, $k, l$, $f_{x_k}$, $f_{x_l}$, and $f_{x_kx_l}$ exist on $U$ and $f_{x_kx_l}$ and $f_{x_lx_k}$ are both continuous at $x \in U$. Then $f_{x_kx_l}(x) = f_{x_lx_k}(x)$.

Finally, by considering the components of $f$ you get the following generalization.

**Corollary 5.15.4** Suppose $U$ is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{F}^m$ has the property that for two indices, $k, l$, $f_{x_k}$, $f_{x_l}$, and $f_{x_kx_l}$ exist on $U$ and $f_{x_kx_l}$ and $f_{x_lx_k}$ are both continuous at $x \in U$. Then $f_{x_kx_l}(x) = f_{x_lx_k}(x)$.

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [3].

**Example 5.15.5** Let

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

From the definition of partial derivatives it follows immediately that $f_x(0, 0) = f_y(0, 0) = 0$. Using the standard rules of differentiation, for $(x, y) \neq (0, 0)$,

$$f_x = y \frac{x^4 - y^4 + 4x^2y^2}{(x^2 + y^2)^2}, \quad f_y = x \frac{x^4 - y^4 - 4x^2y^2}{(x^2 + y^2)^2}$$

Now

$$f_{xy}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = \lim_{y \to 0} \frac{-y^4}{(y^2)^2} = -1$$

while

$$f_{yx}(0, 0) = \lim_{x \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{x} = \lim_{x \to 0} \frac{x^4}{(x^2)^2} = 1$$

showing that although the mixed partial derivatives do exist at $(0, 0)$, they are not equal there.
5.16 Implicit Function Theorem

The implicit function theorem is one of the greatest theorems in mathematics. There are many versions of this theorem. However, I will give a very simple proof valid in finite dimensional spaces.

**Theorem 5.16.1 (implicit function theorem)** Suppose $U$ is an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Let $f : U \to \mathbb{R}^n$ be in $C^1 (U)$ and suppose

$$f (x_0, y_0) = 0, \quad D_1 f (x_0, y_0)^{-1} \in \mathcal{L} (\mathbb{R}^n, \mathbb{R}^n).$$

Then there exist positive constants, $\delta, \eta$, such that for every $y \in B(y_0, \eta)$ there exists a unique $x (y) \in B(x_0, \delta)$ such that

$$f (x (y), y) = 0.$$  \hspace{1cm} (5.16.27)

Furthermore, the mapping, $y \to x (y)$ is in $C^1 (B(y_0, \eta))$.

**Proof:** Let

$$f (x, y) = \begin{pmatrix} f_1 (x, y) \\ f_2 (x, y) \\ \vdots \\ f_n (x, y) \end{pmatrix}.$$  \hspace{1cm} (5.16.26)

Define for $(x^1, \ldots, x^n) \in B(x_0, \delta)^n$ and $y \in B(y_0, \eta)$ the following matrix.

$$J (x^1, \ldots, x^n, y) = \begin{pmatrix} f_{1,x_1} (x^1, y) & \cdots & f_{1,x_n} (x^1, y) \\ \vdots & \ddots & \vdots \\ f_{n,x_1} (x^n, y) & \cdots & f_{n,x_n} (x^n, y) \end{pmatrix}.$$  \hspace{1cm} (5.16.29)

Then by the assumption of continuity of all the partial derivatives, there exists $\delta_0 > 0$ and $\eta_0 > 0$ such that if $\delta < \delta_0$ and $\eta < \eta_0$, it follows that for all $(x^1, \ldots, x^n) \in B(x_0, \delta)$ and $y \in B(y_0, \eta)$,

$$\det (J (x^1, \ldots, x^n, y)) > r > 0.$$  \hspace{1cm} (5.16.28)

and $B(x_0, \delta_0) \times B(y_0, \eta_0) \subseteq U$. Pick $y \in B(y_0, \eta)$ and suppose there exist $x, z \in B(x_0, \delta)$ such that $f (x, y) = f (z, y) = 0$. Consider $f_i$ and let

$$h (t) \equiv f_i (x + t (z - x), y).$$

Then $h (1) = h (0)$ and so by the mean value theorem, $h' (t_i) = 0$ for some $t_i \in (0, 1)$. Therefore, from the chain rule and for this value of $t_i$,

$$h' (t_i) = D f_i (x + t_i (z - x), y) (z - x) = 0.$$  \hspace{1cm} (5.16.29)

Then denote by $x^i$ the vector, $x + t_i (z - x)$. It follows from 5.16.29 that

$$J (x^1, \ldots, x^n, y) (z - x) = 0.$$
and so from \( \| z - x \| = 0 \). Now it will be shown that if \( \eta \) is chosen sufficiently small, then for all \( y \in B(y_0, \eta) \), there exists a unique \( x(y) \in B(x_0, \delta) \) such that \( f(x(y), y) = 0 \).

**Claim:** If \( \eta \) is small enough, then the function, \( h_y(x) \equiv \| f(x, y) \| ^2 \) achieves its minimum value on \( \overline{B(x_0, \delta)} \) at a point of \( B(x_0, \delta) \).

**Proof of claim:** Suppose this is not the case. Then there exists a sequence \( \eta_k \to 0 \) and for some \( \eta_k \) having \( \| y_k - y_0 \| < \eta_k \), the minimum of \( h_{y_k} \) occurs on a point of the boundary of \( B(x_0, \delta) \), \( x_k \) such that \( \| x_0 - x_k \| = \delta \). Now taking a subsequence, still denoted by \( k \), it can be assumed that \( x_k \to x \) with \( \| x - x_0 \| = \delta \) and \( y_k \to y_0 \).

Let \( \varepsilon > 0 \). Then for \( k \) large enough, \( h_{y_k}(x_0) < \varepsilon \) because \( f(x_0, y_0) = 0 \). Therefore, from the definition of \( x_k \), \( h_{y_k}(x_k) < \varepsilon \). Passing to the limit yields \( h_{y_0}(x) \leq \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, it follows that \( h_{y_0}(x) = 0 \) which contradicts the first part of the argument in which it was shown that for \( y \in B(y_0, \eta) \) there is at most one point, \( x \) of \( \overline{B(x_0, \delta)} \) where \( f(x, y) = 0 \). Here two have been obtained, \( x_0 \) and \( x \). This proves the claim.

Choose \( \eta < \eta_0 \) and also small enough that the above claim holds and let \( x(y) \) denote a point of \( B(x_0, \delta) \) at which the minimum of \( h_y \) on \( \overline{B(x_0, \delta)} \) is achieved. Since \( x(y) \) is an interior point, you can consider \( h_y(x + tv) \) for \( |t| \) small and conclude this function of \( t \) has a zero derivative at \( t = 0 \). Thus

\[
Dh_y(x(y))v = 0 = 2f(x(y), y)^TD_xf(x(y), y)v
\]

for every vector \( v \). But from \[5.16.28\] and the fact that \( v \) is arbitrary, it follows \( f(x(y), y) = 0 \). This proves the existence of the function \( y \to x(y) \) such that \( f(x(y), y) = 0 \) for all \( y \in B(y_0, \eta) \).

It remains to verify this function is a \( C^1 \) function. To do this, let \( y_1 \) and \( y_2 \) be points of \( B(y_0, \eta) \). Then as before, consider the \( i^{th} \) component of \( f \) and consider the same argument using the mean value theorem to write

\[
0 = f_i(x(y_1), y_1) - f_i(x(y_2), y_2)
= f_i(x(y_1), y_1) - f_i(x(y_2), y_1) + f_i(x(y_2), y_1) - f_i(x(y_2), y_2)
= D_1f_i(x^1, y_1)(x(y_1) - x(y_2)) + D_2f_i(x(y_2), y_1)(y_1 - y_2).
\]

Therefore,

\[
J(x^1, \cdots, x^n, y_1)(x(y_1) - x(y_2)) = -M(y_1 - y_2)
\]

where \( M \) is the matrix whose \( i^{th} \) row is \( D_2f_i(x(y_2), y) \). Then from \[5.16.28\] there exists a constant, \( C \) independent of the choice of \( y \in B(y_0, \eta) \) such that

\[
\| J(x^1, \cdots, x^n, y) \| < C
\]

whenever \( (x^1, \cdots, x^n) \in \overline{B(x_0, \delta)} \). By continuity of the partial derivatives of \( f \) it also follows there exists a constant, \( C_1 \) such that \( \| D_2f_i(x, y) \| < C_1 \) whenever, \( (x, y) \in \overline{B(x_0, \delta)} \). Hence \( \| M \| \) must also be bounded independent of the
5.16. IMPLICIT FUNCTION THEOREM

choice of \( y_1 \) and \( y_2 \) in \( B (y_0, \eta) \). From \( \text{det} \), it follows there exists a constant, \( C \) such that for all \( y_1, y_2 \) in \( B (y_0, \eta) \),

\[
| x (y_1) - x (y_2) | \leq C | y_1 - y_2 | . \tag{5.16.31}
\]

It follows as in the proof of the chain rule that

\[
o (x (y + v) - x (y)) = o (v) . \tag{5.16.32}
\]

Now let \( y \in B (y_0, \eta) \) and let \( |v| \) be sufficiently small that \( y + v \in B (y_0, \eta) \). Then

\[
0 = f (x (y + v), y + v) - f (x (y), y) = f (x (y + v), y + v) - f (x (y + v), y) + f (x (y + v), y) - f (x (y), y) = D_2 f (x (y + v), y) v + D_1 f (x (y), y) (x (y + v) - x (y)) + o (|x (y + v) - x (y)|) = D_2 f (x (y), y) v + D_1 f (x (y), y) (x (y + v) - x (y)) + o (v).
\]

Therefore,

\[
x (y + v) - x (y) = -D_1 f (x (y), y)^{-1} D_2 f (x (y), y) v + o (v)
\]

which shows that \( D x (y) = -D_1 f (x (y), y)^{-1} D_2 f (x (y), y) \) and \( y \rightarrow D x (y) \) is continuous. This proves the theorem.

In practice, how do you verify the condition, \( D_1 f (x_0, y_0)^{-1} \in \mathcal{L} (\mathbb{R}^n, \mathbb{R}^n) \)?

\[
f (x, y) = \begin{pmatrix} f_1 (x_1, \cdots, x_n, y_1, \cdots, y_n) \\ \vdots \\ f_n (x_1, \cdots, x_n, y_1, \cdots, y_n) \end{pmatrix}.
\]

The matrix of the linear transformation, \( D_1 f (x_0, y_0) \) is then

\[
\begin{pmatrix}
\frac{\partial f_1 (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_1} & \cdots & \frac{\partial f_1 (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_n} \\
\vdots & & \vdots \\
\frac{\partial f_n (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_1} & \cdots & \frac{\partial f_n (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_n}
\end{pmatrix}
\]

and from linear algebra, \( D_1 f (x_0, y_0)^{-1} \in \mathcal{L} (\mathbb{R}^n, \mathbb{R}^n) \) exactly when the above matrix has an inverse. In other words when

\[
\text{det} \begin{pmatrix}
\frac{\partial f_1 (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_1} & \cdots & \frac{\partial f_1 (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_n} \\
\vdots & & \vdots \\
\frac{\partial f_n (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_1} & \cdots & \frac{\partial f_n (x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_n}
\end{pmatrix} \neq 0
\]
at \((x_0,y_0)\). The above determinant is important enough that it is given special notation. Letting \(z = f(x,y)\), the above determinant is often written as

\[
\frac{\partial (z_1, \ldots, z_n)}{\partial (x_1, \ldots, x_n)}.
\]

Of course you can replace \(\mathbb{R}\) with \(F\) in the above by applying the above to the situation in which each \(F\) is replaced with \(R^2\).

**Corollary 5.16.2 (implicit function theorem)** Suppose \(U\) is an open set in \(\mathbb{F}^n \times \mathbb{F}^m\). Let \(f : U \to \mathbb{F}^n\) be in \(C^1(U)\) and suppose

\[
f(x_0, y_0) = 0, \quad D_1 f(x_0, y_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n).
\]

Then there exist positive constants, \(\delta, \eta\), such that for every \(y \in B(y_0, \eta)\) there exists a unique \(x(y) \in B(x_0, \delta)\) such that

\[
f(x(y), y) = 0.
\]

Furthermore, the mapping, \(y \to x(y)\) is in \(C^1(B(y_0, \eta))\).

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [76] and in [3].

**Theorem 5.16.3 (inverse function theorem)** Let \(x_0 \in U \subseteq \mathbb{F}^n\) and let \(f : U \to \mathbb{F}^n\). Suppose

\[
f \text{ is } C^1(U), \quad \text{and } \quad Df(x_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n).
\]

Then there exist open sets, \(W, V\) such that

\[
x_0 \in W \subseteq U, \quad f : W \to V \text{ is one to one and onto}, \quad f^{-1} \text{ is } C^1.
\]

**Proof:** Apply the implicit function theorem to the function

\[
F(x,y) = f(x) - y
\]

where \(y_0 = f(x_0)\). Thus the function \(y \to x(y)\) defined in that theorem is \(f^{-1}\). Now let

\[
W \equiv B(x_0, \delta) \cap f^{-1}(B(y_0, \eta))
\]

and

\[
V \equiv B(y_0, \eta).
\]

This proves the theorem.
5.16. IMPLICIT FUNCTION THEOREM

5.16.1 More Continuous Partial Derivatives

Corollary \ref{corollary:implicit_function_theorem} will now be improved slightly. If \( f \) is \( C^k \), it follows that the function which is implicitly defined is also in \( C^k \), not just \( C^1 \). Since the inverse function theorem comes as a case of the implicit function theorem, this shows that the inverse function also inherits the property of being \( C^k \).

**Theorem 5.16.4** (implicit function theorem) Suppose \( U \) is an open set in \( \mathbb{F}^n \times \mathbb{F}^m \). Let \( f : U \to \mathbb{F}^n \) be in \( C^k (U) \) and suppose
\[
f (x_0, y_0) = 0, \quad D_1 f (x_0, y_0)^{-1} \in \mathcal{L} (\mathbb{F}^n, \mathbb{F}^m). \tag{5.16.39}
\]
Then there exist positive constants, \( \delta, \eta \), such that for every \( y \in B (y_0, \eta) \) there exists a unique \( x (y) \in B (x_0, \delta) \) such that
\[
f (x (y), y) = 0. \tag{5.16.40}
\]
Furthermore, the mapping, \( y \to x (y) \) is in \( C^k (B (y_0, \eta)) \).

**Proof:** From Corollary \ref{corollary:implicit_function_theorem}, \( y \to x (y) \) is \( C^1 \). It remains to show it is \( C^k \) for \( k > 1 \) assuming that \( f \) is \( C^k \). From \ref{corollary:implicit_function_theorem},
\[
\frac{\partial x}{\partial y^q} = -D_1 (x, y)^{-1} \frac{\partial f}{\partial y^q}.
\]
Thus the following formula holds for \( q = 1 \) and \( |\alpha| = q \).
\[
D^\alpha x (y) = \sum_{|\beta| \leq q} M_\beta (x, y) D^\beta f (x, y) \tag{5.16.41}
\]
where \( M_\beta \) is a matrix whose entries are differentiable functions of \( D^\gamma (x) \) for \( |\gamma| < q \) and \( D^\gamma f (x, y) \) for \( |\gamma| \leq q \). This follows easily from the description of \( D_1 (x, y)^{-1} \) in terms of the cofactor matrix and the determinant of \( D_1 (x, y) \). Suppose \ref{corollary:implicit_function_theorem} holds for \( |\alpha| = q < k \). Then by induction, this yields \( x \) is \( C^q \). Then
\[
\frac{\partial D^\alpha x (y)}{\partial y^p} = \sum_{|\beta| \leq |\alpha|} \frac{\partial M_\beta (x, y)}{\partial y^p} D^\beta f (x, y) + M_\beta (x, y) \frac{\partial D^\beta f (x, y)}{\partial y^p}.
\]
By the chain rule \( \frac{\partial M_\beta (x, y)}{\partial y^p} \) is a matrix whose entries are differentiable functions of \( D^\gamma f (x, y) \) for \( |\gamma| \leq q + 1 \) and \( D^\gamma (x) \) for \( |\gamma| < q + 1 \). It follows since \( y^p \) was arbitrary that for any \( |\alpha| = q + 1 \), a formula like \ref{corollary:implicit_function_theorem} holds with \( q \) being replaced by \( q + 1 \). By induction, \( x \) is \( C^k \). This proves the theorem.

As a simple corollary this yields an improved version of the inverse function theorem.

**Theorem 5.16.5** (inverse function theorem) Let \( x_0 \in U \subseteq \mathbb{F}^n \) and let \( f : U \to \mathbb{F}^n \). Suppose for \( k \) a positive integer,
\[
f \text{ is } C^k (U), \text{ and } Df (x_0)^{-1} \in \mathcal{L} (\mathbb{F}^n, \mathbb{F}^n). \tag{5.16.42}
\]
Then there exist open sets, \( W, V \) such that
\[
x_0 \in W \subseteq U, \quad (5.16.43)
\]
\[f : W \to V \text{ is one to one and onto,} \quad (5.16.44)
\]
\[f^{-1} \text{ is } C^k. \quad (5.16.45)
\]

\section*{5.17 The Method Of Lagrange Multipliers}

As an application of the implicit function theorem, consider the method of Lagrange multipliers from calculus. Recall the problem is to maximize or minimize a function subject to equality constraints. Let \( f : U \to \mathbb{R} \) be a \( C^1 \) function where \( U \subseteq \mathbb{R}^n \) and let
\[
g_i (x) = 0, \quad i = 1, \ldots, m \quad (5.17.46)
\]
be a collection of equality constraints with \( m < n \). Now consider the system of nonlinear equations
\[
f(x) = a
\]
\[g_i (x) = 0, \quad i = 1, \ldots, m.
\]
\( x_0 \) is a local maximum if \( f(x_0) \geq f(x) \) for all \( x \) near \( x_0 \) which also satisfies the constraints \( g_i (x) = 0 \). A local minimum is defined similarly. Let \( F : U \times \mathbb{R} \to \mathbb{R}^{m+1} \) be defined by
\[
F(x,a) \equiv \begin{pmatrix}
f(x) - a \\
g_1 (x) \\
\vdots \\
g_m (x)
\end{pmatrix}.
\quad (5.17.47)
\]
Now consider the \( m+1 \times n \) Jacobian matrix,
\[
\begin{pmatrix}
f_{x_1} (x_0) & \cdots & f_{x_n} (x_0) \\
g_{1x_1} (x_0) & \cdots & g_{1x_n} (x_0) \\
\vdots & \ddots & \vdots \\
g_{mx_1} (x_0) & \cdots & g_{mrx_n} (x_0)
\end{pmatrix}.
\]
If this matrix has rank \( m+1 \) then some \( m+1 \times m+1 \) submatrix has nonzero determinant. It follows from the implicit function theorem that there exist \( m+1 \) variables, \( x_{i_1}, \ldots, x_{i_{m+1}} \) such that the system
\[
F(x,a) = 0 \quad (5.17.48)
\]
specifies these \( m+1 \) variables as a function of the remaining \( n-(m+1) \) variables and \( a \) in an open set of \( \mathbb{R}^{n-m} \). Thus there is a solution \((x,a)\) to \( (5.17.48) \) for some \( x \) close to \( x_0 \) whenever \( a \) is in some open interval. Therefore, \( x_0 \) cannot be either a local minimum or a local maximum. It follows that if \( x_0 \) is either a local maximum
or a local minimum, then the above matrix must have rank less than \( m + 1 \) which requires the rows to be linearly dependent. Thus, there exist \( m \) scalars, 
\[
\lambda_1, \ldots, \lambda_m,
\]
and a scalar \( \mu \), not all zero such that
\[
\mu \begin{pmatrix} f_{x_1}(x_0) \\ \vdots \\ f_{x_n}(x_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(x_0) \\ \vdots \\ g_{1x_n}(x_0) \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} g_{mx_1}(x_0) \\ \vdots \\ g_{mx_n}(x_0) \end{pmatrix}.
\] (5.17.49)

If the column vectors
\[
\begin{pmatrix} g_{1x_1}(x_0) \\ \vdots \\ g_{1x_n}(x_0) \end{pmatrix}, \ldots, \begin{pmatrix} g_{mx_1}(x_0) \\ \vdots \\ g_{mx_n}(x_0) \end{pmatrix}
\] (5.17.50)
are linearly independent, then, \( \mu \neq 0 \) and dividing by \( \mu \) yields an expression of the form
\[
\begin{pmatrix} f_{x_1}(x_0) \\ \vdots \\ f_{x_n}(x_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(x_0) \\ \vdots \\ g_{1x_n}(x_0) \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} g_{mx_1}(x_0) \\ \vdots \\ g_{mx_n}(x_0) \end{pmatrix}.
\] (5.17.51)

at every point \( x_0 \) which is either a local maximum or a local minimum. This proves the following theorem.

**Theorem 5.17.1** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R} \) be a \( C^1 \) function. Then if \( x_0 \in U \) is either a local maximum or local minimum of \( f \) subject to the constraints (5.17.46), then (5.17.49) must hold for some scalars \( \mu, \lambda_1, \ldots, \lambda_m \) not all equal to zero. If the vectors in (5.17.50) are linearly independent, it follows that an equation of the form (5.17.51) holds.
Chapter 6

Metric Spaces And General Topological Spaces

6.1 Metric Space

Definition 6.1.1 A metric space is a set, \( X \) and a function \( d : X \times X \to [0, \infty) \) which satisfies the following properties.

\[
\begin{align*}
  d(x, y) &= d(y, x) \\
  d(x, y) &\geq 0 \quad \text{and} \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y \\
  d(x, y) &\leq d(x, z) + d(z, y).
\end{align*}
\]

You can check that \( \mathbb{R}^n \) and \( \mathbb{C}^n \) are metric spaces with \( d(x, y) = |x - y| \). However, there are many others. The definitions of open and closed sets are the same for a metric space as they are for \( \mathbb{R}^n \).

Definition 6.1.2 A set, \( U \) in a metric space is open if whenever \( x \in U \), there exists \( r > 0 \) such that \( B(x, r) \subseteq U \). As before, \( B(x, r) = \{ y : d(x, y) < r \} \). Closed sets are those whose complements are open. A point \( p \) is a limit point of a set, \( S \) if for every \( r > 0 \), \( B(p, r) \) contains infinitely many points of \( S \). A sequence, \( \{x_n\} \) converges to a point \( x \) if for every \( \varepsilon > 0 \) there exists \( N \) such that if \( n \geq N \), then \( d(x, x_n) < \varepsilon \). \( \{x_n\} \) is a Cauchy sequence if for every \( \varepsilon > 0 \) there exists \( N \) such that if \( m, n \geq N \), then \( d(x_n, x_m) < \varepsilon \).

Lemma 6.1.3 In a metric space, \( X \) every ball, \( B(x, r) \) is open. A set is closed if and only if it contains all its limit points. If \( p \) is a limit point of \( S \), then there exists a sequence of distinct points of \( S \), \( \{x_n\} \) such that \( \lim_{n \to \infty} x_n = p \).

Proof: Let \( z \in B(x, r) \). Let \( \delta = r - d(x, z) \). Then if \( w \in B(z, \delta) \),

\[
d(w, x) \leq d(x, z) + d(z, w) < d(x, z) + r - d(x, z) = r.
\]

Therefore, \( B(z, \delta) \subseteq B(x, r) \) and this shows \( B(x, r) \) is open.

The properties of balls are presented in the following theorem.
Theorem 6.1.4 Suppose \((X, d)\) is a metric space. Then the sets \(\{B(x, r) : r > 0, x \in X\}\) satisfy
\[
\bigcup \{B(x, r) : r > 0, x \in X\} = X
\]
(6.1.1)
If \(p \in B(x, r_1) \cap B(z, r_2)\), there exists \(r > 0\) such that
\[
B(p, r) \subseteq B(x, r_1) \cap B(z, r_2).
\]
(6.1.2)

Proof: Observe that the union of these balls includes the whole space, \(X\) so \((6.1.1)\) is obvious. Consider \((6.1.2)\). Let \(p \in B(x, r_1) \cap B(z, r_2)\). Consider
\[
r \equiv \min (r_1 - d(x, p), r_2 - d(z, p))
\]
and suppose \(y \in B(p, r)\). Then
\[
d(y, x) \leq d(y, p) + d(p, x) < r_1 - d(x, p) + d(x, p) = r_1
\]
and so \(B(p, r) \subseteq B(x, r_1)\). By similar reasoning, \(B(p, r) \subseteq B(z, r_2)\). This proves the theorem.

Let \(K\) be a closed set. This means \(K^C \equiv X \setminus K\) is an open set. Let \(p\) be a limit point of \(K\). If \(p \in K^C\), then since \(K^C\) is open, there exists \(B(p, r) \subseteq K^C\). But this contradicts \(p\) being a limit point because there are no points of \(K\) in this ball. Hence all limit points of \(K\) must be in \(K\).

Suppose next that \(K\) contains its limit points. Is \(K^C\) open? Let \(p \in K^C\). Then \(p\) is not a limit point of \(K\). Therefore, there exists \(B(p, r)\) which contains at most finitely many points of \(K\). Since \(p \notin K\), it follows that by making \(r\) smaller if necessary, \(B(p, r)\) contains no points of \(K\). That is \(B(p, r) \subseteq K^C\) showing \(K^C\) is open. Therefore, \(K\) is closed.

Suppose now that \(p\) is a limit point of \(S\). Let \(x_1 \in (S \setminus \{p\}) \cap B(p, 1)\). If \(x_1, \cdots, x_k\) have been chosen, let
\[
r_{k+1} \equiv \min \left\{d(p, x_i), i = 1, \cdots, k, \frac{1}{k+1}\right\}.
\]
Let \(x_{k+1} \in (S \setminus \{p\}) \cap B(p, r_{k+1})\). This proves the lemma.

Lemma 6.1.5 If \(\{x_n\}\) is a Cauchy sequence in a metric space, \(X\) and if some subsequence, \(\{x_{n_k}\}\) converges to \(x\), then \(\{x_n\}\) converges to \(x\). Also if a sequence converges, then it is a Cauchy sequence.

Proof: Note first that \(n_k \geq k\) because in a subsequence, the indices, \(n_1, n_2, \cdots\) are strictly increasing. Let \(\varepsilon > 0\) be given and let \(N\) be such that for \(k > N\), \(d(x, x_{n_k}) < \varepsilon/2\) and for \(m, n > N\), \(d(x_m, x_n) < \varepsilon/2\). Pick \(k > n\). Then if \(n > N\),
\[
d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Finally, suppose \(\lim_{n \to \infty} x_n = x\). Then there exists \(N\) such that if \(n > N\), then \(d(x_n, x) < \varepsilon/2\). It follows that for \(m, n > N\),
\[
d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
6.2 COMPACTNESS IN METRIC SPACE

This proves the lemma.
A useful idea is the idea of distance from a point to a set.

**Definition 6.1.6** Let \((X, d)\) be a metric space and let \(S\) be a nonempty set in \(X\). Then

\[
\text{dist}(x, S) \equiv \inf \{d(x, y) : y \in S\}.
\]

The following lemma is the fundamental result.

**Lemma 6.1.7** The function, \(x \rightarrow \text{dist}(x, S)\) is continuous and in fact satisfies

\[
|\text{dist}(x, S) - \text{dist}(y, S)| \leq d(x, y).
\]

**Proof:** Suppose \(\text{dist}(x, y)\) is as least as large as \(\text{dist}(y, S)\). Then pick \(z \in S\) such that \(d(y, z) \leq \text{dist}(y, S) + \varepsilon\). Then

\[
|\text{dist}(x, S) - \text{dist}(y, S)| = \text{dist}(x, S) - \text{dist}(y, S) \\
\leq d(x, z) - (d(y, z) - \varepsilon) \\
= d(x, z) - d(y, z) + \varepsilon \\
\leq d(x, y) + d(y, z) - d(y, z) + \varepsilon \\
= d(x, y) + \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrary, this proves the lemma.

6.2 Compactness In Metric Space

Many existence theorems in analysis depend on some set being compact. Therefore, it is important to be able to identify compact sets. The purpose of this section is to describe compact sets in a metric space.

**Definition 6.2.1** Let \(A\) be a subset of \(X\). \(A\) is compact if whenever \(A\) is contained in the union of a set of open sets, there exists finitely many of these open sets whose union contains \(A\). (Every open cover admits a finite subcover.) \(A\) is “sequentially compact” means every sequence has a convergent subsequence converging to an element of \(A\).

In a metric space compact is not the same as closed and bounded!

**Example 6.2.2** Let \(X\) be any infinite set and define \(d(x, y) = 1\) if \(x \neq y\) while \(d(x, y) = 0\) if \(x = y\).

You should verify the details that this is a metric space because it satisfies the axioms of a metric. The set \(X\) is closed and bounded because its complement is \(\emptyset\) which is clearly open because every point of \(\emptyset\) is an interior point. (There are none.) Also \(X\) is bounded because \(X = B(x, 2)\). However, \(X\) is clearly not compact because \(\{B(x, \frac{1}{2}) : x \in X\}\) is a collection of open sets whose union contains \(X\) but
since they are all disjoint and nonempty, there is no finite subset of these whose union contains $X$. In fact $B(x, \frac{1}{2}) = \{x\}$.

From this example it is clear something more than closed and bounded is needed. If you are not familiar with the issues just discussed, ignore them and continue.

**Definition 6.2.3** In any metric space, a set $E$ is totally bounded if for every $\varepsilon > 0$ there exists a finite set of points $\{x_1, \cdots, x_n\}$ such that $E \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.

This finite set of points is called an $\varepsilon$ net.

The following proposition tells which sets in a metric space are compact. First here is an interesting lemma.

**Lemma 6.2.4** Let $X$ be a metric space and suppose $D$ is a countable dense subset of $X$. In other words, it is being assumed $X$ is a separable metric space. Consider the open sets of the form $B(d, r)$ where $r$ is a positive rational number and $d \in D$. Denote this countable collection of open sets by $B$. Then every open set is the union of sets of $B$. Furthermore, if $C$ is any collection of open sets, there exists a countable subset, $\{U_n\} \subseteq C$ such that $\bigcup_n U_n = \bigcup C$.

**Proof:** Let $U$ be an open set and let $x \in U$. Let $B(x, \delta) \subseteq U$. Then by density of $D$, there exists $d \in D \cap B(x, \delta/4)$. Now pick $r \in \mathbb{Q} \cap (\delta/4, 3\delta/4)$ and consider $B(d, r)$. Clearly, $B(d, r)$ contains the point $x$ because $r > \delta/4$. Is $B(d, r) \subseteq B(x, \delta)$? if so, this proves the lemma because $x$ was an arbitrary point of $U$. Suppose $z \in B(d, r)$. Then

$$d(z, x) \leq d(z, d) + d(d, x) < r + \frac{\delta}{4} < \frac{3\delta}{4} + \frac{\delta}{4} = \delta$$

Now let $C$ be any collection of open sets. Each set in this collection is the union of countably many sets of $B$. Let $B'$ denote the sets of $B$ which are contained in some set of $C$. Thus $\bigcup B' = \bigcup C$. Then for each $B \in B'$, pick $U_B \in C$ such that $B \subseteq U_B$. Then $\{U_B : B \in B'\}$ is a countable collection of sets of $C$ whose union equals $\bigcup C$. Therefore, this proves the lemma.

**Proposition 6.2.5** Let $(X, d)$ be a metric space. Then the following are equivalent.

\begin{align*}
(X, d) & \text{ is compact,} & (6.2.3) \\
(X, d) & \text{ is sequentially compact,} & (6.2.4) \\
(X, d) & \text{ is complete and totally bounded.} & (6.2.5)
\end{align*}

**Proof:** Suppose (6.2.3) and let $\{x_k\}$ be a sequence. Suppose $\{x_k\}$ has no convergent subsequence. If this is so, then no value of the sequence is repeated more than finitely many times. Also $\{x_k\}$ has no limit point because if it did, there would exist a subsequence which converges. To see this, suppose $p$ is a limit point of $\{x_k\}$. Then in $B(p, 1)$ there are infinitely many points of $\{x_k\}$. Pick one called $x_k_1$. Now if
6.2. COMPACTNESS IN METRIC SPACE

$x_{k_1}, x_{k_2}, \ldots, x_{k_n}$ have been picked with $x_{k_i} \in B(p, 1/i)$, consider $B(p, 1/(n + 1))$. There are infinitely many points of \{x_k\} in this ball also. Pick $x_{k_{n+1}}$ such that $k_{n+1} > k_n$. Then \{x_{k_n}\}_{n=1}^\infty is a subsequence which converges to $p$ and it is assumed this does not happen. Thus \{x_k\} has no limit points. It follows the set

$$C_n = \cup \{x_k : k \geq n\}$$

is a closed set because it has no limit points and if

$$U_n = C_n^C,$$

then

$$X = \cup_{n=1}^\infty U_n$$

but there is no finite subcovering, because no value of the sequence is repeated more than finitely many times. This contradicts compactness of $(X, d)$. Note $x_k$ is not in $U_n$ whenever $k > n$. Thus 6.2.3 implies 6.2.4.

Now suppose 6.2.4 and let \{x_n\} be a Cauchy sequence. Is \{x_n\} convergent? By sequential compactness $x_{n_k} \to x$ for some subsequence. By Lemma 6.1.5 it follows that \{x_n\} also converges to $x$ showing that $(X, d)$ is complete. If $(X, d)$ is not totally bounded, then there exists $\varepsilon > 0$ for which there is no $\varepsilon$ net. Hence there exists a sequence \{x_l\} with $d(x_k, x_l) \geq \varepsilon$ for all $l \neq k$. By Lemma 6.1.5 again, this contradicts 6.2.4 because no subsequence can be a Cauchy sequence and so no subsequence can converge. This shows 6.2.3 implies 6.2.4.

Now suppose 6.2.5. What about 6.2.4? Let \{p_n\} be a sequence and let \{x_i^n\}_{i=1}^{m_n} be a $2^{-n}$ net for $n = 1, 2, \ldots$. Let

$$B_n = B(x_{i_n}^n, 2^{-n})$$

be such that $B_n$ contains $p_k$ for infinitely many values of $k$ and $B_n \cap B_{n+1} \neq \emptyset$. To do this, suppose $B_n$ contains $p_k$ for infinitely many values of $k$. Then one of the sets which intersect $B_n, B(x_{i_n}^{n+1}, 2^{-(n+1)})$ must contain $p_k$ for infinitely many values of $k$ because all these indices of points from \{p_n\} contained in $B_n$ must be accounted for in one of finitely many sets, $B(x_{i_n}^{n+1}, 2^{-(n+1)})$. Thus there exists a strictly increasing sequence of integers, $n_k$ such that

$$p_{n_k} \in B_k.$$

Then if $k \geq l$,

$$d(p_{n_k}, p_l) \leq \sum_{i=l}^{k-1} d(p_{n_{i+1}}, p_{n_i}) < \sum_{i=l}^{k-1} 2^{-(i-1)} < 2^{-(l-2)}.$$

Consequently \{p_{n_k}\} is a Cauchy sequence. Hence it converges because the metric space is complete. This proves 6.2.5.
Now suppose and which have now been shown to be equivalent. Let \( D_n \) be a \( n^{-1} \) net for \( n = 1, 2, \cdots \) and let
\[
D = \bigcup_{n=1}^{\infty} D_n.
\]
Thus \( D \) is a countable dense subset of \((X, d)\).

Now let \( C \) be any set of open sets such that \( \bigcup C \supseteq X \). By Lemma 6.2.4, there exists a countable subset of \( C \),
\[
\tilde{C} = \{U_n\}_{n=1}^{\infty}
\]
such that \( \bigcup \tilde{C} = \bigcup C \). If \( C \) admits no finite subcover, then neither does \( \tilde{C} \) and there exists \( p_n \in X \setminus \bigcup_{k=1}^{n} U_k \). Then since \( X \) is sequentially compact, there is a subsequence \( \{p_{n_k}\} \) such that \( \{p_{n_k}\} \) converges. Say
\[
p = \lim_{k \to \infty} p_{n_k}.
\]
All but finitely many points of \( \{p_{n_k}\} \) are in \( X \setminus \bigcup_{k=1}^{n} U_k \). Therefore \( p \in X \setminus \bigcup_{k=1}^{n} U_k \) for each \( n \). Hence
\[
p \notin \bigcup_{k=1}^{\infty} U_k
\]
contradicting the construction of \( \{U_n\}_{n=1}^{\infty} \) which required that \( \bigcup_{n=1}^{\infty} U_n \supseteq X \). Hence \( X \) is compact. This proves the proposition.

Consider \( \mathbb{R}^n \). In this setting totally bounded and bounded are the same. This will yield a proof of the Heine Borel theorem from advanced calculus.

**Lemma 6.2.6** A subset of \( \mathbb{R}^n \) is totally bounded if and only if it is bounded.

**Proof:** Let \( A \) be totally bounded. Is it bounded? Let \( x_1, \cdots, x_p \) be a 1 net for \( A \). Now consider the ball \( B(0, r+1) \) where \( r > \max (|x_i| : i = 1, \cdots, p) \). If \( z \in A \), then \( z \in B(x_j, 1) \) for some \( j \) and so by the triangle inequality,
\[
|z - 0| \leq |z - x_j| + |x_j| < 1 + r.
\]
Thus \( A \subseteq B(0, r+1) \) and so \( A \) is bounded.

Now suppose \( A \) is bounded and suppose \( A \) is not totally bounded. Then there exists \( \varepsilon > 0 \) such that there is no \( \varepsilon \) net for \( A \). Therefore, there exists a sequence of points \( \{a_i\} \) with \( |a_i - a_j| \geq \varepsilon \) if \( i \neq j \). Since \( A \) is bounded, there exists \( r > 0 \) such that
\[
A \subseteq [-r, r]^n.
\]
(\( x \in [-r, r]^n \) means \( x_i \in [-r, r] \) for each \( i \).) Now define \( S \) to be all cubes of the form
\[
\prod_{k=1}^{n} [a_k, b_k]
\]
where
\[
a_k = -r + i2^{-p}r, \quad b_k = -r + (i + 1)2^{-p}r,
\]
where \( \text{pr} \) is a positive integer.

Now let \( D_{p,m} \) be the union of all cubes \( [a_k, b_k] \) in \( S \) that contain \( p \), where \( m \) is the smallest integer such that \( m \geq 2^p \). Then \( D_{p,m} \) is a \( 2^p \) net for \( A \) that is equivalent to \( \{U_n\}_{n=1}^{\infty} \). Hence \( A \) is totally bounded.
for \( i \in \{0, 1, \cdots, 2^p + 1 - 1\} \). Thus \( S \) is a collection of \((2^p + 1)^n\) non overlapping cubes whose union equals \([-r, r]^n\) and whose diameters are all equal to \(2^{-p}r\sqrt{n}\). Now choose \( p \) large enough that the diameter of these cubes is less than \( \varepsilon \). This yields a contradiction because one of the cubes must contain infinitely many points of \( \{a_i\} \). This proves the lemma.

The next theorem is called the Heine Borel theorem and it characterizes the compact sets in \( \mathbb{R}^n \).

**Theorem 6.2.7** A subset of \( \mathbb{R}^n \) is compact if and only if it is closed and bounded.

**Proof:** Since a set in \( \mathbb{R}^n \) is totally bounded if and only if it is bounded, this theorem follows from Proposition 6.2.5 and the observation that a subset of \( \mathbb{R}^n \) is closed if and only if it is complete. This proves the theorem.

### 6.3 Some Applications Of Compactness

The following corollary is an important existence theorem which depends on compactness.

**Corollary 6.3.1** Let \( X \) be a compact metric space and let \( f : X \to \mathbb{R} \) be continuous. Then \( \max \{ f(x) : x \in X \} \) and \( \min \{ f(x) : x \in X \} \) both exist.

**Proof:** First it is shown \( f(X) \) is compact. Suppose \( C \) is a set of open sets whose union contains \( f(X) \). Then since \( f \) is continuous \( f^{-1}(U) \) is open for all \( U \in C \). Therefore, \( \{ f^{-1}(U) : U \in C \} \) is a collection of open sets whose union contains \( X \). Since \( X \) is compact, it follows finitely many of these, \( \{ f^{-1}(U_1), \cdots, f^{-1}(U_p) \} \) contains \( X \) in their union. Therefore, \( f(X) \subseteq \bigcup_{k=1}^{p} U_k \) showing \( f(X) \) is compact as claimed.

Now since \( f(X) \) is compact, Theorem 6.2.7 implies \( f(X) \) is closed and bounded. Therefore, it contains its inf and its sup. Thus \( f \) achieves both a maximum and a minimum.

**Definition 6.3.2** Let \( X, Y \) be metric spaces and \( f : X \to Y \) a function. \( f \) is uniformly continuous if for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( x_1 \) and \( x_2 \) are two points of \( X \) satisfying \( d(x_1, x_2) < \delta \), it follows that \( d(f(x_1), f(x_2)) < \varepsilon \).

A very important theorem is the following.

**Theorem 6.3.3** Suppose \( f : X \to Y \) is continuous and \( X \) is compact. Then \( f \) is uniformly continuous.

**Proof:** Suppose this is not true and that \( f \) is continuous but not uniformly continuous. Then there exists \( \varepsilon > 0 \) such that for all \( \delta > 0 \) there exist points, \( p_\delta \) and \( q_\delta \) such that \( d(p_\delta, q_\delta) < \delta \) and yet \( d(f(p_\delta), f(q_\delta)) \geq \varepsilon \). Let \( p_n \) and \( q_n \) be the points which go with \( \delta = 1/n \). By Proposition 6.2.5 \( \{p_n\} \) has a convergent
subsequence, \( \{p_{nk}\} \) converging to a point, \( x \in X \). Since \( d(p_n, q_n) < \frac{1}{n} \), it follows that \( q_{nk} \to x \) also. Therefore,

\[
\varepsilon \leq d(f(p_{nk}), f(q_{nk})) \leq d(f(p_{nk}), f(x)) + d(f(x), f(q_{nk}))
\]

but by continuity of \( f \), both \( d(f(p_{nk}), f(x)) \) and \( d(f(x), f(q_{nk})) \) converge to 0 as \( k \to \infty \) contradicting the above inequality. This proves the theorem.

Another important property of compact sets in a metric space concerns the finite intersection property.

Definition 6.3.4 If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.

Theorem 6.3.5 Suppose \( F \) is a collection of compact sets in a metric space, \( X \) which has the finite intersection property. Then there exists a point in their intersection. (\( \bigcap F \neq \emptyset \)).

Proof: First I show each compact set is closed. Let \( K \) be a nonempty compact set and suppose \( p \notin K \). Then for each \( x \in K \), let \( V_x = B(x, d(p, x)/3) \) and \( U_x = B(p, d(p, x)/3) \) so that \( U_x \) and \( V_x \) have empty intersection. Then since \( V \) is compact, there are finitely many \( V_x \) which cover \( K \) say \( V_{x_1}, \ldots, V_{x_n} \). Then let \( U = \cap_{i=1}^n U_{x_i} \). It follows \( p \notin U \) and \( U \) has empty intersection with \( K \). In fact \( U \) has empty intersection with \( \cup_{i=1}^n V_{x_i} \). Since \( U \) is an open set and \( p \in K^C \) is arbitrary, it follows \( K^C \) is an open set.

Consider now the claim about the intersection. If this were not so,

\[
\cup \{ F^C : F \in F \} = X
\]

and so, in particular, picking some \( F_0 \in F \),

\[
\{ F^C : F \in F \}
\]

would be an open cover of \( F_0 \). Since \( F_0 \) is compact, some finite subcover, \( F_1^C, \ldots, F_m^C \) exists. But then

\[
F_0 \subseteq \cup_{k=1}^m F_k^C
\]

which means \( \cap_{k=0}^m F_k = \emptyset \), contrary to the finite intersection property. To see this, note that if \( x \in F_0 \), then it must fail to be in some \( F_k \) and so it is not in \( \cap_{k=0}^m F_k \). Since this is true for every \( x \) it follows \( \cap_{k=0}^m F_k = \emptyset \).

Theorem 6.3.6 Let \( X_i \) be a compact metric space with metric \( d_i \). Then \( \prod_{i=1}^m X_i \) is also a compact metric space with respect to the metric, \( d(x, y) = \max_i (d_i(x_i, y_i)) \).

Proof: This is most easily seen from sequential compactness. Let \( \{x^k\}_{k=1}^\infty \) be a sequence of points in \( \prod_{i=1}^m X_i \). Consider the \( i^{th} \) component of \( x^k \), \( x_i^k \). It follows \( \{x_i^k\} \) is a sequence of points in \( X_i \) and so it has a convergent subsequence. Compactness of \( X_i \) implies there exists a subsequence of \( x^k \), denoted by \( \{x^{k_1}\} \) such that

\[
\lim_{k_1 \to \infty} x^{k_1}_{1} \to x_1 \in X_1.
\]
Now there exists a further subsequence, denoted by \( \{ x^{k_2} \} \) such that in addition to this, \( x_2 \) \( \rightarrow_{m} \) \( x_2 \in X_2 \). After taking \( m \) such subsequences, there exists a subsequence, \( \{ x^l \} \) such that \( lim_{l \rightarrow \infty} x^l = x_i \in X_i \) for each \( i \). Therefore, letting \( x = (x_1, \cdots, x_m) \), \( x^l \rightarrow x \) in \( \prod_{i=1}^{m} X_i \). This proves the theorem.

### 6.4 Ascoli Arzela Theorem

**Definition 6.4.1** Let \((X, d)\) be a complete metric space. Then it is said to be locally compact if \( \overline{B}(x, r) \) is compact for each \( r > 0 \).

Thus if you have a locally compact metric space, then if \( \{a_n\} \) is a bounded sequence, it must have a convergent subsequence.

Let \( K \) be a compact subset of \( \mathbb{R}^n \) and consider the continuous functions which have values in a locally compact metric space, \((X, d)\) where \( d \) denotes the metric on \( X \). Denote this space as \( C(K, X) \).

**Definition 6.4.2** For \( f, g \in C(K, X) \), where \( K \) is a compact subset of \( \mathbb{R}^n \) and \( X \) is a locally compact complete metric space define

\[
\rho_K(f, g) \equiv \sup \left\{ d(f(x), g(x)) : x \in K \right\}.
\]

Then \( \rho_K \) provides a distance which makes \( C(K, X) \) into a metric space.

The Ascoli Arzela theorem is a major result which tells which subsets of \( C(K, X) \) are sequentially compact.

**Definition 6.4.3** Let \( A \subseteq C(K, X) \) for \( K \) a compact subset of \( \mathbb{R}^n \). Then \( A \) is said to be uniformly equicontinuous if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that whenever \( x, y \in K \) with \( |x - y| < \delta \) and \( f \in A \),

\[
d(f(x), f(y)) < \varepsilon.
\]

The set, \( A \) is said to be uniformly bounded if for some \( M < \infty \), and \( a \in X \),

\[
f(x) \in B(a, M)
\]

for all \( f \in A \) and \( x \in K \).

Uniform equicontinuity is like saying that the whole set of functions, \( A \), is uniformly continuous on \( K \) uniformly for \( f \in A \). The version of the Ascoli Arzela theorem I will present here is the following.

**Theorem 6.4.4** Suppose \( K \) is a nonempty compact subset of \( \mathbb{R}^n \) and \( A \subseteq C(K, X) \) is uniformly bounded and uniformly equicontinuous. Then if \( \{f_k\} \subseteq A \), there exists a function, \( f \in C(K, X) \) and a subsequence, \( f_{k_1} \) such that

\[
lim_{l \rightarrow \infty} \rho_K(f_{k_l}, f) = 0.
\]
To give a proof of this theorem, I will first prove some lemmas.

**Lemma 6.4.5** If $K$ is a compact subset of $\mathbb{R}^n$, then there exists $D \equiv \{ x_k \}_{k=1}^{\infty} \subseteq K$ such that $D$ is dense in $K$. Also, for every $\varepsilon > 0$ there exists a finite set of points, $\{ x_1, \ldots, x_m \} \subseteq K$, called an $\varepsilon$ net such that

$$\bigcup_{i=1}^{m} B(x_i, \varepsilon) \supseteq K.$$

**Proof:** For $m \in \mathbb{N}$, pick $x_1^m \in K$. If every point of $K$ is within $1/m$ of $x_1^m$, stop. Otherwise, pick $x_2^m \in K \setminus B(x_1^m, 1/m)$. If every point of $K$ contained in $B(x_1^m, 1/m) \cup B(x_2^m, 1/m)$, stop. Otherwise, pick $x_3^m \in K \setminus (B(x_1^m, 1/m) \cup B(x_2^m, 1/m))$. If every point of $K$ is contained in $B(x_1^m, 1/m) \cup B(x_2^m, 1/m) \cup B(x_3^m, 1/m)$, stop. Otherwise, pick $x_4^m \in K \setminus (B(x_1^m, 1/m) \cup B(x_2^m, 1/m) \cup B(x_3^m, 1/m))$. Continue this way until the process stops, say at $N(m)$. It must stop because if it didn’t, there would be a convergent subsequence due to the compactness of $K$. Ultimately all terms of this convergent subsequence would be closer than $1/m$, violating the manner in which they are chosen. Then $D = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{N(m)} \{ x_k^m \}$. This is countable because it is a countable union of countable sets. If $y \in K$ and $\varepsilon > 0$, then for some $m$, $2/m < \varepsilon$ and so $B(y, \varepsilon)$ must contain some point of $\{ x_k^m \}$ since otherwise, the process stopped too soon. You could have picked $y$. ■

**Lemma 6.4.6** Suppose $D$ is defined above and $\{ g_m \}$ is a sequence of functions of $A$ having the property that for every $x_k \in D$,

$$\lim_{m \to \infty} g_m(x_k) \text{ exists.}$$

Then there exists $g \in C(K, X)$ such that

$$\lim_{m \to \infty} \rho(g_m, g) = 0.$$

**Proof:** Define $g$ first on $D$.

$$g(x_k) \equiv \lim_{m \to \infty} g_m(x_k).$$

Next I show that $\{ g_m \}$ converges at every point of $K$. Let $x \in K$ and let $\varepsilon > 0$ be given. Choose $x_k$ such that for all $f \in A$,

$$d(f(x_k), f(x)) < \frac{\varepsilon}{3}.$$
I can do this by the equicontinuity. Now if $p, q$ are large enough, say $p, q \geq M$,

$$d (g_p (x_k), g_q (x_k)) < \frac{\varepsilon}{3}.$$ 

Therefore, for $p, q \geq M$,

$$d (g_p (x), g_q (x)) \leq d (g_p (x), g_p (x_k)) + d (g_p (x_k), g_q (x_k)) + d (g_q (x_k), g_q (x))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

It follows that $\{g_m (x)\}$ is a Cauchy sequence having values $X$. Therefore, it converges. Let $g(x)$ be the name of the thing it converges to.

Next, I need to verify that the function, $g$ is a continuous function. Let $N$ be large enough that whenever $p, q \geq N$, the above holds. Then for all $x \in K$,

$$d (g_p (x), g_p (x)) \leq \frac{\varepsilon}{3} \leq d (g_q (x), g_p (x)) + d (g_p (x), g_q (x))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $N$ does not depend on the choice of $x$, it follows this sequence $\{g_m (x)\}$ is uniformly Cauchy. That is, for every $\varepsilon > 0$, there exists $N$ such that if $p, q \geq N$, then

$$\rho (g_p, g_q) < \varepsilon.$$

Claim: In a metric space, if $a_n \to a$, then $d (a_n, b) \to d (a, b)$.

Proof of the claim: You note that by the triangle inequality, $d (a_n, b) - d (a, b) \leq d (a_n, a)$ and $d (a, b) - d (a_n, b) \leq d (a_n, a)$ and so

$$|d (a_n, b) - d (a, b)| \leq d (a_n, a).$$
Now let \( p \) satisfy \textbf{6.4.6} for all \( x \) whenever \( p > N \). Also pick \( \delta > 0 \) such that if \( |x - y| < \delta \), then
\[
d (g_p(x), g_p(y)) < \varepsilon.
\]
Then if \( |x - y| < \delta \),
\[
d (g(x), g(y)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
Since \( \varepsilon \) was arbitrary, this shows that \( g \) is continuous.

It only remains to verify that \( \rho (g, g_k) \to 0 \). But this follows from \textbf{6.4.6}.

With these lemmas, it is time to prove \textbf{Theorem 6.4.4}.

\textbf{Proof of Theorem 6.4.4}: Let \( D = \{ x_k \} \) be the countable dense set of \( K \) guaranteed by Lemma \textbf{6.4.5} and let \( \{(1,1), (1,2), (1,3), (1,4), (1,5), \cdots \} \) be a subsequence of \( \mathbb{N} \) such that
\[
\lim_{k \to \infty} f_{(1,k)} (x_1) \text{ exists.}
\]
This is where the local compactness of \( X \) is being used. Now let
\[
\{(2,1), (2,2), (2,3), (2,4), (2,5), \cdots \}
\]
be a subsequence of \( \{(1,1), (1,2), (1,3), (1,4), (1,5), \cdots \} \) which has the property that
\[
\lim_{k \to \infty} f_{(2,k)} (x_2) \text{ exists.}
\]
Thus it is also the case that
\[
f_{(2,k)} (x_1) \text{ converges to } \lim_{k \to \infty} f_{(1,k)} (x_1).
\]
because every subsequence of a convergent sequence converges to the same thing as the convergent sequence. Continue this way and consider the array
\[
f_{(1,1)}, f_{(1,2)}, f_{(1,3)}, f_{(1,4)}, \cdots \text{ converges at } x_1
f_{(2,1)}, f_{(2,2)}, f_{(2,3)}, f_{(2,4)}, \cdots \text{ converges at } x_1 \text{ and } x_2
f_{(3,1)}, f_{(3,2)}, f_{(3,3)}, f_{(3,4)}, \cdots \text{ converges at } x_1, x_2, \text{ and } x_3
\]
Now let \( g_k \equiv f_{(k,k)} \). Thus \( g_k \) is ultimately a subsequence of \( \{f_{(m,k)}\} \) whenever \( k > m \) and therefore, \( \{g_k\} \) converges at each point of \( D \). By Lemma \textbf{6.4.10} it follows there exists \( g \in C(K;X) \) such that
\[
\lim_{k \to \infty} \rho (g, g_k) = 0. \quad \blacksquare
\]
Actually there is an if and only if version of it but the most useful case is what is presented here. The process used to get the subsequence in the proof is called the Cantor diagonalization procedure.
6.5 Another General Version

This will use the characterization of compact metric spaces to give a proof of a general version of the Arzella Ascoli theorem. See Naylor and Sell [11] which is where I saw this general formulation.

Definition 6.5.1 Let \((X,d_X)\) be a compact metric space. Let \((Y,d_Y)\) be another complete metric space. Then \(C(X,Y)\) will denote the continuous functions which map \(X\) to \(Y\). Then \(\rho\) is a metric on \(C(X,Y)\) defined by

\[
\rho(f,g) = \sup_{x \in X} d_Y \left(f(x), g(x)\right).
\]

Theorem 6.5.2 \((C(X,Y), \rho)\) is a complete metric space.

Proof: It is first necessary to show that \(\rho\) is well defined. In this argument, I will just write \(d\) rather than \(d_X\) or \(d_Y\). To show this, note that

\[
x \to d(f(x), g(x))
\]

is a continuous function because \(f, g\) are continuous and

\[
|d(f(x), g(x)) - d(f(y), g(y))| \leq d(f(x), f(y)) + d(g(x), g(y))
\]

This follows from the triangle inequality. Say \(d(f(x), g(x)) \geq d(f(y), g(y))\). Otherwise just replace \(x\) with \(y\) and repeat the argument. Then in this case, it reduces to the claim that

\[
d(f(x), g(x)) \leq d(f(x), f(y)) + d(g(x), g(y)) + d(f(y), g(y))
\]

However, by the triangle inequality, the right side of the above is at least as large as

\[
d(f(x), f(y)) + d(g(x), f(y)) \geq d(f(x), g(x)).
\]

It follows that \(\rho(f,g)\) is just the maximum of a continuous function defined on a compact set. Clearly \(\rho(f,g) = \rho(g,f)\) and

\[
\rho(f,g) + \rho(g,h) = \sup_{x \in X} d(f(x), g(x)) + \sup_{x \in X} d(g(x), h(x))
\]

\[
\geq \sup_{x \in X} (d(f(x), g(x)) + d(g(x), h(x)))
\]

\[
\geq \sup_{x \in X} (d(f(x), h(x))) = \rho(f,h)
\]

so the triangle inequality holds.

It remains to check completeness. Let \(\{f_n\}\) be a Cauchy sequence. Then from the definition, \(\{f_n(x)\}\) is a Cauchy sequence in \(Y\) and so it converges to something called \(f(x)\). I have to verify that \(x \to f(x)\) is continuous. Define

\[
\rho'(f,f_n) = \lim_{m \to \infty} \rho(f_m, f_n).
\]
Then if \( n \) is sufficiently large, \( \rho' (f, f_n) < \varepsilon/3 \). Also,
\[
d (f (x), f_n (x)) = \lim_{m \to \infty} d (f_m (x), f_n (x)) \leq \lim_{m \to \infty} \sup_{n \to \infty} \rho (f_m, f_n) = \rho' (f, f_n) < \frac{\varepsilon}{3} \quad (6.5.7)
\]
Then picking such an \( n \),
\[
d (f (x), f (y)) \leq d (f (x), f_n (x)) + d (f_n (x), f_n (y)) + d_n (f_n (y), f (y)) \\
\leq \rho' (f, f_n) + d (f_n (x), f_n (y)) + \rho' (f, f_n) + \rho' (f, f_n) < \frac{2\varepsilon}{3} + d (f_n (x), f_n (y))
\]
which is less than \( \varepsilon \) provided \( d (x, y) \) is small enough, this by continuity of \( f_n \). Therefore, \( f \) is continuous. By Lemma 6.5.3 this shows that, since \( x \) is arbitrary, \( \rho (f, f_n) < \varepsilon \) whenever \( n \) is large enough. ■

Here is a useful lemma.

**Lemma 6.5.3** Let \( S \) be a totally bounded subset of \((X, d)\) a metric space. Then \( \overline{S} \) is also totally bounded.

**Proof:** Suppose not. Then there exists a sequence \( \{ p_n \} \subseteq \overline{S} \) such that \( d (p_m, p_n) \geq \varepsilon \) for all \( m \neq n \). Now let \( q_n \in B \left( p_n, \frac{\varepsilon}{3} \right) \cap S \). Then it follows that
\[
\frac{\varepsilon}{8} + d (q_n, q_m) + \frac{\varepsilon}{3} \geq d (p_n, q_n) + d (q_n, q_m) + d (q_m, p_m) \geq d (p_n, q_m) \geq \varepsilon
\]
and so \( d (q_n, q_m) > \frac{\varepsilon}{2} \). This contradicts total boundedness of \( S \). ■

Next, here is an important definition.

**Definition 6.5.4** Let \( A \subseteq C (X, Y) \) where \((X, d_X)\) and \((Y, d_Y)\) are metric spaces. Thus \( A \) is a set of continuous functions mapping \( X \) to \( Y \). Then \( A \) is said to be equiuniform if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( d_X (x_1, x_2) < \delta \) then for all \( f \in A \), \( d_Y (f (x_1), f (x_2)) < \varepsilon \). (This is uniform continuity which is uniform in \( A \).) \( A \) is said to be pointwise compact if \( \{ f (x) : f \in A \} \) has compact closure in \( Y \).

Here is the Ascoli Arzela theorem.

**Theorem 6.5.5** Let \((X, d_X)\) be a compact metric space and let \((Y, d_Y)\) be a complete metric space. Thus \((C (X, Y), \rho)\) is a complete metric space. Let \( A \subseteq C (X, Y) \) be pointwise compact and equiuniform. Then \( \overline{A} \) is compact. Here the closure is taken in \((C (X, Y), \rho)\). The converse also holds.

**Proof:** The more useful direction is that the two conditions imply compactness of \( \overline{A} \). I prove this first. Since \( \overline{A} \) is a closed subset of a complete space, it follows that \( \overline{A} \) will be compact if it is totally bounded. In showing this, it follows from Lemma 6.5.3 that it suffices to verify that \( A \) is totally bounded. Suppose this is
not so. Then there exists $\varepsilon > 0$ and a sequence of points of $\mathcal{A}$, $\{f_n\}$ such that

$$\rho (f_n, f_m) \geq \varepsilon$$

whenever $n \neq m$.

By equicontinuity, there exists $\delta > 0$ such that if $d (x, y) < \delta$, then $d (f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{A}$. Let $\{x_i\}_{i=1}^m$ be a $\delta/2$ net for $X$. Since there are only finitely many $x_i$, it follows from pointwise compactness that there exists a subsequence, still denoted by $\{f_n\}$ which converges at each $x_i$. There exists $x_{nm}$ such that

$$\rho (f_n, f_m) - \frac{\varepsilon}{8} < d (f_n (x_{nm}), f_m (x_{nm}))$$

where here $x_i$ is such that $x_{nm} \in B(x_i, \delta)$. From the convergence of $f_n$ at each $x_i$, there exists $N$ such that if $m, n > N$, then for all $x_i,$

$$d (f_n (x_i), f_m (x_i)) < \frac{\varepsilon}{8}.$$  

Now results in the contradiction,

$$\varepsilon - \frac{\varepsilon}{8} < \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8}$$

It follows that $\mathcal{A}$ and hence $\overline{\mathcal{A}}$ is totally bounded. This proves the more important direction.

Next suppose $\overline{\mathcal{A}}$ is compact. Why must $\mathcal{A}$ be pointwise compact and equicontinuous? If it fails to be pointwise compact, then there exists $x \in X$ such that $\{f(x): f \in \mathcal{A}\}$ is not contained in a compact set of $Y$. Thus there exists $\varepsilon > 0$ and a sequence of functions in $\mathcal{A}$ $\{f_n\}$ such that $d (f_n(x), f_m(x)) \geq \varepsilon$. But this implies $\rho (f_m, f_n) \geq \varepsilon$ and so $\overline{\mathcal{A}}$ fails to be totally bounded, a contradiction. Thus $\mathcal{A}$ must be pointwise compact. Now why must it be equicontinuous? If it is not, then for each $n \in \mathbb{N}$ there exists $\varepsilon > 0$ and $x_n, y_n \in X$ such that $d(x_n, y_n) < 1/n$ but for some $f_n \in \mathcal{A}$, $d(f_n(x_n), f_n(y_n)) \geq \varepsilon$. However, by compactness, there exists a subsequence $\{f_{n_k}\}$ such that $\lim_{k \to \infty} \rho (f_{n_k}, f) = 0$ and also that $x_{n_k}, y_{n_k} \rightarrow x \in X$. Hence

$$\varepsilon \leq d (f_{n_k} (x_{n_k}), f_{n_k} (y_{n_k})) \leq d (f_{n_k} (x_{n_k}), f (x_{n_k})) + d (f (x_{n_k}), f (y_{n_k})) + d (f (y_{n_k}), f_{n_k} (y_{n_k}))$$

and now this is a contradiction because each term on the right converges to 0. The middle term converges to 0 because $f(x_{n_k}), f (y_{n_k}) \rightarrow f(x)$. ■
6.6 The Tietze Extension Theorem

It turns out that if $H$ is a closed subset of a metric space, $(X, d)$ and if $f : H \to [a, b]$ is continuous, then there exists $g$ defined on all of $X$ such that $g = f$ on $H$ and $g$ is continuous. This is called the Tietze extension theorem. First it is well to recall continuity in the context of metric space.

**Definition 6.6.1** Let $(X, d)$ be a metric space and suppose $f : X \to Y$ is a function where $(Y, \rho)$ is also a metric space. For example, $Y = \mathbb{R}$. Then $f$ is continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(z)) < \varepsilon$ whenever $d(x, z) < \delta$. As is usual in such definitions, $f$ is said to be continuous if it is continuous at every point of $X$.

The following lemma gives an important example of a continuous real valued function defined on a metric space, $(X, d)$.

**Lemma 6.6.2** Let $(X, d)$ be a metric space and let $S \subseteq X$ be a nonempty subset. Define

$$
dist(x, S) \equiv \inf \{d(x, y) : y \in S\}. 
$$

Then $x \to dist(x, S)$ is a continuous function satisfying the inequality,

$$
|dist(x, S) - dist(y, S)| \leq d(x, y). \tag{6.6.9}
$$

**Proof:** The continuity of $x \to dist(x, S)$ is obvious if the inequality (6.6.9) is established. So let $x, y \in X$. Without loss of generality, assume $dist(x, S) \geq dist(y, S)$ and pick $z \in S$ such that $d(y, z) - \varepsilon < dist(y, S)$. Then

$$
|dist(x, S) - dist(y, S)| = dist(x, S) - dist(y, S) \leq d(x, z) - d(y, z) - \varepsilon \\
\leq d(z, y) + d(x, y) - d(y, z) + \varepsilon = d(x, y) + \varepsilon.
$$

Since $\varepsilon$ is arbitrary, this proves (6.6.9).

**Lemma 6.6.3** Let $H, K$ be two nonempty disjoint closed subsets of a metric space, $(X, d)$. Then there exists a continuous function, $g : X \to [-1, 1]$ such that $g(H) = -1/3$, $g(K) = 1/3$, $g(X) \subseteq [-1/3, 1/3]$.

**Proof:** Let

$$f(x) = \frac{dist(x, H)}{dist(x, H) + dist(x, K)}.$$

The denominator is never equal to zero because if $dist(x, H) = 0$, then $x \in H$ because $H$ is closed. (To see this, pick $h_k \in B(x, 1/k) \cap H$. Then $h_k \to x$ and since $H$ is closed, $x \in H$.) Similarly, if $dist(x, K) = 0$, then $x \in K$ and so the denominator is never zero as claimed. Hence, by Lemma 6.6.2, $f$ is continuous and from its definition, $f = 0$ on $H$ and $f = 1$ on $K$. Now let $g(x) = \frac{2}{3}\left(f(x) - \frac{1}{2}\right)$. Then $g$ has the desired properties.
Definition 6.6.4 For \( f \) a real or complex valued bounded continuous function defined on a metric space, \( M \)

\[
||f||_M \equiv \sup \{ |f(x)| : x \in M \}.
\]

Lemma 6.6.5 Suppose \( M \) is a closed set in \( X \) where \((X,d)\) is a metric space and suppose \( f : M \to [-1,1] \) is continuous at every point of \( M \). Then there exists a function, \( g \) which is defined and continuous on all of \( X \) such that \( ||f - g||_M < \frac{2}{3} \).

Proof: Let \( H = f^{-1}([-1,-1/3]) \), \( K = f^{-1}([1/3,1]) \). Thus \( H \) and \( K \) are disjoint closed subsets of \( M \). Suppose first \( H,K \) are both nonempty. Then by Lemma 6.6.3 there exists \( g \) such that \( g \) is a continuous function defined on all of \( X \) and \( g(H) = -1/3 \), \( g(K) = 1/3 \), and \( g(X) \subseteq [-1/3,1/3] \). It follows \( ||f - g||_M < 2/3 \). If \( H = \emptyset \), then \( f \) has all its values in \([-1/3,1]\) and so letting \( g \equiv 1/3 \), the desired condition is obtained. If \( K = \emptyset \), let \( g \equiv -1/3 \). This proves the lemma.

Lemma 6.6.6 Suppose \( M \) is a closed set in \( X \) where \((X,d)\) is a metric space and suppose \( f : M \to [-1,1] \) is continuous at every point of \( M \). Then there exists a function, \( g \) which is defined and continuous on all of \( X \) such that \( g = f \) on \( M \) and \( g \) has its values in \([-1,1]\).

Proof: Let \( g_1 \) be such that \( g_1(X) \subseteq [-1/3,1/3] \) and \( ||f - g_1||_M \leq \frac{2}{3} \). Suppose \( g_1, \ldots, g_m \) have been chosen such that \( g_j(X) \subseteq [-1/3,1/3] \) and

\[
\left\| f - \sum_{i=1}^{m} \left( \frac{2}{3} \right)^{i-1} g_i \right\|_M < \left( \frac{2}{3} \right)^m . \tag{6.6.10}
\]

Then

\[
\left\| \left( \frac{3}{2} \right)^m \left( f - \sum_{i=1}^{m} \left( \frac{2}{3} \right)^{i-1} g_i \right) \right\|_M \leq 1
\]

and so \( \left( \frac{3}{2} \right)^m \left( f - \sum_{i=1}^{m} \left( \frac{2}{3} \right)^{i-1} g_i \right) \) can play the role of \( f \) in the first step of the proof. Therefore, there exists \( g_{m+1} \) defined and continuous on all of \( X \) such that its values are in \([-1/3,1/3]\) and

\[
\left\| \left( \frac{3}{2} \right)^m \left( f - \sum_{i=1}^{m} \left( \frac{2}{3} \right)^{i-1} g_i \right) - g_{m+1} \right\|_M \leq \frac{2}{3} .
\]

Hence

\[
\left\| \left( f - \sum_{i=1}^{m} \left( \frac{2}{3} \right)^{i-1} g_i \right) - \left( \frac{2}{3} \right)^m g_{m+1} \right\|_M \leq \left( \frac{2}{3} \right)^{m+1} .
\]

It follows there exists a sequence, \( \{g_i\} \) such that each has its values in \([-1/3,1/3]\) and for every \( m \tag{6.6.10} \) holds. Then let

\[
g(x) \equiv \sum_{i=1}^{\infty} \left( \frac{2}{3} \right)^{i-1} g_i(x).
\]
It follows
\[ |g(x)| \leq \sum_{i=1}^{\infty} \left( \frac{2}{3} \right)^{i-1} g_i(x) \leq \sum_{i=1}^{m} \left( \frac{2}{3} \right)^{i-1} \frac{1}{3} \leq 1 \]
and since convergence is uniform, \( g \) must be continuous. The estimate 6.6.10 implies \( f = g \) on \( M \).

The following is the Tietze extension theorem.

**Theorem 6.6.7** Let \( M \) be a closed nonempty subset of a metric space \((X,d)\) and let \( f : M \to [a,b] \) is continuous at every point of \( M \). Then there exists a function, \( g \) continuous on all of \( X \) which coincides with \( f \) on \( M \) such that \( g(X) \subseteq [a,b] \).

**Proof:** Let \( f_1(x) = 1 + \frac{2}{b-a}(f(x) - b) \). Then \( f_1 \) satisfies the conditions of Lemma 6.6.10 and so there exists \( g_1 : X \to [-1,1] \) such that \( g \) is continuous on \( X \) and equals \( f_1 \) on \( M \). Let \( g(x) = (g_1(x) - 1) \left( \frac{b-a}{2} \right) + b \). This works.

### 6.7 Some Simple Fixed Point Theorems

The following is of more interest in the case of normed vector spaces, but there is no harm in stating it in this more general setting. You should verify that the functions described in the following definition are all continuous.

**Definition 6.7.1** Let \( f : X \to Y \) where \((X,d)\) and \((Y,\rho)\) are metric spaces. Then \( f \) is said to be Lipschitz continuous if for every \( x, \hat{x} \in X, \rho(f(x), f(\hat{x})) \leq rd(x, \hat{x}) \). The function is called a contraction map if \( r < 1 \).

The big theorem about contraction maps is the following.

**Theorem 6.7.2** Let \( f : (X,d) \to (X,d) \) be a contraction map and let \((X,d)\) be a complete metric space. Thus Cauchy sequences converge and also \( d(f(x), f(\hat{x})) \leq rd(x, \hat{x}) \) where \( r < 1 \). Then \( f \) has a unique fixed point. This is a point \( x \in X \) such that \( f(x) = x \). Also, if \( x_0 \) is any point of \( X \), then
\[ d(x, x_0) \leq \frac{d(x_0, f(x_0))}{1 - r} \]
Also, for each \( n \),
\[ d(f^n(x_0), x_0) \leq \frac{d(x_0, f(x_0))}{1 - r}, \]
and \( x = \lim_{n \to \infty} f^n(x_0) \).

**Proof:** Pick \( x_0 \in X \) and consider the sequence of iterates of the map,
\[ x_0, f(x_0), f^2(x_0), \ldots \]
We argue that this is a Cauchy sequence. For \( m < n \), it follows from the triangle inequality,

\[
d(m(x_0), n(x_0)) \leq \sum_{k=m}^{n-1} d(f^{k+1}(x_0), f^k(x_0)) \leq \sum_{k=m}^{\infty} r^k d(f(x_0), x_0)
\]

The reason for this last is as follows.

\[
d(f^2(x_0), f(x_0)) \leq r d(f(x_0), x_0)
\]

\[
d(f^3(x_0), f^2(x_0)) \leq r d(f^2(x_0), f(x_0)) \leq r^2 d(f(x_0), x_0)
\]

and so forth. Therefore,

\[
d(f^m(x_0), n(x_0)) \leq d(f(x_0), x_0) \frac{r^m}{1 - r}
\]

which shows that this is indeed a Cauchy sequence. Therefore, there exists \( x \) such that

\[
\lim_{n \to \infty} f^n(x_0) = x
\]

By continuity,

\[
f(x) = f \left( \lim_{n \to \infty} f^n(x_0) \right) = \lim_{n \to \infty} f^{n+1}(x_0) = x.
\]

Also note that this estimate yields

\[
d(x_0, f^n(x_0)) \leq d(x_0, f(x_0)) \frac{1}{1 - r}
\]

Now \( d(x_0, x) \leq d(x_0, f^n(x_0)) + d(f^n(x_0), x) \) and so

\[
d(x_0, x) - d(f^n(x_0), x) \leq \frac{d(x_0, f(x_0))}{1 - r}
\]

Letting \( n \to \infty \), it follows that

\[
d(x_0, x) \leq \frac{d(x_0, f(x_0))}{1 - r}
\]

It only remains to verify that there is only one fixed point. Suppose then that \( x, x' \) are two. Then

\[
d(x, x') = d(f(x), f(x')) \leq rd(x', x)
\]

and so \( d(x, x') = 0 \) because \( r < 1 \).

The above is the usual formulation of this important theorem, but we actually proved a better result.
Corollary 6.7.3 Let $B$ be a closed subset of the complete metric space $(X,d)$ and let $f : B \rightarrow X$ be a contraction map

$$d\left(f(x), f(\hat{x})\right) \leq r d(x, \hat{x}), \quad r < 1.$$ 

Also suppose there exists $x_0 \in B$ such that the sequence of iterates $\{f^n(x_0)\}_{n=1}^{\infty}$ remains in $B$. Then $f$ has a unique fixed point in $B$ which is the limit of the sequence of iterates. This is a point $x \in B$ such that $f(x) = x$. In the case that $B = B(x_0, \delta)$, the sequence of iterates satisfies the inequality

$$d(f^n(x_0), x_0) \leq \frac{d(x_0, f(x_0))}{1 - r}$$

and so it will remain in $B$ if

$$\frac{d(x_0, f(x_0))}{1 - r} < \delta.$$

**Proof:** By assumption, the sequence of iterates stays in $B$. Then, as in the proof of the preceding theorem, for $m < n$, it follows from the triangle inequality,

$$d(f^m(x_0), f^n(x_0)) \leq \sum_{k=m}^{n-1} d(f^{k+1}(x_0), f^k(x_0))$$

$$\leq \sum_{k=m}^{\infty} r^k d(f(x_0), x_0) = \frac{r^m}{1 - r} d(f(x_0), x_0)$$

Hence the sequence of iterates is Cauchy and must converge to a point $x$ in $X$. However, $B$ is closed and so it must be the case that $x \in B$. Then as before,

$$x = \lim_{n \to \infty} f^n(x_0) = \lim_{n \to \infty} f^{n+1}(x_0) = f(\lim_{n \to \infty} f^n(x_0)) = f(x)$$

As to the sequence of iterates remaining in $B$ where $B$ is a ball as described, the inequality above in the case where $m = 0$ yields

$$d(x_0, f^m(x_0)) \leq \frac{1}{1 - r} d(f(x_0), x_0)$$

and so, if the right side is less than $\delta$, then the iterates remain in $B$. As to the fixed point being unique, it is as before. If $x, x'$ are both fixed points in $B$, then $d(x, x') = d(f(x), f(x')) \leq r d(x, x')$ and so $x = x'$. ■

Sometimes you have the contraction depending on a parameter $\lambda$. Then there is a principle of uniform contractions.

Corollary 6.7.4 Suppose $f : X \times \Lambda \rightarrow X$ where $\Lambda$ is a metric space and $X$ is a complete metric space. Suppose $f$ satisfies

1. $d(f(x, \lambda), f(y, \lambda)) \leq rd(x, y)$ for each $\lambda \in \Lambda$. 

2. \( \lambda \to f(x, \lambda) \) is continuous as a map from \( \Lambda \) to \( X \).

Then if \( x(\lambda) \) is the fixed point, it follows that \( \lambda \to x(\lambda) \) is continuous.

**Proof:** Pick \( x_0 \in X \) and consider the above sequence of iterates, \( \{f^n(x, \lambda)\} \).

Let \( \rho \) be the metric on \( \Lambda \). Then there is a fixed point and if \( x(\lambda) \) is this unique fixed point,

\[
d(x(\lambda), x_0) \leq \frac{d(f(x_0, \lambda), x_0)}{1 - r}
\]

In particular, you could start with \( x_0 = x(\mu) \) and conclude that

\[
d(x(\lambda), x(\mu)) \leq \frac{d(f(x(\mu), \lambda), x(\mu))}{1 - r}
\]

\[
\leq \frac{d(f(x(\mu), \lambda), f(x(\mu), \mu))}{1 - r} + \frac{d(f(x(\mu), \mu), x(\mu))}{1 - r}
\]

\[
= \frac{d(f(x(\mu), \lambda), f(x(\mu), \mu))}{1 - r}
\]

Now by continuity of \( \lambda \to f(x, \lambda) \), it follows that if \( \rho(\lambda, \mu) \) is small enough, the above is no larger than

\[
\frac{\varepsilon (1 - r)}{1 - r} = \varepsilon
\]

Hence, if \( \rho(\lambda, \mu) \) is small enough, we have

\[
d(x(\lambda), x(\mu)) < \varepsilon.
\]

This is called the uniform contraction principle.

The contraction mapping theorem has an extremely useful generalization. In order to get a unique fixed point, it suffices to have some power of \( f \) a contraction map.

**Theorem 6.7.5** Let \( f : (X, d) \to (X, d) \) have the property that for some \( n \in \mathbb{N} \), \( f^n \) is a contraction map and let \((X, d)\) be a complete metric space. Then there is a unique fixed point for \( f \). As in the earlier theorem the sequence of iterates \( \{f^n(x_0)\}_{n=1}^{\infty} \) also converges to the fixed point.

**Proof:** From Theorem 6.7.2 there is a unique fixed point for \( f^n \). Thus

\[
f^n(x) = x
\]

Then

\[
f^n(f(x)) = f^{n+1}(x) = f(x)
\]

By uniqueness, \( f(x) = x \).
Now consider the sequence of iterates. Suppose it fails to converge to \( x \). Then there is \( \varepsilon > 0 \) and a subsequence \( n_k \) such that

\[
d(f^{n_k}(x_0), x) \geq \varepsilon
\]

Now \( n_k = p_k n + r_k \) where \( r_k \) is one of the numbers \( \{0, 1, 2, \ldots, n - 1\} \). It follows that there exists one of these numbers which is repeated infinitely often. Call it \( r \) and let the further subsequence continue to be denoted as \( n_k \).

Thus

\[
d(f^{p_k n + r_n}(x_0), x) \geq \varepsilon
\]

In other words,

\[
d(f^{p_k n}(f^r(x_0)), x) \geq \varepsilon
\]

However, from Theorem 6.7.2, as \( k \to \infty \), \( f^{p_k n}(f^r(x_0)) \to x \) which contradicts the above inequality. Hence the sequence of iterates converges to \( x \), as it did for \( f \) a contraction map.

**Definition 6.7.6** Let \( f : (X,d) \to (Y,\rho) \) be a function. Then it is said to be uniformly continuous on \( X \) if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that whenever \( x, \hat{x} \) are two points of \( X \) with \( d(x, \hat{x}) < \delta \), it follows that \( \rho(f(x), f(\hat{x})) < \varepsilon \).

Note the difference between this and continuity. With continuity, the \( \delta \) could depend on \( x \) but here it works for any pair of points in \( X \).

**Lemma 6.7.7** Suppose \( x_n \to x \) and \( y_n \to y \). Then \( d(x_n, y_n) \to d(x, y) \).

**Proof:** Consider the following.

\[
d(x, y) \leq d(x, x_n) + d(x_n, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)
\]

so

\[
d(x, y) - d(x_n, y_n) \leq d(x, x_n) + d(y_n, y)
\]

Similarly

\[
d(x_n, y_n) - d(x, y) \leq d(x, x_n) + d(y_n, y)
\]

and so

\[
|d(x_n, y_n) - d(x, y)| \leq d(x, x_n) + d(y_n, y)
\]

and the right side converges to 0 as \( n \to \infty \).

There is a remarkable result concerning compactness and uniform continuity.

**Theorem 6.7.8** Let \( f : (X,d) \to (Y,\rho) \) be a continuous function and let \( K \) be a compact subset of \( X \). Then the restriction of \( f \) to \( K \) is uniformly continuous.

**Proof:** First of all, \( K \) is a metric space and \( f \) restricted to \( K \) is continuous. Now suppose it fails to be uniformly continuous. Then there exists \( \varepsilon > 0 \) and pairs of points \( x_n, \hat{x}_n \) such that \( d(f(x_n), f(\hat{x}_n)) < 1/n \) but \( \rho(f(x_n), f(\hat{x}_n)) \geq \varepsilon \). Since \( K \) is
compact, it is sequentially compact and so there exists a subsequence, still denoted as \( \{x_n\} \) such that \( x_n \to x \in K \). Then also \( \hat{x}_n \to x \) also and so

\[
\rho(f(x), f(x)) = \lim_{n \to \infty} \rho(f(x_n), f(\hat{x}_n)) \geq \varepsilon
\]

which is a contradiction. Note the use of Lemma 6.7.7 in the equal sign. ■

Next is to consider the meaning of convergence of sequences of functions. There are two main ways of convergence of interest here, pointwise and uniform convergence.

**Definition 6.7.9** Let \( f_n : X \to Y \) where \((X, d), (Y, \rho)\) are two metric spaces. Then \( \{f_n\} \) is said to converge pointwise to a function \( f : X \to Y \) if for every \( x \in X \),

\[
\lim_{n \to \infty} f_n(x) = f(x)
\]

\( \{f_n\} \) is said to converge uniformly if for all \( \varepsilon > 0 \), there exists \( N \) such that if \( n \geq N \),

\[
\sup_{x \in X} \rho(f_n(x), f(x)) < \varepsilon
\]

Here is a well known example illustrating the difference between pointwise and uniform convergence.

**Example 6.7.10** Let \( f_n(x) = x^n \) on the metric space \([0, 1]\). Then this function converges pointwise to

\[
f(x) = \begin{cases} 
0 & \text{on } [0, 1) \\
1 & \text{at } 1
\end{cases}
\]

but it does not converge uniformly on this interval to \( f \).

Note how the target function \( f \) in the above example is not continuous even though each function in the sequence is. The nice thing about uniform convergence is that it takes continuity of the functions in the sequence and imparts it to the target function. It does this for both continuity at a single point and uniform continuity. Thus uniform convergence is a very superior thing.

**Theorem 6.7.11** Let \( f_n : X \to Y \) where \((X, d), (Y, \rho)\) are two metric spaces and suppose each \( f_n \) is continuous at \( x \in X \) and also that \( f_n \) converges uniformly to \( f \) on \( X \). Then \( f \) is also continuous at \( x \). In addition to this, if each \( f_n \) is uniformly continuous on \( X \), then the same is true for \( f \).

**Proof:** Let \( \varepsilon > 0 \) be given. Then

\[
\rho(f(x), f(\hat{x})) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f(\hat{x})) + \rho(f_n(\hat{x}), f(\hat{x}))
\]

By uniform convergence, there exists \( N \) such that both \( \rho(f(x), f_n(x)) \) and \( \rho(f_n(\hat{x}), f(\hat{x})) \) are less than \( \varepsilon/3 \) provided \( n \geq N \). Thus picking such an \( n \),

\[
\rho(f(x), f(\hat{x})) \leq \frac{2\varepsilon}{3} + \rho(f_n(x), f_n(\hat{x}))
\]
Now from the continuity of \( f_n \), there exists \( \delta > 0 \) such that if \( d(x, \hat{x}) < \delta \), then \( \rho(f_n(x), f_n(\hat{x})) < \varepsilon/3 \). Hence, if \( d(x, \hat{x}) < \delta \), then
\[
\rho(f(x), f(\hat{x})) \leq \frac{2\varepsilon}{3} + \rho(f_n(x), f_n(\hat{x})) < \frac{2\varepsilon}{3} + \varepsilon = \varepsilon
\]
Hence, \( f \) is continuous at \( x \).

Next consider uniform continuity. It follows from the uniform convergence that if \( x, \hat{x} \) are any two points of \( X \), then if \( n \geq N \), then picking such an \( n \),
\[
\rho(f(x), f(\hat{x})) \leq \frac{2\varepsilon}{3} + \rho(f_n(x), f_n(\hat{x}))
\]
By uniform continuity of \( f_n \) there exists \( \delta \) such that if \( d(x, \hat{x}) < \delta \), then the term on the right in the above is less than \( \varepsilon/3 \). Hence if \( d(x, \hat{x}) < \delta \), then \( \rho(f(x), f(\hat{x})) < \varepsilon \) and so \( f \) is uniformly continuous as claimed. □

6.8 General Topological Spaces

It turns out that metric spaces are not sufficiently general for some applications. This section is a brief introduction to general topology. In making this generalization, the properties of balls which are the conclusion of Theorem 6.1.4 on Page 144 are stated as axioms for a subset of the power set of a given set which will be known as a basis for the topology. More can be found in [75] and the references listed there.

**Definition 6.8.1** Let \( X \) be a nonempty set and suppose \( \mathcal{B} \subseteq \mathcal{P}(X) \). Then \( \mathcal{B} \) is a basis for a topology if it satisfies the following axioms.

1.) Whenever \( p \in A \cap B \) for \( A, B \in \mathcal{B} \), it follows there exists \( C \in \mathcal{B} \) such that \( p \in C \subseteq A \cap B \).
2.) \( \bigcup \mathcal{B} = X \).

Then a subset, \( U \), of \( X \) is an open set if for every point, \( x \in U \), there exists \( B \in \mathcal{B} \) such that \( x \in B \subseteq U \). Thus the open sets are exactly those which can be obtained as a union of sets of \( \mathcal{B} \). Denote these subsets of \( X \) by the symbol \( \tau \) and refer to \( \tau \) as the topology or the set of open sets.

Note that this is simply the analog of saying a set is open exactly when every point is an interior point.

**Proposition 6.8.2** Let \( X \) be a set and let \( \mathcal{B} \) be a basis for a topology as defined above and let \( \tau \) be the set of open sets determined by \( \mathcal{B} \). Then
\[
\emptyset \in \tau, \ X \in \tau, \hspace{1cm} (6.8.11)
\]
\[
\text{If } C \subseteq \tau, \text{ then } \bigcup C \in \tau \hspace{1cm} (6.8.12)
\]
\[
\text{If } A, B \in \tau, \text{ then } A \cap B \in \tau. \hspace{1cm} (6.8.13)
\]
6.8. GENERAL TOPOLOGICAL SPACES

Proof: If \( p \in \emptyset \) then there exists \( B \in \mathcal{B} \) such that \( p \in B \subseteq \emptyset \) because there are no points in \( \emptyset \). Therefore, \( \emptyset \in \tau \). Now if \( p \in X \), then by part 2) of Definition 6.8.1 \( p \in B \subseteq X \) for some \( B \in \mathcal{B} \) and so \( X \in \tau \).

If \( C \subseteq \tau \), and if \( p \in \cup C \), then there exists a set, \( B \in C \) such that \( p \in B \).

However, \( B \) is itself a union of sets from \( \mathcal{B} \) and so there exists \( C \in \mathcal{B} \) such that \( p \in C \subseteq B \subseteq \cup C \). This verifies 6.8.12.

Finally, if \( A, B \in \tau \) and \( p \in A \cap B \), then since \( A \) and \( B \) are themselves unions of sets of \( \mathcal{B} \), it follows there exists \( A_1, B_1 \in \mathcal{B} \) such that \( A_1 \subseteq A, B_1 \subseteq B \), and \( p \in A_1 \cap B_1 \). Therefore, by 1) of Definition 6.8.1 there exists \( C \in \mathcal{B} \) such that \( p \in C \subseteq A_1 \cap B_1 \subseteq A \cap B \), showing that \( A \cap B \in \tau \) as claimed. Of course if \( A \cap B = \emptyset \), then \( A \cap B \in \tau \). This proves the proposition.

Definition 6.8.3 A set \( X \) together with such a collection of its subsets satisfying 6.8.11-6.8.13 is called a topological space. \( \tau \) is called the topology or set of open sets of \( X \).

Definition 6.8.4 A topological space is said to be Hausdorff if whenever \( p \) and \( q \) are distinct points of \( X \), there exist disjoint open sets \( U, V \) such that \( p \in U, q \in V \). In other words points can be separated with open sets.

\[
\begin{array}{c}
\circ \quad p \\
\circ \\
\text{Hausdorff}
\end{array}
\]

\[
\begin{array}{c}
\circ \quad q \\
\circ \\

U
\end{array}
\]

\[
\begin{array}{c}
\circ \quad V
\end{array}
\]

Definition 6.8.5 A subset of a topological space is said to be closed if its complement is open. Let \( p \) be a point of \( X \) and let \( E \subseteq X \). Then \( p \) is said to be a limit point of \( E \) if every open set containing \( p \) contains a point of \( E \) distinct from \( p \).

Note that if the topological space is Hausdorff, then this definition is equivalent to requiring that every open set containing \( p \) contains infinitely many points from \( E \). Why?

Theorem 6.8.6 A subset, \( E \), of \( X \) is closed if and only if it contains all its limit points.

Proof: Suppose first that \( E \) is closed and let \( x \) be a limit point of \( E \). Is \( x \in E \)? If \( x \notin E \), then \( E^C \) is an open set containing \( x \) which contains no points of \( E \), a contradiction. Thus \( x \in E \).

Now suppose \( E \) contains all its limit points. Is the complement of \( E \) open? If \( x \in E^C \), then \( x \) is not a limit point of \( E \) because \( E \) has all its limit points and so there exists an open set, \( U \) containing \( x \) such that \( U \) contains no point of \( E \) other than \( x \). Since \( x \notin E \), it follows that \( x \in U \subseteq E^C \) which implies \( E^C \) is an open set because this shows \( E^C \) is the union of open sets.
Theorem 6.8.7 If \((X, \tau)\) is a Hausdorff space and if \(p \in X\), then \(\{p\}\) is a closed set.

Proof: If \(x \neq p\), there exist open sets \(U\) and \(V\) such that \(x \in U, p \in V\) and \(U \cap V = \emptyset\). Therefore, \(\{p\}^C\) is an open set so \(\{p\}\) is closed.

Note that the Hausdorff axiom was stronger than needed in order to draw the conclusion of the last theorem. In fact it would have been enough to assume that if \(x \neq y\), then there exists an open set containing \(x\) which does not intersect \(y\).

Definition 6.8.8 A topological space \((X, \tau)\) is said to be regular if whenever \(C\) is a closed set and \(p\) is a point not in \(C\), there exist disjoint open sets \(U\) and \(V\) such that \(p \in U, C \subseteq V\). Thus a closed set can be separated from a point not in the closed set by two disjoint open sets.

![Regular Diagram](image_url)

Definition 6.8.9 The topological space, \((X, \tau)\) is said to be normal if whenever \(C\) and \(K\) are disjoint closed sets, there exist disjoint open sets \(U\) and \(V\) such that \(C \subseteq U, K \subseteq V\). Thus any two disjoint closed sets can be separated with open sets.

![Normal Diagram](image_url)

Definition 6.8.10 Let \(E\) be a subset of \(X\). \(\overline{E}\) is defined to be the smallest closed set containing \(E\).

Lemma 6.8.11 The above definition is well defined.

Proof: Let \(C\) denote all the closed sets which contain \(E\). Then \(C\) is nonempty because \(X \in C\).

\[
(\cap \{A : A \in C\})^C = \cup \{A^C : A \in C\},
\]

an open set which shows that \(\cap C\) is a closed set and is the smallest closed set which contains \(E\).

Theorem 6.8.12 \(\overline{E} = E \cup \{\text{limit points of } E\}\).
Proof: Let \( x \in \overline{E} \) and suppose that \( x \notin E \). If \( x \) is not a limit point either, then there exists an open set, \( U \), containing \( x \) which does not intersect \( E \). But then \( UC \) is a closed set which contains \( E \) which does not contain \( x \), contrary to the definition that \( E \) is the intersection of all closed sets containing \( E \). Therefore, \( x \) must be a limit point of \( E \) after all.

Now \( E \subseteq \overline{E} \) so suppose \( x \) is a limit point of \( E \). Is \( x \in E \)? If \( H \) is a closed set containing \( E \), which does not contain \( x \), then \( H^C \) is an open set containing \( x \) which contains no points of \( E \) other than \( x \) negating the assumption that \( x \) is a limit point of \( E \).

The following is the definition of continuity in terms of general topological spaces. It is really just a generalization of the \( \varepsilon - \delta \) definition of continuity given in calculus.

**Definition 6.8.13** Let \((X, \tau)\) and \((Y, \eta)\) be two topological spaces and let \( f : X \to Y \). \( f \) is continuous at \( x \in X \) if whenever \( V \) is an open set of \( Y \) containing \( f(x) \), there exists an open set \( U \in \tau \) such that \( x \in U \) and \( f(U) \subseteq V \). \( f \) is continuous if \( f^{-1}(V) \in \tau \) whenever \( V \in \eta \).

You should prove the following.

**Proposition 6.8.14** In the situation of Definition 6.8.13 \( f \) is continuous if and only if \( f \) is continuous at every point of \( X \).

**Definition 6.8.15** Let \((X_i, \tau_i)\) be topological spaces. \( \prod_{i=1}^n X_i \) is the Cartesian product. Define a product topology as follows. Let \( B = \prod_{i=1}^n A_i \) where \( A_i \in \tau_i \). Then \( B \) is a basis for the product topology.

**Theorem 6.8.16** The set \( B \) of Definition 6.8.15 is a basis for a topology.

Proof: Suppose \( x \in \prod_{i=1}^n A_i \cap \prod_{i=1}^n B_i \) where \( A_i \) and \( B_i \) are open sets. Say

\[ x = (x_1, \ldots, x_n). \]

Then \( x_i \in A_i \cap B_i \) for each \( i \). Therefore, \( x \in \prod_{i=1}^n A_i \cap \prod_{i=1}^n B_i \in B \) and \( \prod_{i=1}^n A_i \cap B_i \subseteq \prod_{i=1}^n A_i \).

The definition of compactness is also considered for a general topological space. This is given next.

**Definition 6.8.17** A subset, \( E \), of a topological space \((X, \tau)\) is said to be compact if whenever \( \mathcal{C} \subseteq \tau \) and \( E \subseteq \bigcup \mathcal{C} \), there exists a finite subset of \( \mathcal{C} \), \( \{U_1 \cdots U_n\} \), such that \( E \subseteq \bigcup_{i=1}^n U_i \). (Every open covering admits a finite subcovering.) \( E \) is precompact if \( \overline{E} \) is compact. A topological space is called locally compact if it has a basis \( B \), with the property that \( \overline{B} \) is compact for each \( B \in B \).

In general topological spaces there may be no concept of “bounded”. Even if there is, closed and bounded is not necessarily the same as compactness. However, in any Hausdorff space every compact set must be a closed set.
Theorem 6.8.18 If \((X, \tau)\) is a Hausdorff space, then every compact subset must also be a closed set.

Proof: Suppose \(p \notin K\). For each \(x \in X\), there exist open sets, \(U_x\) and \(V_x\) such that
\[ x \in U_x, \ p \in V_x, \]
and
\[ U_x \cap V_x = \emptyset. \]
If \(K\) is assumed to be compact, there are finitely many of these sets, \(U_{x_1}, \ldots, U_{x_m}\) which cover \(K\). Then let \(V \equiv \cap_{i=1}^m V_{x_i}\). It follows that \(V\) is an open set containing \(p\) which has empty intersection with each of the \(U_{x_i}\). Consequently, \(V\) contains no points of \(K\) and is therefore not a limit point of \(K\). This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space.

Definition 6.8.19 Suppose \((X, \tau)\) is a locally compact Hausdorff space. Then let \(\bar{X} \equiv X \cup \{\infty\}\) where \(\infty\) is just the name of some point which is not in \(X\) which is called the point at infinity. A basis for the topology \(\bar{\tau}\) for \(\bar{X}\) is
\[ \tau \cup \{K^C \text{ where } K \text{ is a compact subset of } X\}. \]
The complement is taken with respect to \(\bar{X}\) and so the open sets, \(K^C\) are basic open sets which contain \(\infty\).

The reason this is called a compactification is contained in the next lemma.

Lemma 6.8.20 If \((X, \tau)\) is a locally compact Hausdorff space, then \((\bar{X}, \bar{\tau})\) is a compact Hausdorff space. Also if \(U\) is an open set of \(\bar{\tau}\), then \(U \setminus \{\infty\}\) is an open set of \(\tau\).

Proof: Since \((X, \tau)\) is a locally compact Hausdorff space, it follows \((\bar{X}, \bar{\tau})\) is a Hausdorff topological space. The only case which needs checking is the one of \(p \in X \text{ and } \infty\). Since \((X, \tau)\) is locally compact, there exists an open set of \(\tau\), \(U\) having compact closure which contains \(p\). Then \(p \in U\) and \(\infty \in U^C\) and these are disjoint open sets containing the points, \(p\) and \(\infty\) respectively. Now let \(C\) be an open cover of \(\bar{X}\) with sets from \(\bar{\tau}\). Then \(\infty\) must be in some set, \(U_{\infty}\) from \(C\), which must contain a set of the form \(K^C\) where \(K\) is a compact subset of \(X\). Then there exist sets from \(C, U_1, \ldots, U_r\) which cover \(K\). Therefore, a finite subcover of \(\bar{X}\) is \(U_1, \ldots, U_r, U_{\infty}\).

To see the last claim, suppose \(U\) contains \(\infty\) since otherwise there is nothing to show. Notice that if \(C\) is a compact set, then \(X \setminus C\) is an open set. Therefore, if \(x \in U \setminus \{\infty\}\), and if \(\bar{X} \setminus C\) is a basic open set contained in \(U\) containing \(\infty\), then if \(x\) is in this basic open set of \(\bar{X}\), it is also in the open set \(X \setminus C \subseteq U \setminus \{\infty\}\). If \(x\) is not in any basic open set of the form \(\bar{X} \setminus C\) then \(x\) is contained in an open set of \(\tau\) which is contained in \(U \setminus \{\infty\}\). Thus \(U \setminus \{\infty\}\) is indeed open in \(\tau\).
6.9 CONNECTED SETS

Definition 6.8.21 If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.

Theorem 6.8.22 Let \( \mathcal{K} \) be a set whose elements are compact subsets of a Hausdorff topological space, \((X, \tau)\). Suppose \( \mathcal{K} \) has the finite intersection property. Then \( \emptyset \neq \bigcap \mathcal{K} \).

Proof: Suppose to the contrary that \( \emptyset = \bigcap \mathcal{K} \). Then consider
\[
\mathcal{C} \equiv \{ K^C : K \in \mathcal{K} \}.
\]
It follows \( \mathcal{C} \) is an open cover of \( K_0 \) where \( K_0 \) is any particular element of \( \mathcal{K} \). But then there are finitely many \( K \in \mathcal{K}, K_1, \ldots, K_r \) such that \( K_0 \subseteq \bigcup_{i=1}^r K_i^C \) implying that \( \bigcap_{i=0}^r K_i = \emptyset \), contradicting the finite intersection property.

Lemma 6.8.23 Let \((X, \tau)\) be a topological space and let \( \mathcal{B} \) be a basis for \( \tau \). Then \( K \) is compact if and only if every open cover of basic open sets admits a finite subcover.

Proof: Suppose first that \( X \) is compact. Then if \( \mathcal{C} \) is an open cover consisting of basic open sets, it follows it admits a finite subcover because these are open sets in \( \mathcal{C} \).

Next suppose that every basic open cover admits a finite subcover and let \( \mathcal{C} \) be an open cover of \( X \). Then define \( \mathcal{C} \) to be the collection of basic open sets which are contained in some set of \( \mathcal{C} \). It follows \( \mathcal{C} \) is a basic open cover of \( X \) and so it admits a finite subcover, \( \{ U_1, \ldots, U_p \} \). Now each \( U_i \) is contained in an open set of \( \mathcal{C} \). Let \( O_i \) be a set of \( \mathcal{C} \) which contains \( U_i \). Then \( \{ O_1, \ldots, O_p \} \) is an open cover of \( X \). This proves the lemma.

In fact, much more can be said than Lemma 6.8.23. However, this is all which I will present here.

6.9 Connected Sets

Stated informally, connected sets are those which are in one piece. More precisely,

Definition 6.9.1 A set, \( S \) in a general topological space is separated if there exist sets, \( A, B \) such that
\[
S = A \cup B, \ A, B \neq \emptyset, \text{ and } \overline{A} \cap B = B \cap \overline{A} = \emptyset.
\]
In this case, the sets \( A \) and \( B \) are said to separate \( S \). A set is connected if it is not separated.

One of the most important theorems about connected sets is the following.

Theorem 6.9.2 Suppose \( U \) and \( V \) are connected sets having nonempty intersection. Then \( U \cup V \) is also connected.
Proof: Suppose \( U \cup V = A \cup B \) where \( \overline{A} \cap B = \overline{B} \cap A = \emptyset \). Consider the sets, \( A \cap U \) and \( B \cap U \). Since \[ \overline{(A \cap U)} \cap (B \cap U) = (A \cap U) \cap (\overline{B} \cap U) = \emptyset, \]

It follows one of these sets must be empty since otherwise, \( U \) would be separated. It follows that \( U \) is contained in either \( A \) or \( B \). Similarly, \( V \) must be contained in either \( A \) or \( B \). Since \( U \) and \( V \) have nonempty intersection, it follows that both \( V \) and \( U \) are contained in one of the sets, \( A, B \). Therefore, the other must be empty and this shows \( U \cup V \) cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.

![Diagram](image)

**Theorem 6.9.3** Let \( f : X \to Y \) be continuous where \( X \) and \( Y \) are topological spaces and \( X \) is connected. Then \( f(X) \) is also connected.

Proof: To do this you show \( f(X) \) is not separated. Suppose to the contrary that \( f(X) = A \cup B \) where \( A \) and \( B \) separate \( f(X) \). Then consider the sets, \( f^{-1}(A) \) and \( f^{-1}(B) \). If \( z \in f^{-1}(B) \), then \( f(z) \in B \) and so \( f(z) \) is not a limit point of \( A \). Therefore, there exists an open set, \( U \) containing \( f(z) \) such that \( U \cap A = \emptyset \). But then, the continuity of \( f \) implies that \( f^{-1}(U) \) is an open set containing \( z \) such that \( f^{-1}(U) \cap f^{-1}(A) = \emptyset \). Therefore, \( f^{-1}(B) \) contains no limit points of \( f^{-1}(A) \). Similar reasoning implies \( f^{-1}(A) \) contains no limit points of \( f^{-1}(B) \). It follows that \( X \) is separated by \( f^{-1}(A) \) and \( f^{-1}(B) \), contradicting the assumption that \( X \) was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

**Definition 6.9.4** Let \( S \) be a set and let \( p \in S \). Denote by \( C_p \) the union of all connected subsets of \( S \) which contain \( p \). This is called the connected component determined by \( p \).

**Theorem 6.9.5** Let \( C_p \) be a connected component of a set \( S \) in a general topological space. Then \( C_p \) is a connected set and if \( C_p \cap C_q \neq \emptyset \), then \( C_p = C_q \).
Proof: Let $C$ denote the connected subsets of $S$ which contain $p$. If $C_p = A \cup B$ where
\[
\overline{A} \cap B = \overline{B} \cap A = \emptyset,
\]
then $p$ is in one of $A$ or $B$. Suppose without loss of generality $p \in A$. Then every set of $C$ must also be contained in $A$ also since otherwise, as in Theorem 6.9.2, the set would be separated. But this implies $B$ is empty. Therefore, $C_p$ is connected. From this, and Theorem 6.9.2, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, $I$ is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

**Theorem 6.9.6** A set, $C$ in $\mathbb{R}$ is connected if and only if $C$ is an interval.

Proof: Let $C$ be connected. If $C$ consists of a single point, $p$, there is nothing to prove. The interval is just $[p, p]$. Suppose $p < q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$.

Let $C \cap (-\infty, x) \equiv A$, and $C \cap (x, \infty) \equiv B$. Then $C = A \cup B$ and the sets, $A$ and $B$ separate $C$ contrary to the assumption that $C$ is connected.

Conversely, let $I$ be an interval. Suppose $I$ is separated by $A$ and $B$. Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x < y$. Now define the set,
\[
S \equiv \{ t \in [x, y] : [x, t] \subseteq A \}
\]
and let $l$ be the least upper bound of $S$. Then $l \in \overline{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \overline{B}$, then for some $\delta > 0$,
\[
(l, l + \delta) \cap B = \emptyset
\]
contradicting the definition of $l$ as an upper bound for $S$. Therefore, $l \in \overline{B}$ which implies $l \notin A$ after all, a contradiction. It follows $I$ must be connected.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.

**Theorem 6.9.7** Let $U$ be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\{(a_i, b_i)\}_{i=1}^\infty$ such that $U = \bigcup_{i=1}^\infty (a_i, b_i)$.

Proof: Let $p \in U$ and let $z \in C_p$, the connected component determined by $p$. Since $U$ is open, there exists, $\delta > 0$ such that $(z - \delta, z + \delta) \subseteq U$. It follows from Theorem 6.9.2 that
\[
(z - \delta, z + \delta) \subseteq C_p.
\]
This shows $C_p$ is open. By Theorem 6.9.2, this shows $C_p$ is an open interval, $(a, b)$ where $a, b \in [-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\{(a_i, b_i)\}_{i=1}^\infty$ the set of these connected components. This proves the theorem.
Definition 6.9.8 A topological space, \(E\) is arcwise connected if for any two points, \(p, q \in E\), there exists a closed interval, \([a, b]\) and a continuous function, \(\gamma : [a, b] \to E\) such that \(\gamma(a) = p\) and \(\gamma(b) = q\). \(E\) is locally connected if it has a basis of connected open sets. \(E\) is locally arcwise connected if it has a basis of arcwise connected open sets.

An example of an arcwise connected topological space would be the any subset of \(\mathbb{R}^n\) which is the continuous image of an interval. Locally connected is not the same as connected. A well known example is the following.

\[
\left\{ \left( x, \sin \frac{1}{x} \right) : x \in (0, 1] \right\} \cup \{(0, y) : y \in [-1, 1]\} \tag{6.9.14}
\]

You can verify that this set of points considered as a metric space with the metric from \(\mathbb{R}^2\) is not locally connected or arcwise connected but is connected.

Proposition 6.9.9 If a topological space is arcwise connected, then it is connected.

Proof: Let \(X\) be an arcwise connected space and suppose it is separated. Then \(X = A \cup B\) where \(A, B\) are two separated sets. Pick \(p \in A\) and \(q \in B\). Since \(X\) is given to be arcwise connected, there must exist a continuous function \(\gamma : [a, b] \to X\) such that \(\gamma(a) = p\) and \(\gamma(b) = q\). But then we would have \(\gamma([a, b]) = (\gamma([a, b]) \cap A) \cup (\gamma([a, b]) \cap B)\) and the two sets, \(\gamma([a, b]) \cap A\) and \(\gamma([a, b]) \cap B\) are separated thus showing that \(\gamma([a, b])\) is separated and contradicting Theorem 6.9.6 and Theorem 6.9.3. It follows that \(X\) must be connected as claimed.

Theorem 6.9.10 Let \(U\) be an open subset of a locally arcwise connected topological space, \(X\). Then \(U\) is arcwise connected if and only if \(U\) is connected. Also the connected components of an open set in such a space are open sets, hence arcwise connected.

Proof: By Proposition 6.9.4 it is only necessary to verify that if \(U\) is connected and open in the context of this theorem, then \(U\) is arcwise connected. Pick \(p \in U\). Say \(x \in U\) satisfies \(\mathcal{P}\) if there exists a continuous function, \(\gamma : [a, b] \to U\) such that \(\gamma(a) = p\) and \(\gamma(b) = x\).

\[
A \equiv \{ x \in U \text{ such that } x \text{ satisfies } \mathcal{P} \}. 
\]

If \(x \in A\), there exists, according to the assumption that \(X\) is locally arcwise connected, an open set, \(V\), containing \(x\) and contained in \(U\) which is arcwise connected. Thus letting \(y \in V\), there exist intervals, \([a, b]\) and \([c, d]\) and continuous functions having values in \(U\), \(\gamma, \eta\) such that \(\gamma(a) = p\), \(\gamma(b) = x\), \(\eta(c) = x\), and \(\eta(d) = y\). Then let \(\gamma_1 : [a, b + d - c] \to U\) be defined as

\[
\gamma_1(t) \equiv \begin{cases} 
\gamma(t) & \text{if } t \in [a, b] \\
\eta(t + c - b) & \text{if } t \in [b, b + d - c]
\end{cases}
\]
Then it is clear that $\gamma_1$ is a continuous function mapping $p$ to $y$ and showing that $V \subseteq A$. Therefore, $A$ is open. $A \neq \emptyset$ because there is an open set, $V$ containing $p$ which is contained in $U$ and is arcwise connected.

Now consider $B \equiv U \setminus A$. This is also open. If $B$ is not open, there exists a point $z \in B$ such that every open set containing $z$ is not contained in $B$. Therefore, letting $V$ be one of the basic open sets chosen such that $z \in V \subseteq U$, there exist points of $A$ contained in $V$. But then, a repeat of the above argument shows $z \in A$ also. Hence $B$ is open and so if $B \neq \emptyset$, then $U = B \cup A$ and so $U$ is separated by the two sets, $B$ and $A$ contradicting the assumption that $U$ is connected.

It remains to verify the connected components are open. Let $z \in C_p$ where $C_p$ is the connected component determined by $p$. Then picking $V$ an arcwise connected open set which contains $z$ and is contained in $U$, $C_p \cup V$ is connected and contained in $U$ and so it must also be contained in $C_p$. This proves the theorem.

As an application, consider the following corollary.

**Corollary 6.9.11** Let $f : \Omega \to \mathbb{Z}$ be continuous where $\Omega$ is a connected open set. Then $f$ must be a constant.

**Proof:** Suppose not. Then it achieves two different values, $k$ and $l \neq k$. Then $\Omega = f^{-1}(l) \cup f^{-1}\left(\{m \in \mathbb{Z} : m \neq l\}\right)$ and these are disjoint nonempty open sets which separate $\Omega$. To see they are open, note

$$f^{-1}\left(\{m \in \mathbb{Z} : m \neq l\}\right) = f^{-1}\left(\bigcup_{m \neq l}\left(m - \frac{1}{6}, m + \frac{1}{6}\right)\right)$$

which is the inverse image of an open set.

### 6.10 Exercises

1. Let $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. Show that this is a metric on $\mathbb{R}$.

2. Now consider $\mathbb{R}^n$. Let $\|x\|_\infty \equiv \max \{|x_i|, i = 1, \ldots, n\}$. Define

$$d(x, y) \equiv \|x - y\|_\infty.$$ 

Show that this is a metric on $\mathbb{R}^n$. In the case of $n = 2$, describe the ball $B(0, r)$. **Hint:** First show that $\|x + y\| \leq \|x\| + \|y\|$.

3. Let $C([0, T])$ denote the space of functions which are continuous on $[0, T]$. Define

$$\|f\| \equiv \sup_{t \in [0, T]} |f(t)| = \max_{t \in [0, T]} |f(t)|$$

Verify the following. $\|f + g\| \leq \|f\| + \|g\|$. Then use to show that $d(f, g) \equiv \|f - g\|$ is a metric and that with this metric, $(C([0, T]), d)$ is a metric space.
4. Recall that $[a, b]$ is compact. This was done in single variable advanced calculus. That is, every sequence has a convergent subsequence. (We will go over it in here as well.) Also recall that a sequence of numbers $\{x_n\}$ is a Cauchy sequence means that for every $\varepsilon > 0$ there exists $N$ such that if $m, n > N$, then $|x_n - x_m| < \varepsilon$. First show that every Cauchy sequence is bounded. Next, using the compactness of closed intervals, show that every Cauchy sequence has a convergent subsequence. It is shown later that if this is true, the original Cauchy sequence converges. Thus $\mathbb{R}$ with the usual metric just described is complete because every Cauchy sequence converges.

5. Using the result of the above problem, show that $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a complete metric space. That is, every Cauchy sequence converges. Here $d(x, y) \equiv \|x - y\|_\infty$.

6. Suppose you had $(X_i, d_i)$ is a metric space. Now consider the product space

$$X \equiv \prod_{i=1}^{n} X_i$$

with $d(x, y) = \max \{d(x_i, y_i), i = 1 \cdots n\}$. Would this be a metric space? If so, prove that this is the case.

Does triangle inequality hold? **Hint:** For each $i$,

$$d_i(x_i, z_i) \leq d_i(x_i, y_i) + d_i(y_i, z_i) \leq d(x, y) + d(y, z)$$

Now take max of the two ends.

7. In the above example, if each $(X_i, d_i)$ is complete, explain why $(X, d)$ is also complete.

8. Show that $C([0, T])$ is a complete metric space. That is, show that if $\{f_n\}$ is a Cauchy sequence, then there exists $f \in C([0, T])$ such that $\lim_{n \to \infty} d(f_n, f) = 0$. **Hint:** First, you know that $\{f_n(t)\}$ is a Cauchy sequence for each $t$. Why? Now let $f(t)$ be the name of the thing to which $f_n(t)$ converges. Recall why the uniform convergence implies $t \to f(t)$ is continuous. Give the proof. It was done in single variable advanced calculus. Review and write down proof. Also show that $\|f - f_n\| \to 0$.

9. Let $X$ be a nonempty set of points. Say it has infinitely many points. Define $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Show that this is a metric. Show that in $(X, d)$ every point is open and closed. In fact, show that every set is open and every set is closed. Is this a complete metric space? Explain why. Describe the open balls.

10. Show that the union of any set of open sets is an open set. Show the intersection of any set of closed sets is closed. Let $A$ be a nonempty subset of a metric space $(X, d)$. Then the closure of $A$, written as $\bar{A}$ is defined to be the
intersection of all closed sets which contain \( A \). Show that \( \overline{A} = A \cup A' \). That is, to find the closure, you just take the set and include all limit points of the set.

11. Let \( A' \) denote the set of limit points of \( A \), a nonempty subset of a metric space \((X, d)\). Show that \( \overline{A} = A \cup A' \). That is, to find the closure, you just take the set and include all limit points of the set.

12. A theorem was proved which gave three equivalent descriptions of compactness of a metric space. One of them said the following: A metric space is compact if and only if it is complete and totally bounded. Suppose \((X, d)\) is a complete metric space and \( K \subseteq X \). Then \((K, d)\) is also clearly a metric space having the same metric as \( X \). Show that \((K, d)\) is compact if and only if it is closed and totally bounded. Note the similarity with the Heine Borel theorem on \( \mathbb{R} \).

13. On \( \mathbb{R} \), every bounded set is also totally bounded. Thus the earlier Heine Borel theorem for \( \mathbb{R} \) is obtained.

14. If you have a metric space \((X, d)\) and a compact subset of \((X, d)\), suppose that \( L \) is a closed subset of \( K \). Explain why \( L \) must also be compact. 

15. Show that compactness is a topological property in the following sense. If 

\[(X, d), (Y, \rho)\]

are both metric spaces and \( f : X \to Y \) has the property that \( f \) is one to one, onto, and continuous, and also \( f^{-1} \) is one to one onto and continuous, then
the two metric spaces are compact or not compact together. That is one is compact if and only if the other is.

16. Consider $\mathbb{R}$ the real numbers. Define a distance in the following way.

$$\rho(x, y) \equiv |\arctan(x) - \arctan(y)|$$

Show this is a good enough distance and that the open sets which come from this distance are the same as the open sets which come from the usual distance $d(x, y) = |x - y|$. Explain why this yields that the identity mapping $f(x) = x$ is continuous with continuous inverse as a map from $(\mathbb{R}, d)$ to $(\mathbb{R}, \rho)$. To do this, you show that an open ball taken with respect to one of these is also open with respect to the other. However, $(\mathbb{R}, \rho)$ is not a complete metric space while $(\mathbb{R}, d)$ is. Thus, unlike compactness. Completeness is not a topological property. Hint: To show the lack of completeness of $(\mathbb{R}, \rho)$, consider $x_n = n$.

17. A very useful idea in metric space is the following distance function. Let $(X, d)$ be a metric space and $S \subseteq X, S \neq \emptyset$. Then $\text{dist}(x, S) \equiv \inf \{d(x, y) : y \in S\}$.

Show that this always satisfies

$$|\text{dist}(x, S) - \text{dist}(z, S)| \leq d(x, z)$$

This is a really neat result.

18. If $K$ is a compact subset of $(X, d)$ and $y \notin K$, show that there always exists $x \in K$ such that $d(x, y) = \text{dist}(y, K)$. Give an example in $\mathbb{R}$ to show that this is might not be so if $K$ is not compact.

19. You know that if $f : X \rightarrow X$ for $X$ a complete metric space, then if $d(f(x), f(y)) < rd(x, y)$ it follows that $f$ has a unique fixed point theorem. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(t) = t + \left(1 + e^t\right)^{-1}$$

Show that $|f(t) - f(s)| < |t - s|$, but $f$ has no fixed point.

20. If $(X, d)$ is a metric space, show that there is a bounded metric $\rho$ such that the open sets for $(X, d)$ are the same as those for $(X, \rho)$.

21. Let $(X, d)$ be a metric space where $d$ is a bounded metric. Let $\mathcal{C}$ denote the collection of closed subsets of $X$. For $A, B \in \mathcal{C}$, define

$$\rho(A, B) \equiv \inf \{\delta > 0 : A_{\delta} \supseteq B \text{ and } B_{\delta} \supseteq A\}$$

where for a set $S$,

$$S_{\delta} \equiv \{x : \text{dist}(x, S) \equiv \inf \{d(x, s) : s \in S\} \leq \delta\}.$$ 

Show $x \rightarrow \text{dist}(x, S)$ is continuous and that therefore, $S_{\delta}$ is a closed set containing $S$. Also show that $\rho$ is a metric on $\mathcal{C}$. This is called the Hausdorff metric.
22. Suppose \((X, d)\) is a compact metric space. Show \((C, \rho)\) is a complete metric space. **Hint:** Show first that if \(W_n \downarrow W\) where \(W_n\) is closed, then \(\rho(W_n, W) \to 0\). Now let \(\{A_n\}\) be a Cauchy sequence in \(C\). Then if \(\varepsilon > 0\) there exists \(N\) such that when \(m, n \geq N\), then \(\rho(A_n, A_m) < \varepsilon\). Therefore, for each \(n \geq N\),

\[
(A_n)_{\varepsilon} \supseteq \bigcup_{k=n}^{\infty} A_k.
\]

Let \(A \equiv \cap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\). By the first part, there exists \(N_1 > N\) such that for \(n \geq N_1\),

\[
\rho\left(\bigcup_{k=n}^{\infty} A_k, A\right) < \varepsilon, \quad \text{and} \quad (A_n)_{\varepsilon} \supseteq \bigcup_{k=n}^{\infty} A_k.
\]

Therefore, for such \(n\), \(A_{\varepsilon} \supseteq W_n \supseteq A_n\) and \((W_n)_{\varepsilon} \supseteq (A_n)_{\varepsilon} \supseteq A\) because

\[
(A_n)_{\varepsilon} \supseteq \bigcup_{k=n}^{\infty} A_k \supseteq A.
\]

23. Let \(X\) be a compact metric space. Show \((C, \rho)\) is compact. **Hint:** Let \(D_n\) be a \(2^{-n}\) net for \(X\). Let \(K_n\) denote finite unions of sets of the form \(\overline{B}(p, 2^{-n})\) where \(p \in D_n\). Show \(K_n\) is a \(2^{-(n-1)}\) net for \((C, \rho)\).
Chapter 7

Weierstrass Approximation Theorem

7.1 The Bernstein Polynomials

This short chapter is on the important Weierstrass approximation theorem. It is about approximating an arbitrary continuous function uniformly by a polynomial. It will be assumed only that \( f \) has values in \( \mathbb{C} \) and that all scalars are in \( \mathbb{C} \). First here is some notation.

**Definition 7.1.1** \( \alpha = (\alpha_1, \ldots, \alpha_n) \) for \( \alpha_1 \cdots \alpha_n \) positive integers is called a multi-index. For \( \alpha \) a multi-index, \( |\alpha| \equiv \alpha_1 + \cdots + \alpha_n \) and if \( x \in \mathbb{R}^n \),

\[
x = (x_1, \ldots, x_n),
\]

and \( f \) a function, define \( x^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \).

A polynomial in \( n \) variables of degree \( m \) is a function of the form

\[
p(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha.
\]

Here \( \alpha \) is a multi-index as just described.

The following estimate will be the basis for the Weierstrass approximation theorem. It is actually a statement about the variance of a binomial random variable.

**Lemma 7.1.2** The following estimate holds for \( x \in [0, 1] \).

\[
\sum_{k=0}^{m} \binom{m}{k} (k - mx)^2 x^k (1 - x)^{m-k} \leq \frac{1}{4} m
\]

181
CHAPTER 7. WEIERSTRASS APPROXIMATION THEOREM

Proof: By the Binomial theorem,
\[ \sum_{k=0}^{m} \binom{m}{k} (e^t x)^k (1-x)^{m-k} = (1-x+e^t x)^m. \] (7.1.1)
Differentiating both sides with respect to \( t \) and then evaluating at \( t = 0 \) yields
\[ \sum_{k=0}^{m} \binom{m}{k} k x^k (1-x)^{m-k} = mx. \]
Now doing two derivatives of (7.1.1) with respect to \( t \) yields
\[ \sum_{k=0}^{m} \binom{m}{k} k^2 x^k (1-x)^{m-k} = m (m-1) x^2 + m x. \]
Evaluating this at \( t = 0 \),
\[ \sum_{k=0}^{m} \binom{m}{k} k^2 x^k (1-x)^{m-k} = m (m-1) x^2 + m x. \]
Therefore,
\[ \sum_{k=0}^{m} \binom{m}{k} (k-m) x^k (1-x)^{m-k} = m (m-1) x^2 + m x - 2 m^2 x^2 + m^2 x^2 \]
\[ = m (x - x^2) \leq \frac{1}{4} m. \]
This proves the lemma.

Now for \( x = (x_1, \cdots, x_n) \in [0, 1]^n \) consider the polynomial,
\[ p_m (x) \equiv \sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} \binom{m}{k_1} \binom{m}{k_2} \cdots \binom{m}{k_n} x_1^{k_1} (1-x_1)^{m-k_1} x_2^{k_2} (1-x_2)^{m-k_2} \]
\[ \cdots x_n^{k_n} (1-x_n)^{m-k_n} f \left( \frac{k_1}{m}, \cdots, \frac{k_n}{m} \right). \] (7.1.2)
Also define if \( I \) is a set in \( \mathbb{R}^n \)
\[ ||h||_I \equiv \sup \{ |h(x)| : x \in I \}. \]
Thus \( p_m \) converges uniformly to \( f \) on a set, \( I \) if
\[ \lim_{m \to \infty} ||p_m - f||_I = 0. \]
Also to simplify the notation, let \( k = (k_1, \cdots, k_n) \) where each \( k_i \in [0, m], \frac{k}{m} \equiv \left( \frac{k_1}{m}, \cdots, \frac{k_n}{m} \right), \) and let
\[ \binom{m}{k} \equiv \binom{m}{k_1} \binom{m}{k_2} \cdots \binom{m}{k_n}. \]
Also define

\[ ||k||_\infty \equiv \max \{k_i, i = 1, 2, \ldots, n\} \]

\[ x^k (1 - x)^{m-k} \equiv x_1^{k_1} (1 - x_1)^{m-k_1} x_2^{k_2} (1 - x_2)^{m-k_2} \cdots x_n^{k_n} (1 - x_n)^{m-k_n}. \]

Thus in terms of this notation,

\[ p_m(x) = \sum_{||k||_\infty \leq m} \binom{m}{k} x^k (1 - x)^{m-k} f \left( \frac{k}{m} \right) \]

**Lemma 7.1.3** For \( x \in [0, 1]^n \), \( f \) a continuous function defined on \([0, 1]^n\), and \( p_m \) given in (7.1.2), \( p_m \) converges uniformly to \( f \) on \([0, 1]^n\) as \( m \to \infty \).

**Proof:** The function, \( f \) is uniformly continuous because it is continuous on a compact set. Therefore, there exists \( \delta > 0 \) such that if \( |x - y| < \delta \), then

\[ |f(x) - f(y)| < \varepsilon. \]

Denote by \( G \) the set of \( k \) such that \((k_i - mx_i)^2 < \eta^2 m^2\) for each \( i \) where \( \eta = \delta/\sqrt{n} \). Note this condition is equivalent to saying that for each \( i, |\frac{k_i}{m} - x_i| < \eta \). By the binomial theorem,

\[ \sum_{||k||_\infty \leq m} \binom{m}{k} x^k (1 - x)^{m-k} = 1 \]

and so for \( x \in [0, 1]^n \),

\[ |p_m(x) - f(x)| \leq \sum_{||k||_\infty \leq m} \binom{m}{k} x^k (1 - x)^{m-k} |f \left( \frac{k}{m} \right) - f(x)| \]

\[ \leq \sum_{k \in G} \binom{m}{k} x^k (1 - x)^{m-k} |f \left( \frac{k}{m} \right) - f(x)| \]

\[ + \sum_{k \in G^c} \binom{m}{k} x^k (1 - x)^{m-k} |f \left( \frac{k}{m} \right) - f(x)| \quad (7.1.3) \]

Now for \( k \in G \) it follows that for each \( i \)

\[ |\frac{k_i}{m} - x_i| < \frac{\delta}{\sqrt{n}} \quad (7.1.4) \]

and so \( |f \left( \frac{k}{m} \right) - f(x)| < \varepsilon \) because the above implies \( |\frac{k_i}{m} - x_i| < \delta \). Therefore, the first sum on the right in (7.1.3) is no larger than

\[ \sum_{k \in G} \binom{m}{k} x^k (1 - x)^{m-k} \varepsilon \leq \sum_{||k||_\infty \leq m} \binom{m}{k} x^k (1 - x)^{m-k} \varepsilon = \varepsilon. \]
Letting $M \geq \max \{|f(x)| : x \in [0,1]^n\}$ it follows
\[
|p_m(x) - f(x)| \\
\leq \varepsilon + 2M \sum_{k \in G^C} \binom{m}{k} x^k (1 - x)^{m-k} \\
\leq \varepsilon + 2M \left( \frac{1}{\eta^2 m^2} \right)^n \sum_{k \in G^C} \binom{m}{k} \prod_{j=1}^{n} (k_j - mx_j)^2 x^k (1 - x)^{m-k} \\
\leq \varepsilon + 2M \left( \frac{1}{\eta^2 m^2} \right)^n \sum_{||k||_\infty \leq m} \binom{m}{k} \prod_{j=1}^{n} (k_j - mx_j)^2 x^k (1 - x)^{m-k}
\]
because on $G^C$, 
\[
\frac{(k_j - mx_j)^2}{\eta^2 m^2} < 1, \ j = 1, \cdots, n.
\]
Now by Lemma 7.1.3, 
\[
|p_m(x) - f(x)| \leq \varepsilon + 2M \left( \frac{1}{\eta^2 m^2} \right)^n \left( \frac{m}{4} \right)^n.
\]
Therefore, since the right side does not depend on $x$, it follows 
\[
\lim_{m \to \infty} \sup_{x \in [0,1]^n} ||p_m - f||_{[0,1]^n} \leq \varepsilon
\]
and since $\varepsilon$ is arbitrary, this shows $\lim_{m \to \infty} ||p_m - f||_{[0,1]^n} = 0$. This proves the lemma.

The following is not surprising.

**Lemma 7.1.4** Let $f$ be a continuous function defined on $[-M,M]^n$. Then there exists a sequence of polynomials, \{\(p_m\)\} converging uniformly to $f$ on $[-M,M]^n$.

**Proof:** Let $h(t) = -M + 2Mt$ so $h : [0,1] \to [-M,M]$ and let $h(t) \equiv (h(t_1), \cdots, h(t_n))$. Therefore, $f \circ h$ is a continuous function defined on $[0,1]^n$. From Lemma 7.1.3 there exists a polynomial, $p(t)$ such that $||p_m - f \circ h||_{[0,1]^n} < \frac{1}{m}$. Now for $x \in [-M,M]^n$, $h^{-1}(x) = (h^{-1}(x_1), \cdots, h^{-1}(x_n))$ and so 
\[
||p_m \circ h^{-1} - f||_{[-M,M]^n} = ||p_m - f \circ h||_{[0,1]^n} < \frac{1}{m}.
\]
But $h^{-1}(x) = \frac{x}{2M} + \frac{1}{2}$ and so $p_m$ is still a polynomial. This proves the lemma.

The classical version of the Weierstrass approximation theorem involved showing that a continuous function of one variable defined on a closed and bounded interval is the uniform limit of a sequence of polynomials. This is certainly included as a special case of the above. Now recall the Tietze extension theorem found on Page 160. In the general version about to be presented, the set on which $f$ is defined is just a compact subset of $\mathbb{R}^n$, not the Cartesian product of intervals. For convenience here is the Tietze extension theorem.
7.2. STONE WEIERSTRASS THEOREM

Theorem 7.1.5 Let \( M \) be a closed nonempty subset of a metric space \((X, d)\) and let \( f : M \to [a, b] \) is continuous at every point of \( M \). Then there exists a function, \( g \) continuous on all of \( X \) which coincides with \( f \) on \( M \) such that \( g(X) \subseteq [a, b] \).

The Weierstrass approximation theorem follows.

Theorem 7.1.6 Let \( K \) be a compact set in \( \mathbb{R}^n \) and let \( f \) be a continuous function defined on \( K \). Then there exists a sequence of polynomials \( \{p_m\} \) converging uniformly to \( f \) on \( K \).

Proof: Choose \( M \) large enough that \( K \subseteq [-M, M]^n \) and let \( \tilde{f} \) denote a continuous function defined on all of \([-M, M]^n\) such that \( \tilde{f} = f \) on \( K \). Such an extension exists by the Tietze extension theorem, Theorem 7.1.5 applied to the real and imaginary parts of \( f \). By Lemma 7.1.4 there exists a sequence of polynomials, \( \{p_m\} \) defined on \([-M, M]^n\) such that \( \|f - p_m\|_{[−M, M]^n} \to 0 \). Therefore, \( \|\tilde{f} - p_m\|_K \to 0 \) also. This proves the theorem.

7.2 Stone Weierstrass Theorem

7.2.1 The Case Of Compact Sets

There is a profound generalization of the Weierstrass approximation theorem due to Stone.

Definition 7.2.1 \( A \) is an algebra of functions if \( A \) is a vector space and if whenever \( f, g \in A \) then \( fg \in A \).

To begin with assume that the field of scalars is \( \mathbb{R} \). This will be generalized later. Theorem 7.1.6 implies the following very special case.

Corollary 7.2.2 The polynomials are dense in \( C([a, b]) \).

The next result is the key to the profound generalization of the Weierstrass theorem due to Stone in which an interval will be replaced by a compact or locally compact set and polynomials will be replaced with elements of an algebra satisfying certain axioms.

Corollary 7.2.3 On the interval \([-M, M]\), there exist polynomials \( p_n \) such that \( p_n(0) = 0 \) and

\[
\lim_{n \to \infty} ||p_n - |\cdot||_\infty = 0.
\]

Proof: By Corollary 7.2.2 there exists a sequence of polynomials, \( \{\tilde{p}_n\} \) such that \( \tilde{p}_n \to |\cdot| \) uniformly. Then let \( p_n(t) \equiv \tilde{p}_n(t) - \tilde{p}_n(0) \). This proves the corollary.
Definition 7.2.4  An algebra of functions, \( \mathcal{A} \) defined on \( A \), annihilates no point of \( A \) if for all \( x \in A \), there exists \( g \in \mathcal{A} \) such that \( g(x) \neq 0 \). The algebra separates points if whenever \( x_1 \neq x_2 \), then there exists \( g \in \mathcal{A} \) such that \( g(x_1) \neq g(x_2) \).

The following generalization is known as the Stone Weierstrass approximation theorem.

Theorem 7.2.5  Let \( A \) be a compact topological space and let \( \mathcal{A} \subseteq C(A; \mathbb{R}) \) be an algebra of functions which separates points and annihilates no point. Then \( \mathcal{A} \) is dense in \( C(A; \mathbb{R}) \).

Proof: First here is a lemma.

Lemma 7.2.6  Let \( c_1 \) and \( c_2 \) be two real numbers and let \( x_1 \neq x_2 \) be two points of \( A \). Then there exists a function \( f_{x_1x_2} \) such that

\[
f_{x_1x_2}(x_1) = c_1, \; f_{x_1x_2}(x_2) = c_2.
\]

Proof of the lemma: Let \( g \in \mathcal{A} \) satisfy

\[
g(x_1) \neq g(x_2).
\]

Such a \( g \) exists because the algebra separates points. Since the algebra annihilates no point, there exist functions \( h \) and \( k \) such that

\[
h(x_1) \neq 0, \; k(x_2) \neq 0.
\]

Then let

\[
u \equiv gh - g(x_2) h, \; v \equiv gk - g(x_1) k.
\]

It follows that \( u(x_1) \neq 0 \) and \( u(x_2) = 0 \) while \( v(x_2) \neq 0 \) and \( v(x_1) = 0 \). Let

\[
f_{x_1x_2} = \frac{c_1 u}{u(x_1)} + \frac{c_2 v}{v(x_2)}.
\]

This proves the lemma. Now continue the proof of Theorem 7.2.5.

First note that \( \overline{\mathcal{A}} \) satisfies the same axioms as \( \mathcal{A} \) but in addition to these axioms, \( \overline{\mathcal{A}} \) is closed. The closure of \( \mathcal{A} \) is taken with respect to the usual norm on \( C(A) \),

\[
\|f\|_\infty = \max \{|f(x)| : x \in A\}.
\]

Suppose \( f \in \overline{\mathcal{A}} \) and suppose \( M \) is large enough that

\[
\|f\|_\infty < M.
\]

Using Corollary 7.2.3, let \( p_n \) be a sequence of polynomials such that

\[
\|p_n - f\|_\infty \to 0, \; p_n(0) = 0.
\]
It follows that \( p_n \circ f \in \overline{A} \) and so \(|f| \in \overline{A}\) whenever \( f \in \overline{A} \). Also note that
\[
\max (f, g) = \frac{|f - g| + (f + g)}{2}
\]
\[
\min (f, g) = \frac{(f + g) - |f - g|}{2}.
\]
Therefore, this shows that if \( f, g \in \overline{A} \) then
\[
\max (f, g), \min (f, g) \in \overline{A}.
\]
By induction, if \( f_i, i = 1, 2, \cdots, m \) are in \( \overline{A} \) then
\[
\max (f_i, i = 1, 2, \cdots, m), \min (f_i, i = 1, 2, \cdots, m) \in \overline{A}.
\]
Now let \( h \in C(A; \mathbb{R}) \) and let \( x \in A \). Use Lemma 7.2.6 to obtain \( f_{xy} \), a function of \( \overline{A} \) which agrees with \( h \) at \( x \) and \( y \). Letting \( \varepsilon > 0 \), there exists an open set \( U (y) \) containing \( y \) such that
\[
f_{xy}(z) > h(z) - \varepsilon \quad \text{if} \quad z \in U(y).
\]
Since \( A \) is compact, let \( U (y_1), \cdots, U (y_l) \) cover \( A \). Let
\[
f_x \equiv \max (f_{xy_1}, f_{xy_2}, \cdots, f_{xy_l}).
\]
Then \( f_x \in \overline{A} \) and
\[
f_x(z) > h(z) - \varepsilon
\]
for all \( z \in A \) and \( f_x(x) = h(x) \). This implies that for each \( x \in A \) there exists an open set \( V (x) \) containing \( x \) such that for \( z \in V (x) \),
\[
f_x(z) < h(z) + \varepsilon.
\]
Let \( V (x_1), \cdots, V (x_m) \) cover \( A \) and let
\[
f \equiv \min (f_{x_1}, \cdots, f_{x_m}).
\]
Therefore,
\[
f(z) < h(z) + \varepsilon
\]
for all \( z \in A \) and since \( f_x(z) > h(z) - \varepsilon \) for all \( z \in A \), it follows
\[
f(x) > h(z) - \varepsilon
\]
also and so
\[
|f(z) - h(z)| < \varepsilon
\]
for all \( z \). Since \( \varepsilon \) is arbitrary, this shows \( h \in \overline{A} \) and proves \( \overline{A} = C(A; \mathbb{R}) \). This proves the theorem.
7.2.2 The Case Of Locally Compact Sets

**Definition 7.2.7** Let \((X,\tau)\) be a locally compact Hausdorff space. \(C_0(X)\) denotes the space of real or complex valued continuous functions defined on \(X\) with the property that if \(f \in C_0(X)\), then for each \(\varepsilon > 0\) there exists a compact set \(K\) such that \(|f(x)| < \varepsilon\) for all \(x \notin K\). Define

\[
||f||_\infty = \sup \{|f(x)| : x \in X\}.
\]

**Lemma 7.2.8** For \((X,\tau)\) a locally compact Hausdorff space with the above norm, \(C_0(X)\) is a complete space.

**Proof:** Let \(\left(\tilde{X}, \tilde{\tau}\right)\) be the one point compactification described in Lemma 6.8.20.

\[
D = \left\{f \in C(\tilde{X}) : f(\infty) = 0\right\}.
\]

Then \(D\) is a closed subspace of \(C(\tilde{X})\). For \(f \in C_0(X)\),

\[
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}
\]

and let \(\theta : C_0(X) \to D\) be given by \(\theta f = \tilde{f}\). Then \(\theta\) is one to one and onto and also satisfies \(||f||_\infty = ||\theta f||_\infty\). Now \(D\) is complete because it is a closed subspace of a complete space and so \(C_0(X)\) with \(||\cdot||_\infty\) is also complete. This proves the lemma.

The above refers to functions which have values in \(\mathbb{C}\) but the same proof works for functions which have values in any complete normed linear space.

In the case where the functions in \(C_0(X)\) all have real values, I will denote the resulting space by \(C_0(X;\mathbb{R})\) with similar meanings in other cases.

With this lemma, the generalization of the Stone Weierstrass theorem to locally compact sets is as follows.

**Theorem 7.2.9** Let \(A\) be an algebra of functions in \(C_0(X;\mathbb{R})\) where \((X,\tau)\) is a locally compact Hausdorff space which separates the points and annihilates no point. Then \(A\) is dense in \(C_0(X;\mathbb{R})\).

**Proof:** Let \(\left(\tilde{X}, \tilde{\tau}\right)\) be the one point compactification as described in Lemma 6.8.20. Let \(\tilde{A}\) denote all finite linear combinations of the form

\[
\left\{ \sum_{i=1}^{n} c_i \tilde{f}_i + c_0 : f \in A, \ c_i \in \mathbb{R} \right\}
\]

where for \(f \in C_0(X;\mathbb{R})\),

\[
\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}
\]
7.2. **STONE WEIERSTRASS THEOREM**

Then \( \overline{A} \) is obviously an algebra of functions in \( C \left( \overline{X}; \mathbb{R} \right) \). It separates points because this is true of \( A \). Similarly, it annihilates no point because of the inclusion of \( c_0 \) an arbitrary element of \( \mathbb{R} \) in the definition above. Therefore from Theorem (2.2.3), \( A \) is dense in \( C \left( \overline{X}; \mathbb{R} \right) \). Letting \( f \in C_0 (X; \mathbb{R}) \), it follows \( \tilde{f} \in C \left( \overline{X}; \mathbb{R} \right) \) and so there exists a sequence \( \{h_n\} \subseteq \overline{A} \) such that \( h_n \) converges uniformly to \( f \). Now \( h_n \) is of the form \( \sum_{i=1}^{n} c_i \tilde{f}_i^n + c_0 \) and since \( \tilde{f}(\infty) = 0 \), you can take each \( c_0 = 0 \) and so this has shown the existence of a sequence of functions in \( A \) such that it converges uniformly to \( f \). This proves the theorem.

### 7.2.3 The Case Of Complex Valued Functions

What about the general case where \( C_0 (X) \) consists of complex valued functions and the field of scalars is \( \mathbb{C} \) rather than \( \mathbb{R} \)? The following is the version of the Stone Weierstrass theorem which applies to this case. You have to assume that for \( f \in A \) it follows \( \overline{f} \in A \). Such an algebra is called self adjoint.

**Theorem 7.2.10** Suppose \( A \) is an algebra of functions in \( C_0 (X) \), where \( X \) is a locally compact Hausdorff space, which separates the points, annihilates no point, and has the property that if \( f \in A \), then \( \overline{f} \in A \). Then \( A \) is dense in \( C_0 (X) \).

**Proof:** Let \( \text{Re} \ A \equiv \{ \text{Re} \ f : f \in A \} \), \( \text{Im} \ A \equiv \{ \text{Im} \ f : f \in A \} \). First I will show that \( A = \text{Re} \ A + i \text{Im} \ A = \text{Im} \ A + i \text{Re} \ A \). Let \( f \in A \). Then

\[
f = \frac{1}{2} (f + \overline{f}) + \frac{1}{2} (f - \overline{f}) = \text{Re} \ f + i \text{Im} \ f \in \text{Re} \ A + i \text{Im} \ A
\]

and so \( A \subseteq \text{Re} \ A + i \text{Im} \ A \). Also

\[
f = \frac{1}{2i} (if + i\overline{f}) - \frac{i}{2} (i f + i\overline{f}) = \text{Im} (if) + i \text{Re} (if) \in \text{Im} \ A + i \text{Re} \ A
\]

This proves one half of the desired equality. Now suppose \( h \in \text{Re} \ A + i \text{Im} \ A \). Then \( h = \text{Re} \ g_1 + i \text{Im} \ g_2 \) where \( g_i \in A \). Then since \( \text{Re} \ g_1 = \frac{1}{2} (g_1 + \overline{g_1}) \), it follows \( \text{Re} \ g_1 \in A \). Similarly \( \text{Im} \ g_2 \in A \). Therefore, \( h \in A \). The case where \( h \in \text{Im} \ A + i \text{Re} \ A \) is similar. This establishes the desired equality.

Now \( \text{Re} \ A \) and \( \text{Im} \ A \) are both real algebras. I will show this now. First consider \( \text{Im} \ A \). It is obvious this is a real vector space. It only remains to verify that the product of two functions in \( \text{Im} \ A \) is in \( \text{Im} \ A \). Note that from the first part, \( \text{Re} \ A, \text{Im} \ A \) are both subsets of \( A \) because, for example, if \( u \in \text{Im} \ A \) then \( u + 0 \in \text{Im} \ A + i \text{Re} \ A = A \). Therefore, if \( v, w \in \text{Im} \ A \), both \( iv \) and \( w \) are in \( A \) and so \( \text{Im} (ivw) = vw \) and \( ivw \in \text{Im} \ A \). Similarly, \( \text{Re} \ A \) is an algebra.

Both \( \text{Re} \ A \) and \( \text{Im} \ A \) must separate the points. Here is why: If \( x_1 \neq x_2 \), then there exists \( f \in A \) such that \( f (x_1) \neq f (x_2) \). If \( \text{Im} f (x_1) \neq \text{Im} f (x_2) \), this shows there is a function in \( \text{Im} \ A \), \( \text{Im} f \) which separates these two points. If \( \text{Im} f \) fails to separate the two points, then \( \text{Re} f \) must separate the points and so you could consider \( \text{Im} (if) \) to get a function in \( \text{Im} \ A \) which separates these points. This shows \( \text{Im} \ A \) separates the points. Similarly \( \text{Re} \ A \) separates the points.
Neither $\text{Re} \mathcal{A}$ nor $\text{Im} \mathcal{A}$ annihilate any point. This is easy to see because if $x$ is a point there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. Thus either $\text{Re} \, f(x) \neq 0$ or $\text{Im} \, f(x) \neq 0$. If $\text{Im} \, f(x) = 0$, consider $\text{Im} \, (if) \, (x) = \text{Re} \, f \, (x) \neq 0$. Similarly, $\text{Re} \mathcal{A}$ does not annihilate any point.

It follows from Theorem 7.2.9 that $\text{Re} \mathcal{A}$ and $\text{Im} \mathcal{A}$ are dense in the real valued functions of $C_0(X)$. Let $f \in C_0(X)$. Then there exists $\{h_n\} \subseteq \text{Re} \mathcal{A}$ and $\{g_n\} \subseteq \text{Im} \mathcal{A}$ such that $h_n \to \text{Re} \, f$ uniformly and $g_n \to \text{Im} \, f$ uniformly. Therefore, $h_n + ig_n \in \mathcal{A}$ and it converges to $f$ uniformly. This proves the theorem.

### 7.3 The Holder Spaces

We consider these spaces as spaces of functions defined on an interval $[0,1]$ although one could have $[0,T]$ just as easily. A slightly more general version is in the exercises. They are a very interesting example of spaces which are not separable.

**Definition 7.3.1** Let $p > 1$. Then $f \in C^{1/p}([0,1])$ means that $f \in C([0,1])$ and also

$$
\rho_p(f) \equiv \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/p}} : x, y \in X, x \neq y \right\} < \infty
$$

Then the norm is defined as $\|f\|_{C([0,1])} + \rho_p(f) \equiv \|f\|_{1/p}$.

We leave it as an exercise to verify that $C^{1/p}([0,1])$ is a complete normed linear space. Let $p > 1$. Then $C^{1/p}([0,1])$ is not separable. Define uncountably many functions, one for each $\varepsilon$ where $\varepsilon$ is a sequence of $-1$ and $1$. Thus $\varepsilon_k \in \{-1,1\}$. Thus $\varepsilon \neq \varepsilon'$ if the two sequences differ in at least one slot, one giving $1$ and the other equaling $-1$. Now define

$$
f_{\varepsilon}(t) \equiv \sum_{k=1}^{\infty} \varepsilon_k 2^{-k/p} \sin (2^k \pi t)
$$

Then this is $1/p$ Holder. Let $s < t$.

$$
|f_{\varepsilon}(t) - f_{\varepsilon}(s)| \leq \sum_{k \leq \log_2(t-s)} \left| 2^{-k/p} \sin (2^k \pi t) - 2^{-k/p} \sin (2^k \pi s) \right|
$$

$$
+ \sum_{k > \log_2(t-s)} \left| 2^{-k/p} \sin (2^k \pi t) - 2^{-k/p} \sin (2^k \pi s) \right|
$$

If $t = 1$ and $s = 0$, there is really nothing to show because then the difference equals $0$. There is also nothing to show if $t = s$. From now on, $0 < t - s < 1$. Let $k_0$ be
the largest integer which is less than or equal to $|\log_2 (t - s)| = -\log_2 (t - s)$. Note that $- \log (t - s) > 0$ because $0 < t - s < 1$. Then

$$|f_\varepsilon (t) - f_\varepsilon (s)| \leq \sum_{k \leq k_0} \left| 2^{-k/p} \sin (2^k \pi t) - 2^{-k/p} \sin (2^k \pi s) \right|$$

$$+ \sum_{k > k_0} \left| 2^{-k/p} \sin (2^k \pi t) - 2^{-k/p} \sin (2^k \pi s) \right|$$

$$\leq \sum_{k \leq k_0} 2^{-k/p} 2^k \pi |t - s| + \sum_{k > k_0} 2^{-k/p} 2$$

Now $k_0 \leq -\log_2 (t - s) < k_0 + 1$ and so $-k_0 \geq \log_2 (t - s) \geq -(k_0 + 1)$. Hence

$$2^{-k_0} \geq |t - s| \geq 2^{-k_0} 2^{-1}$$

and so

$$2^{-k_0/p} \geq |t - s|^{1/p} \geq 2^{-k_0/p} 2^{-1/p}$$

Using this in the sums,

$$|f_\varepsilon (t) - f_\varepsilon (s)| \leq |t - s| C_p + \sum_{k > k_0} 2^{-k/p} 2^k \pi |t - s|^{1/p} 2$$

$$\leq |t - s| C_p + \sum_{k > k_0} 2^{-k/p} 2^k \pi (2^{1/p} |t - s|^{1/p}) 2$$

$$\leq |t - s| C_p + \sum_{k > k_0} 2^{-(k-k_0)/p} \left( 2^{1+1/p} \sum_{k=1}^\infty 2^{-k/p} |t - s|^{1/p} \right) 2$$

$$\leq C_p |t - s| + D_p |t - s|^{1/p} \leq C_p |t - s|^{1/p} + D_p |t - s|^{1/p}$$

Thus $f_\varepsilon$ is indeed $1/p$ Holder continuous.

Now consider $\varepsilon \neq \varepsilon'$. Suppose the first discrepancy in the two sequences occurs with $\varepsilon_j$. Thus one is 1 and the other is $-1$. Let $t = \frac{i+1}{2^j+1}, s = \frac{j+1}{2^j+1}$

$$|f_\varepsilon (t) - f_\varepsilon (s) - (f_{\varepsilon'} (t) - f_{\varepsilon'} (s))| =$$

$$\left| \sum_{k=j}^{\infty} \varepsilon_k 2^{-k/p} \sin (2^k \pi t) - \sum_{k=j}^{\infty} \varepsilon_k 2^{-k/p} \sin (2^k \pi s) \right|$$

$$- \left( \sum_{k=j}^{\infty} \varepsilon_k' 2^{-k/p} \sin (2^k \pi t) - \sum_{k=j}^{\infty} \varepsilon_k' 2^{-k/p} \sin (2^k \pi s) \right)$$

Now consider what happens for $k > j$

$$\sin \left( 2^k \pi \frac{i}{2^j+1} \right) = \sin (m\pi) = 0$$
for some integer $m$. Thus the whole mess reduces to

$$\left(\varepsilon_j - \varepsilon'_j\right) 2^{-j/p} \sin \left(\frac{2^j \pi (i+1)}{2j+1}\right) - \left(\varepsilon_j - \varepsilon'_j\right) 2^{-j/p} \sin \left(\frac{2^j \pi i}{2j+1}\right)$$

$$= \left(\varepsilon_j - \varepsilon'_j\right) 2^{-j/p} \sin \left(\frac{\pi (i+1)}{2j+1}\right) - \left(\varepsilon_j - \varepsilon'_j\right) 2^{-j/p} \sin \left(\frac{\pi i}{2j+1}\right)$$

$$= 2 \left(2^{-j/p}\right)$$

In particular, $|t - s| = \frac{1}{2^{j+1}}$ so $2^{1/p} |t - s|^{1/p} = 2^{-j/p}$

$$|f_{\varepsilon} (t) - f_{\varepsilon} (s) - (f_{\varepsilon'} (t) - f_{\varepsilon'} (s))| = 2 \left(2^{1/p}\right) |t - s|^{1/p}$$

which shows that

$$\sup_{0 \leq s \leq t \leq 1} \frac{|f_{\varepsilon} (t) - f_{\varepsilon'} (t) - (f_{\varepsilon} (s) - f_{\varepsilon'} (s))|}{|t - s|^{1/p}} \geq 2^{1/p} (2)$$

Thus there exists a set of uncountably many functions in $C^{1/p} ([0, T])$ and for any two of them $f, g$, you get

$$\|f - g\|_{C^{1/p} ([0, 1])} > 2$$

so $C^{1/p} ([0, 1])$ is not separable.

### 7.4 Exercises

1. Let $(X, \tau), (Y, \eta)$ be topological spaces and let $A \subseteq X$ be compact. Then if $f : X \to Y$ is continuous, show that $f (A)$ is also compact.

2. ↑ In the context of Problem III, suppose $\mathbb{R} = Y$ where the usual topology is placed on $\mathbb{R}$. Show $f$ achieves its maximum and minimum on $A$.

3. Let $V$ be an open set in $\mathbb{R}^n$. Show there is an increasing sequence of compact sets, $K_m$, such that $V = \bigcup_{m=1}^{\infty} K_m$. **Hint:** Let

$$C_m = \left\{ x \in \mathbb{R}^n : \text{dist} (x, V^C) \geq \frac{1}{m} \right\}$$

where

$$\text{dist} (x, S) = \inf \{|y - x| \text{ such that } y \in S\}.$$  

Consider $K_m = C_m \cap \overline{B}(0, m)$.

4. Let $B (X; \mathbb{R}^n)$ be the space of functions $f$, mapping $X$ to $\mathbb{R}^n$ such that

$$\sup \{|f (x)| : x \in X\} < \infty.$$  

Show $B (X; \mathbb{R}^n)$ is a complete normed linear space if

$$\|f\| = \sup \{|f (x)| : x \in X\}.$$
5. Let $\alpha \in [0, 1]$. Define, for $X$ a compact subset of $\mathbb{R}^p$,

$$C^\alpha (X; \mathbb{R}^n) \equiv \{ f \in C(X; \mathbb{R}^n) : \rho_\alpha (f) + ||f|| \equiv ||f||_\alpha < \infty \}$$

where

$$||f|| \equiv \sup \{ |f(x)| : x \in X \}$$

and

$$\rho_\alpha (f) \equiv \sup \{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in X, \ x \neq y \}.$$ 

Show that $(C^\alpha (X; \mathbb{R}^n), ||\cdot||_\alpha)$ is a complete normed linear space.

6. Let $\{f_n\}_{n=1}^\infty \subseteq C^\alpha (X; \mathbb{R}^n)$ where $X$ is a compact subset of $\mathbb{R}^p$ and suppose

$$||f_n||_\alpha \leq M$$

for all $n$. Show there exists a subsequence, $n_k$, such that $f_{n_k}$ converges in $C(X; \mathbb{R}^n)$. The given sequence is called precompact when this happens. (This also shows the embedding of $C^\alpha (X; \mathbb{R}^n)$ into $C(X; \mathbb{R}^n)$ is a compact embedding.) Note that it is likely the case that $C^\alpha (X; \mathbb{R}^n)$ is not separable although it embeds continuously into a nice separable space. In fact, $C^\alpha ([0, T]; \mathbb{R}^n)$ can be shown to not be separable. See Definition 7.3.1 and the discussion which follows it.

7. Use the general Stone Weierstrass approximation theorem to prove Theorem 7.1.6.
Chapter 8

Brouwer Fixed Point Theorem $\mathbb{R}^n$

In this short optional chapter, is an elementary proof of the Brouwer fixed point theorem and a discussion of some of the manipulations which are important regarding simplices. The Brouwer fixed point theorem will be presented later using analytical techniques. This here is an approach based on combinatorics or graph theory. It features the famous Sperner’s lemma.

8.1 Simplices And Triangulations

**Definition 8.1.1** Define an $n$ simplex, denoted by $[x_0, \cdots, x_n]$, to be the convex hull of the $n + 1$ points, $\{x_0, \cdots, x_n\}$ where $\{x_i - x_0\}_{i=1}^n$ are independent. Thus

$$[x_0, \cdots, x_n] \equiv \left\{ \sum_{i=0}^{n} t_i x_i : \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0 \right\}.$$ 

Note that $\{x_j - x_m\}_{j \neq m}$ are also independent.

Since $\{x_i - x_0\}_{i=1}^n$ is independent, the $t_i$ are uniquely determined. If two of them are

$$\sum_{i=0}^{n} t_i x_i = \sum_{i=0}^{n} s_i x_i$$

Then

$$\sum_{i=0}^{n} t_i (x_i - x_0) = \sum_{i=0}^{n} s_i (x_i - x_0)$$

so $t_i = s_i$ for $i \geq 1$ by independence. Since the $s_i$ and $t_i$ sum to 1, it follows that also $s_0 = t_0$. If $n \leq 2$, the simplex is a triangle, line segment, or point. If $n \leq 3$, it is a tetrahedron, triangle, line segment or point. To say that $\{x_i - x_0\}_{i=1}^n$ are
independent is to say that \( \{x_i - x_r\}_{i \neq r} \) are independent for each fixed \( r \). Indeed, if \( x_i - x_r = \sum_{j \neq i, r} c_j (x_j - x_r) \), then you would have

\[ x_i - x_0 + x_0 - x_r = \sum_{j \neq i, r} c_j (x_j - x_0) + \left( \sum_{j \neq i, r} c_j \right) x_0 \]

and it follows that \( x_i - x_0 \) is a linear combination of the \( x_j - x_0 \) for \( j \neq i \), contrary to assumption.

A simplex \( S \) can be triangulated. This means it is the union of smaller sub-simplices such that if \( S_1, S_2 \) are two simplices in the triangulation, with

\[ S_1 \equiv [z_0, \ldots, z_m], \quad S_2 \equiv [z_0^2, \ldots, z_p^2] \]

then

\[ S_1 \cap S_2 = [x_{k_0}, \ldots, x_{k_r}] \]

where \([x_{k_0}, \ldots, x_{k_r}]\) is in the triangulation and

\[ \{x_{k_0}, \ldots, x_{k_r}\} = \{z_0^1, \ldots, z_m^1\} \cap \{z_0^2, \ldots, z_p^2\} \]

or else the two simplices do not intersect.

Does there exist a triangulation in which all sub-simplices have diameter less than \( \varepsilon \)? This is obvious if \( n \leq 2 \). Supposing it to be true for \( n - 1 \), is it also so for \( n \)? The barycenter \( b \) of a simplex \([x_0, \ldots, x_n]\) is just \( \frac{1}{n+1} \sum_i x_i \). This point is not in the convex hull of any of the faces, those simplices of the form \([x_0, \ldots, \hat{x}_k, \ldots, x_n]\) where the hat indicates \( x_k \) has been left out. Thus \([x_0, \ldots, b, \ldots, x_n]\) is a \( n \) simplex also. Now in general, if you have an \( n \) simplex \([x_0, \ldots, x_n]\), its diameter is the maximum of \( |x_k - x_l| \) for all \( k \neq l \). Consider \([b - x_j]\). It equals

\[ \left| \sum_{i=0}^{n} \frac{1}{n+1} (x_i - x_j) \right| = \left| \sum_{i \neq j} \frac{1}{n+1} (x_i - x_j) \right| \leq \frac{n}{n+1} \text{diam} (S) \]

Next consider the \( k^{th} \) face of \( S \) \([x_0, \ldots, \hat{x}_k, \ldots, x_n]\). By induction, it has a triangulation into simplices which each have diameter no more than \( \frac{n}{n+1} \text{diam} (S) \). Let these \( n - 1 \) simplices be denoted by \( \{S_1^k, \ldots, S_m^k\} \). Then the simplices \( \{[S_i^k, b]\} \) \( i = 1, k = 1 \) are a triangulation of \( S \) such that \( \text{diam} ([S_i^k, b]) \) \( i = 1, k = 1 \) is a triangulation of \( S \) such that \( \text{diam} ([S_i^k, b]) \leq \frac{n}{n+1} \text{diam} (S) \). Do for \([S_i^k, b]\) what was just done for \( S \) obtaining a triangulation of \( S \) as the union of what is obtained such that each simplex has diameter no more than \( \frac{n}{n+1} \text{diam} (S) \). Continuing this way shows the existence of the desired triangulation.

### 8.2 Labeling Vertices

Next is a way to label the vertices. Let \( p_0, \ldots, p_n \) be the first \( n + 1 \) prime numbers. All vertices of a simplex \( S = [x_0, \ldots, x_n] \) having \( \{x_k - x_0\}_{k=1}^n \) independent will be labeled with one of these primes. In particular, the vertex \( x_k \) will be labeled as \( p_k \) if the simplex is \([x_0, \ldots, x_n]\). The value of a simplex will be the product of its labels. Triangulate this \( S \). Consider a 1 simplex coming from the original
simplex \([x_{k_1}, x_{k_2}]\), label one end as \(p_{k_1}\) and the other as \(p_{k_2}\). Then label all other vertices of this triangulation which occur on \([x_{k_1}, x_{k_2}]\) either \(p_{k_1}\) or \(p_{k_2}\). Note that by independence of \(\{x_k - x_r\}_{k \neq r}\), this cannot introduce an inconsistency because the segment cannot contain any other vertex of \(S\). Then obviously there will be an odd number of simplices in this triangulation having value \(p_{k_1} p_{k_2}\), that is a \(p_{k_1}\) at one end and a \(p_{k_2}\) at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on \([x_{j_1}, \ldots, x_{j_{k+1}}]\):

\[\{x_{j_1}, \ldots, x_{j_{k+1}}\} \subseteq \{x_0, \ldots, x_n\}\]

any \(k\) simplex for \(k \leq n - 1\), and there is an odd number of simplices from the triangulation having value equal to \(\prod_{i=1}^{k+1} p_{j_i}\) on this simplex. Consider \(\hat{S} = [x_{j_1}, \ldots, x_{j_{k+1}}, x_{j_{k+2}}]\). Then by induction, there is an odd number of \(k\) simplices on the \(s^{th}\) face \([x_{j_1}, \ldots, x_{j_{k+1}}, x_{j_{k+2}}]\) having value \(\prod_{i \neq s} p_{j_i}\). In particular the face \([x_{j_1}, \ldots, x_{j_{k+1}}, \hat{x}_{j_{k+2}}]\) has an odd number of simplices with value \(\prod_{i \leq k+1} p_{j_i}\). Now no simplex in any other face of \(\hat{S}\) can have this value by uniqueness of prime factorization. Label the “interior” vertices, those \(u\) having all \(s_i > 0\) in \(u = \sum_{i=1}^{\beta} s_i x_{j_i}\), (These have not yet been labeled,) with any of the \(p_{j_1}, \ldots, p_{j_{k+2}}\). Pick a simplex on the face \([x_{j_1}, \ldots, x_{j_{k+1}}, \hat{x}_{j_{k+2}}]\) which has value \(\prod_{i \leq k+1} p_{j_i}\) and cross this simplex into \(\hat{S}\). Continue crossing simplices having value \(\prod_{i \leq k+1} p_{j_i}\) which have not been crossed till the process ends. It must end because there are an odd number of these simplices having value \(\prod_{i \leq k+1} p_{j_i}\). If the process leads to the outside of \(\hat{S}\), then one can always enter it again because there are an odd number of simplices with value \(\prod_{i \leq k+1} p_{j_i}\) available and you will have used up an even number. When the process ends, the value of the simplex must be \(\prod_{i=1}^{k+2} p_{j_i}\) because it will have the additional label \(p_{j_{k+2}}\) on a vertex since if not, there will be another way out of the simplex. This identifies a simplex in the triangulation with value \(\prod_{i=1}^{k+2} p_{j_i}\). Then repeat the process with \(\prod_{i \leq k+1} p_{j_i}\) valued simplices on \([x_{j_1}, \ldots, x_{j_{k+1}}, \hat{x}_{j_{k+2}}]\) which have not been crossed. Repeating the process, entering from the outside, cannot deliver a \(\prod_{i=1}^{k+2} p_{j_i}\) valued simplex encountered earlier. This is because you cross faces labeled \(\prod_{i \leq k+1} p_{j_i}\). If the remaining vertex is labeled \(p_{j_i}\) where \(i \neq k + 2\), then this yields exactly one other face to cross. There are two, the one with the first vertex \(p_{j_1}\), and the next one with the new vertex labeled \(p_{j_i}\), substituted for the first vertex having this label. Thus there is either one route in to a simplex or two. Thus, starting at a simplex labeled \(\prod_{i \leq k+1} p_{j_i}\) one can cross faces having this value till one is led to the \(\prod_{i \leq k+1} p_{j_i}\) valued simplex on the selected face of \(\hat{S}\). In other words, the process is one to one in selecting a \(\prod_{i \leq k+1} p_{j_i}\) vertex from crossing such a vertex on the selected face of \(\hat{S}\). Continue doing this, crossing a \(\prod_{i \leq k+1} p_{j_i}\) simplex on the face of \(\hat{S}\) which has not been crossed previously. This identifies an odd number of simplices having value \(\prod_{i=1}^{k+2} p_{j_i}\). These are the ones which are “accessible” from the outside using this process. If there are any which are not accessible from outside, applying the same process starting inside one of these, leads to exactly one other inaccessible simplex with value \(\prod_{i=1}^{k+2} p_{j_i}\). Hence these inaccessible simplices occur in pairs and so there are an odd number of simplices in the triangulation having value \(\prod_{i=1}^{k+2} p_{j_i}\).
We refer to this procedure of labeling as Sperner’s lemma. The system of labeling is well defined thanks to the assumption that \( \{x_k - x_0\}_{k=1}^n \) is independent which implies that \( \{x_k - x_j\}_{k \neq j} \) is also linearly independent. Thus there can be no ambiguity in the labeling of vertices on any “face” the convex hull of some of the original vertices of \( S \). The following is a description of the system of labeling the vertices.

**Lemma 8.2.1** Let \( [x_0, \ldots, x_n] \) be an \( n \) simplex with \( \{x_k - x_0\}_{k=1}^n \) independent, and let the first \( n + 1 \) primes be \( p_0, p_1, \ldots, p_n \). Label \( x_k \) as \( p_k \) and consider a triangulation of this simplex. Labeling the vertices of this triangulation which occur on \( [x_{k_1}, \ldots, x_{k_s}] \) with any of \( p_{k_1}, \ldots, p_{k_s} \), beginning with all \( 1 \) simplices \( [x_{k_1}, x_{k_2}] \) and then \( 2 \) simplices and so forth, there are an odd number of simplices \( [y_{k_1}, \ldots, y_{k_s}] \) of the triangulation contained in \( [x_{k_1}, \ldots, x_{k_s}] \) which have value \( p_{k_1}, \ldots, p_{k_s} \). This for \( s = 1, 2, \ldots, n \).

A combinatorial method

We now give a brief discussion of the system of labeling for Sperner’s lemma from the point of view of counting numbers of faces rather than obtaining them with an algorithm. Let \( p_0, \ldots, p_n \) be the first \( n + 1 \) prime numbers. All vertices of a simplex \( S = [x_0, \ldots, x_n] \) having \( \{x_k - x_0\}_{k=1}^n \) independent will be labeled with one of these primes. In particular, the vertex \( x_k \) will be labeled as \( p_k \). The value of a simplex will be the product of its labels. Triangulate this \( S \). Consider a \( 1 \) simplex coming from the original simplex \( [x_{k_1}, x_{k_2}] \), label one end as \( p_{k_1} \) and the other as \( p_{k_2} \). Then label all other vertices of this triangulation which occur on \( [x_{k_1}, x_{k_2}] \) either \( p_{k_1} \) or \( p_{k_2} \). The assumption of linear independence assures that no other vertex of \( S \) can be in \( [x_{k_1}, x_{k_2}] \) so there will be no inconsistency in the labeling. Then obviously there will be an odd number of simplices in this triangulation having value \( p_{k_1}p_{k_2} \), that is a \( p_k \) at one end and a \( p_k \) at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on \( [x_{j_1}, \ldots, x_{j_{k+1}}] \),

\[
\{x_{j_1}, \ldots, x_{j_{k+1}}\} \subseteq \{x_0, \ldots, x_n\}
\]

any \( k \) simplex for \( k \leq n - 1 \), and there is an odd number of simplices from the triangulation having value equal to \( \prod_{j=1}^{k+1} p_{j} \). Consider \( \hat{S} \equiv [x_{j_1}, \ldots, x_{j_{k+1}}, x_{j_{k+2}}] \). Then by induction, there is an odd number of \( k \) simplices on the \( s \)th face

\[
[x_{j_1}, \ldots, x_{j_s}, \ldots, x_{j_{k+1}}]
\]

having value \( \prod_{i \neq s} p_{j_i} \). In particular the face \( [x_{j_1}, \ldots, x_{j_{k+1}}, \hat{x}_{j_{k+2}}] \) has an odd number of simplices with value \( \prod_{i=1}^{k+2} p_{j_i} := \hat{P}_k \). We want to argue that some simplex in the triangulation which is contained in \( \hat{S} \) has value \( \hat{P}_{k+1} := \prod_{i=1}^{k+2} p_{j_i} \). Let \( Q \) be the number of \( k + 1 \) simplices from the triangulation contained in \( \hat{S} \) which have two faces with value \( \hat{P}_k \) (A \( k + 1 \) simplex has either 1 or 2 \( \hat{P}_k \) faces,) and let \( R \) be the number of \( k + 1 \) simplices from the triangulation contained in \( \hat{S} \) which have exactly one \( \hat{P}_k \) face. These are the ones we want because they have value \( \hat{P}_{k+1} \). Thus the number of faces having value \( \hat{P}_k \) which is described here is \( 2Q + R \). All interior
8.3. THE BROUWER FIXED POINT THEOREM

\( \hat{P}_k \) faces being counted twice by this number. Now we count the total number of \( \hat{P}_k \) faces another way. There are \( P \) of them on the face \([x_{j_1}, \ldots, x_{j_{k+1}}, \hat{x}_{j_{k+2}}]\) and by induction, \( P \) is odd. Then there are \( O \) of them which are not on this face. These faces got counted twice. Therefore,

\[ 2Q + R = P + 2O \]

and so, since \( P \) is odd, so is \( R \). Thus there is an odd number of \( \hat{P}_{k+1} \) simplices in \( \hat{S} \).

We refer to this procedure of labeling as Sperner’s lemma. The system of labeling is well defined thanks to the assumption that \( \{x_k - x_0\}_{k=1}^n \) is independent which implies that \( \{x_k - x_i\}_{k \neq i} \) is also linearly independent. Thus there can be no ambiguity in the labeling of vertices on any “face”, the convex hull of some of the original vertices of \( S \). Sperner’s lemma is now a consequence of this discussion.

8.3 The Brouwer Fixed Point Theorem

\( S \equiv [x_0, \ldots, x_n] \) is a simplex in \( \mathbb{R}^n \). Assume \( \{x_i - x_0\}_{i=1}^n \) are linearly independent. Thus a typical point of \( S \) is of the form

\[ \sum_{i=0}^n t_i x_i \]

where the \( t_i \) are uniquely determined and the map \( x \rightarrow t \) is continuous from \( S \) to the compact set \( \{t \in \mathbb{R}^{n+1} : \sum t_i = 1, t_i \geq 0\} \).

To see this, suppose \( x^k \rightarrow x \) in \( S \). Let \( x^k = \sum_{i=0}^n t_i^k x_i \) with \( x \) defined similarly with \( t_i^k \) replaced with \( t_i \), \( x \equiv \sum_{i=0}^n t_i x_i \). Then

\[ x^k - x_0 = \sum_{i=0}^n t_i^k x_i - \sum_{i=0}^n t_i^k x_0 = \sum_{i=1}^n t_i^k (x_i - x_0) \]

Thus

\[ x^k - x_0 = \sum_{i=1}^n t_i^k (x_i - x_0) , \ x - x_0 = \sum_{i=1}^n t_i (x_i - x_0) \]

Say \( t_i^k \) fails to converge to \( t_i \) for all \( i \geq 1 \). Then there exists a subsequence, still denoted with superscript \( k \) such that for each \( i = 1, \ldots, n \), it follows that \( t_i^k \rightarrow s_i \), where \( s_i \geq 0 \) and some \( s_i \neq t_i \). But then, taking a limit, it follows that

\[ x - x_0 = \sum_{i=1}^n s_i (x_i - x_0) = \sum_{i=1}^n t_i (x_i - x_0) \]

which contradicts independence of the \( x_i - x_0 \). It follows that for all \( i \geq 1, t_i^k \rightarrow t_i \). Since they all sum to 1, this implies that also \( t_0^k \rightarrow t_0 \). Thus the claim about continuity is verified.
Let \( f : S \to S \) be continuous. When doing \( f \) to a point \( x \), one obtains another point of \( S \) denoted as \( \sum_{i=0}^{n} s_i x_i \).

Label \( x_j \) as \( p_j \) where \( p_0, \ldots, p_n \) are the first \( n+1 \) prime numbers. Thus the vertices of \( S \) have been labeled. Next triangulate \( S \) so that all simplices have diameter less than \( \varepsilon \). If \( [y_0, \ldots, y_n] \) is one of these small vertices, each is of the form \( \sum_{i=0}^{n} t_i x_i \) where \( t_i \geq 0 \) and \( \sum_{i} t_i = 1 \). Define \( r_j = s_j/t_j \) if \( t_j > 0 \) and \( \infty \) if \( t_j = 0 \). Then \( p(y_i) \) will be the label placed on \( y_i \). It equals \( p_k \) where \( r_k \) is the smallest of all these extended positive real numbers just described. If there is duplication, pick \( p_k \) where \( k \) is smallest.

Note that this method of labeling does not contradict the original labels placed on the vertices \( x_i \). This is because for \( x_i, t_i = 1 \) and all other \( t_j = 0 \) so the only ratio that is finite will be \( s_i/t_i \). All others are \( \infty \) by definition. As for the vertices which are on the \( k^{th} \) face \( [x_0, \ldots, \hat{x_k}, \ldots, x_n] \), these will be labeled from the list \( \{p_0, \ldots, p_n \} \) because \( t_k = 0 \) for each of these and so \( r_k = \infty \).

By the Sperner’s lemma procedure described above, there are an odd number of simplices having value \( \prod_{i \neq k} p_i \) on the \( k^{th} \) face and an odd number of simplices in the triangulation of \( S \) for which the product of the labels on their vertices equals \( p_0 p_1 \cdot \ldots \cdot p_n \equiv P_n \). We call this the value of the simplex. Thus if \( [y_0, \ldots, y_n] \) is one of these simplices, and \( p(y_i) \) is the label for \( y_i \),

\[
\prod_{i=0}^{n} p(y_i) = \prod_{i=0}^{n} p_i \equiv P_n
\]

What is \( r_k \)? Could it be larger than \( 1 \)? \( r_k \) is certainly finite because at least some \( t_j \neq 0 \) since they sum to \( 1 \). Thus, if \( r_k > 1 \), you would have \( s_k > t_k \). The \( s_j \) sum to \( 1 \) and so some \( s_j < t_j \) since otherwise, the sum of the \( t_j \) equalling \( 1 \) would require the sum of the \( s_j \) to be larger than \( 1 \). Hence \( r_k \) was not really the smallest after all and so \( r_k \leq 1 \). Hence \( s_k \leq t_k \).

Let \( S = \{S_1, \ldots, S_m\} \) denote those simplices whose value is \( P_n \). In other words, if \( \{y_0, \ldots, y_n\} \) are the vertices of one of these simplices, and

\[
y_s = \sum_{i=0}^{n} t^*_i x_i
\]

\( r_k \leq r_j \) for all \( j \neq k \) and \( \{k_0, \ldots, k_n\} = \{0, \ldots, n\} \). Let \( b \) denote the barycenter of \( S_k \).

Now do this for a sequence \( \varepsilon_k \to 0 \) where \( b_k \) is a barycenter as above for a simplex having value \( P_n \). Thus it is the barycenter of a simplex having diameter less than \( \varepsilon_k \) and value \( P_n \). By compactness, there is a subsequence still denoted with the index \( k \) such that \( b_k \to x \). This \( x \) is a fixed point.

Consider this last claim. \( x = \sum_{i=0}^{n} t_i x_i \) and after applying \( f \), the result is

\[
x = \cap_{k=1}^{\infty} \sigma_k
\]

where \( \sigma_k \) is a simplex having vertices \( \{y^k_0, \ldots, y^k_n\} \) and the value of \( [y^k_0, \ldots, y^k_n] \) is \( P_n \). Re ordering these if necessary, we can assume that the label for \( y^k_i = p_i \) which
implies that, as noted above,
\[ \frac{s_i}{t_i} \leq 1, \ s_i \leq t_i \]
the \(i\)th coordinate of \(f(y^k_i)\) with respect to the original vertices of \(S\) decreases and each \(i\) is represented for \(i = \{0, 1, \ldots, n\}\). Thus from the diameters of these \(\sigma_k\) converging to 0,
\[ y^k_i \to x \]
and so the \(i\)th coordinate of \(y^k_i, t^k_i\) must converge to \(t_i\). Hence if the \(i\)th coordinate of \(f(y^k_i)\) is denoted by \(s^k_i\),
\[ s^k_i \leq t^k_i \]
By continuity of \(f\), it follows that \(s^k_i \to s_i\). Thus the above inequality is preserved on taking \(k \to \infty\) and so
\[ 0 \leq s_i \leq t_i \]
this for each \(i\). But these \(s_i\) add to 1 as do the \(t_i\) and so in fact, \(s_i = t_i\) for each \(i\) and so \(f(x) = x\). This proves the following theorem which is the Brouwer fixed point theorem.

**Theorem 8.3.1** Let \(S\) be a simplex \([x_0, \ldots, x_n]\) such that \(\{x_i - x_0\}_{i=1}^n\) are independent. Also let \(f : S \to S\) be continuous. Then there exists \(x \in S\) such that \(f(x) = x\).

**Corollary 8.3.2** Let \(K\) be a closed convex bounded subset of \(\mathbb{R}^n\). Let \(f : K \to K\) be continuous. Then there exists \(x \in K\) such that \(f(x) = x\).

**Proof:** Let \(S\) be a large simplex containing \(K\) and let \(P\) be the projection map onto \(K\). Consider \(g(x) \equiv f(Px)\). Then \(g\) satisfies the necessary conditions for Theorem 8.3.1 and so there exists \(x \in S\) such that \(g(x) = x\). But this says \(x \in K\) and so \(g(x) = f(x)\). \(\blacksquare\)

## 8.4 Invariance Of Domain

As an application of the inverse function theorem Weierstrass approximation theorem and so forth is a simple proof of the important invariance of domain theorem.

**Lemma 8.4.1** Let \(f\) be continuous and map \(\overline{B(p, r)} \subseteq \mathbb{R}^n\) to \(\mathbb{R}^n\). Suppose that for all \(x \in \overline{B(p, r)}\),
\[ |f(x) - x| < \varepsilon r \]
Then it follows that
\[ f(\overline{B(p, r)}) \supseteq B(p, (1 - \varepsilon) r) \]
CHAPTER 8. BROUWER FIXED POINT THEOREM \(\mathbb{R}^{N^*}\)

**Proof:** This is from the Brouwer fixed point theorem, Corollary 8.3.2. Consider for \(y \in B(p, (1 - \varepsilon) r)\)

\[
h(x) \equiv x - f(x) + y
\]

Then \(h\) is continuous and for \(x \in B(p, r)\),

\[
|h(x) - p| = |x - f(x) + y - p| < \varepsilon r + |y - p| < \varepsilon r + (1 - \varepsilon) r = r
\]

Hence \(h : B(p, r) \to B(p, r)\) and so it has a fixed point \(x\) by Corollary 8.3.2. Thus

\[
x - f(x) + y = x
\]

so \(f(x) = y\). \(\blacksquare\)

The notation \(\|f\|_K\) means \(\sup_{x \in K} |f(x)|\).

**Lemma 8.4.2** Let \(K\) be a compact set in \(\mathbb{R}^n\) and let \(g : K \to \mathbb{R}^n, z \in K\). Let \(\delta > 0\). Then there exists a polynomial \(q\) (each component a polynomial) such that

\[
\|q - g\|_K < \delta, \quad q(z) = g(z), \quad Dq(z)^{-1} \text{ exists}
\]

**Proof:** By the Weierstrass approximation theorem, there exists a polynomial \(\hat{q}\) such that

\[
\|\hat{q} - g\|_K < \frac{\delta}{3}
\]

Then define for \(y \in K\)

\[
q(y) \equiv \hat{q}(y) + g(z) - \hat{q}(z)
\]

Then

\[
q(z) = \hat{q}(z) + g(z) - \hat{q}(z) = g(z)
\]

Also

\[
|q(y) - g(y)| \leq |(\hat{q}(y) + g(z) - \hat{q}(z)) - g(y)|
\]

\[
\leq |\hat{q}(y) - g(y)| + |g(z) - \hat{q}(z)| < \frac{2\delta}{3}
\]

and so since \(y\) was arbitrary,

\[
\|q - g\|_K \leq \frac{2\delta}{3} < \delta
\]

If \(Dq(z)^{-1}\) exists, then this is what is wanted. If not, let

\[
0 < \eta < \{||\lambda| : \lambda \text{ is an eigenvalue of } Dq(z), \lambda \neq 0\}
\]

Then if \(\eta\) is small enough, \(q(y)\) could be replaced with \(q(y) + \eta(y - z)\) and the above inequality would be preserved along with \(q(z) = g(z)\) but now \(Dq(z)\) would have no zero eigenvalues and would therefore be invertible. Simply use the modified \(q\). \(\blacksquare\)
8.4. INVARIANCE OF DOMAIN

Lemma 8.4.3 Let \( f : B(p, r) \to \mathbb{R}^n \) where the ball is also in \( \mathbb{R}^n \). Let \( f \) be one to one. Then there exists \( \delta > 0 \) such that

\[
f \left( B(p, r) \right) \supseteq B(f(p), \delta).
\]

Proof: Since \( f \left( B(p, r) \right) \) is compact, it follows that \( f^{-1} : f \left( B(p, r) \right) \to B(p, r) \) is continuous. By Lemma 8.4.2, there exists a polynomial \( q : f \left( B(p, r) \right) \to \mathbb{R}^n \) such that

\[
\|q - f^{-1}\|_{f(B(p, r))} < \varepsilon r, \quad \varepsilon < 1, \quad Dq(f(p))^{-1} \text{ exists, and } q(f(p)) = p.
\]

From the first inequality in the above,

\[
|q(f(x)) - x| = |q(f(x)) - f^{-1}(f(x))| \leq \|q - f^{-1}\|_{f(B(p, r))} < \varepsilon r
\]

By Lemma 8.4.3,

\[
q \circ f \left( B(p, r) \right) \supseteq B(p, (1 - \varepsilon)r)
\]

By the Inverse function theorem, there is an open set containing \( f(p) \) denoted as \( W \) such that on \( W \), \( g \) is one to one and it and its inverse, defined on an open set \( V = g(W) \) both map open sets to open sets. By the construction, \( p \in V \) and so if \( \eta \) is small enough, it follows that \( B(p, \eta) \subseteq B(p, (1 - \varepsilon)r) \cap V \). Thus

\[
q \circ f \left( B(p, r) \right) \supseteq B(p, \eta)
\]

and \( q, q^{-1} \) both map open sets to open sets. Thus \( q^{-1}(B(p, \eta)) \) is an open set containing \( f(p) \). Hence if \( \delta \) is small enough,

\[
f \left( B(p, r) \right) \supseteq B(f(p), \delta) \quad \square
\]

With this lemma, the invariance of domain theorem comes right away. This remarkable theorem states that if \( f : U \to \mathbb{R}^n \) for \( U \) an open set in \( \mathbb{R}^n \) and if \( f \) is one to one, then \( f(U) \) is also an open set in \( \mathbb{R}^n \).

Theorem 8.4.4 Let \( U \) be an open set in \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R}^n \) be one to one and continuous. Then \( f(U) \) is also an open subset in \( \mathbb{R}^n \).

Proof: It suffices to show that if \( p \in U \) then \( f(p) \) is an interior point of \( f(U) \). Let \( B(p, r) \subseteq U \). By Lemma 8.4.3, \( f(U) \supseteq f \left( B(p, r) \right) \supseteq B(f(p), \delta) \) so \( f(p) \) is indeed an interior point of \( f(U) \). \( \square \)
Part II

Real And Abstract Analysis
Chapter 9

Abstract Measure And Integration

9.1  σ Algebras

This chapter is on the basics of measure theory and integration. A measure is a real valued mapping from some subset of the power set of a given set which has values in $[0, \infty]$. Many apparently different things can be considered as measures and also there is an integral defined. By discussing this in terms of axioms and in a very abstract setting, many different topics can be considered in terms of one general theory. For example, it will turn out that sums are included as an integral of this sort. So is the usual integral as well as things which are often thought of as being in between sums and integrals.

Let $\Omega$ be a set and let $\mathcal{F}$ be a collection of subsets of $\Omega$ satisfying

$$\emptyset \in \mathcal{F}, \ \Omega \in \mathcal{F},$$

$$E \in \mathcal{F} \text{ implies } E^C \equiv \Omega \setminus E \in \mathcal{F},$$

If $\{E_n\}_n \subseteq \mathcal{F}$, then $\bigcup_{n=1}^\infty E_n \in \mathcal{F}$. (9.1.2)

**Definition 9.1.1** A collection of subsets of a set, $\Omega$, satisfying Formulas 9.1.1-9.1.2 is called a σ algebra.

As an example, let $\Omega$ be any set and let $\mathcal{F} = \mathcal{P}(\Omega)$, the set of all subsets of $\Omega$ (power set). This obviously satisfies Formulas 9.1.1-9.1.2.

**Lemma 9.1.2** Let $\mathcal{C}$ be a set whose elements are σ algebras of subsets of $\Omega$. Then $\bigcap \mathcal{C}$ is a σ algebra also.

Be sure to verify this lemma. It follows immediately from the above definitions but it is important for you to check the details.
Example 9.1.3 Let \( \tau \) denote the collection of all open sets in \( \mathbb{R}^n \) and let \( \sigma (\tau) \equiv \) intersection of all \( \sigma \) algebras that contain \( \tau \). \( \sigma (\tau) \) is called the \( \sigma \) algebra of Borel sets. In general, for a collection of sets, \( \Sigma \), \( \sigma (\Sigma) \) is the smallest \( \sigma \) algebra which contains \( \Sigma \).

This is a very important \( \sigma \) algebra and it will be referred to frequently as the Borel sets. Attempts to describe a typical Borel set are more trouble than they are worth and it is not easy to do so. Rather, one uses the definition just given in the example. Note, however, that all countable intersections of open sets and countable unions of closed sets are Borel sets. Such sets are called \( G_\delta \) and \( F_\sigma \) respectively.

Definition 9.1.4 Let \( \mathcal{F} \) be a \( \sigma \) algebra of sets of \( \Omega \) and let \( \mu : \mathcal{F} \to [0, \infty] \). \( \mu \) is called a measure if
\[
\mu (\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu (E_i)
\]
whenever the \( E_i \) are disjoint sets of \( \mathcal{F} \). The triple, \( (\Omega, \mathcal{F}, \mu) \) is called a measure space and the elements of \( \mathcal{F} \) are called the measurable sets. \( (\Omega, \mathcal{F}, \mu) \) is a finite measure space when \( \mu (\Omega) < \infty \).

Note that the above definition immediately implies that if \( E_i \in \mathcal{F} \) and the sets \( E_i \) are not necessarily disjoint,
\[
\mu (\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu (E_i).
\]

To see this, let \( F_1 \equiv E_1, F_2 \equiv E_2 \setminus E_1, \ldots, F_n \equiv E_n \setminus \bigcup_{i=1}^{n-1} E_i, \) then the sets \( F_i \) are disjoint sets in \( \mathcal{F} \) and
\[
\mu (\bigcup_{i=1}^{\infty} E_i) = \mu (\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu (F_i) \leq \sum_{i=1}^{\infty} \mu (E_i)
\]
because of the fact that each \( E_i \supseteq F_i \) and so
\[
\mu (E_i) = \mu (F_i) + \mu (E_i \setminus F_i)
\]
which implies \( \mu (E_i) \geq \mu (F_i) \).

The following theorem is the basis for most of what is done in the theory of measure and integration. It is a very simple result which follows directly from the above definition.

Theorem 9.1.5 Let \( \{E_m\}_{m=1}^{\infty} \) be a sequence of measurable sets in a measure space \( (\Omega, \mathcal{F}, \mu) \). Then if \( \cdots E_n \subseteq E_{n+1} \subseteq E_{n+2} \subseteq \cdots \),
\[
\mu (\bigcup_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu (E_n)
\]
and if \( \cdots E_n \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \cdots \) and \( \mu(E_1) < \infty \), then

\[
\mu(\cap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_n). \tag{9.1.5}
\]

Stated more succinctly, \( E_k \uparrow E \) implies \( \mu(E_k) \uparrow \mu(E) \) and \( E_k \downarrow E \) with \( \mu(E_1) < \infty \) implies \( \mu(E_k) \downarrow \mu(E) \).

**Proof:** First note that \( \cap_{i=1}^{\infty} E_i = (\cup_{i=1}^{\infty} E_i^C)^C \in \mathcal{F} \) so \( \cap_{i=1}^{\infty} E_i \) is measurable. Also note that for \( A \) and \( B \) sets of \( \mathcal{F} \), \( A \setminus B \equiv (A^C \cup B)^C \in \mathcal{F} \). To show 9.1.4, note that 9.1.4 is obviously true if \( \mu(E_k) = \infty \) for any \( k \). Therefore, assume \( \mu(E_k) < \infty \) for all \( k \). Thus

\[
\mu(E_{k+1} \setminus E_k) + \mu(E_k) = \mu(E_{k+1})
\]

and so

\[
\mu(E_{k+1} \setminus E_k) = \mu(E_{k+1}) - \mu(E_k).
\]

Also,

\[
\bigcup_{k=1}^{\infty} E_k = E_1 \cup \bigcup_{k=1}^{\infty} (E_{k+1} \setminus E_k)
\]

and the sets in the above union are disjoint. Hence by 9.1.3,

\[
\mu(\cup_{i=1}^{\infty} E_i) = \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} \setminus E_k) = \mu(E_1)
\]

\[
+ \sum_{k=1}^{\infty} \mu(E_{k+1}) - \mu(E_k)
\]

\[
= \mu(E_1) + \lim_{n \to \infty} \sum_{k=1}^{n} \mu(E_{k+1}) - \mu(E_k) = \lim_{n \to \infty} \mu(E_{n+1}).
\]

This shows part 9.1.4.

To verify 9.1.5,

\[
\mu(E_1) = \mu(\cap_{i=1}^{\infty} E_i) + \mu(E_1 \setminus \cap_{i=1}^{\infty} E_i)
\]

since \( \mu(E_1) < \infty \), it follows \( \mu(\cap_{i=1}^{\infty} E_i) < \infty \). Also, \( E_1 \setminus \cap_{i=1}^{\infty} E_i \uparrow E_1 \setminus \cap_{i=1}^{\infty} E_i \) and so by 9.1.4,

\[
\mu(E_1) - \mu(\cap_{i=1}^{\infty} E_i) = \mu(E_1 \setminus \cap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_1 \setminus \cap_{i=1}^{n} E_i)
\]

\[
= \mu(E_1) - \lim_{n \to \infty} \mu(\cap_{i=1}^{n} E_i) = \mu(E_1) - \lim_{n \to \infty} \mu(E_n),
\]

Hence, subtracting \( \mu(E_1) \) from both sides,

\[
\lim_{n \to \infty} \mu(E_n) = \mu(\cap_{i=1}^{\infty} E_i).
\]

This proves the theorem.
It is convenient to allow functions to take the value \( +\infty \). You should think of \( +\infty \), usually referred to as \( \infty \) as something out at the right end of the real line and its only importance is the notion of sequences converging to it. \( x_n \to \infty \) exactly when for all \( l \in \mathbb{R} \), there exists \( N \) such that if \( n \geq N \), then

\[
x_n > l.
\]

This is what it means for a sequence to converge to \( \infty \). Don’t think of \( \infty \) as a number. It is just a convenient symbol which allows the consideration of some limit operations more simply. Similar considerations apply to \( -\infty \) but this value is not of very great interest. In fact the set of most interest is the complex numbers or some vector space. Therefore, this topic is not considered.

**Lemma 9.1.6** Let \( f : \Omega \to (-\infty, \infty] \) where \( F \) is a \( \sigma \) algebra of subsets of \( \Omega \). Then the following are equivalent.

\[
\begin{align*}
&f^{-1}((d, \infty]) \in F \text{ for all finite } d, \\
&f^{-1}((-\infty, d)) \in F \text{ for all finite } d, \\
&f^{-1}([d, \infty]) \in F \text{ for all finite } d, \\
&f^{-1}((-\infty, d]) \in F \text{ for all finite } d, \\
&f^{-1}((a, b)) \in F \text{ for all } a < b, -\infty < a < b < \infty.
\end{align*}
\]

**Proof:** First note that the first and the third are equivalent. To see this, observe

\[
f^{-1}([d, \infty]) = \cap_{n=1}^{\infty} f^{-1}((d - 1/n, \infty]),
\]

and so if the first condition holds, then so does the third.

\[
f^{-1}((d, \infty]) = \cup_{n=1}^{\infty} f^{-1}([d + 1/n, \infty]),
\]

and so if the third condition holds, so does the first.

Similarly, the second and fourth conditions are equivalent. Now

\[
f^{-1}((-\infty, d]) = (f^{-1}((d, \infty]))^{C}
\]

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for \( -\infty < a < b < \infty \),

\[
f^{-1}((a, b)) = f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty]) \in F.
\]

Finally, if the last condition holds,

\[
f^{-1}([d, \infty]) = (\cup_{k=1}^{\infty} f^{-1}((-k + d, d)))^{C} \in F
\]

and so the third condition holds. Therefore, all five conditions are equivalent. This proves the lemma.

This lemma allows for the following definition of a measurable function having values in \( (-\infty, \infty] \).
Definition 9.1.7 Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and let \(f : \Omega \to (\mathbb{R}, \mathcal{B})\). Then \(f\) is said to be measurable if any of the equivalent conditions of Lemma 9.1.6 hold.

When the \(\sigma\) algebra, \(\mathcal{F}\) equals the Borel \(\sigma\) algebra, \(\mathcal{B}\), the function is called Borel measurable. More generally, if \(f : \Omega \to X\) where \(X\) is a topological space, \(f\) is said to be measurable if \(f^{-1}(U) \in \mathcal{F}\) whenever \(U\) is open.

Theorem 9.1.8 Let \(f_n\) and \(f\) be functions mapping \(\Omega\) to \((-\infty, \infty]\) where \(\mathcal{F}\) is a \(\sigma\) algebra of measurable sets of \(\Omega\). Then if \(f_n\) is measurable, and \(f(\omega) = \lim_{n \to \infty} f_n(\omega)\), it follows that \(f\) is also measurable. (Pointwise limits of measurable functions are measurable.)

Proof: First it is shown \(f^{-1}((a, b)) \in \mathcal{F}\). Let \(V_m = (a + \frac{1}{m}, b - \frac{1}{m})\) and \(\overline{V}_m = [a + \frac{1}{m}, b - \frac{1}{m}]\). Then for all \(m\), \(V_m \subseteq (a, b)\) and \((a, b) = \bigcup_{m=1}^{\infty} V_m = \bigcup_{m=1}^{\infty} \overline{V}_m\).

Note that \(V_m \neq \emptyset\) for all \(m\) large enough. Since \(f\) is the pointwise limit of \(f_n\), \(f^{-1}(V_m) \subseteq \{\omega : f_k(\omega) \in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\overline{V}_m)\).

You should note that the expression in the middle is of the form \(\bigcup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m)\).

Therefore,
\[
f^{-1}((a, b)) = \bigcup_{m=1}^{\infty} f^{-1}(V_m) \subseteq \bigcup_{m=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m) \subseteq \bigcup_{m=1}^{\infty} f^{-1}(\overline{V}_m) = f^{-1}((a, b)).
\]

It follows \(f^{-1}((a, b)) \in \mathcal{F}\) because it equals the expression in the middle which is measurable. This shows \(f\) is measurable.

The following theorem considers the case of functions which have values in a metric space. Its proof is similar to the proof of the above.

Theorem 9.1.9 Let \(\{f_n\}\) be a sequence of measurable functions mapping \(\Omega\) to \((X, d)\) where \((X, d)\) is a metric space and \((\Omega, \mathcal{F})\) is a measure space. Suppose also that \(f(\omega) = \lim_{n \to \infty} f_n(\omega)\) for all \(\omega\). Then \(f\) is also a measurable function.

Proof: It is required to show \(f^{-1}(U)\) is measurable for all \(U\) open. Let \(V_m = \{x \in U : \text{dist}(x, U^c) > \frac{1}{m}\}\).

Thus \(V_m \subseteq \{x \in U : \text{dist}(x, U^c) \geq \frac{1}{m}\}\) and \(V_m \subseteq \overline{V}_m \subseteq V_{m+1}\) and \(\cup_m V_m = U\). Then since \(V_m\) is open,
\[
f^{-1}(V_m) = \bigcup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m)
\]
and so
\[
    f^{-1}(U) = \bigcup_{m=1}^{\infty} f^{-1}(V_m) \\
    = \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} f_k^{-1}(V_m) \\
    \subseteq \bigcup_{m=1}^{\infty} f^{-1}(V_m) = f^{-1}(U)
\]
which shows \( f^{-1}(U) \) is measurable. This proves the theorem.

Now here is a simple observation.

**Observation 9.1.10** Let \( f : \Omega \to X \) where \( X \) is some topological space. Suppose
\[
    f(\omega) = \sum_{k=1}^{m} x_k \mathcal{A}_k(\omega)
\]
where each \( x_k \in X \) and the \( A_k \) are disjoint measurable sets. (Such functions are often referred to as simple functions.) The sum means the function has value \( x_k \) on set \( A_k \). Then \( f \) is measurable.

**Proof:** Letting \( U \) be open, \( f^{-1}(U) = \bigcup \{ A_k : x_k \in U \} \), a finite union of measurable sets.

There is also a very interesting theorem due to Kuratowski \([\mathbb{2}]\) which is presented next.

To summarize the proof, you get an increasing sequence of \( 2^{-n} \) nets \( C_n \) and you obtain a corresponding sequence of simple functions \( \{ s_n \} \) such that \( \{ s_n(\omega) \}_{n=1}^{\infty} \) is a Cauchy sequence, and the maximum value of \( x \to \psi(x,\omega) \) for \( x \in C_n \) equals \( \psi(s_n(\omega),\omega) \). Then you let \( f(\omega) = \lim_{n \to \infty} s_n(\omega) \). Thus \( f(\omega) \) is measurable and
\[
    \sup_{x \in E} \psi(x,\omega) \geq \psi(f(\omega),\omega) = \lim_{n \to \infty} \psi(s_n(\omega),\omega) \geq \sup_{x \in C_n} \psi(x,\omega)
\]
Thus, by continuity in the first entry,
\[
    \sup_{x \in E} \psi(x,\omega) \geq \psi(f(\omega),\omega) \geq \lim_{n \to \infty} \psi(s_n(\omega),\omega) \geq \sup_{x \in \bigcup \mathcal{C}_n} \psi(x,\omega) = \sup_{x \in E} \psi(x,\omega)
\]

**Theorem 9.1.11** Let \( E \) be a compact metric space and let \((\Omega,\mathcal{F})\) be a measure space. Suppose \( \psi : E \times \Omega \to \mathbb{R} \) has the property that \( x \to \psi(x,\omega) \) is continuous and \( \omega \to \psi(x,\omega) \) is measurable. Then there exists a measurable function, \( f \) having values in \( E \) such that
\[
    \psi(f(\omega),\omega) = \sup_{x \in E} \psi(x,\omega).
\]
Furthermore, \( \omega \to \psi(f(\omega),\omega) \) is measurable.

**Proof:** Let \( C_1 \) be a \( 2^{-1} \) net of \( E \). Suppose \( C_1, \cdots, C_m \) have been chosen such that \( C_k \) is a \( 2^{-k} \) net and \( C_{i+1} \supseteq C_i \) for all \( i \). Then consider \( E \setminus \bigcup \{ B(x, 2^{-(m+1)}) : x \in C_m \} \).

If this set is empty, let \( C_{m+1} = C_m \). If it is nonempty, let \( \{ y_i \}_{i=1}^{r} \) be a \( 2^{-(m+1)} \) net
for this compact set. Then let $C_{m+1} = C_m \cup \{y_i\}_{i=1}^{r}$. It follows $\{C_m\}_{m=1}^\infty$ satisfies $C_m$ is a $2^{-m}$ net and $C_m \subseteq C_{m+1}$.

Let $\{x_k\}_{k=1}^{m(1)}$ equal $C_1$. Let

$$A_1^1 = \left\{ \omega : \psi (x_1^1, \omega) = \max_k \psi (x_k^1, \omega) \right\}$$

For $\omega \in A_1^1$, define $s_1(\omega) \equiv x_1^1$. Next let

$$A_1^1 = \left\{ \omega \notin A_1^1 : \psi (x_2^1, \omega) = \max_k \psi (x_k^1, \omega) \right\}$$

and let $s_1(\omega) \equiv x_2^1$ on $A_1^1$. Continue in this way to obtain a simple function, $s_1$ such that

$$\psi (s_1(\omega), \omega) = \max \left\{ \psi (x, \omega) : x \in C_1 \right\}$$

and $s_1$ has values in $C_1$.

Suppose $s_1(\omega), s_2(\omega), \ldots, s_m(\omega)$ are simple functions with the property that if $m > 1$,

$$d(s_k(\omega), s_{k+1}(\omega)) < 2^{-k},$$

$$\psi (s_k(\omega), \omega) = \max \left\{ \psi (x, \omega) : x \in C_k \right\}$$

$s_k$ has values in $C_k$

for each $k + 1 \leq m$, only the second and third assertions holding if $m = 1$. Letting $C_m = \{x_k\}_{k=1}^{N}$, it follows $s_m(\omega)$ is of the form

$$s_m(\omega) = \sum_{k=1}^{N} x_k \chi_{A_k}(\omega), \ A_i \cap A_j = \emptyset.$$  \hspace{1cm} (9.1.6)

meaning that $s_m(\omega)$ has value $x_k$ on $A_k$. Denote by $\{y_{1i}\}_{i=1}^{n_1}$ those points of $C_{m+1}$ which are contained in $B(x_1, 2^{-m})$. Letting $A_k$ play the role of $\Omega$ in the first step in which $s_1$ was constructed, for each $\omega \in A_1$ let $s_{m+1}(\omega)$ be a simple function which has one of the values $y_{1i}$ and satisfies

$$\psi (s_{m+1}(\omega), \omega) = \max_{i \leq n_1} \psi (y_{1i}, \omega)$$

for each $\omega \in A_1$. Next let $\{y_{2i}\}_{i=1}^{n_2}$ be those points of $C_{m+1}$ different than $\{y_{1i}\}_{i=1}^{n_1}$ which are contained in $B(x_2, 2^{-m})$. Then define $s_{m+1}(\omega)$ on $A_2$ to have values taken from $\{y_{2i}\}_{i=1}^{n_2}$ and

$$\psi (s_{m+1}(\omega), \omega) = \max_{i \leq n_2} \psi (y_{2i}, \omega)$$

for each $\omega \in A_2$. Continuing this way defines $s_{m+1}$ on all of $\Omega$ and it satisfies

$$d(s_m(\omega), s_{m+1}(\omega)) < 2^{-m} \text{ for all } \omega \in \Omega \hspace{1cm} (9.1.7)$$
It remains to verify
\[ \psi(s_{m+1}(\omega), \omega) = \max \{ \psi(x, \omega) : x \in C_{m+1} \}. \] (9.1.8)
To see this is so, pick \( \omega \in \Omega \). Let
\[ \max \{ \psi(x, \omega) : x \in C_{m+1} \} = \psi(y_{rj}, \omega) \] (9.1.9)
where \( y_{rj} \in \{y_{ri}\}_{i=1}^n \subseteq C_{m+1} \) and out of all the balls \( B(x_l, 2^{-m}) \), let the first one which contains \( y_{rj} \) be \( B(x_k, 2^{-m}) \). Then by the construction, \( s_{m+1}(\omega) = y_{rj} \) because \( \psi(y_{rj}, \omega) \) is at least as large as \( \psi(y_{sj}, \omega) \) for all the other \( y_{sj} \). This and (9.1.9) verifies (9.1.8).

From (9.1.7) it follows \( s_m(\omega) \) converges uniformly on \( \Omega \) to a measurable function, \( f(\omega) \). Then from the construction, \( \psi(f(\omega), \omega) \geq \psi(s_m(\omega), \omega) \) for all \( m \) and \( \omega \). Now pick \( \omega \in \Omega \) and let \( z \) be such that \( \psi(z, \omega) = \max_{x \in E} \psi(x, \omega) \). Letting \( y_k \to z \) where \( y_k \in C_k \), it follows from continuity of \( \psi \) in the first argument that
\[ \max_{x \in E} \psi(x, \omega) = \psi(z, \omega) = \lim_{k \to \infty} \psi(y_k, \omega) \leq \lim_{m \to \infty} \psi(s_m(\omega), \omega) = \psi(f(\omega), \omega) \leq \max_{x \in E} \psi(x, \omega). \]

To show \( \omega \to \psi(f(\omega), \omega) \) is measurable, note that since \( E \) is compact, there exists a countable dense subset, \( D \). Then using continuity of \( \psi \) in the first argument,
\[ \psi(f(\omega), \omega) = \sup_{x \in E} \psi(x, \omega) = \sup_{x \in D} \psi(x, \omega) \]
which equals a measurable function of \( \omega \) because \( D \) is countable. 

**Theorem 9.1.12** Let \( B \) consist of open cubes of the form
\[ Q_x = \prod_{i=1}^n (x_i - \delta, x_i + \delta) \]
where \( \delta \) is a positive rational number and \( x \in \mathbb{Q}^n \). Then every open set in \( \mathbb{R}^n \) can be written as a countable union of open cubes from \( B \). Furthermore, \( B \) is a countable set. 

**Proof:** Let \( U \) be an open set and let \( y \in U \). Since \( U \) is open, \( B(y, r) \subseteq U \) for some \( r > 0 \) and it can be assumed \( r / \sqrt{n} \in \mathbb{Q} \). Let
\[ x \in B \left( y, \frac{r}{10\sqrt{n}} \right) \cap \mathbb{Q}^n \]
and consider the cube, \( Q_x \in B \) defined by
\[ Q_x = \prod_{i=1}^n (x_i - \delta, x_i + \delta) \]
where \( \delta = r/4\sqrt{n} \). The following picture is roughly illustrative of what is taking place.

Then the diameter of \( Q_x \) equals

\[
\left( n \left( \frac{r}{2\sqrt{n}} \right)^2 \right)^{1/2} = \frac{r}{2}
\]

and so, if \( z \in Q_x \), then

\[
|z - y| \leq |z - x| + |x - y| < \frac{r}{2} + \frac{r}{2} = r.
\]

Consequently, \( Q_x \subseteq U \). Now also,

\[
\left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} < \frac{r}{10\sqrt{n}}
\]

and so it follows that for each \( i \),

\[
|x_i - y| < \frac{r}{4\sqrt{n}}
\]

since otherwise the above inequality would not hold. Therefore, \( y \in Q_x \subseteq U \). Now let \( B_U \) denote those sets of \( B \) which are contained in \( U \). Then \( \cup B_U = U \).

To see \( B \) is countable, note there are countably many choices for \( x \) and countably many choices for \( \delta \). This proves the theorem.

Recall that \( g : \mathbb{R}^n \to \mathbb{R} \) is continuous means \( g^{-1}(\text{open set}) = \text{an open set} \). In particular \( g^{-1}((a,b)) \) must be an open set.

**Theorem 9.1.13** Let \( f_i : \Omega \to \mathbb{R} \) for \( i = 1, \cdots, n \) be measurable functions and let \( g : \mathbb{R}^n \to \mathbb{R} \) be continuous where \( f \equiv (f_1 \cdots f_n)^T \). Then \( g \circ f \) is a measurable function from \( \Omega \) to \( \mathbb{R} \).
**Proof:** First it is shown $$(g \circ f)^{-1}((a, b)) \in \mathcal{F}.$$ Now $(g \circ f)^{-1}((a, b)) = f^{-1}(g^{-1}((a, b)))$ and since $g$ is continuous, it follows that $g^{-1}((a, b))$ is an open set which is denoted as $U$ for convenience. Now by Theorem above, it follows there are countably many open cubes, $\{Q_k\}$ such that $$U = \bigcup_{k=1}^{\infty} Q_k$$ where each $Q_k$ is a cube of the form $$Q_k = \prod_{i=1}^{n} (x_i - \delta, x_i + \delta).$$

Now $$f^{-1}\left(\prod_{i=1}^{n} (x_i - \delta, x_i + \delta)\right) = \bigcap_{i=1}^{n} f_i^{-1}\left((x_i - \delta, x_i + \delta)\right) \in \mathcal{F}$$ and so $$(g \circ f)^{-1}((a, b)) = f^{-1}\left(g^{-1}\left((a, b)\right)\right) = f^{-1}(U) = f^{-1}\left(\bigcup_{k=1}^{\infty} Q_k\right) \in \mathcal{F}.$$ This proves the theorem.

**Corollary 9.1.14** Sums, products, and linear combinations of measurable functions are measurable.

**Proof:** To see the product of two measurable functions is measurable, let $g(x, y) = xy$, a continuous function defined on $\mathbb{R}^2$. Thus if you have two measurable functions, $f_1$ and $f_2$ defined on $\Omega$, $$g \circ (f_1, f_2)(\omega) = f_1(\omega) f_2(\omega)$$ and so $\omega \rightarrow f_1(\omega) f_2(\omega)$ is measurable. Similarly you can show the sum of two measurable functions is measurable by considering $g(x, y) = x + y$ and you can show a linear combination of two measurable functions is measurable by considering $g(x, y) = ax + by$. More than two functions can also be considered as well.

The message of this corollary is that starting with measurable real valued functions you can combine them in pretty much any way you want and you end up with a measurable function.

Here is some notation which will be used whenever convenient.

**Definition 9.1.15** Let $f : \Omega \rightarrow [-\infty, \infty]$. Define $$[\alpha < f] \equiv \{ \omega \in \Omega : f(\omega) > \alpha \} \equiv f^{-1}((\alpha, \infty])$$ with obvious modifications for the symbols $[\alpha \leq f], [\alpha \geq f], [\alpha \geq f \geq \beta]$, etc.
Definition 9.1.16 For a set $E$,

$$\mathcal{X}_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{if } \omega \notin E. \end{cases}$$

This is called the characteristic function of $E$. Sometimes this is called the indicator function which I think is better terminology since the term characteristic function has another meaning. Note that this “indicates” whether a point, $\omega$ is contained in $E$. It is exactly when the function has the value 1.

Theorem 9.1.17 (Egoroff) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space,

$$(\mu(\Omega) < \infty)$$

and let $f_n, f$ be complex valued functions such that $\text{Re} f_n, \text{Im} f_n$ are all measurable and

$$\lim_{n \to \infty} f_n(\omega) = f(\omega)$$

for all $\omega \notin E$ where $\mu(E) = 0$. Then for every $\varepsilon > 0$, there exists a set,

$$F \supseteq E, \mu(F) < \varepsilon,$$

such that $f_n$ converges uniformly to $f$ on $F^C$.

Proof: First suppose $E = \emptyset$ so that convergence is pointwise everywhere. It follows then that $\text{Re} f$ and $\text{Im} f$ are pointwise limits of measurable functions and are therefore measurable. Let $E_{km} = \{ \omega \in \Omega : |f_n(\omega) - f(\omega)| \geq 1/m \text{ for some } n > k \}$. Note that

$$|f_n(\omega) - f(\omega)| = \sqrt{(\text{Re} f_n(\omega) - \text{Re} f(\omega))^2 + (\text{Im} f_n(\omega) - \text{Im} f(\omega))^2}$$

and so, By Theorem 9.1.13

$$\left[ |f_n - f| \geq \frac{1}{m} \right]$$

is measurable. Hence $E_{km}$ is measurable because

$$E_{km} = \cup_{n=k+1}^{\infty} \left[ |f_n - f| \geq \frac{1}{m} \right].$$

For fixed $m, \cap_{k=1}^{\infty} E_{km} = \emptyset$ because $f_n$ converges to $f$. Therefore, if $\omega \in \Omega$ there exists $k$ such that if $n > k$, $|f_n(\omega) - f(\omega)| < \frac{1}{m}$ which means $\omega \notin E_{km}$. Note also that

$$E_{km} \supseteq E_{(k+1)m}.$$ 

Since $\mu(E_{1m}) < \infty$, Theorem 9.1.13 on Page 208 implies

$$0 = \mu(\bigcap_{k=1}^{\infty} E_{km}) = \lim_{k \to \infty} \mu(E_{km}).$$
Let \( k(m) \) be chosen such that \( \mu(E_{k(m)} m) < \varepsilon 2^{-m} \) and let

\[
F = \bigcup_{m=1}^{\infty} E_{k(m)} m.
\]

Then \( \mu(F) < \varepsilon \) because

\[
\mu(F) \leq \sum_{m=1}^{\infty} \mu(E_{k(m)} m) < \sum_{m=1}^{\infty} \varepsilon 2^{-m} = \varepsilon
\]

Now let \( \eta > 0 \) be given and pick \( m_0 \) such that \( m_0^{-1} < \eta \). If \( \omega \in F^c \), then

\[
\omega \in \bigcap_{m=1}^{\infty} E_{k(m)}^c m.
\]

Hence \( \omega \in E_{k(m_0)}^c m_0 \) so

\[
|f_n(\omega) - f(\omega)| < 1/m_0 < \eta
\]

for all \( n > k(m_0) \). This holds for all \( \omega \in F^c \) and so \( f_n \) converges uniformly to \( f \) on \( F^c \).

Now if \( E \neq \emptyset \), consider \( \{X_{E^c} f_n\}_{n=1}^{\infty} \). Each \( X_{E^c} f_n \) has real and imaginary parts measurable and the sequence converges pointwise to \( X_E f \) everywhere. Therefore, from the first part, there exists a set of measure less than \( \varepsilon, F \) such that on \( F^c \), \( \{X_{E^c} f_n\} \) converges uniformly to \( X_{E^c} f \). Therefore, on \( (E \cup F)^c \), \( \{f_n\} \) converges uniformly to \( f \). This proves the theorem.

Finally here is a comment about notation.

**Definition 9.1.18** Something happens for \( \mu \) a.e. \( \omega \) said as \( \mu \) almost everywhere, if there exists a set \( E \) with \( \mu(E) = 0 \) and the thing takes place for all \( \omega \notin E \). Thus \( f(\omega) = g(\omega) \) a.e. if \( f(\omega) = g(\omega) \) for all \( \omega \notin E \) where \( \mu(E) = 0 \). A measure space, \((\Omega, \mathcal{F}, \mu)\) is \( \sigma \) finite if there exist measurable sets, \( \Omega_n \) such that \( \mu(\Omega_n) < \infty \) and \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \).

### 9.2 Exercises

1. Let \( \Omega = \mathbb{N} = \{1, 2, \ldots\} \). Let \( \mathcal{F} = \mathcal{P}(\mathbb{N}) \) and let \( \mu(S) = \) number of elements in \( S \). Thus \( \mu(\{1\}) = 1 = \mu(\{2\}), \mu(\{1, 2\}) = 2 \), etc. Show \((\Omega, \mathcal{F}, \mu)\) is a measure space. It is called counting measure. What functions are measurable in this case?

2. Let \( \Omega \) be any uncountable set and let \( \mathcal{F} = \{A \subseteq \Omega : \) either \( A \) or \( A^c \) is countable\}. Let \( \mu(A) = 1 \) if \( A \) is uncountable and \( \mu(A) = 0 \) if \( A \) is countable. Show \((\Omega, \mathcal{F}, \mu)\) is a measure space. This is a well known bad example.
3. Let $\mathcal{F}$ be a $\sigma$ algebra of subsets of $\Omega$ and suppose $\mathcal{F}$ has infinitely many elements. Show that $\mathcal{F}$ is uncountable. **Hint:** You might try to show there exists a countable sequence of disjoint sets of $\mathcal{F}$, $\{A_i\}$. It might be easiest to verify this by contradiction if it doesn’t exist rather than a direct construction. Once this has been done, you can define a map, $\theta$, from $\mathcal{P}(\mathbb{N})$ into $\mathcal{F}$ which is one to one by $\theta(S) = \bigcup_{i \in S} A_i$. Then argue $\mathcal{P}(\mathbb{N})$ is uncountable and so $\mathcal{F}$ is also uncountable.

4. Prove Lemma \ref{lemma9.1.2}.

5. $g$ is Borel measurable if whenever $U$ is open, $g^{-1}(U)$ is Borel. Let $f : \Omega \to \mathbb{R}^n$ and let $g : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{F}$ is a $\sigma$ algebra of sets of $\Omega$. Suppose $f$ is measurable and $g$ is Borel measurable. Show $g \circ f$ is measurable. To say $g$ is Borel measurable means $g^{-1}$ (open set) = (Borel set) where a Borel set is one of those sets in the smallest $\sigma$ algebra containing the open sets of $\mathbb{R}^n$. See Lemma \ref{lemma9.1.2}. **Hint:** You should show, using Theorem \ref{theorem9.1.12} that $f^{-1}$ (open set) $\in \mathcal{F}$.

Now let $S \equiv \{ E \subseteq \mathbb{R}^n : f^{-1}(E) \in \mathcal{F} \}$

By what you just showed, $S$ contains the open sets. Now verify $S$ is a $\sigma$ algebra. Argue that from the definition of the Borel sets, it follows $S$ contains the Borel sets.

6. Let $(\Omega, \mathcal{F})$ be a measure space and suppose $f : \Omega \to \mathbb{C}$. Then $f$ is said to be measurable if

$$f^{-1}(\text{open set}) \in \mathcal{F}.$$  

Show $f$ is measurable if and only if $\text{Re} f$ and $\text{Im} f$ are measurable real-valued functions. Thus it suffices to define a complex valued function to be measurable if the real and imaginary parts are measurable. **Hint:** Argue that $f^{-1}((a,b) + i(c,d)) = (\text{Re} f)^{-1}((a,b)) \cap (\text{Im} f)^{-1}((c,d))$. Then use Theorem \ref{theorem9.1.12} to verify that if $\text{Re} f$ and $\text{Im} f$ are measurable, it follows $f$ is. Conversely, argue that $(\text{Re} f)^{-1}((a,b)) = f^{-1}((a,b) + i\mathbb{R})$ with a similar formula holding for $\text{Im} f$.

7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]$ by

$$\overline{\mu}(A) = \inf \{ \mu(B) : B \supseteq A, \ B \in \mathcal{F} \}.$$  

Show $\overline{\mu}$ satisfies

$$\overline{\mu}(\emptyset) = 0, \text{ if } A \subseteq B, \ \overline{\mu}(A) \leq \overline{\mu}(B),$$

$$\overline{\mu}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \overline{\mu}(A_i), \ \mu(A) = \overline{\mu}(A) \text{ if } A \in \mathcal{F}.$$  

If $\overline{\mu}$ satisfies these conditions, it is called an outer measure. This shows every measure determines an outer measure on the power set.
8. Let \( \{E_i\} \) be a sequence of measurable sets with the property that
\[
\sum_{i=1}^{\infty} \mu(E_i) < \infty.
\]
Let \( S = \{\omega \in \Omega \text{ such that } \omega \in E_i \text{ for infinitely many values of } i\} \). Show \( \mu(S) = 0 \) and \( S \) is measurable. This is part of the Borel Cantelli lemma. **Hint:** Write \( S \) in terms of intersections and unions. Something is in \( S \) means that for every \( n \) there exists \( k > n \) such that it is in \( E_k \). Remember the tail of a convergent series is small.

9. Let \( f_n, f \) be measurable functions. \( f_n \) converges in measure if
\[
\lim_{n \to \infty} \mu(\{x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon\}) = 0
\]
for each fixed \( \varepsilon > 0 \). Prove the theorem of F. Riesz. If \( f_n \) converges to \( f \) in measure, then there exists a subsequence \( \{f_{n_k}\} \) which converges to \( f \) a.e. **Hint:** Choose \( n_1 \) such that
\[
\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.
\]
Choose \( n_2 > n_1 \) such that
\[
\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,
\]
\( n_3 > n_2 \) such that
\[
\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,
\]
etc. Now consider what it means for \( f_{n_k}(x) \) to fail to converge to \( f(x) \). Then use Problem 8.

### 9.3 The Abstract Lebesgue Integral

#### 9.3.1 Preliminary Observations

This section is on the Lebesgue integral and the major convergence theorems which are the reason for studying it. In all that follows \( \mu \) will be a measure defined on a \( \sigma \) algebra \( \mathcal{F} \) of subsets of \( \Omega \). \( 0 \cdot \infty = 0 \) is always defined to equal zero. This is a meaningless expression and so it can be defined arbitrarily but a little thought will soon demonstrate that this is the right definition in the context of measure theory. To see this, consider the zero function defined on \( \mathbb{R} \). What should the integral of this function equal? Obviously, by an analogy with the Riemann integral, it should equal zero. Formally, it is zero times the length of the set or infinity. This is why this convention will be used.
9.3. THE ABSTRACT LEBESGUE INTEGRAL

Lemma 9.3.1 Let \( f(a,b) \in [−\infty, \infty] \) for \( a \in A \) and \( b \in B \) where \( A, B \) are sets. Then

\[
\sup_{a \in A} \sup_{b \in B} f(a, b) = \sup_{b \in B} \sup_{a \in A} f(a, b).
\]

Proof: Note that for all \( a, b \), \( f(a, b) \leq \sup_{b \in B} \sup_{a \in A} f(a, b) \) and therefore, for all \( a \),

\[
\sup_{b \in B} f(a, b) \leq \sup_{b \in B} \sup_{a \in A} f(a, b).
\]

Therefore,

\[
\sup_{a \in A} \sup_{b \in B} f(a, b) \leq \sup_{b \in B} \sup_{a \in A} f(a, b).
\]

Repeating the same argument interchanging \( a \) and \( b \), gives the conclusion of the lemma.

Lemma 9.3.2 If \( \{A_n\} \) is an increasing sequence in \( [−\infty, \infty] \), then

\[
\sup_{n=1}^{\infty} \{A_n\} = \lim_{n \to \infty} A_n.
\]

The following lemma is useful also and this is a good place to put it. First \( \{b_j\}_{j=1}^{\infty} \) is an enumeration of the \( a_{ij} \) if

\[
\cup_{j=1}^{\infty} \{b_j\} = \cup_{i,j} \{a_{ij}\}.
\]

In other words, the countable set, \( \{a_{ij}\}_{i,j=1}^{\infty} \) is listed as \( b_1, b_2, \ldots \).

Lemma 9.3.3 Let \( a_{ij} \geq 0 \). Then \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \). Also if \( \{b_j\}_{j=1}^{\infty} \) is any enumeration of the \( a_{ij} \), then \( \sum_{j=1}^{\infty} b_j = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \).

Proof: First note there is no trouble in defining these sums because the \( a_{ij} \) are all nonnegative. If a sum diverges, it only diverges to \( \infty \) and so \( \infty \) is written as the answer.

\[
\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sup_{n} \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_{ij} = \sup_{n} \lim_{m \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij}
\]

\[
= \sup_{n} \lim_{m \to \infty} \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ij} = \sup_{n} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}.
\]

Interchanging the \( i \) and \( j \) in the above argument the first part of the lemma is proved.

Finally, note that for all \( p \),

\[
\sum_{j=1}^{p} b_j \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}
\]

and so \( \sum_{j=1}^{\infty} b_j \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \). Now let \( m, n > 1 \) be given. Then

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{p} b_j
\]
where \( p \) is chosen large enough that \( \{b_1, \ldots, b_p\} \supseteq \{a_{ij} : i \leq m \text{ and } j \leq n\} \). Therefore, since such a \( p \) exists for any choice of \( m, n \), it follows that for any \( m, n \),

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{\infty} b_j.
\]

Therefore, taking the limit as \( n \to \infty \),

\[
\sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j
\]

and finally, taking the limit as \( m \to \infty \),

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j
\]

proving the lemma.

9.3.2 The Lebesgue Integral Nonnegative Functions

The following picture illustrates the idea used to define the Lebesgue integral to be like the area under a curve.

You can see that by following the procedure illustrated in the picture and letting \( h \) get smaller, you would expect to obtain better approximations to the area under the curve although all these approximations would likely be too small. Therefore, define

\[
\int f \, d\mu \equiv \sup_{h > 0} \sum_{i=1}^{\infty} h \mu ([ih < f])
\]

\(^1\)Note the difference between this picture and the one usually drawn in calculus courses where the little rectangles are upright rather than on their sides. This illustrates a fundamental philosophical difference between the Riemann and the Lebesgue integrals. With the Riemann integral intervals are measured. With the Lebesgue integral, it is inverse images of intervals which are measured.
Lemma 9.3.4 The following inequality holds.

\[
\sum_{i=1}^{\infty} h \mu ([i h < f]) \leq \sum_{i=1}^{\infty} \frac{h}{2} \mu \left( \left[ \frac{i h}{2} < f \right] \right).
\]

Also, it suffices to consider only \( h \) smaller than a given positive number in the above definition of the integral.

Proof:
Let \( N \in \mathbb{N} \).

\[
\sum_{i=1}^{2N} \frac{h}{2} \mu \left( \left[ \frac{i h}{2} < f \right] \right) = \sum_{i=1}^{2N} \frac{h}{2} \mu \left( [i h < 2f] \right) \\
= \sum_{i=1}^{N} \frac{h}{2} \mu \left( [(2i-1) h < 2f] \right) + \sum_{i=1}^{N} \frac{h}{2} \mu \left( [(2i) h < 2f] \right) \\
= \sum_{i=1}^{N} \frac{h}{2} \mu \left( \left[ \frac{(2i-1) h}{2} < f \right] \right) + \sum_{i=1}^{N} \frac{h}{2} \mu \left( [i h < f] \right) \\
\geq \sum_{i=1}^{N} \frac{h}{2} \mu \left( [i h < f] \right) + \sum_{i=1}^{N} \frac{h}{2} \mu \left( [i h < f] \right) = \sum_{i=1}^{N} \frac{h}{2} \mu \left( [i h < f] \right).
\]

Now letting \( N \to \infty \) yields the claim of the lemma.

To verify the last claim, suppose \( M < \int f d\mu \) and let \( \delta > 0 \) be given. Then there exists \( h > 0 \) such that

\[
M < \sum_{i=1}^{\infty} h \mu ([i h < f]) \leq \int f d\mu.
\]

By the first part of this lemma,

\[
M < \sum_{i=1}^{\infty} \frac{h}{2} \mu \left( \left[ \frac{i h}{2} < f \right] \right) \leq \int f d\mu
\]

and continuing to apply the first part,

\[
M < \sum_{i=1}^{\infty} \frac{h}{2^n} \mu \left( \left[ \frac{i h}{2^n} < f \right] \right) \leq \int f d\mu.
\]

Choose \( n \) large enough that \( h/2^n < \delta \). It follows

\[
M < \sup_{\delta > h > 0} \sum_{i=1}^{\infty} h \mu ([i h < f]) \leq \int f d\mu.
\]

Since \( M \) is arbitrary, this proves the last claim.
9.3.3 The Lebesgue Integral For Nonnegative Simple Functions

Definition 9.3.5 A function, $s$, is called simple if it is a measurable real valued function and has only finitely many values. These values will never be $\pm \infty$. Thus a simple function is one which may be written in the form

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega)$$

where the sets, $E_i$ are disjoint and measurable. $s$ takes the value $c_i$ at $E_i$.

Note that by taking the union of some of the $E_i$ in the above definition, you can assume that the numbers, $c_i$ are the distinct values of $s$. Simple functions are important because it will turn out to be very easy to take their integrals as shown in the following lemma.

Lemma 9.3.6 Let $s(\omega) = \sum_{i=1}^{p} a_i \mathcal{X}_{E_i}(\omega)$ be a nonnegative simple function with the $a_i$ the distinct non zero values of $s$. Then

$$\int s \, d\mu = \sum_{i=1}^{p} a_i \mu(E_i). \quad (9.3.11)$$

Also, for any nonnegative measurable function, $f$, if $\lambda \geq 0$, then

$$\int \lambda f \, d\mu = \lambda \int f \, d\mu. \quad (9.3.12)$$

Proof: Consider (9.3.11) first. Without loss of generality, you can assume $0 < a_1 < a_2 < \cdots < a_p$ and that $\mu(E_i) < \infty$. Let $\varepsilon > 0$ be given and let

$$\delta_1 \sum_{i=1}^{p} \mu(E_i) < \varepsilon.$$

Pick $\delta < \delta_1$ such that for $h < \delta$ it is also true that

$$h < \frac{1}{2} \min (a_1, a_2 - a_1, a_3 - a_2, \cdots, a_n - a_{n-1}).$$

Then for $0 < h < \delta$

$$\sum_{k=1}^{\infty} h \mu([s > kh]) = \sum_{k=1}^{\infty} h \sum_{i=k}^{\infty} \mu([ih < s \leq (i+1)h])$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} h \mu([ih < s \leq (i+1)h])$$

$$= \sum_{i=1}^{\infty} ih \mu([ih < s \leq (i+1)h]). \quad (9.3.13)$$
Because of the choice of $h$ there exist positive integers, $i_k$ such that $i_1 < i_2 < \cdots < i_p$ and
\[
\begin{align*}
i_1 h & < a_1 \leq (i_1 + 1) h < \cdots < i_2 h < a_2 < \\
& < (i_2 + 1) h < \cdots < i_p h < a_p \leq (i_p + 1) h
\end{align*}
\]

Then in the sum of \[9.3.13\] the only terms which are nonzero are those for which $i \in \{i_1, i_2, \cdots, i_p\}$. To see this, you might consider the following picture.

\[
\begin{array}{c}
a_3 \\
a_2 \\
a_1
\end{array}
\begin{array}{c}
i_3h \\
i_2h \\
i_1h
\end{array}
\]

When $ih$ and $(i + 1) h$ are both in between two of the $a_i$ the set $[ih < s \leq (i + 1) h]$ must be empty because the only values of the function are one of the $a_i$. At an $i_k$, $i_kh$ is smaller than $a_k$ while $(i_k + 1) h$ is at least as large. Therefore, the set $[ih < s \leq (i + 1) h]$ equals $E_k$ and so
\[
\mu([i_kh < s \leq (i_k + 1) h]) = \mu(E_k).
\]

Therefore,
\[
\sum_{k=1}^{\infty} h \mu([s > kh]) = \sum_{k=1}^{p} i_kh \mu(E_k).
\]

It follows that for all $h$ this small,
\[
\begin{align*}
0 & < \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{\infty} h \mu([s > kh]) \\
& = \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{p} i_kh \mu(E_k) \leq h \sum_{k=1}^{p} \mu(E_k) < \varepsilon.
\end{align*}
\]

Taking the inf for $h$ this small and using Lemma \[9.3.4\],
\[
\begin{align*}
0 & \leq \sum_{k=1}^{p} a_k \mu(E_k) - \sup_{a > h > 0} \sum_{k=1}^{\infty} h \mu([s > kh]) \\
& = \sum_{k=1}^{p} a_k \mu(E_k) - \int sd\mu \leq \varepsilon.
\end{align*}
\]

Since $\varepsilon > 0$ is arbitrary, this proves the first part.
To verify Note the formula is obvious if $\lambda = 0$ because then $[ih < \lambda f] = \emptyset$ for all $i > 0$. Assume $\lambda > 0$. Then

$$
\int \lambda f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h \mu([ih < \lambda f])
$$

$$
= \sup_{h>0} \sum_{i=1}^{\infty} h \mu([ih/\lambda < f])
$$

$$
= \sup_{h>0} \lambda \sum_{i=1}^{\infty} (h/\lambda) \mu([i(h/\lambda) < f])
$$

$$
= \lambda \int f d\mu.
$$

This proves the lemma.

**Lemma 9.3.7** Let the nonnegative simple function, $s$ be defined as

$$
s(\omega) = \sum_{i=1}^{n} c_{i} \chi_{E_{i}}(\omega)
$$

where the $c_{i}$ are not necessarily distinct but the $E_{i}$ are disjoint. It follows that

$$
\int s = \sum_{i=1}^{n} c_{i} \mu(E_{i}).
$$

**Proof:** Let the values of $s$ be $\{a_{1}, \cdots, a_{m}\}$. Therefore, since the $E_{i}$ are disjoint, each $a_{i}$ equal to one of the $c_{j}$. Let $A_{i} \equiv \cup \{E_{j} : c_{j} = a_{i}\}$. Then from Lemma 9.3.6 it follows that

$$
\int s = \sum_{i=1}^{m} a_{i} \mu(A_{i}) = \sum_{i=1}^{m} a_{i} \sum_{j : c_{j} = a_{i}} \mu(E_{j})
$$

$$
= \sum_{i=1}^{m} \sum_{j : c_{j} = a_{i}} c_{j} \mu(E_{j}) = \sum_{i=1}^{n} c_{i} \mu(E_{i}).
$$

This proves the lemma.

Note that $\int s$ could equal $+\infty$ if $\mu(A_{k}) = \infty$ and $a_{k} > 0$ for some $k$, but $\int s$ is well defined because $s \geq 0$. Recall that $0 \cdot \infty = 0$.

**Lemma 9.3.8** If $a, b \geq 0$ and if $s$ and $t$ are nonnegative simple functions, then

$$
\int as + bt = a \int s + b \int t.
$$
9.3. THE ABSTRACT LEBESGUE INTEGRAL

Proof: Let
\[ s(\omega) = \sum_{i=1}^{n} \alpha_i \mathcal{X}_{A_i}(\omega), \quad t(\omega) = \sum_{i=1}^{m} \beta_j \mathcal{X}_{B_j}(\omega) \]
where \( \alpha_i \) are the distinct values of \( s \) and the \( \beta_j \) are the distinct values of \( t \). Clearly \( as + bt \) is a nonnegative simple function because it is measurable and has finitely many values. Also,
\[ (as + bt)(\omega) = \sum_{j=1}^{m} \sum_{i=1}^{n} (a\alpha_i + b\beta_j) \mathcal{X}_{A_i \cap B_j}(\omega) \]
where the sets \( A_i \cap B_j \) are disjoint. By Lemma 9.3.7,
\[ \int as + bt = \sum_{j=1}^{m} \sum_{i=1}^{n} (a\alpha_i + b\beta_j) \mu(A_i \cap B_j) \]
\[ = a \sum_{i=1}^{n} \alpha_i \mu(A_i) + b \sum_{j=1}^{m} \beta_j \mu(B_j) \]
\[ = a \int s + b \int t. \]

This proves the lemma.

9.3.4 Simple Functions And Measurable Functions

There is a fundamental theorem about the relationship of simple functions to measurable functions given in the next theorem.

Theorem 9.3.9 Let \( f \geq 0 \) be measurable. Then there exists a sequence of nonnegative simple functions \( \{s_n\} \) satisfying
\[ 0 \leq s_n(\omega) \quad (9.3.14) \]
\[ \cdots \leq s_n(\omega) \leq s_{n+1}(\omega) \cdots \]
\[ f(\omega) = \lim_{n \to \infty} s_n(\omega) \text{ for all } \omega \in \Omega. \quad (9.3.15) \]

If \( f \) is bounded the convergence is actually uniform.

Proof: Letting \( I \equiv \{\omega : f(\omega) = \infty\} \), define
\[ t_n(\omega) = \sum_{k=0}^{2^n} \frac{k}{n} \mathcal{X}_{[k/n \leq f<(k+1)/n]}(\omega) + n \mathcal{X}_I(\omega). \]

Then \( t_n(\omega) \leq f(\omega) \) for all \( \omega \) and \( \lim_{n \to \infty} t_n(\omega) = f(\omega) \) for all \( \omega \). This is because \( t_n(\omega) = n \) for \( \omega \in I \) and if \( f(\omega) \in [0, \frac{2^n}{n}] \), then
\[ 0 \leq f(\omega) - t_n(\omega) \leq \frac{1}{n}. \quad (9.3.16) \]
Thus whenever \( \omega \notin I \), the above inequality will hold for all \( n \) large enough. Let
\[
s_1 = t_1, \quad s_2 = \max (t_1, t_2), \quad s_3 = \max (t_1, t_2, t_3), \ldots.
\]
Then the sequence \( \{s_n\} \) satisfies
9.3.14 - 9.3.15.

To verify the last claim, note that in this case the term \( n\mathcal{X}_f(\omega) \) is not present. Therefore, for all \( n \) large enough, 9.3.16 holds for all \( \omega \). Thus the convergence is uniform. This proves the theorem.

Although it is not needed here, there is a similar theorem which applies to measurable functions which have values in a separable metric space. In this context, a simple function is one which is of the form
\[
m \sum_{k=1}^{m} x_k \mathcal{X}_{E_k}(\omega)
\]
where the \( E_k \) are disjoint measurable sets and the \( x_k \) are in \( X \).

**Theorem 9.3.10** Let \( (\Omega, \mathcal{F}) \) be a measure space and let \( f : \Omega \to X \) where \( (X,d) \) is a separable metric space. Then \( f \) is a measurable function if and only if there exists a sequence of simple functions, \( \{f_n\} \) such that for each \( \omega \in \Omega \) and \( n \in \mathbb{N} \),
\[
d(f_n(\omega), f(\omega)) \geq d(f_{n+1}(\omega), f(\omega)) \tag{9.3.17}
\]
and
\[
\lim_{n \to \infty} d(f_n(\omega), f(\omega)) = 0. \tag{9.3.18}
\]

**Proof:** Let \( D = \{x_k\}_{k=1}^{\infty} \) be a countable dense subset of \( X \). First suppose \( f \) is measurable. Then since in a metric space every open set is the countable intersection of closed sets, it follows \( f^{-1}(\text{closed set}) \in \mathcal{F} \). Now let \( D_n = \{x_k\}_{k=1}^{n} \). Let
\[
A_1 = \left\{ \omega : d(x_1, f(\omega)) = \min_{k \leq n} d(x_k, f(\omega)) \right\}
\]
That is, \( A_1 \) are those \( \omega \) such that \( f(\omega) \) is approximated best out of \( D_n \) by \( x_1 \).

Why is this a measurable set? It is because \( \omega \to d(x, f(\omega)) \) is a real valued measurable function, being the composition of a continuous function, \( y \to d(x, y) \) and a measurable function, \( \omega \to f(\omega) \). Next let
\[
A_2 = \left\{ \omega \notin A_1 : d(x_2, f(\omega)) = \min_{k \leq n} d(x_k, f(\omega)) \right\}
\]
and continue in this manner obtaining disjoint measurable sets, \( \{A_k\}_{k=1}^{n} \) such that for \( \omega \in A_k \) the best approximation to \( f(\omega) \) from \( D_n \) is \( x_k \). Then
\[
f_n(\omega) = \sum_{k=1}^{n} x_k \mathcal{X}_{A_k}(\omega).
\]
9.3. THE ABSTRACT LEBESGUE INTEGRAL

Note

\[ \min_{k \leq n+1} d(x_k, f(\omega)) \leq \min_{k \leq n} d(x_k, f(\omega)) \]

and so this verifies (9.3.17). It remains to verify (9.3.18).

Let \( \varepsilon > 0 \) be given and pick \( \omega \in \Omega \). Then there exists \( x_n \in D \) such that \( d(x_n, f(\omega)) < \varepsilon \). It follows from the construction that \( d(f_n(\omega), f(\omega)) \leq d(x_n, f(\omega)) < \varepsilon \). This proves the first half.

Now suppose the existence of the sequence of simple functions as described above. Each \( f_n \) is a measurable function because \( f^{-1}(U) = \bigcup \{ A_k : x_k \in U \} \). Therefore, the conclusion that \( f \) is measurable follows from Theorem 9.1.9 on Page 211.

In the context of this more general notion of measurable function having values in a metric space, here is a version of Egoroff’s theorem.

**Theorem 9.3.11** (Egoroff) Let \((\Omega, F, \mu)\) be a finite measure space,

\[ (\mu(\Omega) < \infty) \]

and let \( f_n, f \) be \( X \) valued measurable functions where \( X \) is a separable metric space and for all \( \omega \notin E \) where \( \mu(E) = 0 \)

\[ f_n(\omega) \to f(\omega) \]

Then for every \( \varepsilon > 0 \), there exists a set,

\[ F \supseteq E, \mu(F) < \varepsilon, \]

such that \( f_n \) converges uniformly to \( f \) on \( F^C \).

**Proof:** First suppose \( E = \emptyset \) so that convergence is pointwise everywhere. Let

\[ E_{km} = \{ \omega \in \Omega : d(f_n(\omega), f(\omega)) \geq 1/m \text{ for some } n > k \}. \]

**Claim:** \( [\omega : d(f_n(\omega), f(\omega)) \geq \frac{1}{m}] \) is measurable.

**Proof of claim:** Let \( \{ x_k \}_{k=1}^{\infty} \) be a countable dense subset of \( X \) and let \( r \) denote a positive rational number, \( \mathbb{Q}^+ \). Then

\[
\bigcup_{k \in \mathbb{N}, r \in \mathbb{Q}^+} f_n^{-1}(B(x_k, r)) \cap f^{-1} \left( B \left( x_k, \frac{1}{m} - r \right) \right)
\]

\[ = \left[ d(f, f_n) < \frac{1}{m} \right] \quad (9.3.19) \]

Here is why. If \( \omega \) is in the set on the left, then \( d(f_n(\omega), x_k) < r \) and

\[ d(f(\omega), x_k) < \frac{1}{m} - r. \]

Therefore,

\[ d(f(\omega), f_n(\omega)) < r + \frac{1}{m} - r = \frac{1}{m}. \]
Thus the left side is contained in the right. Now let $\omega$ be in the right side. That is $d(f_n(\omega), f(\omega)) < \frac{1}{m}$. Choose $2r < \frac{1}{m} - d(f_n(\omega), f(\omega))$ and pick $x_k \in B(f_n(\omega), r)$. Then

$$d(f(\omega), x_k) \leq d(f(\omega), f_n(\omega)) + d(f_n(\omega), x_k) < \frac{1}{m} - 2r + r = \frac{1}{m} - r$$

Thus $\omega \in f_n^{-1}(B(x_k, r)) \cap f^{-1}(B(x_k, \frac{1}{m} - r))$ and so $\omega$ is in the left side. Thus the two sets are equal. Now the set on the left in 9.3.19 is measurable because it is a countable union of measurable sets. This proves the claim since

$$\left[ \omega : d(f_n(\omega), f(\omega)) \geq \frac{1}{m} \right]$$

is the complement of this measurable set.

Hence $E_{km}$ is measurable because

$$E_{km} = \bigcup_{n=k+1}^{\infty} \left[ \omega : d(f_n(\omega), f(\omega)) \geq \frac{1}{m} \right].$$

For fixed $m$, $\bigcap_{k=1}^{\infty} E_{km} = \emptyset$ because $f_n(\omega)$ converges to $f(\omega)$. Therefore, if $\omega \in \Omega$ there exists $k$ such that if $n > k$, $|f_n(\omega) - f(\omega)| < \frac{1}{m}$ which means $\omega \notin E_{km}$. Note also that

$$E_{km} \supseteq E_{(k+1)m}.$$

Since $\mu(E_{1m}) < \infty$, Theorem 9.1.5 on Page 208 implies

$$0 = \mu(\bigcap_{k=1}^{\infty} E_{km}) = \lim_{k \to \infty} \mu(E_{km}).$$

Let $k(m)$ be chosen such that $\mu(E_{k(m)m}) < \varepsilon 2^{-m}$ and let

$$F = \bigcup_{m=1}^{\infty} E_{k(m)m}.$$ Then $\mu(F) < \varepsilon$ because

$$\mu(F) \leq \sum_{m=1}^{\infty} \mu(E_{k(m)m}) < \sum_{m=1}^{\infty} \varepsilon 2^{-m} = \varepsilon.$$ Now let $\eta > 0$ be given and pick $m_0$ such that $m_0^{-1} < \eta$. If $\omega \in F^C$, then

$$\omega \in \bigcap_{m=1}^{\infty} E_{k(m)m}^C.$$ Hence $\omega \in E_{k(m_0)m_0}^C$ so

$$d(f(\omega), f_n(\omega)) < 1/m_0 < \eta.$$
for all $n > k(m_0)$. This holds for all $\omega \in F^C$ and so $f_n$ converges uniformly to $f$ on $F^C$.

Now if $E \neq \emptyset$, consider $\{\mathcal{X}^E_{FC} f_n\}_{n=1}^{\infty}$. Then $\mathcal{X}^E_{FC} f_n$ is measurable and the sequence converges pointwise to $\mathcal{X}^E f$ everywhere. Therefore, from the first part, there exists a set of measure less than $\varepsilon, F$ such that on $F^C, \{\mathcal{X}^E_{FC} f_n\}$ converges uniformly to $\mathcal{X}^E f$. Therefore, on $(E \cup F)^C, \{f_n\}$ converges uniformly to $f$. This proves the theorem.

### 9.3.5 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral.

**Theorem 9.3.12 (Monotone Convergence theorem)** Let $f$ have values in $[0, \infty]$ and suppose $\{f_n\}$ is a sequence of nonnegative measurable functions having values in $[0, \infty]$ and satisfying

$$
\lim_{n \to \infty} f_n(\omega) = f(\omega) \text{ for each } \omega.
$$

$$
\cdots f_n(\omega) \leq f_{n+1}(\omega) \cdots
$$

Then $f$ is measurable and

$$
\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.
$$

**Proof:** From Lemmas 9.3.1 and 9.3.2,

$$
\int f \, d\mu = \sup_{h > 0} \sum_{i=1}^{\infty} h \mu([ih < f])
$$

$$
= \sup_{h > 0} \sup_{k} \sum_{i=1}^{k} h \mu([ih < f])
$$

$$
= \sup_{h > 0} \sup_{k} \sum_{i=1}^{\infty} h \mu([ih < f_m])
$$

$$
= \sup_{m} \sup_{k \geq 0} \sum_{i=1}^{\infty} \mu([ih < f_m])
$$

$$
= \sup_{m} \int f_m \, d\mu
$$

$$
= \lim_{m \to \infty} \int f_m \, d\mu.
$$

The third equality follows from the observation that

$$
\lim_{m \to \infty} \mu([ih < f_m]) = \mu([ih < f])
$$
which follows from Theorem 9.1.5 since the sets, \([ih < f_m]\) are increasing in \(m\) and their union equals \([ih < f]\). This proves the theorem.

To illustrate what goes wrong without the Lebesgue integral, consider the following example.

**Example 9.3.13** Let \(\{r_n\}\) denote the rational numbers in \([0, 1]\) and let

\[
 f_n(t) \equiv \begin{cases} 
 1 & \text{if } t \notin \{r_1, \ldots, r_n\} \\
 0 & \text{otherwise}
\end{cases}
\]

Then \(f_n(t) \uparrow f(t)\) where \(f\) is the function which is one on the rationals and zero on the irrationals. Each \(f_n\) is Riemann integrable (why?) but \(f\) is not Riemann integrable. Therefore, you can’t write \(\int f \, dx = \lim_{n \to \infty} \int f_n \, dx\).

A meta-mathematical observation related to this type of example is this. If you can choose your functions, you don’t need the Lebesgue integral. The Riemann integral is just fine. It is when you can’t choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books.

### 9.3.6 Other Definitions

To review and summarize the above, if \(f \geq 0\) is measurable,

\[
\int f \, d\mu \equiv \sup_{h > 0} \sum_{i=1}^{\infty} h \mu ([f > ih])
\]

(9.3.20)

another way to get the same thing for \(\int f \, d\mu\) is to take an increasing sequence of nonnegative simple functions, \(\{s_n\}\) with \(s_n(\omega) \to f(\omega)\) and then by monotone convergence theorem,

\[
\int f \, d\mu = \lim_{n \to \infty} \int s_n
\]

where if \(s_n(\omega) = \sum_{j=1}^{m} c_i X_{E_i}(\omega)\),

\[
\int s_n \, d\mu = \sum_{i=1}^{m} c_i m(E_i).
\]

Similarly this also shows that for such nonnegative measurable function,

\[
\int f \, d\mu = \sup \left\{ \int s : 0 \leq s \leq f, \text{ s simple} \right\}
\]

which is the usual way of defining the Lebesgue integral for nonnegative simple functions in most books. I have done it differently because this approach led to an easier proof of the Monotone convergence theorem. Here is an equivalent definition of the integral. The fact it is well defined has been discussed above.
9.3. THE ABSTRACT LEBESGUE INTEGRAL

Definition 9.3.14 For $s$ a nonnegative simple function,

$$s(\omega) = \sum_{k=1}^{n} c_k X_{E_k}(\omega), \quad \int s = \sum_{k=1}^{n} c_k \mu(E_k).$$

For $f$ a nonnegative measurable function,

$$\int f d\mu = \sup \left\{ \int s : 0 \leq s \leq f, \ s \text{ simple} \right\}.$$

9.3.7 Fatou’s Lemma

Sometimes the limit of a sequence does not exist. There are two more general notions known as lim sup and lim inf which do always exist in some sense. These notions are dependent on the following lemma.

Lemma 9.3.15 Let $\{a_n\}$ be an increasing (decreasing) sequence in $[-\infty, \infty]$. Then $\lim_{n \to \infty} a_n$ exists.

**Proof:** Suppose first $\{a_n\}$ is increasing. Recall this means $a_n \leq a_{n+1}$ for all $n$. If the sequence is bounded above, then it has a least upper bound and so $a_n \to a$ where $a$ is its least upper bound. If the sequence is not bounded above, then for every $l \in \mathbb{R}$, it follows $l$ is not an upper bound and so eventually, $a_n > l$. But this is what is meant by $a_n \to \infty$. The situation for decreasing sequences is completely similar.

Now take any sequence, $\{a_n\} \subseteq [-\infty, \infty]$ and consider the sequence $\{A_n\}$ where

$$A_n \equiv \inf \{a_k : k \geq n\}.$$

Then as $n$ increases, the set of numbers whose inf is being taken is getting smaller. Therefore, $A_n$ is an increasing sequence and so it must converge. Similarly, if $B_n \equiv \sup \{a_k : k \geq n\}$, it follows $B_n$ is decreasing and so $\{B_n\}$ also must converge. With this preparation, the following definition can be given.

Definition 9.3.16 Let $\{a_n\}$ be a sequence of points in $[-\infty, \infty]$. Then define

$$\lim \inf_{n \to \infty} a_n \equiv \lim \inf_{n \to \infty} \{a_k : k \geq n\}$$

and

$$\lim \sup_{n \to \infty} a_n \equiv \lim \sup_{n \to \infty} \{a_k : k \geq n\}.$$

In the case of functions having values in $[-\infty, \infty]$,

$$\left(\lim \inf_{n \to \infty} f_n\right)(\omega) \equiv \lim \inf_{n \to \infty} (f_n(\omega)).$$

A similar definition applies to $\lim \sup_{n \to \infty} f_n$.
Lemma 9.3.17 Let \( \{a_n\} \) be a sequence in \([-\infty, \infty]\). Then \( \lim_{n \to \infty} a_n \) exists if and only if
\[
\lim \inf_{n \to \infty} a_n = \lim \sup_{n \to \infty} a_n
\]
and in this case, the limit equals the common value of these two numbers.

Proof: Suppose first \( \lim_{n \to \infty} a_n = a \in \mathbb{R} \). Then, letting \( \varepsilon > 0 \) be given, \( a_n \in (a - \varepsilon, a + \varepsilon) \) for all \( n \) large enough, say \( n \geq N \). Therefore, both \( \inf \{ a_k : k \geq n \} \) and \( \sup \{ a_k : k \geq n \} \) are contained in \([a - \varepsilon, a + \varepsilon]\) whenever \( n \geq N \). It follows \( \limsup_{n \to \infty} a_n \) and \( \liminf_{n \to \infty} a_n \) are both in \([a - \varepsilon, a + \varepsilon]\), showing
\[
\left| \lim \inf_{n \to \infty} a_n - \lim \sup_{n \to \infty} a_n \right| < 2\varepsilon.
\]
Since \( \varepsilon \) is arbitrary, the two must be equal and they both must equal \( a \). Next suppose \( \lim_{n \to \infty} a_n = \infty \). Then if \( l \in \mathbb{R} \), there exists \( N \) such that for \( n \geq N \),
\[
l \leq a_n
\]
and therefore, for such \( n \),
\[
l \leq \inf \{ a_k : k \geq n \} \leq \sup \{ a_k : k \geq n \}
\]
and this shows, since \( l \) is arbitrary that
\[
\lim \inf_{n \to \infty} a_n = \lim \sup_{n \to \infty} a_n = \infty.
\]
The case for \( -\infty \) is similar.

Conversely, suppose \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = a \). Suppose first that \( a \in \mathbb{R} \). Then, letting \( \varepsilon > 0 \) be given, there exists \( N \) such that if \( n \geq N \),
\[
\sup \{ a_k : k \geq n \} - \inf \{ a_k : k \geq n \} < \varepsilon
\]
therefore, if \( k, m > N \), and \( a_k > a_m \),
\[
|a_k - a_m| = a_k - a_m \leq \sup \{ a_k : k \geq n \} - \inf \{ a_k : k \geq n \} < \varepsilon
\]
showing that \( \{a_n\} \) is a Cauchy sequence. Therefore, it converges to \( a \in \mathbb{R} \), and as in the first part, the \( \liminf \) and \( \limsup \) both equal \( a \). If \( \liminf_{n \to \infty} a_n = \limsup_{n \to \infty} a_n = \infty \), then given \( l \in \mathbb{R} \), there exists \( N \) such that for \( n \geq N \),
\[
\inf_{n>N} a_n > l.
\]
Therefore, \( \lim_{n \to \infty} a_n = \infty \). The case for \( -\infty \) is similar. This proves the lemma.

The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.
Theorem 9.3.18 (Fatou’s lemma) Let \( f_n \) be a nonnegative measurable function with values in \([0, \infty]\). Let \( g(\omega) = \lim \inf_{n \to \infty} f_n(\omega) \). Then \( g \) is measurable and
\[
\int g \, d\mu \leq \lim \inf_{n \to \infty} \int f_n \, d\mu.
\]
In other words,
\[
\int \left( \lim \inf_{n \to \infty} f_n \right) \, d\mu \leq \lim \inf_{n \to \infty} \int f_n \, d\mu
\]

**Proof:** Let \( g_n(\omega) = \inf \{ f_k(\omega) : k \geq n \} \). Then
\[
g_n^{-1}(a, \infty) = \bigcap_{k=n}^{\infty} f_k^{-1}(a, \infty) \in \mathcal{F}.
\]
Thus \( g_n \) is measurable by Lemma 9.1.6 on Page 210. Also \( g(\omega) = \lim_{n \to \infty} g_n(\omega) \) so \( g \) is measurable because it is the pointwise limit of measurable functions. Now the functions \( g_n \) form an increasing sequence of nonnegative measurable functions so the monotone convergence theorem applies. This yields
\[
\int g \, d\mu = \lim_{n \to \infty} \int g_n \, d\mu \leq \lim \inf_{n \to \infty} \int f_n \, d\mu.
\]
The last inequality holding because
\[
\int g_n \, d\mu \leq \int f_n \, d\mu.
\]
(Note that it is not known whether \( \lim_{n \to \infty} \int f_n \, d\mu \) exists.) This proves the theorem.

9.3.8 The Righteous Algebraic Desires Of The Lebesgue Integral

The monotone convergence theorem shows the integral wants to be linear. This is the essential content of the next theorem.

Theorem 9.3.19 Let \( f, g \) be nonnegative measurable functions and let \( a, b \) be nonnegative numbers. Then
\[
\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu.
\]  

**Proof:** By Theorem 9.1.7 on Page 227 there exist sequences of nonnegative simple functions, \( s_n \to f \) and \( t_n \to g \). Then by the monotone convergence theorem and Lemma 9.3.5,
\[
\int (af + bg) \, d\mu = \lim_{n \to \infty} \int as_n + bt_n \, d\mu
\]
\[
= \lim_{n \to \infty} \left( a \int s_n \, d\mu + b \int t_n \, d\mu \right)
\]
\[
= a \int f \, d\mu + b \int g \, d\mu.
\]
As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f + (-g)$ and so you can’t very well expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. This is discussed next.

### 9.4 The Space $L^1$

The functions considered here have values in $\mathbb{C}$, a vector space.

**Definition 9.4.1** Let $(\Omega, S, \mu)$ be a measure space and suppose $f : \Omega \to \mathbb{C}$. Then $f$ is said to be measurable if both $\text{Re}f$ and $\text{Im}f$ are measurable real valued functions.

**Definition 9.4.2** A complex simple function will be a function which is of the form

$$s(\omega) = \sum_{k=1}^{n} c_k \lambda_{E_k}(\omega)$$

where $c_k \in \mathbb{C}$ and $\mu(E_k) < \infty$. For $s$ a complex simple function as above, define

$$I(s) \equiv \sum_{k=1}^{n} c_k \mu(E_k).$$

**Lemma 9.4.3** The definition, 9.4.2, is well defined. Furthermore, $I$ is linear on the vector space of complex simple functions. Also the triangle inequality holds,

$$|I(s)| \leq I(|s|).$$

**Proof:** Suppose $\sum_{k=1}^{n} c_k \lambda_{E_k}(\omega) = 0$. Does it follow that $\sum_{k} c_k \mu(E_k) = 0$?

The supposition implies

$$\sum_{k=1}^{n} \text{Re}c_k \lambda_{E_k}(\omega) = 0, \quad \sum_{k=1}^{n} \text{Im}c_k \lambda_{E_k}(\omega) = 0. \tag{9.4.22}$$

Choose $\lambda$ large and positive so that $\lambda + \text{Re}c_k \geq 0$. Then adding $\sum_{k} \lambda \lambda_{E_k}$ to both sides of the first equation above,

$$\sum_{k=1}^{n} (\lambda + \text{Re}c_k) \lambda_{E_k}(\omega) = \sum_{k=1}^{n} \lambda \lambda_{E_k}$$

and by Lemma 9.4.2 on Page 226 it follows upon taking $f$ of both sides that

$$\sum_{k=1}^{n} (\lambda + \text{Re}c_k) \mu(E_k) = \sum_{k=1}^{n} \lambda \mu(E_k).$$
which implies \( \sum_{k=1}^{n} \text{Re} \, c_k \mu (E_k) = 0 \). Similarly, \( \sum_{k=1}^{n} \text{Im} \, c_k \mu (E_k) = 0 \) and so \( \sum_{k=1}^{n} c_k \mu (E_k) = 0 \). Thus if
\[
\sum_{j} c_j \mathcal{X}_{E_j} = \sum_{k} d_k \mathcal{X}_{F_k}
\]
then \( \sum_{j} c_j \mathcal{X}_{E_j} + \sum_{k} (-d_k) \mathcal{X}_{F_k} = 0 \) and so the result just established verifies \( \sum_{j} c_j \mu (E_j) - \sum_{k} d_k \mu (F_k) = 0 \) which proves \( I \) is well defined.

That \( I \) is linear is now obvious. It only remains to verify the triangle inequality.

Let \( s \) be a simple function,
\[
s = \sum_{j} c_j \mathcal{X}_{E_j}
\]
Then pick \( \theta \in \mathbb{C} \) such that \( \theta I (s) = |I (s)| \) and \( |\theta| = 1 \). Then from the triangle inequality for sums of complex numbers,
\[
|I (s)| = \theta I (s) = I (\theta s) = \sum_{j} \theta c_j \mu (E_j)
\]
\[
= \left| \sum_{j} \theta c_j \mu (E_j) \right| \leq \sum_{j} |\theta c_j| \mu (E_j) = I (|s|).
\]

This proves the lemma.

With this lemma, the following is the definition of \( L^1 (\Omega) \).

**Definition 9.4.4** \( f \in L^1 (\Omega) \) means there exists a sequence of complex simple functions, \( \{s_n\} \) such that

\[
s_n (\omega) \to f (\omega) \text{ for all } \omega \in \Omega
\]
\[
\lim_{m,n \to \infty} I (|s_n - s_m|) = \lim_{m,n \to \infty} \int |s_n - s_m| \, d\mu = 0 \tag{9.4.23}
\]

Then
\[
I (f) \equiv \lim_{n \to \infty} I (s_n). \tag{9.4.24}
\]

**Lemma 9.4.5** Definition 9.4.4 is well defined.

**Proof:** There are several things which need to be verified. First suppose \( I \) is well defined.

Then by Lemma 9.4.3,
\[
|I (s_n) - I (s_m)| = |I (s_n - s_m)| \leq I (|s_n - s_m|)
\]
and for \( m, n \) large enough this last is given to be small so \( \{I (s_n)\} \) is a Cauchy sequence in \( \mathbb{C} \) and so it converges. This verifies the limit in (9.4.24) at least exists. It remains to consider another sequence \( \{t_n\} \) having the same properties as \( \{s_n\} \) and
verifying $I(f)$ determined by this other sequence is the same. By Lemma 9.4.3 and Fatou’s lemma, Theorem 9.3.18 on Page 235,

$$|I(s_n) - I(t_n)| \leq I(|s_n - t_n|) = \int |s_n - t_n| d\mu$$

$$\leq \int |s_n - f| + |f - t_n| d\mu$$

$$\leq \lim \inf_{k \to \infty} \int |s_n - s_k| d\mu + \lim \inf_{k \to \infty} \int |t_n - t_k| d\mu < \varepsilon$$

whenever $n$ is large enough. Since $\varepsilon$ is arbitrary, this shows the limit from using the $t_n$ is the same as the limit from using $s_n$. This proves the lemma.

What if $f$ has values in $[0, \infty)$? Earlier $\int fd\mu$ was defined for such functions and now $I(f)$ has been defined. Are they the same? If so, $I$ can be regarded as an extension of $\int d\mu$ to a larger class of functions.

**Lemma 9.4.6** Suppose $f$ has values in $[0, \infty)$ and $f \in L^1(\Omega)$. Then $f$ is measurable and

$$I(f) = \int fd\mu.$$

**Proof:** Since $f$ is the pointwise limit of a sequence of complex simple functions, $\{s_n\}$ having the properties described in Definition 9.4.4, it follows $f(\omega) = \lim_{n \to \infty} \text{Re } s_n(\omega)$ and so $f$ is measurable. Also

$$\int \left| \text{Re } s_n^+ - \text{Re } s_m^+ \right| d\mu \leq \int \text{Re } s_n - \text{Re } s_m |d\mu \leq \int |s_n - s_m| d\mu$$

where $x^+ = \frac{1}{2} (|x| + x)$, the positive part of the real number, $x$.

Thus there is no loss of generality in assuming $\{s_n\}$ is a sequence of complex simple functions having values in $[0, \infty)$. Then since for such complex simple functions, $I(s) = \int sd\mu$,

$$\left| I(f) - \int fd\mu \right| \leq |I(f) - I(s_n)| + \left| \int s_n d\mu - \int f d\mu \right|$$

$$\leq \varepsilon + \left| \int_{[s_n - f \geq 0]} s_n d\mu - \int_{[s_n - f \geq 0]} f d\mu \right|$$

$$+ \left| \int_{[s_n - f < 0]} s_n d\mu - \int_{[s_n - f < 0]} f d\mu \right|$$

$$\leq \varepsilon + \left| \int_{[s_n - f \geq 0]} (s_n - f) d\mu \right| + \left| \int_{[s_n - f < 0]} (s_n - f) d\mu \right|$$

2The negative part of the real number $x$ is defined to be $x^- = \frac{1}{2} (|x| - x)$. Thus $|x| = x^+ + x^-$ and $x = x^+ - x^-$. 
9.4. THE SPACE $L^1$

\[
\leq \varepsilon + \int_{|s_n - f| \geq 0} |s_n - f| \, d\mu + \int_{|s_n - f| > 0} |s_n - f| \, d\mu = \varepsilon + \int |s_n - f| \, d\mu
\]

whenever $n$ is large enough. But by Fatou’s lemma, Theorem 9.3.18 on Page 235, the last term is no larger than

\[
\lim \inf_{k \to \infty} \int |s_n - s_k| \, d\mu < \varepsilon
\]

whenever $n$ is large enough. Since $\varepsilon$ is arbitrary, this shows $I(f) = \int f \, d\mu$ as claimed.

As explained above, $I$ can be regarded as an extension of $\int d\mu$ so from now on, the usual symbol, $\int f \, d\mu$ will be used. It is now easy to verify $\int d\mu$ is linear on $L^1(\Omega)$.

**Theorem 9.4.7** $\int d\mu$ is linear on $L^1(\Omega)$ and $L^1(\Omega)$ is a complex vector space. If $f \in L^1(\Omega)$, then $\text{Re} \, f$, $\text{Im} \, f$, and $|f|$ are all in $L^1(\Omega)$. Furthermore, for $f \in L^1(\Omega)$,

\[
\int f \, d\mu = \int (\text{Re} \, f)^+ \, d\mu - \int (\text{Re} \, f)^- \, d\mu + i \left( \int (\text{Im} \, f)^+ \, d\mu - \int (\text{Im} \, f)^- \, d\mu \right)
\]

Also the triangle inequality holds,

\[
\left| \int fd\mu \right| \leq \int |f| \, d\mu
\]

**Proof:** First it is necessary to verify that $L^1(\Omega)$ is really a vector space because it makes no sense to speak of linear maps without having these maps defined on a vector space. Let $f, g$ be in $L^1(\Omega)$ and let $a, b \in \mathbb{C}$. Then let $\{s_n\}$ and $\{t_n\}$ be sequences of complex simple functions associated with $f$ and $g$ respectively as described in Definition 9.4.4. Consider $\{as_n + bt_n\}$, another sequence of complex simple functions. Then $as_n(\omega) + bt_n(\omega) \to af(\omega) + bg(\omega)$ for each $\omega$. Also, from Lemma 9.4.3

\[
\int |as_n + bt_n - (as_m + bt_m)| \, d\mu \leq |a| \int |s_n - s_m| \, d\mu + |b| \int |t_n - t_m| \, d\mu
\]

and the sum of the two terms on the right converge to zero as $m, n \to \infty$. Thus $af + bg \in L^1(\Omega)$. Also

\[
\int (af + bg) \, d\mu = \lim_{n \to \infty} \int (as_n + bt_n) \, d\mu = \lim_{n \to \infty} \left( a \int s_n \, d\mu + b \int t_n \, d\mu \right) = a \lim_{n \to \infty} \int s_n \, d\mu + b \lim_{n \to \infty} \int t_n \, d\mu = a \int f \, d\mu + b \int g \, d\mu.
\]
If \( \{s_n\} \) is a sequence of complex simple functions described in Definition 9.4.4 corresponding to \( f \), then \( \{|s_n|\} \) is a sequence of complex simple functions satisfying the conditions of Definition 9.4.4 corresponding to \(|f|\). This is because \(|s_n(\omega)| \rightarrow |f(\omega)|\) and
\[
\int ||s_n| - |s_m|| \, d\mu \leq \int |s_m - s_n| \, d\mu
\]
with this last expression converging to 0 as \( m, n \rightarrow \infty \). Thus \( |f| \in L^1(\Omega) \). Also, by similar reasoning, \( \{\text{Re } s_n\} \) and \( \{\text{Im } s_n\} \) correspond to \( \text{Re } f \) and \( \text{Im } f \) respectively in the manner described by Definition 9.4.4 showing that \( \text{Re } f \) and \( \text{Im } f \) are in \( L^1(\Omega) \).

The formula follows from the observation that
\[
f = (\text{Re } f)^+ - (\text{Re } f)^- + i \left( (\text{Im } f)^+ - (\text{Im } f)^- \right)
\]
and the fact shown first that \( \int d\mu \) is linear.

To verify the triangle inequality, let \( \{s_n\} \) be complex simple functions for \( f \) as in Definition 9.4.4. Then
\[
\left| \int f \, d\mu \right| = \lim_{n \to \infty} \left| \int s_n \, d\mu \right| \leq \lim_{n \to \infty} \int |s_n| \, d\mu = \int |f| \, d\mu.
\]
This proves the theorem.

The following description of \( L^1(\Omega) \) is the version most often used because it is easy to verify the conditions for it.

**Corollary 9.4.8** Let \((\Omega, S, \mu)\) be a measure space and let \( f : \Omega \rightarrow \mathbb{C} \). Then \( f \in L^1(\Omega) \) if and only if \( f \) is measurable and \( \int |f| \, d\mu < \infty \).

**Proof:** Suppose \( f \in L^1(\Omega) \). Then from Definition 9.4.4, it follows both real and imaginary parts of \( f \) are measurable. Just take real and imaginary parts of \( s_n \) and observe the real and imaginary parts of \( f \) are limits of the real and imaginary parts of \( s_n \) respectively. Why is \( \int |f| \, d\mu < \infty \)? It follows from Theorem 9.4.7. Recall why this was so. Let \( \{s_n\} \) be a sequence of simple functions attached to \( f \) as in the definition of what it means to be \( L^1(\Omega) \). Then from the definition of \( I(s) \) for \( s \) simple,
\[
|I(|s_n| - |s_m|)| \leq I(|s_n - s_m|)
\]
which converges to 0. Since \( \{I(|s_n|)\} \) is a Cauchy sequence, it is bounded by a constant \( C \) and also \( \{|s_n|\} \) is a sequence of simple functions of the right sort which converges pointwise to \(|f|\) and so by definition,
\[
\int |f| \, d\mu = I(f) = \lim_{n \to \infty} I(|s_n|) \leq C.
\]
This shows the only if part.
The more interesting part is the if part. Suppose then that $f$ is measurable and $\int |f| \, d\mu < \infty$. Suppose first that $f$ has values in $[0, \infty)$. It is necessary to obtain the sequence of complex simple functions. By Theorem 9.3.9, there exists a sequence of nonnegative simple functions, $\{s_n\}$ such that $s_n(\omega) \uparrow f(\omega)$. Then by the monotone convergence theorem,

$$\lim_{n \to \infty} \int (2f - (f - s_n)) \, d\mu = \int 2f \, d\mu$$

and so

$$\lim_{n \to \infty} \int (f - s_n) \, d\mu = 0.$$ 

Letting $m$ be large enough, it follows $\int (f - s_m) \, d\mu < \varepsilon$ and so if $n > m$

$$\int |s_m - s_n| \, d\mu \leq \int |f - s_m| \, d\mu < \varepsilon.$$ 

Therefore, $f \in L^1(\Omega)$ because $\{s_n\}$ is a suitable sequence.

The general case follows from considering positive and negative parts of real and imaginary parts of $f$. These are each measurable and nonnegative and their integral is finite so each is in $L^1(\Omega)$ by what was just shown. Thus

$$f = \text{Re} f^+ - \text{Re} f^- + i (\text{Im} f^+ - \text{Im} f^-)$$

and so $f \in L^1(\Omega)$. This proves the corollary.

**Theorem 9.4.9 (Dominated Convergence theorem)** Let $f_n \in L^1(\Omega)$ and suppose

$$f(\omega) = \lim_{n \to \infty} f_n(\omega),$$

and there exists a measurable function $g$, with values in $[0, \infty)$,\footnote{Note that, since $g$ is allowed to have the value $\infty$, it is not known that $g \in L^1(\Omega)$.} such that

$$|f_n(\omega)| \leq g(\omega) \text{ and } \int g(\omega) \, d\mu < \infty.$$ 

Then $f \in L^1(\Omega)$ and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$ 

**Proof:** $f$ is measurable by Theorem 9.1.8. Since $|f| \leq g$, it follows that $f \in L^1(\Omega)$ and $|f - f_n| \leq 2g$.

By Fatou’s lemma (Theorem 9.3.9),

$$\int 2g \, d\mu \leq \lim inf_{n \to \infty} \int 2g - |f - f_n| \, d\mu$$

$$= \int 2g \, d\mu - \lim sup_{n \to \infty} \int |f - f_n| \, d\mu.$$
Subtracting $\int g d\mu$,

$$0 \leq -\lim_{n\to\infty} \sup \int |f - f_n| d\mu.$$  

Hence

$$0 \geq \lim_{n\to\infty} \sup \left( \int |f - f_n| d\mu \right) \geq \lim_{n\to\infty} \left( \int g d\mu - \int f d\mu - \int f_n d\mu \right) \geq 0.$$  

This proves the theorem by Lemma 9.3.17 on Page 234 because the lim sup and lim inf are equal.

**Corollary 9.4.10** Suppose $f_n \in L^1(\Omega)$ and $f(\omega) = \lim_{n\to\infty} f_n(\omega)$. Suppose also there exist measurable functions, $g_n, g$ with values in $[0, \infty)$ such that

$$\lim_{n\to\infty} \int g_n d\mu = \int g d\mu,$$

$g_n(\omega) \to g(\omega)$ $\mu$ a.e. and both $\int g_n d\mu$ and $\int g d\mu$ are finite. Also suppose $|f_n(\omega)| \leq g_n(\omega)$. Then

$$\lim_{n\to\infty} \int |f - f_n| d\mu = 0.$$  

**Proof:** It is just like the above. This time $g + g_n - |f - f_n| \geq 0$ and so by Fatou's lemma,

$$\int 2gd\mu - \lim_{n\to\infty} \sup \int |f - f_n| d\mu =$$

$$\lim_{n\to\infty} \inf \int (g_n + g) - \lim_{n\to\infty} \sup \int |f - f_n| d\mu$$

$$= \lim_{n\to\infty} \inf \int ((g_n + g) - |f - f_n|) d\mu \geq \int 2gd\mu$$

and so $-\lim sup_{n\to\infty} \int |f - f_n| d\mu \geq 0$.

**Definition 9.4.11** Let $E$ be a measurable subset of $\Omega$.

$$\int_E f d\mu \equiv \int f \chi_E d\mu.$$  

If $L^1(E)$ is written, the $\sigma$ algebra is defined as

$$\{E \cap A : A \in \mathcal{F}\}$$

and the measure is $\mu$ restricted to this smaller $\sigma$ algebra. Clearly, if $f \in L^1(\Omega)$, then

$$f \chi_E \in L^1(E)$$

and if $f \in L^1(E)$, then letting $\tilde{f}$ be the 0 extension of $f$ off of $E$, it follows $\tilde{f}$ $\in L^1(\Omega)$.  

9.5 Vitali Convergence Theorem

The Vitali convergence theorem is a convergence theorem which in the case of a finite measure space is superior to the dominated convergence theorem.

**Definition 9.5.1** Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and let \(\mathcal{S} \subseteq L^1(\Omega)\). \(\mathcal{S}\) is uniformly integrable if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that for all \(f \in \mathcal{S}\)

\[
|\int_E f d\mu| < \varepsilon \text{ whenever } \mu(E) < \delta.
\]

**Lemma 9.5.2** If \(\mathcal{S}\) is uniformly integrable, then \(|\mathcal{S}| \equiv \{|f| : f \in \mathcal{S}\}\) is uniformly integrable. Also \(\mathcal{S}\) is uniformly integrable if \(\mathcal{S}\) is finite.

**Proof:** Let \(\varepsilon > 0\) be given and suppose \(\mathcal{S}\) is uniformly integrable. First suppose the functions are real valued. Let \(\delta\) be such that if \(\mu(E) < \delta\), then

\[
\int_E f d\mu < \frac{\varepsilon}{2}
\]

for all \(f \in \mathcal{S}\). Let \(\mu(E) < \delta\). Then if \(f \in \mathcal{S}\),

\[
\int_E |f| d\mu \leq \int_{E \cap [f \leq 0]} (-f) d\mu + \int_{E \cap [f > 0]} f d\mu = \int_{E \cap [f \leq 0]} f d\mu + \int_{E \cap [f > 0]} f d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

In general, if \(\mathcal{S}\) is a uniformly integrable set of complex valued functions, the inequalities,

\[
\left| \int_E \text{Re } f d\mu \right| \leq \left| \int_E f d\mu \right|, \quad \left| \int_E \text{Im } f d\mu \right| \leq \left| \int_E f d\mu \right|,
\]

imply \(\text{Re } \mathcal{S} \equiv \{\text{Re } f : f \in \mathcal{S}\}\) and \(\text{Im } \mathcal{S} \equiv \{\text{Im } f : f \in \mathcal{S}\}\) are also uniformly integrable. Therefore, applying the above result for real valued functions to these sets of functions, it follows \(|\mathcal{S}|\) is uniformly integrable also.

For the last part, is suffices to verify a single function in \(L^1(\Omega)\) is uniformly integrable. To do so, note that from the dominated convergence theorem,

\[
\lim_{R \to \infty} \int_{|f| > R} |f| d\mu = 0.
\]

Let \(\varepsilon > 0\) be given and choose \(R\) large enough that \(\int_{|f| > R} |f| d\mu < \frac{\varepsilon}{2}\). Now let \(\mu(E) < \frac{\varepsilon}{2R}\). Then

\[
\int_E |f| d\mu = \int_{E \cap [f \leq R]} |f| d\mu + \int_{E \cap [f > R]} |f| d\mu < R \mu(E) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
This proves the lemma.

The following theorem is Vitali’s convergence theorem.

**Theorem 9.5.3** Let \( \{f_n\} \) be a uniformly integrable set of complex valued functions, \( \mu(\Omega) < \infty \), and \( f_n(x) \to f(x) \) a.e. where \( f \) is a measurable complex valued function. Then \( f \in L^1(\Omega) \) and

\[
\lim_{n \to \infty} \int_\Omega |f_n - f| d\mu = 0. \tag{9.5.25}
\]

**Proof:** First it will be shown that \( f \in L^1(\Omega) \). By uniform integrability, there exists \( \delta > 0 \) such that if \( \mu(E) < \delta \), then

\[
\int_E |f_n| d\mu < 1
\]

for all \( n \). By Egoroff’s theorem, there exists a set, \( E \) of measure less than \( \delta \) such that on \( E^C \), \( \{f_n\} \) converges uniformly. Therefore, for \( p \) large enough, and \( n > p \),

\[
\int_{E^C} |f_p - f_n| d\mu < 1
\]

which implies

\[
\int_{E^C} |f_n| d\mu < 1 + \int_\Omega |f_p| d\mu.
\]

Then since there are only finitely many functions, \( f_n \) with \( n \leq p \), there exists a constant, \( M_1 \) such that for all \( n \),

\[
\int_{E^C} |f_n| d\mu < M_1.
\]

But also,

\[
\int_\Omega |f_m| d\mu = \int_{E^C} |f_m| d\mu + \int_E |f_m| d\mu \\
\leq M_1 + 1 = M.
\]

Therefore, by Fatou’s lemma,

\[
\int_\Omega |f| d\mu \leq \lim \inf_{n \to \infty} \int |f_n| d\mu \leq M,
\]

showing that \( f \in L^1 \) as hoped.

Now \( \mathcal{S} \cup \{f\} \) is uniformly integrable so there exists \( \delta_1 > 0 \) such that if \( \mu(E) < \delta_1 \), then \( \int_E |g| d\mu < \varepsilon/3 \) for all \( g \in \mathcal{S} \cup \{f\} \). By Egoroff’s theorem, there exists a set, \( F \) with \( \mu(F) < \delta_1 \) such that \( f_n \) converges uniformly to \( f \) on \( F^C \). Therefore, there exists \( N \) such that if \( n > N \), then

\[
\int_{F^C} |f - f_n| d\mu < \frac{\varepsilon}{3}.
\]
It follows that for $n > N$,
\[
\int_{\Omega} |f - f_n| \, d\mu \leq \int_{F^c} |f - f_n| \, d\mu + \int_F |f| \, d\mu + \int_F |f_n| \, d\mu < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]
which verifies \textit{J35.24}.

9.6 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \cdots \}$ and $\mu(S) =$ number of elements in $S$. If $f : \Omega \to \mathbb{C}$ what is meant by $\int f \, d\mu$? Which functions are in $L^1(\Omega)$? Which functions are measurable?

2. Show that for $f \geq 0$ and measurable, $\int f \, d\mu \equiv \lim_{h \to 0^+} \sum_{i=1}^{\infty} h \mu([ih < f])$.

3. For the measure space of Problem \textit{J35.2}, give an example of a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function $f$, such that inequality is obtained in Fatou's lemma.

4. Fill in all the details of the proof of Lemma \textit{J35.2}.

5. Let $\sum_{i=1}^{\infty} c_i \chi_{E_i}(\omega) = s(\omega)$ be a nonnegative simple function for which the $c_i$ are the distinct nonzero values. Show with the aid of the monotone convergence theorem that the two definitions of the Lebesgue integral given in the chapter are equivalent.

6. Suppose $(\Omega, \mu)$ is a finite measure space and $\mathcal{G} \subseteq L^1(\Omega)$. Show $\mathcal{G}$ is uniformly integrable and bounded in $L^1(\Omega)$ if there exists an increasing function $h$ which satisfies
\[
\lim_{t \to \infty} \frac{h(t)}{t} = \infty, \quad \sup \left\{ \int_{\Omega} h(|f|) \, d\mu : f \in \mathcal{G} \right\} < \infty.
\]
$\mathcal{G}$ is bounded if there is some number, $M$ such that
\[
\int |f| \, d\mu \leq M
\]
for all $f \in \mathcal{G}$.

7. Let $\{a_n\}, \{b_n\}$ be sequences in $[-\infty, \infty]$ and $a \in \mathbb{R}$. Show
\[
\lim_{n \to \infty} \inf (a - a_n) = a - \lim_{n \to \infty} \sup a_n.
\]
This was used in the proof of the Dominated convergence theorem. Also show
\[
\lim_{n \to \infty} \sup (-a_n) = -\lim_{n \to \infty} \inf (a_n)
\]
246  

CHAPTER 9. ABSTRACT MEASURE AND INTEGRATION  

\[ \lim \sup_{n \to \infty} (a_n + b_n) \leq \lim \sup_{n \to \infty} a_n + \lim \sup_{n \to \infty} b_n \]

provided no sum is of the form \( \infty - \infty \). Also show strict inequality can hold in the inequality. State and prove corresponding statements for \( \lim \inf \).

8. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and suppose \(f, g : \Omega \to (-\infty, \infty] \) are measurable. Prove the sets

\[ \{ \omega : f(\omega) < g(\omega) \} \text{ and } \{ \omega : f(\omega) = g(\omega) \} \]

are measurable. **Hint:** The easy way to do this is to write

\[ \{ \omega : f(\omega) < g(\omega) \} = \bigcup_{r \in \mathbb{Q}} [f < r] \cap [g > r]. \]

Note that \(l(x, y) = x - y\) is not continuous on \((-\infty, \infty]\) so the obvious idea doesn’t work.

9. Let \(\{f_n\}\) be a sequence of real or complex valued measurable functions. Let

\[ S = \{ \omega : \{f_n(\omega)\} \text{ converges} \}. \]

Show \(S\) is measurable. **Hint:** You might try to exhibit the set where \(f_n\) converges in terms of countable unions and intersections using the definition of a Cauchy sequence.

10. Let \((\Omega, \mathcal{S}, \mu)\) be a measure space and let \(f\) be a nonnegative measurable function defined on \(\Omega\). Also let \(\phi : [0, \infty) \to [0, \infty)\) be strictly increasing and have a continuous derivative and \(\phi(0) = 0\). Suppose \(f\) is bounded and that \(0 \leq \phi(f(\omega)) \leq M\) for some number, \(M\). Show that

\[ \int_{\Omega} \phi(f) \, d\mu = \int_0^\infty \phi'(s) \mu([s < f]) \, ds, \]

where the integral on the right is the ordinary improper Riemann integral. **Hint:** First note that \(s \to \phi'(s) \mu([s < f])\) is Riemann integrable because \(\phi'\) is continuous and \(s \to \mu([s < f])\) is a nonincreasing function, hence Riemann integrable. From the second description of the Lebesgue integral and the assumption that \(\phi(f(\omega)) \leq M\), argue that for \([M/h]\) the greatest integer less than \(M/h\),

\[ \int_{\Omega} \phi(f) \, d\mu = \sup_{h > 0} \sum_{i=1}^{[M/h]} h\mu([ih < \phi(f)]) \]

\[ = \sup_{h > 0} \sum_{i=1}^{[M/h]} h\mu([\phi^{-1}(ih) < f]) \]

\[ = \sup_{h > 0} \sum_{i=1}^{[M/h]} \frac{h}{\Delta_i} \mu([\phi^{-1}(ih) < f])] \]
where $\Delta_i = (\phi^{-1}(ih) - \phi^{-1}((i-1)h))$. Now use the mean value theorem to write

$$\Delta_i = (\phi^{-1})'(t_i) h$$

$$= \frac{1}{\phi'(\phi^{-1}(t_i))} h$$

for some $t_i$ between $(i-1)h$ and $ih$. Therefore, the right side is of the form

$$\sup_{h} \left\{ \frac{[M/h]}{\sum_{i=1}^{[M/h]} \phi'(\phi^{-1}(t_i)) \Delta_i \mu \left( \left[ \phi^{-1}(ih) < f \right] \right) } \right\}$$

where $\phi^{-1}(t_i) \in (\phi^{-1}((i-1)h), \phi^{-1}(ih))$. Argue that if $t_i$ were replaced with $ih$, this would be a Riemann sum for the Riemann integral

$$\int_0^{\phi^{-1}(M)} \phi'(t) \mu \left( \left[ t < f \right] \right) dt = \int_0^{\infty} \phi'(t) \mu \left( \left[ t < f \right] \right) dt.$$

11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f_n$ converges uniformly to $f$ and that $f_n$ is in $L^1(\Omega)$. When is

$$\lim_{n \to \infty} \int f_n d\mu = \int f d\mu?$$

12. Suppose $u_n(t)$ is a differentiable function for $t \in (a, b)$ and suppose that for $t \in (a, b), |

$$|u_n(t)|, |u'_n(t)| < K_n$$

where $\sum_{n=1}^{\infty} K_n < \infty$. Show

$$\left( \sum_{n=1}^{\infty} u_n(t) \right)' = \sum_{n=1}^{\infty} u'_n(t).$$

**Hint:** This is an exercise in the use of the dominated convergence theorem and the mean value theorem.

13. Show that $\{ \sum_{i=1}^{\infty} 2^{-n} \mu([i2^{-n} < f]) \}$ for $f$ a nonnegative measurable function is an increasing sequence. Could you define

$$\int f d\mu \equiv \lim_{n \to \infty} \sum_{i=1}^{\infty} 2^{-n} \mu \left( \left[ i2^{-n} < f \right] \right)$$

and would it be equivalent to the above definitions of the Lebesgue integral?
14. Suppose \( \{f_n\} \) is a sequence of nonnegative measurable functions defined on a measure space, \((\Omega, \mathcal{F}, \mu)\). Show that

\[
\int \sum_{k=1}^{\infty} f_k \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.
\]

**Hint:** Use the monotone convergence theorem along with the fact the integral is linear.
Chapter 10

The Construction Of Measures

10.1 Outer Measures

What are some examples of measure spaces? In this chapter, a general procedure is discussed called the method of outer measures. It is due to Caratheodory (1918). This approach shows how to obtain measure spaces starting with an outer measure. This will then be used to construct measures determined by positive linear functionals.

Definition 10.1.1 Let $\Omega$ be a nonempty set and let $\mu : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ satisfy

\[
\mu(\emptyset) = 0,
\]

If $A \subseteq B$, then $\mu(A) \leq \mu(B)$,

\[
\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i).
\]

Such a function is called an outer measure. For $E \subseteq \Omega$, $E$ is $\mu$ measurable if for all $S \subseteq \Omega$,

\[
\mu(S) = \mu(S \setminus E) + \mu(S \cap E).
\] (10.1.1)

To help in remembering 10.1.1, think of a measurable set, $E$, as a process which divides a given set into two pieces, the part in $E$ and the part not in $E$ as in 10.1.1. In the Bible, there are four incidents recorded in which a process of division resulted in more stuff than was originally present. Measurable sets are exactly

---

1 Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^3$ into five disjoint pieces and
those for which no such miracle occurs. You might think of the measurable sets as the nonmiraculous sets. The idea is to show that they form a σ algebra on which the outer measure, \( \mu \), is a measure.

First here is a definition and a lemma.

**Definition 10.1.2** \( (\mu|S)(A) \equiv \mu(S \cap A) \) for all \( A \subseteq \Omega \). Thus \( \mu|S \) is the name of a new outer measure, called \( \mu \) restricted to \( S \).

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.

**Lemma 10.1.3** If \( A \) is \( \mu \) measurable, then \( A \) is \( \mu|S \) measurable.

**Proof:** Suppose \( A \) is \( \mu \) measurable. It is desired to to show that for all \( T \subseteq \Omega \),

\[
(\mu|S)(T) = (\mu|S)(T \cap A) + (\mu|S)(T \setminus A).
\]

Thus it is desired to show

\[
\mu(S \cap T) = \mu(T \cap A \cap S) + \mu(T \cap S \cap A^C).
\] (10.1.2)

But (10.1.2) holds because \( A \) is \( \mu \) measurable. Apply Definition 10.1.1 to \( S \cap T \) instead of \( S \).

If \( A \) is \( \mu|S \) measurable, it does not follow that \( A \) is \( \mu \) measurable. Indeed, if you believe in the existence of non measurable sets, you could let \( A = S \) for such a \( \mu \) non measurable set and verify that \( S \) is \( \mu|S \) measurable.

The next theorem is the main result on outer measures. It is a very general result which applies whenever one has an outer measure on the power set of any set. This theorem will be referred to as Caratheodory’s procedure in the rest of the book.

**Theorem 10.1.4** The collection of \( \mu \) measurable sets, \( S \), forms a σ algebra and

\[
\text{If } F_i \in S, F_i \cap F_j = \emptyset, \text{ then } \mu(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i). \tag{10.1.3}
\]

If \( \cdots F_n \subseteq F_{n+1} \subseteq \cdots \), then if \( F = \bigcup_{n=1}^{\infty} F_n \) and \( F_n \in S \), it follows that

\[
\mu(F) = \lim_{n \to \infty} \mu(F_n). \tag{10.1.4}
\]

If \( \cdots F_n \supseteq F_{n+1} \supseteq \cdots \), and if \( F = \cap_{n=1}^{\infty} F_n \) for \( F_n \in S \) then if \( \mu(F_1) < \infty \),

\[
\mu(F) = \lim_{n \to \infty} \mu(F_n). \tag{10.1.5}
\]

Also, \((S, \mu)\) is complete. By this it is meant that if \( F \in S \) and if \( E \subseteq \Omega \) with \( \mu(E \setminus F) + \mu(F \setminus E) = 0 \), then \( E \in S \).

assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.
Proof: First note that $\emptyset$ and $\Omega$ are obviously in $\mathcal{S}$. Now suppose $A, B \in \mathcal{S}$. I will show $A \setminus B \equiv A \cap B^C$ is in $\mathcal{S}$. To do so, consider the following picture.

Since $\mu$ is subadditive,

$$
\mu(S) \leq \mu(S \cap A \cap B^C) + \mu(A \cap B \cap S) + \mu(S \cap B \cap A^C) + \mu(S \cap A^C \cap B^C).
$$

Now using $A, B \in \mathcal{S},$

$$
\mu(S) \leq \mu(S \cap A \cap B^C) + \mu(S \cap A \cap B) + \mu(S \cap A^C \cap B^C) + \mu(S \cap A^C \cap B^C) + \mu(S \cap A^C \cap B^C)
$$

$$
= \mu(S \cap A) + \mu(S \cap A^C) = \mu(S)
$$

It follows equality holds in the above. Now observe using the picture if you like that

$$(A \cap B \cap S) \cup (S \cap B \cap A^C) \cup (S \cap A^C \cap B^C) = S \setminus (A \setminus B)$$

and therefore,

$$
\mu(S) = \mu(S \cap A \cap B^C) + \mu(A \cap B \cap S) + \mu(S \cap B \cap A^C) + \mu(S \cap A^C \cap B^C)
$$

$$
\geq \mu(S \cap (A \setminus B)) + \mu(S \setminus (A \setminus B)).
$$

Therefore, since $S$ is arbitrary, this shows $A \setminus B \in \mathcal{S}$.

Since $\Omega \in \mathcal{S}$, this shows that $A \in \mathcal{S}$ if and only if $A^C \in \mathcal{S}$. Now if $A, B \in \mathcal{S}$, $A \cup B = (A^C \cap B^C)^C = (A^C \setminus B)^C \in \mathcal{S}$. By induction, if $A_1, \cdots, A_n \in \mathcal{S}$, then so is $\bigcup_{i=1}^n A_i$. If $A, B \in \mathcal{S}$, with $A \cap B = \emptyset$,

$$
\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \setminus A) = \mu(A) + \mu(B).
$$
By induction, if $A_i \cap A_j = \emptyset$ and $A_i \in \mathcal{S}$, $\mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$.

Now let $A = \cup_{i=1}^{\infty} A_i$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\sum_{i=1}^{\infty} \mu(A_i) \geq \mu(A) \geq \mu(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i).$$

Since this holds for all $n$, you can take the limit as $n \to \infty$ and conclude,

$$\sum_{i=1}^{\infty} \mu(A_i) = \mu(A)$$

which establishes 10.1.3. Part 10.1.4 follows from part 10.1.3 just as in the proof of Theorem 9.1.5 on Page 208. That is, letting $F_0 \equiv \emptyset$, use part 10.1.3 to write

$$\mu(F) = \mu(\cup_{k=1}^{\infty} (F_k \setminus F_{k-1})) = \sum_{k=1}^{\infty} \mu(F_k \setminus F_{k-1})$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} (\mu(F_k) - \mu(F_{k-1})) = \lim_{n \to \infty} \mu(F_n).$$

In order to establish 10.1.5, let the $F_n$ be as given there. Then from what was just shown,

$$\mu(F_1 \setminus F_n) + \mu(F_n) = \mu(F_1)$$

Then, since $(F_1 \setminus F_n)$ increases to $(F_1 \setminus F)$, 10.1.3 implies

$$\lim_{n \to \infty} (\mu(F_1 \setminus F_n)) = \lim_{n \to \infty} (\mu(F_1) - \mu(F_n)) = \mu(F_1 \setminus F).$$

Now I don’t know whether $F \in \mathcal{S}$ and so all that can be said is that

$$\mu(F_1 \setminus F) + \mu(F) \geq \mu(F_1)$$

but this implies

$$\mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F).$$

Hence

$$\lim_{n \to \infty} (\mu(F_1) - \mu(F_n)) = \mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$$

which implies

$$\lim_{n \to \infty} \mu(F_n) \leq \mu(F).$$

But since $F \subseteq F_n$,

$$\mu(F) \leq \lim_{n \to \infty} \mu(F_n)$$

and this establishes 10.1.5. Note that it was assumed $\mu(F_1) < \infty$ because $\mu(F_1)$ was subtracted from both sides.
It remains to show $S$ is closed under countable unions. Recall that if $A \in S$, then $A^C \in S$ and $S$ is closed under finite unions. Let $A_i \in S, A = \bigcup_{i=1}^{\infty} A_i, B_n = \bigcup_{i=1}^{n} A_i$. Then

$$
\mu(S) = \mu(S \cap B_n) + \mu(S \setminus B_n)
$$

(10.1.6)

By Lemma 10.1.3, $B_n$ is $(\mu|S)$ measurable and so is $B^C_n$. I want to show $\mu(S) \geq \mu(S \setminus A) + \mu(S \cap A)$. If $\mu(S) = \infty$, there is nothing to prove. Assume $\mu(S) < \infty$. Then apply Parts 10.1.4 and 10.1.5 to the outer measure, $\mu|S$ in 10.1.6 and let $n \to \infty$. Thus

$$
B_n \uparrow A, \ B^C_n \downarrow A^C
$$

and this yields

$$
\mu(S) = (\mu|S)(A) + (\mu|S)(A^C) = \mu(S \cap A) + \mu(S \setminus A).
$$

Therefore $A \in S$ and this proves Parts 10.1.4 and 10.1.5. It remains to prove the last assertion about the measure being complete.

Let $F \in S$ and let $\mu(E \setminus F) + \mu(F \setminus E) = 0$. Consider the following picture.

Then referring to this picture and using $F \in S$,

$$
\mu(S) \leq \mu(S \cap E) + \mu(S \setminus E)
$$

$$
\leq \mu(S \cap E \cap F) + \mu((S \cap E) \setminus F) + \mu(S \setminus F) + \mu(F \setminus E)
$$

$$
\leq \mu(S \cap F) + \mu(E \setminus F) + \mu(S \setminus F) + \mu(F \setminus E)
$$

$$
= \mu(S \cap F) + \mu(S \setminus F) = \mu(S)
$$

Hence $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$ and so $E \in S$. This shows that $(S, \mu)$ is complete and proves the theorem.

Completeness usually occurs in the following form. $E \subseteq F \in S$ and $\mu(F) = 0$. Then $E \in S$.

**Proposition 10.1.5** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\bar{\mu}$ be the outer measure determined by $\mu$. Also denote as $\bar{\mathcal{F}}$, the $\sigma$ algebra of $\bar{\mu}$ measurable sets. Thus $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ is a complete measure space in which $\bar{\mathcal{F}} \supseteq \mathcal{F}$ and $\bar{\mu} = \mu$ on $\mathcal{F}$. Also, in this situation, if $\bar{\mu}(E) = 0$, then $E \in \bar{\mathcal{F}}$. No new sets are obtained if $(\Omega, \mathcal{F}, \mu)$ is already complete.
CHAPTER 10. THE CONSTRUCTION OF MEASURES

Proof: All that remains to show is the last claim. But this is obvious because
if $S$ is a set,

$$
\bar{\mu}(S) \leq \bar{\mu}(S \cap E) + \bar{\mu}(S \setminus E) \\
\leq \bar{\mu}(E) + \bar{\mu}(S \setminus E) \\
= \bar{\mu}(S \setminus E) \leq \bar{\mu}(S)
$$

and so all inequalities are equal signs.

Suppose now that $(\Omega, \mathcal{F}, \mu)$ is complete. Let $F \in \bar{\mathcal{F}}$. Then there exists $E \supseteq F$ such that $\mu(E) = \bar{\mu}(F)$. This is obvious if $\bar{\mu}(F) = \infty$. Otherwise, let $E_n \supseteq F, \bar{\mu}(F) + \frac{1}{n} > \mu(E_n)$. Just let $E = \cap_n E_n$. Now $\bar{\mu}(E \setminus F) = 0$. Now also, there exists a set of $\mathcal{F}$ called $W$ such that $\mu(W) = 0$ and $W \supseteq E \setminus F$. Thus $E \setminus F \subseteq W$, a set of measure zero. Hence by completeness of $(\Omega, \mathcal{F}, \mu)$, it must be the case that $E \setminus F = E \cap F^C = G \in \mathcal{F}$. Then taking complements of both sides, $E^C \cup F = G^C \in \mathcal{F}$. Now take intersections with $E$. $F \in E \cap G^C \in \mathcal{F}$. □

In the case of a Hausdorff topological space, the following lemma gives conditions under which the $\sigma$ algebra of $\mu$ measurable sets for an outer measure $\mu$ contains the Borel sets. In words, it assumes the outer measure is inner regular on open sets and outer regular on all sets. Also it assumes you can approximate the measure of an open set with a compact set and the measure of a compact set with an open set.

Lemma 10.1.6 Let $\Omega$ be a Hausdorff space and suppose $\mu$ is an outer measure satisfying $\mu$ is finite on compact sets and the following conditions,

1. $\mu(E) = \inf \{\mu(V), V \supseteq E, V \text{ open}\}$ for all $E$. (Outer regularity.)
2. For every open set $V, \mu(V) = \sup \{\mu(K) : K \subseteq V, K \text{ compact}\}$ (Inner regularity on open sets.)
3. If $A, B$ are compact disjoint sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Then the following hold.

1. If $\varepsilon > 0$ and if $K$ is compact, there exists $V$ open such that $V \supseteq K$ and $\mu(V \setminus K) < \varepsilon$

2. If $\varepsilon > 0$ and if $V$ is open with $\mu(V) < \infty$, there exists a compact subset $K$ of $V$ such that $\mu(V \setminus K) < \varepsilon$

3. Then the $\mu$ measurable sets $\mathcal{S}$ contain the Borel sets and also $\mu$ is inner regular on every open set and for every $E \in \mathcal{S}$ with $\mu(E) < \infty$. Here $\mathcal{S}$ consists of those subsets of $\Omega$ $E$ with the property that for any subset $S$ of $\Omega$,

$$
\mu(S) = \mu(S \cap E) + \mu(S \cap E^C)
$$
Proof: First we establish $\mathfrak{I}$ and $\mathfrak{L}$ and use them to establish the last assertion. Consider $\mathfrak{L}$. Suppose it is not true. Then there exists an open set $V$ having $\mu (V) < \infty$ but for all $K \subseteq V, \mu (V \setminus K) \geq \varepsilon$ for some $\varepsilon > 0$. By inner regularity on open sets, there exists $K_1 \subseteq V, K_1$ compact, such that $\mu (K_1) \geq \varepsilon/2$. Now by assumption, $\mu (V \setminus K_1) \geq \varepsilon$ and so by inner regularity on open sets again, there exists compact $K_2 \subseteq V \setminus K_1$ such that $\mu (K_2) \geq \varepsilon/2$. Continuing this way, there is a sequence of disjoint compact sets contained in $V \{K_i\}$ such that $\mu (K_i) \geq \varepsilon/2$.

Now this is an obvious contradiction because by $\mathfrak{K}$,

$$\mu (V) \geq \mu (\bigcup_{i=1}^{n} K_i) = \sum_{i=1}^{n} \mu (K_i) \geq n \frac{\varepsilon}{2}$$

for each $n$, contradicting $\mu (V) < \infty$.

Next consider $\mathfrak{L}$. By outer regularity, there exists an open set $W \supseteq K$ such that $\mu (W) < \mu (K) + 1$. By $\mathfrak{L}$, there exists compact $K_1 \subseteq W \setminus K$ such that $\mu ((W \setminus K) \setminus K_1) < \varepsilon$. Then consider $V \equiv W \setminus K_1$. This is an open set containing $K$ and from what was just shown,

$$\mu ((W \setminus K_1) \setminus K) = \mu ((W \setminus K) \setminus K_1) < \varepsilon.$$ 

Now consider the last assertion.

Define

$$\mathcal{S}_1 = \{ E \in \mathcal{P} (\Omega) : E \cap K \in \mathcal{S} \}$$

for all compact $K$.

First it will be shown the compact sets are in $\mathcal{S}$. From this it will follow the closed sets are in $\mathcal{S}_1$. Then you show $\mathcal{S}_1 = \mathcal{S}$. Thus $\mathcal{S}_1 = \mathcal{S}$ is a $\sigma$ algebra and so it contains the Borel sets. Finally you show the inner regularity assertion.

Claim 1: Compact sets are in $\mathcal{S}$.

Proof of claim: Let $V$ be an open set with $\mu (V) < \infty$. I will show that for $C$ compact,

$$\mu (V) \geq \mu (V \setminus C) + \mu (V \cap C).$$

Here is a diagram to help keep things straight.
By \( \mathcal{A} \) there exists a compact set \( K \subseteq V \setminus C \) such that
\[
\mu((V \setminus C) \setminus K) < \varepsilon.
\]
and a compact set \( H \subseteq V \) such that
\[
\mu(V \setminus H) < \varepsilon.
\]
Thus \( \mu(V) \leq \mu(V \setminus H) + \mu(H) < \varepsilon + \mu(H) \). Then
\[
\mu(V) \leq \mu(H) + \varepsilon \leq \mu(H \cap C) + \mu(H \setminus C) + \varepsilon
\]
\[
\leq \mu(V \cap C) + \mu(V \setminus C) + \varepsilon \leq \mu(H \cap C) + \mu(K) + 3\varepsilon
\]
By \( \mathcal{B} \),
\[
= \mu(H \cap C) + \mu(K) + 3\varepsilon = \mu((H \cap C) \cup K) + 3\varepsilon \leq \mu(V) + 3\varepsilon.
\]
Since \( \varepsilon \) is arbitrary, this shows that
\[
\mu(V) = \mu(V \setminus C) + \mu(V \cap C). \tag{10.1.7}
\]

Of course \( \mathcal{C} \) is exactly what needs to be shown for arbitrary \( S \) in place of \( V \). It suffices to consider only \( S \) having \( \mu(S) < \infty \). If \( S \subseteq \Omega \), with \( \mu(S) < \infty \), let \( V \supseteq S, \mu(S) + \varepsilon > \mu(V) \). Then from what was just shown, if \( C \) is compact,
\[
\varepsilon + \mu(S) > \mu(V) = \mu(V \setminus C) + \mu(V \cap C)
\]
\[
\geq \mu(S \setminus C) + \mu(S \cap C).
\]
Since \( \varepsilon \) is arbitrary, this shows the compact sets are in \( S \). This proves the claim.

As discussed above, this verifies the closed sets are in \( \mathcal{S}_1 \) because if \( H \) is closed and \( C \) is compact, then \( H \cap C \in \mathcal{S} \). If \( \mathcal{S}_1 \) is a \( \sigma \) algebra, this will show that \( \mathcal{S}_1 \) contains the Borel sets. Thus I first show \( \mathcal{S}_1 \) is a \( \sigma \) algebra.

To see that \( \mathcal{S}_1 \) is closed with respect to taking complements, let \( E \in \mathcal{S}_1 \) and \( K \) a compact set.
\[
K = (E^C \cap K) \cup (E \cap K).
\]
Then from the fact, just established, that the compact sets are in \( \mathcal{S} \),
\[
E^C \cap K = K \setminus (E \cap K) \in \mathcal{S}.
\]
$S_1$ is closed under countable unions because if $K$ is a compact set and $E_n \in S_1$,

$$K \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} K \cap E_n \in S$$

because it is a countable union of sets of $S$. Thus $S_1$ is a $\sigma$ algebra.

Therefore, if $E \in S$ and $K$ is a compact set, just shown to be in $S$, it follows $K \cap E \in S$ because $S$ is a $\sigma$ algebra which contains the compact sets and so $S_1 \supseteq S$. It remains to verify $S_1 \subseteq S$. Recall that

$$S_1 \equiv \{ E : E \cap K \in S \text{ for all } K \text{ compact} \}$$

Let $E \in S_1$ and let $V$ be an open set with $\mu(V) < \infty$ and choose $K \subseteq V$ such that $\mu(V \setminus K) < \varepsilon$. Then since $E \in S_1$, it follows $E \cap K, E^C \cap K \in S$ and so

$$\mu(V) \leq \mu(V \setminus E) + \mu(V \cap E) \leq \mu(K \setminus E) + \mu(K \cap E) + 2\varepsilon = \mu(K) + 2\varepsilon \leq \mu(V) + 3\varepsilon$$

Since $\varepsilon$ is arbitrary, this shows

$$\mu(V) = \mu(V \setminus E) + \mu(V \cap E)$$

which would show $E \in S$ if $V$ were an arbitrary set.

Now let $S \subseteq \Omega$ be such an arbitrary set. If $\mu(S) = \infty$, then

$$\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$$

If $\mu(S) < \infty$, let

$$V \supseteq S, \mu(S) + \varepsilon \geq \mu(V)$$

Then

$$\mu(S) + \varepsilon \geq \mu(V) = \mu(V \setminus E) + \mu(V \cap E) \geq \mu(S \setminus E) + \mu(S \cap E)$$

Since $\varepsilon$ is arbitrary, this shows that $E \in S$ and so $S_1 = S$. Thus $S \supseteq$ Borel sets as claimed.

From $\mu$ is inner regular on all open sets. It remains to show that

$$\mu(F) = \sup\{ \mu(K) : K \subseteq F \}$$

for all $F \in S$ with $\mu(F) < \infty$. It might help to refer to the following crude picture to keep things straight. It also might not help. I am not sure. In the picture, the green marks the boundary of $V$ while red marks $U$ and black marks $F$ and $V^C \cap K$. This last set is as shown because $K$ is a compact subset of $U$ such that $\mu(U \setminus K) < \varepsilon$. 

![Diagram](image-url)
Let \( \mu(F) < \infty \) and let \( U \) be an open set, \( U \supseteq F \), \( \mu(U) < \infty \). Let \( V \) be open, \( V \supseteq U \setminus F \), and

\[
\mu(V \setminus (U \setminus F)) < \varepsilon.
\]

(This can be obtained as follows, because \( \mu \) is a measure on \( \mathcal{S} \).

\[
\mu(V) = \mu(U \setminus F) + \mu(V \setminus (U \setminus F))
\]

Thus from the outer regularity of \( \mu \), above, there exists \( V \) such that it contains \( U \setminus F \) and

\[
\mu(U \setminus F) + \varepsilon > \mu(V).
\]

and so

\[
\mu(V \setminus (U \setminus F)) = \mu(V) - \mu(U \setminus F) < \varepsilon.
\]

Also,

\[
V \setminus (U \setminus F) = V \cap (U \cap F^C)^C
\]

\[
= V \cap [U^C \cup F]
\]

\[
= (V \cap F) \cup (V \cap U^C)
\]

\[
\supseteq V \cap F
\]

and so

\[
\mu(V \cap F) \leq \mu(V \setminus (U \setminus F)) < \varepsilon.
\]

Since \( V \supseteq U \cap F^C \), \( V^C \subseteq U^C \cup F \) so \( U \cap V^C \subseteq U \cap F = F \). Hence \( U \cap V^C \) is a subset of \( F \). Now let \( K \subseteq U \), \( \mu(U \setminus K) < \varepsilon \). Thus \( K \cap V^C \) is a compact subset of \( F \) and

\[
\mu(F) = \mu(V \cap F) + \mu(F \setminus V)
\]

\[
< \varepsilon + \mu(F \setminus V) \leq \varepsilon + \mu(U \cap V^C) \leq 2\varepsilon + \mu(K \cap V^C).
\]

Since \( \varepsilon \) is arbitrary, this proves the second part of the lemma. \( \blacksquare \)

Where do outer measures come from? One way to obtain an outer measure is to start with a measure \( \mu \), defined on a \( \sigma \) algebra of sets, \( \mathcal{S} \), and use the following definition of the outer measure induced by the measure.

**Definition 10.1.7** Let \( \mu \) be a measure defined on a \( \sigma \) algebra of sets, \( \mathcal{S} \subseteq \mathcal{P}(\Omega) \). Then the outer measure induced by \( \mu \), denoted by \( \overline{\mu} \) is defined on \( \mathcal{P}(\Omega) \) as

\[
\overline{\mu}(E) = \inf\{\mu(F) : F \in \mathcal{S} \text{ and } F \supseteq E\}.
\]

A measure space, \( (\mathcal{S}, \Omega, \mu) \) is \( \sigma \) finite if there exist measurable sets, \( \Omega_i \) with \( \mu(\Omega_i) < \infty \) and \( \Omega = \cup_{i=1}^{\infty} \Omega_i \).

You should prove the following lemma.

**Lemma 10.1.8** If \( (\mathcal{S}, \Omega, \mu) \) is \( \sigma \) finite then there exist disjoint measurable sets, \( \{B_n\} \) such that \( \mu(B_n) < \infty \) and \( \cup_{n=1}^{\infty} B_n = \Omega \).
The following lemma deals with the outer measure generated by a measure which is \(\sigma\) finite. It says that if the given measure is \(\sigma\) finite and complete then no new measurable sets are gained by going to the induced outer measure and then considering the measurable sets in the sense of Caratheodory.

**Lemma 10.1.9** Let \((\Omega, \mathcal{S}, \mu)\) be any measure space and let \(\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]\) be the outer measure induced by \(\mu\). Then \(\overline{\mu}\) is an outer measure as claimed and if \(\overline{\mathcal{S}}\) is the set of \(\overline{\mu}\) measurable sets in the sense of Caratheodory, then \(\overline{\mathcal{S}} \supseteq \mathcal{S}\) and \(\overline{\mu} = \mu\) on \(\mathcal{S}\). Furthermore, if \(\mu\) is \(\sigma\) finite and \((\Omega, \mathcal{S}, \mu)\) is complete, then \(\overline{\mathcal{S}} = \mathcal{S}\).

**Proof:** It is easy to see that \(\overline{\mu}\) is an outer measure. Let \(E \in \mathcal{S}\). The plan is to show \(E \in \overline{\mathcal{S}}\) and \(\overline{\mu}(E) = \mu(E)\). To show this, let \(S \subseteq \Omega\) and then show

\[
\overline{\mu}(S) \geq \overline{\mu}(S \cap E) + \overline{\mu}(S \setminus E).
\] (10.1.8)

This will verify that \(E \in \mathcal{S}\). If \(\overline{\mu}(S) = \infty\), there is nothing to prove, so assume \(\overline{\mu}(S) < \infty\). Thus there exists \(T \in \mathcal{S}\), \(T \supseteq S\), and

\[
\overline{\mu}(S) > \mu(T) - \varepsilon = \mu(T \cap E) + \mu(T \setminus E) - \varepsilon \\
\geq \overline{\mu}(T \cap E) + \overline{\mu}(T \setminus E) - \varepsilon \\
\geq \overline{\mu}(S \cap E) + \overline{\mu}(S \setminus E) - \varepsilon.
\]

Since \(\varepsilon\) is arbitrary, this proves \(\overline{\mu}(S) \leq \overline{\mu}(E)\) and verifies \(\mathcal{S} \subseteq \overline{\mathcal{S}}\). Now if \(E \in \mathcal{S}\) and \(V \supseteq E\) with \(V \in \mathcal{S}\), \(\mu(E) \leq \mu(V)\). Hence, taking inf, \(\mu(E) \leq \overline{\mu}(E)\). But also \(\mu(E) \geq \overline{\mu}(E)\) since \(E \in \mathcal{S}\) and \(E \supseteq E\). Hence

\[
\overline{\mu}(E) \leq \mu(E) \leq \overline{\mu}(E).
\]

Next consider the claim about not getting any new sets from the outer measure in the case the measure space is \(\sigma\) finite and complete.

Suppose first \(F \in \overline{\mathcal{S}}\) and \(\overline{\mu}(F) < \infty\). Then there exists \(E \in \mathcal{S}\) such that \(E \supseteq F\) and \(\mu(E) = \overline{\mu}(F)\). Since \(\overline{\mu}(F) < \infty\),

\[
\overline{\mu}(E \setminus F) = \mu(E) - \overline{\mu}(F) = 0.
\]

Then there exists \(D \supseteq E \setminus F\) such that \(D \in \mathcal{S}\) and \(\mu(D) = \overline{\mu}(E \setminus F) = 0\). Then by completeness of \(\mathcal{S}\), it follows \(E \setminus F \in \mathcal{S}\) and so

\[
E = (E \setminus F) \cup F
\]

Hence \(F = E \setminus (E \setminus F) \in \mathcal{S}\). In the general case where \(\overline{\mu}(F)\) is not known to be finite, let \(\mu(B_n) < \infty\), with \(B_n \cap B_m = \emptyset\) for all \(n \neq m\) and \(\cup_n B_n = \Omega\). Apply what was just shown to \(F \cap B_n\), obtaining each of these is in \(\mathcal{S}\). Then \(F = \cup_n F \cap B_n \in \mathcal{S}\). This proves the Lemma.

Usually \(\Omega\) is not just a set. It is also a topological space. It is very important to consider how the measure is related to this topology. The following definition tells what it means for a measure to be regular.
Definition 10.1.10 Let $\mu$ be a measure on a $\sigma$ algebra $S$, of subsets of $\Omega$, where $(\Omega, \tau)$ is a topological space. $\mu$ is a Borel measure if $S$ contains all Borel sets. $\mu$ is called outer regular if $\mu$ is Borel and for all $E \in S$,

$$\mu(E) = \inf\{\mu(V) : V \text{ is open and } V \supseteq E\}.$$ 

$\mu$ is called inner regular if $\mu$ is Borel and

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, \text{ and } K \text{ is compact}\}.$$ 

If the measure is both outer and inner regular, it is called regular.

There is an interesting situation in which regularity is obtained automatically. To save on words, let $B(E)$ denote the $\sigma$ algebra of Borel sets in $E$, a closed subset of $\mathbb{R}^n$. It is a very interesting fact that every finite measure on $B(E)$ must be regular.

Lemma 10.1.11 Let $\mu$ be a finite measure defined on $B(E)$ where $E$ is a closed subset of $\mathbb{R}^n$. Then for every $F \in B(E)$,

$$\mu(F) = \sup\{\mu(K) : K \subseteq F, \text{ and } K \text{ is closed}\}$$

$$\mu(F) = \inf\{\mu(V) : V \supseteq F, \text{ and } V \text{ is open}\}.$$ 

Proof: For convenience, I will call a measure which satisfies the above two conditions “almost regular”. It would be regular if closed were replaced with compact. First note every open set is the countable union of compact sets and every closed set is the countable intersection of open sets. Here is why. Let $V$ be an open set and let

$$K_k \equiv \{x \in V : \text{dist}(x, V^c) \geq 1/k\}.$$ 

Then clearly the union of the $K_k$ equals $V$ and each is closed because $x \to \text{dist}(x, S)$ is always a continuous function whenever $S$ is any nonempty set. Next, for $K$ closed let

$$V_k \equiv \{x \in E : \text{dist}(x, K) < 1/k\}.$$ 

Clearly the intersection of the $V_k$ equals $K$. Therefore, letting $V$ denote an open set and $K$ a closed set,

$$\mu(V) = \sup\{\mu(K) : K \subseteq V \text{ and } K \text{ is closed}\}$$

$$\mu(K) = \inf\{\mu(V) : V \supseteq K \text{ and } V \text{ is open}\}.$$ 

Also since $V$ is open and $K$ is closed,

$$\mu(V) = \inf\{\mu(U) : U \supseteq V \text{ and } V \text{ is open}\}$$

$$\mu(K) = \sup\{\mu(L) : L \subseteq K \text{ and } L \text{ is closed}\}.$$ 

In words, $\mu$ is almost regular on open and closed sets. Let

$$\mathcal{F} \equiv \{F \in B(E) \text{ such that } \mu \text{ is almost regular on } F\}.$$
10.1. OUTER MEASURES

Then $\mathcal{F}$ contains the open sets. I want to show $\mathcal{F}$ is a $\sigma$ algebra and then it will follow $\mathcal{F} = \mathcal{B}(E)$.

First I will show $\mathcal{F}$ is closed with respect to complements. Let $F \in \mathcal{F}$. Then since $\mu$ is finite and $F$ is inner regular, there exists $K \subseteq F$ such that $\mu(F \setminus K) < \varepsilon$. But $K^C \cap F^C = F \setminus K$ and so $\mu(K^C \setminus F^C) < \varepsilon$ showing that $F^C$ is outer regular. I have just approximated the measure of $F^C$ with the measure of $K^C$, an open set containing $F^C$. A similar argument works to show $F^C$ is inner regular. You start with $V \supseteq F$ such that $\mu(V \setminus F) < \varepsilon$, note $F^C \cap V^C = V \setminus F$, and then conclude $\mu(F^C \setminus V^C) < \varepsilon$, thus approximating $F^C$ with the closed subset, $V^C$.

Next I will show $\mathcal{F}$ is closed with respect to taking countable unions. Let $\{F_k\}$ be a sequence of sets in $\mathcal{F}$. Then since $F_k \in \mathcal{F}$, there exist $K_k \subseteq F_k$ and $\mu(F_k \setminus K_k) < \varepsilon/2^{k+1}$. First choose $m$ large enough that

$$\mu((\bigcup_{k=1}^\infty F_k) \setminus (\bigcup_{k=1}^m F_k)) < \frac{\varepsilon}{2}. $$

Then

$$\mu((\bigcup_{k=1}^m F_k) \setminus (\bigcup_{k=1}^m K_k)) \leq \sum_{k=1}^m \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}$$

and so

$$\mu((\bigcup_{k=1}^\infty F_k) \setminus (\bigcup_{k=1}^m K_k)) \leq \mu((\bigcup_{k=1}^\infty F_k) \setminus (\bigcup_{k=1}^m K_k)) + \mu((\bigcup_{k=1}^m F_k) \setminus (\bigcup_{k=1}^m K_k)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\mu$ is outer regular on $F_k$, there exists $V_k$ such that $\mu(V_k \setminus F_k) < \varepsilon/2^k$. Then

$$\mu((\bigcup_{k=1}^\infty V_k) \setminus (\bigcup_{k=1}^\infty F_k)) \leq \sum_{k=1}^\infty \mu(V_k \setminus F_k) < \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon$$

and this completes the demonstration that $\mathcal{F}$ is a $\sigma$ algebra. This proves the lemma.

**Theorem 10.1.12** Let $\mu$ be a finite measure defined on $\mathcal{B}(E)$ where $E$ is a closed subset of $\mathbb{R}^n$. Then $\mu$ is regular.

**Proof:** From Lemma 10.1.11 $\mu$ is outer regular. Now let $F \in \mathcal{B}(E)$. Then since $\mu$ is finite, there exists $K \subseteq F$ such that $K$ is closed, $K \subseteq F$, and

$$\mu(F) < \mu(K) + \varepsilon.$$
CHAPTER 10. THE CONSTRUCTION OF MEASURES

Then let \( K_k \equiv K \cap B(0,k) \). Thus \( K_k \) is a closed and bounded, hence compact set and \( \bigcup_{k=1}^{\infty} K_k = K \). Therefore, for all \( k \) large enough,

\[
\mu(F) < \mu(K_k) + \varepsilon < \sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ compact} \} + \varepsilon \leq \mu(F) + \varepsilon
\]

Since \( \varepsilon \) was arbitrary, it follows

\[
\sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ compact} \} = \mu(F).
\]

This proves the theorem.

It will be assumed in what follows that \((\Omega, \tau)\) is a locally compact Hausdorff space. This means it is Hausdorff: If \( p, q \in \Omega \) such that \( p \neq q \), there exist open sets, \( U_p \) and \( U_q \) containing \( p \) and \( q \) respectively such that \( U_p \cap U_q = \emptyset \) and Locally compact: There exists a basis of open sets for the topology, \( \mathcal{B} \) such that for each \( U \in \mathcal{B}, \overline{U} \) is compact. Recall \( \mathcal{B} \) is a basis for the topology if \( \bigcup \mathcal{B} = \Omega \) and if every open set in \( \tau \) is the union of sets of \( \mathcal{B} \). Also recall a Hausdorff space is normal if whenever \( H \) and \( C \) are two closed sets, there exist disjoint open sets, \( U_H \) and \( U_C \) containing \( H \) and \( C \) respectively. A regular space is one which has the property that if \( p \) is a point not in \( H \), a closed set, then there exist disjoint open sets, \( U_p \) and \( U_H \) containing \( p \) and \( H \) respectively.

10.2 Urysohn’s lemma

Urysohn’s lemma which characterizes normal spaces is a very important result which is useful in general topology and in the construction of measures. Because it is somewhat technical a proof is given for the part which is needed.

**Theorem 10.2.1** (Urysohn) Let \((X, \tau)\) be normal and let \( H \subseteq U \) where \( H \) is closed and \( U \) is open. Then there exists \( g : X \to [0,1] \) such that \( g \) is continuous, \( g(x) = 1 \) on \( H \) and \( g(x) = 0 \) if \( x \notin U \).

**Proof:** Let \( D \equiv \{r_n\}_{n=1}^{\infty} \) be the rational numbers in \((0,1] \). Choose \( V_{r_1} \) an open set such that

\[
H \subseteq V_{r_1} \subseteq \overline{V}_{r_1} \subseteq U.
\]

This can be done by applying the assumption that \( X \) is normal to the disjoint closed sets, \( H \) and \( U^C \), to obtain open sets \( V \) and \( W \) with

\[
H \subseteq V, \ U^C \subseteq W, \text{ and } V \cap W = \emptyset.
\]

Then

\[
H \subseteq V \subseteq \overline{V}, \ \overline{V} \cap U^C = \emptyset
\]

and so let \( V_{r_1} = V \).
Suppose \( V_1, \ldots, V_k \) have been chosen and list the rational numbers \( r_1, \ldots, r_k \) in order,
\[
r_{l_1} < r_{l_2} < \cdots < r_{l_k} \quad \text{for} \quad \{l_1, \cdots, l_k\} = \{1, \cdots, k\}.
\]
If \( r_{k+1} > r_{l_k} \) then letting \( p = r_{l_k} \), let \( V_{rk+1} \) satisfy
\[
\overline{V}_p \subseteq V_{rk+1} \subseteq V_{rk+1} \subseteq U.
\]
If \( r_{k+1} \in (r_{l_i}, r_{l_{i+1}}) \), let \( p = r_{l_i} \) and let \( q = r_{l_{i+1}} \). Then let \( V_{rk+1} \) satisfy
\[
\overline{V}_p \subseteq V_{rk+1} \subseteq V_{rk+1} \subseteq V_q.
\]
If \( r_{k+1} < r_{l_1} \), let \( p = r_{l_1} \) and let \( V_{rk+1} \) satisfy
\[
H \subseteq V_{rk+1} \subseteq V_{rk+1} \subseteq V_p.
\]
Thus there exist open sets \( V_r \) for each \( r \in \mathbb{Q} \cap (0, 1) \) with the property that if \( r < s \),
\[
H \subseteq V_r \subseteq V_r \subseteq V_s \subseteq V_s \subseteq U.
\]
Now let
\[
f(x) = \min (\inf \{t \in D : x \in V_t\}, 1) , \quad f(x) \equiv 1 \quad \text{if} \quad x \notin \bigcup_{t \in D} V_t.
\]
(Recall \( D = \mathbb{Q} \cap (0, 1] \)) I claim \( f \) is continuous.
\[
f^{-1} ([0, a]) = \bigcup \{V_t : t < a, t \in D\},
\]
an open set.

Next consider \( x \in f^{-1} ([0, a]) \) so \( f(x) \leq a \). If \( t > a \), then \( x \in V_t \) because if not, then
\[
f(x) = \inf \{t \in D : x \in V_t\} > a.
\]
Thus
\[
f^{-1} ([0, a]) \subseteq \bigcap \{V_t : t > a\} = \bigcap \overline{V}_t : t > a\}
\]
which is a closed set. If \( x \in \bigcap \overline{V}_t : t > a \), then \( x \in \bigcap V_t : t > a \) and so \( f(x) \leq a \).

If \( a = 1 \), \( f^{-1} ([0, 1]) = f^{-1} ([0, a]) = X \). Therefore,
\[
f^{-1} ((a, 1)) = X \setminus f^{-1} ([0, a]) = \text{open set}.
\]
It follows \( f \) is continuous. Clearly \( f(x) = 0 \) on \( H \). If \( x \in U^C \), then \( x \notin V_t \) for any \( t \in D \) so \( f(x) = 1 \) on \( U^C \).

In any metric space there is a much easier proof of the conclusion of Urysohn’s lemma which applies.

**Lemma 10.2.2** Let \( S \) be a nonempty subset of a metric space, \((X,d)\). Define
\[
f(x) = \text{dist} (x, S) \equiv \inf \{d(x, y) : y \in S\}.
\]
Then \( f \) is continuous.
**Proof:** Consider $|f(x) - f(x_1)|$ and suppose without loss of generality that $f(x_1) \geq f(x)$. Then choose $y \in S$ such that $f(x) + \varepsilon > d(x,y)$. Then

$$|f(x_1) - f(x)| = f(x_1) - f(x) \leq f(x_1) - d(x,y) + \varepsilon$$

$$\leq d(x_1,y) - d(x,y) + \varepsilon$$

$$\leq d(x_1) + d(x,y) - d(x,y) + \varepsilon$$

$$= d(x_1, x) + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, it follows that $|f(x_1) - f(x)| \leq d(x_1, x)$ and this proves the lemma.

**Theorem 10.2.3 (Urysohn’s lemma for metric space) Let $H$ be a closed subset of an open set, $U$ in a metric space, $(X,d)$. Then there exists a continuous function, $g : X \to [0,1]$ such that $g(x) = 1$ for all $x \in H$ and $g(x) = 0$ for all $x \notin U$.

**Proof:** If $x \notin C$, a closed set, then $\text{dist}(x,C) > 0$ because if not, there would exist a sequence of points of $C$ converging to $x$ and it would follow that $x \in C$. Therefore, $\text{dist}(x,H) + \text{dist}(x,U^C) > 0$ for all $x \in X$. Now define a continuous function, $g$ as

$$g(x) = \frac{\text{dist}(x,U^C)}{\text{dist}(x,H) + \text{dist}(x,U^C)}.$$ 

It is easy to see this verifies the conclusion of the theorem and this proves the theorem.

**Theorem 10.2.4** Every compact Hausdorff space is normal.

**Proof:** First it is shown that $X$, is regular. Let $H$ be a closed set and let $p \notin H$. Then for each $h \in H$, there exists an open set $U_h$ containing $p$ and an open set $V_h$, containing $h$ such that $U_h \cap V_h = \emptyset$. Since $H$ must be compact, it follows there are finitely many of the sets $V_h$, $V_{h1} \cdots V_{hn}$ such that $H \subseteq \bigcup_{i=1}^{n} V_{hi}$. Then letting $U = \cap_{i=1}^{n} U_{hi}$, and $V = \cup_{i=1}^{n} V_{hi}$, it follows that $p \in U$, $H \in V$ and $U \cap V = \emptyset$. Thus $X$ is regular as claimed.

Next let $K$ and $H$ be disjoint nonempty closed sets. Using regularity of $X$, for every $k \in K$, there exists an open set $U_k$ containing $k$ and an open set $V_k$ containing $H$ such that these two open sets have empty intersection. Thus $H \cap \overline{U}_k = \emptyset$. Finitely many of the $U_k$, $U_{k1}, \cdots, U_{kp}$ cover $K$ and so $\bigcup_{i=1}^{p} \overline{U}_k$ is a closed set which has empty intersection with $H$. Therefore, $K \subseteq \bigcup_{i=1}^{p} U_k$ and $H \subseteq (\bigcup_{i=1}^{p} \overline{U}_k)^C$. This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space discussed earlier. However, it is reviewed here for the sake of convenience or in case you have not read the earlier treatment.

**Definition 10.2.5** Suppose $(X,\tau)$ is a locally compact Hausdorff space. Then let $\overline{X} \equiv X \cup \{\infty\}$ where $\infty$ is just the name of some point which is not in $X$ which is
called the point at infinity. A basis for the topology $\bar{\tau}$ for $\bar{X}$ is

$$\tau \cup \{K^C \text{ where } K \text{ is a compact subset of } X\}.$$  

The complement is taken with respect to $\bar{X}$ and so the open sets, $K^C$ are basic open sets which contain $\infty$.

The reason this is called a compactification is contained in the next lemma.

Lemma 10.2.6 If $(X, \tau)$ is a locally compact Hausdorff space, then $\left(\bar{X}, \bar{\tau}\right)$ is a compact Hausdorff space. Also if $U$ is an open set of $\bar{\tau}$, then $U \setminus \{\infty\}$ is an open set of $\tau$.

Proof: Since $(X, \tau)$ is a locally compact Hausdorff space, it follows that $\left(\bar{X}, \bar{\tau}\right)$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and $\infty$. Since $(X, \tau)$ is locally compact, there exists an open set of $\tau$, $U$ having compact closure which contains $p$. Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, $p$ and $\infty$ respectively. Now let $C$ be an open cover of $X$ with sets from $\bar{\tau}$. Then $\infty$ must be in some set, $U_\infty$ from $C$, which must contain a set of the form $K^C$ where $K$ is a compact subset of $X$. Then there exist sets from $C$, $U_1, \cdots, U_r$, which cover $K$. Therefore, a finite subcover of $\bar{X}$ is $U_1, \cdots, U_r, U_\infty$.

To see the last claim, suppose $U$ contains $\infty$ since otherwise there is nothing to show. Notice that if $C$ is a compact set, then $X \setminus C$ is an open set. Therefore, if $x \in U \setminus \{\infty\}$, and if $\bar{X} \setminus C$ is a basic open set contained in $U$ containing $\infty$, then if $x$ is in this basic open set of $\bar{X}$, it is also in the open set $X \setminus C \subseteq U \setminus \{\infty\}$. If $x$ is not in any basic open set of the form $\bar{X} \setminus C$ then $x$ is contained in an open set of $\tau$ which is contained in $U \setminus \{\infty\}$. Thus $U \setminus \{\infty\}$ is indeed open in $\tau$.

Theorem 10.2.7 Let $X$ be a locally compact Hausdorff space, and let $K$ be a compact subset of the open set $V$. Then there exists a continuous function, $f : X \rightarrow [0, 1]$, such that $f$ equals 1 on $K$ and $\{x : f(x) \neq 0\} \equiv \text{spt}(f)$ is a compact subset of $V$.

Proof: Let $\bar{X}$ be the space just described. Then $K$ and $V$ are respectively closed and open in $\bar{\tau}$. By Theorem 10.2.3 there exist open sets in $\bar{\tau}$, $U$, and $W$ such that $K \subseteq U, \infty \in V^C \subseteq W$, and $U \cap W = U \cap (W \setminus \{\infty\}) = \emptyset$. 

![Diagram of compactification](image-url)
Thus $W \setminus \{\infty\}$ is an open set in the original topological space which contains $V^C$, $U$ is an open set in the original topological space which contains $K$, and $W \setminus \{\infty\}$ and $U$ are disjoint.

Now for each $x \in K$, let $U_x$ be a basic open set whose closure is compact and such that

$$x \in U_x \subseteq U.$$ 

Thus $U_x$ must have empty intersection with $V^C$ because the open set, $W \setminus \{\infty\}$ contains no points of $U_x$. Since $K$ is compact, there are finitely many of these sets, $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ which cover $K$. Now let $H \equiv \bigcup_{i=1}^n U_{x_i}$.

Claim: $H = \bigcup_{i=1}^n U_{x_i}$

Proof of claim: Suppose $p \in H$. If $p \notin \bigcup_{i=1}^n U_{x_i}$ then if follows $p \notin U_{x_i}$ for each $i$. Therefore, there exists an open set, $R_i$ containing $p$ such that $R_i$ contains no other points of $U_{x_i}$. Therefore, $R \equiv \bigcap_{i=1}^n R_i$ is an open set containing $p$ which contains no other points of $\bigcup_{i=1}^n U_{x_i} = W$, a contradiction. Therefore, $H \subseteq \bigcup_{i=1}^n U_{x_i}$.

On the other hand, if $p \in U_{x_i}$ then $p$ is obviously in $H$ so this proves the claim.

From the claim, $K \subseteq H \subseteq H \subseteq V$ and $H$ is compact because it is the finite union of compact sets. By Urysohn’s lemma, there exists $f_1$ continuous on $H$ which has values in $[0,1]$ such that $f_1$ equals 1 on $K$ and equals 0 off $H$. Let $f$ denote the function which extends $f_1$ to be 0 off $H$. Then for $\alpha > 0$, the continuity of $f_1$ implies there exists $U$ open in the topological space such that

$$f^{-1}((-\infty, \alpha)) = f_1^{-1}((-\infty, \alpha)) \cup \overline{H}^C = (U \cap \overline{H}) \cup \overline{H}^C = U \cup \overline{H}^C$$

an open set. If $\alpha \leq 0$,

$$f^{-1}((-\infty, \alpha)) = \emptyset$$

an open set. If $\alpha > 0$, there exists an open set $U$ such that

$$f^{-1}((\alpha, \infty)) = f_1^{-1}((\alpha, \infty)) = U \cap \overline{H} = U \cap H$$

because $U$ must be a subset of $H$ since by definition $f = 0$ off $H$. If $\alpha \leq 0$, then

$$f^{-1}((\alpha, \infty)) = X,$$

an open set. Thus $f$ is continuous and $\text{spt}(f) \subseteq H$, a compact subset of $V$. This proves the theorem.

In fact, the conclusion of the above theorem could be used to prove that the topological space is locally compact. However, this is not needed here.

In case you would like a more elementary proof which does not use the one point compactification idea, here is such a proof.

**Theorem 10.2.8** Let $X$ be a locally compact Hausdorff space, and let $K$ be a compact subset of the open set $V$. Then there exists a continuous function, $f : X \to [0,1]$, such that $f$ equals 1 on $K$ and $\{x : f(x) \neq 0\} \equiv \text{spt}(f)$ is a compact subset of $V$. 
Proof: To begin with, here is a claim. This claim is obvious in the case of a metric space but requires some proof in this more general case.

Claim: If $k \in K$ then there exists an open set $U_k$ containing $k$ such that $\overline{U_k}$ is contained in $V$.

Proof of claim: Since $X$ is locally compact, there exists a basis of open sets whose closures are compact, $\mathcal{U}$. Denote by $\mathcal{C}$ the set of all $U \in \mathcal{U}$ which contain $k$ and let $\mathcal{C}'$ denote the set of all closures of these sets of $\mathcal{C}$ intersected with the closed set $V^C$. Thus $\mathcal{C}'$ is a collection of compact sets. I will argue that there are finitely many of the sets of $\mathcal{C}'$ which have empty intersection. If not, then $\mathcal{C}'$ has the finite intersection property and so there exists a point $p$ in all of them. Since $X$ is a Hausdorff space, there exist disjoint basic open sets from $\mathcal{U}$, $A,B$ such that $k \in A$ and $p \in B$. Therefore, $p \notin \overline{A}$ contrary to the above requirement that $p$ be in all such sets. It follows there are sets $A_1, \ldots, A_m$ in $\mathcal{C}$ such that

$$V^C \cap \overline{A_1} \cap \cdots \cap \overline{A_m} = \emptyset$$

Let $U_k = A_1 \cap \cdots \cap A_m$. Then $\overline{U_k} \subseteq \overline{A_1} \cap \cdots \cap \overline{A_m}$ and so it has empty intersection with $V^C$. Thus it is contained in $V$. Also $\overline{U_k}$ is a closed subset of the compact set $\overline{A_1}$ so it is compact. This proves the claim.

Now to complete the proof of the theorem, since $K$ is compact, there are finitely many $U_k$ of the sort just described which cover $K$, $U_k_1, \ldots, U_k_r$. Let

$$H = \cup_{i=1}^r U_i$$

so it follows

$$H = \cup_{i=1}^r \overline{U_i}$$

and so $K \subseteq H \subseteq \overline{H} \subseteq V$ and $\overline{H}$ is a compact set. By Urysohn’s lemma, there exists $f_1$ continuous on $\overline{H}$ which has values in $[0,1]$ such that $f_1$ equals 1 on $K$ and equals 0 off $H$. Let $f$ denote the function which extends $f_1$ to be 0 off $\overline{H}$. Then for $\alpha > 0$, the continuity of $f_1$ implies there exists $U$ open in the topological space such that

$$f^{-1}((-\infty, \alpha)) = f_1^{-1}((-\infty, \alpha)) \cup \overline{H^C} = (U \cap \overline{H}) \cup \overline{H^C} = U \cup \overline{H^C}$$

an open set. If $\alpha \leq 0$,

$$f^{-1}((-\infty, \alpha)) = \emptyset$$

an open set. If $\alpha > 0$, there exists an open set $U$ such that

$$f^{-1}((\alpha, \infty)) = f_1^{-1}((\alpha, \infty)) = U \cap \overline{H} = U \cap H$$

because $U$ must be a subset of $H$ since by definition $f = 0$ off $H$. If $\alpha \leq 0$, then

$$f^{-1}((\alpha, \infty)) = X,$$

an open set. Thus $f$ is continuous and $\text{spt}(f) \subseteq \overline{H}$, a compact subset of $V$. This proves the theorem.
Definition 10.2.9 Define \( \text{spt}(f) \) (support of \( f \)) to be the closure of the set \( \{ x : f(x) \neq 0 \} \). If \( V \) is an open set, \( C_c(V) \) will be the set of continuous functions \( f \), defined on \( \Omega \) having \( \text{spt}(f) \subseteq V \). Thus in Theorem 10.2.7 or 10.2.8, \( f \in C_c(V) \).

Definition 10.2.10 If \( K \) is a compact subset of an open set, \( V \), then \( K \prec \phi \prec V \) if \( \phi \in C_c(V) \), \( \phi(K) = \{ 1 \} \), \( \phi(\Omega) \subseteq [0, 1] \), where \( \Omega \) denotes the whole topological space considered. Also for \( \phi \in C_c(\Omega) \), \( K \prec \phi \prec V \) if \( \phi(\Omega) \subseteq [0, 1] \) and \( \phi(K) = 1 \).

Theorem 10.2.11 (Partition of unity) Let \( K \) be a compact subset of a locally compact Hausdorff topological space satisfying Theorem 10.2.7 or 10.2.8 and suppose \( K \subseteq V = \bigcup_{i=1}^{n} V_i \), \( V_i \) open. Then there exist \( \psi_i \prec V_i \) with \( \sum_{i=1}^{n} \psi_i(x) = 1 \) for all \( x \in K \).

Proof: Let \( K_1 = K \setminus \bigcup_{i=2}^{n} V_i \). Thus \( K_1 \) is compact and \( K_1 \subseteq V_1 \). Let \( K_2 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1 \) with \( \overline{W}_1 \) compact. To obtain \( W_1 \), use Theorem 10.2.7 or 10.2.8 to get \( f \) such that \( K \prec f \prec V_1 \) and let \( W_1 = \{ x : f(x) \neq 0 \} \). Thus \( W_1, V_2, \cdots, V_n \) covers \( K \) and \( \overline{W}_1 \subseteq V_1 \). Let \( K_2 = K \setminus (\bigcup_{i=3}^{n} V_i \cup W_1) \). Then \( K_2 \) is compact and \( K_2 \subseteq V_2 \). Let \( K_2 \subseteq W_2 \subseteq \overline{W}_2 \subseteq V_2 \) compact. Continue this way finally obtaining \( W_1, \cdots, W_n \), \( K \subseteq W_1 \cup \cdots \cup W_n \), and \( \overline{W}_i \subseteq V_i \), \( \overline{W}_i \) compact. Now let \( \overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i \), \( \overline{U}_i \) compact.

By Theorem 10.2.7 or 10.2.8, let \( U_i \prec \phi_i \prec V_i \), \( \bigcup_{i=1}^{n} \overline{W}_i \prec \gamma \prec \bigcup_{i=1}^{n} \overline{U}_i \). Define \( \psi_i(x) = \begin{cases} \gamma(x)\phi_i(x)/\sum_{j=1}^{n} \phi_j(x) & \text{if } \sum_{j=1}^{n} \phi_j(x) \neq 0, \\ 0 & \text{if } \sum_{j=1}^{n} \phi_j(x) = 0. \end{cases} \)

If \( x \) is such that \( \sum_{j=1}^{n} \phi_j(x) = 0 \), then \( x \notin \bigcup_{i=1}^{n} \overline{U}_i \). Consequently \( \gamma(y) = 0 \) for all \( y \) near \( x \) and so \( \psi_i(y) = 0 \) for all \( y \) near \( x \). Hence \( \psi_i \) is continuous at such \( x \).
If $\sum_{j=1}^{n} \phi_j(x) \neq 0$, this situation persists near $x$ and so $\psi_i$ is continuous at such points. Therefore $\psi_i$ is continuous. If $x \in K$, then $\gamma(x) = 1$ and so $\sum_{j=1}^{n} \psi_j(x) = 1$. Clearly $0 \leq \psi_i(x) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The following corollary won’t be needed immediately but is of considerable interest later.

**Corollary 10.2.12** If $H$ is a compact subset of $V_i$, there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Theorem 10.2.11.

**Proof:** Keep $V_i$ the same but replace $V_j$ with $\overline{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on $H$.

### 10.3 Positive Linear Functionals

**Definition 10.3.1** Let $(\Omega, \tau)$ be a topological space. $L : C_c(\Omega) \to \mathbb{C}$ is called a positive linear functional if $L$ is linear,

$$L(af_1 + bf_2) = aLf_1 + bLf_2,$$

and if $Lf \geq 0$ whenever $f \geq 0$.

**Theorem 10.3.2** (Riesz representation theorem) Let $(\Omega, \tau)$ be a locally compact Hausdorff space and let $L$ be a positive linear functional on $C_c(\Omega)$. Then there exists a $\sigma$ algebra $\mathcal{S}$ containing the Borel sets and a unique measure $\mu$, defined on $\mathcal{S}$, such that

\begin{align*}
\mu & \text{ is complete,} \\
\mu(K) & < \infty \text{ for all } K \text{ compact,} \\
\mu(F) & = \sup\{\mu(K) : K \subseteq F, K \text{ compact}\},
\end{align*}

for all $F$ open and for all $F \in \mathcal{S}$ with $\mu(F) < \infty$,

$$\mu(F) = \inf\{\mu(V) : V \supseteq F, V \text{ open}\}$$

for all $F \in \mathcal{S}$, and

$$\int fd\mu = Lf \text{ for all } f \in C_c(\Omega).$$

The plan is to define an outer measure and then to show that it, together with the $\sigma$ algebra of sets measurable in the sense of Caratheodory, satisfies the conclusions of the theorem. Always, $K$ will be a compact set and $V$ will be an open set.

**Definition 10.3.3** $\mu(V) \equiv \sup\{Lf : f \ll V\}$ for $V$ open, $\mu(\emptyset) = 0$. $\mu(E) \equiv \inf\{\mu(V) : V \supseteq E\}$ for arbitrary sets $E$. 

Lemma 10.3.4 \( \mu \) is a well-defined outer measure.

**Proof:** First it is necessary to verify \( \mu \) is well defined because there are two descriptions of it on open sets. Suppose then that \( \mu_1(V) = \inf \{ \mu(U) : U \supseteq V \text{ and } U \text{ is open} \} \). It is required to verify that \( \mu_1(V) = \mu(V) \) where \( \mu \) is given as \( \sup \{ Lf : f \prec V \} \). If \( U \supseteq V \), then \( \mu(U) \geq \mu(V) \) directly from the definition. Hence from the definition of \( \mu_1 \), it follows \( \mu_1(V) \geq \mu(V) \). On the other hand, \( V \supseteq V \) and so \( \mu_1(V) \leq \mu(V) \). This verifies \( \mu \) is well defined.

It remains to show that \( \mu \) is an outer measure. Let \( V = \bigcup_{i=1}^{\infty} V_i \) and let \( f \prec V \). Then \( \text{spt}(f) \subseteq \bigcup_{i=1}^{n} V_i \) for some \( n \). Let \( \psi_i \prec V_i, \sum_{i=1}^{n} \psi_i = 1 \) on \( \text{spt}(f) \).

\[
Lf = \sum_{i=1}^{n} L(f\psi_i) \leq \sum_{i=1}^{n} \mu(V_i) \leq \sum_{i=1}^{\infty} \mu(V_i).
\]

Hence

\[
\mu(V) \leq \sum_{i=1}^{\infty} \mu(V_i)
\]

since \( f \prec V \) is arbitrary. Now let \( E = \bigcup_{i=1}^{\infty} E_i \). Is \( \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) \)? Without loss of generality, it can be assumed \( \mu(E_i) < \infty \) for each \( i \) since if not so, there is nothing to prove. Let \( V_i \supseteq E_i \) with \( \mu(E_i) + \epsilon 2^{-i} > \mu(V_i) \).

\[
\mu(E) \leq \mu(\bigcup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq \epsilon + \sum_{i=1}^{\infty} \mu(E_i).
\]

Since \( \epsilon \) was arbitrary, \( \mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) \) which proves the lemma.

**Lemma 10.3.5** Let \( K \) be compact, \( g \geq 0, g \in C_c(\Omega) \), and \( g = 1 \) on \( K \). Then \( \mu(K) \leq Lg \). Also \( \mu(K) < \infty \) whenever \( K \) is compact.

**Proof:** Let \( \alpha \in (0,1) \) and \( V_\alpha = \{ x : g(x) > \alpha \} \) so \( V_\alpha \supseteq K \) and let \( h \prec V_\alpha \).

Then \( h \leq 1 \) on \( V_\alpha \) while \( g\alpha^{-1} \geq 1 \) on \( V_\alpha \) and so \( g\alpha^{-1} \geq h \) which implies \( L(g\alpha^{-1}) \geq Lh \) and that therefore, since \( L \) is linear,

\[
Lg \geq \alpha Lh.
\]

Since \( h \prec V_\alpha \) is arbitrary, and \( K \subseteq V_\alpha \),

\[
Lg \geq \alpha \mu(V_\alpha) \geq \alpha \mu(K).
\]
Letting $\alpha \uparrow 1$ yields $Lg \geq \mu(K)$. This proves the first part of the lemma. The second assertion follows from this and Theorem 10.2.7. If $K$ is given, let

$$K \prec g \prec \Omega$$

and so from what was just shown, $\mu(K) \leq Lg < \infty$. This proves the lemma.

**Lemma 10.3.6** If $A$ and $B$ are disjoint compact subsets of $\Omega$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

**Proof:** By Theorem 10.2.7 or 10.2.8, there exists $h \in C_c(\Omega)$ such that $A \prec h \prec B^C$. Let $U_1 = h^{-1}((\frac{1}{2}, 1]), V_1 = h^{-1}([0, \frac{1}{2}))$. Then $A \subseteq U_1, B \subseteq V_1$ and $U_1 \cap V_1 = \emptyset$.

From Lemma 10.3.5 $\mu(A \cup B) < \infty$ and so there exists an open set, $W$ such that

$$W \supseteq A \cup B, \mu(A \cup B) + \varepsilon > \mu(W).$$

Now let $U = U_1 \cap W$ and $V = V_1 \cap W$. Then

$$U \supseteq A, V \supseteq B, U \cap V = \emptyset, \text{ and } \mu(A \cup B) + \varepsilon \geq \mu(W) \geq \mu(U \cup V).$$

Let $A \prec f \prec U, B \prec g \prec V$. Then by Lemma 10.3.5

$$\mu(A \cup B) + \varepsilon \geq \mu(U \cup V) \geq L(f + g) - Lf + Lg \geq \mu(A) + \mu(B).$$

Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

From Lemma 10.3.5 the following lemma is obtained.

**Lemma 10.3.7** Let $f \in C_c(\Omega), f(\Omega) \subseteq [0, 1]$. Then $\mu(spt(f)) \geq Lf$. Also, every open set, $V$ satisfies

$$\mu(V) = \sup \{\mu(K) : K \subseteq V\}.$$

**Proof:** Let $V \supseteq spt(f)$ and let $spt(f) \prec g \prec V$. Then $Lf \leq Lg \leq \mu(V)$ because $f \leq g$. Since this holds for all $V \supseteq spt(f)$, $Lf \leq \mu(spt(f))$ by definition of $\mu$. 

![Diagram](image-url)
Finally, let \( V \) be open and let \( l < \mu(V) \). Then from the definition of \( \mu \), there exists \( f \prec V \) such that \( L(f) > l \). Therefore, \( l < \mu(\text{spt}(f)) \leq \mu(V) \) and so this shows the claim about inner regularity of the measure on an open set.

At this point, the conditions of Lemma 10.1.6 have been verified. Thus \( S \) contains the Borel sets and \( \mu \) is inner regular on sets of \( S \) having finite measure.

It remains to show \( \mu \) satisfies Lemma 10.3.11.

**Lemma 10.3.8** \( \int f \, d\mu = Lf \) for all \( f \in C_c(\Omega) \).

**Proof:** Let \( f \in C_c(\Omega) \), \( f \) real-valued, and suppose \( f(\Omega) \subseteq [a, b] \). Choose \( t_0 < a \) and let \( t_0 < t_1 < \cdots < t_n = b \), \( t_i - t_{i-1} < \varepsilon \). Let

\[
E_i = f^{-1}((t_{i-1}, t_i)) \cap \text{spt}(f). \tag{10.3.12}
\]

Note that \( \bigcup_{i=1}^n E_i \) is a closed set, and in fact

\[
\bigcup_{i=1}^n E_i = \text{spt}(f) \tag{10.3.13}
\]

since \( \Omega = \bigcup_{i=1}^n f^{-1}((t_{i-1}, t_i)) \). Let \( V_i \supseteq E_i \), \( V_i \) is open and let \( V_i \) satisfy

\[
f(x) < t_i + \varepsilon \text{ for all } x \in V_i, \tag{10.3.14}
\]

\[
\mu(V_i \setminus E_i) < \varepsilon/n.
\]

By Theorem 10.2.11 there exists \( h_i \in C_c(\Omega) \) such that

\[
h_i \prec V_i, \sum_{i=1}^n h_i(x) = 1 \text{ on } \text{spt}(f).
\]

Now note that for each \( i \),

\[
f(x)h_i(x) \leq h_i(x)(t_i + \varepsilon).
\]

(If \( x \in V_i \), this follows from Lemma 10.3.11. If \( x \notin V_i \) both sides equal 0.) Therefore,

\[
L f = L \left( \sum_{i=1}^n f h_i \right) \leq L \left( \sum_{i=1}^n h_i (t_i + \varepsilon) \right)
\]

\[
= \sum_{i=1}^n (t_i + \varepsilon) L(h_i)
\]

\[
= \sum_{i=1}^n (|t_0| + t_i + \varepsilon) L(h_i) - |t_0| L \left( \sum_{i=1}^n h_i \right).
\]

Now note that \( |t_0| + t_i + \varepsilon \geq 0 \) and so from the definition of \( \mu \) and Lemma 10.3.11, this is no larger than

\[
\sum_{i=1}^n (|t_0| + t_i + \varepsilon) \mu(V_i) - |t_0| \mu(\text{spt}(f))
\]
10.3. POSITIVE LINEAR FUNCTIONALS

\[ \leq \sum_{i=1}^{n} (|t_0| + t_i + \varepsilon) (\mu(E_i) + \varepsilon/n) - |t_0|\mu(\text{spt}(f)) \]

\[ \leq |t_0|\sum_{i=1}^{n} \mu(E_i) + |t_0|\varepsilon + \sum_{i=1}^{n} t_i\mu(E_i) + \varepsilon(|t_0| + |b|) \]

\[ \sum_{i=1}^{n} t_i \frac{\varepsilon}{n} + \varepsilon \sum_{i=1}^{n} \mu(E_i) + \varepsilon^2 - |t_0|\mu(\text{spt}(f)). \]

From (10.3.13) and (10.3.12), the first and last terms cancel. Therefore this is no larger than

\[ (2|t_0| + |b| + \mu(\text{spt}(f)) + \varepsilon)\varepsilon \]

\[ + \sum_{i=1}^{n} t_{i-1}\mu(E_i) + \varepsilon\mu(\text{spt}(f)) + \sum_{i=1}^{n} (|t_0| + |b|) \frac{\varepsilon}{n} \]

\[ \leq \int fd\mu + (2|t_0| + |b| + 2\mu(\text{spt}(f)) + \varepsilon)\varepsilon + (|t_0| + |b|)\varepsilon \]

Since \( \varepsilon > 0 \) is arbitrary,

\[ Lf \leq \int fd\mu \quad (10.3.15) \]

for all \( f \in C_c(\Omega) \), \( f \) real. Hence equality holds in (10.3.15) because \( L(-f) \leq -\int fd\mu \) so \( L(f) \geq \int fd\mu \). Thus \( Lf = \int fd\mu \) for all \( f \in C_c(\Omega) \). Just apply the result for real functions to the real and imaginary parts of \( f \). This proves the Lemma.

This gives the existence part of the Riesz representation theorem.

It only remains to prove uniqueness. Suppose both \( \mu_1 \) and \( \mu_2 \) are measures on \( S \) satisfying the conclusions of the theorem. Then if \( K \) is compact and \( V \supseteq K \), let \( K \prec f \prec V \). Then

\[ \mu_1(K) \leq \int fd\mu_1 = Lf = \int fd\mu_2 \leq \mu_2(V). \]

Thus \( \mu_1(K) \leq \mu_2(K) \) for all \( K \). Similarly, the inequality can be reversed and so it follows the two measures are equal on compact sets. By the assumption of inner regularity on open sets, the two measures are also equal on all open sets. By outer regularity, they are equal on all sets of \( S \). This proves the theorem.

An important example of a locally compact Hausdorff space is any metric space in which the closures of balls are compact. For example, \( \mathbb{R}^n \) with the usual metric is an example of this. Not surprisingly, more can be said in this important special case.

**Theorem 10.3.9** Let \((\Omega, \tau)\) be a metric space in which the closures of the balls are compact and let \( L \) be a positive linear functional defined on \( C_c(\Omega) \). Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 10.2.7 or 10.2.8 and in addition the property that \( \mu \) is regular. The same conclusion follows if \((\Omega, \tau)\) is a compact Hausdorff space.
Theorem 10.3.10 Let \((\Omega, \tau)\) be a metric space in which the closures of the balls are compact and let \(L\) be a positive linear functional defined on \(C_c(\Omega)\). Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 10.2.7 or 10.2.8 and in addition the property that \(\mu\) is regular. The same conclusion follows if \((\Omega, \tau)\) is a compact Hausdorff space.

**Proof:** Let \(\mu\) and \(S\) be as described in Theorem 10.3.2. The outer regularity comes automatically as a conclusion of Theorem 10.3.2. It remains to verify inner regularity. Let \(F \in S\) and let \(l < k < \mu(F)\). Now let \(z \in \Omega\) and \(\Omega_n = B(z,n)\) for \(n \in \mathbb{N}\). Thus \(F \cap \Omega_n \uparrow F\). It follows that for \(n\) large enough,

\[
k < \mu(F \cap \Omega_n) \leq \mu(F).
\]

Since \(\mu(F \cap \Omega_n) < \infty\) it follows there exists a compact set, \(K\) such that \(K \subseteq F \cap \Omega_n \subseteq F\) and

\[
l < \mu(K) \leq \mu(F).
\]

This proves inner regularity. In case \((\Omega, \tau)\) is a compact Hausdorff space, the conclusion of inner regularity follows from Theorem 10.3.2. This proves the theorem.

The proof of the above yields the following corollary.

**Corollary 10.3.11** Let \((\Omega, \tau)\) be a locally compact Hausdorff space and suppose \(\mu\) defined on a \(\sigma\) algebra, \(S\) represents the positive linear functional \(L\) where \(L\) is defined on \(C_c(\Omega)\) in the sense of Theorem 10.2.7 or 10.2.8. Suppose also that there exist \(\Omega_n \in S\) such that \(\Omega = \bigcup_{n=1}^{\infty} \Omega_n\) and \(\mu(\Omega_n) < \infty\). Then \(\mu\) is regular.

The following is on the uniqueness of the \(\sigma\) algebra in some cases.

**Definition 10.3.12** Let \((\Omega, \tau)\) be a locally compact Hausdorff space and let \(L\) be a positive linear functional defined on \(C_c(\Omega)\) such that the complete measure defined by the Riesz representation theorem for positive linear functionals is inner regular. Then this is called a Radon measure. Thus a Radon measure is complete, and regular.

**Corollary 10.3.13** Let \((\Omega, \tau)\) be a locally compact Hausdorff space which is also \(\sigma\) compact meaning \(\Omega = \bigcup_{n=1}^{\infty} \Omega_n\), \(\Omega_n\) is compact, and let \(L\) be a positive linear functional defined on \(C_c(\Omega)\). Then if \((\mu_1, S_1)\), and \((\mu_2, S_2)\) are two Radon measures, together with their \(\sigma\) algebras which represent \(L\) then the two \(\sigma\) algebras are equal and the two measures are equal.

**Proof:** Suppose \((\mu_1, S_1)\) and \((\mu_2, S_2)\) both work. It will be shown the two measures are equal on every compact set. Let \(K\) be compact and let \(V\) be an open set containing \(K\). Then let \(K \prec f \prec V\). Then

\[
\mu_1(K) = \int_K d\mu_1 \leq \int f d\mu_1 = L(f) = \int f d\mu_2 \leq \mu_2(V).
\]
Therefore, taking the infimum over all $V$ containing $K$ implies $\mu_1 (K) \leq \mu_2 (K)$. Reversing the argument shows $\mu_1 (K) = \mu_2 (K)$. This also implies the two measures are equal on all open sets because they are both inner regular on open sets. It is being assumed the two measures are regular. Now let $F \in \mathcal{S}_1$ with $\mu_1 (F) < \infty$. Then there exist sets, $H,G$ such that $H \subseteq F \subseteq G$ such that $H$ is the countable union of compact sets and $G$ is a countable intersection of open sets such that $\mu_1 (G) = \mu_1 (H)$ which implies $\mu_1 (G \setminus H) = 0$. Now $G \setminus H$ can be written as the countable intersection of sets of the form $V_k \setminus K_k$ where $V_k$ is open, $\mu_1 (V_k) < \infty$ and $K_k$ is compact. From what was just shown, $\mu_2 (V_k \setminus K_k) = \mu_1 (V_k \setminus K_k)$ so it follows $\mu_2 (G \setminus H) = 0$ also. Since $\mu_2$ is complete, and $G$ and $H$ are in $\mathcal{S}_2$, it follows $F \in \mathcal{S}_2$ and $\mu_2 (F) = \mu_1 (F)$. Now for arbitrary $F$ possibly having $\mu_1 (F) = \infty$, consider $F \cap \Omega_n$. From what was just shown, this set is in $\mathcal{S}_2$ and $\mu_2 (F \cap \Omega_n) = \mu_1 (F \cap \Omega_n)$. Taking the union of these $F \cap \Omega_n$ gives $F \in \mathcal{S}_2$ and also $\mu_1 (F) = \mu_2 (F)$. This shows $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Similarly, $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

The following lemma is often useful.

**Lemma 10.3.14** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where $\Omega$ is a topological space. Suppose $\mu$ is a Radon measure and $f$ is measurable with respect to $\mathcal{F}$. Then there exists a Borel measurable function, $g$, such that $g = f$ a.e.

**Proof:** Assume without loss of generality that $f \geq 0$. Then let $s_n \uparrow f$ pointwise. Say

$$s_n (\omega) = \sum_{k=1}^{p_n} c_k^n \chi_{E_k^n} (\omega)$$

where $E_k^n \in \mathcal{F}$. By the outer regularity of $\mu$, there exists a Borel set, $F_k^n \supseteq E_k^n$ such that $\mu (F_k^n) = \mu (E_k^n)$. In fact $F_k^n$ can be assumed to be a $G_\delta$ set. Let

$$t_n (\omega) = \sum_{k=1}^{p_n} c_k^n \chi_{F_k^n} (\omega).$$

Then $t_n$ is Borel measurable and $t_n (\omega) = s_n (\omega)$ for all $\omega \notin N_n$ where $N_n \in \mathcal{F}$ is a set of measure zero. Now let $N \equiv \cup_{n=1}^{\infty} N_n$. Then $N$ is a set of measure zero and if $\omega \notin N$, then $t_n (\omega) \rightarrow f (\omega)$. Let $N' \supseteq N$ where $N'$ is a Borel set and $\mu (N') = 0$. Then $t_n \chi_{(N')^c}$ converges pointwise to a Borel measurable function, $g$, and $g (\omega) = f (\omega)$ for all $\omega \notin N'$. Therefore, $g = f$ a.e. and this proves the lemma.

### 10.4 One Dimensional Lebesgue Measure

To obtain one dimensional Lebesgue measure, you use the positive linear functional $L$ given by

$$Lf = \int f (x) \, dx$$
whenever \( f \in C_c(\mathbb{R}) \). Lebesgue measure, denoted by \( m \) is the measure obtained from the Riesz representation theorem such that
\[
\int f \, dm = Lf = \int f(x) \, dx.
\]
From this it is easy to verify that
\[
m((a, b]) = m((a, b)) = b - a. \tag{10.4.16}
\]
This will be done in general a little later but for now, consider the following picture of functions, \( f^k \) and \( g^k \). Note that \( f^k \leq \chi_{(a,b]} \leq \chi_{[a,b]} \leq g^k \).

Then considering lower sums and upper sums in the inequalities on the ends,
\[
\left( b - a - \frac{2}{k} \right) \leq \int f^k \, dx = \int f^k \, dm \leq m((a, b)) \leq m([a,b]) \\
= \int \chi_{[a,b]} \, dm \leq \int g^k \, dm = \int g^k \, dx \leq \left( b - a + \frac{2}{k} \right).
\]
From this the claim in [10.4.16] follows.

### 10.5 One Dimensional Lebesgue Stieltjes Measure

This is just a generalization of Lebesgue measure. Instead of the functional,
\[
Lf \equiv \int f(x) \, dx, \quad f \in C_c(\mathbb{R}),
\]
you use the functional
\[
Lf \equiv \int f(x) \, dF(x), \quad f \in C_c(\mathbb{R}),
\]
where \( F \) is an increasing function defined on \( \mathbb{R} \). By Theorem [3.3.4] this functional is easily seen to be well defined. Therefore, by the Riesz representation theorem there exists a unique Radon measure \( \mu \) representing the functional. Thus
\[
\int \int f \, d\mu = \int fdF
\]
for all \( f \in C_c(\mathbb{R}) \). Now consider what this measure does to intervals. To begin with, consider what it does to the closed interval, \([a, b]\). The following picture may help.

![Diagram](image)

In this picture \( \{a_n\} \) increases to \( a \) and \( b_n \) decreases to \( b \). Also suppose \( a, b \) are points of continuity of \( F \). Therefore,

\[
F(b) - F(a) \leq Lf_n = \int_\mathbb{R} f_n d\mu \leq F(b_n) - F(a_n)
\]

Passing to the limit and using the dominated convergence theorem, this shows

\[
\mu([a, b]) = F(b) - F(a) = F(b^+) - F(a^-).
\]

Next suppose \( a, b \) are arbitrary, maybe not points of continuity of \( F \). Then letting \( a_n \) and \( b_n \) be as in the above picture which are points of continuity of \( F \),

\[
\mu([a, b]) = \lim_{n \to \infty} \mu([a_n, b_n]) = \lim_{n \to \infty} F(b_n) - F(a_n)
\]

\[
= F(b^+) - F(a^-).
\]

In particular \( \mu(a) = F(a^+) - F(a^-) \) and so

\[
\mu((a, b)) = F(b^+) - F(a^-) - (F(a^+) - F(a^-))
\]

\[
= F(b^-) - F(a^+)
\]

This shows what \( \mu \) does to intervals. This is stated as the following proposition.

**Proposition 10.5.1** Let \( \mu \) be the measure representing the functional

\[
Lf \equiv \int fdF, f \in C_c(\mathbb{R})
\]

for \( F \) an increasing function defined on \( \mathbb{R} \). Then

\[
\mu([a, b]) = F(b^+) - F(a^-)
\]

\[
\mu((a, b)) = F(b^-) - F(a^+)
\]

\[
\mu(a) = F(a^+) - F(a^-).
\]
Observation 10.5.2 Note that all the above would work as well if

\[ Lf \equiv \int f dF, \; f \in C_c([0, \infty)) \]

where \( F \) is continuous at 0 and \( \nu \) is the measure representing this functional. This is because you could just extend \( F(x) \) to equal \( F(0) \) for \( x \leq 0 \) and apply the above to the extended \( F \). In this case, \( \nu([0,b]) = F(b^+) - F(0) \).

10.6 The Distribution Function

There is an interesting connection between the Lebesgue integral of a nonnegative function with something called the distribution function.

Definition 10.6.1 Let \( f \geq 0 \) and suppose \( f \) is measurable. The distribution function is the function defined by

\[ t \mapsto \mu([t < f]). \]

Lemma 10.6.2 If \( \{f_n\} \) is an increasing sequence of functions converging pointwise to \( f \) then

\[ \mu([f > t]) = \lim_{n \to \infty} \mu([f_n > t]) \]

Proof: The sets, \([f_n > t]\) are increasing and their union is \([f > t]\) because if \( f(\omega) > t \), then for all \( n \) large enough, \( f_n(\omega) > t \) also. Therefore, the desired conclusion follows from properties of measures.

Lemma 10.6.3 Suppose \( s \geq 0 \) is a measurable simple function,

\[ s(\omega) \equiv \sum_{k=1}^{n} a_k X_{E_k}(\omega) \]

where the \( a_k \) are the distinct nonzero values of \( s, 0 < a_1 < a_2 < \cdots < a_n \). Suppose \( \phi \) is a \( C^1 \) function defined on \([0, \infty)\) which has the property that \( \phi(0) = 0, \phi'(t) > 0 \) for all \( t \). Then

\[ \int_0^\infty \phi'(t) \mu([s > t]) \, dm = \int \phi(s) \, d\mu. \]

Proof: First note that if \( \mu(E_k) = \infty \) for any \( k \) then both sides equal \( \infty \) and so without loss of generality, assume \( \mu(E_k) < \infty \) for all \( k \). Letting \( a_0 \equiv 0 \), the left
10.6. THE DISTRIBUTION FUNCTION

side equals

\[
\sum_{k=1}^{n} \int_{a_{k-1}}^{a_{k}} \phi'(t) \mu([s > t]) \, dm(t) = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_{k}} \phi'(t) \sum_{i=k}^{n} \mu(E_i) \, dm
\]

\[
= \sum_{k=1}^{n} \sum_{i=k}^{n} \mu(E_i) \int_{a_{k-1}}^{a_{k}} \phi'(t) \, dm
\]

\[
= \sum_{i=1}^{n} \sum_{k=1}^{i} \mu(E_i) (\phi(a_k) - \phi(a_{k-1}))
\]

\[
= \sum_{i=1}^{n} \mu(E_i) \phi(a_i) = \int \phi(s) \, d\mu. \quad \blacksquare
\]

With this lemma the next theorem which is the main result follows easily.

**Theorem 10.6.4** Let \( f \geq 0 \) be measurable and let \( \phi \) be a \( C^1 \) function defined on \([0, \infty)\) which satisfies \( \phi'(t) > 0 \) for all \( t > 0 \) and \( \phi(0) = 0 \). Then

\[
\int \phi(f) \, d\mu = \int_{0}^{\infty} \phi'(t) \mu([f > t]) \, dm.
\]

**Proof:** By Theorem [9.3.9](#) on Page 227 there exists an increasing sequence of nonnegative simple functions, \( \{s_n\} \) which converges pointwise to \( f \). By the monotone convergence theorem and Lemma 10.6.2,

\[
\int \phi(f) \, d\mu = \lim_{n \to \infty} \int \phi(s_n) \, d\mu = \lim_{n \to \infty} \int_{0}^{\infty} \phi'(t) \mu([s_n > t]) \, dm
\]

\[
= \int_{0}^{\infty} \phi'(t) \mu([f > t]) \, dm. \quad \blacksquare
\]

This theorem can be generalized to a situation in which \( \phi \) is only increasing and continuous. In the generalization I will replace the symbol \( \phi \) with \( F \) to coincide with earlier notation.

**Lemma 10.6.5** Suppose \( s \geq 0 \) is a measurable simple function,

\[
s(\omega) \equiv \sum_{k=1}^{n} a_k \mathcal{A}_{E_k}(\omega)
\]

where the \( a_k \) are the distinct nonzero values of \( s, a_1 < a_2 < \cdots < a_n \). Suppose \( F \) is an increasing function defined on \([0, \infty), F(0) = 0, F \) being continuous at 0 from the right and continuous at every \( a_k \). Then letting \( \mu \) be a measure and \((\Omega, \mathcal{F}, \mu)\) a measure space,

\[
\int_{[0, \infty]} \mu([s > t]) \, d\nu = \int_{\Omega} F(s) \, d\mu.
\]
where the integral on the left is the Lebesgue integral for the measure \( \nu \) given as the Radon measure representing the functional

\[
\int_0^\infty g dF
\]

for \( g \in C_c([0, \infty)) \).

**Proof:** This follows from the following computation and Proposition 10.5.1.

Since \( F \) is continuous at 0 and the values \( a_k \),

\[
\int_0^\infty \mu ([s > t]) d\nu(t) = \sum_{k=1}^n \int_{(a_{k-1}, a_k]} \mu ([s > t]) d\nu(t)
\]

\[
= \sum_{k=1}^n \int_{(a_{k-1}, a_k]} \sum_{j=1}^n \mu(E_j) dF(t) = \sum_{j=1}^n \mu(E_j) \sum_{k=1}^n \nu((a_{k-1}, a_k])
\]

\[
= \sum_{j=1}^n \mu(E_j) \sum_{k=1}^n (F(a_k) - F(a_{k-1})) = \sum_{j=1}^n \mu(E_j) F(a_j) \equiv \int_\Omega F(s) d\mu \quad \blacksquare
\]

Now here is the generalization to nonnegative measurable \( f \).

**Theorem 10.6.6** Let \( f \geq 0 \) be measurable with respect to \( \mathcal{F} \) where \( (\Omega, \mathcal{F}, \mu) \) a measure space, and let \( F \) be an increasing continuous function defined on \([0, \infty)\) and \( F(0) = 0 \). Then

\[
\int_\Omega F(f) d\mu = \int_{[0, \infty]} \mu ([f > t]) d\nu(t)
\]

where \( \nu \) is the Radon measure representing

\[
L_g = \int_0^\infty g dF
\]

for \( g \in C_c([0, \infty)) \).

**Proof:** By Theorem 9.2.9 on Page 227 there exists an increasing sequence of nonnegative simple functions, \( \{s_n\} \) which converges pointwise to \( f \). By the monotone convergence theorem and Lemma 10.6.3,

\[
\int_\Omega F(f) d\mu = \lim_{n \to \infty} \int_\Omega F(s_n) d\mu = \lim_{n \to \infty} \int_{[0, \infty]} \mu ([s_n > t]) d\nu
\]

\[
= \int_{[0, \infty]} \mu ([f > t]) d\nu \quad \blacksquare
\]
Note that the function \( t \to \mu([f > t]) \) is a decreasing function. Therefore, one can make sense of an improper Riemann Stieltjes integral
\[
\int_0^\infty \mu([f > t]) \, dF(t).
\]

With more work, one can have this equal to the corresponding Lebesgue integral above.

## 10.7 Good Lambda Inequality

There is a very interesting and important inequality called the good lambda inequality (I am not sure if there is a bad lambda inequality.) which follows from the above theory of distribution functions. It involves the inequality

\[
\mu([f > \beta \lambda] \cap [g \leq r \delta \lambda]) \leq \phi(\delta) \mu([f > \lambda])
\]

for \( \beta > 1 \), nonnegative functions \( f, g \) and is supposed to hold for all small positive \( \delta \) and \( \phi(\delta) \to 0 \) as \( \delta \to 0 \). Note the left side is small when \( g \) is large and \( f \) is small. The inequality involves dominating an integral involving \( f \) with one involving \( g \) as described below. As above, \( \nu \) is the measure which comes from the functional \( \int g \, dF \) for \( g \in C_c(\mathbb{R}) \).

**Theorem 10.7.1** Let \((\Omega, \mathcal{F}, \mu)\) be a finite measure space and let \( F \) be a continuous increasing function defined on \([0, \infty)\) such that \( F(0) = 0 \). Suppose also that for all \( \alpha > 1 \), there exists a constant \( C_\alpha \) such that for all \( x \in [0, \infty) \),

\[
F(\alpha x) \leq C_\alpha F(x).
\]

Also suppose \( f, g \) are nonnegative measurable functions and there exists \( \beta > 1, 0 < r \leq 1 \), such that for all \( \lambda > 0 \) and \( 1 > \delta > 0 \),

\[
\mu([f > \beta \lambda] \cap [g \leq r \delta \lambda]) \leq \phi(\delta) \mu([f > \lambda]) \quad (10.7.17)
\]

where \( \lim_{\delta \to 0^+} \phi(\delta) = 0 \) and \( \phi \) is increasing. Under these conditions, there exists a constant \( C \) depending only on \( \beta, \phi, r \) such that

\[
\int_{\Omega} F(f(\omega)) \, d\mu(\omega) \leq C \int_{\Omega} F(g(\omega)) \, d\mu(\omega).
\]

**Proof:** Let \( \beta > 1 \) be as given above. First suppose \( f \) is bounded.

\[
\int_{\Omega} F(f) \, d\mu = \int_{\Omega} F\left(\frac{f}{\beta}\right) \, d\mu \leq C_\beta \int_{\Omega} F\left(\frac{f}{\beta^2}\right) \, d\mu
\]

\[
= C_\beta \int_0^\infty \mu([f > \beta \lambda]) \, d\nu
\]
Now using the given inequality,
\[ = C_\beta \int_0^\infty \mu ([f > \beta \lambda] \cap [g \leq r\delta \lambda]) \, d\nu \\
+ C_\beta \int_0^\infty \mu ([f > \beta \lambda] \cap [g > r\delta \lambda]) \, d\nu \]
\[ \leq C_\beta \phi (\delta) \int_0^\infty \mu ([f > \lambda]) \, d\nu + C_\beta \int_0^\infty \mu ([g > r\delta \lambda]) \, d\nu \]
\[ \leq C_\beta \phi (\delta) \int_\Omega F(f) \, d\mu + C_\beta \int_\Omega F\left(\frac{g}{r\delta}\right) \, d\mu \]

Now choose \( \delta \) small enough that \( C_\beta \phi (\delta) < \frac{1}{2} \) and then subtract the first term on the right in the above from both sides. It follows from the properties of \( F \) again that
\[ \frac{1}{2} \int_\Omega F(f) \, d\mu \leq C_\beta C_{r\delta}^{-1} \int_\Omega F(g) \, d\mu. \]

This establishes the inequality in the case where \( f \) is bounded.

In general, let \( f_n = \min (f, n) \). Then for \( n \leq \lambda \), the inequality
\[ \mu ([f > \beta \lambda] \cap [g \leq r\delta \lambda]) \leq \phi (\delta) \mu ([f > \lambda]) \]
holds with \( f \) replaced with \( f_n \), because both sides equal 0 thanks to \( \beta > 1 \). If \( n > \lambda \), then \( [f > \lambda] = [f_n > \lambda] \) and so the inequality still holds because in this case,
\[ \mu ([f_n > \beta \lambda] \cap [g \leq r\delta \lambda]) \leq \mu ([f > \beta \lambda] \cap [g \leq r\delta \lambda]) \]
\[ \leq \phi (\delta) \mu ([f > \lambda]) = \phi (\delta) \mu ([f_n > \lambda]) \]

Therefore, \textbf{10.7.17} is valid with \( f \) replaced with \( f_n \). Now pass to the limit as \( n \to \infty \) and use the monotone convergence theorem. 

\section*{10.8 The Ergodic Theorem}

I am putting this theorem here because it seems to fit in well with the material of this chapter.

In this section \((\Omega, \mathcal{F}, \mu)\) will be a finite measure space. This means that \( \mu (\Omega) < \infty \). The mapping, \( T : \Omega \to \Omega \) will satisfy the following condition.
\[ T(A), T^{-1}(A) \in \mathcal{F} \text{ whenever } A \in \mathcal{F}, \text{ } T \text{ is one to one.} \quad (10.8.18) \]

For example, you could have \( T \) a homeomorphism on some topological space \( X \) and the \( \sigma \) algebra could be the Borel sets.

\textbf{Lemma 10.8.1} If \( T \) satisfies \textbf{10.7.18}, then \( f \circ T \) is measurable whenever \( f \) is measurable.
10.8. THE ERGODIC THEOREM

Proof: Let $U$ be an open set. Then

$$(f \circ T)^{-1}(U) = T^{-1}(f^{-1}(U)) \in \mathcal{F}$$

by \ref{10.8.18}. \hfill \Box

Now suppose that in addition to \ref{10.8.18}, $T$ also satisfies

$$\mu(T^{-1}A) = \mu(A),$$

for all $A \in \mathcal{F}$. In words, $T^{-1}$ is measure preserving. Note that also

$$\mu(TA) = \mu(T^{-1}TA) = \mu(A)$$

so also $T$ is measure preserving. Then for $T$ satisfying \ref{10.8.18} and \ref{10.8.19}, we have the following simple lemma.

Lemma 10.8.2 If $T$ satisfies \ref{10.8.18} and \ref{10.8.19} then whenever $f$ is nonnegative and measurable,

$$\int_{\Omega} f(\omega) d\mu = \int_{\Omega} f(T\omega) d\mu.$$ \hfill (10.8.20)

Also \ref{10.8.20} holds whenever $f \in L^1(\Omega)$.

Proof: Let $f \geq 0$ and $f$ is measurable. Let $A \in \mathcal{F}$. Then from \ref{10.8.19},

$$\int_{\Omega} X_A(\omega) d\mu = \mu(A) = \mu(T^{-1}(A)) = \int_{\Omega} X_{T^{-1}(A)}(\omega) d\mu = \int_{\Omega} X_A(T(\omega)) d\mu.$$

It follows that whenever $s$ is a simple function,

$$\int s(\omega) d\mu = \int s(T\omega) d\mu$$

If $f \geq 0$ and measurable, Theorem \ref{9.3.9} on Page 227, implies there exists an increasing sequence of simple functions, $\{s_n\}$ converging pointwise to $f$. Then the result follows from monotone convergence theorem. Splitting $f \in L^1$ into real and imaginary parts we apply this to the positive and negative parts of these and obtain \ref{10.8.20} in this case also. \hfill \Box

Definition 10.8.3 A measurable function $f$, is said to be invariant if

$$f(T\omega) = f(\omega).$$

A set, $A \in \mathcal{F}$ is said to be invariant if $X_A$ is an invariant function. Thus a set is invariant if and only if $T^{-1}A = A$. ($X_A(T\omega) = X_{T^{-1}(A)}(\omega)$ so to say that $X_A$ is invariant is to say that $T^{-1}A = A$.)
The following theorem, the individual ergodic theorem, is the main result. Define 
\( T^0(\omega) = \omega \). Let 
\[
S_n f(\omega) \equiv \sum_{k=1}^{n} f(T^{k-1}\omega), \quad S_0 f(\omega) \equiv 0.
\]
Also define the following maximal type function \( M_\infty f(\omega) \)
\[
M_\infty f(\omega) \equiv \sup\{S_k f(\omega) : 0 \leq k\} \tag{10.8.21}
\]
and let
\[
M_n f(\omega) \equiv \sup\{S_k f(\omega) : 0 \leq k \leq n\} \tag{10.8.22}
\]
Then one can prove the following interesting lemma.

**Lemma 10.8.4** Let \( f \in L^1(\mu) \) where \( f \) has real values. Then \( \int_{[M_\infty f > 0]} f \, d\mu \geq 0 \).

**Proof:** First note that \( M_n f(\omega) \geq 0 \) for all \( n \) and \( \omega \). This follows easily from the observation that by definition, \( S_0 f(\omega) = 0 \) and so \( M_n f(\omega) \) is at least as large. There is certainly something to show here because the integrand is not known to be nonnegative. The integral involves \( f \) not \( M_\infty f \).

Let \( T^* h \equiv h \circ T \). Thus \( T^* \) is linear and maps measurable functions to measurable functions by Lemma 10.8.1. It is also clear that if \( h \geq 0 \), then \( T^* h \geq 0 \) also. Therefore, for large \( k \leq n \),
\[
S_k f(\omega) \equiv \sum_{j=1}^{k} f(T^{j-1}\omega) = f(\omega) + \sum_{j=2}^{k} f(T^{j-1}\omega)
\]
\[
= f(\omega) + T^* \sum_{j=1}^{k-1} f(T^{j-1}\omega) \quad \text{(factored out } T^*)
\]
\[
= f(\omega) + T^* S_{k-1} f(\omega) \leq f(\omega) + T^* M_n f
\]
and so, taking the supremum for \( k \leq n \),
\[
M_n f(\omega) \leq f(\omega) + T^* M_n f(\omega).
\]
Now since \( M_n f \geq 0 \),
\[
\int_{\Omega} M_n f(\omega) \, d\mu = \int_{[M_n f > 0]} M_n f(\omega) \, d\mu
\]
\[
\leq \int_{[M_n f > 0]} f(\omega) \, d\mu + \int_{\Omega} T^* M_n f(\omega) \, d\mu
\]
\[
= \int_{[M_n f > 0]} f(\omega) \, d\mu + \int_{\Omega} M_n f(\omega) \, d\mu
\]
by Lemma 10.8.3. It follows that
\[
\int_{[M_n f > 0]} f(\omega) \, d\mu \geq 0
\]
10.8. THE ERGODIC THEOREM

for each $n$. Also, since $M_n f(\omega) \to M_\infty f(\omega)$, the following pointwise convergence holds.

$$X_{[M_n f > 0]}(\omega) f(\omega) \to X_{[M_\infty f > 0]}(\omega) f(\omega)$$

Since $f$ is in $L^1$, the dominated convergence theorem implies

$$\int_{[M_\infty f > 0]} f(\omega) d\mu = \lim_{n \to \infty} \int_{[M_n f > 0]} f(\omega) d\mu \geq 0.$$  \hfill \blacksquare

**Theorem 10.8.5** Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T : \Omega \to \Omega$ satisfy \[H.8.26\] and \[H.8.27\], $T^{-1}$ is measure preserving and $T^{-1}$ maps $\mathcal{F}$ to $\mathcal{F}$ and $T$ is one to one. Then if $f \in L^1(\Omega)$ having real or complex values and

$$S_n f(\omega) \equiv \sum_{k=1}^{n} f(T^{k-1}\omega), \quad S_0 f(\omega) \equiv 0, \quad (10.8.23)$$

it follows there exists a set of measure zero $N$, and an invariant function $g$ such that for all $\omega \notin N$,

$$\lim_{n \to \infty} \frac{1}{n} S_n f(\omega) = g(\omega). \quad (10.8.24)$$

and also

$$\lim_{n \to \infty} \frac{1}{n} S_n f = g \text{ in } L^1(\Omega)$$

**Proof:** To begin with, we assume $f$ has real values. Now if $A$ is an invariant set, $X_A(T^m \omega) = X_A(\omega)$ and so

$$S_n(X_A f)(\omega) \equiv \sum_{k=1}^{n} f(T^{k-1}\omega) X_A(T^{k-1}\omega) = \sum_{k=1}^{n} f(T^{k-1}\omega) X_A(\omega)$$

$$= X_A(\omega) \sum_{k=1}^{n} f(T^{k-1}\omega) = X_A(\omega) S_n f(\omega).$$

Therefore, for such an invariant set,

$$M_n(X_A f)(\omega) = X_A(\omega) M_n f(\omega), \quad M_\infty(X_A f)(\omega) = X_A(\omega) M_\infty f(\omega). \quad (10.8.25)$$

Let $-\infty < a < b < \infty$ and define

$$N_{ab} \equiv \left\{ -\infty < \lim_{n \to \infty} \inf \frac{1}{n} S_n f(\omega) < a < b < \lim_{n \to \infty} \sup \frac{1}{n} S_n f(\omega) < \infty \right\} \quad (10.8.26)$$

Observe that from the definition,

$$\lim_{n \to \infty} \inf \frac{1}{n} S_n f(\omega) = \lim_{n \to \infty} \inf \frac{1}{n} S_n f(T\omega)$$

and

$$\lim_{n \to \infty} \sup \frac{1}{n} S_n f(\omega) = \lim_{n \to \infty} \sup \frac{1}{n} S_n f(T\omega).$$
Thus if $\omega \in N_{ab}$, it follows that $T\omega \in N_{ab}$ and if $T\omega \in N_{ab}$, then so is $\omega$. Thus $N_{ab}$ is an invariant set. Also, if $\omega \in N_{ab}$, then

$$a - \liminf_{n \to \infty} \frac{1}{n} S_n f (\omega) = \limsup_{n \to \infty} \left( a - \frac{1}{n} S_n f (\omega) \right) > 0$$

and

$$\limsup_{n \to \infty} \left( \frac{1}{n} S_n f (\omega) - b \right) > 0$$

It follows that

$$N_{ab} \subseteq \left[ M_{\infty} (f - b) > 0 \right] \cap \left[ M_{\infty} (a - f) > 0 \right].$$

Consequently, since $N_{ab}$ is invariant, argued above,

$$X_{N_{ab}} M_{\infty} (f - b) = M_{\infty} (X_{N_{ab}} (f - b))$$

and so from Lemma 10.8.4

$$\int_{N_{ab}} (f (\omega) - b) \, d\mu = \int_{[X_{N_{ab}} M_{\infty} (f - b) > 0]} X_{N_{ab}} (\omega) (f (\omega) - b) \, d\mu$$

$$= \int_{[M_{\infty} (X_{N_{ab}} (f - b)) > 0]} X_{N_{ab}} (\omega) (f (\omega) - b) \, d\mu \geq 0 \quad (10.8.27)$$

and

$$\int_{N_{ab}} (a - f (\omega)) \, d\mu = \int_{[X_{N_{ab}} M_{\infty} (a - f) > 0]} X_{N_{ab}} (\omega) (a - f (\omega)) \, d\mu$$

$$= \int_{[M_{\infty} (X_{N_{ab}} (a - f)) > 0]} X_{N_{ab}} (\omega) (a - f (\omega)) \, d\mu \geq 0 \quad (10.8.28)$$

It follows that

$$a \mu (N_{ab}) \geq \int_{N_{ab}} f \, d\mu \geq b \mu (N_{ab}). \quad (10.8.29)$$

Since $a < b$, it follows that $\mu (N_{ab}) = 0$.

Now let

$$N = \bigcup \left\{ N_{ab} : a < b, \ a, b \in \mathbb{Q} \right\}.$$

It follows that $\mu (N) = 0$. Now $T N_{a,b} = N_{a,b}$ and so

$$T (N) = \bigcup_{a,b} T (N_{a,b}) = \bigcup_{a,b} N_{a,b} = N.$$

Thus, $T^n N = N$ for all $n \in \mathbb{N}$. For $\omega \notin N$, let $\lim_{n \to \infty} \frac{1}{n} S_n f (\omega)$ exists. Now let

$$g (\omega) \equiv \begin{cases} 0 & \text{if } \omega \in N \\ \lim_{n \to \infty} \frac{1}{n} S_n f (\omega) & \text{if } \omega \notin N \end{cases}.$$

Then it is clear $g$ satisfies the conditions of the theorem because if $\omega \in N$, then $T \omega \in N$ also and so in this case, $g (T \omega) = g (\omega) \equiv 0$. On the other hand, if $\omega \notin N$, then

$$g (T \omega) = \lim_{n \to \infty} \frac{1}{n} S_n f (T \omega) = \lim_{n \to \infty} \frac{1}{n} S_n f (\omega) = g (\omega).$$
Which shows that \( g \) is invariant. Also, from Lemma 10.8.2,

\[
\int \Omega |g| \, d\mu \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \int \Omega |f(T^{k-1}\omega)| \, d\mu = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \int \Omega f(T^{k-1}\omega) \, d\mu = \|f\|_{L^1}
\]

so \( g \in L^1(\Omega, \mu) \).

The last claim about convergence in \( L^1 \) follows from the Vitali convergence theorem if we verify the sequence, \( \left\{ \frac{1}{n}S_n f \right\}_{n=1}^{\infty} \) is uniformly integrable. To see this is the case, we know \( f \in L^1(\Omega) \) and so if \( \varepsilon > 0 \) is given, there exists \( \delta > 0 \) such that whenever \( B \in \mathcal{F} \) and \( \mu(B) \leq \delta \), then \( \left| \int_B f(\omega) \, d\mu \right| < \varepsilon \). Taking \( \mu(A) < \delta \), it follows

\[
\left| \int A \frac{1}{n}S_n f(\omega) \, d\mu \right| = \left| \frac{1}{n} \sum_{k=1}^{n} \int A f(T^{k-1}\omega) \, d\mu \right| = \left| \frac{1}{n} \sum_{k=1}^{n} \int \Omega X_A(\omega) f(T^{k-1}\omega) \, d\mu \right|
\]

\[
= \left| \frac{1}{n} \sum_{k=1}^{n} \int \Omega X_A(T^{k-1}T^{-1}) f(T^{k-1}\omega) \, d\mu \right|
\]

\[
= \left| \frac{1}{n} \sum_{k=1}^{n} \int \Omega X_A(T^{-1}) f(\omega) \, d\mu \right|
\]

\[
= \left| \frac{1}{n} \sum_{k=1}^{n} \int \Omega X_A(T^{-k}) f(\omega) \, d\mu \right| < \frac{1}{n} \sum_{k=1}^{n} \varepsilon = \varepsilon
\]

because \( \mu(T^{k-1}A) = \mu(A) \) by assumption. This proves the above sequence is uniformly integrable and so, by the Vitali convergence theorem,

\[
\lim_{n \to \infty} \int \Omega \left| \frac{1}{n}S_n f - g \right| \, d\mu = 0.
\]

This proves the theorem in the case the function has real values. In the case where \( f \) has complex values, apply the above result to the real and imaginary parts of \( f \).

\[\blacksquare\]

**Definition 10.8.6** The above mapping \( T \) is ergodic if the only invariant sets have measure 0 or 1.

If the map, \( T \) is ergodic, the following corollary holds.

**Corollary 10.8.7** In the situation of Theorem 10.8.5, if \( T \) is ergodic, then

\[g(\omega) = \int f(\omega) \, d\mu\]

for a.e. \( \omega \).
CHAPTER 10. THE CONSTRUCTION OF MEASURES

Proof: Let \( g \) be the function of Theorem 10.8.5 and let \( R_1 \) be a rectangle in \( \mathbb{R}^2 = \mathbb{C} \) of the form \([-a, a] \times [-a, a] \) such that \( g^{-1}(R_1) \) has measure greater than 0. This set is invariant because the function, \( g \) is invariant and so it must have measure 1. Divide \( R_1 \) into four equal rectangles, \( R'_1, R'_2, R'_3, R'_4 \). Then one of these, renamed \( R_2 \) has the property that \( g^{-1}(R_2) \) has positive measure. Therefore, since the set is invariant, it must have measure 1. Continue in this way obtaining a sequence of closed rectangles, \( \{R_i\} \) such that the diameter of \( R_i \) converges to zero and \( g^{-1}(R_i) \) has measure 1. Then let \( c = \bigcap_{i=1}^{\infty} R_i \). We know \( \mu(g^{-1}(c)) = \lim_{n \to \infty} \mu(g^{-1}(R_i)) = 1 \). It follows that \( g(\omega) = c \) for a.e. \( \omega \). Now from Theorem 

\[ c = \int cd\mu = \lim_{n \to \infty} \frac{1}{n} \int S_n fd\mu = \int f d\mu. \]

10.9 Product Measures

Let \((X, \mathcal{S}, \mu)\) and \((Y, \mathcal{T}, \nu)\) be two complete measure spaces. In this section consider the problem of defining a product measure, \( \mu \times \nu \) which is defined on a \( \sigma \)-algebra of sets of \( X \times Y \) such that \( (\mu \times \nu)(E \times F) = \mu(E) \nu(F) \) whenever \( E \in \mathcal{S} \) and \( F \in \mathcal{T} \).

I found the following approach to product measures in [44] and they say they got it from [47].

**Definition 10.9.1** Let \( \mathcal{R} \) denote the set of countable unions of sets of the form \( A \times B \), where \( A \in \mathcal{S} \) and \( B \in \mathcal{T} \). (Sets of the form \( A \times B \) are referred to as measurable rectangles) and also let

\[ \rho(A \times B) = \mu(A) \nu(B) \quad (10.9.30) \]

More generally, define

\[ \rho(E) \equiv \int \int \chi_E(x,y) \, d\mu d\nu \quad (10.9.31) \]

whenever \( E \) is such that \( x \to \chi_E(x,y) \) is \( \mu \) measurable for all \( y \) \quad (10.9.32)

and \( y \to \int \chi_E(x,y) \, d\mu \) is \( \nu \) measurable. \quad (10.9.33)

Note that if \( E = A \times B \) as above, then

\[ \int \int \chi_E(x,y) \, d\mu d\nu = \int \int \chi_A(x) \chi_B(y) \, d\mu d\nu = \mu(A) \nu(B) = \rho(E) \]

and so there is no contradiction between (10.9.31) and (10.9.30).
10.9. PRODUCT MEASURES

The first goal is to show that for $Q \in \mathcal{R}$, and $\nu$ both hold. That is, $x \to X_Q(x, y)$ is $\mu$ measurable for all $y$ and $y \to \int X_Q(x, y) \, d\mu$ is $\nu$ measurable. This is done so that it is possible to speak of $\rho(Q)$. The following lemma will be the fundamental result which will make this possible. First here is a picture.

Lemma 10.9.2 Given $C \times D$ and $\{A_i \times B_i\}_{i=1}^n$, there exist finitely many disjoint rectangles, $\{C'_k \times D'_k\}_{i=1}^p$ such that none of these sets intersect any of the $A_i \times B_i$, each set is contained in $C \times D$ and

$$(\cup_{i=1}^n A_i \times B_i) \cup (\cup_{k=1}^p C'_k \times D'_k) = (C \times D) \cup (\cup_{i=1}^m A_i \times B_i).$$

Proof: From the above picture, you see that

$$(C \times D) \setminus (A_1 \times B_1) = C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)$$

and these last two sets are disjoint, have empty intersection with $A_1 \times B_1$, and

$$(C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)) \cup (\cup_{i=1}^n A_i \times B_i) = (C \times D) \cup (\cup_{i=1}^m A_i \times B_i)$$

Now suppose disjoint sets, $\{\tilde{C}_i \times \tilde{D}_i\}_{i=1}^m$ have been obtained, each being a subset of $C \times D$ such that

$$(\cup_{i=1}^n A_i \times B_i) \cup (\cup_{k=1}^m \tilde{C}_k \times \tilde{D}_k) = (\cup_{i=1}^n A_i \times B_i) \cup (C \times D)$$

and for all $k$, $\tilde{C}_k \times \tilde{D}_k$ has empty intersection with each set of $\{A_i \times B_i\}_{i=1}^p$. Then using the same procedure, replace each of $\tilde{C}_k \times \tilde{D}_k$ with finitely many disjoint rectangles such that none of these intersect $A_{p+1} \times B_{p+1}$ while preserving the union of all the sets involved. The process stops when you have gotten to $n$. This proves the lemma.

Lemma 10.9.3 If $Q = \cup_{i=1}^\infty A_i \times B_i \in \mathcal{R}$, then there exist disjoint sets, of the form $A'_i \times B'_i$ such that $Q = \cup_{i=1}^\infty A'_i \times B'_i$, each $A'_i \times B'_i$ is a subset of some $A_i \times B_i$, and $A'_i \in S$ while $B'_i \in T$. Also, the intersection of finitely many sets of $\mathcal{R}$ is a set of $\mathcal{R}$. For $\rho$ defined in (10.9.1), it follows that $\mu$ and $\nu$ hold for any element of $\mathcal{R}$. Furthermore,

$$\rho(Q) = \sum_i \mu(A'_i) \nu(B'_i) = \sum_i \rho(A'_i \times B'_i).$$
Proof: Let \( Q \) be given as above. Let \( A'_1 \times B'_1 = A_1 \times B_1 \). By Lemma 10.9.2, it is possible to replace \( A_2 \times B_2 \) with finitely many disjoint rectangles, \( \{ A'_i \times B'_i \}_{i=2}^{m_2} \), such that none of these rectangles intersect \( A'_1 \times B'_1 \), each is a subset of \( A_2 \times B_2 \), and

\[
\bigcup_{i=1}^\infty A_i \times B_i = \left( \bigcup_{i=1}^{m_2} A'_i \times B'_i \right) \cup \left( \bigcup_{k=3}^\infty A_k \times B_k \right)
\]

Now suppose disjoint rectangles, \( \{ A'_i \times B'_i \}_{i=1}^{m_p} \) have been obtained such that each rectangle is a subset of \( A_k \times B_k \) for some \( k \leq p \) and

\[
\bigcup_{i=1}^\infty A_i \times B_i = \left( \bigcup_{i=1}^{m_p} A'_i \times B'_i \right) \cup \left( \bigcup_{k=p+1}^\infty A_k \times B_k \right)
\]

By Lemma 10.9.2 again, there exist disjoint rectangles \( \{ A'_i \times B'_i \}_{i=1}^{m_{p+1}} \) such that each is contained in \( A_{p+1} \times B_{p+1} \), none have intersection with any of \( \{ A'_i \times B'_i \}_{i=1}^{m_p} \) and

\[
\bigcup_{i=1}^\infty A_i \times B_i = \left( \bigcup_{i=1}^{m_{p+1}} A'_i \times B'_i \right) \cup \left( \bigcup_{k=p+2}^\infty A_k \times B_k \right)
\]

Note that no change is made in \( \{ A'_i \times B'_i \}_{i=1}^{m_p} \). Continuing this way proves the existence of the desired sequence of disjoint rectangles, each of which is a subset of at least one of the original rectangles and such that

\[
Q = \bigcup_{i=1}^\infty A'_i \times B'_i.
\]

It remains to verify \( x \to X_Q (x, y) \) is \( \mu \) measurable for all \( y \) and

\[
y \to \int X_Q (x, y) \, d\mu
\]

is \( \nu \) measurable whenever \( Q \in \mathcal{R} \). Let \( Q \equiv \bigcup_{i=1}^\infty A_i \times B_i \in \mathcal{R} \). Then by the first part of this lemma, there exists \( \{ A'_i \times B'_i \}_{i=1}^{\infty} \) such that the sets are disjoint and

\[
\bigcup_{i=1}^\infty A'_i \times B'_i = Q.
\]

Therefore, since the sets are disjoint,

\[
X_Q (x, y) = \sum_{i=1}^\infty X_{A'_i \times B'_i} (x, y) = \sum_{i=1}^\infty X_{A'_i} (x) \cdot X_{B'_i} (y).
\]

It follows \( x \to X_Q (x, y) \) is measurable. Now by the monotone convergence theorem,

\[
\int X_Q (x, y) \, d\mu = \int \sum_{i=1}^\infty X_{A'_i} (x) \cdot X_{B'_i} (y) \, d\mu
\]

\[
= \sum_{i=1}^\infty X_{B'_i} (y) \int X_{A'_i} (x) \, d\mu
\]

\[
= \sum_{i=1}^\infty X_{B'_i} (y) \mu (A'_i).
\]
It follows \( y \to \int X_Q(x, y) \, d\mu \) is measurable and so by the monotone convergence theorem again,

\[
\int \int X_Q(x, y) \, d\mu \, d\nu = \int \sum_{i=1}^{\infty} X_{B'_i}(y) \, \mu(A'_i) \, d\nu
\]

\[
= \sum_{i=1}^{\infty} \int X_{B'_i}(y) \, \mu(A'_i) \, d\nu
\]

\[
= \sum_{i=1}^{\infty} \nu(B'_i) \, \mu(A'_i).
\]  

(10.9.34)

This shows the measurability conditions, \(10.9.32\) and \(10.9.33\) hold for \(Q \in \mathcal{R}\) and also establishes the formula for \(\rho(Q)\), \(10.9.34\).

If \(\bigcup_i A_i \times B_i\) and \(\bigcup_j C_j \times D_j\) are two sets of \(\mathcal{R}\), then their intersection is

\[
\bigcup_i \bigcup_j (A_i \cap C_j) \times (B_i \cap D_j)
\]

a countable union of measurable rectangles. Thus finite intersections of sets of \(\mathcal{R}\) are in \(\mathcal{R}\). This proves the lemma.

Now note that from the definition of \(\mathcal{R}\) if you have a sequence of elements of \(\mathcal{R}\) then their union is also in \(\mathcal{R}\). The next lemma will enable the definition of an outer measure.

**Lemma 10.9.4** Suppose \(\{R_i\}_{i=1}^{\infty}\) is a sequence of sets of \(\mathcal{R}\) then

\[
\rho \left( \bigcup_{i=1}^{\infty} R_i \right) \leq \sum_{i=1}^{\infty} \rho(R_i).
\]

**Proof:** Let \(R_i = \bigcup_{j=1}^{\infty} A'_i \times B'_j\). Using Lemma 10.9.3, let \(\{A'_m \times B'_m\}_{m=1}^{\infty}\) be a sequence of disjoint rectangles each of which is contained in some \(A'_j \times B'_j\) for some \(i, j\) such that

\[
\bigcup_{i=1}^{\infty} R_i = \bigcup_{m=1}^{\infty} A'_m \times B'_m.
\]

Now define

\[
S_i \equiv \{ m : A'_m \times B'_m \subseteq A'_j \times B'_j \text{ for some } j \}.
\]

It is not important to consider whether some \(m\) might be in more than one \(S_i\). The important thing to notice is that

\[
\bigcup_{m \in S_i} A'_m \times B'_m \subseteq \bigcup_{j=1}^{\infty} A'_j \times B'_j = R_i.
\]

Then by Lemma 10.9.3,

\[
\rho \left( \bigcup_{i=1}^{\infty} R_i \right) = \sum_{m} \rho(A'_m \times B'_m)
\]

\[
\leq \sum_{i=1}^{\infty} \sum_{m \in S_i} \rho(A'_m \times B'_m)
\]

\[
\leq \sum_{i=1}^{\infty} \rho \left( \bigcup_{m \in S_i} A'_m \times B'_m \right) \leq \sum_{i=1}^{\infty} \rho(R_i).
\]
This proves the lemma.

So far, there is no measure and no \( \sigma \) algebra. However, the next step is to define
an outer measure which will lead to a measure on a \( \sigma \) algebra of measurable sets
from the Caratheodory procedure. When this is done, it will be shown that this
measure can be computed using \( \rho \) which implies the important Fubini theorem.

Now it is possible to define an outer measure.

**Definition 10.9.5** For \( S \subseteq X \times Y \), define
\[
(\overline{\mu \times \nu})(S) \equiv \inf \{ \rho(R) : S \subseteq R, R \in \mathcal{R} \}.
\]  
(10.9.35)

The following proposition is interesting but is not needed in the development
which follows. It gives a different description of \( (\overline{\mu \times \nu}) \).

**Proposition 10.9.6** \( (\overline{\mu \times \nu})(S) = \inf \{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : S \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i \} \)

**Proof:** Let \( \lambda(S) = \inf \{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : S \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i \} \). Suppose \( S \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i = Q \in \mathcal{R} \). Then by Lemma 10.9.7, \( Q = \bigcup_{i=1}^{\infty} A_i \times B_i \) where these rectangles are disjoint. Thus by this lemma, \( \rho(Q) = \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) \geq \lambda(S) \) and so \( \lambda(S) \leq (\overline{\mu \times \nu})(S) \). If \( \lambda(S) = \infty \), this shows \( \lambda(S) = (\overline{\mu \times \nu})(S) \). Suppose then that \( \lambda(S) < \infty \) and \( \lambda(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) \) where \( Q = \bigcup_{i=1}^{\infty} A_i \times B_i \supseteq S \). Then by Lemma 10.9.7 again, \( \bigcup_{i=1}^{\infty} A_i \times B_i = \bigcup_{i=1}^{\infty} A_i' \times B_i' \) where the primed rectangles are disjoint, each is a subset of some \( A_i \times B_i \) and so

\[
\lambda(S) + \varepsilon \geq \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) \geq \sum_{i=1}^{\infty} \mu(A_i') \nu(B_i') = \rho(Q) \geq (\overline{\mu \times \nu})(S).
\]

Since \( \varepsilon \) is arbitrary, this shows \( \lambda(S) \geq (\overline{\mu \times \nu})(S) \) and this proves the proposition.

**Lemma 10.9.7** \( \overline{\mu \times \nu} \) is an outer measure on \( X \times Y \) and for \( R \in \mathcal{R} \)
\[
(\overline{\mu \times \nu})(R) = \rho(R).
\]  
(10.9.36)

**Proof:** First consider 10.9.6. Since \( R \supseteq \mathcal{R} \), it follows \( \rho(R) \geq (\overline{\mu \times \nu})(R) \). On
the other hand, if \( Q \in \mathcal{R} \) and \( Q \supseteq R \), then \( \rho(Q) \geq \rho(R) \) and so, taking the infimum
on the left yields \( (\overline{\mu \times \nu})(R) \geq \rho(R) \). This shows 10.9.6.

It is necessary to show that if \( S \subseteq T \), then
\[
(\overline{\mu \times \nu})(S) \leq (\overline{\mu \times \nu})(T),
\]  
(10.9.37)
\[
(\overline{\mu \times \nu})(\bigcup_{i=1}^{\infty} S_i) \leq \sum_{i=1}^{\infty} (\overline{\mu \times \nu})(S_i).
\]  
(10.9.38)

To do this, note that 10.9.7 is obvious. To verify 10.9.6, note that it is obvious
if \( (\overline{\mu \times \nu})(S_i) = \infty \) for any \( i \). Therefore, assume \( (\overline{\mu \times \nu})(S_i) < \infty \). Then letting
\( \varepsilon > 0 \) be given, there exist \( R_i \in \mathcal{R} \) such that
\[
(\overline{\mu \times \nu})(S_i) + \frac{\varepsilon}{2^i} > \rho(R_i), \ R_i \supseteq S_i.
\]
Then by Lemma 10.9.4 and the observation that \( \bigcup_{i=1}^{\infty} R_i \in \mathcal{R} \),

\[
(\mu \times \nu) \left( \bigcup_{i=1}^{\infty} S_i \right) \leq (\mu \times \nu) \left( \bigcup_{i=1}^{\infty} R_i \right) = \rho \left( \bigcup_{i=1}^{\infty} R_i \right) \leq \sum_{i=1}^{\infty} \rho(R_i) \leq \sum_{i=1}^{\infty} \left( (\mu \times \nu)(S_i) + \frac{\varepsilon}{2^i} \right) = \left( \sum_{i=1}^{\infty} (\mu \times \nu)(S_i) \right) + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this proves the lemma.

By Caratheodory’s procedure, it follows there is a \( \sigma \)-algebra of subsets of \( X \times Y \), denoted here by \( S \times T \) such that \( (\mu \times \nu) \) is a complete measure on this \( \sigma \)-algebra.

The first thing to note is that every rectangle is in this \( \sigma \)-algebra.

**Lemma 10.9.8** Every rectangle is \( (\mu \times \nu) \) measurable.

**Proof:** Let \( S \subseteq X \times Y \). The following inequality must be established.

\[
(\mu \times \nu)(S) \geq (\mu \times \nu)(S \cap (A \times B)) + (\mu \times \nu)(S \setminus (A \times B)). \tag{10.9.39}
\]

The following claim will be used to establish this inequality.

**Claim:** Let \( P, A \times B \in \mathcal{R} \). Then

\[
\rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) = \rho(P).
\]

**Proof of the claim:** From Lemma 10.9.3, \( P = \bigcup_{i=1}^{\infty} A'_i \times B'_i \) where the \( A'_i \times B'_i \) are disjoint. Therefore,

\[
P \cap (A \times B) = \bigcup_{i=1}^{\infty} (A \cap A'_i) \times (B \cap B'_i)
\]

while

\[
P \setminus (A \times B) = \bigcup_{i=1}^{\infty} (A'_i \setminus A) \times B'_i \cup \bigcup_{i=1}^{\infty} (A \setminus A'_i) \times (B'_i \setminus B).
\]

Since all of the sets in the above unions are disjoint,

\[
\rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) =
\]

\[
\int \int \sum_{i=1}^{\infty} \chi_{(A \cap A'_i)}(x) \chi_{B \cap B'_i}(y) \, d\mu d\nu + \int \int \sum_{i=1}^{\infty} \chi_{(A'_i \setminus A)}(x) \chi_{B'_i \setminus B}(y) \, d\mu d\nu
\]

\[
+ \int \int \sum_{i=1}^{\infty} \chi_{A \cap A'_i}(x) \chi_{B'_i \setminus B}(y) \, d\mu d\nu
\]
\[ \sum_{i=1}^{\infty} \mu(A \cap A_i') \nu(B \cap B_i') + \mu(A \setminus A_i') \nu(B_i') + \mu(A \cap A_i') \nu(B_i') = \mu(A \setminus A_i') \nu(B_i') + \mu(A_i') \nu(B_i') = \rho(P). \]

This proves the claim.

Now continuing to verify \[10.9.39\], without loss of generality, \((\mu \times \nu)(S)\) can be assumed finite. Let \(P \supseteq S\) for \(P \in \mathcal{R}\) and

\[ (\mu \times \nu)(S) + \varepsilon > \rho(P). \]

Then from the claim,

\[ (\mu \times \nu)(S) + \varepsilon > \rho(P) = \rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) \geq (\mu \times \nu)(S \cap (A \times B)) + (\mu \times \nu)(S \setminus (A \times B)). \]

Since \(\varepsilon > 0\) this shows \(A \times B\) is \(\mu \times \nu\) measurable as claimed.

**Lemma 10.9.9** Let \(\mathcal{R}_1\) be defined as the set of all countable intersections of sets of \(\mathcal{R}\). Then if \(S \subseteq X \times Y\), there exists \(R \in \mathcal{R}_1\) for which it makes sense to write \(\rho(R)\) because \([10.9.32\) and \([10.9.33\) hold such that

\[ (\mu \times \nu)(S) = \rho(R). \] (10.9.40)

Also, every element of \(\mathcal{R}_1\) is \(\mu \times \nu\) measurable.

**Proof:** Consider \([10.9.41\). Let \(S \subseteq X \times Y\). If \((\mu \times \nu)(S) = \infty\), let \(R = X \times Y\) and it follows \(\rho(X \times Y) = \infty = (\mu \times \nu)(S)\). Assume then that \((\mu \times \nu)(S) < \infty\).

Therefore, there exists \(P_n \in \mathcal{R}\) such that \(P_n \supseteq S\) and

\[ (\mu \times \nu)(S) \leq \rho(P_n) < (\mu \times \nu)(S) + 1/n. \] (10.9.41)

Let \(Q_n = \cap_{i=1}^{n} P_i \in \mathcal{R}\). Define

\[ P \equiv \cap_{i=1}^{\infty} Q_i \supseteq S. \]

Then \([10.9.42\) holds with \(Q_n\) in place of \(P_n\). It is clear that

\[ x \rightarrow X_P(x, y) \text{ is } \mu \text{ measurable} \]

because this function is the pointwise limit of functions for which this is so. It remains to consider whether \(y \rightarrow \int X_P(x, y) \, d\mu\) is \(\nu\) measurable. First observe \(Q_n \supseteq Q_{n+1}, X_{Q_n} \leq X_{P_n}\), and

\[ \rho(Q_1) = \rho(P_1) = \int \int X_{P_1}(x, y) \, d\mu \nu < \infty. \] (10.9.42)
Therefore, there exists a set of $\nu$ measure 0, $N$, such that if $y \not\in N$, then
\[
\int X_{P_1} (x,y) \, d\mu < \infty.
\]
It follows from the dominated convergence theorem that
\[
\lim_{n \to \infty} X_{N_C} (y) \int X_{Q_n} (x,y) \, d\mu = X_{N_C} (y) \int X_P (x,y) \, d\mu
\]
and so
\[
y \to X_{N_C} (y) \int X_P (x,y) \, d\mu
\]
is also measurable. By completeness of $\nu$,
\[
y \to \int X_P (x,y) \, d\mu
\]
must also be $\nu$ measurable and so it makes sense to write
\[
\int \int X_P (x,y) \, d\mu \, d\nu
\]
for every $P \in \mathcal{R}_1$. Also, by the dominated convergence theorem,
\[
\int \int X_P (x,y) \, d\mu \, d\nu = \int X_{N_C} (y) \int X_P (x,y) \, d\mu \, d\nu
\]
\[
= \lim_{n \to \infty} \int X_{N_C} (y) \int X_{Q_n} (x,y) \, d\mu \, d\nu
\]
\[
= \lim_{n \to \infty} \int \int X_{Q_n} (x,y) \, d\mu \, d\nu
\]
\[
= \lim_{n \to \infty} \rho (Q_n) \in [ (\mu \times \nu) (S), (\mu \times \nu) (S) + 1/n ]
\]
for all $n$. Therefore,
\[
\rho (P) = \int \int X_P (x,y) \, d\mu \, d\nu = (\mu \times \nu) (S).
\]

The sets of $\mathcal{R}_1$ are $\mu \times \nu$ measurable because these sets are countable intersections of countable unions of rectangles and Lemma [10.9.8] verifies the rectangles are $\mu \times \nu$ measurable. This proves the Lemma.

The following theorem is the main result.

**Theorem 10.9.10** Let $E \subseteq X \times Y$ be $\mu \times \nu$ measurable and suppose $(\mu \times \nu) (E) < \infty$. Then
\[
x \to X_E (x,y) \text{ is } \mu \text{ measurable a.e. } y.
\]
Modifying $X_E$ on a set of measure zero, it is possible to write
\[
\int X_E (x,y) \, d\mu.
\]
The function,
\[ y \mapsto \int_X E(x,y) \, d\mu \]
is \( \nu \) measurable and
\[ (\mu \times \nu)(E) = \int \int_X E(x,y) \, d\mu d\nu. \]

Similarly,
\[ (\mu \times \nu)(E) = \int \int_X E(x,y) \, d\nu d\mu. \]

**Proof:** By Lemma [10.9.9], there exists \( R \in \mathcal{R}_1 \) such that
\[ \rho(R) = (\mu \times \nu)(E), \quad R \supseteq E. \]

Therefore, since \( R \) is \( \mu \times \nu \) measurable and \( \rho(R) = (\mu \times \nu)(R) \), it follows
\[ (\mu \times \nu)(R \setminus E) = 0. \]

By Lemma [10.9.9] again, there exists \( P \supseteq R \setminus E \) with \( P \in \mathcal{R}_1 \) and
\[ \rho(P) = (\mu \times \nu)(R \setminus E) = 0. \]

Thus
\[
\int \int_X X_P(x,y) \, d\mu d\nu = 0. \tag{10.9.43}
\]

Since \( P \in \mathcal{R}_1 \) Lemma [10.9.44] implies \( x \mapsto X_P(x,y) \) is \( \mu \) measurable and it follows from the above there exists a set of \( \nu \) measure zero, \( N \) such that if \( y \notin N \), then \( \int X_P(x,y) \, d\mu = 0. \) Therefore, by completeness of \( \nu \),
\[ x \mapsto X_{NC}(y) X_{R \setminus E}(x,y) \]
is \( \mu \) measurable and
\[
\int X_{NC}(y) X_{R \setminus E}(x,y) \, d\mu = 0. \tag{10.9.44}
\]

Now also
\[ X_{NC}(y) X_R(x,y) = X_{NC}(y) X_{R \setminus E}(x,y) + X_{NC}(y) X_E(x,y) \tag{10.9.45} \]
and this shows that
\[ x \mapsto X_{NC}(y) X_E(x,y) \]
is \( \mu \) measurable because it is the difference of two functions with this property. Then by [10.9.44] it follows
\[
\int X_{NC}(y) X_E(x,y) \, d\mu = \int X_{NC}(y) X_R(x,y) \, d\mu.
\]
10.9. PRODUCT MEASURES

The right side of this equation equals a $\nu$ measurable function and so the left side which equals it is also a $\nu$ measurable function. It follows from completeness of $\nu$ that $y \to \int X_E (x, y) \, d\mu$ is $\nu$ measurable because for $y$ outside of a set of $\nu$ measure zero, $N$ it equals $\int X_R (x, y) \, d\mu$. Therefore,

$$
\int \int X_E (x, y) \, d\mu \, d\nu = \int \int X_{NC} (y) X_E (x, y) \, d\mu \, d\nu
= \int \int X_{NC} (y) X_R (x, y) \, d\mu \, d\nu
= \int \int X_R (x, y) \, d\mu \, d\nu
= \rho (R) = \left( \mu \times \nu \right) (E).
$$

In all the above there would be no change in writing $d\nu d\mu$ instead of $d\mu d\nu$. The same result would be obtained. This proves the theorem.

Now let $f : X \times Y \to [0, \infty]$ be $\mu \times \nu$ measurable and

$$
\int fd (\mu \times \nu) < \infty. \quad (10.9.46)
$$

Let $s (x, y) \equiv \sum_{i=1}^{m} c_i X_{E_i} (x, y)$ be a nonnegative simple function with $c_i$ being the nonzero values of $s$ and suppose

$$
0 \leq s \leq f.
$$

Then from the above theorem,

$$
\int sd (\mu \times \nu) = \int \int sd\mu \, d\nu
$$

In which

$$
\int sd\mu = \int X_{NC} (y) \, sd\mu
$$

for $N$ a set of $\nu$ measure zero such that $y \to \int X_{NC} (y) \, sd\mu$ is $\nu$ measurable. This follows because $\mu \times \nu$ implies $(\mu \times \nu) (E_i) < \infty$. Now let $s_n \uparrow f$ where $s_n$ is a nonnegative simple function and

$$
\int s_n d (\mu \times \nu) = \int \int X_{NC} (y) s_n (x, y) \, d\mu \, d\nu
$$

where

$$
y \to \int X_{NC} (y) s_n (x, y) \, d\mu
$$

is $\nu$ measurable. Then let $N \equiv \cup_{n=1}^{\infty} N_n$. It follows $N$ is a set of $\nu$ measure zero. Thus

$$
\int s_n d (\mu \times \nu) = \int \int X_{NC} (y) s_n (x, y) \, d\mu \, d\nu
$$
and letting \( n \to \infty \), the monotone convergence theorem implies

\[
\int f \, d(\mu \times \nu) = \int \int X_N C(y) f(x, y) \, d\mu d\nu = \int \int f(x, y) \, d\mu d\nu
\]

because of completeness of the measures, \( \mu \) and \( \nu \). This proves Fubini’s theorem.

**Theorem 10.9.11 (Fubini)** Let \((X, \mathcal{S}, \mu)\) and \((Y, \mathcal{T}, \nu)\) be complete measure spaces and let

\[
(\mu \times \nu)(E) \equiv \inf \left\{ \int \int X_R (x, y) \, d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}
\]

where \( A_i \in \mathcal{S} \) and \( B_i \in \mathcal{T} \). Then \( \mu \times \nu \) is an outer measure on the subsets of \( X \times Y \) and the \( \sigma \)-algebra of \( \mu \times \nu \) measurable sets, \( \mathcal{S} \times \mathcal{T} \), contains all measurable rectangles. If \( f \geq 0 \) is a \( \mu \times \nu \) measurable function satisfying

\[
\int_{X \times Y} f \, d(\mu \times \nu) < \infty,
\]

then

\[
\int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \int_X f \, d\mu d\nu,
\]

where the iterated integral on the right makes sense because for \( \nu \) a.e. \( y, x \to f(x, y) \) is \( \mu \) measurable and \( y \to \int f(x, y) \, d\mu \) is \( \nu \) measurable. Similarly,

\[
\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \int_Y f \, d\mu d\nu.
\]

In the case where \((X, \mathcal{S}, \mu)\) and \((Y, \mathcal{T}, \nu)\) are both \( \sigma \)-finite, it is not necessary to assume 10.9.47.

**Corollary 10.9.12 (Fubini)** Let \((X, \mathcal{S}, \mu)\) and \((Y, \mathcal{T}, \nu)\) be complete measure spaces such that \((X, \mathcal{S}, \mu)\) and \((Y, \mathcal{T}, \nu)\) are both \( \sigma \)-finite and let

\[
(\mu \times \nu)(E) \equiv \inf \left\{ \int \int X_R (x, y) \, d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}
\]

where \( A_i \in \mathcal{S} \) and \( B_i \in \mathcal{T} \). Then \( \mu \times \nu \) is an outer measure. If \( f \geq 0 \) is a \( \mu \times \nu \) measurable function then

\[
\int_{X \times Y} f \, d(\mu \times \nu) = \int_Y \int_X f \, d\mu d\nu,
\]

\[\text{Recall this is the same as}\]

\[
\inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : E \subseteq \bigcup_{i=1}^{\infty} A_i \times B_i \right\}
\]

in which the \( A_i \) and \( B_i \) are measurable.
where the iterated integral on the right makes sense because for \( \nu \) a.e. \( y, x \to f(x, y) \) is \( \mu \) measurable and \( y \to \int f(x, y) \, d\mu \) is \( \nu \) measurable. Similarly,

\[
\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \int_Y f \, d\nu d\mu.
\]

**Proof:** Let \( \cup_{n=1}^{\infty} X_n = X \) and \( \cup_{n=1}^{\infty} Y_n = Y \) where \( X_n \in \mathcal{S}, Y_n \in \mathcal{T} \), \( X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1} \) for all \( n \) and \( \mu(X_n) < \infty, \nu(Y_n) < \infty \). From Theorem applied to \( X_n, Y_n \) and \( f \equiv \min(f, m) \),

\[
\int_{X_n \times Y_n} f_m \, d(\mu \times \nu) = \int_{Y_n} \int_{X_n} f_m \, d\mu d\nu.
\]

Now take \( m \to \infty \) and use the monotone convergence theorem to obtain

\[
\int_{X \times Y} f \, d(\mu \times \nu) = \int_{Y_n} \int_{X_n} f \, d\mu d\nu.
\]

Then use the monotone convergence theorem again letting \( n \to \infty \) to obtain the desired conclusion. The argument for the other order of integration is similar.

**Corollary 10.9.13** If \( f \in L^1(X \times Y) \), then

\[
\int f \, d(\mu \times \nu) = \int \int f(x, y) \, d\mu d\nu = \int \int f(x, y) \, d\mu d\nu.
\]

If \( \mu \) and \( \nu \) are \( \sigma \) finite, then if \( f \) is \( \mu \times \nu \) measurable having complex values and either \( \int |f| \, d\mu d\nu < \infty \) or \( \int |f| \, d\mu d\nu < \infty \), then \( \int |f| \, d(\mu \times \nu) < \infty \) so \( f \in L^1(X \times Y) \).

**Proof:** Without loss of generality, it can be assumed that \( f \) has real values. Then

\[
f = \frac{|f| + f - (|f| - f)}{2}
\]

and both \( f^+ = \frac{|f| + f}{2} \) and \( f^- = \frac{|f| - f}{2} \) are nonnegative and are less than \( |f| \). Therefore, \( \int g d(\mu \times \nu) < \infty \) for \( g = f^+ \) and \( g = f^- \) so the above theorem applies and

\[
\int f \, d(\mu \times \nu) = \int f^+ d(\mu \times \nu) - \int f^- d(\mu \times \nu)
= \int \int f^+ d\mu d\nu - \int \int f^- d\mu d\nu
= \int \int f d\mu d\nu.
\]

It remains to verify the last claim. Suppose \( s \) is a simple function,

\[
s(x, y) \equiv \sum_{i=1}^{m} c_i X_{E_i} \leq |f|(x, y)
\]
where the \( c_i \) are the nonzero values of \( s \). Then

\[
\sum c_i X_{R_n} \leq |f| X_{R_n}
\]

where \( R_n = X_n \times Y_n \) where \( X_n \uparrow X \) and \( Y_n \uparrow Y \) with \( \mu(X_n) < \infty \) and \( \nu(Y_n) < \infty \). It follows, since the nonzero values of \( sX_{R_n} \) are achieved on sets of finite measure,

\[
\int sX_{R_n} d(\mu \times \nu) = \int \int sX_{R_n} d\mu d\nu.
\]

Letting \( n \to \infty \) and applying the monotone convergence theorem, this yields

\[
\int s d(\mu \times \nu) = \int \int s d\mu d\nu. \tag{10.9.48}
\]

Now let \( s_n \uparrow |f| \) where \( s_n \) is a nonnegative simple function. From 10.9.48,

\[
\int s_n d(\mu \times \nu) = \int \int s_n d\mu d\nu.
\]

Letting \( n \to \infty \) and using the monotone convergence theorem, yields

\[
\int |f| d(\mu \times \nu) = \int \int |f| d\mu d\nu < \infty
\]

10.10 Alternative Treatment Of Product Measure

10.10.1 Monotone Classes And Algebras

Measures are defined on \( \sigma \) algebras which are closed under countable unions. It is for this reason that the theory of measure and integration is so useful in dealing with limits of sequences. However, there is a more basic notion which involves only finite unions and differences.

**Definition 10.10.1** \( \mathcal{A} \) is said to be an algebra of subsets of a set, \( Z \) if \( Z \in \mathcal{A} \), \( \emptyset \in \mathcal{A} \), and when \( E, F \in \mathcal{A} \), \( E \cup F \) and \( E \setminus F \) are both in \( \mathcal{A} \).

It is important to note that if \( \mathcal{A} \) is an algebra, then it is also closed under finite intersections. This is because \( E \cap F = (E^C \cup F^C)^C \in \mathcal{A} \) since \( E^C = Z \setminus E \in \mathcal{A} \) and \( F^C = Z \setminus F \in \mathcal{A} \). Note that every \( \sigma \) algebra is an algebra but not the other way around.

Something satisfying the above definition is called an algebra because union is like addition, the set difference is like subtraction and intersection is like multiplication. Furthermore, only finitely many operations are done at a time and so there is nothing like a limit involved.

How can you recognize an algebra when you see one? The answer to this question is the purpose of the following lemma.
Lemma 10.10.2 Suppose $\mathcal{R}$ and $\mathcal{E}$ are subsets of $\mathcal{P}(\mathbb{Z})^3$ such that $\mathcal{E}$ is defined as the set of all finite disjoint unions of sets of $\mathcal{R}$. Suppose also
\[ \emptyset, Z \in \mathcal{R} \]
\[ A \cap B \in \mathcal{R} \quad \text{whenever} \quad A, B \in \mathcal{R}, \]
\[ A \setminus B \in \mathcal{E} \quad \text{whenever} \quad A, B \in \mathcal{R}. \]
Then $\mathcal{E}$ is an algebra of sets of $\mathbb{Z}$.

Proof: Note first that if $A \in \mathcal{R}$, then $A^C \in \mathcal{E}$ because $A^C = Z \setminus A$.
Now suppose that $E_1$ and $E_2$ are in $\mathcal{E}$,
\[ E_1 = \bigcup_{i=1}^m R_i, \quad E_2 = \bigcup_{j=1}^n R_j \]
where the $R_i$ are disjoint sets in $\mathcal{R}$ and the $R_j$ are disjoint sets in $\mathcal{R}$. Then
\[ E_1 \cap E_2 = \bigcup_{i=1}^m \bigcup_{j=1}^n R_i \cap R_j \]
which is clearly an element of $\mathcal{E}$ because no two of the sets in the union can intersect and by assumption they are all in $\mathcal{R}$. Thus by induction, finite intersections of sets of $\mathcal{E}$ are in $\mathcal{E}$. Consider the difference of two elements of $\mathcal{E}$ next. If $E = \bigcup_{i=1}^n R_i \in \mathcal{E}$,
\[ E^C = \bigcap_{i=1}^n R_i^C = \text{finite intersection of sets of } \mathcal{E} \]
which was just shown to be in $\mathcal{E}$. Now, if $E_1, E_2 \in \mathcal{E}$,
\[ E_1 \setminus E_2 = E_1 \cap E_2^C \in \mathcal{E} \]
from what was just shown about finite intersections.
Finally consider finite unions of sets of $\mathcal{E}$. Let $E_1$ and $E_2$ be sets of $\mathcal{E}$. Then
\[ E_1 \cup E_2 = (E_1 \setminus E_2) \cup E_2 \in \mathcal{E} \]
because $E_1 \setminus E_2$ consists of a finite disjoint union of sets of $\mathcal{R}$ and these sets must be disjoint from the sets of $\mathcal{R}$ whose union yields $E_2$ because $(E_1 \setminus E_2) \cap E_2 = \emptyset$. This proves the lemma.

The following corollary is particularly helpful in verifying the conditions of the above lemma.

Corollary 10.10.3 Let $(Z_1, \mathcal{R}_1, \mathcal{E}_1)$ and $(Z_2, \mathcal{R}_2, \mathcal{E}_2)$ be as described in Lemma 12.2.14. Then $(Z_1 \times Z_2, \mathcal{R}, \mathcal{E})$ also satisfies the conditions of Lemma 12.2.14 if $\mathcal{R}$ is defined as
\[ \mathcal{R} \equiv \{ R_1 \times R_2 : R_i \in \mathcal{R}_i \} \]
and
\[ \mathcal{E} \equiv \{ \text{finite disjoint unions of sets of } \mathcal{R} \}. \]
Consequently, $\mathcal{E}$ is an algebra of sets.

$^3$Set of all subsets of $\mathbb{Z}$
**Proof:** It is clear $\emptyset, Z_1 \times Z_2 \in R$. Let $A \times B$ and $C \times D$ be two elements of $R$. 

$$A \times B \cap C \times D = A \cap C \times B \cap D \in R$$

by assumption.

$$A \times B \setminus (C \times D) =$$

$$\bigcup_{E \in \mathcal{E}_2} A \times (B \setminus D) \cup \bigcup_{E \in \mathcal{E}_1} (A \setminus C) \times (D \cap B)$$

$$= (A \times Q) \cup (P \times R)$$

where $Q \in \mathcal{E}_2$, $P \in \mathcal{E}_1$, and $R \in \mathcal{R}_2$.

Since $A \times Q$ and $P \times R$ do not intersect, it follows the above expression is in $E$ because each of these terms are. This proves the corollary.

**Definition 10.10.4** $\mathcal{M} \subseteq \mathcal{P}(Z)$ is called a **monotone class** if

a.) $\cdots E_n \supseteq E_{n+1} \cdots$, $E = \bigcap_{n=1}^{\infty} E_n$, and $E_n \in \mathcal{M}$, then $E \in \mathcal{M}$.  
b.) $\cdots E_n \subseteq E_{n+1} \cdots$, $E = \bigcup_{n=1}^{\infty} E_n$, and $E_n \in \mathcal{M}$, then $E \in \mathcal{M}$.  

(In simpler notation, $E_n \downarrow E$ and $E_n \in \mathcal{M}$ implies $E \in \mathcal{M}$. $E_n \uparrow E$ and $E_n \in \mathcal{M}$ implies $E \in \mathcal{M}$.)

**Theorem 10.10.5** (Monotone Class theorem) Let $\mathcal{A}$ be an algebra of subsets of $Z$ and let $\mathcal{M}$ be a monotone class containing $\mathcal{A}$. Then $\mathcal{M} \supseteq \sigma(\mathcal{A})$, the smallest $\sigma$-algebra containing $\mathcal{A}$.

**Proof:** Consider all monotone classes which contain $\mathcal{A}$, and take their intersection. The result is still a monotone class which contains $\mathcal{A}$ and is therefore the smallest monotone class containing $\mathcal{A}$. Therefore, assume without loss of generality that $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$ because if it is shown the smallest monotone class containing $\mathcal{A}$ contains $\sigma(\mathcal{A})$, then the given monotone class does also. To avoid more notation, let $\mathcal{M}$ denote this smallest monotone class.

The plan is to show $\mathcal{M}$ is a $\sigma$-algebra. It will then follow $\mathcal{M} \supseteq \sigma(\mathcal{A})$ because $\sigma(\mathcal{A})$ is defined as the intersection of all $\sigma$ algebras which contain $\mathcal{A}$. For $A \in \mathcal{A}$, define

$$\mathcal{M}_A = \{ B \in \mathcal{M} \text{ such that } A \cup B \in \mathcal{M} \}.$$  

Clearly $\mathcal{M}_A$ is a monotone class containing $\mathcal{A}$. Hence $\mathcal{M}_A \supseteq \mathcal{M}$ because $\mathcal{M}$ is the smallest such monotone class. But by construction, $\mathcal{M}_A \subseteq \mathcal{M}$. Therefore,
$M = M_A$. This shows that $A \cup B \in M$ whenever $A \in A$ and $B \in M$. Now pick $B \in M$ and define

$$M_B \equiv \{ D \in M \text{ such that } D \cup B \in M \}.$$ 

It was just shown that $A \subseteq M_B$. It is clear that $M_B$ is a monotone class. Thus by a similar argument, $M_B = M$ and it follows that $D \cup B \in M$ whenever $D \in M$ and $B \in M$. This shows $M$ is closed under finite unions.

Next consider the difference of two sets. Let $A \in A$

$$M_A \equiv \{ B \in M \text{ such that } B \setminus A \text{ and } A \setminus B \in M \}.$$ 

Then $M_A$, is a monotone class containing $A$. As before, $M = M_A$. Thus $B \setminus A$ and $A \setminus B$ are both in $M$ whenever $A \in A$ and $B \in M$. Now pick $A \in M$ and consider

$$M_A \equiv \{ B \in M \text{ such that } B \setminus A \text{ and } A \setminus B \in M \}.$$ 

It was just shown $M_A$ contains $A$. Now $M_A$ is a monotone class and so $M_A = M$ as before.

Thus $M$ is both a monotone class and an algebra. Hence, if $E \in M$ then

$$Z \setminus E \in M.$$ 

Next consider the question of whether $M$ is a $\sigma$-algebra. If $E_i \in M$ and $F_n = \bigcup_{i=1}^{n} E_i$, then $F_n \in M$ and $F_n \uparrow \bigcup_{i=1}^{\infty} E_i$. Since $M$ is a monotone class, $\bigcup_{i=1}^{\infty} E_i \in M$ and so $M$ is a $\sigma$-algebra. This proves the theorem.

### 10.10.2 Product Measure

**Definition 10.10.6** Let $(X, S, \mu)$ and $(Y, F, \lambda)$ be two measure spaces. A measurable rectangle is a set $A \times B \subseteq X \times Y$ where $A \in S$ and $B \in F$. An elementary set will be any subset of $X \times Y$ which is a finite union of disjoint measurable rectangles. $S \times F$ will denote the smallest $\sigma$ algebra of sets in $\mathcal{P}(X \times Y)$ containing all elementary sets.

**Example 10.10.7** It follows from Lemma [12.1.2] or more easily from Corollary [12.1.3] that the elementary sets form an algebra.

**Definition 10.10.8** Let $E \subseteq X \times Y$,

$$E_x = \{ y \in Y : (x, y) \in E \},$$

$$E^y = \{ x \in X : (x, y) \in E \}.$$ 

These are called the $x$ and $y$ sections.
Theorem 10.10.9 If $E \in \mathcal{S} \times \mathcal{F}$, then $E_x \in \mathcal{F}$ and $E^y \in \mathcal{S}$ for all $x \in X$ and $y \in Y$.

Proof: Let

$$\mathcal{M} = \{E \subseteq \mathcal{S} \times \mathcal{F} \text{ such that for all } x \in X, \ E_x \in \mathcal{F},$$

and for all $y \in Y, \ E^y \in \mathcal{S} \}. \]

Then $\mathcal{M}$ contains all measurable rectangles. If $E_i \in \mathcal{M}$,

$$(\cup_{i=1}^{\infty} E_i)_x = \cup_{i=1}^{\infty} (E_i)_x \in \mathcal{F}.$$ 

Similarly,

$$(\cup_{i=1}^{\infty} E_i)^y = \cup_{i=1}^{\infty} E_i^y \in \mathcal{S}.$$ 

It follows $\mathcal{M}$ is closed under countable unions.

If $E \in \mathcal{M}$,

$$(E^C)_x = (E_x)^C \in \mathcal{F}.$$ 

Similarly, $(E^C)^y \in \mathcal{S}$. Thus $\mathcal{M}$ is closed under complementation. Therefore $\mathcal{M}$ is a $\sigma$-algebra containing the elementary sets. Hence, $\mathcal{M} \supseteq \mathcal{S} \times \mathcal{F}$ because $\mathcal{S} \times \mathcal{F}$ is the smallest $\sigma$ algebra containing these elementary sets. But $\mathcal{M} \subseteq \mathcal{S} \times \mathcal{F}$ by definition and so $\mathcal{M} = \mathcal{S} \times \mathcal{F}$. This proves the theorem.

It follows from Lemma 12.1.2 that the elementary sets form an algebra because clearly the intersection of two measurable rectangles is a measurable rectangle and

$$(A \times B) \setminus (A_0 \times B_0) = (A \setminus A_0) \times B \cup (A \cap A_0) \times (B \setminus B_0),$$ 

an elementary set.

Theorem 10.10.10 If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ are both finite measure spaces $\mu(X)$, $\lambda(Y) < \infty$, then for every $E \in \mathcal{S} \times \mathcal{F}$,

a.) $x \to \lambda(E_x)$ is $\mu$ measurable, $y \to \mu(E^y)$ is $\lambda$ measurable

b.) $\int_X \lambda(E_x)d\mu = \int_Y \mu(E^y)d\lambda$. 


10.10. ALTERNATIVE TREATMENT OF PRODUCT MEASURE

Proof: Let

\[ \mathcal{M} = \{ E \in \mathcal{S} \times \mathcal{F} \text{ such that both } a. \text{ and } b. \text{ hold} \} . \]

Since \( \mu \) and \( \lambda \) are both finite, the monotone convergence and dominated convergence theorems imply \( \mathcal{M} \) is a monotone class.

Next I will argue \( \mathcal{M} \) contains the elementary sets. Let

\[ E = \bigcup_{i=1}^{n} A_i \times B_i \]

where the measurable rectangles, \( A_i \times B_i \) are disjoint. Then

\[ \lambda (E_x) = \int_Y \chi_E (x, y) \, d\lambda = \sum_{i=1}^{n} \int_Y \chi_{A_i \times B_i} (x, y) \, d\lambda \]

which is clearly \( \mu \) measurable. Furthermore,

\[ \int_X \lambda (E_x) \, d\mu = \int_X \sum_{i=1}^{n} \chi_{A_i} (x) \lambda (B_i) \, d\mu = \sum_{i=1}^{n} \mu (A_i) \lambda (B_i) . \]

Similarly,

\[ \int_Y \mu (E_y) \, d\lambda = \sum_{i=1}^{n} \mu (A_i) \lambda (B_i) \]

and \( y \to \mu (E_y) \) is \( \lambda \) measurable and this shows \( \mathcal{M} \) contains the algebra of elementary sets. By the monotone class theorem, \( \mathcal{M} = \mathcal{S} \times \mathcal{F} \). This proves the theorem.

One can easily extend this theorem to the case where the measure spaces are \( \sigma \) finite.

Theorem 10.10.11 If \( (X, \mathcal{S}, \mu) \) and \( (Y, \mathcal{F}, \lambda) \) are both \( \sigma \) finite measure spaces, then for every \( E \in \mathcal{S} \times \mathcal{F} \),

a.) \( x \to \lambda (E_x) \) is \( \mu \) measurable, \( y \to \mu (E_y) \) is \( \lambda \) measurable.

b.) \( \int_X \lambda (E_x) \, d\mu = \int_Y \mu (E_y) \, d\lambda . \)

Proof: Let \( X = \bigcup_{n=1}^{\infty} X_n, Y = \bigcup_{n=1}^{\infty} Y_n \) where,

\[ X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1}, \mu (X_n) < \infty, \lambda (Y_n) < \infty. \]

Let

\[ \mathcal{S}_n = \{ A \cap X_n : A \in \mathcal{S} \}, \mathcal{F}_n = \{ B \cap Y_n : B \in \mathcal{F} \} . \]

Thus \( (X_n, \mathcal{S}_n, \mu) \) and \( (Y_n, \mathcal{F}_n, \lambda) \) are both finite measure spaces.

Claim: If \( E \in \mathcal{S} \times \mathcal{F} \), then \( E \cap (X_n \times Y_n) \in \mathcal{S}_n \times \mathcal{F}_n \).

Proof: Let

\[ \mathcal{M}_n = \{ E \in \mathcal{S} \times \mathcal{F} : E \cap (X_n \times Y_n) \in \mathcal{S}_n \times \mathcal{F}_n \} . \]
Clearly $M_n$ contains the algebra of elementary sets. It is also clear that $M_n$ is a monotone class. Thus $M_n = S \times F$.

Now let $E \in S \times F$. By Theorem 10.10.11,

$$\int_{X_n} \lambda((E \cap (X_n \times Y_n))_x) d\mu = \int_{Y_n} \mu((E \cap (X_n \times Y_n))^y) d\lambda \quad (10.10.49)$$

where the integrands are measurable. Also

$$(E \cap (X_n \times Y_n))_x = \emptyset$$

if $x \notin X_n$ and a similar observation holds for the second integrand in $10.10.49$ if $y \notin Y_n$. Therefore,

$$\int_{X} \lambda((E \cap (X_n \times Y_n))_x) d\mu = \int_{X_n} \lambda((E \cap (X_n \times Y_n))_x) d\mu$$

$$= \int_{Y_n} \mu((E \cap (X_n \times Y_n))^y) d\lambda$$

$$= \int_{Y} \mu((E \cap (X_n \times Y_n))^y) d\lambda.$$

Then letting $n \to \infty$, the monotone convergence theorem implies b.) and the measurability assertions of a.) are valid because

$$\lambda(E_x) = \lim_{n \to \infty} \lambda((E \cap (X_n \times Y_n))_x)$$

$$\mu(E^y) = \lim_{n \to \infty} \mu((E \cap (X_n \times Y_n))^y).$$

This proves the theorem.

This theorem makes it possible to define product measure.

**Definition 10.10.12** For $E \in S \times F$ and $(X, S, \mu), (Y, F, \lambda) \sigma$ finite, $(\mu \times \lambda)(E) \equiv \int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.

This definition is well defined because of Theorem 10.10.11.

**Theorem 10.10.13** If $A \in S$, $B \in F$, then $(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$, and $\mu \times \lambda$ is a measure on $S \times F$ called product measure.

**Proof:** The first assertion about the measure of a measurable rectangle was established above. Now suppose $\{E_i\}_{i=1}^\infty$ is a disjoint collection of sets of $S \times F$. Then using the monotone convergence theorem along with the observation that
10.10. ALTERNATIVE TREATMENT OF PRODUCT MEASURE

\((E_i)_x \cap (E_j)_x = \emptyset,\)

\((\mu \times \lambda) (\bigcup_{i=1}^{\infty} E_i) = \int_X \lambda((\bigcup_{i=1}^{\infty} E_i)_x) d\mu\)

\(= \int_X \lambda((\bigcup_{i=1}^{\infty} E_i)_x) d\mu = \int_X \sum_{i=1}^{\infty} \lambda((E_i)_x) d\mu\)

\(= \sum_{i=1}^{\infty} \int_X \lambda((E_i)_x) d\mu\)

\(= \sum_{i=1}^{\infty} (\mu \times \lambda)(E_i)\)

This proves the theorem.

The next theorem is one of several theorems due to Fubini and Tonelli. These theorems all have to do with interchanging the order of integration in a multiple integral.

**Theorem 10.10.14** Let \(f : X \times Y \to [0, \infty]\) be measurable with respect to \(\mathcal{S} \times \mathcal{F}\) and suppose \(\mu\) and \(\lambda\) are \(\sigma\) finite. Then

\[\int_{X \times Y} f d(\mu \times \lambda) = \int_X \int_Y f(x, y) d\lambda d\mu = \int_Y \int_X f(x, y) d\mu d\lambda\]  \hspace{1cm} (10.10.50)

and all integrals make sense.

**Proof:** For \(E \in \mathcal{S} \times \mathcal{F},\)

\[\int_Y \chi_E(x, y) d\lambda = \lambda(E_x), \quad \int_X \chi_E(x, y) d\mu = \mu(E_y).\]

Thus from Definition 10.10.34, 10.10.35 holds if \(f = \chi_E\). It follows that 10.10.34 holds for every nonnegative simple function. By Theorem 9.3.9 on Page 227, there exists an increasing sequence, \(\{f_n\}\), of simple functions converging pointwise to \(f\). Then

\[\int_Y f(x, y) d\lambda = \lim_{n \to \infty} \int_Y f_n(x, y) d\lambda,\]

\[\int_X f(x, y) d\mu = \lim_{n \to \infty} \int_X f_n(x, y) d\mu.\]

This follows from the monotone convergence theorem. Since \(x \to \int_Y f_n(x, y) d\lambda\) is measurable with respect to \(\mathcal{S}\), it follows that \(x \to \int_Y f(x, y) d\lambda\) is also measurable with respect to \(\mathcal{S}\). A similar conclusion can be drawn about \(y \to \int_X f(x, y) d\mu\). Thus the two iterated integrals make sense. Since 10.10.35 holds for \(f_n\), another application of the Monotone Convergence theorem shows 10.10.34 holds for \(f\). This proves the theorem.
Corollary 10.10.15 Let $f : X \times Y \to \mathbb{C}$ be $\mathcal{S} \times \mathcal{F}$ measurable. Suppose either $\int_X \int_Y |f| \, d\mu \, d\lambda < \infty$ or $\int_Y \int_X |f| \, d\mu \, d\lambda < \infty$. Then $f \in L^1(X \times Y, \mu \times \lambda)$ and

$$\int_{X \times Y} f \, d(\mu \times \lambda) = \int_X \int_Y f \, d\mu \, d\lambda = \int_Y \int_X f \, d\mu \, d\lambda \quad (10.10.51)$$

with all integrals making sense.

Proof: Suppose first that $f$ is real valued. Apply Theorem 10.10.14 to $f^+$ and $f^-$. \[ (10.10.51) \] follows from observing that $f = f^+ - f^-$; and that all integrals are finite. If $f$ is complex valued, consider real and imaginary parts. This proves the corollary.

Suppose $f$ is product measurable. From the above discussion, and breaking $f$ down into a sum of positive and negative parts of real and imaginary parts and then using Theorem \[ (9.3.9) \] on approximation by simple functions, it follows that whenever $f$ is $\mathcal{S} \times \mathcal{F}$ measurable, $x \to f(x, y)$ is $\mu$ measurable, $y \to f(x, y)$ is $\lambda$ measurable.

10.11 Completion Of Measures

Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space. Then it is always possible to enlarge the $\sigma$ algebra and define a new measure $\overline{\mu}$ on this larger $\sigma$ algebra such that $(\Omega, \mathcal{F}, \overline{\mu})$ is a complete measure space. Recall this means that if $N \subseteq N' \in \mathcal{F}$ and $\mu(N') = 0$, then $N \in \mathcal{F}$. The following theorem is the main result. The new measure space is called the completion of the measure space.

Theorem 10.11.1 Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$ finite measure space. Then there exists a unique measure space, $(\Omega, \overline{\mathcal{F}}, \overline{\mu})$ satisfying

1. $(\Omega, \mathcal{F}, \overline{\mu})$ is a complete measure space.
2. $\overline{\mu} = \mu$ on $\mathcal{F}$
3. $\mathcal{F} \subseteq \overline{\mathcal{F}}$
4. For every $E \in \overline{\mathcal{F}}$ there exists $G \in \mathcal{F}$ such that $G \supseteq E$ and $\mu(G) = \overline{\mu}(E)$.
5. For every $E \in \overline{\mathcal{F}}$ there exists $F \in \mathcal{F}$ such that $F \subseteq E$ and $\mu(F) = \overline{\mu}(E)$.

Also for every $E \in \overline{\mathcal{F}}$ there exist sets $G, F \in \mathcal{F}$ such that $G \supseteq E \supseteq F$ and

$$\mu(G \setminus F) = \overline{\mu}(G \setminus F) = 0 \quad (10.11.52)$$

Proof: First consider the claim about uniqueness. Suppose $(\Omega, \mathcal{F}_1, \nu_1)$ and $(\Omega, \mathcal{F}_2, \nu_2)$ both work and let $E \in \mathcal{F}_1$. Also let $\mu(\Omega_n) < \infty$, $\cdots \Omega_n \subseteq \Omega_{n+1} \cdots$, and $\cup_{n=1}^{\infty} \Omega_n = \Omega$. Define $E_n \equiv E \cap \Omega_n$. Then pick $G_n \supseteq E_n \supseteq F_n$ such that $\mu(G_n) = \nu_1(E_n \cap \Omega_n) = \nu_2(E_n \cap \Omega_n)$. Then

$$\lim_{n \to \infty} \mu(\Omega_n \setminus E_n) = \lim_{n \to \infty} \nu_1(E_n \setminus \Omega_n) = \lim_{n \to \infty} \nu_2(E_n \setminus \Omega_n) = 0$$

By Theorem 10.11.1 we may assume $(\Omega, \mathcal{F}, \mu)$ is a complete measure space.
10.11. COMPLETION OF MEASURES

\( \mu (F_n) = \nu_1 (E_n) \). It follows \( \mu (G_n \setminus F_n) = 0 \). Then letting \( G = \cup_n G_n, F = \cup_n F_n \), it follows \( G \supseteq E \supseteq F \) and

\[
\mu (G \setminus F) \leq \mu (\cup_n (G_n \setminus F_n)) \\
\leq \sum_n \mu (G_n \setminus F_n) = 0.
\]

It follows that \( \nu_2 (G \setminus F) = 0 \) also. Now \( E \setminus F \subseteq G \setminus F \) and since \( (\Omega, \mathcal{F}_2, \nu_2) \) is complete, it follows \( E \setminus F \in \mathcal{F}_2 \). Since \( F \in \mathcal{F}_2 \), it follows \( E \setminus F \subseteq G \setminus F \) and since \( (\Omega, \mathcal{F}, \nu) \) is complete, it follows \( E \setminus F \in \mathcal{F} \).

Thus \( \nu_1 (E \setminus F) = \nu_2 (E \setminus F) \). Now it only remains to verify \( \nu_1 = \nu_2 \). Thus let \( E \setminus F \subseteq G \setminus F \) and let \( G \) and \( F \) be as just described. Since \( \nu_i = \mu \) on \( F \),

\[
\mu (F) \leq \nu_1 (E) \leq \nu_1 (G \setminus F) + \nu_1 (F) = \nu_1 (E \setminus F) + \nu_1 (F).
\]

Similarly \( \nu_2 (E) = \mu (F) \). This proves uniqueness. The construction has also verified \( 10.11.52 \).

Next define an outer measure, \( \overline{\mu} \) on \( \mathcal{P} (\Omega) \) as follows. For \( S \subseteq \Omega \),

\[ \overline{\mu} (S) = \inf \{ \mu (E) : E \in \mathcal{F} \} \]

Then it is clear \( \overline{\mu} \) is increasing. It only remains to verify \( \overline{\mu} \) is subadditive. Then let \( S = \cup_{i=1}^{\infty} S_i \). If any \( \overline{\mu} (S_i) = \infty \), there is nothing to prove so suppose \( \overline{\mu} (S_i) < \infty \) for each \( i \). Then there exist \( E_i \in \mathcal{F} \) such that \( E_i \supseteq S_i \) and

\[ \overline{\mu} (S_i) + \varepsilon / 2^i > \mu (E_i) \]

Then

\[
\overline{\mu} (S) = \overline{\mu} (\cup_i S_i) \\
\leq \mu (\cup_i E_i) \leq \sum_i \mu (E_i) \\
\leq \sum_i (\overline{\mu} (S_i) + \varepsilon / 2^i) = \sum_i \overline{\mu} (S_i) + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this verifies \( \overline{\mu} \) is subadditive and is an outer measure as claimed.

Denote by \( \mathcal{F} \) the \( \sigma \) algebra of measurable sets in the sense of Caratheodory. Then it follows from the Caratheodory procedure, Theorem \( 10.1.4 \) on Page 250, that \( (\Omega, \mathcal{F}, \overline{\mu}) \) is a complete measure space. This verifies \( 11 \).

Now let \( E \in \mathcal{F} \). Then from the definition of \( \overline{\mu} \), it follows

\[
\overline{\mu} (E) = \inf \{ \mu (F) : F \in \mathcal{F} \text{ and } F \supseteq E \} \leq \mu (E).
\]

If \( F \supseteq E \) and \( F \in \mathcal{F} \), then \( \mu (F) \geq \mu (E) \) and so \( \mu (E) \) is a lower bound for all such \( \mu (F) \) which shows that

\[
\overline{\mu} (E) = \inf \{ \mu (F) : F \in \mathcal{F} \text{ and } F \supseteq E \} \geq \mu (E).
\]
This verifies 3.

Next consider 3. Let \( E \in \mathcal{F} \) and let \( S \) be a set. I must show

\[
\mu(S) \geq \mu(S \setminus E) + \mu(S \cap E).
\]

If \( \mu(S) = \infty \) there is nothing to show. Therefore, suppose \( \mu(S) < \infty \). Then from the definition of \( \mu \),

\[
\mu(S) \leq \mu(S \setminus E) + \mu(S \cap E) \leq \mu(G \setminus E) + \mu(G \cap E) = \mu(G) = \mu(S).
\]

This verifies 3.

Claim 4 comes by the definition of \( \mu \) as used above. The only other case is when \( \mu(S) = \infty \). However, in this case, you can let \( G = \Omega \).

It only remains to verify 5. Let the \( \Omega_n \) be as described above and let \( E \in \mathcal{F} \) such that \( E \subseteq \Omega_n \). By 4 there exists \( H \in \mathcal{F} \) such that \( H \subseteq \Omega_n, H \supseteq \Omega_n \setminus E \), and

\[
\mu(H) = \mu(\Omega_n \setminus E). \quad (10.11.53)
\]

Then let \( F = \Omega_n \cap H^C \). It follows \( F \subseteq E \) and

\[
E \setminus F = E \cap F^C = E \cap (H \cup \Omega_n^C) = E \cap H = H \setminus (\Omega_n \setminus E)
\]

Hence from 10.11.53

\[
\mu(E \setminus F) = \mu(H \setminus (\Omega_n \setminus E)) = 0.
\]

It follows

\[
\mu(E) = \mu(F) = \mu(F).
\]

In the case where \( E \in \mathcal{F} \) is arbitrary, not necessarily contained in some \( \Omega_n \), it follows from what was just shown that there exists \( F_n \in \mathcal{F} \) such that \( F_n \subseteq E \cap \Omega_n \) and

\[
\mu(F_n) = \mu(E \cap \Omega_n).
\]

Letting \( F = \cup_n F_n \)

\[
\mu(E \setminus F) = \mu(\cup_n (E \cap \Omega_n \setminus F_n)) \leq \sum_n \mu(E \cap \Omega_n \setminus F_n) = 0.
\]

Therefore, \( \mu(E) = \mu(F) \) and this proves 5. This proves the theorem.

Now here is an interesting theorem about complete measure spaces.
**Theorem 10.11.2** Let \((\Omega, \mathcal{F}, \mu)\) be a complete measure space and let \(f \leq g \leq h\) be functions having values in \([0, \infty]\). Suppose also that \(f(\omega) = h(\omega)\) a.e. \(\omega\) and that \(f\) and \(h\) are measurable. Then \(g\) is also measurable. If \((\Omega, \mathcal{F}, \mu)\) is the completion of a \(\sigma\) finite measure space \((\Omega, \mathcal{F}, \mu)\) as described above in Theorem 10.11.1 then if \(f\) is measurable with respect to \(\mathcal{F}\) having values in \([0, \infty]\), it follows there exists \(g\) measurable with respect to \(\mathcal{F}\), \(g \leq f\), and a set \(N \in \mathcal{F}\) with \(\mu(N) = 0\) and \(g = f\) on \(N^C\). There also exists \(h\) measurable with respect to \(\mathcal{F}\) such that \(h \geq f\), and a set of measure zero, \(M \in \mathcal{F}\) such that \(f = h\) on \(M^C\).

**Proof:** Let \(\alpha \in \mathbb{R}\).

\[
[f > \alpha] \subseteq [g > \alpha] \subseteq [h > \alpha]
\]

Thus

\[
[g > \alpha] = [f > \alpha] \cup ([g > \alpha] \setminus [f > \alpha])
\]

and \([g > \alpha] \setminus [f > \alpha]\) is a measurable set because it is a subset of the set of measure zero,

\[
[h > \alpha] \setminus [f > \alpha].
\]

Now consider the last assertion. By Theorem 10.11.1 on Page 321 there exists an increasing sequence of nonnegative simple functions, \(\{s_n\}\) measurable with respect to \(\mathcal{F}\) which converges pointwise to \(f\). Letting

\[
s_n(\omega) = \sum_{k=1}^{m_n} c_{nk} \mathcal{X}_{E_{nk}}(\omega) \tag{10.11.54}
\]

be one of these simple functions, it follows from Theorem 10.11.1 there exist sets, \(F^n_k \in \mathcal{F}\) such that \(F^n_k \subseteq E_{nk}\) and \(\mu(F^n_k) = \mu(E_{nk})\). Then let

\[
t_n(\omega) = \sum_{k=1}^{m_n} c_{nk} \mathcal{X}_{F^n_k}(\omega).
\]

Thus \(t_n = s_n\) off a set of measure zero, \(N_n \in \mathcal{F}, t_n \leq s_n\). Let \(N' = \cap_n N_n\). Then by Theorem 10.11.1 again, there exists \(N \in \mathcal{F}\) such that \(N \supseteq N'\) and \(\mu(N) = 0\). Consider the simple functions,

\[
s'_n(\omega) = t_n(\omega) \mathcal{X}_{N^C}(\omega).
\]

It is an increasing sequence so let \(g(\omega) = \lim_{n \to \infty} s'_n(\omega)\). It follows \(g\) is measurable with respect to \(\mathcal{F}\) and equals \(f\) off \(N\).

Finally, to obtain the function, \(h \geq f\), in 10.11 use Theorem 10.11 to obtain the existence of \(F^n_k \in \mathcal{F}\) such that \(F^n_k \supseteq E_{nk}\) and \(\mu(F^n_k) = \mu(E_{nk})\). Then let

\[
t_n(\omega) = \sum_{k=1}^{m_n} c_{nk} \mathcal{X}_{F^n_k}(\omega).
\]
Thus \( t_n = s_n \) off a set of measure zero, \( M_n \in \mathcal{F}, \) \( t_n \geq s_n, \) and \( t_n \) is measurable with respect to \( \mathcal{F}. \) Then define

\[
  s'_n = \max_{k \leq n} t_n.
\]

It follows \( s'_n \) is an increasing sequence of \( \mathcal{F} \) measurable nonnegative simple functions. Since each \( s'_n \geq s_n, \) it follows that if \( h(\omega) = \lim_{n \to \infty} s'_n(\omega), \) then \( h(\omega) \geq f(\omega). \) Also if \( h(\omega) > f(\omega), \) then \( \omega \in \cup_n M_n \equiv M', \) a set of \( \mathcal{F} \) having measure zero. By Theorem 10.11.1, there exists \( M \supseteq M' \) such that \( M \in \mathcal{F} \) and \( \mu(M) = 0. \) It follows \( h = f \) off \( M. \) This proves the theorem.

10.12 Another Version Of Product Measures

10.12.1 General Theory

Given two finite measure spaces, \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{S}, \nu)\), there is a way to define a \( \sigma \) algebra of subsets of \( X \times Y, \) denoted by \( \mathcal{F} \times \mathcal{S} \) and a measure, denoted by \( \mu \times \nu \) defined on this \( \sigma \) algebra such that

\[
  \mu \times \nu(A \times B) = \mu(A) \nu(B)
\]

whenever \( A \in \mathcal{F} \) and \( B \in \mathcal{S}. \) This is naturally related to the concept of iterated integrals similar to what is used in calculus to evaluate a multiple integral. The approach is based on something called a \( \pi \) system, [34].

**Definition 10.12.1** Let \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{S}, \nu)\) be two measure spaces. A measurable rectangle is a set of the form \( A \times B \) where \( A \in \mathcal{F} \) and \( B \in \mathcal{S}. \)

**Definition 10.12.2** Let \( \Omega \) be a set and let \( \mathcal{K} \) be a collection of subsets of \( \Omega. \) Then \( \mathcal{K} \) is called a \( \pi \) system if \( \emptyset, \Omega \in \mathcal{K} \) and whenever \( A, B \in \mathcal{K}, \) it follows \( A \cap B \in \mathcal{K}. \)

Obviously an example of a \( \pi \) system is the set of measurable rectangles because

\[
  A \times B \cap A' \times B' = (A \cap A') \times (B \cap B').
\]

The following is the fundamental lemma which shows these \( \pi \) systems are useful. This lemma is due to Dynkin.

**Lemma 10.12.3** Let \( \mathcal{K} \) be a \( \pi \) system of subsets of \( \Omega, \) a set. Also let \( \mathcal{G} \) be a collection of subsets of \( \Omega \) which satisfies the following three properties.

1. \( \mathcal{K} \subseteq \mathcal{G} \)
2. If \( A \in \mathcal{G}, \) then \( A^C \in \mathcal{G} \)
3. If \( \{A_i\}_{i=1}^\infty \) is a sequence of disjoint sets from \( \mathcal{G} \) then \( \cup_{i=1}^\infty A_i \in \mathcal{G}. \)

Then \( \mathcal{G} \supseteq \sigma(\mathcal{K}), \) where \( \sigma(\mathcal{K}) \) is the smallest \( \sigma \) algebra which contains \( \mathcal{K}. \)
10.12. ANOTHER VERSION OF PRODUCT MEASURES

Proof: First note that if

\[ \mathcal{H} \equiv \{ \mathcal{G} : \text{all hold} \} \]

then \( \cap \mathcal{H} \) yields a collection of sets which also satisfies \( \text{H - K} \). Therefore, I will assume in the argument that \( \mathcal{G} \) is the smallest collection satisfying \( \text{H - K} \). Let \( A \in \mathcal{K} \) and define

\[ \mathcal{G}_A \equiv \{ B \in \mathcal{G} : A \cap B \in \mathcal{G} \} . \]

I want to show \( \mathcal{G}_A \) satisfies \( \text{H - K} \) because then it must equal \( \mathcal{G} \) since \( \mathcal{G} \) is the smallest collection of subsets of \( \Omega \) which satisfies \( \text{H - K} \). This will give the conclusion that for \( A \in \mathcal{K} \) and \( B \in \mathcal{G} \), \( A \cap B \in \mathcal{G} \). From this it will follow very easily that \( \mathcal{G} \) is a \( \sigma \) algebra which will imply it contains \( \sigma (\mathcal{K}) \). Now here are the details of the argument.

Since \( \mathcal{K} \) is given to be a \( \pi \) system, \( \mathcal{K} \subseteq \mathcal{G}_A \). Property \( \mathcal{K} \) is obvious because if \( \{ B_i \} \) is a sequence of disjoint sets in \( \mathcal{G}_A \), then

\[ A \cap \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A \cap B_i \in \mathcal{G} \]

because \( A \cap B_i \in \mathcal{G} \) and the property \( \mathcal{K} \) of \( \mathcal{G} \).

It remains to verify Property \( \mathcal{G} \) so let \( B \in \mathcal{G}_A \). I need to verify that \( B^C \in \mathcal{G}_A \). In other words, I need to show that \( A \cap B^C \in \mathcal{G} \). However,

\[ A \cap B^C = (A^C \cup (A \cap B))^C \in \mathcal{G} \]

Here is why. Since \( B \in \mathcal{G}_A \), \( A \cap B \in \mathcal{G} \) and since \( A \in \mathcal{K} \subseteq \mathcal{G} \) it follows \( A^C \in \mathcal{G} \) by assumption \( \mathcal{G} \). It follows from assumption \( \mathcal{K} \) the union of the disjoint sets, \( A^C \) and \( (A \cap B) \) is in \( \mathcal{G} \) and then from \( \mathcal{G} \) the complement of their union is in \( \mathcal{G} \). Thus \( \mathcal{G}_A \) satisfies \( \text{H - K} \) and this implies since \( \mathcal{G} \) is the smallest such, that \( \mathcal{G}_A \supseteq \mathcal{G} \). However, \( \mathcal{G}_A \) is constructed as a subset of \( \mathcal{G} \). This proves that for every \( B \in \mathcal{G} \) and \( A \in \mathcal{K} \), \( A \cap B \in \mathcal{G} \). Now pick \( B \in \mathcal{G} \) and consider

\[ \mathcal{G}_B \equiv \{ A \in \mathcal{G} : A \cap B \in \mathcal{G} \} . \]

I just proved \( \mathcal{K} \subseteq \mathcal{G}_B \). The other arguments are identical to show \( \mathcal{G}_B \) satisfies \( \text{H - K} \) and is therefore equal to \( \mathcal{G} \). This shows that whenever \( A, B \in \mathcal{G} \) it follows \( A \cap B \in \mathcal{G} \).

This implies \( \mathcal{G} \) is a \( \sigma \) algebra. To show this, all that is left is to verify \( \mathcal{G} \) is closed under countable unions because then it follows \( \mathcal{G} \) is a \( \sigma \) algebra. Let \( \{ A_i \} \subseteq \mathcal{G} \). Then let \( A_{n+1} = A_1 \) and

\[ A_{n+1}' = A_{n+1} \setminus \bigcup_{i=1}^{n} A_i \]

because finite intersections of sets of \( \mathcal{G} \) are in \( \mathcal{G} \). Since the \( A_i \) are disjoint, it follows

\[ \cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} A_i' \in \mathcal{G} \]
Therefore, $\mathcal{G} \supset \sigma(\mathcal{K})$ and this proves the Lemma. 

With this lemma, it is easy to define product measure.

Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two finite measure spaces. Define $\mathcal{K}$ to be the set of measurable rectangles, $A \times B$, $A \in \mathcal{F}$ and $B \in \mathcal{S}$. Let

$$\mathcal{G} \equiv \left\{ E \subseteq X \times Y : \int_Y \int_X \chi_E d\mu d\nu = \int_X \int_Y \chi_E d\nu d\mu \right\}$$

(10.12.55)

where in the above, part of the requirement is for all integrals to make sense.

Then $\mathcal{K} \subseteq \mathcal{G}$. This is obvious.

Next I want to show that if $E \in \mathcal{G}$ then $E^C \in \mathcal{G}$. Observe $\chi_{E^C} = 1 - \chi_E$ and so

$$\int_Y \int_X \chi_{E^C} d\mu d\nu = \int_Y \int_X (1 - \chi_E) d\mu d\nu = \int_X \int_Y (1 - \chi_E) d\nu d\mu = \int_X \int_Y \chi_{E^C} d\nu d\mu$$

which shows that if $E \in \mathcal{G}$, then $E^C \in \mathcal{G}$.

Next I want to show $\mathcal{G}$ is closed under countable unions of disjoint sets of $\mathcal{G}$. Let $\{A_i\}$ be a sequence of disjoint sets from $\mathcal{G}$. Then

$$\int_Y \int_X \chi_{\bigcup_{i=1}\infty A_i} d\mu d\nu = \int_Y \int_X \sum_{i=1}\infty \chi_{A_i} d\mu d\nu = \int_Y \sum_{i=1}\infty \int_X \chi_{A_i} d\mu d\nu = \sum_{i=1}\infty \int_Y \int_X \chi_{A_i} d\mu d\nu = \int_X \sum_{i=1}\infty \int_Y \chi_{A_i} d\nu d\mu = \int_X \int_Y \sum_{i=1}\infty \chi_{A_i} d\nu d\mu = \int_X \int_Y \chi_{\bigcup_{i=1}\infty A_i} d\nu d\mu,$$

(10.12.56)

the interchanges between the summation and the integral depending on the monotone convergence theorem. Thus $\mathcal{G}$ is closed with respect to countable disjoint unions.
From Lemma 10.12.3, $G \supseteq \sigma(K)$. Also the computation in 10.12.5 implies that on $\sigma(K)$ one can define a measure, denoted by $\mu \times \nu$ and that for every $E \in \sigma(K)$,

$$(\mu \times \nu)(E) = \int_Y \int_X X_E d\mu d\nu = \int_X \int_Y X_E d\nu d\mu. \quad (10.12.57)$$

Now here is Fubini's theorem.

**Theorem 10.12.4** Let $f : X \times Y \rightarrow [0, \infty]$ be measurable with respect to the $\sigma$ algebra, $\sigma(K)$ just defined and let $\mu \times \nu$ be the product measure of 10.12.5 where $\mu$ and $\nu$ are finite measures on $(X, F)$ and $(Y, S)$ respectively. Then

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f d\nu d\mu = \int_X \int_Y f d\nu d\mu.$$ 

**Proof:** Let $\{s_n\}$ be an increasing sequence of $\sigma(K)$ measurable simple functions which converges pointwise to $f$. The above equation holds for $s_n$ in place of $f$ from what was shown above. The final result follows from passing to the limit and using the monotone convergence theorem. $\Box$

The symbol, $F \times S$ denotes $\sigma(K)$.

Of course one can generalize right away to measures which are only $\sigma$ finite.

**Theorem 10.12.5** Let $f : X \times Y \rightarrow [0, \infty]$ be measurable with respect to the $\sigma$ algebra, $\sigma(K)$ just defined and let $\mu \times \nu$ be the product measure of 10.12.5 where $\mu$ and $\nu$ are $\sigma$ finite measures on $(X, F)$ and $(Y, S)$ respectively. Then

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f d\nu d\mu = \int_X \int_Y f d\nu d\mu.$$ 

**Proof:** Since the measures are $\sigma$ finite, there exist increasing sequences of sets, $\{X_n\}$ and $\{Y_n\}$ such that $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. Then $\mu$ and $\nu$ restricted to $X_n$ and $Y_n$ respectively are finite. Then from Theorem 10.12.4,

$$\int_{Y_n} \int_{X_n} f d\nu d\mu = \int_{X_n} \int_{Y_n} f d\nu d\mu$$

Passing to the limit yields

$$\int_Y \int_X f d\nu d\mu = \int_X \int_Y f d\nu d\mu$$

whenever $f$ is as above. In particular, you could take $f = \chi_E$ where $E \in F \times S$ and define

$$(\mu \times \nu)(E) \equiv \int_Y \int_X \chi_E d\mu d\nu = \int_X \int_Y \chi_E d\nu d\mu.$$ 

Then just as in the proof of Theorem 10.12.4, the conclusion of this theorem is obtained. This proves the theorem.

It is also useful to note that all the above holds for $\prod_{i=1}^n X_i$ in place of $X \times Y$. You would simply modify the definition of $G$ in 10.12.5 including all permutations for the iterated integrals and for $K$ you would use sets of the form $\prod_{i=1}^n A_i$ where $A_i$ is measurable. Everything goes through exactly as above. Thus the following is obtained.
CHAPTER 10. THE CONSTRUCTION OF MEASURES

Theorem 10.12.6 Let \( \{(X_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n \) be \( \sigma \) finite measure spaces and let \( \prod_{i=1}^n \mathcal{F}_i \) denote the smallest \( \sigma \) algebra which contains the measurable boxes of the form \( \prod_{i=1}^n A_i \), where \( A_i \in \mathcal{F}_i \). Then there exists a measure, \( \lambda \) defined on \( \prod_{i=1}^n \mathcal{F}_i \) such that if \( f: \prod_{i=1}^n X_i \to [0, \infty] \) is \( \prod_{i=1}^n \mathcal{F}_i \) measurable, and \((i_1, \ldots, i_n)\) is any permutation of \((1, \ldots, n)\), then

\[
\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}
\]

10.12.2 Completion Of Product Measure Spaces

Using Theorem 10.12.6 it is easy to give a generalization to yield a theorem for the completion of product spaces.

Theorem 10.12.7 Let \( \{(X_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n \) be \( \sigma \) finite measure spaces and let \( \prod_{i=1}^n \mathcal{F}_i \) denote the smallest \( \sigma \) algebra which contains the measurable boxes of the form \( \prod_{i=1}^n A_i \), where \( A_i \in \mathcal{F}_i \). Then there exists a measure, \( \lambda \) defined on \( \prod_{i=1}^n \mathcal{F}_i \) such that if \( f: \prod_{i=1}^n X_i \to [0, \infty] \) is \( \prod_{i=1}^n \mathcal{F}_i \) measurable, and \((i_1, \ldots, i_n)\) is any permutation of \((1, \ldots, n)\), then

\[
\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}
\]

Let \( \left( \prod_{i=1}^n X_i, \prod_{i=1}^n \mathcal{F}_i, \lambda \right) \) denote the completion of this product measure space and let

\[
f: \prod_{i=1}^n X_i \to [0, \infty]
\]

be \( \prod_{i=1}^n \mathcal{F}_i \) measurable. Then there exists \( N \in \prod_{i=1}^n \mathcal{F}_i \) such that \( \lambda(N) = 0 \) and a nonnegative function, \( f_1 \) measurable with respect to \( \prod_{i=1}^n \mathcal{F}_i \) such that \( f_1 = f \) off \( N \) and if \((i_1, \ldots, i_n)\) is any permutation of \((1, \ldots, n)\), then

\[
\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f_1 d\mu_{i_1} \cdots d\mu_{i_n}.
\]

Furthermore, \( f_1 \) may be chosen to satisfy either \( f_1 \leq f \) or \( f_1 \geq f \).

**Proof:** This follows immediately from Theorem 10.12.4 and Theorem 10.12.6. By the second theorem, there exists a function \( f_1 \geq f \) such that \( f_1 = f \) for all \((x_1, \ldots, x_n) \notin N\), a set of \( \prod_{i=1}^n \mathcal{F}_i \) having measure zero. Then by Theorem 10.12.4 and Theorem 10.12.6

\[
\int f d\lambda = \int f_1 d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f_1 d\mu_{i_1} \cdots d\mu_{i_n}.
\]

Since \( f_1 = f \) off a set of measure zero, I will dispense with the subscript. Also it is customary to write

\[
\lambda = \mu_1 \times \cdots \times \mu_n
\]
10.12. ANOTHER VERSION OF PRODUCT MEASURES

and

\[ \mathcal{X} = \mu_1 \times \cdots \times \mu_n. \]

Thus in more standard notation, one writes

\[ \int f \, d(\mu_1 \times \cdots \times \mu_n) = \int_{X_1} \cdots \int_{X_n} f \, d\mu_1 \cdots d\mu_n. \]

This theorem is often referred to as Fubini’s theorem. The next theorem is also called this.

**Corollary 10.12.8** Suppose \( f \in L^1 \left( \prod_{i=1}^n X_i, \prod_{i=1}^n \mathcal{F}_i, \mu_1 \times \cdots \times \mu_n \right) \) where each \( X_i \) is a \( \sigma \)-finite measure space. Then if \( (i_1, \cdots, i_n) \) is any permutation of \( (1, \cdots, n) \), it follows

\[ \int f \, d(\mu_1 \times \cdots \times \mu_n) = \int_{X_1} \cdots \int_{X_n} f \, d\mu_{i_1} \cdots d\mu_{i_n}. \]

**Proof:** Just apply Theorem 10.12.7 to the positive and negative parts of the real and imaginary parts of \( f \). This proves the theorem.

Here is another easy corollary.

**Corollary 10.12.9** Suppose in the situation of Corollary 10.12.8, \( f = f_1 \) off \( N \), a set of \( \prod_{i=1}^n \mathcal{F}_i \) having \( \mu_1 \times \cdots \times \mu_n \) measure zero and that \( f_1 \) is a complex valued function measurable with respect to \( \prod_{i=1}^n \mathcal{F}_i \). Suppose also that for some permutation of \( (1, 2, \cdots, n), (j_1, \cdots, j_n) \)

\[ \int_{X_{j_1}} \cdots \int_{X_{j_n}} |f_1| \, d\mu_{j_1} \cdots d\mu_{j_n} < \infty. \]

Then

\[ f \in L^1 \left( \prod_{i=1}^n X_i, \prod_{i=1}^n \mathcal{F}_i, \mu_1 \times \cdots \times \mu_n \right) \]

and the conclusion of Corollary 10.12.8 holds.

**Proof:** Since \( |f_1| \) is \( \prod_{i=1}^n \mathcal{F}_i \) measurable, it follows from Theorem 10.12.7 that

\[ \infty > \int_{X_{j_1}} \cdots \int_{X_{j_n}} |f_1| \, d\mu_{j_1} \cdots d\mu_{j_n} \]

\[ = \int |f_1| \, d(\mu_1 \times \cdots \times \mu_n) \]

\[ = \int f_1^* \, d(\mu_1 \times \cdots \times \mu_n) \]

\[ = \int f \, d(\mu_1 \times \cdots \times \mu_n). \]

Thus \( f \in L^1 \left( \prod_{i=1}^n X_i, \prod_{i=1}^n \mathcal{F}_i, \mu_1 \times \cdots \times \mu_n \right) \) as claimed and the rest follows from Corollary 10.12.8. This proves the corollary.

The following lemma is also useful.
Lemma 10.12.10  Let \((X, \mathcal{F}, \mu)\) and \((Y, \mathcal{S}, \nu)\) be \(\sigma\) finite complete measure spaces and suppose \(f \geq 0\) is \(\mathcal{F} \times \mathcal{S}\) measurable. Then for a.e. \(x,\)
\[
y \to f(x, y)
\]
is \(\mathcal{S}\) measurable. Similarly for a.e. \(y,\)
\[
x \to f(x, y)
\]
is \(\mathcal{F}\) measurable.

Proof: By Theorem\[10.11.2\], there exist \(\mathcal{F} \times \mathcal{S}\) measurable functions, \(g\) and \(h\) and a set, \(N \in \mathcal{F} \times \mathcal{S}\) of \(\mu \times \lambda\) measure zero such that \(g \leq f \leq h\) and for \((x, y) \notin N\), it follows that \(g(x, y) = h(x, y)\). Then
\[
\int_X \int_Y g d\nu d\mu = \int_X \int_Y h d\nu d\mu
\]
and so for a.e. \(x,\)
\[
\int_Y g d\nu = \int_Y h d\nu.
\]
Then it follows that for these values of \(x, g(x, y) = h(x, y)\) and so by Theorem\[10.11.2\] again and the assumption that \((Y, \mathcal{S}, \nu)\) is complete, \(y \to f(x, y)\) is \(\mathcal{S}\) measurable. The other claim is similar. This proves the lemma.

10.13 Disturbing Examples

There are examples which help to define what can be expected of product measures and Fubini type theorems. Three such examples are given in Rudin\[102\]. Some of the theorems given above are more general than those in this reference but the same examples are still useful for showing that the hypotheses of the above theorems are all necessary.

Example 10.13.1  Let \(\{a_n\}\) be an increasing sequence of numbers in \((0, 1)\) which converges to 1. Let \(g_n \in C_c(a_n, a_{n+1})\) such that \(\int g_n dx = 1\). Now for \((x, y) \in [0, 1) \times [0, 1)\) define
\[
f(x, y) = \sum_{k=1}^{\infty} g_n(y) \left( g_n(x) - g_{n+1}(x) \right).
\]
Note this is actually a finite sum for each such \((x, y)\). Therefore, this is a continuous function on \([0, 1) \times [0, 1)\). Now for a fixed \(y,\)
\[
\int_0^1 f(x, y) dx = \sum_{k=1}^{\infty} g_n(y) \int_0^1 \left( g_n(x) - g_{n+1}(x) \right) dx = 0
\]
10.13. DISTURBING EXAMPLES

showing that \( \int_{0}^{1} \int_{0}^{1} f(x, y) \, dy \, dx = \int_{0}^{1} 0 \, dy = 0 \). Next fix \( x \).

\[ \int_{0}^{1} f(x, y) \, dy = \sum_{k=1}^{\infty} (g_n(x) - g_{n+1}(x)) \int_{0}^{1} g_n(y) \, dy = g_1(x). \]

Hence \( \int_{0}^{1} \int_{0}^{1} f(x, y) \, dy \, dx = \int_{0}^{1} g_1(x) \, dx = 1 \). The iterated integrals are not equal.
Note the function, \( g \), is not nonnegative even though it is measurable. In addition, neither \( \int_{0}^{1} \int_{0}^{1} |f(x, y)| \, dy \, dx \) nor \( \int_{0}^{1} \int_{0}^{1} |f(x, y)| \, dy \, dx \) is finite and so you can’t apply Corollary 10.13.14. The problem here is the function is not nonnegative and is not absolutely integrable.

Example 10.13.2 This time let \( \mu = m \), Lebesgue measure on \([0, 1]\) and let \( \nu \) be counting measure on \([0, 1]\), in this case, the \( \sigma \)-algebra is \( P([0, 1]) \). Let \( l \) denote the line segment in \([0, 1] \times [0, 1]\) which goes from \((0, 0)\) to \((1, 1)\). Thus \( l = (x, x) \) where \( x \in [0, 1] \). Consider the outer measure of \( l \) in \( m \times \nu \). Let \( l \subseteq \bigcup_k A_k \times B_k \) where \( A_k \)

is Lebesgue measurable and \( B_k \) is a subset of \([0, 1]\). Let \( B = \{ k \in \mathbb{N} : \nu(B_k) = \infty \} \). If \( m(\bigcup_{k \in B} A_k) \) has measure zero, then there are uncountably many points of \([0, 1]\) outside of \( \bigcup_{k \in B} A_k \). For \( p \) one of these points, \( (p, p) \in A_i \times B_i \) and \( i \notin B \). Thus each of these points is in \( \bigcup_{i \notin B} B_i \), a countable set because these \( B_i \) are each finite. But this is a contradiction because there need to be uncountably many of these points as just indicated. Thus \( m(A_k) > 0 \) for some \( k \in B \) and so \( m \times \nu(A_k \times B_k) = \infty \).

It follows \( m \times \nu(l) = \infty \) and so \( l \) is \( m \times \nu \) measurable. Thus \( \int X_i(x, y) \, dm \times \nu = \infty \) and so you cannot apply Fubini’s theorem, Theorem 10.13.14. Since \( \nu \) is not \( \sigma \) finite, you cannot apply the corollary to this theorem either. Thus there is no contradiction to the above theorem in the following observation.

\[ \int \int X_i(x, y) \, d\nu \, dm = \int 1 \, dm = 1, \quad \int \int X_i(x, y) \, dm \, d\nu = \int 0 \, d\nu = 0. \]

The problem here is that you have neither \( \int f \, dm \times \nu < \infty \) not \( \sigma \) finite measure spaces.

The next example is far more exotic. It concerns the case where both iterated integrals make perfect sense but are unequal. In 1877 Cantor conjectured that the cardinality of the real numbers is the next size of infinity after countable infinity. This hypothesis is called the continuum hypothesis and it has never been proved or disproved. Assuming this continuum hypothesis will provide the basis for the following example. It is due to Sierpinski.

Example 10.13.3 Let \( X \) be an uncountable set. It follows from the well ordering theorem which says every set can be well ordered which is presented in the appendix that \( X \) can be well ordered. Let \( \omega \in X \) be the first element of \( X \) which is

\[ \text{In 1940 it was shown by Godel that the continuum hypothesis cannot be disproved. In 1963 it was shown by Cohen that the continuum hypothesis cannot be proved. These assertions are based on the axiom of choice and the Zermelo Frankel axioms of set theory. This topic is far outside the scope of this book and this is only a hopefully interesting historical observation.} \]
preceded by uncountably many points of \( X \). Let \( \Omega \) denote \( \{ x \in X : x < \omega \} \). Then \( \Omega \) is uncountable but there is no smaller uncountable set. Thus by the continuum hypothesis, there exists a one to one and onto mapping, \( j \) which maps \([0,1]\) onto \( \Omega \). Thus, for \( x \in [0,1] \), \( j(x) \) is preceded by countably many points. Let \( Q \equiv \{ (x,y) \in [0,1]^2 : j(x) < j(y) \} \) and let \( f(x,y) = \chi_Q(x,y) \). Then

\[
\int_0^1 f(x,y) dy = 1, \quad \int_0^1 f(x,y) dx = 0
\]

In each case, the integrals make sense. In the first, for fixed \( x \), \( f(x,y) = 1 \) for all but countably many \( y \), so the function of \( y \) is Borel measurable. In the second where \( y \) is fixed, \( f(x,y) = 0 \) for all but countably many \( x \). Thus

\[
\int_0^1 \int_0^1 f(x,y) dy dx = 1, \quad \int_0^1 \int_0^1 f(x,y) dx dy = 0.
\]

The problem here must be that \( f \) is not \( m \times m \) measurable.

### 10.14 Exercises

1. Let \( \Omega = \mathbb{N} \), the natural numbers and let \( d(p,q) = |p - q| \), the usual distance in \( \mathbb{R} \). Show that \((\Omega,d)\) the closures of the balls are compact. Now let

\[
\Lambda f = \sum_{k=1}^\infty f(k)
\]

whenever \( f \in C_c(\Omega) \). Show this is a well defined positive linear functional on the space \( C_c(\Omega) \). Describe the measure of the Riesz representation theorem which results from this positive linear functional. What if \( \Lambda(f) = f(1) \)? What measure would result from this functional? Which functions are measurable?

2. Verify that \( \mu \) defined in Lemma 10.13 is an outer measure.

3. Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be increasing and right continuous. Let \( \Lambda f = \int f dF \) where the integral is the Riemann Stieltjes integral of \( f \). Show the measure \( \mu \) from the Riesz representation theorem satisfies

\[
\mu([a,b]) = F(b) - F(a-), \quad \mu((a,b]) = F(b) - F(a),
\]

\[
\mu([a,a]) = F(a) - F(a-).
\]

4. Let \( \Omega \) be a metric space with the closed balls compact and suppose \( \mu \) is a measure defined on the Borel sets of \( \Omega \) which is finite on compact sets. Show there exists a unique Radon measure, \( \overline{\mu} \) which equals \( \mu \) on the Borel sets.

5. Random vectors are measurable functions, \( X \), mapping a probability space, \((\Omega,P,\mathcal{F})\) to \( \mathbb{R}^n \). Thus \( X(\omega) \in \mathbb{R}^n \) for each \( \omega \in \Omega \) and \( P \) is a probability measure defined on the sets of \( \mathcal{F} \), a \( \sigma \) algebra of subsets of \( \Omega \). For \( E \) a Borel set in \( \mathbb{R}^n \), define

\[
\mu(E) = P(X^{-1}(E)) \equiv \text{probability that } X \in E.
\]
Show this is a well defined measure on the Borel sets of $\mathbb{R}^n$ and use Problem 4 to obtain a Radon measure, $\lambda_X$ defined on a σ algebra of sets of $\mathbb{R}^n$ including the Borel sets such that for $E$ a Borel set, $\lambda_X(E) =$ Probability that $(X \in E)$.

6. Suppose $X$ and $Y$ are metric spaces having compact closed balls. Show

$$(X \times Y, d_{X \times Y})$$

is also a metric space which has the closures of balls compact. Here

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) \equiv \max(d(x_1, x_2), d(y_1, y_2)).$$

Let

$$A \equiv \{E \times F : E \text{ is a Borel set in } X, F \text{ is a Borel set in } Y\}.$$ 

Show $\sigma(A)$, the smallest σ algebra containing $A$ contains the Borel sets. **Hint:** Show every open set in a metric space which has closed balls compact can be obtained as a countable union of compact sets. Next show this implies every open set can be obtained as a countable union of open sets of the form $U \times V$ where $U$ is open in $X$ and $V$ is open in $Y$.

7. Suppose $(\Omega, S, \mu)$ is a measure space which may not be complete. Could you obtain a complete measure space, $(\Omega, \mathcal{S}, \mu_1)$ by simply letting $S$ consist of all sets of the form $E$ where there exists $F \in S$ such that $(F \setminus E) \cup (E \setminus F) \subseteq N$ for some $N \in S$ which has measure zero and then let $\mu(E) = \mu_1(F)$?

8. If $\mu$ and $\nu$ are Radon measures defined on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, show $\mu \times \nu$ is also a radon measure on $\mathbb{R}^{n+m}$. **Hint:** Show the $\mu \times \nu$ measurable sets include the open sets using the observation that every open set in $\mathbb{R}^{n+m}$ is the countable union of sets of the form $U \times V$ where $U$ and $V$ are open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Next verify outer regularity by considering $A \times B$ for $A, B$ measurable. Argue sets of $\mathcal{R}$ defined above have the property that they can be approximated in measure from above by open sets. Then verify the same is true of sets of $\mathcal{R}_1$. Finally conclude using an appropriate lemma that $\mu \times \nu$ is inner regular as well.

9. Let $(\Omega, S, \mu)$ be a σ finite measure space and let $f : \Omega \to [0, \infty)$ be measurable. Define

$$A \equiv \{(x, y) : y < f(x)\}$$

Verify that $A$ is $\overline{\mu \times m}$ measurable. Show that

$$\int f \, d\mu = \int \int \mathcal{X}_A(x, y) \, d\mu \, dm = \int \mathcal{X}_A \, d(\mu \times m).$$
Chapter 11

Lebesgue Measure

11.1 Basic Properties

Definition 11.1.1 Define the following positive linear functional for \( f \in C_c(\mathbb{R}^n) \).

\[
\Lambda f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.
\]

Then the measure representing this functional is Lebesgue measure.

The following lemma will help in understanding Lebesgue measure.

Lemma 11.1.2 Every open set in \( \mathbb{R}^n \) is the countable disjoint union of half open boxes of the form

\[
\prod_{i=1}^{n}(a_i, a_i + 2^{-k}]
\]

where \( a_i = l2^{-k} \) for some integers, \( l, k \). The sides of these boxes are of equal length. One could also have half open boxes of the form

\[
\prod_{i=1}^{n}(a_i, a_i + 2^{-k})
\]

and the conclusion would be unchanged.

Proof: Let

\[ C_k = \{ \text{All half open boxes} \prod_{i=1}^{n}(a_i, a_i + 2^{-k}) \text{ where} \]

\[ a_i = l2^{-k} \text{ for some integer } l \} \]

Thus \( C_k \) consists of a countable disjoint collection of boxes whose union is \( \mathbb{R}^n \). This is sometimes called a tiling of \( \mathbb{R}^n \). Think of tiles on the floor of a bathroom and
you will get the idea. Note that each box has diameter no larger than $2^{-k}\sqrt{n}$. This
is because if
\[ x, y \in \prod_{i=1}^{n} (a_i, a_i + 2^{-k}], \]
then $|x_i - y_i| \leq 2^{-k}$. Therefore,
\[ |x - y| \leq \left( \sum_{i=1}^{n} (2^{-k})^2 \right)^{1/2} = 2^{-k}\sqrt{n}. \]

Let $U$ be open and let $B_1 \equiv$ all sets of $C_1$ which are contained in $U$. If $B_1, \cdots, B_k$
have been chosen, $B_{k+1} \equiv$ all sets of $C_{k+1}$ contained in
\[ U \setminus \left( \cup_{i=1}^{k} B_i \right). \]

Let $B_{\infty} = \cup_{i=1}^{\infty} B_i$. In fact $\cup B_{\infty} = U$. Clearly $\cup B_{\infty} \subseteq U$ because every box of every $B_i$
is contained in $U$. If $p \in U$, let $k$ be the smallest integer such that $p$ is contained
in a box from $C_k$ which is also a subset of $U$. Thus
\[ p \in \cup B_k \subseteq \cup B_{\infty}. \]

Hence $B_{\infty}$ is the desired countable disjoint collection of half open boxes whose union
is $U$. The last assertion about the other type of half open rectangle is obvious. This
proves the lemma.

Now what does Lebesgue measure do to a rectangle, $\prod_{i=1}^{n} (a_i, b_i]$?

**Lemma 11.1.3** Let $R = \prod_{i=1}^{n} [a_i, b_i]$, $R_0 = \prod_{i=1}^{n} (a_i, b_i)$. Then
\[ m_n(R_0) = m_n(R) = \prod_{i=1}^{n} (b_i - a_i). \]

**Proof:** Let $k$ be large enough that
\[ a_i + 1/k < b_i - 1/k \]
for $i = 1, \cdots, n$ and consider functions $g^k_i$ and $f^k_i$ having the following graphs.

Let
\[ g^k(x) = \prod_{i=1}^{n} g^k_i(x_i), \quad f^k(x) = \prod_{i=1}^{n} f^k_i(x_i). \]
Then by elementary calculus along with the definition of $\Lambda$,
\[
\prod_{i=1}^{n} (b_i - a_i + 2/k) \geq \Lambda g^k = \int g^k dm_n \geq m_n(R) \geq m_n(R_0)
\]
\[
\geq \int f^k dm_n = \Lambda f^k \geq \prod_{i=1}^{n} (b_i - a_i - 2/k).
\]
Letting $k \to \infty$, it follows that
\[
m_n(R) = m_n(R_0) = \prod_{i=1}^{n} (b_i - a_i).
\]
This proves the lemma.

**Lemma 11.1.4** Let $U$ be an open or closed set. Then $m_n(U) = m_n(x + U)$.

**Proof:** By Lemma 11.1.2, there is a sequence of disjoint half open rectangles, \{\(R_i\)\} such that $\cup_i R_i = U$. Therefore, $x + U = \cup_i (x + R_i)$ and the $x + R_i$ are also disjoint rectangles which are identical to the $R_i$ but translated. From Lemma 11.1.3, $m_n(U) = \sum_i m_n(R_i) = \sum_i m_n(x + R_i) = m_n(x + U)$.

It remains to verify the lemma for a closed set. Let $H$ be a closed bounded set first. Then $H \subseteq B(0, R)$ for some $R$ large enough. First note that $x + H$ is a closed set. Thus
\[
m_n(B(x, R)) = m_n(x + H) + m_n((B(0, R) + x) \setminus (x + H))
\]
\[
= m_n(x + H) + m_n((B(0, R) \setminus H) + x)
\]
\[
= m_n(x + H) + m_n((B(0, R) \setminus H))
\]
\[
= m_n(B(0, R)) - m_n(H) + m_n(x + H)
\]
\[
= m_n(B(x, R)) - m_n(H) + m_n(x + H)
\]
the last equality because of the first part of the lemma which implies $m_n(B(x, R)) = m_n(B(0, R))$. Therefore, $m_n(x + H) = m_n(H)$ as claimed. If $H$ is not bounded, consider $H_m \equiv B(0, m) \cap H$. Then $m_n(x + H_m) = m_n(H_m)$. Passing to the limit as $m \to \infty$ yields the result in general.

**Theorem 11.1.5** Lebesgue measure is translation invariant. That is
\[
m_n(E) = m_n(x + E)
\]
for all $E$ Lebesgue measurable.

**Proof:** Suppose $m_n(E) < \infty$. By regularity of the measure, there exist sets $G, H$ such that $G$ is a countable intersection of open sets, $H$ is a countable union of compact sets, $m_n(G \setminus H) = 0$, and $G \supseteq E \supseteq H$. Now $m_n(G) = m_n(G + x)$ and
Let $m_n(H) = m_n(H + x)$ which follows from Lemma \ref{lemma:translation} applied to the sets which are either intersected to form $G$ or unioned to form $H$. Now

$$x + H \subseteq x + E \subseteq x + G$$

and both $x + H$ and $x + G$ are measurable because they are either countable unions or countable intersections of measurable sets. Furthermore,

$$m_n(x + G \setminus x + H) = m_n(x + G) - m_n(x + H) = m_n(G) - m_n(H) = 0$$

and so by completeness of the measure, $x + E$ is measurable. It follows

$$m_n(E) = m_n(H) = m_n(x + H) \leq m_n(x + E) \leq m_n(x + G) = m_n(G) = m_n(E).$$

If $m_n(E)$ is not necessarily less than infinity, consider $E_m = B(0, m) \cap E$. Then $m_n(E_m) = m_n(E_m + x)$ by the above. Letting $m \to \infty$ it follows $m_n(E) = m_n(E + x)$. This proves the theorem.

**Corollary 11.1.6** Let $D$ be an $n \times n$ diagonal matrix and let $U$ be an open set. Then

$$m_n(DU) = |\det(D)| m_n(U).$$

**Proof:** If any of the diagonal entries of $D$ equals 0 there is nothing to prove because then both sides equal zero. Therefore, it can be assumed none are equal to zero. Suppose these diagonal entries are $k_1, \ldots, k_n$. From Lemma \ref{lemma:translation} there exist half open boxes, $\{R_i\}$ having all sides equal such that $U = \cup_i R_i$. Suppose one of these is $R_i = \prod_{j=1}^n (a_j, b_j)$, where $b_j - a_j = l_i$. Then $DR_i = \prod_{j=1}^n I_j$ where $I_j = (k_j a_j, k_j b_j)$ if $k_j > 0$ and $I_j = [k_j b_j, k_j a_j]$ if $k_j < 0$. Then the rectangles, $DR_i$ are disjoint because $D$ is one to one and their union is $DU$. Also,

$$m_n(DR_i) = \prod_{j=1}^n |k_j| l_i = |\det D| m_n(R_i).$$

Therefore,

$$m_n(DU) = \sum_{i=1}^\infty m_n(DR_i) = |\det(D)| \sum_{i=1}^\infty m_n(R_i) = |\det(D)| m_n(U).$$

and this proves the corollary.

From this the following corollary is obtained.

**Corollary 11.1.7** Let $M > 0$. Then $m_n(B(a, Mr)) = M^n m_n(B(0, r))$.

**Proof:** By Lemma \ref{lemma:translation} there is no loss of generality in taking $a = 0$. Let $D$ be the diagonal matrix which has $M$ in every entry of the main diagonal so $|\det(D)| = M^n$. Note that $DB(0, r) = B(0, Mr)$. By Corollary \ref{corollary:scaling} $m_n(B(0, Mr)) = m_n(DB(0, r)) = M^n m_n(B(0, r))$. 

11.2. THE VITALI COVERING THEOREM

There are many norms on $\mathbb{R}^n$. Other common examples are

\[ ||x||_\infty \equiv \max\{ |x_k| : x = (x_1, \cdots, x_n) \} \]

or

\[ ||x||_p \equiv \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \]

With $||\cdot||$ any norm for $\mathbb{R}^n$ you can define a corresponding ball in terms of this norm.

\[ B(a, r) \equiv \{ x \in \mathbb{R}^n \text{ such that } ||x - a|| < r \} \]

It follows from general considerations involving metric spaces presented earlier that these balls are open sets. Therefore, Corollary 11.1.7 has an obvious generalization.

**Corollary 11.1.8** Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Then for $M > 0$, $m_n(B(a, Mr)) = M^n m_n(B(0, r))$ where these balls are defined in terms of the norm $||\cdot||$.

### 11.2 The Vitali Covering Theorem

The Vitali covering theorem is concerned with the situation in which a set is contained in the union of balls. You can imagine that it might be very hard to get disjoint balls from this collection of balls which would cover the given set. However, it is possible to get disjoint balls from this collection of balls which have the property that if each ball is enlarged appropriately, the resulting enlarged balls do cover the set. When this result is established, it is used to prove another form of this theorem in which the disjoint balls do not cover the set but they only miss a set of measure zero.

Recall the Hausdorff maximal principle, Theorem 1.4.2 on Page 32 which is proved to be equivalent to the axiom of choice in the appendix. For convenience, here it is:

**Theorem 11.2.1** (Hausdorff Maximal Principle) Let $F$ be a nonempty partially ordered set. Then there exists a maximal chain.

I will use this Hausdorff maximal principle to give a very short and elegant proof of the Vitali covering theorem. This follows the treatment in Evans and Gariepy [44] which they got from another book. I am not sure who first did it this way but it is very nice because it is so short. In the following lemma and theorem, the balls will be either open or closed and determined by some norm on $\mathbb{R}^n$. When pictures are drawn, I shall draw them as though the norm is the usual norm but the results are unchanged for any norm. Also, I will write (in this section only) $B(a, r)$ to indicate a set which satisfies

\[ \{ x \in \mathbb{R}^n : ||x - a|| < r \} \subseteq B(a, r) \subseteq \{ x \in \mathbb{R}^n : ||x - a|| \leq r \} \]

and $\hat{B}(a, r)$ to indicate the usual ball but with radius 5 times as large,

\[ \{ x \in \mathbb{R}^n : ||x - a|| < 5r \} . \]
Lemma 11.2.2  Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ and let $\mathcal{F}$ be a collection of balls determined by this norm. Suppose

$$\infty > M \equiv \sup \{ r : B(p, r) \in \mathcal{F} \} > 0$$

and $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

\begin{align*}
&\text{if } B(p, r) \in \mathcal{G} \text{ then } r > k, \\
&\text{if } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset,
\end{align*}

$\mathcal{G}$ is maximal with respect to $11.2.1$ and $11.2.2$.

Note that if there is no ball of $\mathcal{F}$ which has radius larger than $k$ then $\mathcal{G} = \emptyset$.

Proof: Let $\mathcal{H} = \{ B \subseteq \mathcal{F} \text{ such that } 11.2.1 \text{ and } 11.2.2 \text{ hold} \}$. If there are no balls with radius larger than $k$ then $\mathcal{H} = \emptyset$ and you let $\mathcal{G} = \emptyset$. In the other case, $\mathcal{H} \neq \emptyset$ because there exists $B(p, r) \in \mathcal{F}$ with $r > k$. In this case, partially order $\mathcal{H}$ by set inclusion and use the Hausdorff maximal principle (see the appendix on set theory) to let $\mathcal{C}$ be a maximal chain in $\mathcal{H}$. Clearly $\cup \mathcal{C}$ satisfies $11.2.1$ and $11.2.2$ because if $B_1$ and $B_2$ are two balls from $\cup \mathcal{C}$ then since $\mathcal{C}$ is a chain, it follows there is some element of $\mathcal{C}, \mathcal{B}$ such that both $B_1$ and $B_2$ are elements of $\mathcal{B}$ and $\mathcal{B}$ satisfies $11.2.1$ and $11.2.2$. If $\cup \mathcal{C}$ is not maximal with respect to these two properties, then $\mathcal{C}$ was not a maximal chain because then there would exist $B \supseteq \cup \mathcal{C}$, that is, $B$ contains $\mathcal{C}$ as a proper subset and $\{ \mathcal{C}, \mathcal{B} \}$ would be a strictly larger chain in $\mathcal{H}$. Let $\mathcal{G} = \cup \mathcal{C}$.

Theorem 11.2.3  (Vitali) Let $\mathcal{F}$ be a collection of balls and let

$$A \equiv \cup \{ B : B \in \mathcal{F} \}.$$

Suppose

$$\infty > M \equiv \sup \{ r : B(p, r) \in \mathcal{F} \} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup \{ \tilde{B} : B \in \mathcal{G} \}.$$

Proof: Using Lemma 11.2.2, there exists $\mathcal{G}_1 \subseteq \mathcal{F} \equiv \mathcal{F}_0$ which satisfies

\begin{align*}
&\text{if } B(p, r) \in \mathcal{G}_1 \text{ implies } r > \frac{M}{2}, \\
&\text{if } B_1, B_2 \in \mathcal{G}_1 \text{ implies } B_1 \cap B_2 = \emptyset.
\end{align*}

$\mathcal{G}_1$ is maximal with respect to $11.2.1$ and $11.2.2$.

Suppose $\mathcal{G}_1, \ldots, \mathcal{G}_m$ have been chosen, $m \geq 1$. Let

$$\mathcal{F}_m \equiv \{ B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \cup \{ \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m \} \}.$$
11.3. THE VITALI COVERING THEOREM (ELEMENTARY VERSION)

Using Lemma 11.2.2, there exists \( G_{m+1} \subseteq F_m \) such that

\[
B(p, r) \in G_{m+1} \text{ implies } r > \frac{M}{2^{m+1}}, \tag{11.2.5}
\]

\[
B_1, B_2 \in G_{m+1} \text{ implies } B_1 \cap B_2 = \emptyset, \tag{11.2.6}
\]

\( G_{m+1} \) is a maximal subset of \( F_m \) with respect to (11.2.5) and (11.2.6).

Note it might be the case that \( G_{m+1} = \emptyset \) which happens if \( F_m = \emptyset \). Define

\[
\mathcal{G} \equiv \bigcup_{k=1}^{\infty} G_k.
\]

Thus \( \mathcal{G} \) is a collection of disjoint balls in \( F \). I must show \( \{ \hat{B} : B \in \mathcal{G} \} \) covers \( A \).

Let \( x \in B(p, r) \in F \) and let

\[
\frac{M}{2^m} < r \leq \frac{M}{2^{m-1}}.
\]

Then \( B(p, r) \) must intersect some set, \( B(p_0, r_0) \in G_1 \cup \cdots \cup G_m \) since otherwise, \( G_m \) would fail to be maximal. Then \( r_0 > \frac{M}{2^m} \) because all balls in \( G_1 \cup \cdots \cup G_m \) satisfy this inequality.

Then for \( x \in B(p, r) \), the following chain of inequalities holds because \( r \leq \frac{M}{2^m} \) and \( r_0 > \frac{M}{2^m} \)

\[
|x - p_0| \leq |x - p| + |p - p_0| \leq r + r_0 + r \\
\leq \frac{2M}{2^{m-1}} + r_0 = \frac{4M}{2^m} + r_0 < 5r_0.
\]

Thus \( B(p, r) \subseteq \hat{B}(p_0, r_0) \) and this proves the theorem.

11.3 The Vitali Covering Theorem (Elementary Version)

The proof given here is from Basic Analysis [2]. It first considers the case of open balls and then generalizes to balls which may be neither open nor closed or closed.
Lemma 11.3.1 Let \( \mathcal{F} \) be a countable collection of balls satisfying
\[
\infty > M \equiv \sup \{ r : B(p, r) \in \mathcal{F} \} > 0
\]
and let \( k \in (0, \infty) \). Then there exists \( \mathcal{G} \subseteq \mathcal{F} \) such that
\[
\text{If } B(p, r) \in \mathcal{G} \text{ then } r > k, \quad \text{(11.3.7)}
\]
\[
\text{If } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \quad \text{(11.3.8)}
\]
\( \mathcal{G} \) is maximal with respect to 11.3.7 and 11.3.8.

Proof: If no ball of \( \mathcal{F} \) has radius larger than \( k \), let \( \mathcal{G} = \emptyset \). Assume therefore, that some balls have radius larger than \( k \). Let \( \mathcal{F} = \{ B_i \}_{i=1}^\infty \). Now let \( B_{n_1} \) be the first ball in the list which has radius greater than \( k \). If every ball having radius larger than \( k \) intersects this one, then stop. The maximal set is just \( B_{n_1} \). Otherwise, let \( B_{n_2} \) be the next ball having radius larger than \( k \) which is disjoint from \( B_{n_1} \). Continue this way obtaining \( \{ B_{n_i} \}_{i=1}^\infty \), a finite or infinite sequence of disjoint balls having radius larger than \( k \). Then let \( \mathcal{G} = \{ B_{n_i} \} \). To see that \( \mathcal{G} \) is maximal with respect to 11.3.7 and 11.3.8, suppose \( B \in \mathcal{F} \), \( B \) has radius larger than \( k \), and \( \mathcal{G} \cup \{ B \} \) satisfies 11.3.7 and 11.3.8. Then at some point in the process, \( B \) would have been chosen because it would be the ball of radius larger than \( k \) which has the smallest index. Therefore, \( B \in \mathcal{G} \) and this shows \( \mathcal{G} \) is maximal with respect to 11.3.7 and 11.3.8.

For the next lemma, for an open ball, \( B = B(x, r) \), denote by \( \bar{B} \) the open ball, \( B(x, 4r) \).

Lemma 11.3.2 Let \( \mathcal{F} \) be a collection of open balls, and let
\[
A \equiv \cup \{ B : B \in \mathcal{F} \}.
\]
Suppose
\[
\infty > M \equiv \sup \{ r : B(p, r) \in \mathcal{F} \} > 0.
\]
Then there exists \( \mathcal{G} \subseteq \mathcal{F} \) such that \( \mathcal{G} \) consists of disjoint balls and
\[
A \subseteq \cup \{ \bar{B} : B \in \mathcal{G} \}.
\]

Proof: Without loss of generality assume \( \mathcal{F} \) is countable. This is because there is a countable subset of \( \mathcal{F} \), \( \mathcal{F}' \) such that \( \cup \mathcal{F}' = A \). To see this, consider the set of balls having rational radii and centers having all components rational. This is a countable set of balls and you should verify that every open set is the union of balls of this form. Therefore, you can consider the subset of this set of balls consisting of those which are contained in some open set of \( \mathcal{F} \), \( G \) so \( \cup G = A \) and use the axiom of choice to define a subset of \( \mathcal{F} \) consisting of a single set from \( \mathcal{F} \) containing each set of \( G \). Then this is \( \mathcal{F}' \). The union of these sets equals \( A \). Then consider \( \mathcal{F}' \) instead of \( \mathcal{F} \). Therefore, assume at the outset \( \mathcal{F} \) is countable. By Lemma 11.3.1, there exists \( \mathcal{G}_1 \subseteq \mathcal{F} \) which satisfies 11.3.7, 11.3.8, and 11.3.9 with \( k = \frac{2M}{3} \).
11.3. THE VITALI COVERING THEOREM (ELEMENTARY VERSION) 331

Suppose \( \mathcal{G}_1, \cdots, \mathcal{G}_{m-1} \) have been chosen for \( m \geq 2 \). Let

\[
\mathcal{F}_m = \{ B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \bigcup_{j=1}^{m-1} \mathcal{G}_j \} \quad \text{and using Lemma 11.3.1, let } \mathcal{G}_m \text{ be a maximal collection of disjoint balls from } \mathcal{F}_m \text{ with the property that each ball has radius larger than } (\frac{2}{3})^m M. \text{ Let } \mathcal{G} \equiv \bigcup_{k=1}^{\infty} \mathcal{G}_k. \text{ Let } x \in B(p, r) \in \mathcal{F}. \text{ Choose } m \text{ such that }
\[
(\frac{2}{3})^m M < r \leq (\frac{2}{3})^{m-1} M.
\]

Then \( B(p, r) \) must have nonempty intersection with some ball from \( \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m \) because if it didn’t, then \( \mathcal{G}_m \) would fail to be maximal. Denote by \( B(p_0, r_0) \) a ball in \( \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m \) which has nonempty intersection with \( B(p, r) \). Thus

\[
r_0 > (\frac{2}{3})^m M.
\]

Consider the picture, in which \( w \in B(p_0, r_0) \cap B(p, r) \).

Then

\[
|x - p_0| \leq |x - p| + |p - w| + |w - p_0| < r_0 \leq \frac{2}{3} r_0 \leq (\frac{2}{3})^{m-1} M + r_0 < 2 (\frac{3}{2})^m r_0 + r_0 = 4r_0.
\]

This proves the lemma since it shows \( B(p, r) \subseteq B(p_0, 4r_0) \).

With this Lemma consider a version of the Vitali covering theorem in which the balls do not have to be open. A ball centered at \( x \) of radius \( r \) will denote something which contains the open ball, \( B(x, r) \) and is contained in the closed ball, \( \overline{B(x, r)} \). Thus the balls could be open or they could contain some but not all of their boundary points.

**Definition 11.3.3** Let \( B \) be a ball centered at \( x \) having radius \( r \). Denote by \( \overline{B} \) the open ball, \( B(x, 5r) \).
Theorem 11.3.4 (Vitali) Let $F$ be a collection of balls, and let $A \equiv \bigcup \{ B : B \in F \}$.

Suppose $\infty > M \equiv \sup \{ r : B(p, r) \in F \} > 0$.

Then there exists $G \subseteq F$ such that $G$ consists of disjoint balls and $A \subseteq \bigcup \{ \tilde{B} : B \in G \}$.

Proof: For $B$ one of these balls, say $B(x, r) \supseteq B \supseteq B(x, r)$, denote by $B_1 = B(x, 5r)$. Let $F_1 \equiv \{ B_1 : B \in F \}$ and let $A_1$ denote the union of the balls in $F_1$. Apply Lemma 11.3.2 to $F_1$ to obtain

$$A_1 \subseteq \bigcup \{ \tilde{B}_1 : B_1 \in G_1 \}$$

where $G_1$ consists of disjoint balls from $F_1$. Now let $G \equiv \{ B \in F : B_1 \in G_1 \}$. Thus $G$ consists of disjoint balls from $F$ because they are contained in the disjoint open balls, $G_1$. Then

$$A \subseteq A_1 \subseteq \bigcup \{ \tilde{B}_1 : B_1 \in G_1 \} = \bigcup \{ \tilde{B} : B \in G \}$$

because for $B_1 = B(x, 5r)$, it follows $\tilde{B}_1 = B(x, 5r) = \tilde{B}$. This proves the theorem.

11.4 Vitali Coverings

There is another version of the Vitali covering theorem which is also of great importance. In this one, balls from the original set of balls almost cover the set, leaving out only a set of measure zero. It is like packing a truck with stuff. You keep trying to fill in the holes with smaller and smaller things so as to not waste space. It is remarkable that you can avoid wasting any space at all when you are dealing with balls of any sort provided you can use arbitrarily small balls.

Definition 11.4.1 Let $F$ be a collection of balls that cover a set, $E$, which have the property that if $x \in E$ and $\varepsilon > 0$, then there exists $B \in F$, diameter of $B < \varepsilon$ and $x \in B$. Such a collection covers $E$ in the sense of Vitali.

In the following covering theorem, $m_n$ denotes the outer measure determined by $n$ dimensional Lebesgue measure.

Theorem 11.4.2 Let $E \subseteq \mathbb{R}^n$ and suppose $0 < m_n(E) < \infty$ where $m_n$ is the outer measure determined by $m_n$, $n$ dimensional Lebesgue measure, and let $F$ be a collection of closed balls of bounded radii such that $F$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $F$, $\{ B_j \}_{j=1}^\infty$, such that $m_n(E \setminus \bigcup_{j=1}^\infty B_j) = 0$. 

**Proof:** From the definition of outer measure there exists a Lebesgue measurable set, \( E_1 \supseteq E \) such that \( m_n(E_1) = \overline{m}_n(E) \). Now by outer regularity of Lebesgue measure, there exists \( U \), an open set which satisfies
\[
m_n(E_1) > (1 - 10^{-n})m_n(U), \quad U \supseteq E_1.
\]

Each point of \( E \) is contained in balls of \( F \) of arbitrarily small radii and so there exists a covering of \( E \) with balls of \( F \) which are themselves contained in \( U \). Therefore, by the Vitali covering theorem, there exist disjoint balls, \( \{B_j\}_{j=1}^{\infty} \subseteq F \) such that
\[
E \subseteq \bigcup_{j=1}^{\infty} \hat{B}_j, \quad B_j \subseteq U.
\]

Therefore,
\[
m_n(E_1) = \overline{m}_n(E) \leq m_n\left( \bigcup_{j=1}^{\infty} \hat{B}_j \right) \leq \sum_j m_n(\hat{B}_j) = 5^n \sum_j m_n(B_j) = 5^n m_n\left( \bigcup_{j=1}^{\infty} B_j \right)
\]

Then \( E_1 \) and \( \bigcup_{j=1}^{\infty} B_j \) are contained in \( U \) and so
\[
m_n(E_1) > (1 - 10^{-n})m_n(U)
\]
\[
\geq (1 - 10^{-n})[m_n(E_1 \setminus \bigcup_{j=1}^{\infty} B_j) + m_n(\bigcup_{j=1}^{\infty} B_j)] = m_n(E_1)
\]
\[
\geq (1 - 10^{-n})[m_n(E_1 \setminus \bigcup_{j=1}^{\infty} B_j) + 5^{-n} \overline{m}_n(E)].
\]

and so
\[
(1 - (1 - 10^{-n})5^{-n}) m_n(E_1) \geq (1 - 10^{-n})m_n(E_1 \setminus \bigcup_{j=1}^{\infty} B_j)
\]

which implies
\[
m_n(E_1 \setminus \bigcup_{j=1}^{\infty} B_j) \leq \frac{(1 - (1 - 10^{-n})5^{-n})}{(1 - 10^{-n})} m_n(E_1)
\]
Now a short computation shows
\[ 0 < \frac{(1 - (1 - 10^{-n}) 5^{-n})}{(1 - 10^{-n})} < 1 \]
Hence, denoting by \( \theta_n \) a number such that
\[ \frac{(1 - (1 - 10^{-n}) 5^{-n})}{(1 - 10^{-n})} < \theta_n < 1, \]
\[ \overline{m}_n(E \setminus \bigcup_{j=1}^{N_1} B_j) \leq m_n(E_1 \setminus \bigcup_{j=1}^{N_1} B_j) < \theta_n m_n(E_1) = \theta_n \overline{m}_n(E) \]
Now using Theorem 9.1.5 on Page 208 there exists \( N_1 \) large enough that
\[ \theta_n \overline{m}_n(E) \geq m_n(E_1 \setminus \bigcup_{j=1}^{N_1} B_j) \geq \overline{m}_n(E \setminus \bigcup_{j=1}^{N_1} B_j) \quad (11.4.10) \]
Let \( F_1 = \{ B \in F : B_j \cap B = \emptyset, \ j = 1, \cdots, N_1 \} \). If \( E \setminus \bigcup_{j=1}^{N_1} B_j = \emptyset \), then \( F_1 = \emptyset \) and
\[ \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_1} B_j \right) = 0 \]
Therefore, in this case let \( B_k = \emptyset \) for all \( k > N_1 \). Consider the case where
\[ E \setminus \bigcup_{j=1}^{N_1} B_j \neq \emptyset. \]
In this case, since the balls are closed and \( F \) is a Vitali cover, \( F_1 \neq \emptyset \) and covers \( E \setminus \bigcup_{j=1}^{N_1} B_j \) in the sense of Vitali. Repeat the same argument, letting \( E \setminus \bigcup_{j=1}^{N_1} B_j \) play the role of \( E \). (You pick a different \( E_1 \) whose measure equals the outer measure of \( E \setminus \bigcup_{j=1}^{N_1} B_j \) and proceed as before.) Then choosing \( B_j \) for \( j = N_1 + 1, \cdots, N_2 \) as in the above argument,
\[ \theta_n \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_1} B_j \right) \geq \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_2} B_j \right) \]
and so from (11.4.10),
\[ \theta^2_n \overline{m}_n(E) \geq \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_2} B_j \right). \]
Continuing this way
\[ \theta^k_n \overline{m}_n(E) \geq \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_k} B_j \right). \]
If it is ever the case that \( E \setminus \bigcup_{j=1}^{N_k} B_j = \emptyset \), then as in the above argument,
\[ \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_k} B_j \right) = 0. \]
Otherwise, the process continues and
\[ \overline{m}_n \left( E \setminus \bigcup_{j=1}^{\infty} B_j \right) \leq \overline{m}_n \left( E \setminus \bigcup_{j=1}^{N_k} B_j \right) \leq \theta^k_n \overline{m}_n(E) \]
for every \( k \in \mathbb{N} \). Therefore, the conclusion holds in this case also. This proves the Theorem.
There is an obvious corollary which removes the assumption that \( 0 < \overline{m}_n(E) \).
Corollary 11.4.3 Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by $m_n$, $n$ dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m_n}(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.

**Proof:** If $0 = \overline{m_n}(E)$ you simply pick any ball from $\mathcal{F}$ for your collection of disjoint balls. It is also not hard to remove the assumption that $\overline{m_n}(E) < \infty$.

Corollary 11.4.4 Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m_n}(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.

**Proof:** Let $R_m \equiv (-m, m)^n$ be the open rectangle having sides of length $2m$ which is centered at $0$ and let $R_0 = \emptyset$. Let $H_m \equiv \overline{R_m} \setminus R_m$. Since both $\overline{R_m}$ and $R_m$ have the same measure, $(2m)^n$, it follows $m_n(H_m) = 0$. Now for all $k \in \mathbb{N}$, $R_k \subseteq \overline{R_k} \subseteq R_{k+1}$. Consider the disjoint open sets, $U_k \equiv R_{k+1} \setminus \overline{R_k}$. Thus $\mathbb{R}^n = \bigcup_{k=0}^\infty U_k \cup N$ where $N$ is a set of measure zero equal to the union of the $H_k$. Let $\mathcal{F}_k$ denote those balls of $\mathcal{F}$ which are contained in $U_k$ and let $E_k \equiv U_k \cap E$. Then from Theorem 11.4.4, there exists a sequence of disjoint balls, $D_k \equiv \{B^k_i\}_{i=1}^\infty$ of $\mathcal{F}_k$ such that $\overline{m_n}(E_k \setminus \bigcup_{j=1}^\infty B^k_j) = 0$. Letting $\{B_i\}_{i=1}^\infty$ be an enumeration of all the balls of $\bigcup D_k$, it follows that

$$\overline{m_n}(E \setminus \bigcup_{j=1}^\infty B_j) \leq m_n(N) + \sum_{k=1}^\infty \overline{m_n}(E_k \setminus \bigcup_{j=1}^\infty B^k_j) = 0.$$ 

Also, you don’t have to assume the balls are closed.

Corollary 11.4.5 Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m_n}(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.

**Proof:** Let $\overline{\mathcal{F}}$ be the collection of closures of balls in $\mathcal{F}$. Then $\overline{\mathcal{F}}$ covers $E$ in the sense of Vitali and so from Corollary 11.4.3 there exists a sequence of disjoint closed balls from $\overline{\mathcal{F}}$ satisfying $\overline{m_n}(E \setminus \bigcup_{i=1}^\infty B_i) = 0$. Now boundaries of the balls, $B_i$ have measure zero and so $\{B_i\}$ is a sequence of disjoint open balls satisfying $\overline{m_n}(E \setminus \bigcup_{i=1}^\infty B_i) = 0$. The reason for this is that

$$(E \setminus \bigcup_{i=1}^\infty B_i) \setminus (E \setminus \bigcup_{i=1}^\infty \overline{B}_i) \subseteq \bigcup_{i=1}^\infty \overline{B}_i \setminus \bigcup_{i=1}^\infty B_i \subseteq \bigcup_{i=1}^\infty \overline{B}_i \setminus B_i,$$

da set of measure zero. Therefore, 

$$E \setminus \bigcup_{i=1}^\infty B_i \subseteq (E \setminus \bigcup_{i=1}^\infty \overline{B}_i) \cup \bigcup_{i=1}^\infty \overline{B}_i \setminus B_i$$

and so

$$\overline{m_n}(E \setminus \bigcup_{i=1}^\infty B_i) \leq \overline{m_n}(E \setminus \bigcup_{i=1}^\infty \overline{B}_i) + m_n(\bigcup_{i=1}^\infty \overline{B}_i \setminus B_i)$$

$$= \overline{m_n}(E \setminus \bigcup_{i=1}^\infty B_i) = 0.$$
This implies you can fill up an open set with balls which cover the open set in the sense of Vitali.

**Corollary 11.4.6** Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{F}$ be a collection of closed or even open balls of bounded radii contained in $U$ such that $\mathcal{F}$ covers $U$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $m_n(U \setminus \bigcup_{j=1}^{\infty} B_j) = 0$.

### 11.5 Change Of Variables For Linear Maps

To begin with certain kinds of functions map measurable sets to measurable sets. It will be assumed that $U$ is an open set in $\mathbb{R}^n$ and that $h: U \to \mathbb{R}^n$ satisfies

$$Dh(x) \text{ exists for all } x \in U, \quad (11.5.11)$$

**Lemma 11.5.1** Let $h$ satisfy (11.5.11). If $T \subseteq U$ and $m_n(T) = 0$, then $m_n(h(T)) = 0$.

**Proof:** Let

$$T_k \equiv \{x \in T : ||Dh(x)|| < k\}$$

and let $\varepsilon > 0$ be given. Now by outer regularity, there exists an open set, $V$, containing $T_k$ which is contained in $U$ such that $m_n(V) < \varepsilon$. Let $x \in T_k$. Then by differentiability,

$$h(x + v) = h(x) + Dh(x)v + o(v)$$

and so there exist arbitrarily small $r_x < 1$ such that $B(x, 5r_x) \subseteq V$ and whenever $|v| \leq r_x$, $|o(v)| < k|v|$. Thus

$$h\left(B(x, r_x)\right) \subseteq B\left(h(x), 2kr_x\right).$$

From the Vitali covering theorem there exists a countable disjoint sequence of these sets, $\{B(x_i, r_i)\}_{i=1}^{\infty}$ such that $\{B(x_i, 5r_i)\}_{i=1}^{\infty} = \{\widehat{B}_i\}_{i=1}^{\infty}$ covers $T_k$. Then letting $\overline{m_n}$ denote the outer measure determined by $m_n$,

$$\overline{m_n}(h(T_k)) \leq \overline{m_n}\left(h\left(\bigcup_{i=1}^{\infty} \widehat{B}_i\right)\right)$$

$$\leq \sum_{i=1}^{\infty} \overline{m_n}\left(h\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^{\infty} m_n(B(h(x_i), 2kr_x))$$

$$= \sum_{i=1}^{\infty} m_n(B(x_i, 2kr_x)) = (2k)^n \sum_{i=1}^{\infty} m_n(B(x_i, r_x))$$

$$\leq (2k)^n m_n(V) \leq (2k)^n \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows $m_n(h(T_k)) = 0$. Now

$$m_n(h(T)) = \lim_{k \to \infty} m_n(h(T_k)) = 0.$$

This proves the lemma.
Lemma 11.5.2 Let \( h \) satisfy \([\text{Lemma 11.5.4}]\). If \( S \) is a Lebesgue measurable subset of \( U \), then \( h(S) \) is Lebesgue measurable.

Proof: Let \( S_k = S \cap B(0, k), k \in \mathbb{N} \). By inner regularity of Lebesgue measure, there exists a set, \( F \), which is the countable union of compact sets and a set \( T \) with \( m_n(T) = 0 \) such that

\[
F \cup T = S_k.
\]

Then \( h(F) \subseteq h(S_k) \subseteq h(F) \cup h(T) \). By continuity of \( h \), \( h(F) \) is a countable union of compact sets and so it is Borel. By Lemma \([\text{Lemma 11.5.4}]\), \( m_n(h(T)) = 0 \) and so \( h(S_k) \) is Lebesgue measurable because of completeness of Lebesgue measure. Now \( h(S) = \bigcup_{k=1}^{\infty} h(S_k) \) and so it is also true that \( h(S) \) is Lebesgue measurable. This proves the lemma.

In particular, this proves the following corollary.

Corollary 11.5.3 Suppose \( A \) is an \( n \times n \) matrix. Then if \( S \) is a Lebesgue measurable set, it follows \( AS \) is also a Lebesgue measurable set.

Lemma 11.5.4 Let \( R \) be unitary \((R^* R = RR^* = I)\) and let \( V \) be an open or closed set. Then \( m_n(RV) = m_n(V) \).

Proof: First assume \( V \) is a bounded open set. By Corollary \([\text{Lemma 11.5.4}]\) there is a disjoint sequence of closed balls, \( \{B_i\} \) such that \( V = \bigcup_{i=1}^{\infty} B_i \cup N \) where \( m_n(N) = 0 \). Denote by \( x_i \) the center of \( B_i \) and let \( r_i \) be the radius of \( B_i \). Then by Lemma \([\text{Lemma 11.5.4}]\)

\[
m_n(RV) = \sum_{i=1}^{\infty} m_n(RB_i).
\]

Now by invariance of translation of Lebesgue measure, this equals

\[
\sum_{i=1}^{\infty} m_n(RB_i - Rx_i) = \sum_{i=1}^{\infty} m_n(RB(0, r_i)).
\]

Since \( R \) is unitary, it preserves all distances and so \( RB(0, r_i) = B(0, r_i) \) and therefore,

\[
m_n(RV) = \sum_{i=1}^{\infty} m_n(B(0, r_i)) = \sum_{i=1}^{\infty} m_n(B_i) = m_n(V).
\]

This proves the lemma in the case that \( V \) is bounded. Suppose now that \( V \) is just an open set. Let \( V_k = V \cap B(0, k) \). Then \( m_n(RV_k) = m_n(V_k) \). Letting \( k \to \infty \), this yields the desired conclusion. This proves the lemma in the case that \( V \) is open.

Suppose now that \( H \) is a closed and bounded set. Let \( B(0,R) \supseteq H \). Then letting \( B = B(0,R) \) for short,

\[
m_n(RH) = m_n(RB) - m_n(R(B \setminus H)) = m_n(B) - m_n(B \setminus H) = m_n(H).
\]

In general, let \( H_m = H \cap B(0,m) \). Then from what was just shown, \( m_n(RH_m) = m_n(H_m) \). Now let \( m \to \infty \) to get the conclusion of the lemma in general. This proves the lemma.

Lemma 11.5.5 Let \( E \) be Lebesgue measurable set in \( \mathbb{R}^n \) and let \( R \) be unitary. Then \( m_n(RE) = m_n(E) \).
Proof: First suppose $E$ is bounded. Then there exist sets, $G$ and $H$ such that $H \subseteq E \subseteq G$ and $H$ is the countable union of closed sets while $G$ is the countable intersection of open sets such that $m_n (G \setminus H) = 0$. By Lemma 11.5.4 applied to these sets whose union or intersection equals $H$ or $G$ respectively, it follows

$$m_n (RG) = m_n (G) = m_n (H) = m_n (RH).$$

Therefore,

$$m_n (H) = m_n (RH) \leq m_n (RE) \leq m_n (RG) = m_n (G) = m_n (E) = m_n (H).$$

In the general case, let $E_m = E \cap B (0, m)$ and apply what was just shown and let $m \to \infty$.

Lemma 11.5.6 Let $V$ be an open or closed set in $\mathbb{R}^n$ and let $A$ be an $n \times n$ matrix. Then $m_n (AV) = |\det (A)| m_n (V)$.

Proof: Let $RU$ be the right polar decomposition (Theorem 4.13.6 on Page 99) of $A$ and let $V$ be an open set. Then from Lemma 11.5.4,

$$m_n (AV) = m_n (RUV) = m_n (UV).$$

Now $U = Q^* D Q$ where $D$ is a diagonal matrix such that $|\det (D)| = |\det (A)|$ and $Q$ is unitary. Therefore,

$$m_n (AV) = m_n (Q^* D Q V) = m_n (D Q V).$$

Now $Q V$ is an open set and so by Corollary 11.1.6 on Page 326 and Lemma 11.5.4,

$$m_n (AV) = |\det (D)| m_n (Q V) = |\det (D)| m_n (V) = |\det (A)| m_n (V).$$

This proves the lemma in case $V$ is open.

Now let $H$ be a closed set which is also bounded. First suppose $|\det (A)| = 0$. Then letting $V$ be an open set containing $H$,

$$m_n (AH) \leq m_n (AV) = |\det (A)| m_n (V) = 0$$

which shows the desired equation is obvious in the case where $|\det (A)| = 0$. Therefore, assume $A$ is one to one. Since $H$ is bounded, $H \subseteq B (0, R)$ for some $R > 0$. Then letting $B = B (0, R)$ for short,

$$m_n (AH) = m_n (AB) - m_n (A (B \setminus H))$$

$$= |\det (A)| m_n (B) - |\det (A)| m_n (B \setminus H) = |\det (A)| m_n (H).$$

If $H$ is not bounded, apply the result just obtained to $H_m = H \cap \overline{B (0, m)}$ and then let $m \to \infty$.

With this preparation, the main result is the following theorem.
11.6. CHANGE OF VARIABLES FOR $C^1$ FUNCTIONS

**Theorem 11.5.7** Let $E$ be Lebesgue measurable set in $\mathbb{R}^n$ and let $A$ be an $n \times n$ matrix. Then $m_n (AE) = |\det (A)| m_n (E)$.

**Proof:** First suppose $E$ is bounded. Then there exist sets, $G$ and $H$ such that $H \subseteq E \subseteq G$ and $H$ is the countable union of closed sets while $G$ is the countable intersection of open sets such that $m_n (G \setminus H) = 0$. By Lemma 11.5.6 applied to these sets whose union or intersection equals $H$ or $G$ respectively, it follows

$$m_n (AG) = |\det (A)| m_n (G) = |\det (A)| m_n (H) = m_n (AH).$$

Therefore,

$$|\det (A)| m_n (E) = |\det (A)| m_n (H) = m_n (AH) \leq m_n (AE) \leq m_n (AG) = |\det (A)| m_n (G) = |\det (A)| m_n (E).$$

In the general case, let $E_m = E \cap B (0, m)$ and apply what was just shown and let $m \to \infty$.

### 11.6 Change Of Variables For $C^1$ Functions

In this section theorems are proved which generalize the above to $C^1$ functions. More general versions can be seen in Kuttler [75], Kuttler [76], and Rudin [102]. There is also a very different approach to this theorem given in [75]. The more general version in [75] follows [102] and both are based on the Brouwer fixed point theorem and a very clever lemma presented in Rudin [102]. The proof will be based on a sequence of easy lemmas.

**Lemma 11.6.1** Let $U$ and $V$ be bounded open sets in $\mathbb{R}^n$ and let $h, h^{-1}$ be $C^1$ functions such that $h (U) = V$. Also let $f \in C_c (V)$. Then

$$\int_V f (y) \, dm_n = \int_U f (h (x)) |\det (Dh (x))| \, dm_n$$

**Proof:** First note $h^{-1} (\text{spt} (f))$ is a closed subset of the bounded set, $U$ and so it is compact. Thus $x \to f (h (x)) |\det (Dh (x))|$ is bounded and continuous.

Let $x \in U$. By the assumption that $h$ and $h^{-1}$ are $C^1$,

$$h (x + v) - h (x) = Dh (x) \, v + o (v)$$

$$= Dh (x) \, (v + Dh^{-1} (h (x)) \, o (v))$$

$$= Dh (x) \, (v + o (v))$$

and so if $r > 0$ is small enough then $B (x, r)$ is contained in $U$ and

$$h (B (x, r)) - h (x) =$$

$$h (x + B (0, r)) - h (x) \subseteq Dh (x) \, (B (0, (1 + \varepsilon) r)).$$

(11.6.12)
Making \( r \) still smaller if necessary, one can also obtain
\[
|f(y) - f(h(x))| < \varepsilon
\]
for any \( y \in h(B(x, r)) \) and also
\[
|f(h(x_1))| \left| \det (Dh(x_1)) \right| - f(h(x)) \left| \det (Dh(x)) \right| < \varepsilon
\]
whenever \( x_1 \in B(x, r) \). The collection of such balls is a Vitali cover of \( U \). By Corollary 11.6.1, there is a sequence of disjoint closed balls \( \{B_i\} \) such that \( U = \bigcup_{i=1}^{\infty} B_i \cup N \) where \( m_n(N) = 0 \). Denote by \( x_i \) the center of \( B_i \) and \( r_i \) the radius. Then by Lemma 11.5.1, the monotone convergence theorem, and 11.6.14 - 11.6.15,
\[
\int_V f(y) \, dm_n = \sum_{i=1}^{\infty} \int_{h(B_i)} f(y) \, dm_n
\]
\[
\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} \int_{h(B_i)} f(h(x_i)) \, dm_n
\]
\[
\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(h(x_i)) m_n(h(B_i))
\]
\[
\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(h(x_i)) m_n(Dh(x_i)) (B(0, (1 + \varepsilon) r_i))
\]
\[
= \varepsilon m_n(V) + (1 + \varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(h(x_i)) |\det (Dh(x_i))| \, dm_n
\]
\[
\leq \varepsilon m_n(V) + (1 + \varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(h(x_i)) |\det (Dh(x_i))| \, dm_n + \varepsilon m_n(B_i)
\]
\[
\leq \varepsilon m_n(V) + (1 + \varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(h(x_i)) |\det (Dh(x_i))| \, dm_n + (1 + \varepsilon)^n \varepsilon m_n(U)
\]
\[
= \varepsilon m_n(V) + (1 + \varepsilon)^n \int_U f(h(x)) |\det (Dh(x))| \, dm_n + (1 + \varepsilon)^n \varepsilon m_n(U)
\]
Since \( \varepsilon > 0 \) is arbitrary, this shows
\[
\int_V f(y) \, dm_n \leq \int_U f(h(x)) |\det (Dh(x))| \, dm_n
\]
whenever \( f \in C_c(V) \). Now \( \forall x \rightarrow f(h(x)) |\det (Dh(x))| \) is in \( C_c(U) \) and so using the same argument with \( U \) and \( V \) switching roles and replacing \( h \) with \( h^{-1} \),
\[
\int_U f(h(x)) |\det (Dh(x))| \, dm_n
\]
\[
\leq \int_V f(h(h^{-1}(y))) |\det (Dh(h^{-1}(y)))| |\det (Dh^{-1}(y))| \, dm_n
\]
\[
= \int_V f(y) \, dm_n
\]
by the chain rule. This with 11.6.15 proves the lemma.

The next task is to relax the assumption that \( f \) is continuous.

**Corollary 11.6.2** Let \( U \) and \( V \) be bounded open sets in \( \mathbb{R}^n \) and let \( h, h^{-1} \) be \( C^1 \) functions such that \( h(U) = V \) and \( |\det(Dh(x))| \) is bounded. Also let \( E \subseteq V \) be measurable. Then
\[
\int_V \chi_E(y) \, dm_n = \int_U \chi_E(h(x)) |\det (Dh(x))| \, dm_n.
\]
11.6. CHANGE OF VARIABLES FOR $C^1$ FUNCTIONS

**Proof:** By regularity, there exist compact sets, $K_k$ and open sets $G_k$ such that

$$K_k \subseteq E \subseteq G_k$$

and $m_n(G_k \setminus K_k) < 2^{-k}$. By Theorem 10.2.7, there exist $f_k$ such that $K_k \prec f_k \prec G_k$. Then $f_k(y) \to \mathcal{X}_E(y)$ a.e. because if $y$ is such that convergence fails, it must be the case that $y$ is in $G_k \setminus K_k$ for infinitely many $k$ and $\sum_k m_n(G_k \setminus K_k) < \infty$. This set equals

$$N = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} G_k \setminus K_k$$

and so for each $m \in \mathbb{N}$

$$m_n(N) \leq m_n(\cup_{k=m}^{\infty} G_k \setminus K_k) \leq \sum_{k=m}^{\infty} m_n(G_k \setminus K_k) < \sum_{k=m}^{\infty} 2^{-k} = 2^{-(m-1)}$$

showing $m_n(N) = 0$.

Then $f_k(h(x))$ must converge to $\mathcal{X}_E(h(x))$ for all $x \notin h^{-1}(N)$, a set of measure zero by Lemma 11.5.1. Thus

$$\int_V f_k(y) \, dm_n = \int_U f_k(h(x)) \, |\det(Dh(x))| \, dm_n.$$ 

By the dominated convergence theorem using a dominating function, $\mathcal{X}_V$ in the integral on the left and $\mathcal{X}_U |\det(Dh)|$ on the right, it follows

$$\int_V \mathcal{X}_E(y) \, dm_n = \int_U \mathcal{X}_E(h(x)) \, |\det(Dh(x))| \, dm_n.$$ 

This proves the corollary.

You don’t need to assume the open sets are bounded.

**Corollary 11.6.3** Let $U$ and $V$ be open sets in $\mathbb{R}^n$ and let $h, h^{-1}$ be $C^1$ functions such that $h(U) = V$. Also let $E \subseteq V$ be measurable. Then

$$\int_V \mathcal{X}_E(y) \, dm_n = \int_U \mathcal{X}_E(h(x)) \, |\det(Dh(x))| \, dm_n.$$ 

**Proof:** For each $x \in U$, there exists $r_x$ such that $B(x, r_x) \subseteq U$ and $r_x < 1$. Then by the mean value inequality Theorem 10.5.3, it follows $h(B(x, r_x))$ is also bounded. This is a Vitali cover of $U$ and so by Corollary 11.4.6 there is a sequence of these balls, $\{B_i\}$ such that they are disjoint, $h(B_i)$ is also bounded and

$$m_n(U \setminus \cup_i B_i) = 0.$$
CHAPTER 11. LEBESGUE MEASURE

It follows from Lemma 11.5.1 that $h(U \cup_i B_i)$ also has measure zero. Then from Corollary 11.6.2,

$$
\int_V \mathcal{X}_E(y) \, dm_n = \sum_i \int_{h(B_i)} \mathcal{X}_{E \cap h(B_i)}(y) \, dm_n
$$

$$=
\sum_i \int_{B_i} \mathcal{X}_E(h(x)) |\det (Dh(x))| \, dm_n
$$

$$=
\int_U \mathcal{X}_E(h(x)) |\det (Dh(x))| \, dm_n.
$$

This proves the corollary.

With this corollary, the main theorem follows.

Theorem 11.6.4 Let $U$ and $V$ be open sets in $\mathbb{R}^n$ and let $h, h^{-1}$ be $C^1$ functions such that $h(U) = V$. Then if $g$ is a nonnegative Lebesgue measurable function,

$$
\int_V g(y) \, dy = \int_U g(h(x)) |\det (Dh(x))| \, dx. \quad (11.6.16)
$$

Proof: From Corollary 11.6.3 holds for any nonnegative simple function in place of $g$. In general, let $\{s_k\}$ be an increasing sequence of simple functions which converges to $g$ pointwise. Then from the monotone convergence theorem

$$
\int_V g(y) \, dy = \lim_{k \to \infty} \int_V s_k \, dy = \lim_{k \to \infty} \int_U s_k(h(x)) |\det (Dh(x))| \, dx
$$

$$=
\int_U g(h(x)) |\det (Dh(x))| \, dx.
$$

This proves the theorem.

This is a pretty good theorem but it isn’t too hard to generalize it. In particular, it is not necessary to assume $h^{-1}$ is $C^1$.

Lemma 11.6.5 (Sard) Let $U$ be an open set in $\mathbb{R}^n$ and let $h : U \to \mathbb{R}^n$ be $C^1$. Let

$$Z \equiv \{x \in U : \det Dh(x) = 0\}.
$$

Then $m_n(h(Z)) = 0$.

Proof: Let $Z_k$ denote those points $x$ of $Z$ such that $|Dh(x)| \leq k$ and such that $|x| < k$. Let $\varepsilon > 0$ be given. For $x \in Z_k$,

$$h(x + v) = h(x) + Dh(x)v + o(v)
$$

and so whenever $r$ is small enough,

$$h(x + B(0,r)) = h(B(x,r)) \subseteq h(x) + Dh(x)B(0,r) + B(0,r\varepsilon)$$

and so whenever $r$ is small enough,
Note \( Dh(x) B(0, r) \) is contained in an \( n - 1 \) dimensional subspace of \( \mathbb{R}^n \) due to the fact \( Dh(x) \) has rank less than \( n \). Now let \( Q \) denote an orthogonal transformation preserving all distances,

\[
QQ^* = Q^* Q = I,
\]

such that

\[
QDh(x) B(0, r) \subseteq \mathbb{R}^{n-1}.
\]

Then

\[
Qh(B(x, r)) \subseteq Qh(x) + QDh(x) B(0, r) + B(0, r\varepsilon)
\]

and by translation invariance of Lebesgue measure,

\[
m_n(Qh(B(x, r))) \leq m_n(QDh(x) B(0, r) + B(0, r\varepsilon))
\]

\[
\leq (||QDh(x)|| (2r + 2r\varepsilon))^{n-1} 2r \varepsilon = C (1 + \varepsilon)^{n-1} m_n(B(0, r)) \varepsilon
\]

These balls give a Vitali cover of \( Z_k \) and so there exists a disjoint sequence of them \( \{B_i\} \), each contained in \( B(0,k) \) which covers \( Z_k \) except for a set of measure zero which is mapped by \( h \) to a set of measure zero. Therefore using Theorem 11.5.7,

\[
m_n(h(Z_k)) = m_n(h(\bigcup_{i=1}^{\infty} B_i)) \leq \sum_{i=1}^{\infty} m_n(h(B_i))
\]

\[
= \sum_{i=1}^{\infty} m_n(h(B_i)) \leq C (1 + \varepsilon)^{n-1} \varepsilon \sum_{i=1}^{\infty} m_n(B_i) \leq C (1 + \varepsilon)^{n-1} \varepsilon m_n(B(0,k))
\]

and since \( \varepsilon \) is arbitrary, this shows \( m_n(h(Z_k)) = 0 \). Now

\[
m_n(h(Z)) = \lim_{k \to \infty} m_n(h(Z_k)) = 0.
\]

This proves the lemma.

With this important lemma, here is a generalization of Theorem 11.5.4.

**Theorem 11.6.6** Let \( U \) be an open set and let \( h \) be a \( 1 - 1, C^1 \) function with values in \( \mathbb{R}^n \). Then if \( g \) is a nonnegative Lebesgue measurable function,

\[
\int_{h(U)} g(y) \, dy = \int_{U} g(h(x)) |\det(Dh(x))| \, dx. \tag{11.6.17}
\]

**Proof:** Let \( Z = \{x : \det(Dh(x)) = 0\} \). Then by the inverse function theorem, \( h^{-1} \) is \( C^1 \) on \( h(U \setminus Z) \) and \( h(U \setminus Z) \) is an open set. Therefore, from Lemma 11.6.5 and Theorem 11.6.4,

\[
\int_{h(U)} g(y) \, dy = \int_{h(U \setminus Z)} g(y) \, dy = \int_{U \setminus Z} g(h(x)) |\det(Dh(x))| \, dx
\]

\[
= \int_{U} g(h(x)) |\det(Dh(x))| \, dx.
\]

This proves the theorem.

Of course the next generalization considers the case when \( h \) is not even one to one.
11.7 Mappings Which Are Not One To One

Now suppose $h$ is only $C^1$, not necessarily one to one. For $U_+ \equiv \{x \in U : |\det Dh(x)| > 0\}$ and $Z$ the set where $|\det Dh(x)| = 0$, Lemma 11.6.5 implies $m_n(h(Z)) = 0$. For $x \in U_+$, the inverse function theorem implies there exists an open set $B_x$ such that $x \in B_x \subseteq U_+$, $h$ is one to one on $B_x$.

Let $\{B_i\}$ be a countable subset of $\{B_x\}_{x \in U_+}$ such that $U_+ = \bigcup_{i=1}^{\infty} B_i$. Let $E_1 = B_1$. If $E_1, \ldots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \bigcup_{i=1}^{k} E_i$. Thus

$$\bigcup_{i=1}^{\infty} E_i = U_+, \quad h \text{ is one to one on } E_i, \quad E_i \cap E_j = \emptyset,$$

and each $E_i$ is a Borel set contained in the open set $B_i$. Now define

$$n(y) \equiv \sum_{i=1}^{\infty} \chi_{h(E_i)}(y) + \chi_{h(Z)}(y).$$

The set, $h(E_i), h(Z)$ are measurable by Lemma 11.5.2. Thus $n(\cdot)$ is measurable.

**Lemma 11.7.1** Let $F \subseteq h(U)$ be measurable. Then

$$\int_{h(U)} n(y)X_F(y)dy = \int_U X_F(h(x))|\det Dh(x)|dx.$$

**Proof:** Using Lemma 11.6.5 and the Monotone Convergence Theorem or Fubini’s Theorem,

$$\int_{h(U)} n(y)X_F(y)dy = \int_{h(U)} \left( \sum_{i=1}^{\infty} \chi_{h(E_i)}(y) + \chi_{h(Z)}(y) \right) X_F(y)dy$$

$$= \sum_{i=1}^{\infty} \int_{h(U)} \chi_{h(E_i)}(y)X_F(y)dy$$

$$= \sum_{i=1}^{\infty} \int_{h(U) \cap h(E_i)} X_F(y)dy$$

$$= \sum_{i=1}^{\infty} \int_{h(B_i) \cap h(E_i)} X_F(y)dy$$
\[ = \sum_{i=1}^{\infty} \int_{h(B_i)} X_{h(y)}(y) X_F(y) dy \]
\[ = \sum_{i=1}^{\infty} \int_{B_i} X_{E_i}(x) X_F(h(x)) |\det D h(x)| dx \]
\[ = \sum_{i=1}^{\infty} \int_{U} X_{E_i}(x) X_F(h(x)) |\det D h(x)| dx \]
\[ = \int_{U} X_F(h(x)) |\det D h(x)| dx = \int_{U} X_F(h(x)) |\det D h(x)| dx. \]

This proves the lemma.

**Definition 11.7.2** For \( y \in h(U) \), define a function, \#, according to the formula
\[ \#(y) \equiv \text{number of elements in } h^{-1}(y). \]

Observe that
\[ \#(y) = n(y) \text{ a.e.} \quad (11.7.18) \]
because \( n(y) = \#(y) \) if \( y \notin h(Z) \), a set of measure 0. Therefore, \# is a measurable function.

**Theorem 11.7.3** Let \( g \geq 0 \), \( g \) measurable, and let \( h \) be \( C^1(U) \). Then
\[ \int_{h(U)} \#(y) g(y) dy = \int_{U} g(h(x)) |\det D h(x)| dx. \quad (11.7.19) \]

**Proof:** From **11.7.18** and Lemma **11.7.1**, \( 11.7.19 \) holds for all \( g \), a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, \( g \), with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields \( 11.7.19 \) for an arbitrary nonnegative measurable function, \( g \). This proves the theorem.

### 11.8 Lebesgue Measure And Iterated Integrals

The following is the main result.

**Theorem 11.8.1** Let \( f \geq 0 \) and suppose \( f \) is a Lebesgue measurable function defined on \( \mathbb{R}^n \) and \( \int_{\mathbb{R}^n} f dm_n < \infty \). Then
\[ \int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k. \]
This will be accomplished by Fubini’s theorem, Theorem 10.9.11 and the following lemma.

**Lemma 11.8.2** \( m_k \times m_{n-k} = m_n \) on the \( m_n \) measurable sets.

**Proof:** First of all, let \( R = \prod_{i=1}^n (a_i, b_i] \) be a measurable rectangle and let \( R_k = \prod_{i=1}^k (a_i, b_i], R_{n-k} = \prod_{i=k+1}^n (a_i, b_i] \). Then by Fubini’s theorem,

\[
\int \chi_R d(m_k \times m_{n-k}) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \chi_{R_k} \chi_{R_{n-k}} dm_k dm_{n-k}
\]

and so \( m_k \times m_{n-k} \) and \( m_n \) agree on every half open rectangle. By Lemma 11.1.2 these two measures agree on every open set. Now if \( K \) is a compact set, then \( K = \bigcap_{k=1}^{\infty} U_k \) where \( U_k \) is the open set, \( K + B(0, \frac{1}{k}) \). Another way of saying this is \( U_k \equiv \{ x : \text{dist}(x, K) < \frac{1}{k} \} \) which is obviously open because \( x \to \text{dist}(x, K) \) is a continuous function. Since \( K \) is the countable intersection of these decreasing open sets, each of which has finite measure with respect to either of the two measures, it follows that \( m_k \times m_{n-k} \) and \( m_n \) agree on all the compact sets. Now let \( E \) be a bounded Lebesgue measurable set. Then there are sets, \( H \) and \( G \) such that \( H \) is a countable union of compact sets, \( G \) a countable intersection of open sets, \( H \subseteq E \subseteq G \), and \( m_n(G \setminus H) = 0 \). Then from what was just shown about compact and open sets, the two measures agree on \( G \) and on \( H \). Therefore,

\[
m_n(H) = \frac{m_k \times m_{n-k}(H)}{m_{n-k}(G)} \leq \frac{m_k \times m_{n-k}(E)}{m_{n-k}(E)} = m_n(H)
\]

By completeness of the measure space for \( m_k \times m_{n-k} \), it follows \( E \) is \( m_k \times m_{n-k} \) measurable and

\[
\overline{m_k \times m_{n-k}}(E) = m_n(E).
\]

This proves the lemma.

You could also show that the two \( \sigma \) algebras are the same. However, this is not needed for the lemma or the theorem.

**Proof of Theorem 11.8.3** By the lemma and Fubini’s theorem, Theorem 10.9.11,

\[
\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^n} f d(\overline{m_k \times m_{n-k}}) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.
\]

**Corollary 11.8.3** Let \( f \) be a nonnegative real valued measurable function. Then

\[
\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.
\]
11.9. SPHERICAL COORDINATES IN P DIMENSIONS

Proof: Let \( S_p = \{ x \in \mathbb{R}^n : 0 \leq f(x) \leq p \} \cap B(0, p) \). Then \( \int_{\mathbb{R}^n} f X_{S_p} dm_n < \infty \). Therefore, from Theorem 11.8.1,

\[
\int_{\mathbb{R}^n} f X_{S_p} dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} X_{S_p} f dm_{n-k} dm_k.
\]

Now let \( p \to \infty \) and use the Monotone convergence theorem and the Fubini Theorem 10.9.11 on Page 298.

Not surprisingly, the following corollary follows from this.

Corollary 11.8.4 Let \( f \in L^1(\mathbb{R}^n) \) where the measure is \( m_n \). Then

\[
\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.
\]

Proof: Apply Corollary 11.8.3 to the positive and negative parts of the real and imaginary parts of \( f \).

11.9 Spherical Coordinates In \( p \) Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

\[
\begin{align*}
y_1 &= \rho \cos \theta \\
y_2 &= \rho \sin \theta
\end{align*}
\]

where \( \rho > 0 \) and \( \theta \in \mathbb{R} \). Thus these transformation equations are not one to one but they are one to one on \((0, \infty) \times [0, 2\pi)\). Here I am writing \( \rho \) in place of \( r \) to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing \( y_1 \) and \( y_2 \) instead of the more usual \( x \) and \( y \). Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.

From this picture, you see that \( y_3 = \rho \cos \phi_1 \). Also the distance between \((y_1, y_2)\) and \((0, 0)\) is \( \rho \sin (\phi_1) \). Therefore, using polar coordinates to write \((y_1, y_2)\) in terms of \( \theta \) and this distance,

\[
\begin{align*}
y_1 &= \rho \sin \phi_1 \cos \theta, \\
y_2 &= \rho \sin \phi_1 \sin \theta, \\
y_3 &= \rho \cos \phi_1.
\end{align*}
\]
where $\phi_1 \in \mathbb{R}$ and the transformations are one to one if $\phi_1$ is restricted to be in $[0, \pi]$. What was done is to replace $\rho$ with $\rho \sin \phi_1$ and then to add in $y_3 = \rho \cos \phi_1$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:

![Diagram showing spherical coordinates](image)

From this picture, you see that $y_4 = \rho \cos \phi_2$. Also the distance between $(y_1, y_2, y_3)$ and $(0, 0, 0)$ is $\rho \sin (\phi_2)$. Therefore, using polar coordinates to write $(y_1, y_2, y_3)$ in terms of $\theta, \phi_1$, and this distance,

\[
\begin{align*}
y_1 &= \rho \sin \phi_2 \sin \phi_1 \cos \theta, \\
y_2 &= \rho \sin \phi_2 \sin \phi_1 \sin \theta, \\
y_3 &= \rho \sin \phi_2 \cos \phi_1, \\
y_4 &= \rho \cos \phi_2
\end{align*}
\]

where $\phi_2 \in \mathbb{R}$ and the transformations will be one to one if $\phi_2, \phi_1 \in (0, \pi), \theta \in (0, 2\pi), \rho \in (0, \infty)$.

Continuing this way, given spherical coordinates in $\mathbb{R}^p$, to get the spherical coordinates in $\mathbb{R}^{p+1}$, you let $y_{p+1} = \rho \cos \phi_{p-1}$ and then replace every occurrence of $\rho$ with $\rho \sin \phi_{p-1}$ to obtain $y_1 \cdots y_p$ in terms of $\phi_1, \phi_2, \cdots, \phi_{p-1}, \theta$, and $\rho$.

It is always the case that $\rho$ measures the distance from the point in $\mathbb{R}^p$ to the origin in $\mathbb{R}^p$, $\mathbf{0}$. Each $\phi_i \in \mathbb{R}$ and the transformations will be one to one if each $\phi_i \in (0, \pi)$, and $\theta \in (0, 2\pi)$. Denote by $h_p(\rho, \vec{\phi}, \theta)$ the above transformation.

It can be shown using math induction and geometric reasoning that these coordinates map $\prod_{i=1}^{p-2} (0, \pi) \times (0, 2\pi) \times (0, \infty)$ one to one onto an open subset of $\mathbb{R}^p$ which is everything except for the set of measure zero $\Psi_p(N)$ where $N$ results from having some $\phi_i$ equal to 0 or $\pi$ or for $\rho = 0$ or for $\theta$ equal to either $2\pi$ or 0. Each of these are sets of Lebesgue measure zero and so their union is also a set of measure zero. You can see that $h_p\left(\prod_{i=1}^{p-2} (0, \pi) \times (0, 2\pi) \times (0, \infty)\right)$ omits the union of the coordinate axes except for maybe one of them. This is not important to the integral because it is just a set of measure zero.

**Theorem 11.9.1** Let $\mathbf{y} = h_p(\vec{\phi}, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^p$. Then letting $A = \prod_{i=1}^{p-2} (0, \pi) \times (0, 2\pi)$, it follows $h$ maps $A \times (0, \infty)$ one to one onto all of $\mathbb{R}^p$ except a set of measure zero given by $h_p(N)$ where $N$ is the set of measure zero

\[
(\bar{A} \times [0, \infty)) \setminus (A \times (0, \infty))
\]
11.9. SPHERICAL COORDINATES IN $P$ DIMENSIONS

Also $\left| \det D\mathbf{h}_p\left(\vec{\phi}, \theta, \rho\right) \right|$ will always be of the form

$$\left| \det D\mathbf{h}_p\left(\vec{\phi}, \theta, \rho\right) \right| = \rho^{p-1} \Phi\left(\vec{\phi}, \theta\right).$$ (11.9.20)

where $\Phi$ is a continuous function of $\vec{\phi}$ and $\theta$. Then if $f$ is nonnegative and Lebesgue measurable,

$$\int_{\mathbb{R}^p} f(y) \, dm_p = \int_{h_p(A)} f(y) \, dm_p = \int_A f\left(h_p\left(\vec{\phi}, \theta, \rho\right)\right) \rho^{p-1} \Phi\left(\vec{\phi}, \theta\right) \, dm_p$$ (11.9.21)

Furthermore whenever $f$ is Borel measurable and nonnegative, one can apply Fubini’s theorem and write

$$\int_{\mathbb{R}^p} f(y) \, dy = \int_{(0, \infty)} \int_A f\left(h_p\left(\vec{\phi}, \theta, \rho\right)\right) \Phi\left(\vec{\phi}, \theta\right) \, d\vec{\phi} d\theta d\rho$$ (11.9.22)

where here $d\vec{\phi} d\theta$ denotes $dm_{p-1}$ on $A$. The same formulas hold if $f \in L^1(\mathbb{R}^p)$.

**Proof:** Formula 11.9.20 is obvious from the definition of the spherical coordinates because in the matrix of the derivative, there will be a $\rho$ in $p - 1$ columns. The first claim is also clear from the definition and math induction or from the geometry of the above description. It remains to verify 11.9.21 and 11.9.22. It is clear $h_p$ maps $A \times [0, \infty)$ onto $\mathbb{R}^p$. Since $h_p$ is differentiable, it maps sets of measure zero to sets of measure zero. Then

$$\mathbb{R}^p = h_p\left(N \cup A \times (0, \infty)\right) = h_p\left(N\right) \cup h_p\left(A \times (0, \infty)\right),$$

the union of a set of measure zero with $h_p\left(A \times (0, \infty)\right)$. Therefore, from the change of variables formula,

$$\int_{\mathbb{R}^p} f(y) \, dm_p = \int_{h_p(A \times (0, \infty))} f(y) \, dm_p$$

$$= \int_{A \times (0, \infty)} f\left(h_p\left(\vec{\phi}, \theta, \rho\right)\right) \rho^{p-1} \Phi\left(\vec{\phi}, \theta\right) \, dm_p$$

which proves 11.9.21. This formula continues to hold if $f$ is in $L^1(\mathbb{R}^p)$. Finally, if $f \geq 0$ or in $L^1(\mathbb{R}^n)$ and is Borel measurable, then it is $\mathcal{F}^p$ measurable as well. Recall that $\mathcal{F}^p$ includes the smallest $\sigma$-algebra which contains products of open intervals. Hence $\mathcal{F}^p$ includes the Borel sets $\mathcal{B}(\mathbb{R}^p)$. Thus from the definition of $m_p$

$$\int_{A \times (0, \infty)} f\left(h_p\left(\vec{\phi}, \theta, \rho\right)\right) \rho^{p-1} \Phi\left(\vec{\phi}, \theta\right) \, dm_p$$

$$= \int_{(0, \infty)} \int_A f\left(h_p\left(\vec{\phi}, \theta, \rho\right)\right) \rho^{p-1} \Phi\left(\vec{\phi}, \theta\right) \, dm_{p-1} \, dm$$

$$= \int_{(0, \infty)} \rho^{p-1} \int_A f\left(h_p\left(\vec{\phi}, \theta, \rho\right)\right) \Phi\left(\vec{\phi}, \theta\right) \, dm_{p-1} \, dm$$

$^1$Actually it is only a function of the first but this is not important in what follows.
Now the claim about \( f \in L^1 \) follows routinely from considering the positive and negative parts of the real and imaginary parts of \( f \) in the usual way. ■

Note that the above equals
\[
\int_{A \times [0, \infty)} f \left( h_\rho (\tilde{\phi}, \theta, \rho) \right) \rho^{p-1} \Phi (\tilde{\phi}, \theta) \, dm_p
\]
and the iterated integral is also equal to
\[
\int_{[0, \infty)} \rho^{p-1} \int_{A} f \left( h_\rho (\tilde{\phi}, \theta, \rho) \right) \Phi (\tilde{\phi}, \theta) \, dm_{p-1} \, dm
\]
because the difference is just a set of measure zero.

**Notation 11.9.2** Often this is written differently. Note that from the spherical coordinate formulas, \( f \left( h (\tilde{\phi}, \theta, \rho) \right) = f (\rho \omega) \) where \( |\omega| = 1 \). Letting \( S^{p-1} \) denote the unit sphere, \( \{ \omega \in \mathbb{R}^p : |\omega| = 1 \} \), the inside integral in the above formula is sometimes written as
\[
\int_{S^{p-1}} f (\rho \omega) \, d\sigma
\]
where \( \sigma \) is a measure on \( S^{p-1} \). See [75] for another description of this measure. It isn’t an important issue here. Either formula or the formula
\[
\int_0^\infty \rho^{p-1} \left( \int_{S^{p-1}} f (\rho \omega) \, d\sigma \right) \, d\rho
\]
will be referred to as polar coordinates and is very useful in establishing estimates. Here \( \sigma \left( S^{p-1} \right) \equiv \int_{A} \Phi (\tilde{\phi}, \theta) \, dm_{p-1} \).

**Example 11.9.3** For what values of \( s \) is the integral \( \int_{B(0,R)} \left( 1 + |x|^2 \right)^s \, dy \) bounded independent of \( R \)? Here \( B(0,R) \) is the ball, \( \{ x \in \mathbb{R}^p : |x| \leq R \} \).

I think you can see immediately that \( s \) must be negative but exactly how negative? It turns out it depends on \( p \) and using polar coordinates, you can find just exactly what is needed. From the polar coordinates formula above,
\[
\int_{B(0,R)} \left( 1 + |x|^2 \right)^s \, dy = \int_0^R \int_{S^{p-1}} (1 + \rho^2)^s \rho^{p-1} \, d\sigma \, d\rho
\]
\[
= C_p \int_0^R (1 + \rho^2)^s \rho^{p-1} \, d\rho
\]
Now the very hard problem has been reduced to considering an easy one variable problem of finding when
\[
\int_0^R \rho^{p-1} (1 + \rho^2)^s \, d\rho
\]
is bounded independent of \( R \). You need \( 2s + (p - 1) < -1 \) so you need \( s < -p/2 \).
11.10 The Brouwer Fixed Point Theorem

This seems to be a good place to present a short proof of one of the most important of all fixed point theorems. There are many approaches to this but one of the easiest and shortest I have ever seen is the one in Dunford and Schwartz [42]. This is what is presented here. In Evans [45] there is a different proof which depends on integration theory. A good reference for an introduction to various kinds of fixed point theorems is the book by Smart [107]. This book also gives an entirely different approach to the Brouwer fixed point theorem.

The proof given here is based on the following lemma. Recall that the mixed partial derivatives of a \( C^2 \) function are equal. In the following lemma, and elsewhere, a comma followed by an index indicates the partial derivative with respect to the indicated variable. Thus, \( f_{,j} \) will mean \( \frac{\partial f}{\partial x^j} \). Also, write \( Dg \) for the Jacobian matrix which is the matrix of \( Dg \) taken with respect to the usual basis vectors in \( \mathbb{R}^n \). Recall that for \( A \) an \( n \times n \) matrix, \( \text{cof} (A)_{ij} \) is the determinant of the matrix which results from deleting the \( i \)th row and the \( j \)th column and multiplying by \( (-1)^{i+j} \). The following lemma is proved earlier. See Lemma 14.3.1.

**Lemma 11.10.1** \( \text{Let } g : U \to \mathbb{R}^n \text{ be } C^2 \text{ where } U \text{ is an open subset of } \mathbb{R}^n. \) Then

\[
\sum_{j=1}^{n} \text{cof} (Dg)_{ij,j} = 0,
\]

where here \( (Dg)_{ij} \equiv g_{,ij} \equiv \frac{\partial g_i}{\partial x_j} \). Also, \( \text{cof} (Dg)_{ij} = \frac{\partial \det(Dg)}{\partial g_{ij}} \).

To prove the Brouwer fixed point theorem, first consider a version of it valid for \( C^2 \) mappings. This is the following lemma.

**Lemma 11.10.2** \( \text{Let } B_r = B(0, r) \text{ and suppose } g \text{ is a } C^2 \text{ function defined on } \mathbb{R}^n \text{ which maps } B_r \text{ to } B_r. \text{ Then } g(x) = x \text{ for some } x \in B_r. \)

**Proof:** Suppose not. Then \( |g(x) - x| \) must be bounded away from zero on \( B_r \). Let \( a(x) \) be the larger of the two roots of the equation,

\[
|x + a(x)(x - g(x))|^2 = r^2. \tag{11.10.23}
\]

Thus

\[
a(x) = \frac{-(x, (x - g(x)))}{x - g(x)} + \frac{\sqrt{(x, (x - g(x)))^2 + (r^2 - |x|^2)(x - g(x))^2}}{|x - g(x)|^2}, \tag{11.10.24}
\]

The expression under the square root sign is always nonnegative and it follows from the formula that \( a(x) \geq 0 \). Therefore, \( (x, (x - g(x))) \geq 0 \) for all \( x \in B_r \).

The reason for this is that \( a(x) \) is the larger zero of a polynomial of the form

\[
p(z) = |x|^2 + z^2 |x - g(x)|^2 - 2z(x, x - g(x)) \text{ and from the formula above, it is}
\]
nonnegative. \(-2(x, x - g(x))\) is the slope of the tangent line to \(p(z)\) at \(z = 0.\) If \(x \neq 0,\) then \(|x|^2 > 0\) and so this slope needs to be negative for the larger of the two zeros to be positive. If \(x = 0,\) then \((x, x - g(x)) = 0.\)

Now define for \(t \in [0, 1],\)

\[
f(t, x) = x + ta(x)(x - g(x)).
\]

The important properties of \(f(t, x)\) and \(a(x)\) are that

\[
a(x) = 0 \text{ if } |x| = r. \tag{11.10.25}
\]

and

\[
|f(t, x)| = r \text{ for all } |x| = r. \tag{11.10.26}
\]

These properties follow immediately from (11.10.24) and the above observation that for \(x \in B_r,\) it follows \((x, (x - g(x))) \geq 0.\)

Also from (11.10.24), \(a\) is a \(C^2\) function near \(B_r.\) This is obvious from (11.10.24) as long as \(|x| < r.\) However, even if \(|x| = r\) it is still true. To show this, it suffices to verify the expression under the square root sign is positive. If this expression were not positive for some \(|x| = r,\) then \((x, (x - g(x))) = 0.\) Then also, since \(g(x) \neq x,\)

\[
\left|\frac{g(x) + x}{2}\right| < r
\]

and so

\[
r^2 > \left(x, \frac{g(x) + x}{2}\right) = \frac{1}{2} (x, g(x)) + \frac{r^2}{2} = \frac{|x|^2}{2} + \frac{r^2}{2} = r^2,
\]

a contradiction. Therefore, the expression under the square root in (11.10.24) is always positive near \(B_r\) and so \(a\) is a \(C^2\) function near \(B_r\) as claimed because the square root function is \(C^2\) away from zero.

Now define

\[
I(t) = \int_{B_r} \det(D_2f(t, x)) \, dx.
\]

Then

\[
I(0) = \int_{B_r} dx = m_n(B_r) > 0. \tag{11.10.27}
\]

Using the dominated convergence theorem one can differentiate \(I(t)\) as follows.

\[
I'(t) = \int_{B_r} \sum_{ij} \frac{\partial \det(D_2f(t, x))}{\partial f_{i,j}} \frac{\partial f_{i,j}}{\partial t} \, dx
\]

\[
= \int_{B_r} \sum_{ij} \text{cof}(D_2f)_{ij} \frac{\partial (a(x)(x_i - g_i(x)))}{\partial x_j} \, dx.
\]
11.10. THE BROUWER FIXED POINT THEOREM

Now from \( a(x) = 0 \) when \( |x| = r \) and so integration by parts and Lemma 14.3.1 yields

\[
I'(t) = \int_{B_r} \sum_i \text{cof} (D^2f)_{ij} \frac{\partial (a(x) (x_i - g_i(x)))}{\partial x_j} dx
= - \int_{B_r} \sum_i \text{cof} (D^2f)_{ij,j} a(x) (x_i - g_i(x)) dx = 0.
\]

Therefore, \( I(1) = I(0) \). However, from 11.10.23 it follows that for \( t = 1 \),
\[
\sum_i f_i f_i = r^2
\]
and so, \( \sum_i f_i f_i = 0 \) which implies since \(|f(1, x)| = r\) by 11.10.23 that \( \det (f_{i,j}) = \det (D^2f(1, x)) = 0 \) and so \( I(1) = 0 \), a contradiction to 11.10.27 since \( I(1) = I(0) \).

**Theorem 11.10.3** Let \( B_r \) be the above closed ball and let \( f : B_r \to B_r \) be continuous. Then there exists \( x \in B_r \) such that \( f(x) = x \).

**Proof:** Let \( f_k(x) \equiv \frac{f(x)}{1 + k\rho} \). Thus \( ||f_k - f|| < \frac{r}{1 + k} \) where
\[
||h|| \equiv \max \{|h(x)| : x \in B_r\}.
\]

Using the Weierstrass approximation theorem, there exists a polynomial \( g_k \) such that \( ||g_k - f_k|| < \frac{r}{1 + k} \). Then if \( x \in B_r \), it follows
\[
|g_k(x)| \leq |g_k(x) - f_k(x)| + |f_k(x)| < \frac{r}{1 + k} + \frac{kr}{1 + k} = r
\]
and so \( g_k \) maps \( B_r \) to \( B_r \). By Lemma 11.10.2 each of these \( g_k \) has a fixed point, \( x_k \) such that \( g_k(x_k) = x_k \). The sequence of points, \( \{x_k\} \) is contained in the compact set, \( B_r \) and so there exists a convergent subsequence still denoted by \( \{x_k\} \) which converges to a point, \( x \in B_r \). Then
\[
|f(x) - x| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(x_k)| + |f_k(x_k) - g_k(x_k)| + |x_k - x|
\]
\[
\leq \frac{r}{1 + k} + |f(x) - f(x_k)| + \frac{r}{1 + k} + |x_k - x|.
\]
Now let \( k \to \infty \) in the right side to conclude \( f(x) = x \). 

It is not surprising that the ball does not need to be centered at \( 0 \).

**Corollary 11.10.4** Let \( f : B(a, r) \to B(a, r) \) be continuous. Then there exists \( x \in \overline{B(a, r)} \) such that \( f(x) = x \).
Proof: Let \( g : B_r \to B_r \) be defined by \( g(y) = f(y + a) - a \). Then \( g \) is a continuous map from \( B_r \) to \( B_r \). Therefore, there exists \( y \in B_r \) such that \( g(y) = y \). Therefore, \( f(y + a) - a = y \) and so letting \( x = y + a \), \( f \) also has a fixed point as claimed.

11.11 The Brouwer Fixed Point Theorem Another Proof

This proof is also based on Lemma 14.3.1. I found this proof of the Brouwer fixed point theorem or one close to it in Evans [45]. It is even shorter than the proof just presented. I think it might be easier to remember also. It is also based on Lemma 14.3.1 which is stated next for convenience.

Lemma 11.11.1 Let \( g : U \to \mathbb{R}^n \) be \( C^2 \) where \( U \) is an open subset of \( \mathbb{R}^n \). Then

\[
\sum_{j=1}^{n} \text{cof} (Dg)_{ij,j} = 0,
\]

where here \( (Dg)_{ij} = g_{ij} = \frac{\partial g_i}{\partial x_j} \). Also, \( \text{cof} (Dg)_{ij} = \frac{\partial \det(Dg)}{\partial g_{ij}} \).

Definition 11.11.2 Let \( h \) be a function defined on an open set, \( U \subseteq \mathbb{R}^n \). Then \( h \in C^k(U) \) if there exists a function \( g \) defined on an open set, \( W \) containing \( U \) such that \( g = h \) on \( U \) and \( g \) is \( C^k(W) \).

Lemma 11.11.3 There does not exist \( h \in C^2\left(\overline{B(0,R)}\right) \) such that \( h : \overline{B(0,R)} \to \partial B(0,R) \) which also has the property that \( h(x) = x \) for all \( x \in \partial B(0,R) \). Such a function is called a retraction.

Proof: Suppose such an \( h \) exists. Let \( \lambda \in [0,1] \) and let \( p_\lambda(x) = x + \lambda(h(x) - x) \). This function, \( p_\lambda \), is a homotopy of the identity map and the retraction, \( h \). Let

\[
I(\lambda) = \int_{B(0,R)} \det(Dp_\lambda(x)) \, dx.
\]

Then using the dominated convergence theorem,

\[
I'(\lambda) = \int_{B(0,R)} \sum_{i,j} \frac{\partial \det(Dp_\lambda(x))}{\partial p_{\lambda ij}(x)} \frac{\partial p_{\lambda ij}(x)}{\partial \lambda} \, dx
= \int_{B(0,R)} \sum_{i} \sum_{j} \frac{\partial \det(Dp_\lambda(x))}{\partial p_{\lambda ij}} (h_i(x) - x_i)_j \, dx
= \int_{B(0,R)} \sum_{i} \sum_{j} \text{cof}(Dp_\lambda(x))_{ij} (h_i(x) - x_i)_j \, dx
\]
Now by assumption, $h_i(x) = x_i$ on $\partial B(0, R)$ and so one can integrate by parts and write
\[
I'(\lambda) = -\sum_i \int_{B(0,R)} \sum_j \text{cof} (Dp_\lambda(x))_{ij,j} (h_i(x) - x_i) \, dx = 0.
\]
Therefore, $I(\lambda)$ equals a constant. However,
\[
I(0) = m_n(B(0,R)) > 0
\]
but
\[
I(1) = \int_{B(0,1)} \det(Dh(x)) \, dm_n = \int_{\partial B(0,1)} \#(y) \, dm_n = 0
\]
because from polar coordinates or other elementary reasoning, $m_n(\partial B(0, 1)) = 0$.

The following is the Brouwer fixed point theorem for $C^2$ maps.

**Lemma 11.11.4** If $h \in C^2 \left( \overline{B(0, R)} \right)$ and $h : \overline{B(0, R)} \to \overline{B(0, R)}$, then $h$ has a fixed point, $x$ such that $h(x) = x$.

**Proof:** Suppose the lemma is not true. Then for all $x$, $|x - h(x)| \neq 0$. Then define
\[
g(x) = h(x) + \frac{x - h(x)}{|x - h(x)|} t(x)
\]
where $t(x)$ is nonnegative and is chosen such that $g(x) \in \partial B(0, R)$.

This mapping is illustrated in the following picture.

![Diagram](image-url)

If $x \to t(x)$ is $C^2$ near $\overline{B(0, R)}$, it will follow $g$ is a $C^2$ retraction onto $\partial B(0, R)$ contrary to Lemma 11.11.3. Thus $t(x)$ is the nonnegative solution to
\[
H(x, t) = |h(x)|^2 + 2 \left( h(x), \frac{x - h(x)}{|x - h(x)|} \right) t + t^2 = R^2 \tag{11.11.28}
\]
Then
\[
H_t(x, t) = 2 \left( h(x), \frac{x - h(x)}{|x - h(x)|} \right) + 2t.
\]
If this is nonzero for all \( x \) near \( B(0, R) \), it follows from the implicit function theorem that \( t \) is a \( C^2 \) function of \( x \). Then from (11.11.28)

\[
2t = -2 \left( h(x), \frac{x - h(x)}{|x - h(x)|} \right)
\]

\[
\pm \sqrt{4 \left( h(x), \frac{x - h(x)}{|x - h(x)|} \right)^2 - 4 \left( |h(x)|^2 - R^2 \right)}
\]

and so

\[
H_t(x, t) = 2t + 2 \left( h(x), \frac{x - h(x)}{|x - h(x)|} \right)
\]

\[
= \pm \sqrt{4 \left( R^2 - |h(x)|^2 \right) + 4 \left( h(x), \frac{x - h(x)}{|x - h(x)|} \right)^2}
\]

If \( |h(x)| < R \), this is nonzero. If \( |h(x)| = R \), then it is still nonzero unless

\[
(h(x), x - h(x)) = 0.
\]

But this cannot happen because the angle between \( h(x) \) and \( x - h(x) \) cannot be \( \pi/2 \). Alternatively, if the above equals zero, you would need

\[
(h(x), x) = |h(x)|^2 = R^2
\]

which cannot happen unless \( x = h(x) \) which is assumed not to happen. Therefore, \( x \rightarrow t(x) \) is \( C^2 \) near \( B(0, R) \) and so \( g(x) \) given above contradicts Lemma 11.11.3.

Now it is easy to prove the Brouwer fixed point theorem.

**Theorem 11.11.5** Let \( f : B(0, R) \rightarrow B(0, R) \) be continuous. Then \( f \) has a fixed point.

**Proof:** For \( f \) continuous, define \( \|f\| = \max \left\{ |f(x)| : x \in B(0, R) \right\} \). Let

\[
f_k(x) = \frac{1}{1 + (1/k)} f(x)
\]

Then by Weierstrass approximation theorem, there exists \( g_k \) a function in \( C^2 \left( \overline{B(0, R)} \right) \) such that

\[
\|f_k - g_k\| < R/(k + 1)
\]

Then

\[
\|g_k\| \leq \|f_k\| + \frac{R}{k + 1} \leq \frac{R}{1 + (1/k)} + \frac{R}{k + 1} = R
\]

Then \( g_k \) has a fixed point \( x_k \) and so

\[
|f(x_k) - x_k| \leq |f(x_k) - g_k(x_k)| + |g_k(x_k) - x_k|
\]

\[
= |f(x_k) - g_k(x_k)| < \frac{R}{1 + k}
\]
There is a subsequence, still called \( \{x_k\} \) which converges to \( x \in \overline{B(0, R)} \) and so, passing to a limit, one gets that
\[
|f(x) - x| \leq 0
\]
so \( f \) has a fixed point. □

### 11.12 Invariance Of Domain

This principal says that if \( f : U \subseteq \mathbb{R}^n \to f(U) \subseteq \mathbb{R}^n \) where \( U \) is open and \( f \) is one to one and continuous, then \( f(U) \) is also open. To do this, we first prove the following lemma. I found something like this on the web. I liked it a lot because it shows how the Brouwer fixed point theorem implies the invariance of domain. The other ways I know about involve degree theory or some sort of algebraic topology. I will give two proofs of the following lemma, the first being somewhat more informal than the second.

**Lemma 11.12.1** Let \( B \) be a closed ball in \( \mathbb{R}^n \) centered at \( a \) which has radius \( r \). Let \( f : B \to \mathbb{R}^n \). Then \( f(a) \) is an interior point of \( f(B) \).

**Proof:** Since \( f(B) \) is compact and \( f \) is one to one, \( f^{-1} \) is continuous on \( f(B) \). Use Tietze extension theorem on components of \( f^{-1} \) or some such thing to obtain \( g : \mathbb{R}^n \to \mathbb{R}^n \) such that \( g \) is continuous and equals \( f^{-1} \) on \( f(B) \). Then multiply by a suitable truncation function to get \( g \) uniformly continuous on \( \mathbb{R}^n \).

Suppose \( f(a) \) is not an interior point of \( f(B) \). Then there exists \( c_k \to f(a) \) but \( c_k \notin f(B) \). In the picture, let \( C_k \) be a sphere whose radius is
\[
2|c_k - f(a)|
\]

Let \( \hat{g}_k \) be \( C^1 \) and let it satisfy
\[
\|\hat{g}_k - g\|_{f(B) \cup D} = \max_{y \in f(B) \cup D} |\hat{g}_k(y) - g(y)| < \varepsilon_k
\]
\( \varepsilon_k \) is very small, \( \varepsilon_k \to 0 \). How small will be considered later. Here \( D \) is a large closed disk which contains all of the spheres \( C_k \) considered above. The idea is to have a large compact set which includes everything of interest below.
To get $\mathbf{g}_k$, you could use the Weierstrass approximation theorem, Theorem [a]. An easier way involving convolution will be presented in the next chapter. Also let $a \notin \mathbf{g}_k(C_k)$. This is no problem. $C_k$ has measure zero and so $\mathbf{g}(C_k)$ also has measure zero thanks to the assumption that $\mathbf{g}$ is $C^1$ and Lemma [b]. Therefore, you could simply add a small enough nonzero vector to $\mathbf{g}$ to preserve the above inequality of $\mathbf{g}$ and $g$ so that $\hat{g}(C_k)$ no longer contains $a$. That is, replace $g$ with $\hat{g} + a - b$ where $|a - b|$ is very small but $b \notin \mathbf{g}(C_k)$.

There is a set $\Sigma_k$ consisting of that part of $f(B)$ which is outside of the sphere $C_k$ in the picture along with the sphere $C_k$ itself. By construction, $\hat{g}_k$ misses $a$ on $C_k$. As to the other part of $\Sigma_k$, $g$ misses $a$ on this part, because $f$ is one to one and so $f^{-1}$ is also. Now we will squash the part of $f(B)$ inside $C_k$ onto $C_k$ while leaving the rest of $f(B)$ unchanged.

Let $\Phi_k$ be defined on $f(B)$

$$
\Phi_k(y) = \max \left( \frac{2|c_k - f(a)|}{|y - c_k|}, 1 \right) (y - c_k) + c_k
$$

This $\Phi_k$ squashes the part of $f(B)$ inside $C_k$ to $C_k$ and leaves the rest of $f(B)$ unchanged. Thus

$$
\Phi_k : f(B) \to f(B) \cap \{|y : |y - c_k| \geq 2|c_k - f(a)|\} \cup C_k
$$

a compact set. Now $\|\hat{g}_k \circ \Phi_k - g\|_{f(B)} \to 0$ and $g$ misses $a$ on the part of $f(B)$ outside of $C_k$. In the above, we chose $\hat{g}_k$ so close to $g$ that it also misses $a$ on the part of $f(B)$ which is outside of $C_k$. Then by construction, $\hat{g}_k$ misses $a$ on $C_k$ and so in fact $\hat{g}_k \circ \Phi_k$ misses $a$ on $f(B)$. Now consider $a + x - \hat{g}_k(\Phi_k(f(x)))$ for $x \in B$.

$$
|a + x - \hat{g}_k(\Phi_k(f(x))) - a| = |x - \hat{g}(\Phi_k(f(x)))|
$$

$$
= |g(f(x)) - \hat{g}_k(\Phi_k(f(x)))|
$$

For $f(x)$ outside of $C_k$, we could have chosen $\hat{g}_k$ such that $\|g - \hat{g}_k\|_{f(B)} < \frac{r}{2}$ and this was indeed done. When $f(x)$ is inside $C_k$, then eventually, for large $k$, both $g(f(x)), \hat{g}_k(\Phi_k(f(x)))$ are close to $g(f(a))$. To see this,

$$
\|\hat{g}_k(\Phi_k(f(x))) - g(\Phi_k(f(x)))\| + \|g(\Phi_k(f(x))) - g(f(x))\|
$$

$$
\leq \|\hat{g}_k - g\|_{f(B)} + \|g(\Phi_k(f(x))) - g(f(x))\|
$$

the last term being small for large $k$, and so for large $k$, $x \to a + x - \hat{g}_k(\Phi_k(f(x)))$ maps $B$ to $B$ and so by Brouwer fixed point theorem, it has a fixed point and hence $\hat{g}_k(\Phi_k(f(x))) = a$ contrary to what was argued above. Hence, $f(a)$ must be an interior point after all. ■

Now here is the same lemma with the details.

**Lemma 11.12.2** Let $B$ be a closed ball in $\mathbb{R}^n$ centered at $a$ which has radius $r$. Let $f : B \to \mathbb{R}^n$. Then $f(a)$ is an interior point of $f(B)$.
11.12. INVARIANCE OF DOMAIN

Proof: Since \( f(B) \) is compact and \( f \) is one to one, \( f^{-1} \) is continuous on \( f(B) \). Use Tietze extension theorem on components of \( f^{-1} \) or some such thing to obtain \( g: \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( g \) is continuous and equals \( f^{-1} \) on \( f(B) \).

Suppose \( f(a) \) is not an interior point of \( f(B) \). Then for every \( \varepsilon > 0 \) there exists \( c_{\varepsilon} \in B(f(a), \varepsilon) \setminus f(B) \). So fix \( \varepsilon \) small and refer to \( c_{\varepsilon} \) as \( c \). \( \varepsilon \) will be so small that

\[
|g(y) - g(f(a))| < \frac{r}{10} \quad \text{for} \quad y \in B(f(a), 4\varepsilon)
\]

There is \( \delta > 0 \) such that if \( |x - \hat{x}| < \delta \), then

\[
|f(x) - f(\hat{x})| < \varepsilon \quad (11.12.29)
\]

Let \( \hat{g} \) be \( C^1 \) and on \( f(B) \), let it satisfy

\[
\|\hat{g} - g\|_{f(B)} = \max_{y \in f(B)} |\hat{g}(y) - g(y)| < \min\left(\frac{r}{10}, \frac{\delta}{2}\right)
\]

To get \( \hat{g}_k \), you could use the Weierstrass approximation theorem, Theorem [7.2.4]. An easier way involving convolution will be presented in the next chapter. Also let \( a \notin \hat{g}(\partial B(c, 2\varepsilon)) \). This is no problem. \( \partial B(a, 2\varepsilon) \) has measure zero and so \( \hat{g}(\partial B(c, \varepsilon)) \) also has measure zero thanks to the assumption that \( \hat{g} \) is \( C^1 \) and Lemma [11.5.1]. Therefore, you could simply add a small enough nonzero vector to \( \hat{g} \) to preserve the above inequality of \( \hat{g} \) and \( g \) so that \( \hat{g}(\partial B(c, 2\varepsilon)) \) no longer contains \( a \). That is, replace \( \hat{g} \) with \( \hat{g} + a - b \) where \( |a - b| \) is very small but \( b \notin \hat{g}(\partial B(c, 2\varepsilon)) \). A summary of the rest of the argument is contained in the following picture in which the sphere has radius 2\( \varepsilon \).

![Diagram](image-url)

There is a set \( \Sigma \) consisting of that part of \( f(B) \) which is outside of the sphere \( C \) in the picture along with the sphere \( C \) itself. By construction, \( \hat{g} \) misses \( a \) on \( C \). As to the other part of \( \Sigma \), that \( g \) misses \( a \) on this part, follows from the assumption that \( f \) is one to one and so \( f^{-1} \) is also. Then \( \hat{g} \) missing \( a \) follows from \( \hat{g} \) being close enough to \( g \). We define a continuous mapping \( \Phi \) which maps \( f(B) \) to this set \( \Sigma \). This map squishes that part of \( f(B) \) which is inside \( C \) onto \( C \) and does nothing to the part of \( f(B) \) which is outside of \( C \). This is where \( c \) not in \( f(B) \) but close to \( f(B) \) is used. Then we argue that \( \hat{g} \circ \Phi \) is continuous and close to \( g \) and misses \( a \). It will be close to \( g \) and \( \hat{g} \) because of the above assumption that everything inside
$C$ is close to $f(a)$ and $g$ and $\hat{g}$ are continuous. This will yield an easy contradiction from a use of the Brouwer fixed point theorem.

Now let $\Phi$ be defined on $f(B)$

$$\Phi(y) = \max \left( \frac{2\varepsilon}{|y-c|}, 1 \right) (y-c) + c$$

If $|y-c| \geq 2\varepsilon$, then $\Phi(y) = y$. If $|y-c| < 2\varepsilon$, then $\Phi(y) = \frac{2\varepsilon}{|y-c|} (y-c) + c$ and so

$$|\Phi(y) - c| = 2\varepsilon \frac{|y-c|}{|y-c|} = 2\varepsilon$$

Note that this function is well defined because $c \notin f(B)$. Thus

$$\Phi : f(B) \to f(B) \cap [y : |y-c| \geq 2\varepsilon] \cup \partial B(c, 2\varepsilon) \equiv \Sigma$$

a compact set. Now the interesting thing about this set $\Sigma$ is this. For $y \in \Sigma$, $\hat{g}(y) \neq a$. Why is this? It is because by construction, $a \notin \hat{g}(\partial B(c, 2\varepsilon))$. What if $y \in f(B) \cap [y : |y-c| \geq 2\varepsilon]$, the other set in $\Sigma$? Could $\hat{g}(y) = a$? If so, then

$$|g(y) - a| \leq |g(y) - \hat{g}(y)| + |\hat{g}(y) - a| \leq \frac{\delta}{2}$$

and so by (11.12.29),

$$|y - f(a)| < \varepsilon$$

But $|y-c| \geq 2\varepsilon$ and so

$$|y - f(a)| \geq |y-c| - |c - f(a)| \geq 2\varepsilon - |c - f(a)| \geq 2\varepsilon - \varepsilon = \varepsilon$$

which contradicts the above inequality.

Therefore, $\hat{g}(y) \neq a$ for any $y \in \Sigma$. So consider $\hat{g}(\Phi(y))$ for $y \in f(B)$.

$$|\hat{g}(\Phi(y)) - g(y)| \leq |\hat{g}(\Phi(y)) - g(\Phi(y))| + |g(\Phi(y)) - g(y)|$$

$$\leq \frac{r}{10} + |g(\Phi(y)) - g(f(a)) - (g(y) - g(f(a)))|$$

(11.12.30)

This last equals 0 if $|y-c| \geq 2\varepsilon$. On the other hand, if $|y-c| < 2\varepsilon$,

$$y \in B(f(a), 2\varepsilon), \Phi(y) \in \partial B(c, 2\varepsilon)$$

so both $y, \Phi(y)$ are in $B(f(a), 4\varepsilon)$ and so this last term in (11.12.29) is no larger than $|g(\Phi(y) - g(f(a)))| + |g(y) - g(f(a))| < \frac{r}{10} + \frac{r}{10}$ and so for all $y \in f(B)$,

$$|\hat{g}(\Phi(y)) - g(y)| \leq \frac{3r}{10}$$

Now note that for $x \in B$, from what was just shown,

$$|\hat{g}(\Phi(f(x))) - x| = |\hat{g}(\Phi(f(x))) - g(f(x))| \leq \frac{3r}{10}$$
11.12. INVARIANCE OF DOMAIN

It follows that for every \( x \in B, \ a + x - \hat{g} (\Phi (f(x))) \in B \) and so by the Brouwer fixed point theorem, there is a fixed point \( x \) and hence

\[
a + x - \hat{g} (\Phi (f(x))) = x
\]

so \( \hat{g} (\Phi (f(x))) = a \) contrary to what was just shown that there is no solution to \( \hat{g} (y) = a \) for \( y \in \Sigma \).

With the lemma, it is easy to prove the invariance of domain theorem which is as follows.

**Theorem 11.12.3** Let \( U \) be an open set in \( \mathbb{R}^n \) and let \( f : U \to f(U) \subseteq \mathbb{R}^n \). Then \( f(U) \) is also an open set in \( \mathbb{R}^n \).

**Proof:** For \( a \in U \), let \( a \in B_a \subseteq U \), where \( B_a \) is a closed ball centered at \( a \). Then from Lemma 11.12.2 \( f(a) \in V_{f(a)} \) an open subset of \( f(B_a) \). Hence \( f(U) = \bigcup_{a \in U} V_{f(a)} \) which is open. \( \blacksquare \)
Chapter 12

Some Extension Theorems

12.1 Algebras

First of all, here is the definition of an algebra and theorems which tell how to recognize one when you see it. An algebra is like a $\sigma$ algebra except it is only closed with respect to finite unions.

**Definition 12.1.1** $\mathcal{A}$ is said to be an algebra of subsets of a set, $Z$ if $Z \in \mathcal{A}$, $\emptyset \in \mathcal{A}$, and when $E, F \in \mathcal{A}$, $E \cup F$ and $E \setminus F$ are both in $\mathcal{A}$.

It is important to note that if $\mathcal{A}$ is an algebra, then it is also closed under finite intersections. This is because $E \cap F = (E^C \cup F^C)^C \in \mathcal{A}$ since $E^C = Z \setminus E \in \mathcal{A}$ and $F^C = Z \setminus F \in \mathcal{A}$. Note that every $\sigma$ algebra is an algebra but not the other way around.

Something satisfying the above definition is called an algebra because union is like addition, the set difference is like subtraction and intersection is like multiplication. Furthermore, only finitely many operations are done at a time and so there is nothing like a limit involved.

How can you recognize an algebra when you see one? The answer to this question is the purpose of the following lemma.

**Lemma 12.1.2** Suppose $\mathcal{R}$ and $\mathcal{E}$ are subsets of $\mathcal{P}(Z)$ such that $\mathcal{E}$ is defined as the set of all finite disjoint unions of sets of $\mathcal{R}$. Suppose also

\begin{align*}
\emptyset, Z & \in \mathcal{R} \\
A \cap B & \in \mathcal{R} \quad \text{whenever } A, B \in \mathcal{R}, \\
A \setminus B & \in \mathcal{E} \quad \text{whenever } A, B \in \mathcal{R}.
\end{align*}

Then $\mathcal{E}$ is an algebra of sets of $Z$. 

\footnotetext{1}{Set of all subsets of $Z$}
**Proof:** Note first that if $A \in \mathcal{R}$, then $A^C \in \mathcal{E}$ because $A^C = Z \setminus A$.

Now suppose that $E_1$ and $E_2$ are in $\mathcal{E}$,

\[ E_1 = \bigcup_{i=1}^{m} R_i, \quad E_2 = \bigcup_{j=1}^{n} R_j \]

where the $R_i$ are disjoint sets in $\mathcal{R}$ and the $R_j$ are disjoint sets in $\mathcal{R}$. Then

\[ E_1 \cap E_2 = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} R_i \cap R_j \]

which is clearly an element of $\mathcal{E}$ because no two of the sets in the union can intersect and by assumption they are all in $\mathcal{R}$. Thus by induction, finite intersections of sets of $\mathcal{E}$ are in $\mathcal{E}$. Consider the difference of two elements of $\mathcal{E}$ next.

If $E = \bigcup_{i=1}^{n} R_i \in \mathcal{E}$,

\[ E^C = \bigcap_{i=1}^{n} R_i^C = \text{finite intersection of sets of } \mathcal{E} \]

which was just shown to be in $\mathcal{E}$. Now, if $E_1, E_2 \in \mathcal{E}$,

\[ E_1 \setminus E_2 = E_1 \cap E_2^C \in \mathcal{E} \]

from what was just shown about finite intersections.

Finally consider finite unions of sets of $\mathcal{E}$. Let $E_1$ and $E_2$ be sets of $\mathcal{E}$. Then

\[ E_1 \cup E_2 = (E_1 \setminus E_2) \cup E_2 \in \mathcal{E} \]

because $E_1 \setminus E_2$ consists of a finite disjoint union of sets of $\mathcal{R}$ and these sets must be disjoint from the sets of $\mathcal{R}$ whose union yields $E_2$ because $(E_1 \setminus E_2) \cap E_2 = \emptyset$. This proves the lemma.

The following corollary is particularly helpful in verifying the conditions of the above lemma.

**Corollary 12.1.3** Let $(Z_1, \mathcal{R}_1, \mathcal{E}_1)$ and $(Z_2, \mathcal{R}_2, \mathcal{E}_2)$ be as described in Lemma [12.1.2](#). Then $(Z_1 \times Z_2, \mathcal{R}, \mathcal{E})$ also satisfies the conditions of Lemma [12.1.2](#) if $\mathcal{R}$ is defined as

\[ \mathcal{R} \equiv \{ R_1 \times R_2 : R_i \in \mathcal{R}_i \} \]

and

\[ \mathcal{E} \equiv \{ \text{finite disjoint unions of sets of } \mathcal{R} \}. \]

Consequently, $\mathcal{E}$ is an algebra of sets.

**Proof:** It is clear $\emptyset, Z_1 \times Z_2 \in \mathcal{R}$. Let $A \times B$ and $C \times D$ be two elements of $\mathcal{R}$.

\[ A \times B \cap C \times D = A \cap C \times B \cap D \in \mathcal{R} \]

by assumption.

\[
A \times B \setminus (C \times D) = \\
\bigcup_{\epsilon \in \mathcal{E}_2} \bigcup_{\epsilon \in \mathcal{E}_1} \bigcup_{\epsilon \in \mathcal{R}_2} \\
A \times (B \setminus D) \cup (A \setminus C) \times (D \cap B)
\]
12.2. CARATHEODORY EXTENSION THEOREM

\[ \mathcal{E} = (A \times Q) \cup (P \times R) \]

where \( Q \in \mathcal{E}_2, P \in \mathcal{E}_1, \) and \( R \in \mathcal{R}_2. \)

Since \( A \times Q \) and \( P \times R \) do not intersect, it follows the above expression is in \( \mathcal{E} \) because each of these terms are. This proves the corollary.

12.2 Caratheodory Extension Theorem

The Caratheodory extension theorem is a fundamental result which makes possible the consideration of measures on infinite products among other things. The idea is that if a finite measure defined only on an algebra is trying to be a measure, then in fact it can be extended to a measure.

**Definition 12.2.1** Let \( \mathcal{E} \) be an algebra of sets of \( \Omega \) and let \( \mu_0 \) be a finite measure on \( \mathcal{E} \). This means \( \mu_0 \) is finitely additive and if \( E_i, E \) are sets of \( \mathcal{E} \) with the \( E_i \) disjoint and

\[ E = \bigcup_{i=1}^{\infty} E_i, \]

then

\[ \mu_0(E) = \sum_{i=1}^{\infty} \mu_0(E_i) \]

while \( \mu_0(\Omega) < \infty. \)

In this definition, \( \mu_0 \) is trying to be a measure and acts like one whenever possible. Under these conditions, \( \mu_0 \) can be extended uniquely to a complete measure, \( \mu \), defined on a \( \sigma \) algebra of sets containing \( \mathcal{E} \) such that \( \mu \) agrees with \( \mu_0 \) on \( \mathcal{E} \). The following is the main result.

**Theorem 12.2.2** Let \( \mu_0 \) be a measure on an algebra of sets, \( \mathcal{E} \), which satisfies \( \mu_0(\Omega) < \infty \). Then there exists a complete measure space \( (\Omega, \mathcal{S}, \mu) \) such that

\[ \mu(E) = \mu_0(E) \]

for all \( E \in \mathcal{E} \). Also if \( \nu \) is any such measure which agrees with \( \mu_0 \) on \( \mathcal{E} \), then \( \nu = \mu \) on \( \sigma(\mathcal{E}) \), the \( \sigma \) algebra generated by \( \mathcal{E} \).
Proof: Define an outer measure as follows.

\[ \mu(S) \equiv \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : S \subseteq \bigcup_{i=1}^{\infty} E_i, E_i \in \mathcal{E} \right\} \]

Claim 1: \( \mu \) is an outer measure.

Proof of Claim 1: Let \( S \subseteq \bigcup_{i=1}^{\infty} S_i \) and let \( S_i \subseteq \bigcup_{j=1}^{\infty} E_{ij} \), where

\[ \mu(S_i) + \frac{\varepsilon}{2^i} \geq \sum_{j=1}^{\infty} \mu(E_{ij}). \]

Then

\[ \mu(S) \leq \sum_{i} \sum_{j} \mu(E_{ij}) = \sum_{i} \left( \mu(S_i) + \frac{\varepsilon}{2^i} \right) = \sum_{i} \mu(S_i) + \varepsilon. \]

Since \( \varepsilon \) is arbitrary, this shows \( \mu \) is an outer measure as claimed.

By the Caratheodory procedure, there exists a unique \( \sigma \)-algebra, \( \mathcal{S} \), consisting of the \( \mu \)-measurable sets such that

\[ (\Omega, \mathcal{S}, \mu) \]

is a complete measure space. It remains to show \( \mu \) extends \( \mu_0 \).

Claim 2: If \( \mathcal{S} \) is the \( \sigma \)-algebra of \( \mu \)-measurable sets, \( \mathcal{S} \supseteq \mathcal{E} \) and \( \mu = \mu_0 \) on \( \mathcal{E} \).

Proof of Claim 2: First observe that if \( A \in \mathcal{E} \), then \( \mu(A) \leq \mu_0(A) \) by definition. Letting

\[ \mu(A) + \varepsilon > \sum_{i=1}^{\infty} \mu_0(E_i), \quad S = \bigcup_{i=1}^{\infty} E_i \supseteq A, \quad E_i \in \mathcal{E}, \]

it follows

\[ \mu(A) + \varepsilon > \sum_{i=1}^{\infty} \mu_0(E_i \cap A) \geq \mu_0(A) \]

since \( A = \bigcup_{i=1}^{\infty} E_i \cap A \). Therefore, \( \mu = \mu_0 \) on \( \mathcal{E} \).

Consider the assertion that \( \mathcal{E} \subseteq \mathcal{S} \). Let \( A \in \mathcal{E} \) and let \( S \subseteq \Omega \) be any set. There exist sets \( \{E_i\} \subseteq \mathcal{E} \) such that \( \bigcup_{i=1}^{\infty} E_i \supseteq S \) but

\[ \mu(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(E_i). \]

Then

\[ \mu(S) \leq \mu(S \cap A) + \mu(S \setminus A) \]
\[ \leq \mu(\bigcup_{i=1}^{\infty} E_i \setminus A) + \mu(\bigcup_{i=1}^{\infty} (E_i \cap A)) \]
\[ \leq \sum_{i=1}^{\infty} \mu(E_i \setminus A) + \sum_{i=1}^{\infty} \mu(E_i \cap A) = \sum_{i=1}^{\infty} \mu(E_i) < \mu(S) + \varepsilon. \]
12.2. CARATHEODORY EXTENSION THEOREM

Since $\varepsilon$ is arbitrary, this shows $A \in \mathcal{S}$.

This has proved the existence part of the theorem. To verify uniqueness, let

$$
\mathcal{G} \equiv \{ E \in \sigma(\mathcal{E}) : \mu(E) = \nu(E) \}.
$$

Then $\mathcal{G}$ is given to contain $\mathcal{E}$ and is obviously closed with respect to countable disjoint unions and complements. Therefore by Lemma 10.12.3, $\mathcal{G} \supseteq \sigma(\mathcal{E})$ and this proves the lemma.

The following lemma is also very significant.

**Lemma 12.2.3** Let $M$ be a metric space with the closed balls compact and suppose $\mu$ is a measure defined on the Borel sets of $M$ which is finite on compact sets. Then there exists a unique Radon measure, $\overline{\mu}$ which equals $\mu$ on the Borel sets. In particular $\mu$ must be both inner and outer regular on all Borel sets.

**Proof:** Define a positive linear functional, $\Lambda(f) = \int f \, d\mu$. Let $\overline{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all $f \in C_0(M)$,

$$
\int f \, d\mu = \int f \, d\overline{\mu}.
$$

If $V$ is an open set, let $\{f_n\}$ be a sequence of continuous functions in $C_0(M)$ which is increasing and converges to $\chi_V$ pointwise. Then applying the monotone convergence theorem,

$$
\int \chi_V \, d\mu = \mu(V) = \int \chi_V \, d\overline{\mu} = \overline{\mu}(V)
$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a,n)$ be a ball of radius $n$ and let $E$ be a Borel set contained in this ball. Then by regularity of $\overline{\mu}$ there exist sets $F, G$ such that $G$ is a countable intersection of open sets and $F$ is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\overline{\mu}(G \setminus F) = 0$. Now $\mu(G) = \overline{\mu}(G)$ and $\mu(F) = \overline{\mu}(F)$. Thus

$$
\overline{\mu}(G \setminus F) + \overline{\mu}(F) = \overline{\mu}(G)
$$

and so $\mu(G \setminus F) = \overline{\mu}(G \setminus F)$. It follows

$$
\mu(E) = \mu(F) = \overline{\mu}(F) = \overline{\mu}(G) = \overline{\mu}(E).
$$

If $E$ is an arbitrary Borel set, then

$$
\mu(E \cap B(a,n)) = \overline{\mu}(E \cap B(a,n))
$$

and letting $n \to \infty$, this yields $\mu(E) = \overline{\mu}(E)$. 
12.3 The Tychonoff Theorem

Sometimes it is necessary to consider infinite Cartesian products of topological spaces. When you have finitely many topological spaces in the product and each is compact, it can be shown that the Cartesian product is compact with the product topology. It turns out that the same thing holds for infinite products but you have to be careful how you define the topology. The first thing likely to come to mind by analogy with finite products is not the right way to do it.

First recall the Hausdorff maximal principle.

**Theorem 12.3.1 (Hausdorff maximal principle)** Let $F$ be a nonempty partially ordered set. Then there exists a maximal chain.

The main tool in the study of products of compact topological spaces is the Alexander subbasis theorem which is presented next. Recall a set is compact if every basic open cover admits a finite subcover. This was pretty easy to prove. However, there is a much smaller set of open sets called a subbasis which has this property. The proof of this result is much harder.

**Definition 12.3.2** $S \subseteq \tau$ is called a subbasis for the topology $\tau$ if the set $B$ of finite intersections of sets of $S$ is a basis for the topology, $\tau$.

**Theorem 12.3.3** Let $(X, \tau)$ be a topological space and let $S \subseteq \tau$ be a subbasis for $\tau$. Then if $H \subseteq X$, $H$ is compact if and only if every open cover of $H$ consisting entirely of sets of $S$ admits a finite subcover.

**Proof:** The only if part is obvious because the subbasic sets are themselves open.

If every basic open cover admits a finite subcover then the set in question is compact. Suppose then that $H$ is a subset of $X$ having the property that subbasic open covers admit finite subcovers. Is $H$ compact? Assume this is not so. Then what was just observed about basic covers implies there exists a basic open cover of $H, \mathcal{O}$, which admits no finite subcover. Let $F$ be defined as

$$\{\mathcal{O} : \mathcal{O} \text{ is a basic open cover of } H \text{ which admits no finite subcover}\}.$$ 

The assumption is that $F$ is nonempty. Partially order $F$ by set inclusion and use the Hausdorff maximal principle to obtain a maximal chain, $\mathcal{C}$, of such open covers and let

$$\mathcal{D} = \cup \mathcal{C}.$$ 

If $\mathcal{D}$ admits a finite subcover, then since $\mathcal{C}$ is a chain and the finite subcover has only finitely many sets, some element of $\mathcal{C}$ would also admit a finite subcover, contrary to the definition of $F$. Therefore, $\mathcal{D}$ admits no finite subcover. If $\mathcal{D}'$ properly contains $\mathcal{D}$ and $\mathcal{D}'$ is a basic open cover of $H$, then $\mathcal{D}'$ has a finite subcover of $H$ since otherwise, $\mathcal{C}$ would fail to be a maximal chain, being properly contained in $\mathcal{C} \cup \{\mathcal{D}'\}$. Every set of $\mathcal{D}$ is of the form

$$U = \cap_{i=1}^{m} B_i, \ B_i \in S.$$
because they are all basic open sets. If it is the case that for all \( U \in \mathcal{D} \) one of the \( B_i \) is found in \( \mathcal{D} \), then replace each such \( U \) with the subbasic set from \( \mathcal{D} \) containing it. But then this would be a subbasic open cover of \( H \) which by assumption would admit a finite subcover contrary to the properties of \( \mathcal{D} \). Therefore, one of the sets of \( \mathcal{D} \), denoted by \( U \), has the property that

\[
U = \cap_{i=1}^m B_i, \quad B_i \in S
\]

and no \( B_i \) is in \( \mathcal{D} \). Thus \( \mathcal{D} \cup \{B_i\} \) admits a finite subcover, for each of the above \( B_i \) because it is strictly larger than \( \mathcal{D} \). Let this finite subcover corresponding to \( B_i \) be denoted by

\[
V_i^1, \ldots, V_{m_i}^i, B_i
\]

Consider

\[
\{U, V_j^i, j = 1, \ldots, m_i, i = 1, \ldots, m\}.
\]

If \( p \in H \setminus \bigcup \{V_j^i\} \), then \( p \in B_i \) for each \( i \) and so \( p \in U \). This is therefore a finite subcover of \( \mathcal{D} \) contradicting the properties of \( \mathcal{D} \). Therefore, \( \mathcal{F} \) must be empty and this proves the theorem.

**Definition 12.3.4** Let \( I \) be a set and suppose for each \( i \in I \), \( (X_i, \tau_i) \) is a nonempty topological space. The Cartesian product of the \( X_i \), denoted by \( \prod_{i \in I} X_i \), consists of the set of all choice functions defined on \( I \) which select a single element of each \( X_i \). Thus \( f \in \prod_{i \in I} X_i \) means for every \( i \in I \), \( f(i) \in X_i \). The axiom of choice says \( \prod_{i \in I} X_i \) is nonempty. Let

\[
P_j(A) = \prod_{i \in I} B_i
\]

where \( B_i = X_i \) if \( i \neq j \) and \( B_j = A \). A subbasis for a topology on the product space consists of all sets \( P_j(A) \) where \( A \in \tau_j \). (These sets have an open set from the topology of \( X_j \) in the \( j^{th} \) slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields \( \prod_{i \in I} X_i \). Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by \( \prod \tau_i \).

**Proposition 12.3.5** The product topology is the smallest topology \( \tau \) for \( X = \prod_{i \in I} X_i \) such that each \( \pi_i \) is continuous. Here \( \pi_i \) is defined in the following manner. For \( x \in X \), \( \pi_i(x) \equiv x_i \). Thus \( \pi_i \) delivers the \( i^{th} \) entry of \( x \).

**Proof:** If each \( \pi_i \) is continuous, then for \( A \in \tau_i \), \( \pi_i^{-1}(A) \) must be in \( \tau \). However, \( \pi_i^{-1}(A) = P_j(A) \) having \( A \) in the \( i^{th} \) slot and \( X_j \) in every other. Therefore, \( \tau \) must contain the sets \( P_j(A) \). Since it must be a topology, it must also contain all finite intersections of these sets. Thus the topology \( \tau \) must contain the product topology described in the above definition. Is it any larger? No, because if it were, it would not be the smallest topology making the coordinate maps continuous, due to the observation that these coordinate maps are indeed continuous with respect to the product topology. ■
It is tempting to define a basis for a topology to be sets of the form \( \prod_{i \in I} A_i \) where \( A_i \) is open in \( X_i \). This is not the same thing at all. Note that the basis just described has at most finitely many slots filled with an open set which is not the whole space. The thing just mentioned in which every slot may be filled by a proper open set is called the box topology and there exist people who are interested in it.

The Alexander subbasis theorem is used to prove the Tychonoff theorem which says that if each \( X_i \) is a compact topological space, then in the product topology, \( \prod_{i \in I} X_i \) is also compact.

**Theorem 12.3.6** If \( (X_i, \tau_i) \) is compact, then so is \( (\prod_{i \in I} X_i, \tau) \) where \( \tau \) is the product topology.

**Proof:** By the Alexander subbasis theorem, the theorem will be proved if every subbasic open cover admits a finite subcover. Therefore, let \( O \) be a subbasic open cover of \( X \equiv \prod_{i \in I} X_i \). Let

\[
O_j = \{ Q \in O : \pi_i Q = X_i \text{ for } i \neq j \}
\]

\[
\pi_j O_j = \{ \pi_j Q : Q \in O_j \}
\]

Thus \( O_j \) are those sets of \( O \) which might have a proper open subset of \( X_j \) in the \( j^{th} \) position. If each \( \pi_j O_j \) fails to cover \( X_j \), then there exists

\[
f \in \prod_{j \in I} X_j \setminus \cup \pi_j O_j
\]

Now \( f \) is contained in some open set from \( O \) which must be in some \( O_j \). Hence \( \pi_j f = f(j) \in \cup \pi_j O_j \) but this does not happen. Hence for some \( j, \pi_j O_j \) must cover \( X_j \).

\[
X_j = \cup \pi_j O_j
\]

and so by compactness of \( X_j \), there exist \( A_1, \ldots, A_m \), sets in \( \tau_j \) such that \( X_j \subseteq \cup_{k=1}^m A_k \) and letting \( \pi_j U_k = A_k \) for \( U_k \in O_j \), \( \{U_k\}^m_{k=1} \) covers \( \prod_{i \in I} X_i \). By the Alexander subbasis theorem this proves \( \prod_{i \in I} X_i \) is compact. \( \blacksquare \)

### 12.4 Kolmogorov Extension Theorem

Let a subbasis for \( [-\infty, \infty] \) be sets of the form \( [-\infty, a) \) and \( (a, \infty] \). Thus with this subbasis, \( [-\infty, \infty] \) is a compact Hausdorff space. Also let \( M_t \equiv [-\infty, \infty]^{n_t} \) where \( n_t \) is a positive integer and endow this product with the product topology so that \( M_t \) is also a compact Hausdorff space.

I will denote a totally ordered index set, (Like \( \mathbb{R} \)) and the interest will be in building a measure on the product space, \( \prod_{t \in I} M_t \). By the well ordering principle, you can always put an order on any index set so this order is no restriction, but we do not insist on a well order and in fact, index sets of great interest are \( \mathbb{R} \) or \( [0, \infty) \). Also for \( X \) a topological space, \( \mathcal{B}(X) \) will denote the Borel sets.
12.4. KOLMOGOROV EXTENSION THEOREM

**Notation 12.4.1** The symbol $J$ will denote a finite subset of $I, J = (t_1, \cdots, t_n)$, the $t_i$ taken in order. $E_J$ will denote a set which has a set $E_t$ of $B(M_t)$ in the $t^{th}$ position for $t \in J$ and for $t \notin J$, the set in the $t^{th}$ position will be $M_t$. $K_J$ will denote a set which has a compact set in the $t^{th}$ position for $t \in J$ and for $t \notin J$, the set in the $t^{th}$ position will be $M_t$. Thus $K_J$ is compact in the product topology of $\Omega \equiv \prod_{t \in I} M_t$. Also denote by $R_J$ the sets $E_J$ and $R$ the union of the $R_J$. Let $E_J$ denote finite disjoint unions of sets of $R_J$ and let $E$ denote finite disjoint unions of sets of $R$. Thus if $F$ is a set of $E$, there exists $J$ such that $F$ is a finite disjoint union of sets of $R_J$. For $F \in \Omega$, denote by $\pi_J(F)$ the set $\prod_{t \in J} F_t$ where $F = \prod_{t \in I} F_t$.

**Lemma 12.4.2** The sets, $E, E_J$ defined above form an algebra of sets of $\prod_{t \in I} M_t$.

**Proof:** First consider $R_J$. If $A, B \in R_J$, then $A \cap B \in R_J$ also. Is $A \setminus B$ a finite disjoint union of sets of $R_J$? It suffices to verify that $\pi_J(A \setminus B)$ is a finite disjoint union of $\pi_J(R_J)$. Let $|J|$ denote the number of indices in $J$. If $|J| = 1$, then it is obvious that $\pi_J(A \setminus B)$ is a finite disjoint union of sets of $\pi_J(R_J)$. In fact, letting $J = (t)$ and the $t^{th}$ entry of $A$ is $A$ and the $t^{th}$ entry of $B$ is $B$, then the $t^{th}$ entry of $A \setminus B$ is $A \setminus B$, a Borel set of $M_t$, a finite disjoint union of Borel sets of $M_t$.

Suppose then that for $A, B$ sets of $R_J$, $\pi_J(A \setminus B)$ is a finite disjoint union of sets of $\pi_J(R_J)$ for $|J| \leq n$, and consider $J = (t_1, \cdots, t_n, t_{n+1})$. Let the $t_i^{th}$ entry of $A$ and $B$ be respectively $A_i$ and $B_i$. It follows that $\pi_J(A \setminus B)$ has the following in the entries for $J$

$$(A_1 \times A_2 \times \cdots \times A_n \times A_{n+1}) \setminus (B_1 \times B_2 \times \cdots \times B_n \times B_{n+1})$$

Letting $A$ represent $A_1 \times A_2 \times \cdots \times A_n$ and $B$ represent $B_1 \times B_2 \times \cdots \times B_n$, this is of the form

$$A \times (A_{n+1} \setminus B_{n+1}) \cup (A \setminus B) \times (A_{n+1} \cap B_{n+1})$$

By induction, $(A \setminus B)$ is the finite disjoint union of sets of $R(t_1, \cdots, t_n)$. Therefore, the above is the finite disjoint union of sets of $R_J$. It follows that $E_J$ is an algebra.

Now suppose $A, B \in R$. Then for some finite set $J$, both are in $R_J$. Then from what was just shown,

$$A \setminus B \in E_J \subseteq E, A \cap B \in R.$$

By Lemma 10.10.2 on page 301 this shows $E$ is an algebra. ■

With this preparation, here is the Kolmogorov extension theorem. In the statement and proof of the theorem, $F_t, G_t$, and $E_t$ will denote Borel sets. Any list of indices from $I$ will always be assumed to be taken in order. Thus, if $J \subseteq I$ and $J = (t_1, \cdots, t_n)$, it will always be assumed $t_1 < t_2 < \cdots < t_n$.

**Theorem 12.4.3** For each finite set

$$J = (t_1, \cdots, t_n) \subseteq I,$$
suppose there exists a Borel probability measure, \( \nu_J = \nu_{t_1 \cdots t_n} \) defined on the Borel sets of \( \prod_{t \in J} M_t \) such that the following consistency condition holds. If

\[
(t_1, \cdots, t_n) \subseteq (s_1, \cdots, s_p),
\]

then

\[
\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p} (G_{s_1} \times \cdots \times G_{s_p}) \tag{12.4.1}
\]

where if \( s_i = t_j \), then \( G_{s_i} = F_{t_j} \) and if \( s_i \) is not equal to any of the indices, \( t_k \), then \( G_{s_i} = M_{s_i} \). Then for \( \mathcal{E} \) defined in Definition 12.4.1, there exists a probability measure, \( P \) and a \( \sigma \)-algebra \( \mathcal{F} = \sigma(\mathcal{E}) \) such that

\[
\left( \prod_{t \in I} M_t, P, \mathcal{F} \right)
\]

is a probability space. Also there exist measurable functions, \( X_s : \prod_{t \in I} M_t \to M_s \) defined as

\[
X_s \equiv x_s
\]

for each \( s \in I \) such that for each \( (t_1 \cdots t_n) \subseteq I \),

\[
\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = P ([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}])
\]

\[
= P \left( (X_{t_1}, \cdots, X_{t_n}) \in \prod_{j=1}^n F_{t_j} \right) = P \left( \prod_{t \in I} F_t \right) \tag{12.4.2}
\]

where \( F_t = M_t \) for every \( t \notin \{t_1 \cdots t_n\} \) and \( F_{t_k} \) is a Borel set. Also if \( f \) is a non-negative function of finitely many variables, \( x_{t_1}, \cdots, x_{t_n} \), measurable with respect to \( \mathcal{B} \left( \prod_{j=1}^n M_{t_j} \right) \), then \( f \) is also measurable with respect to \( \mathcal{F} \) and

\[
\int_{M_{t_1} \times \cdots \times M_{t_n}} f (x_{t_1}, \cdots, x_{t_n}) \, d\nu_{t_1 \cdots t_n}
\]

\[
= \int_{\prod_{t \in I} M_t} f (x_{t_1}, \cdots, x_{t_n}) \, dP \tag{12.4.3}
\]

**Proof:** Let \( \mathcal{E} \) be the algebra of sets defined in Definition 12.4.1. I want to define a measure on \( \mathcal{E} \). For \( F \in \mathcal{E} \), there exists \( J \) such that \( F \) is the finite disjoint unions of sets of \( \mathcal{R}_J \). Define

\[
P_0 (F) \equiv \nu_J (\pi_J (F))
\]

Then \( P_0 \) is well defined because of the consistency condition on the measures \( \nu_J \). \( P_0 \) is clearly finitely additive because the \( \nu_J \) are measures and one can pick \( J \) as large as desired to include all \( t \) where there may be something other than \( M_t \). Also, from the definition,

\[
P_0 (\Omega) = P_0 \left( \prod_{t \in I} M_t \right) = \nu_{t_1} (M_{t_1}) = 1.
\]
Next I will show \( P_0 \) is a finite measure on \( \mathcal{E} \). After this it is only a matter of using the Caratheodory extension theorem to get the existence of the desired probability measure \( P \).

**Claim:** Suppose \( E^n \) is in \( \mathcal{E} \) and suppose \( E^n \downarrow \emptyset \). Then \( P_0 \left( E^n \right) \downarrow 0 \).

**Proof of the claim:** If not, there exists a sequence such that although \( E^n \downarrow \emptyset \), \( P_0 \left( E^n \right) \downarrow \varepsilon > 0 \). Let \( E^n \in \mathcal{E}_{J_n} \). Thus it is a finite disjoint union of sets of \( \mathcal{R}_{J_n} \). By regularity of the measures \( \nu_{J_n} \), there exists a compact set \( K_{J_n} \subseteq E^n \) such that

\[
\nu_{J_n} \left( \pi_{J_n} \left( K_{J_n} \right) \right) + \frac{\varepsilon}{2^{n+2}} > \nu_{J_n} \left( \pi_{J_n} \left( E^n \right) \right)
\]

Thus

\[
P_0 \left( K_{J_n} \right) + \frac{\varepsilon}{2^{n+2}} = \nu_{J_n} \left( \pi_{J_n} \left( K_{J_n} \right) \right) + \frac{\varepsilon}{2^{n+2}} > \nu_{J_n} \left( \pi_{J_n} \left( E^n \right) \right) \equiv P_0 \left( E^n \right)
\]

The interesting thing about these \( K_{J_n} \) is: they have the finite intersection property. Here is why.

\[
\varepsilon \leq P_0 \left( \bigcap_{k=1}^{m} K_{J_k} \right) + P_0 \left( E^n \setminus \bigcap_{k=1}^{m} K_{J_k} \right) \\
\leq P_0 \left( \bigcap_{k=1}^{m} K_{J_k} \right) + P_0 \left( \bigcup_{k=1}^{m} E^k \setminus K_{J_k} \right) \\
< P_0 \left( \bigcap_{k=1}^{m} K_{J_k} \right) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+2}} < P_0 \left( \bigcap_{k=1}^{m} K_{J_k} \right) + \varepsilon / 2,
\]

and so \( P_0 \left( \bigcap_{k=1}^{m} K_{J_k} \right) > \varepsilon / 2 \). Now this yields a contradiction, because this finite intersection property implies the intersection of all the \( K_{J_k} \) is nonempty, contradicting \( E^n \downarrow \emptyset \) since each \( K_{J_n} \) is contained in \( E^n \).

With the claim, it follows \( P_0 \) is a measure on \( \mathcal{E} \). Here is why: If \( E = \bigcup_{k=1}^{\infty} E^k \) where \( E, E^k \in \mathcal{E} \), then \( E \setminus \bigcup_{k=1}^{\infty} E^k \downarrow \emptyset \) and so

\[
P_0 \left( \bigcup_{k=1}^{\infty} E^k \right) \rightarrow P_0 \left( E \right).
\]

Hence if the \( E_k \) are disjoint, \( P_0 \left( \bigcup_{k=1}^{\infty} E^k \right) = \sum_{k=1}^{\infty} P_0 \left( E_k \right) \rightarrow P_0 \left( E \right) \). Thus for disjoint \( E_k \) having \( \bigcup_k E_k = E \in \mathcal{E} \),

\[
P_0 \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} P_0 \left( E_k \right).
\]

Now to conclude the proof, apply the Caratheodory extension theorem to obtain \( P \) a probability measure which extends \( P_0 \) to a \( \sigma \) algebra which contains \( \sigma (\mathcal{E}) \) the sigma algebra generated by \( \mathcal{E} \) with \( P = P_0 \) on \( \mathcal{E} \). Thus for \( E_J \in \mathcal{E} \), \( P \left( E_J \right) = P_0 \left( E_J \right) = \nu_J \left( P_J E_J \right) \).

Next, let \( \left( \prod_{t \in T} M_t, \mathcal{F}, P \right) \) be the probability space and for \( x \in \prod_{t \in T} M_t \) let \( X_t \left( x \right) = x_t \), the \( t^{th} \) entry of \( x \). It follows \( X_t \) is measurable (also continuous) because if \( U \) is open in \( M_t \), then \( X_t^{-1} \left( U \right) \) has a \( U \) in the \( t^{th} \) slot and \( M_s \) everywhere else for \( s \neq t \). Thus inverse images of open sets are measurable. Also, letting \( J \) be a finite
subset of \( I \) and for \( J = (t_1, \cdots, t_n) \), and \( F_{t_1}, \cdots, F_{t_n} \) Borel sets in \( M_{t_1} \cdots M_{t_n} \) respectively, it follows \( F_J \), where \( F_J \) has \( F_{t_i} \) in the \( t_i \) entry, is in \( \mathcal{E} \) and therefore,

\[
P ([X_{t_1} \in F_{t_1}] \cap [X_{t_2} \in F_{t_2}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]) = \nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n})
\]

Finally consider the claim about the integrals. Suppose \( f (x_{t_1}, \cdots, x_{t_n}) = \mathcal{X}_F \) where \( F \) is a Borel set of \( \prod_{t \in J} M_t \) where \( J = (t_1, \cdots, t_n) \). To begin with suppose

\[
F = F_{t_1} \times \cdots \times F_{t_n} \tag{12.4.4}
\]

where each \( F_{t_j} \) is in \( \mathcal{B} (M_{t_j}) \). Then

\[
\int_{M_{t_1} \times \cdots \times M_{t_n}} \mathcal{X}_F (x_{t_1}, \cdots, x_{t_n}) \, d\nu_{t_1 \cdots t_n} = \nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n})
\]

where \( F_t = M_t \) if \( t \notin J \). Let \( K \) denote sets, \( F \) of the sort in 12.4.3. It is clearly a \( \pi \) system. Now let \( \mathcal{G} \) denote those sets \( F \) in \( \mathcal{B} (\prod_{t \in J} M_t) \) such that 12.4.3 holds. Thus \( \mathcal{G} \supset K \). It is clear that \( \mathcal{G} \) is closed with respect to countable disjoint unions and complements. Hence \( \mathcal{G} \supset \sigma (K) \) but \( \sigma (K) = \mathcal{B} (\prod_{t \in J} M_t) \) because every open set in \( \prod_{t \in J} M_t \) is the countable union of rectangles like 12.4.3 in which each \( F_{t_i} \) is open. Therefore, 12.4.3 holds for every \( F \in \mathcal{B} (\prod_{t \in J} M_t) \).

Passing to simple functions and then using the monotone convergence theorem yields the final claim of the theorem.

The next task is to consider the case where \( M_t = (-\infty, \infty)^n_t \). To consider this case, here is a lemma which will allow this case to be deduced from the above theorem. In this lemma, \( M_t' \equiv [-\infty, \infty]^n_t \).

**Lemma 12.4.4** Let \( J \) be a finite subset of \( I \). Then \( U \) is a Borel set in \( \prod_{t \in J} M_t \) if and only if there exists a Borel set, \( U' \) in \( \prod_{t \in J} M_t' \) such that \( U = U' \cap \prod_{t \in J} M_t \).

**Proof:** A subbasis for the topology for \( [-\infty, \infty] \) is sets of the form \( [-\infty, a) \) and \( (a, \infty] \). Also a subbasis for the topology of \( [-\infty, \infty]^n \) is sets of the form \( \prod_{i=1}^n [-\infty, a_i] \) and \( \prod_{i=1}^n (a_i, \infty] \). Similarly, a subbasis for the topology of \( (-\infty, \infty)^n \) consists of sets of the form \( \prod_{i=1}^n (-\infty, a_i] \) and \( \prod_{i=1}^n [a_i, \infty) \). Thus the basic open sets of \( \prod_{t \in J} M_t \) are of the form \( U' \cap \prod_{t \in J} M_t \) where \( U' \) is a basic open set in \( \prod_{t \in J} M_t' \). It follows the open sets of \( \prod_{t \in J} M_t \) are of the form \( U' \cap \prod_{t \in J} M_t \) where \( U' \) is open in \( \prod_{t \in J} M_t' \). Let \( F \) denote those Borel sets of \( \prod_{t \in J} M_t \) which are of the
form $U' \cap \prod_{t \in J} M_t$ for $U'$ a Borel set in $\prod_{t \in J} M'_t$. Then as just shown, $F$ contains the $\pi$ system of open sets in $\prod_{t \in J} M_t$. Let $G'$ denote those Borel sets of $\prod_{t \in J} M_t$ which are of the desired form. It is clearly closed with respect to complements and countable disjoint unions. Hence $G'$ equals the Borel sets of $\prod_{t \in J} M_t$. 

Maybe this diagram will help to keep the argument straight.

Now here is the Kolmogorov extension theorem in the desired form. However, a more general version is given later where $M_t$ is just a Polish space (complete separable metric space).

**Theorem 12.4.5 (Kolmogorov extension theorem)** For each finite set

$$J = (t_1, \cdots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \cdots t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ for $M_t = \mathbb{R}^{n_t}$ for $n_t$ an integer, such that the following consistency condition holds. If

$$(t_1, \cdots , t_n) \subseteq (s_1, \cdots , s_p),$$

then

$$\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p} (G_{s_1} \times \cdots \times G_{s_p})$$

(12.4.6)

where if $s_i = t_j$, then $G_{s_i} = F_{t_j}$ and if $s_i$ is not equal to any of the indices, $t_k$, then $G_{s_i} = M_{s_i}$. Then for $E$ defined as in Definition 12.4.1, adjusted so that $\pm \infty$ never appears as any endpoint of any interval, there exists a probability measure, $P$ and a $\sigma$ algebra $F = \sigma(E)$ such that

$$(\prod_{t \in I} M_t, P, F)$$

is a probability space. Also there exist measurable functions, $X_s : \prod_{t \in I} M_t \rightarrow M_s$ defined as

$$X_s x \equiv x_s$$

for each $s \in I$ such that for each $(t_1 \cdots t_n) \subseteq I$,

$$\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = P ([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}])$$

$$= P \left( (X_{t_1}, \cdots , X_{t_n}) \in \prod_{j=1}^n F_{t_j} \right) = P \left( \prod_{t \in I} F_t \right)$$

(12.4.7)
where \( F_t = M_t \) for every \( t \not\in \{ t_1, \cdots, t_n \} \) and \( F_{t_i} \) is a Borel set. Also if \( f \) is a non-negative function of finitely many variables, \( x_{t_1}, \cdots, x_{t_n} \), measurable with respect to \( \mathcal{B} \left( \prod_{j=1}^{n} M_{t_j} \right) \), then \( f \) is also measurable with respect to \( \mathcal{F} \) and

\[
\int_{M_{t_1} \times \cdots \times M_{t_n}} f(x_{t_1}, \cdots, x_{t_n}) \, d\nu_{t_1 \cdots t_n} = \int_{\prod_{t \in I} M_t} f(x_{t_1}, \cdots, x_{t_n}) \, dP \tag{12.4.8}
\]

**Proof:** Using Lemma \[12.4.4\], extend each measure, \( \nu_J \) to \( M'_t \), defined by adding in the points \( \pm \infty \) at the ends, by letting \( \nu_J(E) \equiv \nu_J(E \cap \prod_{t \in I} M_t) \) for all \( E \in \mathcal{B} \left( \prod_{t \in I} M'_t \right) \). Then apply Theorem \[12.4.3\] to these extended measures and use the definition of the extensions of each \( \nu_J \) to replace each \( M'_t \) with \( M_t \) everywhere it occurs.

As a special case, you can obtain a version of product measure for possibly infinitely many factors. Suppose in the context of the above theorem that \( \nu_t \) is a probability measure defined on the Borel sets of \( M_t \equiv \mathbb{R}^{n_t} \) for \( n_t \) a positive integer, and let the measures, \( \nu_{t_1 \cdots t_n} \) be defined on the Borel sets of \( \prod_{i=1}^{n} M_{t_i} \) by

\[
\nu_{t_1 \cdots t_n}(E) \equiv \text{product measure } (\nu_{t_1} \times \cdots \times \nu_{t_n})(E).
\]

Then these measures satisfy the necessary consistency condition and so the Kolmogorov extension theorem given above can be applied to obtain a measure \( P \) defined on a \( (\prod_{t \in I} M_t, \mathcal{F}) \) and measurable functions \( X_s : \prod_{t \in I} M_t \to M_s \) such that for \( F_{t_i} \), a Borel set in \( M_{t_i} \),

\[
P \left( (X_{t_1}, \cdots, X_{t_n}) \in \prod_{i=1}^{n} F_{t_i} \right) = \nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) = \nu_{t_1}(F_{t_1}) \cdots \nu_{t_n}(F_{t_n}) \tag{12.4.9}
\]

In particular, \( P(X_t \in F_t) = \nu_t(F_t) \). Then \( P \) in the resulting probability space,

\[
\left( \prod_{t \in I} M_t, \mathcal{F}, P \right)
\]

will be denoted as \( \prod_{t \in I} \nu_t \). This proves the following theorem which describes an infinite product measure.

**Theorem 12.4.6** Let \( M_t \) for \( t \in I \) be given as in Theorem \[12.4.5\] and let \( \nu_t \) be a Borel probability measure defined on the Borel sets of \( M_t \). Then there exists a measure \( P \) and a \( \sigma \) algebra \( \mathcal{F} = \sigma(\mathcal{E}) \) where \( \mathcal{E} \) is given in Definition \[12.4.7\] such that \( (\prod_{t \in I} M_t, \mathcal{F}, P) \) is a probability space satisfying \[12.4.8\] whenever each \( F_{t_i} \) is a Borel set of \( M_{t_i} \). This probability measure is sometimes denoted as \( \prod_{t \in I} \nu_t \).
12.5 Exercises

1. Let \((X, S, \mu)\) and \((Y, F, \lambda)\) be two finite measure spaces. A subset of \(X \times Y\) is called a measurable rectangle if it is of the form \(A \times B\) where \(A \in S\) and \(B \in F\). A subset of \(X \times Y\) is called an elementary set if it is a finite disjoint union of measurable rectangles. Denote this set of functions by \(E\). Show that \(E\) is an algebra of sets.

2. For \(A \in \sigma(E)\), the smallest \(\sigma\) algebra containing \(E\), show that \(x \rightarrow \chi_A(x,y)\) is \(\mu\) measurable and that

\[
y \rightarrow \int \chi_A(x,y) \, d\mu
\]

is \(\lambda\) measurable. Show similar assertions hold for \(y \rightarrow \chi_A(x,y)\) and

\[
x \rightarrow \int \chi_A(x,y) \, d\lambda
\]

and that

\[
\int \int \chi_A(x,y) \, d\mu \, d\lambda = \int \int \chi_A(x,y) \, d\lambda \, d\mu. \tag{12.5.10}
\]

Hint: Let \(M \equiv \{ A \in \sigma(E) : \text{[12.5.10] holds}\}\) along with all relevant measurability assertions. Show \(M\) contains \(E\) and is a monotone class. Then apply the Theorem 10.10.5.

3. For \(A \in \sigma(E)\) define \((\mu \times \lambda)(A) \equiv \int \int \chi_A(x,y) \, d\mu \, d\lambda\). Show that \((\mu \times \lambda)\) is a measure on \(\sigma(E)\) and that whenever \(f \geq 0\) is measurable with respect to \(\sigma(E)\),

\[
\int_{X \times Y} f \, (\mu \times \lambda) = \int \int f(x,y) \, d\mu \, d\lambda = \int \int f(x,y) \, d\lambda \, d\mu.
\]

This is a common approach to Fubini's theorem.

4. Generalize the above version of Fubini's theorem to the case where the measure spaces are only \(\sigma\) finite.

5. Suppose now that \(\mu\) and \(\lambda\) are both complete \(\sigma\) finite measures. Let \((\mu \times \lambda)\) denote the completion of this measure. Let the larger measure space be \((X \times Y, \sigma(E), (\mu \times \lambda))\). Thus if \(E \in \sigma(E)\), it follows there exists a set \(A \in \sigma(E)\) such that \(E \cup N = A\) where \((\mu \times \lambda)(N) = 0\). Now argue that for \(\lambda\) a.e. \(y, x \rightarrow \chi_N(x,y)\) is measurable because it is equal to zero \(\mu\) a.e. and \(\mu\) is complete. Therefore,

\[
\int \int \chi_N(x,y) \, d\mu \, d\lambda
\]
makes sense and equals zero. Use to argue that for $\lambda$ a.e. $y, x \to \mathcal{X}_E(x, y)$ is $\mu$ measurable and equals $\int \mathcal{X}_A(x, y) \, d\mu$. Then by completeness of $\lambda, y \to \int \mathcal{X}_E(x, y) \, d\mu$ is $\lambda$ measurable and
\[
\int \int \mathcal{X}_A(x, y) \, d\mu d\lambda = \int \int \mathcal{X}_E(x, y) \, d\mu d\lambda = (\mu \times \lambda)(E).
\]
Similarly
\[
\int \int \mathcal{X}_E(x, y) \, d\lambda d\mu = (\mu \times \lambda)(E).
\]
Use this to give a generalization of the above Fubini theorem. Prove that if $f$ is measurable with respect to the $\sigma$ algebra, $\sigma(E)$ and nonnegative, then
\[
\int_{X \times Y} f \, d(\mu \times \lambda) = \int \int f(x, y) \, d\mu d\lambda = \int \int f(x, y) \, d\lambda d\mu
\]
where the iterated integrals make sense.
Chapter 13

The $L^p$ Spaces

13.1 Basic Inequalities And Properties

One of the main applications of the Lebesgue integral is to the study of various sorts of functions space. These are vector spaces whose elements are functions of various types. One of the most important examples of a function space is the space of measurable functions whose absolute values are $p^{th}$ power integrable where $p \geq 1$. These spaces, referred to as $L^p$ spaces, are very useful in applications. In the chapter $(\Omega, S, \mu)$ will be a measure space.

**Definition 13.1.1** Let $1 \leq p < \infty$. Define

$$L^p(\Omega) \equiv \{ f : f \text{ is measurable and } \int_{\Omega} |f(\omega)|^p d\mu < \infty \}$$

In terms of the distribution function,

$$L^p (\Omega) = \{ f : f \text{ is measurable and } \int_{0}^{\infty} pt^{p-1} \mu (|f| > t) \, dt < \infty \}$$

For each $p > 1$ define $q$ by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Often one uses $p'$ instead of $q$ in this context.

$L^p (\Omega)$ is a vector space and has a norm. This is similar to the situation for $\mathbb{R}^n$ but the proof requires the following fundamental inequality.

**Theorem 13.1.2** (Holder’s inequality) If $f$ and $g$ are measurable functions, then if $p > 1$,

$$\int |f| |g| d\mu \leq \left( \int |f|^p d\mu \right)^{\frac{1}{p}} \left( \int |g|^q d\mu \right)^{\frac{1}{q}}. \quad (13.1.1)$$

**Proof:** First here is a proof of Young’s inequality.
Lemma 13.1.3 If $p > 1$, and $0 \leq a, b$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Proof: Consider the following picture:

\[
\begin{align*}
    & x \\
    & b \\
    & x = t^{p-1} \\
    & t = x^{q-1} \\
    & a \\
    & t
\end{align*}
\]

From this picture, the sum of the area between the $x$ axis and the curve added to the area between the $t$ axis and the curve is at least as large as $ab$. Using beginning calculus, this is equivalent to the following inequality.

\[
ab \leq \int_0^a t^{p-1} dt + \int_0^b x^{q-1} dx = \frac{a^p}{p} + \frac{b^q}{q}.
\]

The above picture represents the situation which occurs when $p > 2$ because the graph of the function is concave up. If $2 \geq p > 1$ the graph would be concave down or a straight line. You should verify that the same argument holds in these cases just as well. In fact, the only thing which matters in the above inequality is that the function $x = t^{p-1}$ be strictly increasing.

Note equality occurs when $a^p = b^q$.

Here is an alternate proof.

Lemma 13.1.4 For $a, b \geq 0$,

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}
\]

and equality occurs when if and only if $a^p = b^q$.

Proof: If $b = 0$, the inequality is obvious. Fix $b > 0$ and consider

\[
f(a) \equiv \frac{a^p}{p} + \frac{b^q}{q} - ab.
\]

Then $f'(a) = a^{p-1} - b$. This is negative when $a < b^{1/(p-1)}$ and is positive when $a > b^{1/(p-1)}$. Therefore, $f$ has a minimum when $a = b^{1/(p-1)}$. In other words, when $a^p = b^{p/(p-1)} = b^q$ since $1/p + 1/q = 1$. Thus the minimum value of $f$ is

\[
\frac{b^q}{p} + \frac{b^q}{q} - b^{1/(p-1)}b = b^q - b^q = 0.
\]

It follows $f \geq 0$ and this yields the desired inequality.
Proof of Holder’s inequality: If either $\int |f|^p d\mu$ or $\int |g|^p d\mu$ equals $\infty$, the inequality is obviously valid because $\infty \geq$ anything. If either $\int |f|^p d\mu$ or $\int |g|^p d\mu$ equals 0, then $f = 0$ a.e. or that $g = 0$ a.e. and so in this case the left side of the inequality equals 0 and so the inequality is therefore true. Therefore assume both $\int |f|^p d\mu$ and $\int |g|^p d\mu$ are less than $\infty$ and not equal to 0. Let

$$\left( \int |f|^p d\mu \right)^{1/p} = I(f)$$

and let $\left( \int |g|^p d\mu \right)^{1/q} = I(g)$. Then using the lemma,

$$\int \frac{|f|}{I(f)} \frac{|g|}{I(g)} d\mu \leq \frac{1}{p} \int \frac{|f|^p}{I(f)^p} d\mu + \frac{1}{q} \int \frac{|g|^q}{I(g)^q} d\mu = 1.$$ 

Hence,

$$\int |f| |g| d\mu \leq I(f) I(g) = \left( \int |f|^p d\mu \right)^{1/p} \left( \int |g|^q d\mu \right)^{1/q}.$$ 

This proves Holder’s inequality.

The following lemma will be needed.

**Lemma 13.1.5** Suppose $x, y \in \mathbb{C}$. Then

$$|x + y|^p \leq 2^{p-1} (|x|^p + |y|^p).$$

**Proof:** The function $f(t) = t^p$ is concave up for $t \geq 0$ because $p > 1$. Therefore, the secant line joining two points on the graph of this function must lie above the graph of the function. This is illustrated in the following picture.

Now as shown above,

$$\left( \frac{|x| + |y|}{2} \right)^p \leq \frac{|x|^p + |y|^p}{2}$$

which implies

$$|x + y|^p \leq (|x| + |y|)^p \leq 2^{p-1} (|x|^p + |y|^p)$$

and this proves the lemma.

Note that if $y = \phi(x)$ is any function for which the graph of $\phi$ is concave up, you could get a similar inequality by the same argument.
Corollary 13.1.6 (Minkowski inequality) Let $1 \leq p < \infty$. Then
\[
\left( \int |f + g|^p \, d\mu \right)^{1/p} \leq \left( \int |f|^p \, d\mu \right)^{1/p} + \left( \int |g|^p \, d\mu \right)^{1/p}.
\]

**Proof:** If $p = 1$, this is obvious because it is just the triangle inequality. Let $p > 1$. Without loss of generality, assume
\[
\left( \int |f|^p \, d\mu \right)^{1/p} + \left( \int |g|^p \, d\mu \right)^{1/p} < \infty
\]
and $(\int |f + g|^p \, d\mu)^{1/p} \not= 0$ or there is nothing to prove. Therefore, using the above lemma,
\[
\int |f + g|^p \, d\mu \leq 2^{p-1} \left( \int |f|^p + |g|^p \, d\mu \right) < \infty.
\]
Now $|f(\omega) + g(\omega)|^p \leq |f(\omega) + g(\omega)|^{p-1} (|f(\omega)| + |g(\omega)|)$. Also, it follows from the definition of $p$ and $q$ that $p - 1 = \frac{p}{q}$. Therefore, using this and Holder’s inequality,
\[
\int |f + g|^p \, d\mu \leq \int |f + g|^{p-1} |f| \, d\mu + \int |f + g|^{p-1} |g| \, d\mu
\]
\[
= \int |f + g|^\frac{p}{q} |f| \, d\mu + \int |f + g|^\frac{p}{q} |g| \, d\mu
\]
\[
\leq (\int |f + g|^p \, d\mu)^{\frac{1}{q}} (\int |f|^p \, d\mu)^{\frac{1}{p}} + (\int |f + g|^p \, d\mu)^{\frac{1}{q}} (\int |g|^p \, d\mu)^{\frac{1}{p}}.
\]
Dividing both sides by $(\int |f + g|^p \, d\mu)^{\frac{1}{q}}$ yields \[\text{(13.1.2)}\] This proves the corollary.

The following follows immediately from the above.

Corollary 13.1.7 Let $f_i \in L^p(\Omega)$ for $i = 1, 2, \cdots, n$. Then
\[
\left( \int \left| \sum_{i=1}^{n} f_i \right|^p \, d\mu \right)^{1/p} \leq \sum_{i=1}^{n} \left( \int |f_i|^p \, d\mu \right)^{1/p}.
\]

This shows that if $f, g \in L^p$, then $f + g \in L^p$. Also, it is clear that if $a$ is a constant and $f \in L^p$, then $af \in L^p$ because
\[
\int |af|^p \, d\mu = |a|^p \int |f|^p \, d\mu < \infty.
\]
Thus $L^p$ is a vector space and

a.) $(\int |f|^p \, d\mu)^{1/p} \geq 0$, $(\int |f|^p \, d\mu)^{1/p} = 0$ if and only if $f = 0$ a.e.
b.) \((\int |af|^p \, d\mu)^{1/p} = |a| \left(\int |f|^p \, d\mu\right)^{1/p}\) if \(a\) is a scalar.

c.) \(\left(\int |f+g|^p \, d\mu\right)^{1/p} \leq \left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p}\).

\(f \rightarrow \left(\int |f|^p \, d\mu\right)^{1/p}\) would define a norm if \(\int |f|^p \, d\mu = 0\) implied \(f = 0\). Unfortunately, this is not so because if \(f = 0\) a.e. but is nonzero on a set of measure zero, \(\left(\int |f|^p \, d\mu\right)^{1/p} = 0\) and this is not allowed. However, all the other properties of a norm are available and so a little thing like a set of measure zero will not prevent the consideration of \(L^p\) as a normed vector space if two functions in \(L^p\) which differ only on a set of measure zero are considered the same. That is, an element of \(L^p\) is really an equivalence class of functions where two functions are equivalent if they are equal a.e. With this convention, here is a definition.

**Definition 13.1.8** Let \(f \in L^p(\Omega)\). Define

\[
|||f|||_p \equiv ||f||_{L^p} \equiv \left(\int |f|^p \, d\mu\right)^{1/p}.
\]

Then with this definition and using the convention that elements in \(L^p\) are considered to be the same if they differ only on a set of measure zero, \(||f||_p\) is a norm on \(L^p(\Omega)\) because if \(||f||_p = 0\) then \(f = 0\) a.e. and so \(f\) is considered to be the zero function because it differs from 0 only on a set of measure zero.

The following is an important definition.

**Definition 13.1.9** A complete normed linear space is called a Banach space.

\(L^p\) is a Banach space. This is the next big theorem.

**Theorem 13.1.10** The following hold for \(L^p(\Omega)\)

a.) \(L^p(\Omega)\) is complete.

b.) If \(\{f_n\}\) is a Cauchy sequence in \(L^p(\Omega)\), then there exists \(f \in L^p(\Omega)\) and a subsequence which converges a.e. to \(f \in L^p(\Omega)\), and \(||f_n - f||_p \to 0\).

**Proof:** Let \(\{f_n\}\) be a Cauchy sequence in \(L^p(\Omega)\). This means that for every \(\varepsilon > 0\) there exists \(N\) such that if \(n, m \geq N\), then \(||f_n - f_m||_p < \varepsilon\). Now select a subsequence as follows. Let \(n_1\) be such that \(||f_n - f_m||_p < 2^{-1}\) whenever \(n, m \geq n_1\).

---

1These spaces are named after Stefan Banach, 1892-1945. Banach spaces are the basic item of study in the subject of functional analysis and will be considered later in this book.

There is a recent biography of Banach, R. Katuža, *The Life of Stefan Banach*, (A. Kostant and W. Wołczyński, translators and editors) Birkhauser, Boston (1996). More information on Banach can also be found in a recent short article written by Douglas Henderson who is in the department of chemistry and biochemistry at BYU.

Banach was born in Austria, worked in Poland and died in the Ukraine but never moved. This is because borders kept changing. There is a rumor that he died in a German concentration camp which is apparently not true. It seems he died after the war of lung cancer.

He was an interesting character. He hated taking examinations so much that he did not receive his undergraduate university degree. Nevertheless, he did become a professor of mathematics due to his important research. He and some friends would meet in a cafe called the Scottish cafe where they wrote on the marble table tops until Banach’s wife supplied them with a notebook which became the "Scottish notebook" and was eventually published.
Let \( n_2 \) be such that \( n_2 > n_1 \) and \( \| f_n - f_m \|_p < 2^{-2} \) whenever \( n, m \geq n_2 \). If \( n_1, \ldots, n_k \) have been chosen, let \( n_{k+1} > n_k \) and whenever \( n, m \geq n_{k+1} \), \( \| f_n - f_m \|_p < 2^{-(k+1)} \). The subsequence just mentioned is \( \{ f_{n_k} \} \). Thus, \( \| f_{n_k} - f_{n_{k+1}} \|_p < 2^{-k} \). Let

\[
g_{k+1} = f_{n_{k+1}} - f_{n_k}.
\]

Then by the corollary to Minkowski’s inequality,

\[
\infty > \sum_{k=1}^{\infty} \| g_{k+1} \|_p \geq \sum_{k=1}^{m} \| g_{k+1} \|_p \geq \left( \sum_{k=1}^{m} \| g_{k+1} \|_p \right)^p
\]

for all \( m \). It follows that

\[
\int \left( \sum_{k=1}^{m} \| g_{k+1} \|_p \right)^p \, d\mu \leq \left( \sum_{k=1}^{\infty} \| g_{k+1} \|_p \right)^p < \infty \quad (13.1.3)
\]

for all \( m \) and so the monotone convergence theorem implies that the sum up to \( m \) in \( 13.1.3 \) can be replaced by a sum up to \( \infty \). Thus,

\[
\int \left( \sum_{k=1}^{\infty} \| g_{k+1} \|_p \right)^p \, d\mu < \infty
\]

which requires

\[
\sum_{k=1}^{\infty} \| g_{k+1} \|_p < \infty \quad \text{a.e. } x.
\]

Therefore, \( \sum_{k=1}^{\infty} g_{k+1}(x) \) converges for a.e. \( x \) because the functions have values in a complete space, \( \mathbb{C} \), and this shows the partial sums form a Cauchy sequence. Now let \( x \) be such that this sum is finite. Then define

\[
f(x) \equiv f_{n_1}(x) + \sum_{k=1}^{\infty} g_{k+1}(x) = \lim_{m \to \infty} f_{n_m}(x)
\]

since \( \sum_{k=1}^{m} g_{k+1}(x) = f_{n_{m+1}}(x) - f_{n_1}(x) \). Therefore there exists a set, \( E \) having measure zero such that

\[
\lim_{k \to \infty} f_{n_k}(x) = f(x)
\]

for all \( x \notin E \). Redefine \( f_{n_k} \) to equal 0 on \( E \) and let \( f(x) = 0 \) for \( x \in E \). It then follows that \( \lim_{k \to \infty} f_{n_k}(x) = f(x) \) for all \( x \). By Fatou’s lemma, and the Minkowski inequality,

\[
\| f - f_{n_k} \|_p = \left( \int \| f - f_{n_k} \|^p \, d\mu \right)^{1/p} \leq \quad \leq \quad \leq
\]

\[
\lim_{m \to \infty} \left( \int \| f_m - f_{n_k} \|^p \, d\mu \right)^{1/p} = \lim_{m \to \infty} \| f_m - f_{n_k} \|_p \leq
\]
13.1. BASIC INEQUALITIES AND PROPERTIES

\[ \lim_{m \to \infty} \inf \sum_{j=k}^{m-1} \|f_{n+1} - f_n\|_p \leq \sum_{i=k}^{\infty} \|f_{n+i} - f_n\|_p \leq 2^{-(k-1)}. \]  

(13.1.4)

Therefore, \( f \in L^p(\Omega) \) because

\[ \|f\|_p \leq \|f - f_{n_k}\|_p + \|f_{n_k}\|_p < \infty, \]

and \( \lim_{k \to \infty} \|f_{n_k} - f\|_p = 0 \). This proves b).

This has shown \( f_{n_k} \) converges to \( f \) in \( L^p(\Omega) \). It follows the original Cauchy sequence also converges to \( f \) in \( L^p(\Omega) \). This is a general fact that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence. You should give a proof of this. This proves the theorem.

In working with the \( L^p \) spaces, the following inequality also known as Minkowski’s inequality is very useful. It is similar to the Minkowski inequality for sums. To see this, replace the integral, \( \int_X \) with a finite summation sign and you will see the usual Minkowski inequality or rather the version of it given in Corollary [13.1.7].

To prove this theorem first consider a special case of it in which technical considerations which shed no light on the proof are excluded.

**Lemma 13.1.11** Let \((X, \mathcal{S}, \mu)\) and \((Y, \mathcal{F}, \lambda)\) be finite complete measure spaces and let \( f \) be \( \mu \times \lambda \) measurable and uniformly bounded. Then the following inequality is valid for \( p \geq 1 \).

\[ \left( \int_Y \left( \int_X |f(x,y)|^p d\mu \right)^\frac{1}{p} d\lambda \right) \geq \left( \int_Y \left( \int_X |f(x,y)|^p d\mu \right)^p d\lambda \right)^\frac{1}{p}. \]  

(13.1.5)

**Proof:** Since \( f \) is bounded and \( \mu(X), \lambda(Y) < \infty \),

\[ \left( \int_Y \left( \int_X |f(x,y)|^p d\mu \right)^\frac{1}{p} d\lambda \right) < \infty. \]

Let

\[ J(y) = \int_X |f(x,y)| d\mu. \]

Note there is no problem in writing this for a.e. \( y \) because \( f \) is product measurable. Then by Fubini’s theorem,

\[ \int_Y \left( \int_X |f(x,y)| d\mu \right)^p d\lambda = \int_Y J(y)^{p-1} \int_X |f(x,y)| d\mu d\lambda = \int_X \int_Y J(y)^{p-1} |f(x,y)| d\lambda d\mu \]
Now apply Holder’s inequality in the last integral above and recall \( p - 1 = \frac{p}{q} \). This yields

\[
\int_Y \left( \int_X |f(x,y)| d\mu \right)^p d\lambda \\
\leq \int_X \left( \int_Y J(y)^p d\lambda \right)^{\frac{1}{q}} \left( \int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \\
= \left( \int_Y J(y)^p d\lambda \right)^{\frac{1}{q}} \int_X \left( \int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \\
= \left( \int_Y \left( \int_X |f(x,y)| d\mu \right)^p d\lambda \right)^{\frac{1}{q}} \int_X \left( \int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu. \tag{13.1.6}
\]

Therefore, dividing both sides by the first factor in the above expression,

\[
\left( \int_Y \left( \int_X |f(x,y)| d\mu \right)^p d\lambda \right)^{\frac{1}{q}} \leq \int_X \left( \int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu. \tag{13.1.7}
\]

Note that (13.1.7) holds even if the first factor of (13.1.6) equals zero. This proves the lemma.

Now consider the case where \( f \) is not assumed to be bounded and where the measure spaces are \( \sigma \) finite.

**Theorem 13.1.12** Let \((X, S, \mu)\) and \((Y, F, \lambda)\) be \( \sigma \)-finite measure spaces and let \( f \) be product measurable. Then the following inequality is valid for \( p \geq 1 \).

\[
\int_X \left( \int_Y |f(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \geq \left( \int_Y \left( \int_X |f(x,y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}}. \tag{13.1.8}
\]

**Proof:** Since the two measure spaces are \( \sigma \) finite, there exist measurable sets, \( X_m \) and \( Y_k \) such that \( X_m \subseteq X_{m+1} \) for all \( m \), \( Y_k \subseteq Y_{k+1} \) for all \( k \), and \( \mu(X_m), \lambda(Y_k) < \infty \). Now define

\[
f_n(x,y) = \begin{cases} f(x,y) & \text{if } |f(x,y)| \leq n \\ n & \text{if } |f(x,y)| > n. \end{cases}
\]

Thus \( f_n \) is uniformly bounded and product measurable. By the above lemma,

\[
\int_{X_m} \left( \int_{Y_k} |f_n(x,y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \geq \left( \int_{Y_k} \left( \int_{X_m} |f_n(x,y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}}. \tag{13.1.9}
\]

Now observe that \( |f_n(x,y)| \) increases in \( n \) and the pointwise limit is \( |f(x,y)| \). Therefore, using the monotone convergence theorem in (13.1.9) yields the same inequality with \( f \) replacing \( f_n \). Next let \( k \to \infty \) and use the monotone convergence theorem again to replace \( Y_k \) with \( Y \). Finally let \( m \to \infty \) in what is left to obtain (13.1.8). This proves the theorem.
Note that the proof of this theorem depends on two manipulations, the interchange of the order of integration and Holder’s inequality. Note that there is nothing to check in the case of double sums. Thus if $a_{ij} \geq 0$, it is always the case that

$$\left( \sum_{j} \left( \sum_{i} a_{ij} \right)^p \right)^{1/p} \leq \sum_{i} \left( \sum_{j} a_{ij}^p \right)^{1/p}$$

because the integrals in this case are just sums and $(i, j) \rightarrow a_{ij}$ is measurable.

The $L^p$ spaces have many important properties.

### 13.2 Density Considerations

**Theorem 13.2.1** Let $p \geq 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a measure space. Then the simple functions are dense in $L^p(\Omega)$.

**Proof:** Recall that a function, $f$, having values in $\mathbb{R}$ can be written in the form $f = f^+ - f^-$ where

$$f^+ = \max(0, f), \quad f^- = \max(0, -f).$$

Therefore, an arbitrary complex valued function, $f$ is of the form

$$f = \text{Re } f^+ - \text{Re } f^- + i(\text{Im } f^+ - \text{Im } f^-).$$

If each of these nonnegative functions is approximated by a simple function, it follows $f$ is also approximated by a simple function. Therefore, there is no loss of generality in assuming at the outset that $f \geq 0$.

Since $f$ is measurable, Theorem 9.3.9 implies there is an increasing sequence of simple functions, $\{s_n\}$, converging pointwise to $f(x)$. Now

$$|f(x) - s_n(x)| \leq |f(x)|.$$

By the Dominated Convergence theorem,

$$0 = \lim_{n \to \infty} \int |f(x) - s_n(x)|^p d\mu.$$

Thus simple functions are dense in $L^p$.

Recall that for $\Omega$ a topological space, $C_c(\Omega)$ is the space of continuous functions with compact support in $\Omega$. Also recall the following definition.

**Definition 13.2.2** Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose $(\Omega, \tau)$ is also a topological space. Then $(\Omega, \mathcal{S}, \mu)$ is called a regular measure space if the $\sigma$ algebra of Borel sets is contained in $\mathcal{S}$ and for all $E \in \mathcal{S}$,

$$\mu(E) = \inf \{\mu(V) : V \supseteq E \text{ and } V \text{ open} \}.$$
and if \( \mu(E) < \infty \),

\[
\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact} \}
\]

and \( \mu(K) < \infty \) for any compact set, \( K \).

For example Lebesgue measure is an example of such a measure. More generally these measures are often referred to as Radon measures.

**Lemma 13.2.3** Let \( \Omega \) be a metric space in which the closed balls are compact and let \( K \) be a compact subset of \( V \), an open set. Then there exists a continuous function \( f : \Omega \to [0,1] \) such that \( f(x) = 1 \) for all \( x \in K \) and \( \text{spt}(f) \) is a compact subset of \( V \). That is, \( K \prec f \prec V \).

**Proof:** Let \( K \subseteq W \subseteq \overline{W} \subseteq V \) and \( \overline{W} \) is compact. To obtain this list of inclusions consider a point in \( K, x \), and take \( B(x, r_x) \) a ball containing \( x \) such that \( B(x, r_x) \) is a compact subset of \( V \). Next use the fact that \( K \) is compact to obtain the existence of a list, \( \{B(x_i, r_{x_i}/2)\}_{i=1}^m \) which covers \( K \). Then let

\[
W \equiv \bigcup_{i=1}^m B\left(x_i, \frac{r_{x_i}}{2}\right).
\]

It follows since this is a finite union that

\[
\overline{W} = \bigcup_{i=1}^m B\left(x_i, \frac{r_{x_i}}{2}\right)
\]

and so \( \overline{W} \), being a finite union of compact sets is itself a compact set. Also, from the construction

\[
W \subseteq \bigcup_{i=1}^m B\left(x_i, r_{x_i}\right).
\]

Define \( f \) by

\[
f(x) = \frac{\text{dist}(x, \overline{W})}{\text{dist}(x, K) + \text{dist}(x, \overline{W})}.
\]

It is clear that \( f \) is continuous if the denominator is always nonzero. But this is clear because if \( x \in \overline{W} \) there must be a ball \( B(x, r) \) such that this ball does not intersect \( K \). Otherwise, \( x \) would be a limit point of \( K \) and since \( K \) is closed, \( x \in K \). However, \( x \notin K \) because \( K \subseteq W \).

It is not necessary to be in a metric space to do this. You can accomplish the same thing using Urysohn’s lemma.

**Theorem 13.2.4** Let \( (\Omega, S, \mu) \) be a regular measure space as in Definition 13.2.2 where the conclusion of Lemma 13.2.3 holds. Then \( C_c(\Omega) \) is dense in \( L^p(\Omega) \).

**Proof:** First consider a measurable set, \( E \) where \( \mu(E) < \infty \). Let \( K \subseteq E \subseteq V \) where \( \mu(V \setminus K) < \varepsilon \). Now let \( K \prec h \prec V \). Then

\[
\int |h - \chi_E|^p \, d\mu \leq \int \chi_{V \setminus K}^p \, d\mu = \mu(V \setminus K) < \varepsilon.
\]
It follows that for each $s$ a simple function in $L^p(\Omega)$, there exists $h \in C_c(\Omega)$ such that $\|s - h\|_p < \varepsilon$. This is because if

$$s(x) = \sum_{i=1}^{m} c_i \mathcal{X}_{E_i}(x)$$

is a simple function in $L^p$ where the $c_i$ are the distinct nonzero values of $s$ each $\mu(E_i) < \infty$ since otherwise $s \notin L^p$ due to the inequality

$$\int |s|^p \ d\mu \geq |c_i|^p \ \mu(E_i).$$

By Theorem 13.3.1, simple functions are dense in $L^p(\Omega)$, and so this proves the Theorem.

### 13.3 Separability

**Theorem 13.3.1** For $p \geq 1$ and $\mu$ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $\|f - g\|_p < \varepsilon$.

**Proof:** Let $Q$ be all functions of the form $c \mathcal{X}_{[a,b)}$ where

$$[a, b) \equiv [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

and both $a_i, b_i$ are rational, while $c$ has rational real and imaginary parts. Let $\mathcal{D}$ be the set of all finite sums of functions in $Q$. Thus, $\mathcal{D}$ is countable. In fact $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n, \mu)$. To prove this it is necessary to show that for every $f \in L^p(\mathbb{R}^n, \mu)$, there exists an element of $\mathcal{D}$, $s$ such that $\|s - f\|_p < \varepsilon$. If it can be shown that for every $g \in C_c(\mathbb{R}^n)$ there exists $h \in \mathcal{D}$ such that $\|g - h\|_p < \varepsilon$, then this will suffice because if $f \in L^p(\mathbb{R}^n)$ is arbitrary, Theorem 13.3.1 implies there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p \leq \frac{\varepsilon}{2}$ and then there would exist $h \in C_c(\mathbb{R}^n)$ such that $\|h - g\|_p < \frac{\varepsilon}{2}$. By the triangle inequality,

$$\|f - h\|_p \leq \|h - g\|_p + \|g - f\|_p < \varepsilon.$$ 

Therefore, assume at the outset that $f \in C_c(\mathbb{R}^n)$.

Let $\mathcal{P}_m$ consist of all sets of the form $[a, b) \equiv \prod_{i=1}^{n} [a_i, b_i)$ where $a_i = j 2^{-m}$ and $b_i = (j + 1) 2^{-m}$ for $j$ an integer. Thus $\mathcal{P}_m$ consists of a tiling of $\mathbb{R}^n$ into half open rectangles having diameters $2^{-m} n^2$. There are countably many of these rectangles; so, let $\mathcal{P}_m = \{[a_i, b_i)\}_{i=1}^{\infty}$ and $\mathbb{R}^n = \cup_{i=1}^{\infty} [a_i, b_i)$. Let $c^m_i$ be complex numbers with rational real and imaginary parts satisfying

$$|f(a_i) - c^m_i| < 2^{-m},$$

$$|c^m_i| \leq |f(a_i)|.$$ (13.3.10)
Let
\[ s_m(x) = \sum_{i=1}^{\infty} c_i^m \chi_{[a_i, b_i]}(x). \]

Since \( f(a_i) = 0 \) except for finitely many values of \( i \), the above is a finite sum. Then \( s_m \in D \). If \( s_m \) converges uniformly to \( f \) then it follows \( \|s_m - f\|_p \to 0 \) because \( |s_m| \leq |f| \) and so

\[
\|s_m - f\|_p = \left( \int |s_m - f|^p \, d\mu \right)^{1/p} = \left( \int_{\text{spt}(f)} |s_m - f|^p \, d\mu \right)^{1/p} \leq (\varepsilon m_n(\text{spt}(f)))^{1/p}
\]

even when \( m \) is large enough.

Since \( f \in C_c(\mathbb{R}^n) \) it follows that \( f \) is uniformly continuous and so given \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |x - y| < \delta \), \( |f(x) - f(y)| < \varepsilon/2 \). Now let \( m \) be large enough that every box in \( P_m \) has diameter less than \( \delta \) and also that \( 2^{-m} < \varepsilon/2 \). Then if \( [a_i, b_i] \) is one of these boxes of \( P_m \), and \( x \in [a_i, b_i] \),

\[
|f(x) - f(a_i)| < \varepsilon/2
\]

and

\[
|f(a_i) - c_i^m| < 2^{-m} < \varepsilon/2.
\]

Therefore, using the triangle inequality, it follows that

\[
|f(x) - c_i^m| = |s_m(x) - f(x)| < \varepsilon
\]

and since \( x \) is arbitrary, this establishes uniform convergence. This proves the theorem.

Here is an easier proof if you know the Weierstrass approximation theorem.

**Theorem 13.3.2** For \( p \geq 1 \) and \( \mu \) a Radon measure, \( L^p(\mathbb{R}^n, \mu) \) is separable. Recall this means there exists a countable set, \( D \), such that if \( f \in L^p(\mathbb{R}^n, \mu) \) and \( \varepsilon > 0 \), there exists \( g \in D \) such that \( \|f - g\|_p < \varepsilon \).

**Proof:** Let \( P \) denote the set of all polynomials which have rational coefficients. Then \( P \) is countable. Let \( \tau_k \in C_c((- (k + 1), (k + 1))^{\mathbb{n}}) \) such that \( [-k,k]^{\mathbb{N}} \prec \tau_k \prec (- (k + 1), (k + 1))^{\mathbb{n}} \). Let \( D_k \) denote the functions which are of the form, \( p \tau_k \) where \( p \in P \). Thus \( D_k \) is also countable. Let \( D = \cup_{k=1}^{\infty} D_k \). It follows each function in \( D \) is in \( C_c(\mathbb{R}^n) \) and so it in \( L^p(\mathbb{R}^n, \mu) \). Let \( f \in L^p(\mathbb{R}^n, \mu) \). By regularity of \( \mu \) there exists \( g \in C_c(\mathbb{R}^n) \) such that \( \|f - g\|_{L^p(\mathbb{R}^n, \mu)} < \frac{\varepsilon}{3} \). Let \( k \) be such that \( \text{spt}(g) \subseteq (-k, k)^n \). Now by the Weierstrass approximation theorem there exists a polynomial \( q \) such that

\[
\|g - q\|_{[-(k+1),k+1]^n} \leq \sup \{ |g(x) - q(x)| : x \in [- (k + 1), (k + 1)]^n \} \leq \frac{\varepsilon}{3\mu((- (k + 1), k + 1)^n)}.
\]
13.4. CONTINUITY OF TRANSLATION

It follows
\[ ||g - \tau_k q||_{[-(k+1),k+1]^n} = ||\tau_k g - \tau_k q||_{[-(k+1),k+1]^n} < \frac{\varepsilon}{3\mu((-k+1), k+1)^n}. \]

Without loss of generality, it can be assumed this polynomial has all rational coefficients. Therefore, \( \tau_k q \in \mathcal{D} \).

\[ ||g - \tau_k q||_{L^p(\mathbb{R}^n)} = \int_{-(k+1),k+1} |g(x) - \tau_k (x) q(x)|^p \, d\mu \leq \left( \frac{\varepsilon}{3\mu((-k+1), k+1)^n} \right)^p \mu((-k+1), k+1)^n < \left( \frac{\varepsilon}{3} \right)^p. \]

It follows
\[ ||f - \tau_k q||_{L^p(\mathbb{R}^n, \mu)} \leq ||f - g||_{L^p(\mathbb{R}^n, \mu)} + ||g - \tau_k q||_{L^p(\mathbb{R}^n, \mu)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon. \]

This proves the theorem.

**Corollary 13.3.3** Let \( \Omega \) be any \( \mu \) measurable subset of \( \mathbb{R}^n \) and let \( \mu \) be a Radon measure. Then \( L^p(\Omega, \mu) \) is separable. Here the \( \sigma \) algebra of measurable sets will consist of all intersections of measurable sets with \( \Omega \) and the measure will be \( \mu \) restricted to these sets.

**Proof:** Let \( \tilde{\mathcal{D}} \) be the restrictions of \( \mathcal{D} \) to \( \Omega \). If \( f \in L^p(\Omega) \), let \( F \) be the zero extension of \( f \) to all of \( \mathbb{R}^n \). Let \( \varepsilon > 0 \) be given. By Theorem 13.3.1 or 13.3.2 there exists \( s \in \mathcal{D} \) such that \( ||F - s||_p < \varepsilon \). Thus
\[ ||s - f||_{L^p(\Omega, \mu)} \leq ||s - F||_{L^p(\mathbb{R}^n, \mu)} < \varepsilon \]
and so the countable set \( \tilde{\mathcal{D}} \) is dense in \( L^p(\Omega) \).

### 13.4 Continuity Of Translation

**Definition 13.4.1** Let \( f \) be a function defined on \( U \subseteq \mathbb{R}^n \) and let \( w \in \mathbb{R}^n \). Then \( f_w \) will be the function defined on \( w + U \) by
\[ f_w(x) = f(x - w). \]

**Theorem 13.4.2** (Continuity of translation in \( L^p \)) Let \( f \in L^p(\mathbb{R}^n) \) with the measure being Lebesgue measure. Then
\[ \lim_{||w|| \to 0} ||f_w - f||_p = 0. \]
Proof: Let \( \varepsilon > 0 \) be given and let \( g \in C_c(\mathbb{R}^n) \) with \( \|g - f\|_p < \frac{\varepsilon}{3} \). Since Lebesgue measure is translation invariant \( (m_n(w + E) = m_n(E)) \),
\[
\|g_w - f_w\|_p = \|g - f\|_p < \frac{\varepsilon}{3}.
\]
You can see this from looking at simple functions and passing to the limit or you could use the change of variables formula to verify it.

Therefore
\[
\|f - f_w\|_p \leq \|f - g\|_p + \|g - g_w\|_p + \|g_w - f_w\| < \frac{2\varepsilon}{3} + \|g - g_w\|_p.
\]

But \( \lim_{|w| \to 0} g_w(x) = g(x) \) uniformly in \( x \) because \( g \) is uniformly continuous. Now let \( B \) be a large ball containing \( \text{spt}(g) \) and let \( \delta_1 \) be small enough that \( B(x, \delta) \subseteq B \) whenever \( x \in \text{spt}(g) \). If \( \varepsilon > 0 \) is given there exists \( \delta < \delta_1 \) such that if \( |w| < \delta \), it follows that \( |g(x - w) - g(x)| < \varepsilon/3 \left(1 + m_n(B)^{1/p}\right) \). Therefore,
\[
\|g - g_w\|_p = \left( \int_B |g(x) - g(x - w)|^p dm_n \right)^{1/p} \\
\leq \frac{\varepsilon}{3} \left(1 + m_n(B)^{1/p}\right) < \frac{\varepsilon}{3}.
\]

Therefore, whenever \( |w| < \delta \), it follows \( \|g - g_w\|_p < \frac{\varepsilon}{3} \) and so from \( 13.4.11 \) \( \|f - f_w\|_p < \varepsilon \). This proves the theorem.

13.5 Mollifiers And Density Of Smooth Functions

Definition 13.5.1 Let \( U \) be an open subset of \( \mathbb{R}^n \). \( C_c^\infty(U) \) is the vector space of all infinitely differentiable functions which equal zero for all \( x \) outside of some compact set contained in \( U \). Similarly, \( C_{mc}^m(U) \) is the vector space of all functions which are \( m \) times continuously differentiable and whose support is a compact subset of \( U \).

Example 13.5.2 Let \( U = B(z, 2r) \)
\[
\psi(x) = \begin{cases} 
\exp \left( \left( \frac{|x - z|^2 - r^2}{r^2} \right)^{-1} \right) & \text{if } |x - z| < r, \\
0 & \text{if } |x - z| \geq r.
\end{cases}
\]
Then a little work shows \( \psi \in C_c^\infty(U) \). Note that if \( z = 0 \) then \( \psi(x) = \psi(-x) \). The following also is easily obtained.

Lemma 13.5.3 Let \( U \) be any open set. Then \( C_c^\infty(U) \neq \emptyset \).
13.5. MOLLIFIERS AND DENSITY OF SMOOTH FUNCTIONS

Proof: Pick \( z \in U \) and let \( r \) be small enough that \( B(z, 2r) \subseteq U \). Then let \( \psi \in C_c^\infty(B(z, 2r)) \subseteq C_c^\infty(U) \) be the function of the above example.

**Definition 13.5.4** Let \( U = \{x \in \mathbb{R}^n : |x| < 1\} \). A sequence \( \{\psi_m\} \subseteq C_c^\infty(U) \) is called a mollifier (This is sometimes called an approximate identity if the differentiability is not included.) if

\[
\psi_m(x) \geq 0, \quad \psi_m(x) = 0, \quad \text{if } |x| \geq \frac{1}{m},
\]

and \( \int \psi_m(x) = 1 \). Sometimes it may be written as \( \{\psi_x\} \) where \( \psi_x \) satisfies the above conditions except \( \psi_x(x) = 0 \) if \( |x| > \varepsilon \). In other words, \( \varepsilon \) takes the place of \( 1/m \) and in everything that follows \( \varepsilon \to 0 \) instead of \( m \to \infty \).

As before, \( \int f(x, y) \, d\mu(y) \) will mean \( y \) is fixed and the function \( y \to f(x, y) \) is being integrated. To make the notation more familiar, \( dx \) is written instead of \( dm_n(x) \).

**Example 13.5.5** Let

\[
\psi \in C_c^\infty(B(0, 1)) \quad (B(0, 1) = \{x : |x| < 1\})
\]

with \( \psi(x) \geq 0 \) and \( \int \psi \, dm = 1 \). Let \( \psi_m(x) = c_m \psi(mx) \) where \( c_m \) is chosen in such a way that \( \int \psi_m \, dm = 1 \). By the change of variables theorem \( c_m = m^n \).

**Definition 13.5.6** A function, \( f \), is said to be in \( L^1_{loc}(\mathbb{R}^n, \mu) \) if \( f \) is \( \mu \) measurable and if \( |f| \chi_K \in L^1(\mathbb{R}^n, \mu) \) for every compact set, \( K \). Here \( \mu \) is a Radon measure on \( \mathbb{R}^n \). Usually \( \mu = m_n \), Lebesgue measure. When this is so, write \( L^1_{loc}(\mathbb{R}^n) \) or \( L^p(\mathbb{R}^n) \), etc. If \( f \in L^1_{loc}(\mathbb{R}^n, \mu) \), and \( g \in C_c(\mathbb{R}^n) \),

\[
f * g(x) \equiv \int f(y) g(x - y) \, d\mu.
\]

The following lemma will be useful in what follows. It says that one of these very unregular functions in \( L^1_{loc}(\mathbb{R}^n, \mu) \) is smoothed out by convolving with a mollifier.

**Lemma 13.5.7** Let \( f \in L^1_{loc}(\mathbb{R}^n, \mu) \), and \( g \in C_c^\infty(\mathbb{R}^n) \). Then \( f * g \) is an infinitely differentiable function. Here \( \mu \) is a Radon measure on \( \mathbb{R}^n \).

**Proof:** Consider the difference quotient for calculating a partial derivative of \( f * g \).

\[
\frac{f * g(x + te_j) - f * g(x)}{t} = \int f(y) \frac{g(x + te_j - y) - g(x - y)}{t} \, d\mu(y).
\]

Using the fact that \( g \in C_c^\infty(\mathbb{R}^n) \), the quotient,

\[
\frac{g(x + te_j - y) - g(x - y)}{t},
\]
is uniformly bounded. To see this easily, use Theorem 5.13.4 on Page 128 to get the existence of a constant, \( M \) depending on

\[
\max \{ ||Dg(x)|| : x \in \mathbb{R}^n \}
\]

such that

\[
|g(x + te_j - y) - g(x - y)| \leq M |t|
\]

for any choice of \( x \) and \( y \). Therefore, there exists a dominating function for the integrand of the above integral which is of the form \( C |f(y)| \chi_K \) where \( K \) is a compact set depending on the support of \( g \). It follows the limit of the difference quotient above passes inside the integral as \( t \to 0 \) and

\[
\frac{\partial}{\partial x_j} (f * g)(x) = \int f(y) \frac{\partial}{\partial x_j} g(x - y) \, d\mu(y).
\]

Now letting \( \frac{\partial}{\partial x_j} g \) play the role of \( g \) in the above argument, partial derivatives of all orders exist. A similar use of the dominated convergence theorem shows all these partial derivatives are also continuous. This proves the lemma.

**Theorem 13.5.8** Let \( K \) be a compact subset of an open set, \( U \). Then there exists a function, \( h \in C_\infty(U) \), such that \( h(x) = 1 \) for all \( x \in K \) and \( h(x) \in [0, 1] \) for all \( x \).

**Proof:** Let \( r > 0 \) be small enough that \( K + B(0, 3r) \subseteq U \). The symbol, \( K + B(0, 3r) \) means

\[
\{ k + x : k \in K \text{ and } x \in B(0, 3r) \}.
\]

Thus this is simply a way to write

\[
\cup \{ B(k, 3r) : k \in K \}.
\]

Think of it as fattening up the set, \( K \). Let \( K_r = K + B(0, r) \). A picture of what is happening follows.

Consider \( \chi_{K_r} * \psi_m \) where \( \psi_m \) is a mollifier. Let \( m \) be so large that \( \frac{1}{m} < r \). Then from the definition of what is meant by a convolution, and using that \( \psi_m \) has support in \( B(0, \frac{1}{m}) \), \( \chi_{K_r} * \psi_m = 1 \) on \( K \) and that its support is in \( K + B(0, 3r) \). Now using Lemma 13.5.7, \( \chi_{K_r} * \psi_m \) is also infinitely differentiable. Therefore, let \( h = \chi_{K_r} * \psi_m \).

The following corollary will be used later.
Corollary 13.5.9 Let $K$ be a compact set in $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^\infty$ be an open cover of $K$. Then there exist functions, $\psi_k \in C_\infty^c(U_i)$ such that $\psi_i \prec U_i$ and for all $x \in K$,
\[ \sum_{i=1}^{\infty} \psi_i(x) = 1. \]
If $K_1$ is a compact subset of $U_1$ there exist such functions such that also $\psi_1(x) = 1$ for all $x \in K_1$.

Proof: This follows from a repeat of the proof of Theorem 10.2.11 on Page 268, replacing the lemma used in that proof with Theorem 13.5.8.

Note that in the last conclusion of above corollary, the set $U_1$ could be replaced with $U_i$ for any fixed $i$ by simply renumbering.

Theorem 13.5.10 For each $p \geq 1$, $C_\infty^c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Here the measure is Lebesgue measure.

Proof: Let $f \in L^p(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given. Choose $g \in C_\alpha(\mathbb{R}^n)$ such that $\|f - g\|_p < \frac{\varepsilon}{2}$. This can be done by using Theorem 13.2.4. Now let
\[ g_m(x) = g \ast \psi_m(x) \equiv \int g(x - y) \psi_m(y) \, dm_n(y) = \int g(y) \psi_m(x - y) \, dm_n(y) \]
where $\{\psi_m\}$ is a mollifier. It follows from Lemma 13.5.2 that $g_m \in C_\infty^c(\mathbb{R}^n)$. It vanishes if $x \notin \text{spt}(g) + B(0, \frac{1}{m})$.

\[ \|g - g_m\|_p = \left( \int |g(x) - \int g(x - y) \psi_m(y) \, dm_n(y)|^p \, dm_n(x) \right)^{\frac{1}{p}} \leq \left( \int \left( \int |g(x) - g(x - y)| \psi_m(y) \, dm_n(y) \right)^p \, dm_n(x) \right)^{\frac{1}{p}} \leq \int \left( \int |g(x) - g(x - y)| \, dm_n(x) \right)^{\frac{1}{p}} \psi_m(y) \, dm_n(y) \]
\[ = \int_{B(0, \frac{1}{m})} \|g - g_m\|_p \psi_m(y) \, dm_n(y) < \frac{\varepsilon}{2} \]
whenever $m$ is large enough thanks to the uniform continuity of $g$. Theorem 13.2.4 was used to obtain the third inequality. There is no measurability problem because the function
\[ (x, y) \rightarrow |g(x) - g(x - y)| \psi_m(y) \]
is continuous. Thus when $m$ is large enough,
\[ \|f - g_m\|_p \leq \|f - g\|_p + \|g - g_m\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]
This proves the theorem.
This is a very remarkable result. Functions in $L^p(\mathbb{R}^n)$ don’t need to be continuous anywhere and yet every such function is very close in the $L^p$ norm to one which is infinitely differentiable having compact support. The same result holds for $L^p(U)$ for $U$ an open set. This is the next corollary.

**Corollary 13.5.11** Let $U$ be an open set. For each $p \geq 1$, $C^\infty_c(U)$ is dense in $L^p(U)$. Here the measure is Lebesgue measure.

**Proof:** Let $f \in L^p(U)$ and let $\varepsilon > 0$ be given. Choose $g \in C^\infty_c(U)$ such that $\|f - g\|_p < \frac{\varepsilon}{2}$. This is possible because Lebesgue measure restricted to the open set, $U$ is regular. Thus the existence of such a $g$ follows from Theorem 13.2.4. Now let

$$g_m(x) = g * \psi_m(x) \equiv \int g(x - y) \psi_m(y) \, dm_n(y) = \int g(y) \psi_m(x - y) \, dm_n(y)$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 13.5.7 $g_m \in C^\infty_c(U)$ for all $m$ sufficiently large. It vanishes if $x \notin \text{spt}(g) + B(0, \frac{1}{m})$. Then

$$\|g - g_m\|_p = \left( \int |g(x) - \int g(x - y) \psi_m(y) \, dm_n(y)|^p \, dm_n(x) \right)^{\frac{1}{p}} \leq \left( \int \left( \int |g(x) - g(x - y)| \psi_m(y) \, dm_n(y) \right|^p \, dm_n(x) \right)^{\frac{1}{p}} \leq \int B(0, \frac{1}{m}) \|g - g_m\|_p \psi_m(y) \, dm_n(y) < \frac{\varepsilon}{2}$$

whenever $m$ is large enough thanks to uniform continuity of $g$. Theorem 13.1.12 was used to obtain the third inequality. There is no measurability problem because the function

$$(x, y) \rightarrow |g(x) - g(x - y)| \psi_m(y)$$

is continuous. Thus when $m$ is large enough,

$$\|f - g_m\|_p \leq \|f - g\|_p + \|g - g_m\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the corollary.

Another thing should probably be mentioned. If you have had a course in complex analysis, you may be wondering whether these infinitely differentiable functions having compact support have anything to do with analytic functions which also have infinitely many derivatives. The answer is no! Recall that if an analytic function has a limit point in the set of zeros then it is identically equal to zero. Thus these functions in $C^\infty_c(\mathbb{R}^n)$ are not analytic. This is a strictly real analysis phenomenon and has absolutely nothing to do with the theory of functions of a complex variable.
13.6 Exercises

1. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set
$$E - E = \{x - y : x \in E, y \in E\}.$$
Show that $E - E$ contains an interval. **Hint:** Let
$$f(x) = \int \chi_E(t)\chi_E(x + t)dt.$$
Note $f$ is continuous at 0 and $f(0) > 0$ and use continuity of translation in $L^p$.

2. Establish the inequality $||fg||_r \leq ||f||_p ||g||_q$ whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

3. Let $(\Omega, S, \mu)$ be counting measure on $\mathbb{N}$. Thus $\Omega = \mathbb{N}$ and $S = \mathcal{P}(\mathbb{N})$ with $\mu(S) =$ number of things in $S$. Let $1 \leq p \leq q$. Show that in this case,
$$L^1(\mathbb{N}) \subseteq L^p(\mathbb{N}) \subseteq L^q(\mathbb{N}).$$
**Hint:** This is real easy if you consider what $\int_{\Omega} f d\mu$ equals. How are the norms related?

4. Consider the function, $f(x, y) = \frac{x^{p-1}}{py} + \frac{y^{q-1}}{qx}$ for $x, y > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show directly that $f(x, y) \geq 1$ for all such $x, y$ and show this implies $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.

5. Give an example of a sequence of functions in $L^p(\mathbb{R})$ which converges to zero in $L^p$ but does not converge pointwise to 0. Does this contradict the proof of the theorem that $L^p$ is complete?

6. Let $K$ be a bounded subset of $L^p(\mathbb{R}^n)$ and suppose that there exists $G$ such that $G$ is compact with
$$\int_{\mathbb{R}^n \setminus G} |u(x)|^p dx < \varepsilon^p$$
and for all $\varepsilon > 0$, there exist a $\delta > 0$ and such that if $|h| < \delta$, then
$$\int |u(x + h) - u(x)|^p dx < \varepsilon^p$$
for all $u \in K$. Show that $K$ is precompact in $L^p(\mathbb{R}^n)$. **Hint:** Let $\phi_k$ be a mollifier and consider $K_k \equiv \{u * \phi_k : u \in K\}$.
Verify the conditions of the Ascoli Arzela theorem for these functions defined on $\overline{G}$ and show there is an $\varepsilon$ net for each $\varepsilon > 0$. Can you modify this to let an arbitrary open set take the place of $\mathbb{R}^n$?
7. Let \((\Omega, d)\) be a metric space and suppose also that \((\Omega, \mathcal{S}, \mu)\) is a regular measure space such that \(\mu(\Omega) < \infty\) and let \(f \in L^1(\Omega)\) where \(f\) has complex values. Show that for every \(\varepsilon > 0\), there exists an open set of measure less than \(\varepsilon\), denoted here by \(V\) and a continuous function, \(g\) defined on \(\Omega\) such that \(f = g\) on \(V^c\). Thus, aside from a set of small measure, \(f\) is continuous. If \(|f(\omega)| \leq M\), show that it can be assumed that \(|g(\omega)| \leq M\). This is called Lusin’s theorem. **Hint:** Use Theorems 13.2.7 and 13.1.10 to obtain a sequence of functions in \(C_c(\Omega)\), \(\{g_n\}\) which converges pointwise a.e. to \(f\) and then use Egoroff’s theorem to obtain a small set, \(W\) of measure less than \(\varepsilon/2\) such that convergence is uniform on \(W^c\). Now let \(F\) be a closed subset of \(W^c\) such that \(\mu(W^c \setminus F) < \varepsilon/2\). Let \(V = F^c\). Thus \(\mu(V) < \varepsilon\) and on \(F = V^c\), the convergence of \(\{g_n\}\) is uniform showing that the restriction of \(f\) to \(V^c\) is continuous. Now use the Tietze extension theorem.

8. Let \(\phi_m \in C_c^\infty(\mathbb{R}^n), \phi_m(x) \geq 0, \text{and} \int_{\mathbb{R}^n} \phi_m(y)dy = 1\) with
\[
\lim_{m \to \infty} \sup \{|x| : x \in \text{spt} (\phi_m)\} = 0.
\]
Show if \(f \in L^p(\mathbb{R}^n), \lim_{m \to \infty} f \ast \phi_m = f\) in \(L^p(\mathbb{R}^n)\).

9. Let \(\phi : \mathbb{R} \to \mathbb{R}\) be convex. This means
\[
\phi(\lambda x + (1 - \lambda)y) \leq \lambda \phi(x) + (1 - \lambda)\phi(y)
\]
whenever \(\lambda \in [0, 1]\). Verify that if \(x < y < z\), then \(\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(x)}{z - x}\) and that \(\frac{\phi(z) - \phi(y)}{z - y} \leq \frac{\phi(z) - \phi(x)}{z - x}\). Show if \(s \in \mathbb{R}\) there exists \(\lambda\) such that \(\phi(s) \leq \phi(t) + \lambda(s - t)\) for all \(t\). Show that if \(\phi\) is convex, then \(\phi\) is continuous.

10. \(\dagger\) Prove Jensen’s inequality. If \(\phi : \mathbb{R} \to \mathbb{R}\) is convex, \(\mu(\Omega) = 1\), and \(f : \Omega \to \mathbb{R}\) is in \(L^1(\Omega)\), then \(\phi(\int_{\Omega} f \, du) \leq \int_{\Omega} \phi(f) \, du\). **Hint:** Let \(s = \int_{\Omega} f \, du\) and use Problem 3.

11. Let \(\frac{1}{p} + \frac{1}{p'} = 1, p > 1, \text{ let } f \in L^p(\mathbb{R}), \ g \in L^{p'}(\mathbb{R})\). Show \(f \ast g\) is uniformly continuous on \(\mathbb{R}\) and \(|(f \ast g)(x)| \leq ||f||_{L^p} ||g||_{L^{p'}}\). **Hint:** You need to consider why \(f \ast g\) exists and then this follows from the definition of convolution and continuity of translation in \(L^p\).

12. \(B(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx, \Gamma(p) = \int_0^\infty e^{-t} t^{p-1} \, dt\) for \(p, q > 0\). The first of these is called the beta function, while the second is the gamma function. Show a.) \(\Gamma(p + 1) = p \Gamma(p)\); b.) \(\Gamma(p) \Gamma(q) = B(p,q) \Gamma(p+q)\).

13. Let \(f \in C_c(0,\infty)\) and define \(F(x) = \frac{1}{x} \int_0^x f(t) \, dt\). Show
\[
||F||_{L^p(0,\infty)} \leq \frac{p}{p - 1} ||f||_{L^p(0,\infty)} \quad \text{whenever} \quad p > 1.
\]
13.6. EXERCISES

**Hint:** Argue there is no loss of generality in assuming \( f \geq 0 \) and then assume this is so. Integrate \( \int_0^\infty |F(x)|^p dx \) by parts as follows:

\[
\int_0^\infty F^p dx = \frac{\text{show} = 0}{\infty} - p \int_0^\infty xF^{p-1}F' dx.
\]

Now show \( xF' = f - F \) and use this in the last integral. Complete the argument by using Holder’s inequality and \( p - 1 = p/q \).

14. ↑ Now suppose \( f \in L^p(0, \infty), p > 1, \) and \( f \) not necessarily in \( C_c(0, \infty) \). Show that \( F(x) = \frac{1}{2} \int_0^x f(t)dt \) still makes sense for each \( x > 0 \). Show the inequality of Problem 13 is still valid. This inequality is called Hardy’s inequality. **Hint:** To show this, use the above inequality along with the density of \( C_c(0, \infty) \) in \( L^p(0, \infty) \).

15. Suppose \( f, g \geq 0 \). When does equality hold in Holder’s inequality?

16. Prove Vitali’s Convergence theorem: Let \( \{f_n\} \) be uniformly integrable and complex valued, \( \mu(\Omega) < \infty, f_n(x) \to f(x) \) a.e. where \( f \) is measurable. Then \( f \in L^1 \) and \( \lim_{n \to \infty} \int_\Omega |f_n - f| d\mu = 0 \). **Hint:** Use Egoroff’s theorem to show \( \{f_n\} \) is a Cauchy sequence in \( L^1(\Omega) \). This yields a different and easier proof than what was done earlier. See Theorem 4.5.3 on Page 244.

17. ↑ Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces but there exist examples where the Vitali convergence theorem works and the dominated convergence theorem does not.

18. ↑ Suppose \( \mu(\Omega) < \infty, \{f_n\} \subseteq L^1(\Omega) \), and

\[
\int_\Omega h(|f_n|) d\mu < C
\]

for all \( n \) where \( h \) is a continuous, nonnegative function satisfying

\[
\lim_{t \to \infty} \frac{h(t)}{t} = \infty.
\]

Show \( \{f_n\} \) is uniformly integrable. In applications, this often occurs in the form of a bound on \( ||f_n||_p \).

19. ↑ Sometimes, especially in books on probability, a different definition of uniform integrability is used than that presented here. A set of functions, \( \mathcal{G}, \) defined on a finite measure space, \( (\Omega, \mathcal{S}, \mu) \) is said to be uniformly integrable if for all \( \varepsilon > 0 \) there exists \( \alpha > 0 \) such that for all \( f \in \mathcal{G}, \)

\[
\int_{|f| \geq \alpha} |f| d\mu \leq \varepsilon.
\]
CHAPTER 13. THE L^P SPACES

Show that this definition is equivalent to the definition of uniform integrability given earlier in Definition 9.5.1 on Page 243 with the addition of the condition that there is a constant, C < ∞ such that

\[ \int |f| \, d\mu \leq C \]

for all f ∈ ℘.

20. f ∈ L^∞(Ω, μ) if there exists a set of measure zero, E, and a constant C < ∞ such that |f(x)| ≤ C for all x ∈ E.

\[ \|f\|_\infty \equiv \inf \{ C : |f(x)| \leq C \text{ a.e.} \} \]

Show |||f|||_\infty is a norm on L^∞(Ω, μ) provided f and g are identified if f(x) = g(x) a.e. Show L^∞(Ω, μ) is complete. Hint: You might want to show that |||f|||_\infty > ε has measure zero so |||f|||_\infty is the smallest number at least as large as |f(x)| for a.e. x. Thus |||f|||_\infty is one of the constants, C in the above.

21. Suppose f ∈ L^∞ ∩ L^1. Show \( \lim_{p \to \infty} \|f\|_{L^p} = \|f\|_\infty \). Hint:

\[ (\|f\|_\infty - \varepsilon)^p \mu(\{ |f| > \|f\|_\infty - \varepsilon \}) \leq \int_{\{ |f| > \|f\|_\infty - \varepsilon \}} |f|^p \, d\mu \leq \int |f|^p \, d\mu = \int |f|^{p-1} |f| \, d\mu \leq \|f\|^{p-1}_\infty \int |f| \, d\mu. \]

Now raise both ends to the 1/p power and take lim inf and lim sup as \( p \to \infty \). You should get \( \|f\|_\infty - \varepsilon \leq \lim inf \|f\|_p \leq \lim sup \|f\|_p \leq \|f\|_\infty \).

22. Suppose \( \mu(\Omega) < \infty \). Show that if 1 ≤ p < q, then L^q(Ω) ⊆ L^p(Ω). Hint Use Holder’s inequality.

23. Show L^1(\mathbb{R}) \nsubseteq L^2(\mathbb{R}) and L^2(\mathbb{R}) \nsubseteq L^1(\mathbb{R}) if Lebesgue measure is used. Hint: Consider \( 1/\sqrt{x} \) and \( 1/x \).

24. Suppose that \( \theta \in [0, 1] \) and \( r, s, q > 0 \) with

\[ \frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}. \]

show that

\[ (\int |f|^q d\mu)^{1/q} \leq ((\int |f|^r d\mu)^{1/r})^\theta ((\int |f|^s d\mu)^{1/s})^{1-\theta}. \]

If \( q, r, s \geq 1 \) this says that

\[ \|f\|_q \leq \|f\|_r^\theta \|f\|_s^{1-\theta}. \]
Using this, show that
\[ \ln \left( \|f\|_q \right) \leq \theta \ln \left( \|f\|_r \right) + (1 - \theta) \ln \left( \|f\|_s \right). \]

Hint:
\[ \int |f|^q d\mu = \int |f|^\theta |f|^\theta (1 - \theta) d\mu. \]

Now note that 1 = \frac{\theta q}{r} + \frac{q(1-\theta)}{s} and use Holder’s inequality.

25. Suppose \( f \) is a function in \( L^1(\mathbb{R}) \) and \( f \) is infinitely differentiable. Is \( f' \in L^1(\mathbb{R}) \)?

Hint: What if \( \phi \in C^\infty_c(0,1) \) and \( f(x) = \phi(2^n(x - n)) \) for \( x \in (n, n + 1) \), \( f(x) = 0 \) if \( x < 0 \)?
Chapter 14

Stone’s Theorem And Partitions Of Unity

This section is devoted to Stone’s theorem which says that a metric space is paracompact, defined below. See [38] for this which is where I read it. First is the definition of what is meant by a refinement.

Definition 14.0.1 Let $S$ be a topological space. We say that a collection of sets $D$ is a refinement of an open cover $\mathcal{S}$, if every set of $D$ is contained in some set of $\mathcal{S}$. An open refinement would be one in which all sets are open, with a similar convention holding for the term “closed refinement”.

Definition 14.0.2 We say that a collection of sets $D$, is locally finite if for all $p \in S$, there exists $V$ an open set containing $p$ such that $V$ has nonempty intersection with only finitely many sets of $D$.

Definition 14.0.3 We say $S$ is paracompact if it is Hausdorff and for every open cover $\mathcal{S}$, there exists an open refinement $\mathcal{D}$ such that $\mathcal{D}$ is locally finite and $\mathcal{D}$ covers $S$.

Theorem 14.0.4 If $\mathcal{D}$ is locally finite then

$$\bigcup\{D : D \in \mathcal{D}\} = \bigcup\{D : D \in \mathcal{D}\}.$$ 

Proof: It is clear the left side is a subset of the right. Let $p$ be a limit point of

$$\bigcup\{D : D \in \mathcal{D}\}$$

and let $p \in V$, an open set intersecting only finitely many sets of $\mathcal{D}$, $D_1...D_n$. If $p$ is not in any of $D_i$ then $p \in W$ where $W$ is some open set which contains no points of $\bigcup_{i=1}^n D_i$. Then $V \cap W$ contains no points of any set of $\mathcal{D}$ and this contradicts the assumption that $p$ is a limit point of

$$\bigcup\{D : D \in \mathcal{D}\}.$$
Thus $p \in \overline{D_i}$ for some $i$. ■

We say $\mathcal{S} \subseteq \mathcal{P}(S)$ is countably locally finite if

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

and each $\mathcal{S}_n$ is locally finite. The following theorem appeared in the 1950’s. It will be used to prove Stone’s theorem.

**Theorem 14.0.5** Let $S$ be a regular topological space. (If $p \in U$ open, then there exists an open set $V$ such that $p \in \overline{V} \subseteq U$.) The following are equivalent

1.) Every open covering of $S$ has a refinement that is open, covers $S$ and is countably locally finite.

2.) Every open covering of $S$ has a refinement that is locally finite and covers $S$. (The sets in refinement maybe not open.)

3.) Every open covering of $S$ has a refinement that is closed, locally finite, and covers $S$. (Sets in refinement are closed.)

4.) Every open covering of $S$ has a refinement that is open, locally finite, and covers $S$. (Sets in refinement are open.)

**Proof:**

1.) $\Rightarrow$ 2.)

Let $\mathcal{S}$ be an open cover of $S$ and let $\mathcal{B}$ be an open countably locally finite refinement

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$$

where $\mathcal{B}_n$ is an open refinement of $\mathcal{S}$ and $\mathcal{B}_n$ is locally finite. For $B \in \mathcal{B}_n$, let

$$E_n(B) = B \setminus \bigcup_{k<n} (\bigcup \{B : B \in \mathcal{B}_k\}).$$

Thus, in words, $E_n(B)$ consists of points in $B$ which are not in any set from any $\mathcal{B}_k$ for $k < n$.

**Claim:** $\{E_n(B) : n \in \mathbb{N}, B \in \mathcal{B}_n\}$ is locally finite.

**Proof of the claim:** Let $p \in S$. Then $p \in B_0 \in \mathcal{B}_n$ for some $n$. Let $V$ be open, $p \in V$, and $V$ intersects only finitely many sets of $\mathcal{B}_1 \cup \ldots \cup \mathcal{B}_n$. Then consider $B_0 \cap V$. If $m > n$,

$$(B_0 \cap V) \cap E_m(B) \subseteq \left[ \bigcup_{k<m} (\bigcup \{B : B \in \mathcal{B}_k\}) \right] \subseteq B_0^C.$$

In words, $E_m(B)$ has nothing in it from any of the $\mathcal{B}_k$ for $k < m$. In particular, it has nothing in it from $B_0$. Thus $(B_0 \cap V) \cap E_m(B) = \emptyset$ for $m > n$. Thus $p \in B_0 \cap V$ which intersects only finitely many sets of $\mathcal{S}$, no more than those intersected by $V$.

This establishes the claim.

**Claim:** $\{E_n(B) : n \in \mathbb{N}, B \in \mathcal{B}_n\}$ covers $S$.

**Proof:** Let $p \in S$ and let $n = \min \{k \in \mathbb{N} : p \in B$ for some $B \in \mathcal{B}_k\}$. Let $p \in B \in \mathcal{B}_n$. Then $p \in E_n(B)$. 


The two claims show that 1.\(\Rightarrow\) 2).

2.\(\Rightarrow\) 3.)
Let \(\mathcal{S}\) be an open cover and let
\[
\mathcal{G} \equiv \{U : U \text{ is open and } \overline{U} \subseteq V \in \mathcal{S} \text{ for some } V \in \mathcal{S}\}.
\]
Then since \(\mathcal{S}\) is regular, \(\mathcal{G}\) covers \(\mathcal{S}\). (If \(p \in \mathcal{S}\), then \(p \in U \subseteq \overline{U} \subseteq V \in \mathcal{S}\).) By 2.), \(\mathcal{G}\) has a locally finite refinement \(\mathcal{C}\), covering \(\mathcal{S}\). Consider
\[
\{E : E \in \mathcal{C}\}.
\]
This collection of closed sets covers \(\mathcal{S}\) and is locally finite because if \(p \in \mathcal{S}\), there exists \(V, p \in V, \) and \(V\) has nonempty intersections with only finitely many elements of \(\mathcal{C}\), say \(E_1, \ldots, E_n\). If \(\overline{E} \cap V \neq \emptyset\), then \(E \cap V \neq \emptyset\) and so \(V\) intersects only \(E_1, \ldots, E_n\). This shows 2.)\(\Rightarrow\) 3.)

3.\(\Rightarrow\) 4.) Here is a table of symbols with a short summary of their meaning.

<table>
<thead>
<tr>
<th>Open covering</th>
<th>Locally finite refinement</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathcal{S}) original covering</td>
<td>(\mathcal{B}) by 3. can be closed refinement</td>
</tr>
<tr>
<td>(\mathfrak{F}) open interectors</td>
<td>(\mathcal{C}) closed refinement</td>
</tr>
</tbody>
</table>

Let \(\mathcal{S}\) be an open cover and let \(\mathcal{B}\) be a locally finite refinement which covers \(\mathcal{S}\). By 3.) we can take \(\mathcal{B}\) to be a closed refinement but this is not important here. Let
\[
\mathfrak{F} \equiv \{U : U \text{ is open and } U \text{ intersects only finitely many sets of } \mathcal{B}\}.
\]
Then \(\mathfrak{F}\) covers \(\mathcal{S}\) because \(\mathcal{B}\) is locally finite. If \(p \in \mathcal{S}\), then there exists an open set \(U\) containing \(p\) which intersects only finitely many sets of \(\mathcal{B}\). Thus \(p \in U \in \mathfrak{F}\). By 3., \(\mathfrak{F}\) has a locally finite closed refinement \(\mathcal{C}\), which covers \(\mathcal{S}\). Define for \(B \in \mathcal{B}\)
\[
\mathcal{C}(B) \equiv \{C \in \mathcal{C} : C \cap B = \emptyset\}
\]
Thus these closed sets \(C\) do not intersect \(B\) and so \(B\) is in their complement. We use \(\mathcal{C}(B)\) to fatten up \(B\). Let
\[
E(B) \equiv (\cup\{C : C \in \mathcal{C}(B)\})^C.
\]
In words, \(E(B)\) is the complement of the union of all closed sets of \(\mathcal{C}\) which do not intersect \(B\). Thus \(E(B) \supseteq B\), and has fattened up \(B\). Then since \(\mathcal{C}(B)\) is locally finite, \(E(B)\) is an open set by Theorem \[\text{[14.4.3.]}\]. Now let \(F(B)\) be defined such that for \(B \in \mathcal{B}\),
\[
B \subseteq F(B) \in \mathcal{S}
\]
(by definition \(B\) is in some set of \(\mathcal{S}\)), and let
\[
\mathcal{L} = \{E(B) \cap F(B) : B \in \mathcal{B}\}
\]
The intersection with \(F(B)\) is to ensure that \(\mathcal{L}\) is a refinement of \(\mathcal{S}\). The important thing to notice is that if \(C \in \mathcal{C}\) intersects \(E(B)\), then it must also intersect
If not, you could include it in the list of closed sets which do not intersect $B$ and whose complement is $E(B)$. Thus $E(B)$ would be too large.

**Claim:** $\mathcal{L}$ covers $S$.

This claim is obvious because if $p \in S$ then $p \in B$ for some $B \in \mathcal{B}$. Hence

$$p \in E(B) \cap F(B) \in \mathcal{L}.$$  

**Claim:** $\mathcal{L}$ is locally finite and a refinement of $\mathcal{G}$.

**Proof:** It is clear $\mathcal{L}$ is a refinement of $\mathcal{G}$ because every set of $\mathcal{L}$ is a subset of a set of $\mathcal{G}$, $F(B)$. Let $p \in S$. There exists an open set $W$, such that $p \in W$ and $W$ intersects only $C_1, \ldots, C_n$, elements of $\mathcal{C}$. Hence $W \subseteq \bigcup_{i=1}^n C_i$ since $\mathcal{C}$ covers $S$.

The following theorem is Stone’s theorem.

**Theorem 14.0.6** If $S$ is a metric space then $S$ is paracompact (Every open cover has a locally finite open refinement also an open cover.)

**Proof:** Let $\mathcal{G}$ be an open cover. Well order $\mathcal{G}$. For $B \in \mathcal{G}$,

$$B_n \equiv \{x \in B : \text{dist} \left( x, B^c \right) < \frac{1}{2^n} \}, \ n = 1, 2, \ldots$$

Thus $B_n$ is contained in $B$ but approximates it up to $2^{-n}$. Let

$$E_n(B) = B_n \setminus \cup \{D : D \prec B \text{ and } D \neq B\}$$

where $\prec$ denotes the well order. If $B, D \in \mathcal{G}$, then one is first in the well order. Let $D \prec B$. Then from the construction, $E_n(B) \subseteq D^c$ and $E_n(D)$ is further than $1/2^n$ from $D^c$. Hence, assuming neither set is empty,

$$\text{dist} \left( E_n(B), E_n(D) \right) \geq 2^{-n}$$
for all $B, D \in \mathcal{S}$. Fatten up $E_n(B)$ as follows.

$$\widehat{E_n(B)} \equiv \{B (x, 8^{-n}) : x \in E_n(B)\}.$$

Thus $\widehat{E_n(B)} \subseteq B$ and

$$\text{dist} \left( \widehat{E_n(B)}, \widehat{E_n(D)} \right) \geq \frac{1}{2^n} - 2 \left( \frac{1}{8} \right)^n \equiv \delta_n > 0.$$

It follows that the collection of open sets

$$\{\widehat{E_n(B)} : B \in \mathcal{S}\} \equiv \mathcal{B}_n$$

is locally finite. In fact, $B \left( p, \frac{\delta_n}{2} \right)$ cannot intersect more than one of them. In addition to this,

$$S \subseteq \bigcup \{\widehat{E_n(B)} : n \in \mathbb{N}, B \in \mathcal{S}\}$$

because if $p \in S$, let $B$ be the first set in $\mathcal{S}$ to contain $p$. Then $p \in E_n(B)$ for $n$ large enough because it will not be in anything deleted. Thus this is an open countably locally finite refinement. Thus 1.) in the above theorem is satisfied.

\section{14.1 Partitions Of Unity And Stone’s Theorem}

First observe that if $S$ is a nonempty set, then $\text{dist} (x, S)$ satisfies $|\text{dist} (x, S) - \text{dist} (y, S)| \leq d (x, y)$. To see this,

$$|\text{dist} (x, S) - \text{dist} (y, S)| \leq d (x, y)$$

To see this, say $\text{dist} (x, S)$ is the larger of the two. Then there exists $z \in S$ such that

$$\text{dist} (y, S) \geq d (y, z) - \varepsilon$$

It follows that

$$|\text{dist} (x, S) - \text{dist} (y, S)| = \text{dist} (x, S) - \text{dist} (y, S)$$

$$\leq \text{dist} (x, S) - (d (y, z) - \varepsilon)$$

$$\leq d (x, z) - d (y, z) + \varepsilon$$

$$\leq d (x, y) + d (y, z) - d (y, z) + \varepsilon = d (x, y) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, this shows the desired conclusion.

\textbf{Theorem 14.1.1} Let $S$ be a metric space and let $\mathcal{S}$ be any open cover of $S$. Then there exists a set $\mathfrak{F}$, an open refinement of $\mathcal{S}$, and functions $\{\phi_F : F \in \mathfrak{F}\}$ such that

$$\phi_F : S \to [0, 1]$$
\( \phi_F \) is continuous
\( \phi_F (x) \) equals 0 for all but finitely many \( F \in \mathcal{F} \)
\[ \sum \{ \phi_F (x) : F \in \mathcal{F} \} = 1 \text{ for all } x \in S. \]

Each \( \phi_F \) is locally Lipschitz continuous which means that for each \( z \) there is an open set \( W \) containing \( z \) for which, if \( x, y \in W \), then there is a constant \( K \) such that
\[ |\phi_F (x) - \phi_F (y)| \leq K d(x, y). \]

**Proof:** By Stone’s theorem, there exists a locally finite refinement \( \mathcal{F} \) covering \( S \). For \( F \in \mathcal{F} \)
\[ g_F (x) \equiv \text{dist} (x, F_C) \]
Let
\[ \phi_F (x) \equiv (\sum \{ g_F (x) : F \in \mathcal{F} \})^{-1} g_F (x). \]
Now
\[ \sum \{ g_F (x) : F \in \mathcal{F} \} \]
is a continuous function because if \( x \in S \), then there exists an open set \( W \) with \( x \in W \) and \( W \) has nonempty intersection with only finitely many sets of \( F \in \mathcal{F} \). Then for \( y \in W \),
\[ \sum \{ g_F (y) : F \in \mathcal{F} \} = \sum_{i=1}^{n} g_{F_i} (y). \]
Since \( \mathcal{F} \) is a cover of \( S \),
\[ \sum \{ g_F (x) : F \in \mathcal{F} \} \neq 0 \]
for any \( x \in S \). Hence \( \phi_F \) is continuous. This also shows \( \phi_F (x) = 0 \) for all but finitely many \( F \in \mathcal{F} \). It is obvious that
\[ \sum \{ \phi_F (x) : F \in \mathcal{F} \} = 1 \]
from the definition.

Let \( z \in S \). Then there is an open set \( W \) containing \( z \) such that \( W \) has nonempty intersection with only finitely many \( F \in \mathcal{F} \). Thus for \( y, x \in W \),
\[ |\phi_{F_j} (x) - \phi_{F_j} (y)| \leq \left| \frac{g_{F_j} (x) \sum_{i=1}^{n} g_{F_i} (y) - g_{F_j} (y) \sum_{i=1}^{n} g_{F_i} (x)}{\sum_{i=1}^{n} g_{F_i} (x) \sum_{i=1}^{n} g_{F_i} (y)} \right| \]
If \( F \) is not one of these \( F_i \), then \( g_F (x) = \phi_F (x) = \phi_F (y) = g_F (y) = 0 \). Thus there is nothing to show for these. It suffices to consider the ones above. Restricting \( W \) if necessary, we can assume that for \( x \in W \),
\[ \sum_{F} g_F (x) = \sum_{i=1}^{n} g_{F_i} (x) > \delta > 0, \quad g_{F_j} (x) < \Delta < \infty, \quad j \leq n \]
Then, simplifying the above, and letting \( x, y \in W \), for each \( j \leq n \),
\[
\left| \phi_{F_j}(x) - \phi_{F_j}(y) \right| \leq \frac{1}{\delta^2} \left| g_{F_j}(x) \sum_F g_F(y) - g_{F_j}(y) \sum_F g_F(x) \right|
\]
\[
\leq \frac{1}{\delta^2} \Delta |g_{F_j}(x) - g_{F_j}(y)| + \frac{1}{\delta^2} \Delta \sum_{i=1}^{n} |g_{F_i}(y) - g_{F_i}(x)|
\]
\[
\leq \frac{\Delta}{\delta^2} d(x, y) + \frac{\Delta}{\delta^2} nd(x, y) = (n + 1) \frac{\Delta}{\delta^2} d(x, y)
\]
Thus on this set \( W \) containing \( z \), all \( \phi_F \) are Lipschitz continuous with Lipschitz constant \((n + 1) \frac{\Delta}{\delta^2}\).

The functions described above are called a partition of unity subordinate to the open cover \( \mathcal{G} \). A useful observation is contained in the following corollary.

**Corollary 14.1.2** Let \( S \) be a metric space and let \( \mathcal{G} \) be any open cover of \( S \). Then there exists a set \( \mathfrak{F} \), an open refinement of \( \mathcal{G} \), and functions \( \{ \phi_F : F \in \mathfrak{F} \} \) such that
\[
\phi_F : S \to [0, 1]
\]
\( \phi_F \) is continuous
\( \phi_F(x) \) equals 0 for all but finitely many \( F \in \mathfrak{F} \)
\( \sum \{ \phi_F(x) : F \in \mathfrak{F} \} = 1 \) for all \( x \in S \).

Each \( \phi_F \) is Lipschitz continuous. If \( U \in \mathcal{G} \) and \( H \) is a closed subset of \( U \), the partition of unity can be chosen such that each \( \phi_F = 0 \) on \( H \) except for one which equals 1 on \( H \).

**Proof:** Just change your open cover to consist of \( U \) and \( V \setminus H \) for each \( V \in \mathcal{G} \). Then every function but one equals 0 on \( H \) and so exactly one of them equals 1 on \( H \).

**14.2 An Extension Theorem, Retracts**

**Lemma 14.2.1** Let \( A \) be a closed set in a metric space and let \( x_n \notin A, x_n \to a_0 \in A \) and \( a_n \in A \) such that \( d(a_n, x_n) < 6 \text{dist}(x_n, A) \). Then \( a_n \to a_0 \).

**Proof:** By assumption,
\[
d(a_n, a_0) \leq d(a_n, x_n) + d(x_n, a_0) < 6 \text{dist}(x_n, A) + d(x_n, a_0)
\]
\[
\leq 6d(x_n, a_0) + d(x_n, a_0) = 7d(x_n, a_0)
\]
and this converges to 0.
Note that there was nothing magic about 6 in the above. Another number would work as well.

In the proof of the following theorem, you get a covering of $A^c$ with open balls $B$ such that for each of these balls, there exists $a \in A$ such that for all $x \in B$, $\|x - a\| \leq 6 \text{dist}(x, A)$. The 6 is not important. Any other constant with this property would work. Then you use Stone’s theorem.

A Banach space is a normed vector space which is also a complete metric space where the metric comes from the norm.

$$d(x, y) = \|x - y\|$$

Thus you can add things in a Banach space. Much more will be considered about Banach spaces a little later.

**Definition 14.2.2** A Banach space is a complete normed linear space. If you have a subset $B$ of a Banach space, then $\text{conv}(B)$ denotes the smallest closed convex set which contains $B$. It can be obtained by taking the intersection of all closed convex sets containing $B$. Recall that a set $C$ is convex if whenever $x, y \in C$, then so is $\lambda x + (1 - \lambda) y$ for all $\lambda \in [0, 1]$. Note how this makes sense in a vector space but maybe not in a general metric space.

In the following theorem, we have in mind both $X$ and $Y$ are Banach spaces, but this is not needed in the proof. All that is needed is that $X$ is a metric space and $Y$ a normed linear space or possibly something more general in which it makes sense to do addition and scalar multiplication.

**Theorem 14.2.3** Let $A$ be a closed subset of a metric space $X$ and let $F : A \to Y$, $Y$ a normed linear space. Then there exists an extension of $F$ denoted as $\hat{F}$ such that $\hat{F}$ is defined on all of $X$ and agrees with $F$ on $A$. It has values in $\text{conv}(F(A))$, the convex hull of $F(A)$.

**Proof:** For each $c \notin A$, let $B_c$ be a ball contained in $A^c$ centered at $c$ where distance of $c$ to $A$ is at least $\text{diam}(B_c)$.
So for \( x \in B_c \) what about \( \text{dist} (x, A) \)? How does it compare with \( \text{dist} (c, A) \)?

\[
\text{dist} (c, A) \leq d (c, x) + \text{dist} (x, A) \\
\leq \frac{1}{2} \text{diam} (B_c) + \text{dist} (x, A) \\
\leq \frac{1}{2} \text{dist} (c, A) + \text{dist} (x, A)
\]

so

\[
\text{dist} (c, A) \leq 2 \text{dist} (x, A)
\]

Now the following is also valid. Letting \( x \in B_c \) be arbitrary, it follows from the assumption on the diameter that there exists \( a_0 \in A \) such that

\[
d (c, a_0) < 2 \text{dist} (c, A)
\]

Then

\[
d (x, a_0) \leq \sup_{y \in B_c} d (y, a_0) \leq \sup_{y \in B_c} (d (y, c) + d (c, a_0)) \leq \frac{\text{diam} (B_c)}{2} + 2 \text{dist} (c, A)
\]

\[
\leq \frac{\text{dist} (c, A)}{2} + 2 \text{dist} (c, A) < 3 \text{dist} (c, A)
\]

It follows from (14.2.1)

\[
d (x, a_0) \leq 3 \text{dist} (c, A) \leq 6 \text{dist} (x, A)
\]

Thus for any \( x \in B_c \), there is an \( a_0 \in A \) such that \( d (x, a_0) \) is bounded by a fixed multiple of the distance from \( x \) to \( A \).

By Stone’s theorem, there is a locally finite open refinement \( R \). These are open sets each of which is contained in one of the balls just mentioned such that each of these balls is the union of sets of \( R \). Thus \( R \) is a locally finite cover of \( A^C \). Since \( x \in A^C \) is in one of those balls, it was just shown that there exists \( a_R \in A \) such that for all \( x \in R \in \mathcal{R} \) we have \( d (x, a_R) \leq 6 \text{dist} (x, A) \). Of course there may be more than one because \( R \) might be contained in more than one of those special balls. One \( a_R \) is chosen for each \( R \in \mathcal{R} \).

Now let \( \phi_R (x) \equiv \text{dist} (x, R^C) \). Then let

\[
\hat{F} (x) \equiv \begin{cases} 
F (x) & \text{for } x \in A \\
\sum_{R \in \mathcal{R}} F (a_R) \frac{\phi_R (x)}{\sum_{R \in \mathcal{R}} \phi_R (x)} & \text{for } x \notin A
\end{cases}
\]

The sum in the bottom is always finite because the covering is locally finite. Also, this sum is never 0 because \( \mathcal{R} \) is a covering. Also \( \hat{F} \) has values in \( \text{conv} (F (K)) \). It only remains to verify that \( \hat{F} \) is continuous. It is clearly so on the interior of \( A \) thanks to continuity of \( F \). It is also clearly continuous on \( A^C \) because the functions \( \phi_R \) are continuous. So it suffices to consider \( x_n \to a \in \partial A \subseteq A \) where \( x_n \notin A \) and see whether \( F (a) = \lim_{n \to \infty} \hat{F} (x_n) \).

Suppose this does not happen. Then there is a sequence converging to some \( a \in \partial A \) and \( \varepsilon > 0 \) such that

\[
\varepsilon \leq \| \hat{F} (a) - \hat{F} (x_n) \| \quad \text{all } n
\]
For \( x_n \in R \), it was shown above that \( d (x_n, a_{R_n}) \leq 6 \text{dist} (x_n, A) \). By the above Lemma \[14.2.1\], it follows that \( a_{R_n} \to a \) and so \( F(a_{R_n}) \to F(a) \).

\[
\varepsilon \leq \left\| \hat{F}(a) - \hat{F}(x_n) \right\| \leq \sum_{R \in R} \frac{\| F(a_{R_n}) - F(a) \| \phi_R(x_{R_n})}{\sum_{R \in R} \phi_R(x_{R_n})}
\]

By local finiteness of the cover, each \( x_n \) involves only finitely many \( R \) Thus, in this limit process, there are countably many \( R \) involved \( \{R_j\}_{j=1}^{\infty} \). Thus one can apply Fatou’s lemma.

\[
\varepsilon \leq \lim \inf_{n \to \infty} \left\| \hat{F}(a) - \hat{F}(x_n) \right\|
\]

\[
\leq \sum_{j=1}^{\infty} \lim \inf_{n \to \infty} \left\| F(a_{R_j,n}) - F(a) \right\| \frac{\phi_{R_j}(x_{R_j,n})}{\sum_{j=1}^{\infty} \phi_{R_j}(x_{R_j,n})}
\]

\[
\leq \sum_{j=1}^{\infty} \lim \inf_{n \to \infty} \left\| F(a_{R_j,n}) - F(a) \right\| = 0 \quad \blacksquare
\]

The last step is needed because you lose local finiteness as you approach \( \partial A \). Note that the only thing needed was that \( X \) is a metric space. The addition takes place in \( Y \) so it needs to be a vector space. Did it need to be complete? No, this was not used. Nor was completeness of \( X \) used. The main interest here is in Banach spaces, but the result is more general than that.

It also appears that \( \hat{F} \) is locally Lipschitz on \( A^C \).

**Definition 14.2.4** Let \( S \) be a subset of \( X \), a Banach space. Then it is a retract if there exists a continuous function \( R : X \to S \) such that \( Rs = s \) for all \( s \in S \). This \( R \) is a retraction. More generally, \( S \subseteq T \) is called a retract of \( T \) if there is a continuous \( R : T \to S \) such that \( Rs = s \) for all \( s \in S \).

**Theorem 14.2.5** Let \( K \) be closed and convex subset of \( X \) a Banach space. Then \( K \) is a retract.

**Proof:** By Theorem \[14.2.4\], there is a continuous function \( \hat{I} \) extending \( I \) to all of \( X \). Then also \( \hat{I} \) has values in \( \text{conv}(IK) = \text{conv}(K) = K \). Hence \( \hat{I} \) is a continuous function which does what is needed. It maps everything into \( K \) and keeps the points of \( K \) unchanged. \( \blacksquare \)

Sometimes people call the set a retraction also or the function which does the job a retraction. This seems like strange thing to call it because a retraction is the act of repudiating something you said earlier. Nevertheless, I will call it that. Note that if \( S \) is a retract of the whole metric space \( X \), then it must be a retract of every set which contains \( S \).

### 14.3 Something Which Is Not A Retract

The next lemma is a fundamental result which will be used to develop the Brouwer degree. It will also be used to give a short proof of the Brouwer fixed point theorem.
14.3. SOMETHING WHICH IS NOT A RETRACT

in the exercises. This major fixed point theorem is probably the most fundamental theorem in nonlinear analysis. The proof outlined in the exercises is from [15].

**Lemma 14.3.1** Let \( g : U \to \mathbb{R}^n \) be \( C^2 \) where \( U \) is an open subset of \( \mathbb{R}^n \). Then

\[
\sum_{j=1}^{n} \text{cof} \left( Dg \right)_{ij,j} = 0,
\]

where here \( (Dg)_{ij} \equiv g_{ij} \equiv \frac{\partial g_i}{\partial x_j} \). Also, \( \text{cof} \left( Dg \right)_{ij} = \frac{\partial \det(Dg)}{\partial g_{ij}} \).

**Proof:** From the cofactor expansion theorem,

\[
\det(Dg) = \sum_{i=1}^{n} g_{i,j} \text{cof}(Dg)_{ij}
\]

and so

\[
\frac{\partial \det(Dg)}{\partial g_{ij}} = \text{cof}(Dg)_{ij} \quad (14.3.2)
\]

which shows the last claim of the lemma. Also

\[
\delta_{kj} \det(Dg) = \sum_i g_{i,k} \left( \text{cof}(Dg) \right)_{ij} \quad (14.3.3)
\]

because if \( k \neq j \) this is just the cofactor expansion of the determinant of a matrix in which the \( k^{th} \) and \( j^{th} \) columns are equal. Differentiate [14.3.2] with respect to \( x_j \) and sum on \( j \). This yields

\[
\sum_{r,s,j} \delta_{kj} \frac{\partial \left( \det(Dg) \right)}{\partial g_{rs}} g_{r,sj} = \sum_{ij} g_{i,k} \left( \text{cof}(Dg) \right)_{ij} + \sum_{ij} g_{i,k} \text{cof}(Dg)_{ij,j}.
\]

Hence, using \( \delta_{kj} = 0 \) if \( j \neq k \) and [14.3.3],

\[
\sum_{rs} \left( \text{cof}(Dg) \right)_{rs} g_{r,sk} = \sum_{rs} g_{r,ks} \left( \text{cof}(Dg) \right)_{rs} + \sum_{ij} g_{i,k} \text{cof}(Dg)_{ij,j}.
\]

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

\[
\sum_i g_{i,k} \left( \sum_j \left( \text{cof}(Dg) \right)_{ij,j} \right) = 0.
\]

If \( \det(g_{i,k}) \neq 0 \) so that \( (g_{i,k}) \) is invertible, this shows \( \sum_j \left( \text{cof}(Dg) \right)_{ij,j} = 0 \). If \( \det(Dg) = 0 \), let

\[
g_k = g + \varepsilon_k I
\]

where \( \varepsilon_k \to 0 \) and \( \det(Dg + \varepsilon_k I) \equiv \det(Dg_k) \neq 0 \). Then

\[
\sum_j \left( \text{cof}(Dg) \right)_{ij,j} = \lim_{k \to \infty} \sum_j \left( \text{cof}(Dg_k) \right)_{ij,j} = 0 \]
Definition 14.3.2 Let $h$ be a function defined on an open set, $U \subseteq \mathbb{R}^n$. Then $h \in C^k(U)$ if there exists a function $g$ defined on an open set, $W$ containing $U$ such that $g = h$ on $U$ and $g$ is $C^k(W)$.

Lemma 14.3.3 There does not exist $h \in C^2(B(0,R))$ such that $h : B(0,R) \rightarrow \partial B(0,R)$ which also has the property that $h(x) = x$ for all $x \in \partial B(0,R)$. That is, there is no retraction of $B(0,R)$ to $\partial B(0,R)$.

Proof: Suppose such an $h$ exists. Let $\lambda \in [0,1]$ and let $p_\lambda(x) \equiv x + \lambda (h(x) - x)$. This function, $p_\lambda$ is a homotopy of the identity map and the retraction, $h$. Let

$$ I(\lambda) = \int_{B(0,R)} \det(Dp_\lambda(x)) \, dx. $$

Then using the dominated convergence theorem,

$$ I'(\lambda) = \int_{B(0,R)} \sum_{i,j} \frac{\partial \det(Dp_\lambda(x))}{\partial p_{\lambda ij}(x)} \, dx $$

$$ = \int_{B(0,R)} \sum_i \sum_j \frac{\partial \det(Dp_\lambda(x))}{\partial p_{\lambda ij}} (h_i(x) - x_i)_j \, dx $$

$$ = \int_{B(0,R)} \sum_i \sum_j \text{cof}(Dp_\lambda(x))_{ij} (h_i(x) - x_i)_j \, dx $$

Now by assumption, $h_i(x) = x_i$ on $\partial B(0,R)$ and so one can integrate by parts and write

$$ I'(\lambda) = - \sum_i \int_{B(0,R)} \sum_j \text{cof}(Dp_\lambda(x))_{ij} (h_i(x) - x_i)_j \, dx = 0. $$

Therefore, $I(\lambda)$ equals a constant. However,

$$ I(0) = m_n(B(0,R)) > 0 $$

but

$$ I(1) = \int_{B(0,R)} \det(Dh(x)) \, dm_n = \int_{\partial B(0,R)} # (y) \, dm_n = 0 $$

because from polar coordinates or other elementary reasoning, $m_n(\partial B(0,1)) = 0$. 

The last formula uses the change of variables formula for functions which are not one to one. In this formula, $#(y)$ equals the number of $x$ such that $h(x) = y$. To see this is so in case you have not seen this, note that $h$ is $C^1$ and so the inverse function theorem from advanced calculus applies. Thus

$$ \int_{B(0,R)} \det(Dh(x)) \, dm_n = \int_{\det(Dh(x)) > 0} \det(Dh(x)) \, dm_n + \int_{\det(Dh(x)) < 0} \det(Dh(x)) \, dm_n $$
Thus $h$ is locally one to one on the two open sets $[\det(Dh(x)) > 0], [\det(Dh(x)) < 0]$. Now use inverse function theorem and change of variables for one to one $h$ to verify that both of these integrals equal 0. You cover $[\det(Dh(x)) > 0]$ with countably many balls on which $h$ is one to one and then use change of variables for each of these integrals over $[\det(Dh(x)) > 0]$ intersected with this ball.

The following is the Brouwer fixed point theorem for $C^2$ maps.

**Lemma 14.3.4** If $h \in C^2\left(B(0,R)\right)$ and $h : B(0,R) \to \overline{B}(0,R)$, then $h$ has a fixed point, $x$ such that $h(x) = x$.

**Proof:** Suppose the lemma is not true. Then for all $x$, $|x - h(x)| \neq 0$. Then define

$$g(x) = h(x) + \frac{x - h(x)}{|x - h(x)|}t(x)$$

where $t(x)$ is nonnegative and is chosen such that $g(x) \in \partial B(0,R)$. This mapping is illustrated in the following picture.

If $x \to t(x)$ is $C^2$ near $\overline{B}(0,R)$, it will follow $g$ is a $C^2$ retraction onto $\partial B(0,R)$ contrary to Lemma 14.3.3. Thus $t(x)$ is the nonnegative solution to

$$H(x,t) \equiv |h(x)|^2 + 2 \left(h(x), \frac{x - h(x)}{|x - h(x)|}\right) t + t^2 = R^2$$  (14.3.4)

Then

$$H_t(x,t) = 2 \left(h(x), \frac{x - h(x)}{|x - h(x)|}\right) + 2t.$$  

If this is nonzero for all $x$ near $\overline{B}(0,R)$, it follows from the implicit function theorem that $t$ is a $C^2$ function of $x$. Then from 14.3.3

$$2t = -2 \left(h(x), \frac{x - h(x)}{|x - h(x)|}\right)$$

$$\pm \sqrt{4 \left(h(x), \frac{x - h(x)}{|x - h(x)|}\right)^2 - 4 \left(|h(x)|^2 - R^2\right)}$$

and so

$$H_t(x,t) = 2t + 2 \left(h(x), \frac{x - h(x)}{|x - h(x)|}\right)$$

$$\pm \sqrt{4 \left(R^2 - |h(x)|^2\right) + 4 \left(h(x), \frac{x - h(x)}{|x - h(x)|}\right)^2}$$
If $|h(x)| < R$, this is nonzero. If $|h(x)| = R$, then it is still nonzero unless

$$(h(x), x - h(x)) = 0.$$ 

But this cannot happen because the angle between $h(x)$ and $x - h(x)$ cannot be $\pi/2$. Alternatively, if the above equals zero, you would need

$$(h(x), x) = |h(x)|^2 = R^2$$

which cannot happen unless $x = h(x)$ which is assumed not to happen. Therefore, $x \to t(x)$ is $C^2$ near $\overline{B(0, R)}$ and so $g(x)$ given above contradicts Lemma [4.23].

Then the Brouwer fixed point theorem is as follows.

**Theorem 14.3.5** Let $f : \overline{B(0, R)} \to \overline{B(0, R)}$ be continuous, this being a ball in $\mathbb{R}^p$. Then it has a fixed point $x \in \overline{B(0, R)}$ such that $f(x) = x$.

**Proof:** You can extend $f$ to assume it is defined on all of $\mathbb{R}^p$, $f(\mathbb{R}^p) \subseteq \overline{B(0, R)}$, the convex hull of $\overline{B(0, R)}$. Then letting $\{\psi_n\}$ be a mollifier, let $f_n \equiv f \ast \psi_n$. Thus

$$|f_n(x)| = \left| \int_{\mathbb{R}^p} f(t) \psi_n(x - t) \, dt \right| \leq \int_{\mathbb{R}^p} |f(t)| \psi_n(x - t) \, dt \leq R \int_{\mathbb{R}^p} \psi_n(x - t) \, dt = R$$

and so the restriction of $f_n$ to $\overline{B(0, R)}$ is $C^2\left(\overline{B(0, R)}\right)$. Therefore, there exists $x_n \in \overline{B(0, R)}$ such that $f_n(x_n) = x_n$. The functions $f_n$ converge uniformly to $f$ on $\overline{B(0, R)}$.

$$|f(x) - f_n(x)| = \left| \int_{\overline{B(0, R)}} (f(x) - f(x - t)) \psi_n(t) \, dt \right| \leq \int_{\overline{B(0, R)}} |f(x) - f(x - t)| \psi_n(t) \, dt < \varepsilon$$

provided $n$ is large enough, this for every $x \in \overline{B(0, R)}$, this by uniform continuity of $f$ on $\overline{B(0, R + 1)}$. There exists a subsequence, still called $\{x_n\}$ which converges to $x \in \overline{B(0, R)}$. Then using the uniform convergence of $f_n$ to $f$,

$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f_n(x_n) = \lim_{n \to \infty} x_n = x \quad \blacksquare$$

**Definition 14.3.6** A nonempty topological space $A$ is said to have the fixed point property if every continuous mapping $f : A \to A$ has a fixed point.

### 14.4 Exercises

1. Suppose you have a Banach space $X$ and a set $A \subseteq X$. Suppose $A$ is a retract of $B$ where $B$ has the fixed point property. By this is meant that $A \subseteq B$ and there is a continuous function $f : B \to A$ such that $f$ equals the identity on $A$. Show that it follows that then $A$ also has the fixed point property.
2. Show that the fixed point property is a topological property. That is, if you have \( A, B \) two topological spaces and there is a continuous one to one onto mapping \( f : A \to B \) which has continuous inverse, then the two topological spaces either both have the fixed point property or neither one does.

3. The Brouwer fixed point theorem says that every closed ball in \( \mathbb{R}^n \) centered at 0 has the fixed point property. Show that it follows that every bounded convex closed set in \( \mathbb{R}^n \) has the fixed point property. **Hint:** You know that the closed convex set is a retract of \( \mathbb{R}^n \). Now if it is also a bounded set, then you could enclose it in \( B(0, r) \) for some large enough \( r \).

4. Convex closed sets in \( \mathbb{R}^n \) are retracts. Are there other examples of retracts not considered by Theorem 14.2.3?

5. In \( \mathbb{R}^2 \), consider an annulus, \( \{ x : 1 \leq |x| \leq 2 \} \). Show that this set does not have the fixed point property. Could it be a retract of \( \mathbb{R}^2 \)?

6. Does \( \{ x \in \mathbb{R}^n : |x| = 1 \} \) have the fixed point property?

7. Suppose you have a closed subset \( H \) of \( X \) a metric space and suppose also that \( C \) is an open cover of \( H \). Show there is another open cover \( \hat{C} \) such that the closure of each open set in \( \hat{C} \) is contained in some set of \( C \). **Hint:** You might want to use the fact that metric space is normal.

8. If \( H \) is a closed nonempty subset of \( \mathbb{R}^n \) and \( C \) is an open cover of \( H \), show that there is a refined open cover such that each of the new open sets are bounded. In the partition of unity result obtained above, applied to \( H \) show that the functions in the partition of unity can be assumed to be infinitely differentiable with compact support.

9. Check that the conclusion of Theorem 14.2.3 applies for \( X \) just a metric space. Then apply it to give another proof of the Tietze extension theorem.

10. Suppose you have that \( h_k : B \to B \) for \( B \) a compact set and each \( h_k \) has a fixed point. Suppose also that \( h_k \) converges to \( h \) uniformly on \( B \). Then \( h \) also has a fixed point. Verify this.

11. The Brouwer fixed point theorem is a finite dimensional creature. Consider a separable Hilbert space \( H \) with a complete orthonormal basis \( \{ e_k \}^{\infty}_{k=1} \). Then define the following map. For \( x = \sum^{\infty}_{i=1} x_i e_i \), define \( L(\sum^{\infty}_{i=1} x_i e_i) \equiv \sum^{\infty}_{i=1} x_i e_{i+1} \). Now let \( f(x) \equiv \frac{1}{2} (1 - \|x\|_H) e_1 + Lx \). Verify that \( f : B(0, 1) \to B(0, 1) \) is continuous and yet it has no fixed point. This example is in $[52]$. 

Chapter 15

Banach Spaces

15.1 Theorems Based On Baire Category

15.1.1 Baire Category Theorem

Some examples of Banach spaces that have been discussed up to now are $\mathbb{R}^n$, $\mathbb{C}^n$, and $L^p(\Omega)$. Theorems about general Banach spaces are proved in this chapter. The main theorems to be presented here are the uniform boundedness theorem, the open mapping theorem, the closed graph theorem, and the Hahn Banach Theorem. The first three of these theorems come from the Baire category theorem which is about to be presented. They are topological in nature. The Hahn Banach theorem has nothing to do with topology. Banach spaces are all normed linear spaces and as such, they are all metric spaces because a normed linear space may be considered as a metric space with $d(x,y) \equiv ||x - y||$. You can check that this satisfies all the axioms of a metric. As usual, if every Cauchy sequence converges, the metric space is called complete.

Definition 15.1.1 A complete normed linear space is called a Banach space.

The following remarkable result is called the Baire category theorem. To get an idea of its meaning, imagine you draw a line in the plane. The complement of this line is an open set and is dense because every point, even those on the line, are limit points of this open set. Now draw another line. The complement of the two lines is still open and dense. Keep drawing lines and looking at the complements of the union of these lines. You always have an open set which is dense. Now what if there were countably many lines? The Baire category theorem implies the complement of the union of these lines is dense. In particular it is nonempty. Thus you cannot write the plane as a countable union of lines. This is a rather rough description of this very important theorem. The precise statement and proof follow.

Theorem 15.1.2 Let $(X,d)$ be a complete metric space and let $\{U_n\}_{n=1}^\infty$ be a sequence of open subsets of $X$ satisfying $\overline{U_n} = X$ ($U_n$ is dense). Then $D \equiv \cap_{n=1}^\infty U_n$ is a dense subset of $X$. 

419
CHAPTER 15. BANACH SPACES

Proof: Let $p \in X$ and let $r_0 > 0$. I need to show $D \cap B(p, r_0) \neq \emptyset$. Since $U_1$ is dense, there exists $p_1 \in U_1 \cap B(p, r_0)$, an open set. Let $p_1 \in B(p_1, r_1) \subseteq B(p_1, r_1) \subseteq U_1 \cap B(p, r_0)$ and $r_1 < 2^{-1}$. This is possible because $U_1 \cap B(p, r_0)$ is an open set and so there exists $r_1$ such that $B(p_1, 2r_1) \subseteq U_1 \cap B(p, r_0)$. But

$$B(p_1, r_1) \subseteq B(p_1, r_1) \subseteq B(p_1, 2r_1)$$

because $B(p_1, r_1) = \{x \in X : d(x, p) \leq r_1\}$. (Why?)

There exists $p_2 \in U_2 \cap B(p_1, r_1)$ because $U_2$ is dense. Let

$$p_2 \in B(p_2, r_2) \subseteq B(p_2, r_2) \subseteq U_2 \cap B(p_1, r_1) \subseteq U_1 \cap U_2 \cap B(p, r_0).$$

and let $r_2 < 2^{-2}$. Continue in this way. Thus

$$r_n < 2^{-n},$$

$$B(p_n, r_n) \subseteq U_1 \cap U_2 \cap ... \cap U_n \cap B(p, r_0),$$

$$B(p_n, r_n) \subseteq B(p_{n-1}, r_{n-1}).$$

The sequence, $\{p_n\}$ is a Cauchy sequence because all terms of $\{p_k\}$ for $k \geq n$ are contained in $B(p_n, r_n)$, a set whose diameter is no larger than $2^{-n}$. Since $X$ is complete, there exists $p_\infty$ such that

$$\lim_{n \to \infty} p_n = p_\infty.$$ 

Since all but finitely many terms of $\{p_n\}$ are in $B(p_m, r_m)$, it follows that $p_\infty \in B(p_m, r_m)$ for each $m$. Therefore,

$$p_\infty \in \cap_{m=1}^{\infty} B(p_m, r_m) \subseteq \cap_{i=1}^{\infty} U_i \cap B(p, r_0).$$

This proves the theorem.

The following corollary is also called the Baire category theorem.

Corollary 15.1.3 Let $X$ be a complete metric space and suppose $X = \bigcup_{i=1}^{\infty} F_i$ where each $F_i$ is a closed set. Then for some $i$, interior $F_i \neq \emptyset$.

Proof: If all $F_i$ has empty interior, then $F_i^C$ would be a dense open set. Therefore, from Theorem 15.1.2, it would follow that

$$\emptyset = (\bigcup_{i=1}^{\infty} F_i)^C = \cap_{i=1}^{\infty} F_i^C \neq \emptyset.$$
15.1. THEOREMS BASED ON BAIRE CATEGORY

The set $D$ of Theorem 15.1.2 is called a $G_δ$ set because it is the countable intersection of open sets. Thus $D$ is a dense $G_δ$ set.

Recall that a norm satisfies:

a.) $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$.

b.) $\|x + y\| \leq \|x\| + \|y\|$.

c.) $\|cx\| = |c| \|x\|$ if $c$ is a scalar and $x \in X$.

From the definition of continuity, it follows easily that a function is continuous if

$$\lim_{n \to \infty} x_n = x$$

implies

$$\lim_{n \to \infty} f(x_n) = f(x).$$

Theorem 15.1.4 Let $X$ and $Y$ be two normed linear spaces and let $L : X \to Y$ be linear ($L(ax + by) = aL(x) + bL(y)$ for $a, b$ scalars and $x, y \in X$). The following are equivalent

a.) $L$ is continuous at 0

b.) $L$ is continuous

c.) There exists $K > 0$ such that $\|Lx\|_Y \leq K \|x\|_X$ for all $x \in X$ ($L$ is bounded).

Proof: a.)$\Rightarrow$b.) Let $x_n \to x$. It is necessary to show that $Lx_n \to Lx$. But $(x_n - x) \to 0$ and so from continuity at 0, it follows

$$L(x_n - x) = Lx_n - Lx \to 0$$

so $Lx_n \to Lx$. This shows a.) implies b.).

b.)$\Rightarrow$c.) Since $L$ is continuous, $L$ is continuous at 0. Hence $\|Lx\|_Y < 1$ whenever $\|x\|_X \leq \delta$ for some $\delta$. Therefore, suppressing the subscript on the $\||\|$, we have

$$\|L\left(\frac{\delta x}{\|x\|}\right)\| \leq 1.$$ 

Hence

$$\|Lx\| \leq \frac{1}{\delta} \|x\|.$$ 

c.)$\Rightarrow$a.) follows from the inequality given in c.).

Definition 15.1.5 Let $L : X \to Y$ be linear and continuous where $X$ and $Y$ are normed linear spaces. Denote the set of all such continuous linear maps by $\mathcal{L}(X, Y)$ and define

$$\|L\| = \sup\{\|Lx\| : \|x\| \leq 1\}. \quad (15.1.1)$$

This is called the operator norm.
Note that from Theorem 15.1.4 \(|L|\) is well defined because of part c.) of that Theorem.

The next lemma follows immediately from the definition of the norm and the assumption that \(L\) is linear.

**Lemma 15.1.6** With \(|L|\) defined in 15.1.1, \(L(X,Y)\) is a normed linear space. Also \(|Lx| \leq |L| |x|\).

**Proof:** Let \(x \neq 0\) then \(x/|x|\) has norm equal to 1 and so
\[
\left| L \left( \frac{x}{|x|} \right) \right| \leq |L|.
\]
Therefore, multiplying both sides by \(|x|\), \(|Lx| \leq |L| |x|\). This is obviously a linear space. It remains to verify the operator norm really is a norm. First of all, if \(|L| = 0\), then \(Lx = 0\) for all \(|x| \leq 1\). It follows that for any \(x \neq 0\), \(0 = L \left( \frac{x}{|x|} \right)\) and so \(Lx = 0\). Therefore, \(L = 0\). Also, if \(c\) is a scalar,
\[
|cL| = \sup_{|x| \leq 1} |cL(x)| = |c| \sup_{|x| \leq 1} |Lx| = |c||L|.
\]
It remains to verify the triangle inequality. Let \(L, M \in \mathcal{L}(X,Y)\).
\[
|L + M| = \sup_{|x| \leq 1} |(L + M)(x)| \leq \sup_{|x| \leq 1} (|Lx| + |Mx|)
\]
\[
\leq \sup_{|x| \leq 1} |Lx| + \sup_{|x| \leq 1} |Mx| = |L| + |M|.
\]
This shows the operator norm is really a norm as hoped. This proves the lemma.

For example, consider the space of linear transformations defined on \(\mathbb{R}^n\) having values in \(\mathbb{R}^m\). The fact the transformation is linear automatically imparts continuity to it. You should give a proof of this fact. Recall that every such linear transformation can be realized in terms of matrix multiplication.

Thus, in finite dimensions the algebraic condition that an operator is linear is sufficient to imply the topological condition that the operator is continuous. The situation is not so simple in infinite dimensional spaces such as \(C(X;\mathbb{R}^n)\). This explains the imposition of the topological condition of continuity as a criterion for membership in \(\mathcal{L}(X,Y)\) in addition to the algebraic condition of linearity.

**Theorem 15.1.7** If \(Y\) is a Banach space, then \(\mathcal{L}(X,Y)\) is also a Banach space.

**Proof:** Let \(\{L_n\}\) be a Cauchy sequence in \(\mathcal{L}(X,Y)\) and let \(x \in X\).
\[
|L_nx - L_mx| \leq |x| |L_n - L_m|.
\]
Thus \(\{L_nx\}\) is a Cauchy sequence. Let
\[
Lx = \lim_{n \to \infty} L_nx.
\]
Then, clearly, $L$ is linear because if $x_1, x_2$ are in $X$, and $a, b$ are scalars, then

$$L(ax_1 + bx_2) = \lim_{n \to \infty} L_n(ax_1 + bx_2) = \lim_{n \to \infty} (aL_n x_1 + bL_n x_2) = aLx_1 + bLx_2.$$  

Also $L$ is continuous. To see this, note that $\{||L_n||\}$ is a Cauchy sequence of real numbers because $||L_n|| - ||L_m|| \leq ||L_n - L_m||$. Hence there exists $K > \sup\{||L_n|| : n \in \mathbb{N}\}$. Thus, if $x \in X$,

$$||Lx|| = \lim_{n \to \infty} ||L_n x|| \leq K||x||.$$  

This proves the theorem.

15.1.2 Uniform Boundedness Theorem  

The next big result is sometimes called the Uniform Boundedness theorem, or the Banach-Steinhaus theorem. This is a very surprising theorem which implies that for a collection of bounded linear operators, if they are bounded pointwise, then they are also bounded uniformly. As an example of a situation in which pointwise bounded does not imply uniformly bounded, consider the functions $f_\alpha(x) \equiv \lambda(x) x^{-1}$ for $\alpha \in (0, 1)$. Clearly each function is bounded and the collection of functions is bounded at each point of $(0, 1)$, but there is no bound for all these functions taken together. One problem is that $(0, 1)$ is not a Banach space. Therefore, the functions cannot be linear.

**Theorem 15.1.8** Let $X$ be a Banach space and let $Y$ be a normed linear space. Let $\{L_\alpha\}_{\alpha \in \Lambda}$ be a collection of elements of $L(X,Y)$. Then one of the following happens.

- **a.)** $\sup\{||L_\alpha|| : \alpha \in \Lambda\} < \infty$
- **b.)** There exists a dense $G_\delta$ set, $D$, such that for all $x \in D$,

$$\sup\{||L_\alpha x|| : \alpha \in \Lambda\} = \infty.$$  

**Proof:** For each $n \in \mathbb{N}$, define

$$U_n = \{x \in X : \sup\{||L_\alpha x|| : \alpha \in \Lambda\} > n\}.$$  

Then $U_n$ is an open set because if $x \in U_n$, then there exists $\alpha \in \Lambda$ such that

$$||L_\alpha x|| > n.$$  

But then, since $L_\alpha$ is continuous, this situation persists for all $y$ sufficiently close to $x$, say for all $y \in B(x, \delta)$. Then $B(x, \delta) \subseteq U_n$ which shows $U_n$ is open.

Case b.) is obtained from Theorem 15.1.8 if each $U_n$ is dense.

The other case is that for some $n$, $U_n$ is not dense. If this occurs, there exists $x_0$ and $r > 0$ such that for all $x \in B(x_0, r)$, $||L_\alpha x|| \leq n$ for all $\alpha$. Now if $y \in$
\[ B(0, r), \quad x_0 + y \in B(x_0, r) \]. Consequently, for all such \( y \), \( ||L_\alpha(x_0 + y)|| \leq n \). This implies that for all \( \alpha \in \Lambda \) and \( ||y|| < r \),
\[
||L_\alpha y|| \leq n + ||L_\alpha(x_0)|| \leq 2n.
\]
Therefore, if \( ||y|| \leq 1, \quad ||\frac{r}{2}y|| < r \) and so for all \( \alpha \),
\[
||L_\alpha \left( \frac{r}{2}y \right)|| \leq 2n.
\]
Now multiplying by \( r/2 \) it follows that whenever \( ||y|| \leq 1, \quad ||L_\alpha(y)|| \leq 4n/r \). Hence case a.) holds.

15.1.3 Open Mapping Theorem

Another remarkable theorem which depends on the Baire category theorem is the open mapping theorem. Unlike Theorem 15.1.8 it requires both \( X \) and \( Y \) to be Banach spaces.

**Theorem 15.1.9** Let \( X \) and \( Y \) be Banach spaces, let \( L \in \mathcal{L}(X,Y) \), and suppose \( L \) is onto. Then \( L \) maps open sets onto open sets.

To aid in the proof, here is a lemma.

**Lemma 15.1.10** Let \( a \) and \( b \) be positive constants and suppose
\[
B(0, a) \subseteq \overline{L(B(0, b))}.
\]
Then
\[
\overline{L(B(0, b))} \subseteq L(B(0, 2b)).
\]

**Proof of Lemma 15.1.10** Let \( y \in \overline{L(B(0, b))} \). There exists \( x_1 \in B(0, b) \) such that \( ||y - Lx_1|| < \frac{a}{2} \). Now this implies
\[
2y - 2Lx_1 \in B(0, a) \subseteq \overline{L(B(0, b))}.
\]
Thus \( 2y - 2Lx_1 \in \overline{L(B(0, b))} \) just like \( y \) was. Therefore, there exists \( x_2 \in B(0, b) \) such that \( ||2y - 2Lx_1 - Lx_2|| < a/2 \). Hence \( ||4y - 4Lx_1 - 2Lx_2|| < a \), and there exists \( x_3 \in B(0, b) \) such that \( ||4y - 4Lx_1 - 2Lx_2 - Lx_3|| < a/2 \). Continuing in this way, there exist \( x_1, x_2, x_3, x_4, \ldots \) in \( B(0, b) \) such that
\[
||2^n y - \sum_{i=1}^{n} 2^{n-(i-1)} L(x_i)|| < a
\]
which implies
\[
||y - \sum_{i=1}^{n} 2^{-i} L(x_i)|| = ||y - L \left( \sum_{i=1}^{n} 2^{-i} (x_i) \right)|| < 2^{-n} a \quad (15.1.2)
\]
Now consider the partial sums of the series, \( \sum_{i=1}^{\infty} 2^{-(i-1)}x_i \).

\[
\| \sum_{i=m}^{n} 2^{-(i-1)}x_i \| \leq b \sum_{i=m}^{\infty} 2^{-(i-1)} = b 2^{-m+2}.
\]

Therefore, these partial sums form a Cauchy sequence and so since \( X \) is complete, there exists \( x = \sum_{i=1}^{\infty} 2^{-(i-1)}x_i \). Letting \( n \to \infty \) in \ref{15.1.2} yields \( \|y - Lx\| = 0 \). Now

\[
\|x\| = \lim_{n \to \infty} \| \sum_{i=1}^{n} 2^{-(i-1)}x_i \| 
\leq \lim_{n \to \infty} \sum_{i=1}^{n} 2^{-(i-1)}\|x_i\| < \lim_{n \to \infty} \sum_{i=1}^{n} 2^{-(i-1)}b = 2b.
\]

This proves the lemma.

**Proof of Theorem** \ref{15.1.3}: \( Y = \cup_{n=0}^{\infty} L(B(0, n_0)) \). By Corollary \ref{15.1.14}, the set, \( L(B(0, n_0)) \) has nonempty interior for some \( n_0 \). Thus \( B(y, r) \subseteq L(B(0, n_0)) \) for some \( y \) and some \( r > 0 \). Since \( L \) is linear, \( B(-y, r) \subseteq L(B(0, n_0)) \) also. Here is why. If \( z \in B(-y, r) \), then \( -z \in B(y, r) \) and so there exists \( x_n \in B(0, n_0) \) such that \( Lx_n \to z \). Therefore, \( L(-x_n) \to z \) and \( -x_n \in B(0, n_0) \) also. Therefore \( z \in L(B(0, n_0)) \). Then it follows that

\[
B(0, r) \subseteq B(y, r) + B(-y, r) \subseteq \{ y_1 + y_2 : y_1 \in B(y, r) \text{ and } y_2 \in B(-y, r) \} \subseteq L(B(0, 2n_0)).
\]

The reason for the last inclusion is that from the above, if \( y_1 \in B(y, r) \) and \( y_2 \in B(-y, r) \), there exists \( x_n, z_n \in B(0, n_0) \) such that

\[
Lx_n \to y_1, \ Lz_n \to y_2.
\]

Therefore,

\[
\|x_n + z_n\| \leq 2n_0
\]

and so \( (y_1 + y_2) \in L(B(0, 2n_0)). \)

By Lemma \ref{15.1.4}, \( L(B(0, 2n_0)) \subseteq L(B(0, 4n_0)) \) which shows

\[
B(0, r) \subseteq L(B(0, 4n_0)).
\]

Letting \( a = r(4n_0)^{-1} \), it follows, since \( L \) is linear, that \( B(0, a) \subseteq L(B(0, 1)) \). It follows since \( L \) is linear,

\[
L(B(0, r)) \supseteq B(0, ar).
\]

Now let \( U \) be open in \( X \) and let \( x + B(0, r) = B(x, r) \subseteq U \). Using \ref{15.1.3},

\[
L(U) \supseteq L(x + B(0, r)) = Lx + L(B(0, r)) \supseteq Lx + B(0, ar) = B(Lx, ar).
\]
Hence
\[ Lx \in B(Lx, ar) \subseteq L(U). \]
which shows that every point, \( Lx \in LU \), is an interior point of \( LU \) and so \( LU \) is open. This proves the theorem.

This theorem is surprising because it implies that if \(|·|\) and \(||·||\) are two norms with respect to which a vector space \( X \) is a Banach space such that \(|·| \leq K ||·||\), then there exists a constant \( k \), such that \( ||·|| \leq k |·| \). This can be useful because sometimes it is not clear how to compute \( k \) when all that is needed is its existence. To see the open mapping theorem implies this, consider the identity map \( id : (X, ||·||) \rightarrow (X, |·|) \) is continuous and onto. Hence \( id \) is an open map which implies \( id^{-1} \) is continuous. Theorem 15.1.4 gives the existence of the constant \( k \).

15.1.4 Closed Graph Theorem

Definition 15.1.11 Let \( f : D \rightarrow E \). The set of all ordered pairs of the form \( \{(x, f(x)) : x \in D \} \) is called the graph of \( f \).

Definition 15.1.12 If \( X \) and \( Y \) are normed linear spaces, make \( X \times Y \) into a normed linear space by using the norm \( ||(x, y)|| = \max(||x||, ||y||) \) along with component-wise addition and scalar multiplication. Thus \( a(x, y) + b(z, w) \equiv (ax + bz, ay + bw) \).

There are other ways to give a norm for \( X \times Y \). For example, you could define \( ||(x, y)|| = ||x|| + ||y|| \)

Lemma 15.1.13 The norm defined in Definition 15.1.12 on \( X \times Y \) along with the definition of addition and scalar multiplication given there make \( X \times Y \) into a normed linear space.

Proof: The only axiom for a norm which is not obvious is the triangle inequality. Therefore, consider
\[
||(x_1, y_1) + (x_2, y_2)|| = ||(x_1 + x_2, y_1 + y_2)||
\leq \max(||x_1|| + ||x_2||, ||y_1|| + ||y_2||)
\leq \max(||x_1||, ||y_1||) + \max(||x_2||, ||y_2||)
= ||(x_1, y_1)|| + ||(x_2, y_2)||.
\]
It is obvious \( X \times Y \) is a vector space from the above definition. This proves the lemma.

Lemma 15.1.14 If \( X \) and \( Y \) are Banach spaces, then \( X \times Y \) with the norm and vector space operations defined in Definition 15.1.12 is also a Banach space.
**Proof:** The only thing left to check is that the space is complete. But this follows from the simple observation that \( \{(x_n, y_n)\} \) is a Cauchy sequence in \( X \times Y \) if and only if \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences in \( X \) and \( Y \) respectively. Thus if \( \{(x_n, y_n)\} \) is a Cauchy sequence in \( X \times Y \), it follows there exist \( x \) and \( y \) such that \( x_n \to x \) and \( y_n \to y \). But then from the definition of the norm, \( (x_n, y_n) \to (x, y) \).

**Lemma 15.1.15** Every closed subspace of a Banach space is a Banach space.

**Proof:** If \( F \subseteq X \) where \( X \) is a Banach space and \( \{x_n\} \) is a Cauchy sequence in \( F \), then since \( X \) is complete, there exists a unique \( x \in X \) such that \( x_n \to x \). However this means \( x \in F = F \) since \( F \) is closed.

**Definition 15.1.16** Let \( X \) and \( Y \) be Banach spaces and let \( D \subseteq X \) be a subspace. A linear map \( L : D \to Y \) is said to be closed if its graph is a closed subspace of \( X \times Y \). Equivalently, \( L \) is closed if \( x_n \to x \) and \( Lx_n \to y \) implies \( x \in D \) and \( y = Lx \).

Note the distinction between closed and continuous. If the operator is closed the assertion that \( y = Lx \) only follows if it is known that the sequence \( \{Lx_n\} \) converges. In the case of a continuous operator, the convergence of \( \{Lx_n\} \) follows from the assumption that \( x_n \to x \). It is not always the case that a mapping which is closed is necessarily continuous. Consider the function \( f(x) = \tan(x) \) if \( x \) is not an odd multiple of \( \pi/2 \) and \( f(x) \equiv 0 \) at every odd multiple of \( \pi/2 \). Then the graph is closed and the function is defined on \( \mathbb{R} \) but it clearly fails to be continuous. Of course this function is not linear. You could also consider the map,

\[
\frac{d}{dx} : \{y \in C^1([0, 1]) : y(0) = 0\} \equiv D \to C([0, 1]).
\]

where the norm is the uniform norm on \( C([0, 1]), ||y||_\infty \). If \( y \in D \), then

\[
y(x) = \int_0^x y'(t) \, dt.
\]

Therefore, if \( \frac{dy_n}{dx} \to f \in C([0, 1]) \) and if \( y_n \to y \) in \( C([0, 1]) \) it follows that

\[
y_n(x) = \int_0^x \frac{dy_n(t)}{dx} \, dt \quad \downarrow \quad y(x) = \int_0^x f(t) \, dt
\]

and so by the fundamental theorem of calculus \( f(x) = y'(x) \) and so the mapping is closed. It is obviously not continuous because it takes \( y(x) \) and \( y(x) + \frac{\pi}{n} \sin(nx) \) to two functions which are far from each other even though these two functions are very close in \( C([0, 1]) \). Furthermore, it is not defined on the whole space, \( C([0, 1]) \).

The next theorem, the closed graph theorem, gives conditions under which closed implies continuous.
**Theorem 15.1.17** Let $X$ and $Y$ be Banach spaces and suppose $L : X \rightarrow Y$ is closed and linear. Then $L$ is continuous.

**Proof:** Let $G$ be the graph of $L$. $G = \{(x, Lx) : x \in X\}$. By Lemma 15.1.15 it follows that $G$ is a Banach space. Define $P : G \rightarrow X$ by $P(x, Lx) = x$. $P$ maps the Banach space $G$ onto the Banach space $X$ and is continuous and linear. By the open mapping theorem, $P$ maps open sets onto open sets. Since $P$ is also one to one, this says that $P^{-1}$ is continuous. Thus $||P^{-1}|| \leq K||x||$. Hence

$$||Lx|| \leq \max(||x||, ||Lx||) \leq K||x||$$

By Theorem 15.1.4 on Page 421 this shows $L$ is continuous and proves the theorem.

The following corollary is quite useful. It shows how to obtain a new norm on the domain of a closed operator such that the domain with this new norm becomes a Banach space.

**Corollary 15.1.18** Let $L : D \subseteq X \rightarrow Y$ where $X, Y$ are a Banach spaces, and $L$ is a closed operator. Then define a new norm on $D$ by

$$||x||_D \equiv ||x||_X + ||Lx||_Y.$$  

Then $D$ with this new norm is a Banach space.

**Proof:** If $\{x_n\}$ is a Cauchy sequence in $D$ with this new norm, it follows both $\{x_n\}$ and $\{Lx_n\}$ are Cauchy sequences and therefore, they converge. Since $L$ is closed, $x_n \rightarrow x$ and $Lx_n \rightarrow Lx$ for some $x \in D$. Thus $||x_n - x||_D \rightarrow 0$.

**15.2 Hahn Banach Theorem**

The closed graph, open mapping, and uniform boundedness theorems are the three major topological theorems in functional analysis. The other major theorem is the Hahn-Banach theorem which has nothing to do with topology. Before presenting this theorem, here are some preliminaries about partially ordered sets.

**15.2.1 Partially Ordered Sets**

**Definition 15.2.1** Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that

$$x \leq x \text{ for all } x \in \mathcal{F}.$$  

If $x \leq y$ and $y \leq z$ then $x \leq z$.

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $D$ is a chain containing $\mathcal{C}$, $D = \mathcal{C}$.
The most common example of a partially ordered set is the power set of a given set with \( \subseteq \) being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

**Theorem 15.2.2 (Hausdorff Maximal Principle)** Let \( F \) be a nonempty partially ordered set. Then there exists a maximal chain.

### 15.2.2 Gauge Functions And Hahn Banach Theorem

**Definition 15.2.3** Let \( X \) be a real vector space \( \rho : X \to \mathbb{R} \) is called a gauge function if

\[
\rho(x + y) \leq \rho(x) + \rho(y), \quad \rho(ax) = a\rho(x) \text{ if } a \geq 0.
\]  

(15.2.4)

Suppose \( M \) is a subspace of \( X \) and \( z \notin M \). Suppose also that \( f \) is a linear real-valued function having the property that \( f(x) \leq \rho(x) \) for all \( x \in M \). Consider the problem of extending \( f \) to \( M \oplus \mathbb{R}z \) such that if \( F \) is the extended function, \( F(y) \leq \rho(y) \) for all \( y \in M \oplus \mathbb{R}z \) and \( F \) is linear. Since \( F \) is to be linear, it suffices to determine how to define \( F(z) \). Letting \( a > 0 \), it is required to define \( F(z) \) such that the following hold for all \( x, y \in M \).

\[
\frac{f(x)}{F(x) + aF(z)} = F(x + az) \leq \rho(x + az),
\]

\[
\frac{f(y)}{F(y) - aF(z)} = F(y - az) \leq \rho(y - az). \quad (15.2.5)
\]

Now if these inequalities hold for all \( y/a \), they hold for all \( y \) because \( M \) is given to be a subspace. Therefore, multiplying by \( a^{-1} \) implies that what is needed is to choose \( F(z) \) such that for all \( x, y \in M \),

\[
f(x) + F(z) \leq \rho(x + z), \quad f(y) - \rho(y - z) \leq F(z)
\]

and that if \( F(z) \) can be chosen in this way, this will satisfy for all \( x, y \) and the problem of extending \( f \) will be solved. Hence it is necessary to choose \( F(z) \) such that for all \( x, y \in M \)

\[
f(y) - \rho(y - z) \leq F(z) \leq \rho(x + z) - f(x). \quad (15.2.6)
\]
Is there any such number between \( f(y) - \rho(y - z) \) and \( \rho(x + z) - f(x) \) for every pair \( x, y \in M \)? This is where \( f(x) \leq \rho(x) \) on \( M \) and that \( f \) is linear is used. For \( x, y \in M \),

\[
\rho(x + z) - f(x) - [f(y) - \rho(y - z)] \\
= \rho(x + z) + \rho(y - z) - (f(x) + f(y)) \\
\geq \rho(x + y) - f(x + y) \geq 0.
\]

Therefore there exists a number between

\[
\sup \{ f(y) - \rho(y - z) : y \in M \}
\]

and

\[
\inf \{ \rho(x + z) - f(x) : x \in M \}
\]

Choose \( F(z) \) to satisfy 15.2.4. This has proved the following lemma.

**Lemma 15.2.4** Let \( M \) be a subspace of \( X \), a real linear space, and let \( \rho \) be a gauge function on \( X \). Suppose \( f : M \to \mathbb{R} \) is linear, \( z \notin M \), and \( f(x) \leq \rho(x) \) for all \( x \in M \). Then \( f \) can be extended to \( M \oplus \mathbb{R}z \) such that, if \( F \) is the extended function, \( F \) is linear and \( F(x) \leq \rho(x) \) for all \( x \in M \oplus \mathbb{R}z \).

With this lemma, the Hahn Banach theorem can be proved.

**Theorem 15.2.5** (Hahn Banach theorem) Let \( X \) be a real vector space, let \( M \) be a subspace of \( X \), let \( f : M \to \mathbb{R} \) be linear, let \( \rho \) be a gauge function on \( X \), and suppose \( f(x) \leq \rho(x) \) for all \( x \in M \). Then there exists a linear function, \( F : X \to \mathbb{R} \), such that

a.) \( F(x) = f(x) \) for all \( x \in M \)

b.) \( F(x) \leq \rho(x) \) for all \( x \in X \).

**Proof:** Let \( \mathcal{F} = \{(V, g) : V \supseteq M, V \text{ is a subspace of } X, g : V \to \mathbb{R} \text{ is linear, } g(x) = f(x) \text{ for all } x \in M, \text{ and } g(x) \leq \rho(x) \text{ for } x \in V \} \). Then \( (M, f) \in \mathcal{F} \) so \( \mathcal{F} \neq \emptyset \). Define a partial order by the following rule.

\[
(V, g) \leq (W, h)
\]

means

\[
V \subseteq W \text{ and } h(x) = g(x) \text{ if } x \in V.
\]

By Theorem 15.2.4, there exists a maximal chain, \( \mathcal{C} \subseteq \mathcal{F} \). Let \( Y = \cup \{V : (V, g) \in \mathcal{C}\} \) and let \( h : Y \to \mathbb{R} \) be defined by \( h(x) = g(x) \) where \( x \in V \) and \( (V, g) \in \mathcal{C} \). This is well defined because if \( x \in V_1 \) and \( V_2 \) where \( (V_1, g_1) \) and \( (V_2, g_2) \) are both in the chain, then since \( \mathcal{C} \) is a chain, the two element related. Therefore, \( g_1(x) = g_2(x) \).

Also \( h \) is linear because if \( ax + by \in Y \), then \( x \in V_1 \) and \( y \in V_2 \) where \( (V_1, g_1) \) and \( (V_2, g_2) \) are elements of \( \mathcal{C} \). Therefore, letting \( V \) denote the larger of the two \( V_1 \),
and \( g \) be the function that goes with \( V \), it follows \( ax + by \in V \) where \((V, g) \in \mathcal{C}\). Therefore,

\[
 h(ax + by) = g(ax + by) = ag(x) + bg(y) = ah(x) + bh(y).
\]

Also, \( h(x) = g(x) \leq \rho(x) \) for any \( x \in Y \) because for such \( x \), \( x \in V \) where \((V, g) \in \mathcal{C}\).

Is \( Y = X \)? If not, there exists \( z \in X \setminus Y \) and there exists an extension of \( h \) to \( Y \oplus \mathbb{R}z \) using Lemma 15.2.4. Letting \( h \) denote this extended function, contradicts the maximality of \( \mathcal{C} \). Indeed, \( \mathcal{C} \cup \{(Y \oplus \mathbb{R}z, \delta h)\} \) would be a longer chain. This proves the Hahn Banach theorem.

This is the original version of the theorem. There is also a version of this theorem for complex vector spaces which is based on a trick.

### 15.2.3 The Complex Version Of The Hahn Banach Theorem

**Corollary 15.2.6 (Hahn Banach)** Let \( M \) be a subspace of a complex normed linear space, \( X \), and suppose \( f : M \to \mathbb{C} \) is linear and satisfies \( |f(x)| \leq K \|x\| \) for all \( x \in M \). Then there exists a linear function, \( F \), defined on all of \( X \) such that \( F(x) = f(x) \) for all \( x \in M \) and \( |F(x)| \leq K \|x\| \) for all \( x \).

**Proof:** First note \( f(x) = \text{Re} f(x) + i \text{Im} f(x) \) and so

\[
\text{Re} f(ix) + i \text{Im} f(ix) = f(ix) = i f(x) = i (\text{Re} f(x) - \text{Im} f(x)).
\]

Therefore, \( \text{Im} f(x) = -\text{Re} f(ix) \), and

\[
f(x) = \text{Re} f(x) - i \text{Re} f(ix).
\]

This is important because it shows it is only necessary to consider \( \text{Re} f \) in understanding \( f \). Now it happens that \( \text{Re} f \) is linear with respect to real scalars so the above version of the Hahn Banach theorem applies. This is shown next.

If \( c \) is a real scalar

\[
\text{Re} f(cx) - i \text{Re} f(icx) = cf(x) = c \text{Re} f(x) - ic \text{Re} f(ix).
\]

Thus \( \text{Re} f(cx) = c \text{Re} f(x) \). Also,

\[
\text{Re} f(x + y) - i \text{Re} f(i(x + y)) = f(x + y) = f(x) + f(y)
\]

\[
= \text{Re} f(x) - i \text{Re} f(ix) + \text{Re} f(y) - i \text{Re} f(iy).
\]

Equating real parts, \( \text{Re} f(x + y) = \text{Re} f(x) + \text{Re} f(y) \). Thus \( \text{Re} f \) is linear with respect to real scalars as hoped.
Consider $X$ as a real vector space and let $\rho(x) \equiv K||x||$. Then for all $x \in M$,

$$|\Re f(x)| \leq |f(x)| \leq K||x|| = \rho(x).$$

From Theorem 15.2.5, $\Re f$ may be extended to a function, $h$ which satisfies

$$h(ax + by) = ah(x) + bh(y) \text{ if } a, b \in \mathbb{R}$$

$$h(x) \leq K||x|| \text{ for all } x \in X.$$ 

Actually, $|h(x)| \leq K||x||$. The reason for this is that $h(-x) = -h(x) \leq K||-x|| = K||x||$ and therefore, $h(x) \geq -K||x||$. Let

$$F(x) \equiv h(x) - ih(ix).$$

By arguments similar to the above, $F$ is linear.

$$F(ix) = h(ix) - ih(-x)$$

$$= ih(x) + h(ix)$$

$$= i(h(x) - ih(ix)) = iF(x).$$

If $c$ is a real scalar,

$$F(cx) = h(cx) - ih(icx)$$

$$= ch(x) - cih(ix) = cF(x).$$

Now

$$F(x + y) = h(x + y) - ih(i(x + y))$$

$$= h(x) + h(y) - ih(ix) - ih(iy)$$

$$= F(x) + F(y).$$

Thus

$$F((a + ib)x) = F(ax) + F(ibx)$$

$$= aF(x) + ibF(x)$$

$$= (a + ib)F(x).$$

This shows $F$ is linear as claimed.

Now $wF(x) = |F(x)|$ for some $|w| = 1$. Therefore

$$|F(x)| = wF(x) = h(wx) - ih(iwx) = h(wx)$$

$$= |h(wx)| \leq K||wx|| = K||x||.$$

This proves the corollary.
15.2.4 The Dual Space And Adjoint Operators

**Definition 15.2.7** Let $X$ be a Banach space. Denote by $X'$ the space of continuous linear functions which map $X$ to the field of scalars. Thus $X' = \mathcal{L}(X, \mathbb{F})$. By Theorem 15.1.7 on Page 422, $X'$ is a Banach space. Remember with the norm defined on $\mathcal{L}(X, \mathbb{F})$,

$$||f|| = \sup\{|f(x)| : ||x|| \leq 1\}$$

$X'$ is called the dual space.

**Definition 15.2.8** Let $X$ and $Y$ be Banach spaces and suppose $L \in \mathcal{L}(X, Y)$. Then define the adjoint map in $\mathcal{L}(Y', X')$, denoted by $L^*$, by

$$L^*y^*(x) \equiv y^*(Lx)$$

for all $y^* \in Y'$.

The following diagram is a good one to help remember this definition.

$$\begin{array}{c}
L^* \\
X' \leftarrow Y' \\
X \rightarrow Y
\end{array}$$

This is a generalization of the adjoint of a linear transformation on an inner product space. Recall

$$(Ax, y) = (x, A^*y)$$

What is being done here is to generalize this algebraic concept to arbitrary Banach spaces. There are some issues which need to be discussed relative to the above definition. First of all, it must be shown that $L^*y^* \in X'$. Also, it will be useful to have the following lemma which is a useful application of the Hahn Banach theorem.

**Lemma 15.2.9** Let $X$ be a normed linear space and let $x \in X \setminus V$ where $V$ is a closed subspace of $X$. Then there exists $x^* \in X'$ such that $x^*(x) = ||x||$, $x^*(V) = \{0\}$, and

$$||x^*|| \leq \frac{1}{\text{dist}(x, V)}$$

In the case that $V = \{0\}$, $||x^*|| = 1$.

**Proof:** Let $f : \mathbb{F}x + V \to \mathbb{F}$ be defined by $f(\alpha x + v) = \alpha ||x||$. First it is necessary to show $f$ is well defined and continuous. If $\alpha_1 x + v_1 = \alpha_2 x + v_2$ then if $\alpha_1 \neq \alpha_2$, then $x \in V$ which is assumed not to happen so $f$ is well defined. It remains to show $f$ is continuous. Suppose then that $\alpha_n x + v_n \to 0$. It is necessary to show $\alpha_n \to 0$. If this does not happen, then there exists a subsequence, still denoted by $\alpha_n$ such that $|\alpha_n| \geq \delta > 0$. Then $x + (1/\alpha_n) v_n \to 0$ contradicting the assumption
that $x \notin V$ and $V$ is a closed subspace. Hence $f$ is continuous on $Fx + V$. Being a little more careful,

$$||f|| = \sup_{||ax + v|| \leq 1} |f(ax + v)| = \sup_{|\alpha||x| + (v/\alpha)|| \leq 1} |\alpha||x|| = \frac{1}{\text{dist} (x, V)} ||x||$$

By the Hahn Banach theorem, there exists $x^* \in X'$ such that $x^* = f$ on $Fx + V$. Thus $x^*(x) = ||x||$ and also

$$||x^*|| \leq ||f|| = \frac{1}{\text{dist} (x, V)}$$

In case $V = \{0\}$, the result follows from the above or alternatively,

$$||f|| = \sup_{|\alpha x| \leq 1} |f(\alpha x)| = \sup_{|\alpha| |x| \leq 1} |\alpha||x|| = 1$$

and so, in this case, $||x^*|| \leq ||f|| = 1$. Since $x^*(x) = ||x||$ it follows

$$||x^*|| \geq \left| x^* \left( \frac{x}{||x||} \right) \right| = \frac{||x||}{||x||} = 1.$$

Thus $||x^*|| = 1$ and this proves the lemma.

**Theorem 15.2.10** Let $L \in \mathcal{L}(X, Y)$ where $X$ and $Y$ are Banach spaces. Then

a.) $L^* \in \mathcal{L}(Y^*, X^*)$ as claimed and $||L^*|| = ||L||$.

b.) If $L$ maps one to one onto a closed subspace of $Y$, then $L^*$ is onto.

c.) If $L$ maps onto a dense subset of $Y$, then $L^*$ is one to one.

**Proof:** It is routine to verify $L^*y^*$ and $L^*$ are both linear. This follows immediately from the definition. As usual, the interesting thing concerns continuity.

$$||L^*y^*|| = \sup_{||x|| \leq 1} ||L^*y^*(x)|| = \sup_{||x|| \leq 1} ||y^*(Lx)|| \leq ||y^*|| ||L||.$$  

Thus $L^*$ is continuous as claimed and $||L^*|| \leq ||L||$.

By Lemma 15.2.9, there exists $y^*_x \in Y'$ such that $||y^*_x|| = 1$ and $y^*_x(Lx) = ||Lx||$. Therefore,

$$||L^*|| = \sup_{||y^*|| \leq 1} ||L^*y^*|| = \sup_{||y^*|| \leq 1} \sup_{||x|| \leq 1} ||L^*y^*(x)||$$

$$= \sup_{||y^*|| \leq 1} \sup_{||x|| \leq 1} ||y^*(Lx)|| = \sup_{||y^*|| \leq 1} \sup_{||x|| \leq 1} ||y^*(Lx)||$$

$$\geq \sup_{||x|| \leq 1} \sup_{||x|| \leq 1} ||y^*_x(Lx)|| = \sup_{||x|| \leq 1} ||Lx|| = ||L||$$

showing that $||L^*|| \geq ||L||$ and this shows part a.).

If $L$ is one to one and onto a closed subspace of $Y$, then $L(X)$ being a closed subspace of a Banach space, is itself a Banach space and so the open mapping theorem implies $L^{-1} : L(X) \to X$ is continuous. Hence

$$||x|| = ||L^{-1}Lx|| \leq ||L^{-1}|| ||Lx||$$

CHAPTER 15. BANACH SPACES
15.2. **HAHN BANACH THEOREM**

Now let \( x^* \in X' \) be given. Define \( f \in \mathcal{L}(L(X), \mathbb{C}) \) by \( f(Lx) = x^*(x) \). The function, \( f \) is well defined because if \( Lx_1 = Lx_2 \), then since \( L \) is one to one, it follows \( x_1 = x_2 \) and so \( f(L(x_1)) = x^*(x_1) = x^*(x_2) = f(L(x_1)) \). Also, \( f \) is linear because

\[
\begin{align*}
    f(ax_1 + bx_2) & = f(L(ax_1 + bx_2)) \\
    & = x^*(ax_1 + bx_2) \\
    & = ax^*(x_1) + bx^*(x_2) \\
    & = af(L(x_1)) + bf(L(x_2)).
\end{align*}
\]

In addition to this,

\[
|f(Lx)| = |x^*(x)| \leq \|x^*\| \|x\| \leq \|x^*\| \|L^{-1}\| \|Lx\|
\]

and so the norm of \( f \) on \( L(X) \) is no larger than \( \|x^*\| \|L^{-1}\| \). By the Hahn Banach theorem, there exists an extension of \( f \) to an element \( y^* \in Y' \) such that \( \|y^*\| \leq \|x^*\| \|L^{-1}\| \). Then

\[
L^* y^*(x) = y^*(Lx) = f(Lx) = x^*(x)
\]

so \( L^* y^* = x^* \) because this holds for all \( x \). Since \( x^* \) was arbitrary, this shows \( L^* \) is onto and proves b).

Consider the last assertion. Suppose \( L^* y^* = 0 \). Is \( y^* = 0 \)? In other words is \( y^*(y) = 0 \) for all \( y \in Y' \)? Pick \( y \in Y \). Since \( L(X) \) is dense in \( Y \), there exists a sequence, \( \{Lx_n\} \) such that \( Lx_n \to y \). But then by continuity of \( y^* \), \( y^*(y) = \lim_{n \to \infty} y^*(Lx_n) = \lim_{n \to \infty} L^* y^*(x_n) = 0 \). Since \( y^*(y) = 0 \) for all \( y \), this implies \( y^* = 0 \) and so \( L^* \) is one to one.

**Corollary 15.2.11** Suppose \( X \) and \( Y \) are Banach spaces, \( L \in \mathcal{L}(X,Y) \), and \( L \) is one to one and onto. Then \( L^* \) is also one to one and onto.

There exists a natural mapping, called the James map from a normed linear space, \( X \), to the dual of the dual space which is described in the following definition.

**Definition 15.2.12** Define \( J : X \to X'' \) by \( J(x)(x^*) = x^*(x) \).

**Theorem 15.2.13** The map, \( J \), has the following properties.

a.) \( J \) is one to one and linear.

b.) \( \|Jx\| = \|x\| \) and \( \|J\| = 1 \).

c.) \( J(X) \) is a closed subspace of \( X'' \) if \( X \) is complete.

Also if \( x^* \in X' \),

\[
\|x^*\| = \sup \{||x^{**}(x^*)| : ||x^{**}|| \leq 1, x^{**} \in X''\}.
\]

**Proof:**

\[
J(ax + by)(x^*) = x^*(ax + by)
= ax^*(x) + bx^*(y)
= (aJ(x) + bJ(y))(x^*).
\]
Since this holds for all \( x^* \in X' \), it follows that
\[
J(ax + by) = aJ(x) + bJ(y)
\]
and so \( J \) is linear. If \( Jx = 0 \), then by Lemma 15.2.9 there exists \( x^* \) such that \( x^*(x) = ||x|| \) and \( ||x^*|| = 1 \). Then
\[
0 = J(x)(x^*) = x^*(x) = ||x||.
\]
This shows a).

To show b.), let \( x \in X \) and use Lemma 15.2.9 to obtain \( x^* \in X' \) such that \( x^*(x) = ||x|| \) with \( ||x^*|| = 1 \). Then
\[
0 = J(x)(x^*) = x^*(x) = ||x||.
\]

Therefore, \( ||Jx|| = ||x|| \) as claimed. Therefore,
\[
||J|| = \sup\{||Jx|| : ||x|| \leq 1\} = \sup\{||x|| : ||x|| \leq 1\} = 1.
\]
This shows b).

To verify c.), use b.). If \( Jx_n \to y^{**} \in X'' \) then by b.), \( x_n \) is a Cauchy sequence converging to some \( x \in X \) because
\[
||x_n - x_m|| = ||Jx_n - Jx_m||
\]
and \( \{Jx_n\} \) is a Cauchy sequence. Then \( Jx = \lim_{n \to \infty} Jx_n = y^{**} \).

Finally, to show the assertion about the norm of \( x^* \), use what was just shown applied to the James map from \( X' \) to \( X'' \) still referred to as \( J \).
\[
||x^*|| = \sup\{|x^*(x)| : ||x|| \leq 1\} = \sup\{|J(x)(x^*)| : ||Jx|| \leq 1\}
\]
\[
\leq \sup\{|x^{**}(x^*)| : ||x^{**}|| \leq 1\} = \sup\{|J(x^*)(x^{**})| : ||x^{**}|| \leq 1\}
\]
\[
= ||Jx^*|| = ||x^*||.
\]
This proves the theorem.

**Definition 15.2.14** When \( J \) maps \( X \) onto \( X'' \), \( X \) is called reflexive.

It happens the \( L^p \) spaces are reflexive whenever \( p > 1 \). This is shown later.

### 15.3 Uniform Convexity Of \( L^p \)

These terms refer roughly to how round the unit ball is. Here is the definition.

**Definition 15.3.1** A Banach space is uniformly convex if whenever \( ||x_n||, ||y_n|| \leq 1 \) and \( ||x_n + y_n|| \to 2 \), it follows that \( ||x_n - y_n|| \to 0 \).
15.3. UNIFORM CONVEXITY OF $L^p$  

You can show that uniform convexity implies strict convexity. There are various other things which can also be shown. See the exercises for some of these. In this section, it will be shown that the $L^p$ spaces are examples of uniformly convex spaces. This involves some inequalities known as Clarkson’s inequalities. Before presenting these, here are the backwards Hölder inequality and the backwards Minkowski inequality.

**Lemma 15.3.2** Let $0 < p < 1$ and let $f, g$ be measurable functions. Also
\[
\int_{\Omega} |g|^{p/(p-1)} \, d\mu < \infty, \, \int_{\Omega} |f|^p \, d\mu < \infty.
\]

Then the following backwards Hölder inequality holds.
\[
\int_{\Omega} |fg| \, d\mu \geq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^{p/(p-1)} \, d\mu \right)^{(p-1)/p}.
\]

**Proof:** If $\int |fg| \, d\mu = \infty$, there is nothing to prove. Hence assume this is finite. Then
\[
\int |f|^p \, d\mu = \int |g|^{-p} |fg|^p \, d\mu.
\]

This makes sense because, due to the hypothesis on $g$ it must be the case that $g$ equals 0 only on a set of measure zero, since $p/(p-1) < 0$. Then
\[
\int |f|^p \, d\mu \leq \left( \int |fg| \, d\mu \right)^p \left( \int \left( \frac{1}{|g|^p} \right)^{1/(1-p)} \, d\mu \right)^{1-p} = \left( \int |fg| \, d\mu \right)^p \left( \int |g|^{p/(p-1)} \, d\mu \right)^{1-p}.
\]

Now divide and then take the $p^{th}$ root.

Here is the backwards Minkowski inequality.

**Corollary 15.3.3** Let $0 < p < 1$ and suppose $\int |h|^p \, d\mu < \infty$ for $h = f, g$. Then
\[
\left( \int (|f| + |g|)^p \, d\mu \right)^{1/p} \geq \left( \int |f|^p \, d\mu \right)^{1/p} + \left( \int |g|^p \, d\mu \right)^{1/p}.
\]

**Proof:** If $\int (|f| + |g|)^p \, d\mu = 0$ then there is nothing to prove so assume this is not zero.
\[
\int (|f| + |g|)^p \, d\mu = \int (|f| + |g|)^{p-1} (|f| + |g|) \, d\mu.
\]

$(|f| + |g|)^p \leq |f|^p + |g|^p$ and so
\[
\int \left( (|f| + |g|)^{p-1} \right)^{p/(p-1)} \, d\mu < \infty.
\]
Hence the backward Holder inequality applies and it follows that
\[
\int (|f| + |g|)^p \, d\mu = \int (|f| + |g|)^{p-1} |f| \, d\mu + \int (|f| + |g|)^{p-1} |g| \, d\mu
\]
\[
\geq \left( \int (|f| + |g|)^{(p-1)/p} \right)^{p/p} \left[ \left( \int |f|^p \, d\mu \right)^{1/p} + \left( \int |g|^p \, d\mu \right)^{1/p} \right]
\]
\[
= \left( \int (|f| + |g|)^p \right)^{(p-1)/p} \left[ \left( \int |f|^p \, d\mu \right)^{1/p} + \left( \int |g|^p \, d\mu \right)^{1/p} \right]
\]
and so, dividing gives the desired inequality.

Consider the easy Clarkson inequalities.

**Lemma 15.3.4** For any \( p \geq 2 \) the following inequality holds for any \( t \in [0, 1] \),
\[
\left| \frac{1 + t}{2} \right|^p + \left| \frac{1 - t}{2} \right|^p \leq \frac{1}{2} (|t|^p + 1)
\]

**Proof:** It is clear that, since \( p \geq 2 \), the inequality holds for \( t = 0 \) and \( t = 1 \). Thus it suffices to consider only \( t \in (0, 1) \). Let \( x = 1/t \). Then, dividing by \( 1/t^p \), the inequality holds if and only if
\[
\left( \frac{x + 1}{2} \right)^p + \left( \frac{x - 1}{2} \right)^p \leq \frac{1}{2} (1 + x^p)
\]
for all \( x \geq 1 \). Let
\[
f(x) = \frac{1}{2} (1 + x^p) - \left( \left( \frac{x + 1}{2} \right)^p + \left( \frac{x - 1}{2} \right)^p \right)
\]
Then \( f(1) = 0 \) and
\[
f'(x) = \frac{p}{2} x^{p-1} - \left( \frac{p}{2} \left( \frac{x + 1}{2} \right)^{p-1} + \frac{p}{2} \left( \frac{x - 1}{2} \right)^{p-1} \right)
\]
Since \( p - 1 \geq 1 \), by convexity of \( f(x) = x^{p-1} \),
\[
f'(x) \geq \frac{p}{2} x^{p-1} - p \left( \frac{x + 1 + x - 1}{2} \right)^{p-1} = \frac{p}{2} x^{p-1} - p \left( \frac{x}{2} \right)^{p-1} \geq 0
\]
Hence \( f(x) \geq 0 \) for all \( x \geq 1 \).

**Corollary 15.3.5** If \( z, w \in \mathbb{C} \) and \( p \geq 2 \), then
\[
\left| \frac{z + w}{2} \right|^p + \left| \frac{z - w}{2} \right|^p \leq \frac{1}{2} (|z|^p + |w|^p)
\] (15.3.7)
Proof: One of $|w|, |z|$ is larger. Say $|z| \geq |w|$. Then dividing both sides of the proposed inequality by $|z|^p$ it suffices to verify that for all complex $t$ having $|t| \leq 1$,

$$\left| \frac{1 + t^2}{2} \right|^p + \left| \frac{1 - t^2}{2} \right|^p \leq \frac{1}{2} (|t|^p + 1)$$

Say $t = re^{i\theta}$ where $r \leq 1$. Then consider the expression

$$\left| \frac{1 + re^{i\theta}}{2} \right|^p + \left| \frac{1 - re^{i\theta}}{2} \right|^p$$

It is $2^{-p}$ times

$$\left( (1 + r \cos \theta)^2 + r^2 \sin^2(\theta) \right)^{p/2} + \left( (1 - r \cos \theta)^2 + r^2 \sin^2(\theta) \right)^{p/2}$$

$$= \left( 1 + r^2 + 2r \cos \theta \right)^{p/2} + (1 + r^2 - 2r \cos \theta)^{p/2},$$

a continuous periodic function for $\theta \in \mathbb{R}$ which achieves its maximum value when $\theta = 0$. This follows from the first derivative test from calculus. Therefore, for $|t| \leq 1$,

$$\left| \frac{1 + t}{2} \right|^p + \left| \frac{1 - t}{2} \right|^p \leq \left| \frac{1 + t}{2} \right|^p + \left| \frac{1 - t}{2} \right|^p \leq \frac{1}{2} (1 + |t|^p)$$

by the above lemma. ■

With this corollary, here is the easy Clarkson inequality.

**Theorem 15.3.6** Let $p \geq 2$. Then

$$\left\| \frac{f + g}{2} \right\|_{L^p}^p + \left\| \frac{f - g}{2} \right\|_{L^p}^p \leq \frac{1}{2} (\|f\|_{L^p}^p + \|g\|_{L^p}^p)$$

**Proof:** This follows right away from the above corollary.

$$\int_{\Omega} \left| \frac{f + g}{2} \right|^p d\mu + \int_{\Omega} \left| \frac{f - g}{2} \right|^p d\mu \leq \frac{1}{2} \int_{\Omega} (|f|^p + |g|^p) d\mu$$

Now it remains to consider the hard Clarkson inequalities. These pertain to $p < 2$. First is the following elementary inequality.

**Lemma 15.3.7** For $1 < p < 2$, the following inequality holds for all $t \in [0, 1]$.

$$\left| \frac{1 + t}{2} \right|^q + \left| \frac{1 - t}{2} \right|^q \leq \left( \frac{1}{2} + \frac{1}{2} |t|^p \right)^{q/p}$$

where here $1/p + 1/q = 1$ so $q > 2$. 

**Proof:** First note that if \( t = 0 \) or \( 1 \), the inequality holds. Next observe that the map \( s \rightarrow \frac{1 - s}{1 + s} \) maps \((0, 1)\) onto \((0, 1)\). Replace \( t \) with \( \frac{(1 - s)}{(1 + s)} \). Then you get

\[
\left| \frac{1}{s + 1} \right|^q + \left| \frac{s}{s + 1} \right|^q \leq \left( \frac{1}{2} + \frac{1}{2} \left| \frac{1 - s}{s + 1} \right|^p \right)^{q/p}
\]

Multiplying both sides by \((1 + s)^q\), this is equivalent to showing that for all \( s \in (0, 1)\),

\[
1 + s^q \leq ((1 + s)^p)^{q/p} \left( \frac{1}{2} + \frac{1}{2} \left| \frac{1 - s}{s + 1} \right|^p \right)^{q/p}
\]

This is the same as establishing

\[
\frac{1}{2} ((1 + s)^p + (1 - s)^p) - (1 + s^q)^{p-1} \geq 0 \quad (15.3.8)
\]

where \( p - 1 = p/q \) due to the definition of \( q \) above.

\[
\binom{p}{l} = \frac{p(p-1) \cdots (p-k+1)}{l!}, \quad l \geq 1
\]

and \( \binom{p}{0} \equiv 1 \). What is the sign of \( \binom{p}{l} \)? Recall that \( 1 < p < 2 \) so the sign is positive if \( l = 0, l = 1, l = 2 \). What about \( l = 3? \) \( \binom{p}{3} = \frac{p(p-1)(p-2)}{3!} \) so this is negative. Then \( \binom{p}{4} \) is positive. Thus these alternate between positive and negative with \( \binom{p}{2k} > 0 \) for all \( k \). What about \( \binom{p-1}{k} \)? When \( k = 0 \) it is positive. When \( k = 1 \) it is also positive. When \( k = 2 \) it equals \( \frac{(p-1)(p-2)}{2!} < 0 \). Then when \( k = 3 \), \( \binom{p-1}{3} > 0 \). Thus \( \binom{p-1}{k} \) is positive when \( k \) is odd and is negative when \( k \) is even.

Now return to **Lemma.** The left side equals

\[
\frac{1}{2} \left( \sum_{k=0}^{\infty} \binom{p}{k} s^k + \sum_{k=0}^{\infty} \binom{p}{k} (-s)^k \right) - \sum_{k=0}^{\infty} \binom{p-1}{k} s^{qk}.
\]

The first term equals 0. Then this reduces to

\[
\sum_{k=1}^{\infty} \binom{p}{2k} s^{2k} - \binom{p-1}{2k} s^{q2k} - \binom{p-1}{2k-1} s^{q(2k-1)}
\]
From the above observation about the binomial coefficients, the above is larger than

$$
\sum_{k=1}^{\infty} \binom{p}{2k} s^{2k} - \binom{p-1}{2k-1} s^{q(2k-1)}
$$

It remains to show the \(k\)th term in the above sum is nonnegative. Now \(q(2k - 1) > 2k\) for all \(k \geq 1\) because \(q > 2\). Then since \(0 < s < 1\)

$$
\left( \frac{p}{2k} \right) s^{2k} - \left( \frac{p-1}{2k-1} \right) s^{q(2k-1)} \geq s^{2k} \left( \left( \frac{p}{2k} \right) - \left( \frac{p-1}{2k-1} \right) \right)
$$

However, this is nonnegative because it equals

$$
s^{2k} \left( \frac{p(p-1) \cdots (p-2k+1)}{(2k)!} - \frac{(p-1)(p-2) \cdots (p-2k+1)}{(2k-1)!} \right) \geq 0
$$

Now \(q > 2\) and so by the same argument given in proving Corollary 15.3.5 for \(t = re^{i\theta}\), the left side of the above inequality is maximized when \(\theta = 0\). Hence, from Lemma 15.3.7,

$$
\left( \frac{t+1}{2} \right)^q + \left( \frac{1-t}{2} \right)^q \leq \left( \frac{1}{2} |t|^p + \frac{1}{2} \right)^{q/p}
$$

From this the hard Clarkson inequality follows. The two Clarkson inequalities are summarized in the following theorem.

**Corollary 15.3.8** Let \(z, w \in \mathbb{C}\). Then for \(p \in (1, 2)\),

$$
\frac{|z+w|^q}{2} + \frac{|z-w|^q}{2} \leq \left( \frac{1}{2} |z|^p + \frac{1}{2} |w|^p \right)^{q/p}
$$

**Proof:** One of \(|w|, |z|\) is larger. Say \(|w| \geq |z|\). Then dividing by \(|w|^q\), for \(t = z/w\), showing the above inequality is equivalent to showing that for all \(t \in \mathbb{C}\), \(|t| \leq 1\),

$$
\left( \frac{t+1}{2} \right)^q + \left( \frac{1-t}{2} \right)^q \leq \left( \frac{1}{2} |t|^p + \frac{1}{2} \right)^{q/p}
$$

Now \(q > 2\) and so by the same argument given in proving Corollary 15.3.5 for \(t = re^{i\theta}\), the left side of the above inequality is maximized when \(\theta = 0\). Hence, from Lemma 15.3.7,

$$
\left( \frac{t+1}{2} \right)^q + \left( \frac{1-t}{2} \right)^q \leq \left( \frac{1}{2} |t+1|^p + \frac{1}{2} \right)^{q/p} + \left( \frac{1}{2} |1-t|^p + \frac{1}{2} \right)^{q/p}
$$

From this the hard Clarkson inequality follows. The two Clarkson inequalities are summarized in the following theorem.
Theorem 15.3.9 Let \( 2 \leq p \). Then
\[
\left\| \frac{f + g}{2} \right\|_{L^p}^p + \left\| \frac{f - g}{2} \right\|_{L^p}^p \leq \frac{1}{2} (\|f\|_{L^p}^p + \|g\|_{L^p}^p)
\]

Let \( 1 < p < 2 \), then for \( \frac{1}{p} + \frac{1}{q} = 1 \),
\[
\left\| \frac{f + g}{2} \right\|_{L^p}^q + \left\| \frac{f - g}{2} \right\|_{L^p}^q \leq \left( \frac{1}{2} \left( \|f\|_{L^p}^p + \frac{1}{2} \|g\|_{L^p}^p \right) \right) ^{q/p}
\]

Proof: The first was established above.
\[
\left\| \frac{f + g}{2} \right\|_{L^p} ^q + \left\| \frac{f - g}{2} \right\|_{L^p} ^q \leq \left( \int_{\Omega} \left( \frac{f + g}{2} \right) ^p \, d\mu \right) ^{\frac{q}{p}} + \left( \int_{\Omega} \left( \frac{f - g}{2} \right) ^p \, d\mu \right) ^{\frac{q}{p}}
\]
\[
= \left( \int_{\Omega} \left( \left( \frac{f + g}{2} \right) ^q + \left( \frac{f - g}{2} \right) ^q \right) ^{\frac{p}{q}} \, d\mu \right) ^{\frac{q}{p}}
\]
Now \( p/q < 1 \) and so the backwards Minkowski inequality applies. Thus
\[
\leq \left( \int_{\Omega} \left( \left( \frac{f + g}{2} \right) ^q + \left( \frac{f - g}{2} \right) ^q \right) ^{\frac{p}{q}} \, d\mu \right) ^{\frac{q}{p}}
\]
From Corollary 15.3.8,
\[
\leq \left( \int_{\Omega} \left( \left( \frac{1}{2} |f| + \frac{1}{2} |g| \right) ^{p/q} \right) ^{p/q} \, d\mu \right) ^{q/p}
\]
\[
= \left( \int_{\Omega} \left( \frac{1}{2} |f| + \frac{1}{2} |g| \right) ^{q/p} \, d\mu \right) ^{q/p} = \left( \frac{1}{2} \left( \|f\|_{L^p}^p + \frac{1}{2} \|g\|_{L^p}^p \right) \right) ^{q/p} \]  

Now with these Clarkson inequalities, it is not hard to show that all the \( L^p \) spaces are uniformly convex.

Theorem 15.3.10 The \( L^p \) spaces are uniformly convex.

Proof: First suppose \( p \geq 2 \). Suppose \( \|f_n\|_{L^p}, \|g_n\|_{L^p} \leq 1 \) and \( \left\| \frac{f_n + g_n}{2} \right\|_{L^p} \rightarrow 1 \). Then from the first Clarkson inequality,
\[
\left\| \frac{f_n + g_n}{2} \right\|_{L^p} ^p + \left\| \frac{f_n - g_n}{2} \right\|_{L^p} ^p \leq \frac{1}{2} (\|f_n\|_{L^p}^p + \|g_n\|_{L^p}^p) \leq 1
\]
and so \( \|f_n - g_n\|_{L^p} \rightarrow 0 \).
Next suppose \(1 < p < 2\) and \(\|f_n + g_n\|_{L^p} \to 1\). Then from the second Clarkson inequality
\[
\left\| \frac{f_n + g_n}{2} \right\|_{L^p}^q + \left\| \frac{f_n - g_n}{2} \right\|_{L^p}^q \leq \left( \frac{1}{2} \|f_n\|_{L^p}^p + \frac{1}{2} \|g_n\|_{L^p}^p \right)^{q/p} \leq 1
\]
which shows that \(\|f_n - g_n\|_{L^p} \to 0\). ■

### 15.4 Closed Subspaces

**Theorem 15.4.1** Let \(X\) be a Banach space and let \(V = \text{span}\{x_1, \ldots, x_n\}\). Then \(V\) is a closed subspace of \(X\).

**Proof:** Without loss of generality, it can be assumed \(\{x_1, \ldots, x_n\}\) is linearly independent. Otherwise, delete those vectors which are in the span of the others till a linearly independent set is obtained. Let
\[
x = \lim_{p \to \infty} \sum_{k=1}^{n} c_k^p x_k \in V.
\] (15.4.9)

First suppose \(c^p \equiv (c_1^p, \ldots, c_n^p)\) is not bounded in \(F^n\). Then \(d^p \equiv c^p / |c^p|_{F^n}\) is a unit vector in \(F^n\) and so there exists a subsequence, still denoted by \(d^p\) which converges to \(d\) where \(|d| = 1\). Then
\[
0 = \lim_{p \to \infty} \frac{x}{|c^p|} = \lim_{p \to \infty} \sum_{k=1}^{n} d_k^p x_k = \sum_{k=1}^{n} d_k x_k
\]
where \(\sum_k |d_k|^2 = 1\) in contradiction to the linear independence of the \(\{x_1, \ldots, x_n\}\). Hence it must be the case that \(c^p\) is bounded in \(F^n\). Then taking a subsequence, still denoted as \(p\), it can be assumed \(c^p \to c\) and then in (15.4.9) it follows
\[
x = \sum_{k=1}^{n} c_k x_k \in \text{span}\{x_1, \ldots, x_n\}.
\] ■

**Proposition 15.4.2** Let \(E\) be a separable Banach space. Then there exists an increasing sequence of subspaces, \(\{F_n\}\) such that \(\dim(F_{n+1}) - \dim(F_n) \leq 1\) and equals 1 for all \(n\) if the dimension of \(E\) is infinite. Also \(\cup_{n=1}^{\infty} F_n\) is dense in \(E\). In the case where \(E\) is infinite dimensional, \(F_n = \text{span}\{e_1, \ldots, e_n\}\) where for each \(n\)
\[
\text{dist} (e_{n+1}, F_n) \geq \frac{1}{2}
\] (15.4.10)

and defining,
\[
G_k \equiv \text{span}\{\{e_j : j \neq k\}\}
\]
\[
\text{dist} (e_k, G_k) \geq \frac{1}{4}.
\] (15.4.11)
Proof: Since $E$ is separable, so is $\partial B(0,1)$, the boundary of the unit ball. Let $\{w_k\}_{k=1}^\infty$ be a countable dense subset of $\partial B(0,1)$.

Let $e_1 = w_1$. Let $F_1 = \mathbb{F}e_1$. Suppose $F_n$ has been obtained and equals $\text{span}(e_1, \ldots, e_n)$ where $\{e_1, \ldots, e_n\}$ is independent, $\|e_k\| = 1$, and

$$\text{dist}(e_n, \text{span}(e_1, \ldots, e_{n-1})) \geq \frac{1}{2}.$$ 

For each $n$, $F_n$ is closed by Theorem 15.4.1.

If $F_n$ contains $\{w_k\}_{k=1}^\infty$, let $F_m = F_n$ for all $m > n$. Otherwise, pick $w \in \{w_k\}$ to be the point of $\{w_k\}_{k=1}^\infty$ having the smallest subscript which is not contained in $F_n$. Then $w$ is at a positive distance, $\lambda$ from $F_n$ because $F_n$ is closed. Therefore, there exists $y \in F_n$ such that $\lambda \leq ||y - w|| \leq 2\lambda$. Let $e_{n+1} = \frac{w - y}{||w - y||}$. It follows

$$w = ||w - y|| e_{n+1} + y \in \text{span}(e_1, \ldots, e_{n+1}) \equiv F_{n+1}.$$ 

Then if $x \in \text{span}(e_1, \ldots, e_n)$,

$$||e_{n+1} - x|| = \left|\frac{w - y}{||w - y||} - x\right|$$

$$= \left|\frac{w - y}{||w - y||} - \frac{||w - y||}{||w - y||} x\right|$$

$$\geq \frac{1}{2\lambda} ||w - y - \frac{||w - y||}{||w - y||} x||$$

$$\geq \frac{\lambda}{2\lambda} = \frac{1}{2}.$$ 

This has shown the existence of an increasing sequence of subspaces, $\{F_n\}$ as described above. It remains to show the union of these subspaces is dense. First note that the union of these subspaces must contain the $\{w_k\}_{k=1}^\infty$ because if $w_m$ is missing, then it would contradict the construction at the $m^{th}$ step. That one should have been chosen. However, $\{w_k\}_{k=1}^\infty$ is dense in $\partial B(0,1)$. If $x \in E$ and $x \neq 0$, then $\frac{x}{||x||} \in \partial B(0,1)$ then there exists

$$w_m \in \{w_k\}_{k=1}^\infty \subseteq \bigcup_{n=1}^\infty F_n$$

such that $\frac{w_m - x}{||x||} < \frac{\epsilon}{||x||}$. But then

$$|||x|| w_m - x|| < \epsilon$$

and so $||x|| w_m$ is a point of $\bigcup_{n=1}^\infty F_n$ which is within $\epsilon$ of $x$. This proves $\bigcup_{n=1}^\infty F_n$ is dense as desired. 15.4.10 follows from the construction. It remains to verify 15.4.11.

Let $y \in G_k$. Thus for some $n$,

$$y = \sum_{j=1}^{k-1} c_j e_j + \sum_{j=k+1}^n c_j e_j$$
and I need to show $\|y - e_k\| \geq 1/4$. Without loss of generality, $c_n \neq 0$ and $n > k$.

Suppose (15.4.11) does not hold for some such $y$ so that

$$\left\|e_k - \left( \sum_{j=1}^{k-1} c_j e_j + \sum_{j=k+1}^n c_j e_j \right) \right\| < \frac{1}{4}. \tag{15.4.12}$$

Then from the construction,

$$\frac{1}{4} > |c_n| \left\|e_k - \left( \sum_{j=1}^{k-1} (c_j/c_n) e_j + \sum_{j=k+1}^{n-1} (c_j/c_n) e_j + e_n \right) \right\| \geq |c_n| \frac{1}{2}$$

and so $|c_n| < 1/2$. Consider the left side of (15.4.12). By the construction

$$\left\|c_n (e_k - e_n) + (1 - e_n) e_k - \left( \sum_{j=1}^{k-1} c_j e_j + \sum_{j=k+1}^{n-1} c_j e_j \right) \right\| \geq |1 - c_n| - |c_n| \frac{1}{2} \geq 1 - \frac{3}{2} |c_n| > 1 - \frac{3}{2} \frac{1}{2} = \frac{1}{4},$$

a contradiction. This proves the desired estimate. $lacksquare$

### 15.5 Weak And Weak * Topologies

#### 15.5.1 Basic Definitions

Let $X$ be a Banach space and let $X'$ be its dual space. For $A'$ a finite subset of $X'$, denote by $\rho_{A'}$ the function defined on $X$

$$\rho_{A'}(x) \equiv \max_{x^* \in A'} |x^*(x)| \tag{15.5.13}$$

and also let $B_{A'}(x, r)$ be defined by

$$B_{A'}(x, r) \equiv \{ y \in X : \rho_{A'}(y - x) < r \} \tag{15.5.14}$$

Then certain things are obvious. First of all, if $a \in \mathbb{F}$ and $x, y \in X$,

$$\rho_{A'}(x + y) \leq \rho_{A'}(x) + \rho_{A'}(y),$$

$$\rho_{A'}(ax) = |a| \rho_{A'}(x).$$

\(^1\)Actually, all this works in much more general settings than this.
Similarly, letting $A$ be a finite subset of $X$, denote by $\rho_A$ the function defined on $X'$

$$\rho_A(x^*) \equiv \max_{x \in A} |x^*(x)| \tag{15.5.15}$$

and let $B_A(x^*, r)$ be defined by

$$B_A(x^*, r) = \{y^* \in X' : \rho_A(y^* - x^*) < r\}. \tag{15.5.16}$$

It is also clear that

$$\rho_A(x^* + y^*) \leq \rho(x^*) + \rho_A(y^*),$$

$$\rho_A(ax^*) = |a| \rho_A(x^*).$$

**Lemma 15.5.1** The sets, $B_{A'}(x, r)$ where $A'$ is a finite subset of $X'$ and $x \in X$ form a basis for a topology on $X$ known as the weak topology. The sets $B_A(x^*, r)$ where $A$ is a finite subset of $X$ and $x^* \in X'$ form a basis for a topology on $X'$ known as the weak $^*$ topology.

**Proof:** The two assertions are very similar. I will verify the one for the weak topology. The union of these sets, $B_{A'}(x, r)$ for $x \in X$ and $r > 0$ is all of $X$. Now suppose $z$ is contained in the intersection of two of these sets. Say

$$z \in B_{A'}(x, r) \cap B_{A_1'}(x_1, r_1)$$

Then let $C' = A' \cup A_1'$ and let

$$0 < \delta \leq \min \left( r - \rho_{A'}(z - x), r_1 - \rho_{A_1'}(z - x_1) \right).$$

Consider $y \in B_{C'}(z, \delta)$. Then

$$r - \rho_{A'}(z - x) \geq \delta > \rho_{C'}(y - z) \geq \rho_{A'}(y - z)$$

and so

$$r > \rho_{A'}(y - z) + \rho_{A'}(z - x) \geq \rho_{A'}(y - x)$$

which shows $y \in B_{A'}(x, r)$. Similar reasoning shows $y \in B_{A_1'}(x_1, r_1)$ and so

$$B_{C'}(z, \delta) \subseteq B_{A'}(x, r) \cap B_{A_1'}(x_1, r_1).$$

Therefore, the weak topology consists of the union of all sets of the form $B_A(x, r)$.

### 15.5.2 Banach Alaoglu Theorem

Why does anyone care about these topologies? The short answer is that in the weak $^*$ topology, closed unit ball in $X'$ is compact. This is not true in the normal topology. This wonderful result is the Banach Alaoglu theorem. First recall the notion of the product topology, and the Tychonoff theorem, Theorem on Page 370 which are stated here for convenience.
Let $I$ be a set and suppose for each $i \in I$, $(X_i, \tau_i)$ is a nonempty topological space. The Cartesian product of the $X_i$, denoted by $\prod_{i \in I} X_i$, consists of the set of all choice functions defined on $I$ which select a single element of each $X_i$. Thus $f \in \prod_{i \in I} X_i$ means for every $i \in I$, $f(i) \in X_i$. The axiom of choice says $\prod_{i \in I} X_i$ is nonempty. Let

$$P_j(A) = \prod_{i \in I} B_i$$

where $B_i = X_i$ if $i \neq j$ and $B_j = A$. A subbasis for a topology on the product space consists of all sets $P_j(A)$ where $A \in \tau_j$. (These sets have an open set from the topology of $X_j$ in the $j$th slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_i$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\prod \tau_i$.

**Theorem 15.5.3** If $(X_i, \tau_i)$ is compact, then so is $(\prod_{i \in I} X_i, \prod \tau_i)$.

The Banach Alaoglu theorem is as follows.

**Theorem 15.5.4** Let $B'$ be the closed unit ball in $X'$. Then $B'$ is compact in the weak $\ast$ topology.

**Proof:** By the Tychonoff theorem, Theorem 15.5.2

$$P = \prod_{x \in X} B(0, ||x||)$$

is compact in the product topology where the topology on $B(0, ||x||)$ is the usual topology of $F$. Recall $P$ is the set of functions which map a point, $x \in X$ to a point in $B(0, ||x||)$. Therefore, $B' \subseteq P$. Also the basic open sets in the weak $\ast$ topology on $B'$ are obtained as the intersection of basic open sets in the product topology of $P$ to $B'$ and so it suffices to show $B'$ is a closed subset of $P$. Suppose then that $f \in P \setminus B'$. Since $|f(x)| \leq ||x||$ for each $x$, it follows $f$ cannot be linear. There are two ways this can happen. One way is that for some $x, y$

$$f(x + y) \neq f(x) + f(y)$$

for some $x, y \in X$. However, if $g$ is close enough to $f$ at the three points, $x + y, x, \text{ and } y$, the above inequality will hold for $g$ in place of $f$. In other words there is a basic open set containing $f$, such that for all $g$ in this basic open set, $g \notin B'$. A similar consideration applies in case $f(\lambda x) \neq \lambda f(x)$ for some scalar $\lambda$ and $x$. Since $P \setminus B'$ is open, it follows $B'$ is a closed subset of $P$ and is therefore, compact.

Sometimes one can consider the weak $\ast$ topology in terms of a metric space.

**Theorem 15.5.5** If $K \subseteq X'$ is compact in the weak $\ast$ topology and $X$ is separable in the weak topology then there exists a metric, $d$, on $K$ such that if $\tau_d$ is the topology on $K$ induced by $d$ and if $\tau$ is the topology on $K$ induced by the weak $\ast$ topology of $X'$, then $\tau = \tau_d$. Thus one can consider $K$ with the weak $\ast$ topology as a metric space.
CHAPTER 15. BANACH SPACES

Proof: Let $D = \{x_n\}$ be the dense countable subset in $X$. The metric is

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_{x_n}(f - g)}{1 + \rho_{x_n}(f - g)}$$

where $\rho_{x_n}(f) = |f(x_n)|$. Clearly $d(f, g) = d(g, f) \geq 0$. If $d(f, g) = 0$, then this requires $f(x_n) = g(x_n)$ for all $x_n \in D$. Is it the case that $f = g$?

$B_{(f, g)}(x, r)$ contains some $x_n \in D$. Hence

$$\max \{|f(x_n) - f(x)|, |g(x_n) - g(x)|\} < r$$

and $f(x_n) = g(x_n)$. It follows that $|f(x) - g(x)| < 2r$. Since $r$ is arbitrary, this implies $f(x) = g(x)$. It is routine to verify the triangle inequality from the easy to establish inequality,

$$\frac{x}{1 + x} + \frac{y}{1 + y} \geq \frac{x + y}{1 + x + y},$$

valid whenever $x, y \geq 0$. Therefore this is a metric.

Thus there are two topological spaces, $(K, \tau)$ and $(K, d)$, the first being $K$ with the weak * topology and the second being $K$ with this metric. It is clear that if $i$ is the identity map, $i: (K, \tau) \to (K, d)$, then $i$ is continuous. Therefore, sets which are open in $(K, d)$ are open in $(K, \tau)$. Letting $\tau_d$ denote those sets which are open with respect to the metric, $\tau_d \subseteq \tau$.

Now suppose $U \in \tau$. Is $U$ in $\tau_d$? Since $K$ is compact with respect to $\tau$, it follows from the above that $K$ is compact with respect to $\tau_d \subseteq \tau$. Hence $K \setminus U$ is compact with respect to $\tau_d$ and so it is closed with respect to $\tau_d$. Thus $U$ is open with respect to $\tau_d$.

The fact that this set with the weak * topology can be considered a metric space is very significant because if a point is a limit point in a metric space, one can extract a convergent sequence.

Note that if a Banach space is separable, then it is weakly separable.

Corollary 15.5.6 If $X$ is weakly separable and $K \subseteq X'$ is compact in the weak * topology, then $K$ is sequentially compact. That is, if $\{f_n\}_{n=1}^{\infty} \subseteq K$, then there exists a subsequence $f_{n_k}$ and $f \in K$ such that for all $x \in X$,

$$\lim_{k \to \infty} f_{n_k}(x) = f(x).$$

Proof: By Theorem 15.5.5, $K$ is a metric space for the metric described there and it is compact. Therefore by the characterization of compact metric spaces, Proposition 6.2.5 on Page 146, $K$ is sequentially compact. This proves the corollary.

15.5.3 Eberlein Smulian Theorem

Next consider the weak topology. The most interesting results have to do with a reflexive Banach space. The following lemma ties together the weak and weak * topologies in the case of a reflexive Banach space.
Lemma 15.5.7 Let $J : X \to X''$ be the James map

$$Jx(f) \equiv f(x)$$

and let $X$ be reflexive so that $J$ is onto. Then $J$ is a homeomorphism of $(X, \text{ weak topology})$ and $(X'', \text{ weak }^* \text{ topology}).$ This means $J$ is one to one, onto, and both $J$ and $J^{-1}$ are continuous.

Proof: Let $f \in X'$ and let

$$B_f(x,r) \equiv \{ y : |f(x) - f(y)| < r \}.$$ 

Thus $B_f(x,r)$ is a subbasic set for the weak topology on $X.$ I claim that

$$JB_f(x,r) = B_f(Jx,r)$$

where $B_f(Jx,r)$ is a subbasic set for the weak $^*$ topology. If $y \in B_f(x,r),$ then $\|Jy - Jx\| = \|x - y\| < r$ and so $JB_f(x,r) \subseteq B_f(Jx,r).$ Now if $x^{**} \in B_f(Jx,r),$ then since $J$ is reflexive, there exists $y \in X$ such that $Jy = x^{**}$ and so

$$\|y - x\| = \|Jy - Jx\| < r$$

showing that $JB_f(x,r) = B_f(Jx,r).$ A typical subbasic set in the weak $^*$ topology is of the form $B_f(Jx,r).$ Thus $J$ maps the subbasic sets of the weak topology to the subbasic sets of the weak $^*$ topology. Therefore, $J$ is a homeomorphism as claimed.

The following is an easy corollary.

Corollary 15.5.8 If $X$ is a reflexive Banach space, then the closed unit ball is weakly compact.

Proof: Let $B$ be the closed unit ball. Then $B = J^{-1}(B^{**})$ where $B^{**}$ is the unit ball in $X''$ which is compact in the weak $^*$ topology. Therefore $B$ is weakly compact because $J^{-1}$ is continuous.

Corollary 15.5.9 Let $X$ be a reflexive Banach space. If $K \subseteq X$ is compact in the weak topology and $X'$ is separable in the weak $^*$ topology, then there exists a metric $d,$ on $K$ such that if $\tau_d$ is the topology on $K$ induced by $d$ and if $\tau$ is the topology on $K$ induced by the weak topology of $X,$ then $\tau = \tau_d.$ Thus one can consider $K$ with the weak topology as a metric space.

Proof: This follows from Theorem 15.5.5 and Lemma 15.5.7. Lemma 15.5.7 implies $J(K)$ is compact in $X''.$ Then since $X'$ is separable in the weak $^*$ topology, $X$ is separable in the weak topology and so there is a metric, $d''$ on $J(K)$ which delivers the weak $^*$ topology on $J(K).$ Let $d(x,y) \equiv d''(Jx,Jy).$ Then

$$(K,\tau_d) \xrightarrow{J} (J(K),\tau_{d''}) \xrightarrow{id} (J(K),\tau_{weak^*}) \xrightarrow{J^{-1}} (K,\tau_{weak})$$

and all the maps are homeomorphisms.

Here is a useful lemma.
Lemma 15.5.10 Let $Y$ be a closed subspace of a Banach space $X$ and let $y \in X \setminus Y$. Then there exists $x^* \in X'$ such that $x^*(Y) = 0$ but $x^*(y) \neq 0$.

Proof: Define $f (x + \alpha y) \equiv \|y\| \alpha$. Thus $f$ is linear on $Y \oplus Fy$. I claim that $f$ is also continuous on this subspace of $X$. If not, then there exists $x_n + \alpha_n y \to 0$ but $|f (x_n + \alpha_n y)| \geq \varepsilon > 0$ for all $n$. First suppose $|\alpha_n|$ is bounded. Then, taking a further subsequence, we can assume $\alpha_n \to \alpha$. It follows then that $\{x_n\}$ must also converge to some $x \in Y$ since $Y$ is closed. Therefore, in this case, $x + \alpha y = 0$ and so $\alpha = 0$ since otherwise, $y \in Y$. In the other case when $\alpha_n$ is unbounded, you have $(x_n/\alpha_n + y) \to 0$ and so it would require that $y \in Y$ which cannot happen because $Y$ is closed. Hence $f$ is continuous as claimed. It follows that for some $k$,

$$|f (x + \alpha y)| \leq k \|x + \alpha y\|$$

Now apply the Hahn Banach theorem to extend $f$ to $x^* \in X'$.

Next is the Eberlein Smulian theorem which states that a Banach space is reflexive if and only if the closed unit ball is weakly sequentially compact. Actually, only half the theorem is proved here, the more useful only if part. The book by Yoshida has the complete theorem discussed. First here is an interesting lemma for its own sake.

Lemma 15.5.11 A closed subspace of a reflexive Banach space is reflexive.

Proof: Let $Y$ be the closed subspace of the reflexive space, $X$. Consider the following diagram

$$
\begin{array}{ccc}
Y'' & \overset{i^{**}}{\longrightarrow} & X'' \\
Y' & \overset{i^*}{\longrightarrow} & X' \\
Y & \overset{i}{\longrightarrow} & X \\
\end{array}
$$

This diagram follows from Theorem 15.5.11 on Page 450, the theorem on adjoints. Now let $y^{**} \in Y''$. Then $i^{**}y^{**} = J_X (y)$ because $X$ is reflexive. I want to show that $y \in Y$. If it is not in $Y$ then since $Y$ is closed, there exists $x^* \in X'$ such that $x^*(y) \neq 0$ but $x^*(Y) = 0$. Then $i^*x^* = 0$. Hence

$$0 = y^{**} (i^* x^*) = i^{**} y^{**} (x^*) = J_X (y) (x^*) = x^*(y) \neq 0,$$

a contradiction. Hence $y \in Y$. Letting $J_Y$ denote the James map from $Y$ to $Y''$ and $x^* \in X'$,

$$
y^{**} (i^* x^*) = i^{**} y^{**} (x^*) = J_X (y) (x^*) = x^*(y) = J_Y (y) (i^* x^*)
$$

Since $i^*$ is onto, this shows $y^{**} = J_Y (y)$. ■

Theorem 15.5.12 (Eberlein Smulian) The closed unit ball in a reflexive Banach space $X$, is weakly sequentially compact. By this is meant that if $\{x_n\}$ is contained in the closed unit ball, there exists a subsequence, $\{x_{n_k}\}$ and $x \in X$ such that for all $x^* \in X'$,

$$x^*(x_{n_k}) \to x^*(x).$$
15.5. WEAK AND WEAK ∗ TOPOLOGIES

Proof: Let \( \{x_n\} \subseteq B \equiv B(0,1) \). Let \( Y \) be the closure of the linear span of \( \{x_n\} \). Thus \( Y \) is a separable. It is reflexive because it is a closed subspace of a reflexive space so the above lemma applies. By the Banach Alaoglu theorem, the closed unit ball \( B^* \) in \( Y' \) is weak ∗ compact. Also by Theorem 15.5.5, \( B^* \) is a metric space with a suitable metric.

\[
\begin{align*}
B^{**} & \xrightarrow{i'^* 1-1} X'' \\
\text{weakly separable } B^* & \xleftarrow{i^* \text{onto}} Y' \\
\text{separable } B & \xrightarrow{i} Y
\end{align*}
\]

Thus \( B^* \) is complete and totally bounded with respect to this metric and it follows that \( B^* \) with the weak ∗ topology is separable. This implies \( Y' \) is also separable in the weak ∗ topology. To see this, let \( \{y_n^*\} \equiv D \) be a weak ∗ dense set in \( B^* \) and let \( y^* \in Y' \). Let \( p \) be a large enough positive rational number that \( y^*/p \in B^* \). Then if \( A \) is any finite set from \( Y \), there exists \( y_n^* \in D \) such that \( \rho_A(y^*/p - y_n^*) < \frac{\varepsilon}{p} \). It follows \( py_n^* \in B_A(y^*,\varepsilon) \) showing that rational multiples of \( D \) are weak ∗ dense in \( Y' \). Since \( Y \) is reflexive, the weak and weak ∗ topologies on \( Y' \) coincide and so \( Y' \) is weakly separable. Since \( Y' \) is weakly separable, Corollary 15.5.13 implies \( B^{**} \), the closed unit ball in \( Y'' \) is weak ∗ sequentially compact. Then by Lemma 15.5.14 \( B \), the unit ball in \( Y \), is weakly sequentially compact. It follows there exists a subsequence \( x_{n_k} \), of the sequence \( \{x_n\} \) and a point \( x \in Y \), such that for all \( f \in Y' \),

\[
f(x_{n_k}) \to f(x).
\]

Now if \( x^* \in X' \), and \( i \) is the inclusion map of \( Y \) into \( X \),

\[
x^*(x_{n_k}) = i^*x^*(x_{n_k}) \to i^*x^*(x) = x^*(x).
\]

which shows \( x_{n_k} \) converges weakly and this shows the unit ball in \( X \) is weakly sequentially compact.

Corollary 15.5.13 Let \( \{x_n\} \) be any bounded sequence in a reflexive Banach space \( X \). Then there exists \( x \in X \) and a subsequence, \( \{x_{n_k}\} \) such that for all \( x^* \in X' \),

\[
\lim_{k \to \infty} x^*(x_{n_k}) = x^*(x)
\]

Proof: If a subsequence, \( x_{n_k} \) has \( \|x_{n_k}\| \to 0 \), then the conclusion follows. Simply let \( x = 0 \). Suppose then that \( \|x_n\| \) is bounded away from 0. That is, \( \|x_n\| \in [\delta,C] \). Take a subsequence such that \( \|x_{n_k}\| \to a \). Then consider \( x_{n_k} / \|x_{n_k}\| \). By the Eberlein Smulian theorem, this subsequence has a further subsequence, \( x_{n_{k_j}} / \|x_{n_{k_j}}\| \) which converges weakly to \( x \in B \) where \( B \) is the closed unit ball. It follows from routine considerations that \( x_{n_{k_j}} \to ax \) weakly. This proves the corollary.
15.6 Operators With Closed Range

When is $T(X)$ a closed subset of $Y$ for $T \in \mathcal{L}(X,Y)$? One way this happens is when $T = I - C$ for $C$ compact.

**Definition 15.6.1** Let $C \in \mathcal{L}(X,Y)$ where $X,Y$ are two Banach spaces. Then $C$ is called a compact operator if $C$ (bounded set) = (precompact set).

**Lemma 15.6.2** Suppose $C \in \mathcal{L}(X,X)$ is compact. Then $(I - C)(X)$ is closed.

**Proof:** Let $(I - C)x_n \to y$. Let $z_n \in \ker (I - C)$ such that

$$\text{dist}(x_n, \ker (I - C)) \leq \|x_n - z_n\| \leq \left(1 + \frac{1}{n}\right) \text{dist}(x_n, \ker (I - C))$$

**Case 1:** $\|x_n - z_n\| \to \infty$.

In this case, you get $(I - C)(x_n - z_n) \to y$ and so there is a subsequence such that $C \left(\frac{x_n - z_n}{\|x_n - z_n\|}\right)$ converges. Also $\frac{x_n - z_n}{\|x_n - z_n\|}$ converges to the same thing. Let it be called $w$. Thus

$$\frac{x_n - z_n}{\|x_n - z_n\|} \to w, \quad C \frac{x_n - z_n}{\|x_n - z_n\|} \to Cw$$

$$C \left(\frac{x_n - z_n}{\|x_n - z_n\|}\right) \to w \text{ so } Cw = w, \ w \in \ker (I - C)$$

$$\left\|\frac{x_n - z_n}{\|x_n - z_n\|} - w\right\| = \frac{1}{\|x_n - z_n\|} \left\| (x_n - z_n) - \underbrace{w \|x_n - z_n\|}_{\in \ker(I-C)} \right\|$$

$$\geq \frac{1}{\|x_n - z_n\|} \text{dist}(x_n, \ker (I - C))$$

$$\geq \frac{1}{\left(1 + \frac{1}{n}\right) \text{dist}(x_n, \ker (I - C))} \text{dist}(x_n, \ker (I - C))$$

Now passing to a limit,

$$0 \geq \lim_{n \to \infty} \frac{1}{1 + 1/n} = 1$$

so **Case 1** cannot occur.

**Case 2:** A subsequence of $\|x_n - z_n\|$ is bounded.

Let $n$ denote the subscript for the subsequence. Then there is a further subsequence still denoted with $n$ such that $C(x_n - z_n)$ converges. Then also $(x_n - z_n)$ converges because $(I - C)(x_n) = (I - C)(x_n - z_n)$ is given to converge. Let $(x_n - z_n) \to x$. Then

$$y = \lim_{n \to \infty} (I - C)x_n = \lim_{n \to \infty} (I - C)(x_n - z_n) = (I - C)x$$
and so \( y \in (I - C)(X) \) showing that \((I - C)(X)\) is closed. ■

Here is a useful lemma.

**Lemma 15.6.3** Suppose \( W \) and \( V \) are closed subspaces of a Banach space \( X \) and \( V \not
\subseteq W \) (\( V \) is a proper subset of \( W \)) while \((\lambda I - L)(W) \subseteq V, \lambda \neq 0 \). Then there
exists \( w \in W \setminus V \) such that \( \|w\| = 1 \) and

\[
\text{dist}(Lw, LV) \geq 1/2
\]

**Proof:** Let \( w_0 \in W \setminus V \). Then let \( v \in V \) be such that \( \|\lambda w_0 - v\| \leq 2 \text{dist}(\lambda w_0, V) \).

Then let

\[
w = \frac{\lambda w_0 - v}{\|\lambda w_0 - v\|}
\]

It follows that \( \|w\| = 1 \) and is in \( W \setminus V \). Now let \( x \in V \). Then

\[
Lx - Lw = \lambda (x - w) + (L - \lambda I)(x - w)
\]

\[
= \frac{1}{\|\lambda w_0 - v\|} (\lambda x \|\lambda w_0 - v\| + (L - \lambda I)(x - w) \|\lambda w_0 - v\| - \lambda \|\lambda w_0 - v\| w)
\]

\[
= \frac{1}{\|\lambda w_0 - v\|} (\lambda x \|\lambda w_0 - v\| + (L - \lambda I)(x - w) \|\lambda w_0 - v\| - \lambda (\lambda w_0 - v))
\]

\[
= \frac{1}{\|\lambda w_0 - v\|} (\lambda x \|u_0 - v\| + (L - \lambda I)(x - w) \|\lambda w_0 - v\| + \lambda v - \lambda w_0)
\]

Thus

\[
\|Lx - Lw\| \geq \frac{1}{\|\lambda w_0 - v\|} \|\lambda x \|\lambda w_0 - v\| + (L - \lambda I)(x - w) \|\lambda w_0 - v\| + \lambda v - \lambda w_0\|
\]

\[
\geq \frac{1}{2 \text{dist}(\lambda w_0, V)} \text{dist}(\lambda w_0, V) = \frac{1}{2} \quad ■
\]

Here is another fairly elementary lemma a little like the above.

**Lemma 15.6.4** Let \( Y \) be an infinite dimensional Banach space. Then there exists a sequence \( \{x_n\} \) in the unit sphere \( S, \|x_n\| = 1 \), such that \( \|x_n - x_m\| \geq \frac{1}{2} \) whenever \( n \neq m \).

**Proof:** Pick \( x_1 \in S \). Now the span of \( x_1 \) is not everything and so there exists \( u_2 \notin \text{span}(x_1) \). Let \( u_2 \) be a point of \( \text{span}(x_1) \) such that \( \|u_2 - w_2\| \leq 2 \text{dist}(u_2, \text{span}(x_1)) \).

Then \( x_2 = \frac{u_2 - w_2}{\|u_2 - w_2\|} \). Then

\[
\|x_1 - x_2\| = \left\| \frac{\|u_2 - w_2\| x_1 - (u_2 - w_2)}{\|u_2 - w_2\|} \right\| \geq \frac{\text{dist}(u_2, \text{span}(x_1))}{2 \text{dist}(u_2, \text{span}(x_1))} = \frac{1}{2}
\]

Now repeat the argument with \( \text{span}(x_1, x_2) \) in place of \( \text{span}(x_1) \) and continue to get the desired sequence. ■
Lemma 15.6.5 Let L be a compact linear map. Then the eigenspace of L is finite dimensional for each eigenvalue \( \lambda \neq 0 \).

**Proof:** Consider \( (L - \lambda I)^{-1}(0) \cap S \) where S is the unit sphere. The eigenspace is just \( (L - \lambda I)^{-1}(0) \). Let Y be this inverse image. If Y is infinite dimensional, then the above Lemma 15.6.4 applies. There exists \( \{x_n\} \subseteq (L - \lambda I)^{-1}(0) \cap S \) where \( \|x_n - x_m\| \geq 1/2 \) for all \( n \neq m \). Then there is a subsequence, still denoted with subscript \( n \) such that \( \{Lx_n\} \) is a Cauchy sequence. Thus \( Lx_n = \lambda x_n \) and so, since \( \lambda \neq 0 \), it follows that \( \{x_n\} \) is also a Cauchy sequence and converges to some \( x \). But this is impossible because of the construction of the \( \{x_n\} \) which prevents there being any Cauchy sequence. Thus Y must be finite dimensional. \( \blacksquare \)

This lemma is useful in proving the following major spectral theorem about the eigenvalues of a compact operator. I found this theorem in Deimling [45].

**Theorem 15.6.6** Let \( L \in \mathcal{L}(X, X) \) with L compact. Let \( \Lambda \) be the eigenvalues of L. That is \( \lambda \in \Lambda \) means there exists \( x \neq 0 \) such that \( Lx = \lambda x \). It is assumed the field of scalars is \( \mathbb{R} \) or \( \mathbb{C} \). Let \( R_\lambda \equiv L - \lambda I \). Then the following hold.

1. If \( \mu \in \Lambda \) then \( |\mu| \leq \|L\| \), \( \Lambda \) is at most countable and has no limit points other than possibly 0.
2. \( R_\lambda \) is a homeomorphism onto \( X \) whenever \( \lambda \notin \Lambda \cup \{0\} \).
3. For all \( \lambda \in \Lambda \setminus \{0\} \), there exists a smallest \( k = k(\lambda) \),
   
   (a) \( R_\lambda^k X + N (R_\lambda^k) = X \) where \( N (R_\lambda^k) \) is the vectors \( x \) such that \( R_\lambda^k x = 0 \).
   \( R_\lambda^k X \) is closed, \( \dim (N (R_\lambda^k)) < \infty \).
   
   (b) \( R_\lambda^k X \) and \( N (R_\lambda^k) \) are invariant under \( L \) and \( R_\lambda |_{R_\lambda^k X} \) is a homeomorphism onto \( R_\lambda^k X \).
   
   (c) \( N (R_\mu^k) \subseteq R_\lambda^k X \) for all \( \lambda, \mu \in \Lambda \setminus \{0\} \) where \( \lambda \neq \mu \).

**Proof:** Consider \( \lambda \neq 0 \). The \( N (R_\lambda^k) \) are increasing in \( k \) and \( R_\lambda (N (R_\lambda^{k+1})) \subseteq N (R_\lambda^k) \). This follows from the definition. (It isn’t necessary to assume in most of this that \( \lambda \in \Lambda \), just a nonzero number will do.) Now

\[
R_\lambda = -\lambda \left( I - \frac{1}{\lambda} L \right)
\]

If these things are strictly increasing for infinitely many \( k \), then by Lemma 15.6.5, there is an infinite sequence \( x_k \in N (R_\lambda^{k+1}) \setminus N (R_\lambda^k) \), \( \text{dist} (Lx_k, LN (R_\lambda^k)) \geq 1/2 \). Hence \( \|Lx_k - Lx_{k-1}\| \geq 1/2 \) and this can’t happen because L is compact so \( \{Lx_k\} \) has a Cauchy subsequence. Therefore there exists a smallest \( k \) such that

\[
N (R_\lambda^k) = N (R_\lambda^m), m \geq k
\]

On the other hand, \( \{R_\lambda^k X\} \) are decreasing in \( k \). By similar reasoning using Lemma 15.6.5 and the observation that \( R_\lambda (R_\lambda^k X) \geq R_\lambda^{k+1} X \) (in fact they are equal) it follows that the \( \{R_\lambda^k X\} \) are also eventually constant, say for \( m \geq l \).
Now if you have $y \in N \left( R^k_{\lambda} \right) \cap R^k_{\lambda} X$, then $y = R^k_{\lambda} w$ and also $R^k_{\lambda} y = 0$. Hence $R^k_{\lambda} w = 0$ and so, $w \in N \left( R^k_{\lambda} \right) = N \left( R^k_{\lambda} \right)$ which implies $R^k_{\lambda} w = 0$ and so $y = 0$. It follows $N \left( R^k_{\lambda} \right) \cap R^k_{\lambda} X = \{0\}$.

Now suppose $l > k$. Then there exists $y \in R^{l-1}_{\lambda} \setminus R^l_{\lambda} X$ and so $R_{\lambda} y \in R^l_{\lambda} X = R^l_{\lambda} R^k_{\lambda} X$. So $R_{\lambda} y = R_{\lambda} z$ for some $z \in R^k_{\lambda} X$. Thus $y - z \neq 0$ because $y \notin R^k_{\lambda} X$ but $z$ is. However, $R_{\lambda} (y - z) = 0$ and so

$$(y - z) \in N \left( R^l_{\lambda} \right) \cap R^k_{\lambda} X \subseteq N \left( R^k_{\lambda} \right) \cap R^k_{\lambda} X$$

which cannot happen from the above which showed that $N \left( R^k_{\lambda} \right) \cap R^k_{\lambda} X = \{0\}$. Thus $l \leq k$.

Next suppose $l < k$. Then you would have $R^l_{\lambda} X = R^k_{\lambda} X$ and $N \left( R^l_{\lambda} \right) \supset N \left( R^k_{\lambda} \right)$. Thus there exists $y \in N \left( R^l_{\lambda} \right)$ but not in $N \left( R^k_{\lambda} \right)$. Hence $R^k_{\lambda} y = 0$ but $R^l_{\lambda} y \neq 0$. However, $R^l_{\lambda} y$ is in $R^k_{\lambda} X$ from the definition of $l$ and so there is $u$ such that $R^l_{\lambda} y = R^k_{\lambda} u$. Thus

$$0 = R^k_{\lambda} y = R^{k-l} y = R^k_{\lambda} R^l_{\lambda} y = R^{k-l} R^k_{\lambda} u = R^{2k-l} u$$

Now it follows that $u \in N \left( R^{2k-l}_{\lambda} \right) = N \left( R^k_{\lambda} \right)$. This is a contradiction because it says that $R^k_{\lambda} u = 0$ but right above the displayed equation, we had $R^k_{\lambda} y = R^k_{\lambda} u$ and $R^l_{\lambda} y \neq 0$. Thus, with the above paragraph, $k = l$.

What about the claim that $R_{\lambda}$ restricted to $R^k_{\lambda} X$ is a homeomorphism? It maps $R^k_{\lambda} X$ to $R^{k+1}_{\lambda} X = R^k_{\lambda} X$. Also, if $R_{\lambda} (y) = 0$ for $y \in R^k_{\lambda} X$, then $R^k_{\lambda} y = 0$ also and so $y \in R^k_{\lambda} X \cap N \left( R^k_{\lambda} \right)$. It was shown above that this implies $y = 0$. Thus $R_{\lambda}$ appears to be one to one. By assumption, it is continuous. Also from Lemma 15.6.7.

$R^k_{\lambda} X$ is closed.

This follows from the observation that

$$R^k_{\lambda} = (L - \lambda I)^k = \sum_{j=0}^{k} \binom{k}{j} L^j (-\lambda I)^{k-j} = (-\lambda)^k I + \sum_{j=1}^{k} \binom{k}{j} L^j (-\lambda I)^{k-j}$$

(15.6.17)

which is a multiple of $I - C$ where $C$ is a compact map. Then by the open mapping theorem, it follows that $R_{\lambda}$ is a homeomorphism onto $R^{k+1}_{\lambda} X = R^k_{\lambda} X$.

What about $R^k_{\lambda} X \oplus N \left( R^k_{\lambda} \right) = X$? It only remains to verify that $R^k_{\lambda} X + N \left( R^k_{\lambda} \right) = X$ because the only vector in the intersection was shown to be $0$. Thus if you have $x + y = 0$ where $x$ is in one of these and $y$ in the other, then $x = -y$ so each is in both and hence both are $0$. Pick $x \in X$. Then $R^k_{\lambda} x \in R^k_{\lambda} \left( R^k_{\lambda} X \right) = R^k_{\lambda} X$. Therefore, $R^k_{\lambda} x = R^k_{\lambda} \left( R^k_{\lambda} y \right)$ for some $y$ and so $R^k_{\lambda} \left( x - R^k_{\lambda} y \right) = 0$. Hence

$$x - R^k_{\lambda} y \in N \left( R^k_{\lambda} \right)$$

showing that $x \in R^k_{\lambda} X + N \left( R^k_{\lambda} \right)$.

It is obvious that $R^k_{\lambda} X$ and $N \left( R^k_{\lambda} \right)$ are invariant under $L$. If $\lambda_0 \notin \Lambda \setminus \{0\}$, then $L - \lambda_0 I$ is one to one and so the compactness of $L$ and Lemma 15.6.7 implies that
Thus operators have this property that their image is closed. These are discussed next.

For \( \mu \in \Lambda \), \( Lx = \mu x \) and so \( |\mu|\|x\| \leq \|L\|\|x\| \) so \( |\mu| \leq \|L\| \). Why is \( \Lambda \) at most countable and has only one possible limit point at \( 0 \)? It was shown that \( R_\lambda \) is a homeomorphism when restricted to \( R_\lambda^k X \). It follows that for \( x \in R_\lambda^k X, \|R_\lambda x\| > \delta \|x\| \) for some \( \delta > 0 \), this for every such \( x \in R_\lambda^k X \). Now consider \( \mu \) close to \( \lambda \) and consider \( R_\mu \). Then for \( x \in R_\lambda^k X, \|R_\mu x\| = \|\mu(R_\lambda x)\| \geq \|R_\lambda x\| - |\lambda - \mu| \|x\| > |\lambda - \mu| \|x\| > \frac{\delta}{2} \|x\| \) provided \( |\lambda - \mu| < \delta/2 \). Thus for \( \mu \) close enough to \( \lambda \), \( R_\mu \) is one to one on \( R_\lambda^k X \). But also \( R_\mu \) is one to one on \( \Lambda \). Let's see why this is so. Suppose \( (L - \mu I)x = 0 \) for \( x \in N (R_\lambda^k X) \). Then

\[
0 = (L - \mu I + (\mu - \lambda) I)^k x = (\mu - \lambda)^k x + \sum_{j=1}^{k} \binom{k}{j} (L - \mu I)^j (\mu - \lambda)^{k-j} x
\]

and the second term involving the sum yields 0. Since \( R_\lambda^1 X \oplus N (R_\lambda^k) = X \), this shows that \( (L - \mu I) \) is one to one for \( \mu \) near \( \lambda \). It follows that for \( \mu \) near \( \lambda \), \( R_\mu \) is one to one on \( (L - \mu I) \) to be one to one on \( N (R_\lambda^k) \).

Why is dim \( (N (R_\lambda^k) ) \) \( < \infty \) for each \( \lambda \neq 0 \). This follows from \([15.6.10]\). \( R_\lambda^k \) is a multiple of \( I - C \) for \( C \) a compact operator. Hence this is finite dimensional by Lemma \([15.6.3]\).

What about \( N (R_\mu^k) \subseteq R_\lambda^k X \) for \( \mu \) an eigenvalue different than \( \lambda \)? Say \( R_\mu^k x = 0 \). Then, does it follow that \( x \in R_\lambda^k X \)? From what was just shown

\[
x = y + z, \quad y, z \in R_\lambda^k X, \quad z \in N (R_\lambda^k)
\]

Then

\[
0 = R_\mu^k x = R_\mu^k y + R_\mu^k z
\]

Here \( p = k(\mu) \). This is where it is important that \( \mu \in \Lambda \). However, \( N (R_\lambda^k) \) and \( R_\lambda^k X \) are invariant under \( R_\mu^k \) since it is clear that \( R_\lambda \) and \( R_\mu \) commute. Thus \( R_\mu^k y = -R_\mu^k z \) and \( R_\mu^k y \in R_\lambda^k X, \quad -R_\mu^k z \in N (R_\lambda^k) \) and these are equal. Hence they are both 0. Now it was just shown that \( R_\mu \) is one to one on \( N (R_\lambda^k) \) and so \( z = 0 \). Hence \( x, y \in R_\lambda^k X \).

Note that in the last step, we can’t conclude that \( y = 0 \) because we only know that \( R_\mu \) is one to one on \( R_\lambda^k X \) if \( \mu \) is sufficiently close to \( \lambda \). The above is about compact mappings from a single space to itself. However, there are also mappings which have closed range which map from one space to another. The Fredholm operators have this property that their image is closed. These are discussed next.

Suppose \( T \in \mathcal{L} (X, Y) \). Then \( TX \) is a subspace of \( Y \) and so it has a Hamel basis \( \mathcal{B} \). Extending \( \mathcal{B} \) to a Hamel basis for \( Y \) yields \( \mathcal{C} \). Then \( Y = \text{span} (\mathcal{B}) \oplus \text{span} (\mathcal{C} \setminus \mathcal{B}) \). Thus \( Y = TX \oplus E \). For more on this, see \([22]\).
Definition 15.6.7 Let $T \in \mathcal{L}(X, Y)$. Then this is a Fredholm operator means

1. $\dim(\ker(T)) < \infty$
2. $\dim(E) < \infty$ where $Y = TX \oplus E$

Proposition 15.6.8 Let $T \in \mathcal{L}(X, Y)$. Then $TX$ is closed if and only if there exists $\delta > 0$ such that

$$\|Tx\| \geq \delta \text{dist}(x, \ker(T)).$$

Proof: First suppose $TX$ is closed. Let $\tilde{T} : X/\ker(T) \to Y$ be defined as $\tilde{T}([x]) \equiv Tx$. Then by Theorem 16.7.2, $\tilde{T}$ is one to one and continuous and $X/\ker(T)$ is a Banach space, $\|\tilde{T}\| \leq \|T\|$. Also $\tilde{T}$ has the same range as $T$. Thus $TX$ is the same as $\tilde{T}(X/\ker(T))$ and $\tilde{T} \in \mathcal{L}(X/\ker(T), Y)$. By the open mapping theorem, $\tilde{T}$ is continuous and has continuous inverse. Recall

$$\|[x]\| = \inf \{\|x + z\| : z \in \ker T\} = \text{dist}(x, \ker(T))$$

Then

$$\text{dist}(x, \ker(T)) = \|[x]\| = \|\tilde{T}^{-1}\tilde{T}[x]\| \leq \|\tilde{T}^{-1}\| \|\tilde{T}[x]\| = \|\tilde{T}^{-1}\| \|Tx\|$$

and so,

$$\|Tx\| \geq \delta \text{dist}(x, \ker(T))$$

where $\delta = 1/\|\tilde{T}^{-1}\|$.

Next suppose the inequality holds. Why will $TX$ be closed? Say $\{Tx_n\}$ is a sequence in $TX$ converging to $y$. Then by the inequality,

$$\|Tx_n - Tx_m\| \geq \delta \text{dist}(x_n - x_m, \ker(T)) = \delta \|[x_n] - [x_m]\|_{X/\ker(T)}$$

showing that $\{[x_n]\}$ is a Cauchy sequence in $X/\ker(T)$. Therefore, since this is a Banach space, there exists $[x]$ such that $[x_n] \to [x]$ in $X/\ker(T)$ and so $\tilde{T}([x_n]) \to \tilde{T}([x])$ in $Y$. But this is the same as saying that $T(x_n) \to T(x)$. It follows that $y = Tx$ and so $TX$ is indeed closed.

Theorem 15.6.9 If $T$ is a Fredholm operator, then $TX$ is closed in $Y$.

Proof: Recall that $Y = TX \oplus E$ where $E$ is a closed subspace of $Y$. In fact, $E$ is finite dimensional, but it is only needed that $E$ is closed. Let $T_0 \in \mathcal{L}(X \times E, TX \oplus E)$ be given by

$$T_0(x, e) \equiv Tx + e$$

Let the norm on $X \times E$ be

$$\|(x, e)\|_{X \times E} \equiv \max\{\|x\|_X, \|e\|_E\}$$
Thus \( T_0 (x, e) = 0 \) implies both \( Tx = 0 \) and \( e = 0 \). Thus \( \ker (T_0) = \ker (T) \times \{0\} \). Also, \( T_0 (X \times E) \) is closed in \( Y \) because in fact it is all of \( Y, TX \oplus E \). By Proposition 15.6.8, there exists \( \delta > 0 \) such that
\[
\| T_0 (x, e) \|_Y \geq \delta \text{ dist } ((x, e), \ker (T_0)) = \delta \text{ dist } ((x, e), \ker (T) \times \{0\}) \geq \delta \text{ dist } (x, \ker (T))
\]
Then
\[
\| Tx \|_Y = \| T_0 (x, 0) \|_Y \geq \delta \text{ dist } (x, \ker (T))
\]
and by Proposition 15.6.8, \( TX \) is closed.

Actually, the above proves the following corollary.

**Corollary 15.6.10** If \( TX \oplus E \) is closed in \( Y \) and \( E \) is a closed subspace of \( Y \), then \( TX \) is closed. Here \( T \in \mathcal{L} (X, Y) \).

Note that it appears that \( \dim (\ker (T)) < \infty \) was not really needed.

Let \( B \) be a Hamel basis for \( TX \) and consider \( A = \{ x : Tx \in B \} \). Then this is a linearly independent set of vectors in \( X \). Suppose now that \( \ker (T) = \text{span} (z_1, \ldots, z_n) \) where \( \{z_1, \ldots, z_n\} \) is linearly independent so here the assumption that \( \ker (T) \) has finite dimensions is being used. Then if \( x \in X, Tx \in TX \) and so there are finitely many vectors \( x_i \in A \) such that
\[
Tx = \sum_i c_i Tx_i.
\]
Hence
\[
T \left( x - \sum_i c_i x_i \right) = 0
\]
so
\[
x - \sum_i c_i x_i = \sum_{j=1}^n a_j z_j
\]
Hence \( X = \text{span} (A) + \ker (T) \). In fact, \( \{A, \{z_1, \ldots, z_n\}\} \) is linearly independent as is easily seen and so this is a basis for \( X \). Hence
\[
X = \text{span} (A) \oplus \ker (T) \equiv X_1 \oplus \ker (T)
\]
Is \( X_1 \) closed? Define \( S : TX \to X_1 \) as follows: \( Sy = x \in X_1 \) such that \( Tx = y \).
Since \( T \) is one to one on \( X_1 \), there is only one such \( x \). Is \( S \) continuous? Yes, this is so by the open mapping theorem. It is just the inverse of a continuous one to one linear onto map. Now this reduces to the situation discussed above in Corollary 15.6.10. You have \( S \in \mathcal{L}(TX, X_1) \) and \( S(TX) \oplus \ker (T) \) is all of \( X \) and so it is closed in \( X \). Therefore, \( S(TX) = X_1 \) is closed. This, along with the above proves the following.

**Theorem 15.6.11** Let \( T \in \mathcal{L} (X, Y) \) be a Fredholm operator and suppose \( \ker (T) \) is finite dimensional and that \( Y = TX \oplus E \) where \( E \) is a finite dimensional subspace or more generally closed. Then \( TX \) is closed and also for \( X = X_1 \oplus \ker (T) \), it follows that \( X_1 \) is closed.
15.7 Exercises

1. Is \( \mathbb{N} \) a \( G_\delta \) set? What about \( \mathbb{Q} \)? What about a countable dense subset of a complete metric space?

2. ↑ Let \( f : \mathbb{R} \to \mathbb{C} \) be a function. Define the oscillation of a function in \( B(x,r) \) by \( \omega_r f(x) = \sup\{|f(z) - f(y)| : y, z \in B(x,r)\} \). Define the oscillation of the function at the point, \( x \) by \( \omega f(x) = \lim_{r \to 0} \omega_r f(x) \). Show \( f \) is continuous at \( x \) if and only if \( \omega f(x) = 0 \). Then show the set of points where \( f \) is continuous is a \( G_\delta \) set (try \( U_n = \{x : \omega f(x) < \frac{1}{n}\} \)). Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? **Hint:** Suppose \( D \) is any countable set, \( D = \{d_i\}_{i=1}^\infty \), and define the function, \( f_n(x) \) to equal zero for every \( x \notin \{d_1, \ldots, d_n\} \) and \( 2^{-n} \) for \( x \) in this finite set. Then consider \( g(x) \equiv \sum_{n=1}^\infty f_n(x) \). Show that this series converges uniformly.

3. Let \( f \in C([0,1]) \) and suppose \( f'(x) \) exists. Show there exists a constant, \( K \), such that \( |f(x) - f(y)| \leq K|x - y| \) for all \( y \in [0,1] \). Let \( U_n = \{f \in C([0,1]) \text{ such that for each } x \in [0,1] \text{ there exists } y \in [0,1] \text{ such that } |f(x) - f(y)| > n|x - y|\} \). Show that \( U_n \) is open and dense in \( C([0,1]) \) where for \( f \in C([0,1]) \),

\[
||f|| \equiv \sup \{|f(x) : x \in [0,1]|.
\]

Show that \( \cap_n U_n \) is a dense \( G_\delta \) set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.

4. ↑ Suppose \( f(x) = \sum_{k=1}^\infty u_k(x) \) where the convergence is uniform and each \( u_k \) is a polynomial. Is it reasonable to conclude that \( f'(x) = \sum_{k=1}^\infty u'_k(x) \)? The answer is no. Use Problem 9 and the Weierstrass approximation theorem do show this.

5. Let \( X \) be a normed linear space. We say \( A \subseteq X \) is “weakly bounded” if for each \( x^* \in X' \), \( \sup\{|x^*(x)| : x \in A\} < \infty \), while \( A \) is bounded if \( \sup\{|||x|| : x \in A\} < \infty \). Show \( A \) is weakly bounded if and only if it is bounded.

6. Let \( X \) and \( Y \) be two Banach spaces. Define the norm

\[
|||(x,y)||| \equiv ||x||_X + ||y||_Y.
\]

Show this is a norm on \( X \times Y \) which is equivalent to the norm given in the chapter for \( X \times Y \). Can you do the same for the norm defined for \( p > 1 \) by

\[
|||(x,y)||| \equiv (||x||_X^p + ||y||_Y^p)^{1/p}.
\]

7. Let \( f \) be a \( 2\pi \) periodic locally integrable function on \( \mathbb{R} \). The Fourier series for \( f \) is given by

\[
\sum_{k=-\infty}^{\infty} a_k e^{ikx} \equiv \lim_{n \to \infty} \sum_{k=-n}^{n} a_k e^{ikx} \equiv \lim_{n \to \infty} \sum_{k=-n}^{n} a_k f(x)
\]
CHAPTER 15. BANACH SPACES

where

\[ a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) \, dx. \]

Show

\[ S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y) \, dy \]

where

\[ D_n(t) = \frac{\sin((n + \frac{1}{2})t)}{2\pi \sin(\frac{t}{2})}. \]

Verify that \( \int_{-\pi}^{\pi} D_n(t) \, dt = 1 \). Also show that if \( g \in L^1(\mathbb{R}) \), then

\[ \lim_{a \to \infty} \int_{\mathbb{R}} g(x) \sin(ax) \, dx = 0. \]

This last is called the Riemann Lebesgue lemma. **Hint:** For the last part, assume first that \( g \in C_c^\infty(\mathbb{R}) \) and integrate by parts. Then exploit density of the set of functions in \( L^1(\mathbb{R}) \).

8. ↑It turns out that the Fourier series sometimes converges to the function pointwise. Suppose \( f \) is 2\( \pi \) periodic and Holder continuous. That is \( |f(x) - f(y)| \leq K|x - y|^{\theta} \) where \( \theta \in (0, 1] \). Show that if \( f \) is like this, then the Fourier series converges to \( f \) at every point. Next modify your argument to show that if at every point, \( x \), \( |f(x+) - f(y)| \leq K|x - y|^{\theta} \) for \( y \) close enough to \( x \) and larger than \( x \) and \( |f(x-) - f(y)| \leq K|x - y|^{\theta} \) for every \( y \) close enough to \( x \) and smaller than \( x \), then \( S_n f(x) \to f(x+)+f(x-) \), the midpoint of the jump of the function. **Hint:** Use Problem 4.

9. ↑Let \( Y = \{ f \text{ such that } f \text{ is continuous, defined on } \mathbb{R}, \text{ and } 2\pi \text{ periodic} \}. \) Define \( \|f\|_{Y} = \sup\{|f(x)| : x \in [-\pi, \pi]\} \). Show that \( (Y, \|\|_{Y}) \) is a Banach space. Let \( x \in \mathbb{R} \) and define \( L_n (f) = S_n f(x) \). Show \( L_n \in Y' \) but \( \lim_{n \to \infty} \|L_n\| = \infty \). Show that for each \( x \in \mathbb{R} \), there exists a dense \( G_\delta \) subset of \( Y \) such that for \( f \) in this set, \( |S_n f(x)| \) is unbounded. Finally, show there is a dense \( G_\delta \) subset of \( Y \) having the property that \( |S_n f(x)| \) is unbounded on the rational numbers. **Hint:** To do the first part, let \( f(y) \) approximate \( \text{sgn}(D_n(x-y)) \). Here \( \text{sgn} r = 1 \) if \( r > 0 \), \(-1 \) if \( r < 0 \) and \( 0 \) if \( r = 0 \). This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense \( G_\delta \) sets must also be a dense \( G_\delta \) set.

10. Let \( \alpha \in (0, 1] \). Define, for \( X \) a compact subset of \( \mathbb{R}^p \),

\[ C^\alpha(X; \mathbb{R}^n) \equiv \{ f \in C(X; \mathbb{R}^n) : \rho_\alpha (f) + \|f\| \equiv \|f\|_\alpha < \infty \} \]

where

\[ \|f\| = \sup\{|f(x)| : x \in X\} \]
and
\[ \rho_\alpha (f) = \sup \{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in X, x \neq y \}. \]

Show that \((C^\alpha (X; \mathbb{R}^n), ||\cdot||_\alpha)\) is a complete normed linear space. This is called a Helder space. What would this space consist of if \(\alpha > 1\)?

11. ↑Now recall Problem 10 about the Helder spaces. Let \(X\) be the Helder functions which are periodic of period \(2\pi\). Define \(L_n f (x) = S_n f (x)\) where \(L_n : X \to Y\) for \(Y\) given in Problem 9. Show \(||L_n||\) is bounded independent of \(n\). Conclude that \(L_n f \to f\) in \(Y\) for all \(f \in X\). In other words, for the Helder continuous and \(2\pi\) periodic functions, the Fourier series converges to the function uniformly. **Hint:** \(L_n f (x)\) is given by

\[ L_n f (x) = \int_{-\pi}^{\pi} D_n (y) f (x - y) dy \]

where \(f (x - y) = f (x) + g(x, y)\) where \(|g(x, y)| \leq C |y|^\alpha\). Use the fact the Dirichlet kernel integrates to one to write

\[ \left| \int_{-\pi}^{\pi} D_n (y) f (x - y) dy \right| \leq \int_{-\pi}^{\pi} D_n (y) f (x) dy \]

\[ + C \left| \int_{-\pi}^{\pi} \sin \left( \left( n + \frac{1}{2} \right) y \right) (g(x, y) / \sin (y/2)) dy \right| \]

Show the functions, \(y \to g(x, y) / \sin (y/2)\) are bounded in \(L^1\) independent of \(x\) and get a uniform bound on \(||L_n||\). Now use a similar argument to show \(\{L_n f\}\) is equicontinuous in addition to being uniformly bounded. If \(L_n f\) fails to converge to \(f\) uniformly, then there exists \(\varepsilon > 0\) and a subsequence, \(n_k\) such that \(||L_{n_k} f - f||_\infty \geq \varepsilon\) where this is the norm in \(Y\) or equivalently the sup norm on \([-\pi, \pi]\). By the Arzela Ascoli theorem, there is a further subsequence, \(L_{n_k} f\) which converges uniformly on \([-\pi, \pi]\). But by Problem 9, \(L_{n_k} f (x) \to f (x)\).

12. Let \(X\) be a normed linear space and let \(M\) be a convex open set containing \(0\). Define

\[ \rho(x) = \inf \{ t > 0 : \frac{x}{t} \in M \}. \]

Show \(\rho\) is a gauge function defined on \(X\). This particular example is called a Minkowski functional. It is of fundamental importance in the study of locally convex topological vector spaces. A set, \(M\), is convex if \(\lambda x + (1 - \lambda) y \in M\) whenever \(\lambda \in [0, 1]\) and \(x, y \in M\).

13. ↑The Hahn Banach theorem can be used to establish separation theorems. Let \(M\) be an open convex set containing \(0\). Let \(x \notin M\). Show there exists \(x^* \in X'\) such that \(\text{Re } x^*(x) \geq 1 > \text{Re } x^*(y)\) for all \(y \in M\). **Hint:** If \(y \in M, \rho(y) < 1\).
Show this. If \( x \notin M \), \( \rho(x) \geq 1 \). Try \( f(\alpha x) = \alpha \rho(x) \) for \( \alpha \in \mathbb{R} \). Then extend \( f \) to the whole space using the Hahn Banach theorem and call the result \( F \), show \( F \) is continuous, then fix it so \( F \) is the real part of \( x^* \in X' \).

14. A Banach space is said to be strictly convex if whenever \( ||x|| = ||y|| \) and \( x \neq y \), then
\[
\left\| \frac{x + y}{2} \right\| < ||x||.
\]

\( F : X \to X' \) is said to be a duality map if it satisfies the following: a.) \( ||F(x)|| = ||x|| \). b.) \( F(x)(x) = ||x||^2 \). Show that if \( X' \) is strictly convex, then such a duality map exists. The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space. 

**Hint:** For an arbitrary Banach space, let
\[
F(x) \equiv \left\{ x^* : ||x^*|| \leq ||x|| \text{ and } x^*(x) = ||x||^2 \right\}
\]
Show \( F(x) \neq \emptyset \) by using the Hahn Banach theorem on \( f(\alpha x) = \alpha ||x||^2 \). Next show \( F(x) \) is closed and convex. Finally show that you can replace the inequality in the definition of \( F(x) \) with an equal sign. Now use strict convexity to show there is only one element in \( F(x) \).

15. Prove the following theorem which is an improved version of the open mapping theorem. Let \( X \) and \( Y \) be Banach spaces and let \( A \in L(X,Y) \). Then the following are equivalent.
\[
AX = Y,
\]
\( A \) is an open map.

There exists a constant \( M \) such that for every \( y \in Y \), there exists \( x \in X \) with \( y = Ax \) and
\[
||x|| \leq M ||y||.
\]
Note this gives the equivalence between \( A \) being onto and \( A \) being an open map. The open mapping theorem says that if \( A \) is onto then it is open.

16. Suppose \( D \subseteq X \) and \( D \) is dense in \( X \). Suppose \( L : D \to Y \) is linear and \( ||Lx|| \leq K||x|| \) for all \( x \in D \). Show there is a unique extension of \( L \), \( \tilde{L} \), defined on all of \( X \) with \( ||\tilde{L}x|| \leq K||x|| \) and \( \tilde{L} \) is linear. You do not get uniqueness when you use the Hahn Banach theorem. Therefore, in the situation of this problem, it is better to use this result.

17. \( \uparrow \) A Banach space is uniformly convex if whenever \( ||x_n||, ||y_n|| \leq 1 \) and \( ||x_n + y_n|| \to 2 \), it follows that \( ||x_n - y_n|| \to 0 \). Show uniform convexity implies strict convexity (See Problem 13). 

**Hint:** Suppose it is not strictly convex. Then there exist \( ||x|| \) and \( ||y|| \) both equal to 1 and \( ||x_n + y_n|| = 1 \) consider \( x_n \equiv x \) and \( y_n \equiv y \), and use the conditions for uniform convexity to get a contradiction. It can be shown that \( L^p \) is uniformly convex whenever \( \infty > p > 1 \). See Hewitt and Stromberg or Ray.
18. Show that a closed subspace of a reflexive Banach space is reflexive. \textbf{Hint:} The proof of this is an exercise in the use of the Hahn Banach theorem. Let $X$ be the closed subspace of the reflexive space $X$ and let $y^{**} \in Y''$. Then $i^{**}y^{**} \in X''$ and so $i^{**}y^{**} = Jx$ for some $x \in X$ because $X$ is reflexive. Now argue that $x \in Y$ as follows. If $x \not\in Y$, then there exists $x^*$ such that $x^*(Y) = 0$ but $x^*(x) \neq 0$. Thus, $i^*x^* = 0$. Use this to get a contradiction. When you know that $x = y \in Y$, the Hahn Banach theorem implies $i^*$ is onto $Y'$ and for all $x^* \in X'$,

$$y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = Jx(x^*) = x^*(iy) = i^*x^*(y).$$

19. We say that $x_n$ converges weakly to $x$ if for every $x^* \in X'$, $x^*(x_n) \to x^*(x)$. $x_n \rightharpoonup x$ denotes weak convergence. Show that if $||x_n - x|| \to 0$, then $x_n \to x$.

20. ↑ Show that if $X$ is uniformly convex, then if $x_n \to x$ and $||x_n|| \to ||x||$, it follows $||x_n - x|| \to 0$. \textbf{Hint:} Use Lemma 15.2.9 to obtain $f \in X'$ with $||f|| = 1$ and $f(x) = ||x||$. See Problem 17 for the definition of uniform convexity. Now by the weak convergence, you can argue that if $x \neq 0$, $f(x_n/||x_n||) \to f(x/||x||)$. You also might try to show this in the special case where $||x_n|| = ||x|| = 1$.

21. Suppose $L \in \mathcal{L}(X,Y)$ and $M \in \mathcal{L}(Y,Z)$. Show $ML \in \mathcal{L}(X,Z)$ and that $(ML)^* = L^*M^*$.

22. Let $X$ and $Y$ be Banach spaces and suppose $f \in \mathcal{L}(X,Y)$ is compact. Recall this means that if $B$ is a bounded set in $X$, then $f(B)$ has compact closure in $Y$. Show that $f^*$ is also a compact map. \textbf{Hint:} Take a bounded subset of $Y'$, $S$. You need to show $f^*(S)$ is totally bounded. You might consider using the Ascoli Arzela theorem on the functions of $S$ applied to $f(B)$ where $B$ is the closed unit ball in $X$. 


Chapter 16

Locally Convex Topological Vector Spaces

16.1 Fundamental Considerations

The right context to consider certain topics like separation theorems is in locally convex topological vector spaces, a generalization of normed linear spaces. Let $X$ be a vector space and let $\Psi$ be a collection of functions defined on $X$ such that if $\rho \in \Psi$,

\[
\rho(x + y) \leq \rho(x) + \rho(y),
\]

\[
\rho(ax) = |a| \rho(x) \text{ if } a \in \mathbb{F},
\]

\[
\rho(x) \geq 0,
\]

where $\mathbb{F}$ denotes the field of scalars, either $\mathbb{R}$ or $\mathbb{C}$, assumed to be $\mathbb{C}$ unless otherwise specified. These functions are called seminorms because it is not necessarily true that $x = 0$ when $\rho(x) = 0$. A basis for a topology, $B$, is defined as follows.

**Definition 16.1.1** For $A$ a finite subset of $\Psi$ and $r > 0$,

\[
B_A(x, r) \equiv \{ y \in X : \rho(x - y) < r \text{ for all } \rho \in A \}.
\]

Then

\[
B \equiv \{ B_A(x, r) : x \in X, r > 0, \text{ and } A \subseteq \Psi, A \text{ finite} \}.
\]

That this really is a basis is the content of the next theorem.

**Theorem 16.1.2** $B$ is the basis for a topology.

**Proof:** I need to show that if $B_A(x, r_1)$ and $B_B(y, r_2)$ are two elements of $B$ and if $z \in B_A(x, r_1) \cap B_B(y, r_2)$, then there exists $U \in B$ such that

\[
z \in U \subseteq B_A(x, r_1) \cap B_B(y, r_2).
\]
Let
\[ r = \min \left( \min \{ (r_1 - \rho(z - x)) : \rho \in A \}, \min \{ (r_2 - \rho(z - y)) : \rho \in B \} \right) \]
and consider \( B_{A \cup B}(z, r) \). If \( w \) belongs to this set, then for \( \rho \in A \),
\[ \rho(w - z) < r_1 - \rho(z - x). \]
Hence
\[ \rho(w - x) \leq \rho(w - z) + \rho(z - x) < r_1 \]
for each \( \rho \in A \) and so \( B_{A \cup B}(z, r) \subseteq B_A(x, r_1) \). Similarly, \( B_{A \cup B}(z, r) \subseteq B_B(y, r_2) \).
This proves the theorem.

Let \( \tau \) be the topology consisting of unions of all subsets of \( \mathcal{B} \). Then \((X, \tau)\) is a locally convex topological vector space.

**Theorem 16.1.3** The vector space operations of addition and scalar multiplication are continuous. More precisely,
\[ + : X \times X \to X, \cdot : F \times X \to X \]
are continuous.

**Proof:** It suffices to show \(+^{-1}(B)\) is open in \( X \times X \) and \( \cdot^{-1}(B) \) is open in \( F \times X \) if \( B \) is of the form
\[ B = \{ y \in X : \rho(y - x) < r \} \]
because finite intersections of such sets form the basis \( \mathcal{B} \). (This collection of sets is a subbasis.) Suppose \( u + v \in B \) where \( B \) is described above. Then
\[ \rho(u + v - x) < \lambda r \]
for some \( \lambda < 1 \). Consider
\[ B_{\rho}(u, \delta) \times B_{\rho}(v, \delta). \]
If \((u_1, v_1)\) is in this set, then
\[ \rho(u_1 + v_1 - x) \leq \rho(u + v - x) + \rho(u_1 - u) + \rho(v_1 - v) < \lambda r + 2\delta. \]
Let \( \delta \) be positive but small enough that
\[ 2\delta + \lambda r < r. \]
Thus this choice of \( \delta \) shows that \(+^{-1}(B)\) is open and this shows \( + \) is continuous.

Now suppose \( \alpha z \in B \). Then
\[ \rho(\alpha z - x) < \lambda r < r \]
and consider \( B_{A \cup B}(z, r) \). If \( w \) belongs to this set, then for \( \rho \in A \),
\[ \rho(w - z) < r_1 - \rho(z - x). \]
Hence
\[ \rho(w - x) \leq \rho(w - z) + \rho(z - x) < r_1 \]
for each \( \rho \in A \) and so \( B_{A \cup B}(z, r) \subseteq B_A(x, r_1) \). Similarly, \( B_{A \cup B}(z, r) \subseteq B_B(y, r_2) \).
This proves the theorem.
for some \( \lambda \in (0, 1) \). Let \( \delta > 0 \) be small enough that \( \delta < 1 \) and also
\[
\lambda r + \delta (\rho(z) + 1) + \delta |\alpha| < r.
\]
Then consider \((\beta, w) \in B(\alpha, \delta) \times B_\rho(z, \delta)\).
\[
\rho(\beta w - x) - \rho(\alpha z - x) \leq \rho(\beta w - \alpha z) \leq |
\beta - \alpha| \rho(w) + \rho(w - z)|\alpha| \\
\leq |
\beta - \alpha| (\rho(z) + 1) + \rho(w - z)|\alpha| \leq \delta (\rho(z) + 1) + \delta |\alpha|.
\]
Hence
\[
\rho(\beta w - x) < \lambda r + \delta (\rho(z) + 1) + \delta |\alpha| < r
\]
and so
\[
B(\alpha, \delta) \times B_\rho(z, \delta) \subseteq B^{-1}(B).
\]

This proves the theorem.

**Theorem 16.1.4** Let \( x \) be given and let \( f_x(y) = x + y \). Then \( f_x \) is \( 1 \rightarrow 1 \), onto, and continuous. If \( \alpha \neq 0 \) and \( g_\alpha(x) = \alpha x \), then \( g_\alpha \) is also \( 1 \rightarrow 1 \) onto and continuous.

**Proof**: The assertions about \( 1 \rightarrow 1 \) and onto are obvious. It remains to show \( f_x \) and \( g_\alpha \) are continuous. Let \( B = B_\rho(z, r) \) and consider \( f_x^{-1}(B) \). Then it is easy to see that
\[
f_x^{-1}(B) = B_\rho(z - x, r)
\]
and so \( f_x \) is continuous. To see that \( g_\alpha \) is continuous, note that
\[
g_\alpha^{-1}(B) = B_\rho\left(\frac{z}{\alpha}, \frac{r}{|\alpha|}\right).
\]
This proves the theorem.

As in the case of a normed linear space, the vector space of continuous linear functionals, is denoted by \( X' \).

**Definition 16.1.5** Define, for \( A \) a finite subset of \( \Psi \),
\[
\rho_A(x) = \max\{\rho(x) : \rho \in A\}.
\]

The following theorem is the equivalent to the earlier theorems concerning continuous linear functionals on normed linear spaces.

**Theorem 16.1.6** The following are equivalent for \( f \), a linear function mapping \( X \) to \( \mathbb{F} \).

1. \( f \) is continuous at \( 0 \).
2. For some \( A \subseteq \Psi \), \( A \) finite,
\[
|f(x)| \leq C\rho_A(x)
\]
for all \( x \in X \) where the constant may depend on \( A \) but is independent of \( x \).
3. \( f \) is continuous at \( x \) for all \( x \).
Proof: Clearly \( 16.1.3 \) implies \( 16.1.1 \). Suppose \( 16.1.1 \). Then
\[
0 = f(0) \in B(0, 1) \subseteq F.
\]
Since \( f \) is continuous at 0, 0 \( \in f^{-1}(B(0, 1)) \) and there exists an open set \( V \in \tau \) such that
\[
0 \in V \subseteq f^{-1}(B(0, 1)).
\]
Then \( 0 \in B_A(0, r) \subseteq V \) for some \( r \) and some \( A \subseteq \Psi \), \( A \) finite. Hence
\[
|f(y)| < 1 \text{ if } \rho_A(y) < r.
\]
Since \( f \) is linear
\[
|f(x)| \leq \frac{2}{r} \rho_A(x).
\]
To see this, note that if \( x \neq 0 \), then
\[
\frac{rx}{2\rho_A(x)} \in B_A(0, r)
\]
and so
\[
\frac{|f(rx)|}{2\rho_A(x)} \leq 1
\]
which shows that \( 16.1.1 \) implies \( 16.1.2 \).

Now suppose \( 16.1.2 \) and suppose \( f(x) \in V \), an open set in \( F \). Then
\[
f(x) \in B(f(x), r) \subseteq V
\]
for some \( r > 0 \). Suppose \( \rho_A(x - y) < r(C_A + 1)^{-1} \). Then
\[
|f(x) - f(y)| = |f(x - y)| \leq C_A \rho_A(y - x) < r.
\]
Hence
\[
f\left(B_A\left(x, r(C_A + 1)^{-1}\right)\right) \subseteq B(f(x), r) \subseteq V.
\]
Thus \( f \) is continuous at \( x \). This proves the theorem.

What are some examples of locally convex topological vector spaces? It is obvious that any normed linear space is such an example. More generally, here is a theorem which shows how to make any vector space into a locally convex topological vector space.

**Theorem 16.1.7** Let \( X \) be a vector space and let \( Y \) be a vector space of linear functionals defined on \( X \). For each \( y \in Y \), define
\[
\rho_y(x) \equiv |y(x)|.
\]
Then the collection of seminorms \( \{\rho_y\}_{y \in Y} \) defined on \( X \) makes \( X \) into a locally convex topological vector space and \( Y = X' \).
16.2. SEPARATION THEOREMS

Proof: Clearly \( \{\rho_y\}_{y \in Y} \) is a collection of seminorms defined on \( X \); so, \( X \) supplied with the topology induced by this collection of seminorms is a locally convex topological vector space. Is \( Y = X' \)?

Let \( y \in Y \), let \( U \subseteq \mathbb{F} \) be open and let \( x \in y^{-1}(U) \). Then \( B(y(x), r) \subseteq U \) for some \( r > 0 \). Letting \( A = \{y\} \), it is easy to see from the definition that \( B_A(x, r) \subseteq y^{-1}(U) \) and so \( y^{-1}(U) \) is an open set as desired. Thus, \( Y \subseteq X' \).

Now suppose \( z \in X' \). Then by 16.1.2, there exists a finite subset of \( Y \), \( A \equiv \{y_1, \ldots, y_n\} \), such that

\[ |z(x)| \leq C \rho_A(x). \]

Let

\[ \pi(x) \equiv (y_1(x), \ldots, y_n(x)) \]

and let \( f \) be a linear map from \( \pi(X) \) to \( \mathbb{F} \) defined by

\[ f(\pi x) \equiv z(x). \]

(This is well defined because if \( \pi(x) = \pi(x_1) \), then \( y_i(x) = y_i(x_1) \) for \( i = 1, \ldots, n \) and so

\[ \rho_A(x - x_1) = 0. \]

Thus,

\[ |z(x_1) - z(x)| = |z(x_1 - x)| \leq C \rho_A(x - x_1) = 0. \]

Extend \( f \) to all of \( \mathbb{F}^n \) and denote the resulting linear map by \( F \). Then there exists a vector

\[ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n \]

with \( \alpha_i = F(e_i) \) such that

\[ F(\beta) = \alpha \cdot \beta. \]

Hence for each \( x \in X \),

\[ z(x) = f(\pi x) = F(\pi x) = \sum_{i=1}^{n} \alpha_i y_i(x) \]

and so

\[ z = \sum_{i=1}^{n} \alpha_i y_i \in Y. \]

This proves the theorem.

16.2 Separation Theorems

It will always be assumed that \( X \) is a locally convex topological vector space. A set, \( K \), is said to be convex if whenever \( x, y \in K \),

\[ \lambda x + (1 - \lambda) y \in K \]

for all \( \lambda \in [0, 1] \).
CHAPTER 16. LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

Definition 16.2.1 Let $U$ be an open convex set containing $0$ and define

$$m(x) \equiv \inf\{t > 0 : x/t \in U\}.$$ 

This is called a Minkowski functional.

Proposition 16.2.2 Let $X$ be a locally convex topological vector space. Then $m$ is defined on $X$ and satisfies

$$m(x + y) \leq m(x) + m(y) \quad (16.2.4)$$

$$m(\lambda x) = \lambda m(x) \quad \text{if } \lambda > 0. \quad (16.2.5)$$

Thus, $m$ is a gauge function on $X$.

Proof: Let $x \in X$ be arbitrary. There exists $A \subseteq \Psi$ such that

$$0 \in B_A(0, r) \subseteq U.$$ 

Then

$$\frac{rx}{2\rho_A(x)} \in B_A(0, r) \subseteq U$$

which implies

$$\frac{2\rho_A(x)}{r} \geq m(x). \quad (16.2.6)$$

Thus $m(x)$ is defined on $X$.

Let $x/t \in U$, $y/s \in U$. Then since $U$ is convex,

$$\frac{x+y}{t+s} = \left(\frac{t}{t+s}\right)\left(\frac{x}{t}\right) + \left(\frac{s}{t+s}\right)\left(\frac{y}{s}\right) \in U.$$ 

It follows that

$$m(x + y) \leq t + s.$$ 

Choosing $s, t$ such that $t - \varepsilon < m(x)$ and $s - \varepsilon < m(y)$,

$$m(x + y) \leq m(x) + m(y) + 2\varepsilon.$$ 

Since $\varepsilon$ is arbitrary, this shows $m(x + y) \leq t + s$. It remains to show $m(\lambda x) \leq \lambda m(x)$ for all $\lambda > 0$. Hence

$$m(x) = m(\lambda^{-1} \lambda x) \leq \lambda^{-1} m(\lambda x) \leq \lambda^{-1} \lambda m(x) = m(x)$$

and so

$$\lambda m(x) = m(\lambda x).$$

This proves the proposition.
Lemma 16.2.3 Let $U$ be an open convex set containing $0$ and let $q \notin U$. Then there exists $f \in X'$ such that

$$\text{Re } f(q) > \text{Re } f(x)$$

for all $x \in U$.

**Proof:** Let $m$ be the Minkowski functional just defined and let

$$F(cq) = cm(q)$$

for $c \in \mathbb{R}$. If $c > 0$ then

$$F(cq) = m(cq)$$

while if $c \leq 0$,

$$F(cq) = cm(q) \leq 0 \leq m(cq).$$

By the Hahn Banach theorem, $F$ has an extension, $g$, defined on all of $X$ satisfying

$$g(x + y) = g(x) + g(y), \; g(cx) = cg(x)$$

for all $c \in \mathbb{R}$, and

$$g(x) \leq m(x).$$

Thus, $g(-x) \leq m(-x)$ and so

$$-m(-x) \leq g(x) \leq m(x).$$

It follows as in Lemma 16.2.6 that for some $A \subseteq \Psi$, $A$ finite, and $r > 0$,

$$|g(x)| \leq m(x) + m(-x)$$

$$\leq \frac{2}{r} \rho_A(x) + \frac{2}{r} \rho_A(-x) = \frac{4}{r} \rho_A(x)$$

because

$$\rho_A(-x) = |1| \rho_A(x) = \rho_A(x).$$

Hence $g$ is continuous by Theorem 16.1.6. Now define

$$f(x) \equiv g(x) - ig(ix).$$

Thus $f$ is linear and continuous so $f \in X'$ and $\text{Re } f(x) = g(x)$. But for $x \in U$, Theorem 16.1.6 implies that $x/t \in U$ for some $t < 1$ and so $m(x) < 1$. Since $U$ is convex and $0 \in U$, it follows $q/t \notin U$ if $t < 1$ because if it were,

$$q = t \left( \frac{q}{t} \right) + (1 - t) 0 \in U.$$ 

Therefore, $m(q) \geq 1$ and for $x \in U$,

$$\text{Re } f(x) = g(x) \leq m(x) < 1 \leq m(q) = g(q) = \text{Re } f(q)$$

and this proves the lemma.
**Corollary 16.2.4** Let $U$ be an open nonempty convex set and let $q \notin U$. Then there exists $f \in X'$ such that
\[ f(q) > f(x) \]
for all $x \in U$.

**Proof:** Let $u_0 \in U$ and consider $\hat{U} \equiv U - u_0$. Then $0 \in \hat{U}$ and $q - u_0 \notin \hat{U}$. By separation theorems, Lemma [16.2.3] there exists $f \in X'$ such that
\[ f(q - u_0) > f(x - u_0) \]
for all $x \in U$. Thus $f(q) > f(x)$ for all $x \in U$. ■

**Theorem 16.2.5** Let $K$ be closed and convex in a locally convex topological vector space and let $p \notin K$. Then there exists a real number $c$, and $f \in X'$ such that
\[ \text{Re } f(p) > c > \text{Re } f(k) \]
for all $k \in K$.

**Proof:** Since $K$ is closed, and $p \notin K$, there exists a finite subset of $\Psi, A$, and a positive $r > 0$ such that
\[ K \cap B_A(p, 2r) = \emptyset. \]
Pick $k_0 \in K$ and let
\[ U = K + B_A(0, r) - k_0, \quad q = p - k_0. \]

It follows that $U$ is an open convex set containing 0 and $q \notin U$. Therefore, by Lemma [16.2.3], there exists $f \in X'$ such that
\[ \text{Re } f(p - k_0) = \text{Re } f(q) > \text{Re } f(k + e - k_0) \quad (16.2.7) \]
for all $k \in K$ and $e \in B_A(0, r)$. If $\text{Re } f(e) = 0$ for all $e \in B_A(0, r)$, then $\text{Re } f = 0$ and (16.2.7) could not hold. Therefore, $\text{Re } f(e) > 0$ for some $e \in B_A(0, r)$ and so,
\[ \text{Re } f(p) > \text{Re } f(k) + \text{Re } f(e) \]
for all $k \in K$. Let $c_1 \equiv \sup \{ \text{Re } f(k) : k \in K \}$. Then for all $k \in K$,
\[ \text{Re } f(p) \geq c_1 + \text{Re } f(e) > c_1 + \frac{\text{Re } f(e)}{2} > \text{Re } f(k). \]

Let $c = c_1 + \frac{\text{Re } f(e)}{2}$. ■
Corollary 16.2.6 In the situation of the above theorem, there exist real numbers $c, d$ such that $\text{Re} f(p) > d > c > \text{Re} f(k)$ for all $k \in K$.

Proof: From the theorem, there exists $\hat{c}$ such that $\text{Re} f(p) > \hat{c} > \text{Re} f(k)$ for all $k \in K$. Thus $\text{Re} f(p) > \hat{c} > \sup_{k \in K} \text{Re} f(k)$. Now choose $d, c$ such that $f(p) > d > c > \hat{c} > \sup_{k \in K} \text{Re} f(k)$. Note that if the field of scalars comes from $\mathbb{R}$ rather than $\mathbb{C}$ there is no essential change to the above conclusions. Just eliminate all references to the real part.

16.2.1 Convex Functionals

As an important application, this theorem gives the basis for proving something about lower semicontinuity of functionals.

Definition 16.2.7 Let $X$ be a Banach space and let $\phi : X \to (0, \infty]$ be convex and lower semicontinuous. This means whenever $x \in X$ and $\lim_{n \to \infty} x_n = x$, $\phi(x) \leq \liminf_{n \to \infty} \phi(x_n)$.

Also assume $\phi$ is not identically equal to $\infty$.

Lemma 16.2.8 Let $X,Y$ be two Banach spaces. Then letting $||(x,y)|| \equiv \max(||x||_X,||y||_Y)$, it follows $X \times Y$ is a Banach space and $\phi \in (X \times Y)'$ if and only if there exist $x^* \in X'$ and $y^* \in Y'$ such that $\phi((x,y)) = x^*(x) + y^*(y)$.

The topology coming from this norm is called the strong topology.

Proof: Most of these conclusions are obvious. In particular it is clear $X \times Y$ is a Banach space with the given norm. Let $\phi \in (X \times Y)'$. Also let $\pi_X(x,y) \equiv (x,0)$ and $\pi_Y(x,y) \equiv (0,y)$. Then each of $\pi_X$ and $\pi_Y$ is continuous and $\phi((x,y)) = \phi((x,0)) + \phi((0,y))$.

Thus $\phi \circ \pi_X$ and $\phi \circ \pi_Y$ are both continuous and their sum equals $\phi$. Let $x^*(x) \equiv \phi \circ \pi_X(x,0)$ and let $y^* \equiv \phi \circ \pi_Y(x,0)$. Then it is clear both $x^*$ and $y^*$ are continuous and linear defined on $X$ and $Y$ respectively. Also, if $(x^*, y^*) \in X' \times Y'$, then if $\phi((x,y)) \equiv x^*(x) + y^*(y)$, it follows $\phi \in (X \times Y)'$. This proves the lemma.

Lemma 16.2.9 Let $\phi$ be a functional as described in Definition 16.2.7. Then $\phi$ is lower semicontinuous if and only if the epigraph of $\phi$ is closed in $X \times \mathbb{R}$ with the strong topology. Here the epigraph is defined as $\text{epi}(\phi) \equiv \{(x,y) : y \geq \phi(x)\}$.

In this case the functional is called strongly lower semicontinuous.
CHAPTER 16. LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

Proof: First suppose epi (\( \phi \)) is closed and suppose \( x_n \to x \). Let \( l < \phi (x) \). Then \( (x, l) \notin \text{epi} (\phi) \) and so there exists \( \delta > 0 \) such that if \( |x - y| < \delta \) and \( |\alpha - l| < \delta \), then \( \alpha < \phi (y) \). This implies that if \( |x - y| < \delta \) and \( \alpha < l + \delta \), then the above holds. Therefore, \((x_n, \phi (x_n))\), being in epi (\( \phi \)) cannot satisfy both conditions,

\[
|x_n - x| < \delta, \phi (x_n) < l + \delta.
\]

However, for all \( n \) large enough, the first condition is satisfied. Consequently, for all \( n \) large enough, \( \phi (x_n) \geq l + \delta \). Thus

\[
\lim \inf_{n \to \infty} \phi (x_n) \geq l
\]

and since \( l < \phi (x) \) is arbitrary, it follows

\[
\lim \inf_{n \to \infty} \phi (x_n) \geq \phi (x).
\]

Next suppose the condition about the \( \lim \inf \). If epi (\( \phi \)) is not closed, then there exists \((x, l)\) such that \((x, l)\) is a limit point of points of epi (\( \phi \)). Thus, there exists \((x_n, l_n)\) \( \in \) epi (\( \phi \)) such that \((x_n, l_n) \to (x, l)\) and so

\[
l = \lim \inf_{n \to \infty} l_n \geq \lim \inf_{n \to \infty} \phi (x_n) \geq \phi (x),
\]

contradicting \((x, l)\) \( \notin \) epi (\( \phi \)). This proves the lemma.

Definition 16.2.10 Let \( \phi \) be convex and defined on \( X \), a Banach space. Then \( \phi \) is said to be weakly lower semicontinuous if epi (\( \phi \)) is closed in \( X \times \mathbb{R} \) where a basis for the topology of \( X \times \mathbb{R} \) consists of sets of the form \( U \times (a, b) \) for \( U \) a weakly open set in \( X \).

Theorem 16.2.11 Let \( \phi \) be a lower semicontinuous convex functional as described in Definition 16.2.7 and let \( X \) be a real Banach space. Then \( \phi \) is also weakly lower semicontinuous.

Proof: By Lemma 16.2.9, epi (\( \phi \)) is closed in \( X \times \mathbb{R} \) with the strong topology as well as being convex. Letting \((z, l) \notin \text{epi} (\phi)\), it follows from Theorem 16.2.8 and Lemma 16.2.9 there exists \((x^*, \alpha) \in X' \times \mathbb{R} \) such that for some \( c \)

\[
x^* (z) + \alpha l > c > x^* (x) + \alpha \beta
\]

whenever \( \beta \geq \phi (x) \). Consider \( B_{\{(x^*, \alpha)\}} ((z, l), r) \) where \( r \) is chosen so small that if \((y, \gamma) \in B_{\{(x^*, \alpha)\}} ((z, l), r)\), then

\[
x^* (y) + \alpha \gamma > c.
\]

This shows that the complement of epi (\( \phi \)) is weakly open and this proves the theorem.
Corollary 16.2.12 Let $\phi$ be a lower semicontinuous convex functional as described in Definition 16.2.7 and let $X$ be a real Banach space. Then if $x_n$ converges weakly to $x$, it follows that

\[ \phi(x) \leq \liminf_{n \to \infty} \phi(x_n). \]

Proof: Let $l < \phi(x)$ so that $(x, l) \notin \text{epi}(\phi)$. Then by Theorem 16.2.11 there exists $B \times (-\infty, l + \delta)$ such that $B$ is a weakly open set in $X$ containing $x$ and

\[ B \times (-\infty, l + \delta) \subseteq \text{epi}(\phi)^C. \]

Thus $(x_n, \phi(x_n)) \notin B \times (-\infty, l + \delta)$ for all $n$. However, $x_n \in B$ for all $n$ large enough. Therefore, for those values of $n$, it must be the case that $\phi(x_n) \notin (-\infty, l + \delta)$ and so

\[ \liminf_{n \to \infty} \phi(x_n) \geq l + \delta \geq l \]

which shows, since $l < \phi(x)$ is arbitrary that

\[ \liminf_{n \to \infty} \phi(x_n) \geq \phi(x). \]

This proves the corollary.

The following is a convenient fact which follows from the above.

Proposition 16.2.13 Let $A$ be a linear operator which maps a real normed linear space $(X, \|\cdot\|_X)$ to a real normed linear space $(Y, \|\cdot\|_Y)$. Then $x_n \to x$ strongly implies $Ax_n \to Ax$ if and only if whenever $x_n \to x$ weakly, it follows that $Ax_n \to Ax$ weakly.

Proof: $\Rightarrow$ Define $\phi(x) \equiv f(Ax)$ where $f \in Y'$. Then $\phi$ is convex and continuous. Therefore, if $x_n \to x$ weakly, then

\[ \phi(x) = f(Ax) \leq \liminf_{n \to \infty} f(Ax_n) = \liminf_{n \to \infty} \phi(x_n) \]

Then substituting $-A$ for $A$,

\[ -f(Ax) \leq \liminf_{n \to \infty} f(-Ax_n), \quad f(Ax) \geq \limsup_{n \to \infty} f(Ax_n) \]

which shows that for each $f \in Y'$,

\[ \limsup_{n \to \infty} f(Ax_n) \leq f(Ax) \leq \liminf_{n \to \infty} f(Ax_n) \]

and so the second condition holds.

$\Leftarrow$ By the second condition, $x \to f(Ax)$ satisfies the condition that if $x_n \to x$ weakly, then

\[ f(Ax) = \lim_{n \to \infty} f(Ax_n) \]

If $A$ is not bounded, then there exists $x_n, \|x_n\| \leq 1$ but $\|Ax_n\| \geq n$. It follows that $x_n/\sqrt{n} \to 0$ and so $A\left(\frac{x_n}{\sqrt{n}}\right) \to 0$ weakly. Therefore, $A\left(\frac{x_n}{\sqrt{n}}\right)$ is bounded contrary to the construction which says that $\left\|A\left(\frac{x_n}{\sqrt{n}}\right)\right\| \geq \sqrt{n}$. Since $A$ is bounded, it must be continuous.
16.2.2 More Separation Theorems

There are other separation theorems which can be proved in a similar way. The next theorem considers the separation of an open convex set from a convex set.

**Theorem 16.2.14** Let $A$ and $B$ be disjoint, convex and nonempty sets with $B$ open. Then there exists $f \in X'$ such that

$$\text{Re } f(a) < \text{Re } f(b)$$

for all $a \in A$ and $b \in B$.

**Proof:** Let $b_0 \in B, a_0 \in A$. Then the set

$$B - A + a_0 - b_0$$

is open, convex, contains 0, and does not contain $a_0 - b_0$. By Lemma 16.2.3 there exists $f \in X'$ such that

$$\text{Re } f(a_0 - b_0) > \text{Re } f(b - a + a_0 - b_0)$$

for all $a \in A$ and $b \in B$. Therefore, for all $a \in A, b \in B$,

$$\text{Re } f(b) > \text{Re } f(a).$$

Before giving another separation theorem, here is a lemma.

**Lemma 16.2.15** If $B$ is convex, then $\text{int } (B) \equiv \text{union of all open sets contained in } B$ is convex. Also, if $\text{int } (B) \neq \emptyset$, then $B \subseteq \text{int } (B)$.

**Proof:** Suppose $x, y \in \text{int } (B)$. Then there exists $r > 0$ and a finite set $A \subseteq \Psi$ such that

$$B_A (x, r), B_A (y, r) \subseteq B.$$ Let

$$V \equiv \bigcup_{\lambda \in [0, 1)} \lambda B_A (x, r) + (1 - \lambda) B_A (y, r).$$

Then $V$ is open, $V \subseteq B$, and if $\lambda \in [0, 1]$, then

$$\lambda x + (1 - \lambda) y \in V \subseteq B.$$ Therefore, $\text{int } (B)$ is convex as claimed.

Now let $y \in B$ and $x \in \text{int } (B)$. Let

$$x \in B_A (x, r) \subseteq \text{int } (B)$$

and let $x_\lambda \equiv (1 - \lambda)x + \lambda y$. Define the open cone,

$$C \equiv \bigcup_{\lambda \in [0, 1)} B_A (x_\lambda, (1 - \lambda) r).$$

Thus $C$ is represented in the following picture.
I claim \( C \subseteq B \) as suggested in the picture. To see this, let

\[
z \in B_A(x_\lambda, (1 - \lambda) r), \lambda \in (0, 1).
\]

Then

\[
\rho_A(z - x_\lambda) < (1 - \lambda) r
\]

and so

\[
\rho_A \left( \frac{z}{1 - \lambda} - x - \frac{\lambda y}{1 - \lambda} \right) < r.
\]

Therefore,

\[
\frac{z}{1 - \lambda} - \frac{\lambda y}{1 - \lambda} \in B_A(x, r) \subseteq B.
\]

It follows

\[
(1 - \lambda) \left( \frac{z}{1 - \lambda} - \frac{\lambda y}{1 - \lambda} \right) + \lambda y = z \in B
\]

and so \( C \subseteq B \) as claimed. Now this shows \( x_\lambda \in \text{int}(B) \) and \( \lim_{\lambda \to 1} x_\lambda = y \). Thus, \( y \in \text{int}(B) \) and this proves the lemma.

Note this also shows that \( B = \text{int}(B) \).

**Corollary 16.2.16** Let \( A, B \) be convex, nonempty sets. Suppose \( \text{int}(B) \neq \emptyset \) and \( A \cap \text{int}(B) = \emptyset \). Then there exists \( f \in X' \), \( f \neq 0 \), such that for all \( a \in A \) and \( b \in B \),

\[
\Re f(b) \geq \Re f(a).
\]

**Proof:** By Theorem 16.2.14, there exists \( f \in X' \) such that for all \( b \in \text{int}(B) \), and \( a \in A \),

\[
\Re f(b) > \Re f(a).
\]

Thus, in particular, \( f \neq 0 \). By Lemma 16.2.15, if \( b \in B \) and \( a \in A \),

\[
\Re f(b) \geq \Re f(a).
\]

This proves the theorem.

**Lemma 16.2.17** If \( X \) is a topological Hausdorff space then compact implies closed.
Proof: Let $K$ be compact and suppose $K^C$ is not open. Then there exists $p \in K^C$ such that

$$V_p \cap K \neq \emptyset$$

for all open sets $V_p$ containing $p$. Let

$$\mathcal{C} = \{ (V_p)^C : V_p \text{ is an open set containing } p \}.$$ 

Then $\mathcal{C}$ is an open cover of $K$ because if $q \in K$, there exist disjoint open sets $V_p$ and $V_q$ containing $p$ and $q$ respectively. Thus $q \in (V_p)^C$. This is an example of an open cover of $K$ which has no finite subcover, contradicting the assumption that $K$ is compact. This proves the lemma.

Lemma 16.2.18 If $X$ is a locally convex topological vector space, and if every point is a closed set, then the seminorms and $X'$ separate the points. This means if $x \neq y$, then for some $\rho \in \Psi$,

$$\rho (x - y) \neq 0$$

and for some $f \in X'$,

$$f (x) \neq f (y).$$

In this case, $X$ is a Hausdorff space.

Proof: Let $x \neq y$. Then by Theorem 16.2.5 there exists $f \in X'$ such that $f (x) \neq f (y)$. Thus $X'$ separates the points. Since $f \in X'$, Theorem 16.1.6 implies

$$|f (z)| \leq C\rho_A (z)$$

for some $A$ a finite subset of $\Psi$. Thus

$$0 < |f (x - y)| \leq C\rho_A (x - y)$$

and so $\rho (x - y) \neq 0$ for some $\rho \in A \subseteq \Psi$. Now to show $X$ is Hausdorff, let

$$0 < r < \rho (x - y) 2^{-1}.$$ 

Then the two disjoint open sets containing $x$ and $y$ respectively are

$$B_{\rho} (x, r) \text{ and } B_{\rho} (y, r).$$

This proves the lemma.

16.3 The Weak And Weak* Topologies

The weak and weak * topologies are examples which make the underlying vector space into a topological vector space. This section gives a description of these topologies. Unless otherwise specified, $X$ is a locally convex topological vector space. For $G$ a finite subset of $X'$ define $\delta_G : X \to [0, \infty)$ by

$$\delta_G (x) = \max \{|f (x)| : f \in G\}.$$
16.3. THE WEAK AND WEAK* TOPOLOGIES

Lemma 16.3.1 The functions $\delta_G$ for $G$ a finite subset of $X'$ are seminorms and the sets
\[ B_G (x, r) \equiv \{ y \in X : \delta_G (x - y) < r \} \]
form a basis for a topology on $X$. Furthermore, $X$ with this topology is a locally convex topological vector space. If each point in $X$ is a closed set, then the same is true of $X$ with respect to this new topology.

Proof: It is obvious that the functions $\delta_G$ are seminorms and therefore the proof that the sets $B_G (x, r)$ form a basis for a topology is the same as in Theorem 16.1.2. To see every point is a closed set in this new topology, assuming this is true for $X$ with the original topology, use Lemma 16.2.18 to assert $X'$ separates the points.

Let $x \in X$ and let $y \neq x$. There exists $f \in X'$ such that $f(x) \neq f(y)$. Let $G = \{ f \}$ and consider
\[ B_G (y, |f(x - y)|/2) . \]
Then this open set does not contain $x$. Thus $\{ x \}^C$ is open and so $\{ x \}$ is closed. This proves the Lemma.

This topology for $X$ is called the weak topology for $X$. For $F$ a finite subset of $X$, define $\gamma_F : X' \to [0, \infty)$ by
\[ \gamma_F (f) = \max \{|f(x)| : x \in F\} . \]

Lemma 16.3.2 The functions $\gamma_F$ for $F$ a finite subset of $X$ are seminorms and the sets
\[ B_F (f, r) \equiv \{ g \in X' : \gamma_F (f - g) < r \} \]
form a basis for a topology on $X'$. Furthermore, $X'$ with this topology is a locally convex topological vector space having the property that every point is a closed set.

Proof: The proof is similar to that of Lemma 16.3.1 but there is a difference in the part where every point is shown to be a closed set. Let $f \in X'$ and let $g \neq f$. Thus there exists $x \in X$ such that $f(x) \neq g(x)$. Let $F = \{ x \}$. Then
\[ B_F (g, |(f - g)(x)|/2) \]
contains $g$ but not $f$. Thus $\{ f \}^C$ is open and so $\{ f \}$ is closed. ■

Note that it was not necessary to assume points in $X$ are closed sets to get this.

The topology for $X'$ just described is called the weak * topology. In terms of Theorem 16.2.8 the weak topology is obtained by letting $Y = X'$ in that theorem while the weak * topology is obtained by letting $Y = X$ with the understanding that $X$ is a vector space of linear functionals on $X'$ defined by
\[ x(x^*) \equiv x^* (x) . \]

By Theorem 16.2.8, there is a useful result which follows immediately.
Theorem 16.3.3 Let $K$ be closed and convex in a Banach space $X$. Then it is also weakly closed. Furthermore, if $p \not\in K$, there exists $f \in X'$ such that
\[
\Re f(p) > c > \Re f(k)
\] (16.3.8)
for all $k \in K$. If $K^*$ is closed and convex in the dual of a Banach space, $X'$, then it is also weak * closed.

Proof: By Theorem 16.2.5 there exists $f \in X'$ such that 16.3.8 holds. Therefore, letting $A = \{f\}$, it follows that for $r$ small enough, $B_A(p, r) \cap K = \emptyset$. Thus $K$ is weakly closed. This establishes the first part.

For the second part, the seminorms for the weak * topology are determined from $X$ and the continuous linear functionals are of the form $x^* \mapsto x^*(x)$ where $x \in X$. Thus if $p^* \not\in K^*$, it follows from Theorem 16.2.5 there exists $x \in X$ such that
\[
\Re p^*(x) > c > \Re k^*(x)
\]
for all $k^* \in K^*$. Therefore, letting $A = \{x\}$, $B_A(p^*, r) \cap K^* = \emptyset$ whenever $r$ is small enough and this shows $K^*$ is weak * closed.

16.4 Mean Ergodic Theorem

The following theorem is called the mean ergodic theorem.

Theorem 16.4.1 Let $(\Omega, S, \mu)$ be a finite measure space and let $T : \Omega \to \Omega$ satisfy $T^{-1}(E) \in S, T(E) \in S$ for all $E \in S$. Also suppose for all positive integers, $n$, that
\[
\mu (T^{-n}(E)) \leq K \mu(E).
\]
For $f \in L^p(\Omega)$, and $p > 1$, let
\[
T^* f \equiv f \circ T.
\] (16.4.9)
Then $T^* \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$, the continuous linear mappings form $L^p(\Omega)$ to itself with
\[
||T^{*n}|| \leq K^{1/p}.
\] (16.4.10)
Defining $A_n \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ by
\[
A_n \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^{*k},
\]
there exists $A \in \mathcal{L}(L^p(\Omega), L^p(\Omega))$ such that for all $f \in L^p(\Omega)$,
\[
A_n f \to Af \text{ weakly}
\] (16.4.11)
and $A$ is a projection, $A^2 = A$, onto the space of all $f \in L^p(\Omega)$ such that $T^* f = f$. (The invariant functions.) The norm of $A$ satisfies
\[
||A|| \leq K^{1/p}.
\] (16.4.12)
Proof: To begin with, it follows from simple considerations that
\[
\int |\mathcal{X}_A (T^n (\omega))|^p \, d\mu = \int |\mathcal{X}_{T^{-n}(A)} (\omega)|^p \, d\mu = \mu \left( T^{-n}(A) \right) \leq K \mu (A)
\]
Hence
\[
\| T^n (\mathcal{X}_A) \| \leq K^{1/p} \mu (A)^{1/p} = K^{1/p} \| \mathcal{X}_A \|_{L^p}
\]
Next suppose you have a simple function \( s(\omega) = \sum_{k=1}^n \mathcal{X}_{A_i}(\omega) c_i \) where we assume the \( A_i \) are disjoint. From the above,
\[
\int \left| \sum_{k=1}^n \mathcal{X}_{A_i}(T^n(\omega)) c_i \right|^p \, d\mu = \int \sum_{k=1}^n \mathcal{X}_{A_i}(T^n(\omega))^p |c_i|^p \, d\mu \leq \sum_{k=1}^n K \mu (A_i) |c_i|^p = K \int |s|^p \, d\mu
\]
and so
\[
\| T^n s \| \leq K^{1/p} \| s \|
\]
and so the density of the simple functions implies that \( \| T^n s \| \leq K^{1/p} \).
Next let
\[
M \equiv \left\{ g \in L^p (\Omega) : \| A_n g \|_p \to 0 \right\}
\]
It follows from 16.4.10 that \( M \) is a closed subspace of \( L^p (\Omega) \) containing \( (I - T^*) (L^p (\Omega)) \).
This is shown next.

Claim 1: \( M \) is a closed subspace which contains \( (I - T^*) (L^p (\Omega)) \).
First it is shown that this is true if \( m = 1 \) and then it will be observed that the same argument would work for any positive integer \( m \).
\[
A_n (f - T^* f) = \frac{1}{n} \sum_{k=0}^{n-1} T^k f - T^{k+1} f = \frac{1}{n} \sum_{k=0}^{n-1} T^k f - \frac{1}{n} \sum_{k=1}^{n} T^k f = \frac{1}{n} (f - T^n f)
\]
Hence
\[
\| A_n (f - T^* f) \|_p \leq \frac{1}{n} \left( \| f \|_p + \| T^n f \|_p \right) \leq \frac{1}{n} \left( \| f \|_p + K^{1/p} \| f \|_p \right)
\]
and this clearly converges to 0. In fact, the same argument shows that \( M \) contains \( (I - T^*) (L^p (\Omega)) \) for any \( m \). Now suppose \( g_n \in M \) and \( g_n \to g \). Does it follow that \( g \in M \) also? Note that \( T^m \) is clearly linear. Thus
\[
\| T^m g \| \leq \| T^m g - T^m g_n \| + \| T^m g_n \| \leq K^{1/p} \| g - g_n \| + \| T^m g_n \|
\]
Now pick \( n \) large enough that \( \| g_n - g \| < \varepsilon / (2K^{1/p}) \) so that
\[
\| T^m g \| \leq \frac{\varepsilon}{2} + \| T^m g_n \|
\]
Then for all $m$ large enough, the right side of the above is less than $\varepsilon$ and this shows that $g \in M$. Note that $M$ is also a subspace and so it is a closed subspace.

**Claim 2:** If $A_{n_k} f \to g$ weakly and $A_{m_k} f \to h$ weakly, then $g = h$.

It is first shown that if $\xi \in L^p'(\Omega)$ and $\int \xi gd\mu = 0$ for all $g \in M$, then $\int \xi (g - h) d\mu = 0$.

If $\xi \in L^p'(\Omega)$ is such that $\int \xi gd\mu = 0$ for all $g \in M$, then since $M \supseteq (I - T^{*n}) (L^p(\Omega))$, it follows that for all $k \in L^p(\Omega)$,

$$\int \xi kd\mu = \int (\xi T^{*n}k + \xi (I - T^{*n}) k) d\mu = \int \xi T^{*n}kd\mu$$

and so from the definition of $A_n$ as an average, for such $\xi$,

$$\int \xi kd\mu = \int \xi A_n kd\mu.$$  \hfill(16.4.13)

Since $A_{n_k} f \to g$ weakly and $A_{m_k} f \to h$ weakly. Then (16.4.13) shows that

$$\int \xi gd\mu = \lim_{k \to \infty} \int \xi A_{n_k} f d\mu = \int \xi f d\mu = \lim_{k \to \infty} \int \xi A_{m_k} f d\mu = \int \xi h d\mu.$$ \hfill(16.4.14)

Thus for these special $\xi$, it follows that

$$\int \xi (g - h) d\mu = 0.$$ \hfill(16.4.15)

Next observe that for each fixed $n$, if $n_k \to \infty$,

$$\lim_{k \to \infty} \|T^{*n} A_{n_k} f - A_{n_k} f\| = 0$$ \hfill(16.4.16)

this follows like the arguments given above in Claim 1. Note that if $L \in \mathcal{L}(X, X)$ and $x_n \to x$ weakly in $X$, then for $\phi \in X'$

$$\langle \phi, Lx_n \rangle = \langle L^* \phi, x_n \rangle \to \langle L^* \phi, x \rangle = \langle \phi, Lx \rangle$$

and so $Lx_n \to Lx$ weakly. Therefore, this simple observation along with the above strong convergence (16.4.16) implies

$$T^{*n} g = \text{weak lim}_{k \to \infty} T^{*n} A_{n_k} f = \text{weak lim}_{k \to \infty} A_{n_k} f = g.$$  

Similarly $T^{*n} h = h$ where $A_{m_k} f \to h$ weakly. It follows that $A_n (g - h) = g - h$ so if $g \neq h$, then $g - h \notin M$ because

$$A_n (g - h) \to g - h \neq 0.$$  

It follows that since $M$ is a closed subspace, there exists $\xi \in L^p'(\Omega)$ such that $\int \xi (g - h) d\mu \neq 0$ but $\int \xi kd\mu = 0$ for all $k \in M$, contradicting (16.4.15). This verifies Claim 2.
Now
\[
\|A_n f\|_p = \left( \int \left( \frac{1}{n} \sum_{k=0}^{n-1} f(T^k \omega) \right)^p d\mu \right)^{1/p} \leq \frac{1}{n} \sum_{k=0}^{n-1} \left( \int |f(T^k \omega)|^p d\mu \right)^{1/p}
\]
\[
= \frac{1}{n} \sum_{k=0}^{n-1} \|T^k f\|_p \leq \frac{1}{n} \sum_{k=0}^{n-1} K^{1/p} \|f\|_p = K^{1/p} \|f\|_p
\]
(16.4.17)

Hence, by the Eberlein Smulian theorem, Theorem 15.5.12, in case \( p > 1 \), there is a subsequence for which \( A_n f \) converges weakly in \( L^p(\Omega) \). From the above, it follows that the original sequence must converge. That is, \( A_n f \) converges weakly for each \( f \in L^p(\Omega) \). Let \( Af \) denote this weak limit. Then it is clear that \( A \) is linear because this is true for each \( A_n \). What of the claim about the estimate? From weak lower semicontinuity of the norm, Corollary 16.2.12,
\[
\|Af\|_p \leq \lim \inf_{n \to \infty} \|A_n f\| \leq K^{1/p} \|f\|_p
\]

16.5 The Tychonoff And Schauder Fixed Point Theorems

First we give a proof of the Schauder fixed point theorem which is an infinite dimensional generalization of the Brouwer fixed point theorem. This is a theorem which lives in Banach space. After this, we give a generalization to locally convex topological vector spaces where the theorem is sometimes called Tychonoff’s theorem. First here is an interesting example [52].

Exercise 16.5.1 Let \( B \) be the closed unit ball in a separable Hilbert space \( H \) which is infinite dimensional. Then there exists continuous \( f : B \to B \) which has no fixed point.

Let \( \{e_k\}_{k=1}^\infty \) be a complete orthonormal set in \( H \). Let \( L \in \mathcal{L}(H,H) \) be defined as follows. \( Le_k = e_{k+1} \) and then extend linearly. Then in particular,
\[
L \left( \sum_i x_i e_i \right) = \sum_i x_i e_{i+1}
\]
Then it is clear that \( L \) preserves norms and so it is linear and continuous. Note how this would not work at all if the Hilbert space were finite dimensional. Then define \( f(x) = \frac{1}{2} (1 - \|x\|_H) e_1 + Lx \). Then if \( \|x\| \leq 1 \),
\[
\|f(x)\| = \frac{1}{2} (1 - \|x\|)^2 + \|Lx\|^2 = \frac{1}{2} (1 - \|x\|)^2 + \|x\|^2 = \frac{1}{2} \|x\|^2 + \frac{1}{2} \leq 1
\]
and so \( f : B \to B \) yet has no fixed point because if it did, you would need to have
\[
x = \frac{1}{2} (1 - \|x\|_H) e_1 + Lx
\]
and so
\[ \|x\|^2 = \frac{1}{4}(1 - \|x\|)^2 + \|Lx\|^2 = \frac{1}{4}(1 - \|x\|)^2 + \|Lx\|^2 \]
\[ = \frac{1}{4} + \frac{5}{4}\|x\|^2 - \frac{1}{2}\|x\| \]
\[ \frac{1}{2}\|x\| = \frac{1}{4} + \frac{1}{4}\|x\|^2 \]
this requires \(\|x\| = 1\). But then you would need to have \(x = Lx\) which is not so because if \(x\) is in the closure of the span of \(\{e_i\}_{i=m}^\infty\), such that the first nonzero Fourier coefficient is the \(m^{th}\), then \(Lx\) is in the closure of the span of \(\{e_i\}_{i=m+1}^\infty\).

This shows you need something other than continuity if you want to get a fixed point. This also shows that there is a retraction of \(B\) onto \(\partial B\) in any infinite dimensional separable Hilbert space. You get it the usual way. Take the line from \(x\) to \(f(x)\) and the retraction will be the function which gives the point on \(\partial B\) which is obtained by extending this line till it hits the boundary of \(B\). Thus for Hilbert spaces, those which have \(\partial B\) a retraction of \(B\) are exactly those which are infinite dimensional. The above reference claims this retraction property holds for any infinite dimensional normed linear space. I think it is fairly clear to see from the above example that this is not a surprising assertion. Recall that one of the proofs of the Brouwer fixed point theorem used the non existence of such a retraction, obtained using integration theory, to prove the theorem.

We let \(K\) be a closed convex subset of \(X\) a Banach space and let \(f\) be continuous, \(f : K \rightarrow K\), and \(\overline{f(K)}\) is compact.

**Lemma 16.5.2** For each \(r > 0\) there exists a finite set of points
\[ \{y_1, \ldots, y_n\} \subseteq \overline{f(K)} \]
and continuous functions \(\psi_i\) defined on \(\overline{f(K)}\) such that for \(x \in \overline{f(K)}\),
\[ \sum_{i=1}^{n} \psi_i(x) = 1, \quad (16.5.18) \]
\[ \psi_i(x) = 0 \text{ if } x \notin B(y_i, r), \quad \psi_i(x) > 0 \text{ if } x \in B(y_i, r). \]

If
\[ f_r(x) \equiv \sum_{i=1}^{n} y_i \psi_i(f(x)), \quad (16.5.19) \]
then whenever \(x \in K\),
\[ \|f(x) - f_r(x)\| \leq r. \]

**Proof:** Using the compactness of \(\overline{f(K)}\), there exists
\[ \{y_1, \ldots, y_n\} \subseteq \overline{f(K)} \subseteq K \]
such that
\[ \{ B(y_i, r) \}_{i=1}^n \]
covers \( f(K) \). Let
\[ \phi_i(y) \equiv (r - \|y - y_i\|)^+ \]
Thus \( \phi_i(y) > 0 \) if \( y \in B(y_i, r) \) and \( \phi_i(y) = 0 \) if \( y \notin B(y_i, r) \). For \( x \in f(K) \), let
\[ \psi_i(x) \equiv \phi_i(x) \left( \sum_{j=1}^n \phi_j(x) \right)^{-1} . \]
Then \( 16.5.18 \) is satisfied. Indeed the denominator is not zero because \( x \) is in one of the \( B(y_i, r) \). Thus it is obvious that the sum of these equals 1 on \( K \). Now let \( f_r \) be given by \( 16.5.19 \) for \( x \in K \). For such \( x \),
\[ f(x) - f_r(x) = \sum_{i=1}^n (f(x) - y_i) \psi_i(f(x)) \]
Thus
\[ f(x) - f_r(x) = \sum_{\{i : f(x) \in B(y_i, r)\}} (f(x) - y_i) \psi_i(f(x)) \]
\[ + \sum_{\{i : f(x) \notin B(y_i, r)\}} (f(x) - y_i) \psi_i(f(x)) = \sum_{\{i : f(x) - y_i \in B(0, r)\}} (f(x) - y_i) \psi_i(f(x)) + \sum_{\{i : f(x) \notin B(y_i, r)\}} 0 \psi_i(f(x)) \in B(0, r) \]
because \( 0 \in B(0, r) \), \( B(0, r) \) is convex, and \( 16.5.18 \). It is just a convex combination of things in \( B(0, r) \).

**Note that we could have had the \( y_i \) in \( f(K) \) in addition to being in \( \bar{f(K)} \).** This would make it possible to eliminate the assumption that \( K \) is closed later on. All you really need is that \( K \) is convex.

We think of \( f_r \) as an approximation to \( f \). In fact it is uniformly within \( r \) of \( f \) on \( K \). The next lemma shows that this \( f_r \) has a fixed point. This is the main result and comes from the Brouwer fixed point theorem in \( \mathbb{R}^n \). It is an approximate fixed point.

**Lemma 16.5.3** For each \( r > 0 \), there exists \( x_r \in \text{convex hull of } f(K) \subseteq K \) such that
\[ f_r(x_r) = x_r, \quad \|f_r(x) - f(x)\| < r \text{ for all } x \]
CHAPTER 16. LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

Proof: If \( f_r(x_r) = x_r \) and

\[
x_r = \sum_{i=1}^{n} a_i y_i
\]

for \( \sum_{i=1}^{n} a_i = 1 \) and the \( y_i \) described in the above lemma, we need

\[
f_r(x_r) = \sum_{i=1}^{n} y_i \psi_i \left( f(x_r) \right) = \sum_{j=1}^{n} y_j \psi_j \left( f \left( \sum_{i=1}^{n} a_i y_i \right) \right) = \sum_{j=1}^{n} a_j y_j = x_r.
\]

Also, if this is satisfied, then we have the desired fixed point.

This will be satisfied if for each \( j = 1, \cdots, n, \)

\[
a_j = \psi_j \left( f \left( \sum_{i=1}^{n} a_i y_i \right) \right);
\]

so, let

\[
\Sigma_{n-1} = \left\{ a \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = 1, \ a_i \geq 0 \right\}
\]

and let \( h : \Sigma_{n-1} \to \Sigma_{n-1} \) be given by

\[
h(a)_j = \psi_j \left( f \left( \sum_{i=1}^{n} a_i y_i \right) \right).
\]

Since \( h \) is a continuous function of \( a \), the Brouwer fixed point theorem applies and there exists a fixed point for \( h \) which is a solution to \[16.5.20\].

The following is the Schauder fixed point theorem.

**Theorem 16.5.4** Let \( K \) be a closed and convex subset of \( X \), a normed linear space. Let \( f : K \to K \) be continuous and suppose \( f(K) \) is compact. Then \( f \) has a fixed point.

**Proof:** Recall that \( f(x_r) - f_r(x_r) \in B(0, r) \) and \( f_r(x_r) = x_r \) with \( x_r \in \text{convex hull of } f(K) \subseteq K \).

There is a subsequence, still denoted with subscript \( r \) such that \( f(x_r) \to x \in f(K) \). Note that the fact that \( K \) is convex is what makes \( f \) defined at \( x_r \). \( x_r \) is in the convex hull of \( f(K) \subseteq K \). This is where we use \( K \) convex. Then since \( f_r \) is uniformly close to \( f \), it follows that \( f_r(x_r) = x_r \to x \) also. Thus \( x_r \) converges strongly to \( x \). Therefore,

\[
f(x) = \lim_{r \to 0} f(x_r) = \lim_{r \to 0} f_r(x_r) = \lim_{r \to 0} x_r = x.
\]

We usually have in mind the mapping defined on a Banach space. However, the completeness was never used. Thus the result holds in a normed linear space.

There is a nice corollary of this major theorem which is called the Schaefer fixed point theorem or the Leray Schauder alterative principle [12].
Theorem 16.5.5 Let $f : X \to X$ be a compact map. Then either
1. There is a fixed point for $tf$ for all $t \in [0, 1]$ or
2. For every $r > 0$, there exists a solution to $x = tf(x)$ for $t \in (0, 1)$ such that $\|x\| > r$.

Proof: Suppose there is $t_0 \in [0, 1]$ such that $t_0f$ has no fixed point. Then $t_0 \neq 0$. If $t_0f$ obviously has a fixed point if $t_0 = 0$. Thus $t_0 \in (0, 1]$. Then let $r_M$ be the radial projection onto $B(0, M)$. By Schauder’s theorem there exists $x \in B(0, M)$ such that $t_0r_Mf(x) = x$. Then if $\|f(x)\| \leq M$, $r_M$ has no effect and so $t_0f(x) = x$ which is assumed not to take place. Hence $\|f(x)\| > M$ and so $\|r_Mf(x)\| = M$ so $\|x\| = t_0M$. Also $t_0r_Mf(x) = t_0Mf(x) = x$ and so $\|f(x)\| = t_0r_Mf(x)$, $t = t_01^M < 1$. Since $M$ is arbitrary, it follows that the solutions to $x = tf(x)$ for $t \in (0, 1)$ are unbounded. It was just shown that there is a solution to $x = tf(x), t < 1$ such that $\|x\| = t_0M$ where $M$ is arbitrary. Thus the second of the two alternatives holds.

Next this is considered in the more general setting of locally convex topological vector space. This is the Tychonoff fixed point theorem. In this theorem, $X$ will be a locally convex topological vector space in which every point is a closed set. Let $B$ be the basis described earlier and let $\mathcal{B}_0$ consist of all sets of $B$ which are of the form $B_A(0, r)$ where $A$ is a finite subset of $\Psi$ as described earlier. Note that for $U \in \mathcal{B}_0$, $U = -U$ and $U$ is convex. Also, if $U \in \mathcal{B}_0$, there exists $V \in \mathcal{B}_0$ such that $V + V \subseteq U$

where

$$V + V \equiv \{v_1 + v_2 : v_i \in V\}.$$ 

To see this, note

$$B_A(0, r/2) + B_A(0, r/2) \subseteq B_A(0, r).$$

We let $K$ be a closed convex subset of $X$ and let $f$ be continuous, $f : K \to K$, and $\overline{f(K)}$ is compact.

Lemma 16.5.6 For each $U \in \mathcal{B}_0$, there exists a finite set of points

$$\{y_1 \cdots y_n\} \subseteq \overline{f(K)}$$

and continuous functions $\psi_i$ defined on $\overline{f(K)}$ such that for $x \in \overline{f(K)}$,

$$\sum_{i=1}^{n} \psi_i(x) = 1, \quad (16.5.21)$$

$$\psi_i(x) = 0 \text{ if } x \notin y_i + U, \quad \psi_i(x) > 0 \text{ if } x \in y_i + U.$$ 

If

$$f_U(x) = \sum_{i=1}^{n} y_i \psi_i(f(x)), \quad (16.5.22)$$

then whenever $x \in K$,

$$f(x) - f_U(x) \in U.$$
Proof: Let $U = B_A(0, r)$. Using the compactness of $\overline{f(K)}$, there exists

$$\{y_1 \cdots y_n\} \subseteq \overline{f(K)}$$

such that

$$\{y_i + U\}_{i=1}^n$$

covers $\overline{f(K)}$. Let

$$\phi_i(y) \equiv (r - \rho_A(y - y_i))^+.$$ 

Thus $\phi_i(y) > 0$ if $y \in y_i + U$ and $\phi_i(y) = 0$ if $y \notin y_i + U$. For $x \in \overline{f(K)}$, let

$$\psi_i(x) \equiv \phi_i(x) \left( \sum_{j=1}^n \phi_j(x) \right)^{-1}.$$ 

Then (16.5.21) is satisfied. Now let $f_U$ be given by (16.5.22) for $x \in K$. For such $x$,

$$f(x) - f_U(x) = \sum_{\{i: f(x) - y_i \in U\}} (f(x) - y_i) \psi_i(f(x))$$

$$+ \sum_{\{i: f(x) - y_i \notin U\}} (f(x) - y_i) \psi_i(f(x))$$

$$= \sum_{\{i: f(x) - y_i \in U\}} (f(x) - y_i) \psi_i(f(x))$$

$$\sum_{\{i: f(x) - y_i \in U\}} (f(x) - y_i) \psi_i(f(x)) + \sum_{\{i: f(x) - y_i \notin U\}} 0 \psi_i(f(x)) \in U$$

because $0 \in U$, $U$ is convex, and (16.5.21). □

We think of $f_U$ as an approximation to $f$.

Lemma 16.5.7 For each $U \in B_0$, there exists $x_U \in \text{convex hull of } \overline{f(K)} \subseteq K$ such that

$$f_U(x_U) = x_U.$$ 

Proof: If $f_U(x_U) = x_U$ and

$$x_U = \sum_{i=1}^n a_i y_i$$

for $\sum_{i=1}^n a_i = 1$, we need

$$\sum_{j=1}^n y_j \psi_j \left( f \left( \sum_{i=1}^n a_i y_i \right) \right) = \sum_{j=1}^n a_j y_j.$$
Also, if this is satisfied, then we have the desired fixed point. This will be satisfied if for each \( j = 1, \ldots, n \),

\[
a_j = \psi_j \left( f \left( \sum_{i=1}^{n} a_i y_i \right) \right)
\]

so, let

\[
\Sigma_{n-1} \equiv \left\{ a \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = 1, \ a_i \geq 0 \right\}
\]

and let \( h : \Sigma_{n-1} \to \Sigma_{n-1} \) be given by

\[
h(a)_j \equiv \psi_j \left( f \left( \sum_{i=1}^{n} a_i y_i \right) \right).
\]

Since \( h \) is continuous, the Brouwer fixed point theorem applies and we see there exists a fixed point for \( h \) which is a solution to \( 16.5.23 \).

**Theorem 16.5.8** Let \( K \) be a closed and convex subset of \( X \), a locally convex topological vector space in which every point is closed. Let \( f : K \to K \) be continuous and suppose \( \overline{f(K)} \) is compact. Then \( f \) has a fixed point.

**Proof:** First consider the following claim which will yield a candidate for the fixed point. Recall that \( f(x_U) - f_U(x_U) \in U \) and \( f_U(x_U) = x_U \) with \( x_U \in \text{convex hull of } \overline{f(K)} \subseteq K \).

**Claim:** There exists \( x \in \overline{f(K)} \) with the property that if \( V \in B_0 \), there exists \( U \subseteq V, U \in B_0 \), such that

\[
f(x_U) \in x + V.
\]

**Proof of the claim:** If no such \( x \) exists, then for each \( x \in \overline{f(K)} \), there exists \( V_x \in B_0 \) such that whenever \( U \subseteq V_x \), with \( U \in B_0 \),

\[
f(x_U) \notin x + V_x.
\]

Since \( \overline{f(K)} \) is compact, there exist \( x_1, \ldots, x_n \in \overline{f(K)} \) such that

\[
\{x_i + V_x\}_{i=1}^{n}
\]

cover \( \overline{f(K)} \). Let \( U \in B_0, U \subseteq \cap_{i=1}^{n} V_{x_i} \) and consider \( x_U \).

\[
f(x_U) \in x_i + V_{x_i}
\]

for some \( i \) because these sets cover \( \overline{f(K)} \) and \( f(x_U) \) is something in \( \overline{f(K)} \). But \( U \subseteq V_{x_i} \), a contradiction. This shows the claim.

Now I show \( x \) is the desired fixed point. Let \( W \in B_0 \) and let \( V \in B_0 \) with

\[
V + V + V \subseteq W.
\]
Since \( f \) is continuous at \( x \), there exists \( V_0 \in \mathcal{B}_0 \) such that
\[
V_0 + V_0 \subseteq V
\]
and if
\[
y - x \in V_0 + V_0,
\]
then
\[
f(x) - f(y) \in V.
\]
Using the claim, let \( U \in \mathcal{B}_0, U \subseteq V_0, \) such that
\[
f(x_U) \in x + V_0.
\]
Then
\[
x - x_U = x - f(x_U) + f(x_U) - f_U(x_U) \in V_0 + U
\]
and so
\[
f(x) - x = f(x) - f(x_U) + f(x_U) - f_U(x_U) + f_U(x_U) - x
\]
\[
\subseteq V_0 + V_0 \subseteq V.
\]
Since \( W \in \mathcal{B}_0 \) is arbitrary, it follows from Lemma 16.2.18 that \( f(x) - x = 0 \).

As an example of the usefulness of this fixed point theorem, consider the following application to the theory of ordinary differential equations. In the context of this theorem, \( X = C([0, T] ; \mathbb{R}^n) \), a Banach space with norm given by
\[
\|x\| = \max \{|x(t)| : t \in [0, T]\}.
\]

**Theorem 16.5.9** Let \( f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be continuous and suppose there exists \( L > 0 \) such that for all \( \lambda \in (0, 1) \), if
\[
x' = \lambda f(t, x) , \quad x(0) = x_0 \quad (16.5.24)
\]
for all \( t \in [0, T] \), then \( \|x\| < L \). Then there exists a solution to
\[
x' = f(t, x) , \quad x(0) = x_0 \quad (16.5.25)
\]
for \( t \in [0, T] \).

**Proof:** Let
\[
N_x(t) = \int_t^0 f(s, x(s)) \, ds.
\]
Thus a solution to the initial value problem exists if there exists a solution to
\[
x_0 + N(x) = x.
\]
16.5. THE TYCHONOFF AND SCHAUDER FIXED POINT THEOREMS

Let
\[ m \equiv \max \left\{ |f(t, x)| : (t, x) \in [0, T] \times B(0, L) \right\}, \quad M \equiv |x_0| + mT \]
and let
\[ K \equiv \{ x \in C(0, T; \mathbb{R}^n) \text{ such that } x(0) = x_0 \text{ and } ||x|| \leq M \}. \]

Now define
\[ A x \equiv \begin{cases} x_0 + Nx \quad & \text{if } ||Nx|| \leq M - |x_0|, \\ x_0 + \frac{(M - |x_0|)Nx}{||Nx||} \quad & \text{if } ||Nx|| > M - |x_0|. \end{cases} \]

Then \( A \) is continuous and maps \( X \) to \( K \) because
\[ ||Ax|| \leq |x_0| + ||Nx|| \leq M \text{ if } ||Nx|| \leq M - |x_0| \]
and otherwise,
\[ ||Ax|| \leq |x_0| + \frac{(M - |x_0|)||Nx||}{||Nx||} \leq |x_0| + M - |x_0| = M. \]

Also \( A(K) \) is equicontinuous because
\[ x_0 + Nx(t) - (x_0 + Nx(t_1)) = \int_{t_1}^{t} f(s, x(s)) \, ds \]
and the integrand is bounded. Thus \( A(K) \) is a compact set in \( X \) by the Ascoli Arzela theorem. By the Schauder fixed point theorem, \( A \) has a fixed point, \( x \in K \).

If \( ||N(x)|| > M - |x_0| \), then
\[ x_0 + \lambda N(x) = x \]
where
\[ \lambda = \frac{(M - |x_0|)}{||Nx||} < 1 \]
and so [10.5.24] holds. Therefore, by the assumed estimate on the solutions to [10.5.24], it follows that
\[ ||x|| < L \]
and so \( ||Nx|| \leq mT = M - |x_0| \), a contradiction. Therefore, it must be the case that
\[ ||N(x)|| \leq M - |x_0| \]
which implies that
\[ x_0 + N(x) = x. \]

Since this is equivalent to [10.5.24], this proves the theorem. \( \blacksquare \)

Here is a neater proof which uses the Leray Schauder alternative, also called the Schaefer fixed point theorem presented above.
Theorem 16.5.10 Let $f: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and suppose there exists $L > 0$ such that for all $\lambda \in (0, 1)$, if
\[ x' = \lambda f(t, x), \quad x(0) = x_0 \quad (16.5.26) \]
for all $t \in [0, T]$, then $\|x\| < L$. Then there exists a solution to
\[ x' = f(t, x), \quad x(0) = x_0 \quad (16.5.27) \]
for $t \in [0, T]$.

Proof: Let $F: X \to X$ where $X$ described above.
\[ F_y(t) = \int_0^t f(s, y(s) + x_0) \, ds \]
Let $B$ be a bounded set in $X$. Then $|f(s, y(s) + x_0)|$ is bounded for $s \in [0, T]$ if $y \in B$. Say $|f(s, y(s) + x_0)| \leq C_B$. Hence $F(B)$ is bounded in $X$. Also, for $y \in B, s < t$,
\[ |F_y(t) - F_y(s)| \leq \int_s^t |f(s, y(s) + x_0)| \, ds \leq C_B |t - s| \]
and so $F(B)$ is pre-compact by the Ascoli Arzela theorem. By the Schaefer fixed point theorem, there are two alternatives. Either there are unbounded solutions $y$ to
\[ \lambda F(y) = y \]
for various $\lambda \in (0, 1)$ or for all $\lambda \in [0, 1]$, there is a fixed point for $\lambda F$. In the first case, there would be unbounded $y_\lambda$ solving
\[ y_\lambda(t) = \lambda \int_0^t f(s, y_\lambda(s) + x_0) \, ds \]
Then let $x_\lambda(s) \equiv y_\lambda(s) + x_0$ and you get $\|x_\lambda\|$ also unbounded for various $\lambda \in (0, 1)$. The above implies
\[ x_\lambda(t) - x_0 = \lambda \int_0^t f(s, x_\lambda(s)) \, ds \]
so $x_\lambda' = \lambda f(t, x_\lambda), x_\lambda(0) = x_0$ and these would be unbounded for $\lambda \in (0, 1)$ contrary to the assumption that there exists an estimate for these valid for all $\lambda \in (0, 1)$. Hence the first alternative must hold and hence there is $y \in X$ such that
\[ Fy = y \]
Then letting $x(s) \equiv y(s) + x_0$, it follows that
\[ x(t) - x_0 = \int_0^t f(s, x(s)) \, ds \]
and so $x$ is a solution to the differential equation on $[0, T]$.

Note that existence for solutions to (16.5.26) is not assumed, only estimates of possible solutions. These estimates are called $a$-priori estimates. Also note this is a global existence theorem, not a local one for a solution defined on only a small interval.
16.6 A Variational Principle of Ekeland

Definition 16.6.1 A function \( \phi : X \to (-\infty, \infty] \) is called proper if it is not constantly equal to \( \infty \). Here \( X \) is assumed to be a complete metric space. The function \( \phi \) is lower semicontinuous if

\[
x_n \to x \text{ implies } \phi(x) \leq \lim \inf_{n \to \infty} \phi(x_n)
\]

It is bounded below if there is some constant \( C \) such that \( C \leq \phi(x) \) for all \( x \).

The variational principle of Ekeland is the following theorem [52]. You start with an approximate minimizer \( x_0 \). It says there is \( y_\lambda \) fairly close to \( x_0 \) such that if you subtract a “cone” from the value of \( \phi \) at \( y_\lambda \), then the resulting function is less than \( \phi(x) \) for all \( x \neq y_\lambda \). This cone is like a supporting plane for a convex function but pertains to functions which are certainly not convex.

Theorem 16.6.2 Let \( X \) be a complete metric space and let \( \phi : X \to (-\infty, \infty] \) be proper, lower semicontinuous and bounded below. Let \( x_0 \) be such that

\[
\phi(x_0) \leq \inf_{x \in X} \phi(x) + \varepsilon
\]

Then for every \( \lambda > 0 \) there exists a \( y_\lambda \) such that

1. \( \phi(y_\lambda) \leq \phi(x_0) \)
2. \( d(y_\lambda, x_0) \leq \lambda \)
3. \( \phi(y_\lambda) - \frac{\varepsilon}{\lambda} d(x, y_\lambda) < \phi(x) \) for all \( x \neq y_\lambda \)

To motivate the proof, see the following picture which illustrates the first two steps. The \( S_i \) will be sets in \( X \) but are denoted symbolically by labeling them in \( X \times (-\infty, \infty] \).
Then the end result of this iteration would be a picture like the following.

Thus you would have $\phi(y_\lambda) - \frac{\varepsilon}{\lambda}d(y_\lambda, x) \leq \phi(x)$ for all $x$ which is seen to be what is wanted.

**Proof:** Let $x_1 = x_0$ and define

$$S_1 \equiv \left\{ z \in X : \phi(z) \leq \phi(x_1) - \frac{\varepsilon}{\lambda} d(z, x_1) \right\}$$

Then $S_1$ contains $x_1$ so it is nonempty. It is also clear that $S_1$ is a closed set. This follows from the lower semicontinuity of $\phi$. Let $x_2$ be a point of $S_1$, possibly different than $x_1$ and let

$$S_2 \equiv \left\{ z \in X : \phi(z) \leq \phi(x_2) - \frac{\varepsilon}{\lambda} d(z, x_2) \right\}$$

Continue in this way. Now let there be a sequence of points $\{x_k\}$ such that $x_k \in S_{k-1}$ and define $S_k$ by

$$S_k \equiv \left\{ z \in X : \phi(z) \leq \phi(x_k) - \frac{\varepsilon}{\lambda} d(z, x_k) \right\}$$

where $x_k$ is some point of $S_{k-1}$. Then $x_k$ is a point of $S_k$. Will this yield a nested sequence of nonempty closed sets? Yes, it appears that it would because if $z \in S_k$ then

$$\phi(z) \leq \frac{\varepsilon}{\lambda} d(z, x_k) \leq \left( \phi(x_{k-1}) - \frac{\varepsilon}{\lambda} d(x_{k-1}, x_k) \right) - \frac{\varepsilon}{\lambda} d(z, x_k)$$

$$\leq \phi(x_{k-1}) - \frac{\varepsilon}{\lambda} d(z, x_{k-1})$$

showing that $z$ has what it takes to be in $S_{k-1}$. Thus we would obtain a sequence of nested, nonempty, closed sets according to this scheme.

Now here is how to choose the $x_k \in S_{k-1}$. Let

$$\phi(x_k) < \inf_{x \in S_{k-1}} \phi(x) + \frac{1}{2k}$$

Then for $z \in S_{n+1} \subseteq S_n$,

$$\phi(z) \leq \phi(x_{n+1}) - \frac{\varepsilon}{\lambda} d(z, x_{n+1})$$
and so
\[ \frac{\varepsilon}{\lambda} d(z, x_{n+1}) \leq \phi(x_{n+1}) - \phi(z) \leq \inf_{x \in S_n} \phi(x) + \frac{1}{2^{n+1}} - \phi(z) \]
\[ \leq \phi(z) + \frac{1}{2^{n+1}} - \phi(z) = \frac{1}{2^{n+1}} \]

Thus every \( z \in S_{n+1} \) is within \( \frac{1}{2^{n+1}} \) of the single point \( x_{n+1} \) and so the diameter of \( S_n \) converges to 0 as \( n \to \infty \). By completeness of \( X \), there exists a unique \( y_\lambda \in \cap_{n} S_n \). Then it follows in particular that for \( x_0 = x_1 \) as above,
\[ \phi(y_\lambda) \leq \phi(x_0) - \frac{\varepsilon}{\lambda} d(y_\lambda, x_0) \leq \phi(x_0) \]
which verifies the first of the above conclusions.

As to the second, \( \phi(x_0) \leq \inf_{x \in X} \phi(x) + \varepsilon \) and so, for any \( x \),
\[ \phi(y_\lambda) \leq \phi(x_0) - \frac{\varepsilon}{\lambda} d(y_\lambda, x_0) \leq \phi(x) + \varepsilon - \frac{\varepsilon}{\lambda} d(y_\lambda, x_0) \]
this being true for \( x = y_\lambda \). Hence \( \frac{\varepsilon}{\lambda} d(y_\lambda, x_0) \leq \varepsilon \) and so \( d(y_\lambda, x_0) \leq \lambda \).

Finally consider the third condition. If it does not hold, then there exists \( z \neq y_\lambda \) such that
\[ \phi(y_\lambda) \geq \phi(z) + \frac{\varepsilon}{\lambda} d(z, y_\lambda) \]
so that
\[ \phi(z) \leq \phi(y_\lambda) - \frac{\varepsilon}{\lambda} d(z, y_\lambda) . \]
But then, by the definition of \( y_\lambda \) as being in all the \( S_n \),
\[ \phi(y_\lambda) \leq \phi(x_n) - \frac{\varepsilon}{\lambda} d(x_n, y_\lambda) \]
and so
\[ \phi(z) \leq \phi(x_n) - \frac{\varepsilon}{\lambda} (d(x_n, y_\lambda) + d(z, y_\lambda)) \]
\[ \leq \phi(x_n) - \frac{\varepsilon}{\lambda} d(x_n, z) \]
Since \( n \) is arbitrary, this shows that \( z \in \cap_{n} S_n \) but there is only one element of this intersection and it is \( y_\lambda \) so \( z \) must equal \( y_\lambda \), a contradiction. 

Note how if you make \( \lambda \) very small, you could pick \( \varepsilon \) very small such that the cone looks pretty flat.

### 16.6.1 Cariste Fixed Point Theorem

As mentioned in [72], the above result can be used to prove a fixed point theorem called the Cariste fixed point theorem.
Theorem 16.6.3 Let \( \phi \) be lower semicontinuous, proper, and bounded below on a complete metric space \( X \) and let \( F : X \to \mathcal{P}(X) \) be set valued such that \( F(x) \neq \emptyset \) for all \( x \). Also suppose that for each \( x \in X \), there exists \( y \in F(x) \) such that
\[
\phi(y) \leq \phi(x) - d(x, y)
\]
Then there exists \( x_0 \) such that \( x_0 \in F(x_0) \).

Proof: In the above Ekeland variational principle, let \( \varepsilon = 1 = \lambda \). Then there exists \( x_0 \) such that for all \( y \neq x_0 \)
\[
\phi(x_0) - d(y, x_0) < \phi(y), \quad \text{so} \quad \phi(x_0) < \phi(y) + d(y, x_0)
\]
for all \( y \neq x_0 \).

Suppose \( x_0 \notin F(x_0) \). From the assumption, there is \( y \in F(x_0) \) (so \( y \neq x_0 \)) such that
\[
\phi(y) \leq \phi(x_0) - d(x_0, y)
\]
Since \( y \neq x_0 \), it follows
\[
\phi(y) + d(x_0, y) \leq \phi(x_0) < \phi(y) + d(y, x_0)
\]
a contradiction. Hence \( x_0 \in F(x_0) \) after all. ■

It is a funny theorem. It is easy to prove, but you look at it and wonder what it says. If \( F \) is single valued, you would need to have a function \( \phi \) such that for each \( x \),
\[
\phi(F(x)) \leq \phi(x) - d(x, y)
\]
and if you have such a \( \phi \) then you can assert there is a fixed point for \( F \). Suppose \( F \) is single valued and \( d(Fx, Fy) \leq rd(x, y), 0 < r < 1 \). Of course \( F \) has a fixed point using easier techniques. However, this also follows from this result. Let
\[
\phi(x) = \frac{1}{1-r}d(x, F(x))
\]
Then is it true that for each \( x \), there exists \( y \in F(x) \) such that the inequality holds for all \( x \)? Is
\[
\frac{1}{1-r}d(F(x), F(F(x))) \leq \frac{1}{1-r}d(x, F(x)) - d(x, F(x))
\]
Yes, this is certainly so because the right side reduces to $\frac{1}{\lambda} d(x, F(x))$. Thus this fixed point theorem implies the usual Banach fixed point theorem.

The Ekeland variational principle says that when $\phi$ is lower semicontinuous proper and bounded below, there exists $y$ such that

$$\phi(y) - d(x, y) < \phi(x) \text{ for all } x \neq y$$

In fact this can be proved from the Cariste fixed point theorem. Suppose the EVP does not hold. This would mean that for all $y$ there exists $x \neq y$ such that

$$\phi(y) - d(x, y) \geq \phi(x)$$

Thus, for all $x$ there exists $y \neq x$ such that

$$\phi(x) - d(x, y) \geq \phi(y)$$

The inequality is preserved if $x = y$. Then let

$$F(x) \equiv \{ y \neq x : \phi(x) - d(x, y) \geq \phi(y) \} \neq \emptyset$$

by assumption. This is the hypothesis for the Cariste fixed point theorem. Hence there exists $x_0 \in F(x_0) = \{ y \neq x_0 : \phi(x_0) - d(x_0, y) \geq \phi(y) \}$ but this cannot happen because you can’t have $x_0 \neq x_0$. Thus the Ekeland variational principle must hold after all.

### 16.6.2 A Density Result

There are several applications of the Ekeland variational principle. For more of them, see [72]. One of these is to show that there is a point where $\phi'$ is small assuming $\phi$ is bounded below, lower semicontinuous, and Gateaux differentiable. Here

$$\langle \phi'(x), v \rangle \equiv \lim_{h \to 0} \frac{\phi(x + hv) - \phi(x)}{h}, \phi'(x) \in X'$$

It is sort of an approximate critical point at a point which causes $\phi$ to be near the infimum.

**Theorem 16.6.4** Let $X$ be a Banach space and $\phi : X \to \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Then for every $\varepsilon > 0$ there exists $x \in X$ such that

$$\phi(x_\varepsilon) \leq \inf_{x \in X} \phi(x) + \varepsilon \text{ and } \|\phi'(x_\varepsilon)\|_{X'} \leq \varepsilon$$

**Proof:** From the Ekeland variational principle with $\lambda = 1$, there exists $x_\varepsilon$ such that

$$\phi(x_\varepsilon) \leq \inf_{x \in X} \phi(x) + \varepsilon$$

and for all $x$,

$$\phi(x_\varepsilon) < \phi(x) + \varepsilon \|x - x_\varepsilon\|$$
Then letting $x = x_\varepsilon + hv$ where $\|v\| = 1$,
\[
\phi(x_\varepsilon + hv) - \phi(x_\varepsilon) > -\varepsilon |h|
\]
Let $h < 0$. Then divide by it
\[
\frac{\phi(x_\varepsilon + hv) - \phi(x_\varepsilon)}{h} < \varepsilon
\]
Passing to a limit as $h \to 0$ yields
\[
\langle \phi'(x_\varepsilon), v \rangle \leq \varepsilon
\]
Now $v$ was arbitrary with norm 1 and so
\[
\sup_{\|v\|=1} \langle \phi'(x_\varepsilon), v \rangle = \|\phi'(x_\varepsilon)\| \leq \varepsilon
\]
There is another very interesting application of the Ekeland variational principle.

**Theorem 16.6.5** Let $X$ be a Banach space and $\phi : X \to \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Also suppose there exists $a, c > 0$ such that
\[
a \|x\| - c \leq \phi(x) \text{ for all } x \in X
\]
Then $\{ \phi'(x) : x \in X \}$ is dense in the ball of $X'$ centered at 0 with radius $a$. Here $\phi'(x) \in X'$ and is determined by
\[
\langle \phi'(x), v \rangle \equiv \lim_{h \to 0} \frac{\phi(x + hv) - \phi(x)}{h}
\]

**Proof:** Let $x^* \in X'$, $\|x^*\| \leq a$. Let
\[
\psi(x) = \phi(x) - \langle x^*, x \rangle
\]
This is lower semicontinuous. It is also bounded from below because
\[
\psi(x) \geq \phi(x) - a \|x\| \geq (a \|x\| - c) - a \|x\| = -c
\]
It is also clearly Gateaux differentiable and lower semicontinuous because the piece added in is actually continuous. It is clear that the Gateaux derivative is just $\phi'(x) - x^*$. By Theorem 16.6.4, there exists $x_\varepsilon$ such that
\[
\|\phi'(x_\varepsilon) - x^*\| \leq \varepsilon
\]
Thus this theorem says that if $\phi(x) \geq a \|x\| - c$ where $\phi$ has the nice properties of the theorem it follows that $\phi'(x)$ is dense in $B(0, a)$ in the dual space $X'$. It follows that if for every $a$, there exists $c$ such that
\[
\phi(x) \geq a \|x\| - c \text{ for all } x \in X
\]
then $\{ \phi'(x) : x \in X \}$ is dense in $X'$. This proves the following lemma.
**Lemma 16.6.6** Let $X$ be a Banach space and $\phi : X \to \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Suppose for all $a > 0$ there exists $a \ c > 0$ such that

$$\phi(x) \geq a \|x\| - c \text{ for all } x$$

Then $\{\phi'(x) : x \in X\}$ is dense in $X'$.

If the above holds, then

$$\frac{\phi(x)}{\|x\|} \geq a - \frac{c}{\|x\|}$$

and so, since $a$ is arbitrary, it must be the case that

$$\lim_{\|x\| \to \infty} \frac{\phi(x)}{\|x\|} = \infty. \quad (16.6.29)$$

In fact, this is sufficient. If not, there would exist $a > 0$ such that $\phi(x_n) < a \|x_n\| - n$. Let $-L$ be a lower bound for $\phi(x)$. Then

$$-L + n \leq a \|x_n\|$$

and so $\|x_n\| \to \infty$. Now it follows that

$$a \geq \frac{\phi(x_n)}{\|x_n\|} + \frac{n}{\|x_n\|} \geq \frac{\phi(x_n)}{\|x_n\|} \quad (16.6.30)$$

which is a contradiction to (16.6.29). This proves the following interesting density theorem.

**Theorem 16.6.7** Let $X$ be a Banach space and $\phi : X \to \mathbb{R}$ be Gateaux differentiable, bounded from below, and lower semicontinuous. Also suppose the coercivity condition

$$\lim_{\|x\| \to \infty} \frac{\phi(x)}{\|x\|} = \infty$$

Then $\{\phi'(x) : x \in X\}$ is dense in $X'$. Here $\phi'(x) \in X'$ and is determined by

$$\langle \phi'(x), v \rangle = \lim_{h \to 0} \frac{\phi(x + hv) - \phi(x)}{h}$$

### 16.7 Quotient Spaces

A useful idea is that of a quotient space. It is a way to create another Banach space from a given Banach space and a closed subspace. It generalizes similar concepts which are routine in linear algebra.

**Definition 16.7.1** Let $X$ be a Banach space and let $V$ be a closed subspace of $X$. Then $X/V$ denotes the set of equivalence classes determined by the equivalence relation which says $x \sim y$ means $x - y \in V$. An individual equivalence class will be
denoted by any of the following symbols. \( x + V, [x], \text{ or } [x]_V \). Vector space operations are defined as follows:

\[
(x + V) + y + V \equiv x + y + V
\]

or in other symbols,

\[
[x] + [y] \equiv [x + y]
\]

and for \( \alpha \in \mathbb{F} \),

\[
\alpha [x] \equiv [\alpha x].
\]

Also a norm is defined by

\[
\| [x] \| \equiv \inf \{ \| x + v \| : v \in V \}.
\]

It is left as an exercise to verify the above algebraic operations are well defined. With the above definition, here is the major theorem about quotient spaces.

**Theorem 16.7.2** Let \( X \) be a Banach space and let \( V \) be a closed subspace of \( X \). Then with the above definitions of vector space operations, \( X/V \) is a Banach space. In the case where \( V = \ker (A) \) for \( A \in \mathcal{L}(X,Y) \) for \( Y \) another Banach space, define \( \hat{A} : X/V \to A(X) \subseteq Y \) by \( \hat{A} ([x]) \equiv Ax \). Then \( \hat{A} \) is continuous and \( 1 - 1 \). In fact, \( \| \hat{A} \| \leq \| A \| \).

**Proof:** First of all, consider the claim that the given norm really is a norm. First note that \( \| x + V \| \geq 0 \) and \( \| x + V \| = 0 \) only if \( x \in V \) because \( V \) is closed. Therefore, \( x + V = 0 + V \). Next,

\[
\| \alpha [x] \| = \| \alpha x \| \equiv \inf \{ \| \alpha x + v \| : v \in V \} \leq \inf \{ \| \alpha x + v \| : v \in V \} = |\alpha| \| [x] \|.
\]

Consider the triangle inequality.

\[
\| [x + y] \| = \inf \{ \| x + y + v \| : v \in V \}
\]

\[
\leq \| x + v_1 \| + \| y + v_2 \|
\]

for any choice of \( v_1 \) and \( v_2 \). Therefore, taking the infimum of both sides over \( v_2 \) yields

\[
\| [x + y] \| \leq \| x + v_1 \| + \| y \|
\]

and then taking the infimum over all \( v_1 \) yields

\[
\| [x + y] \| \leq \| [x] \| + \| [y] \|.
\]

Next consider the claim that \( X/V \) is a Banach space. Letting \( \{ [x_n] \} \) be a Cauchy sequence in \( X/V \), I will show a subsequence of this converges to a point in \( X/V \). This is done by defining a suitable sequence in \( X \) and then using completeness of \( X \). By
choosing a subsequence, it can be assumed that $||[x_n] - [x_{n+1}]|| < 2^{-n}$. Let $z_1 = x_1$. Then choose $v_2 \in V$ such that $||x_2 + v_2 - z_1|| < 2^{-1}$. Let $z_2 = x_2 + v_2$. Suppose \{z_1, \ldots, z_n\} have been chosen, each having the property that $[z_k] = [x_k]$ and such that $||z_k - z_{k+1}|| < 2^{-k}$. Then let $v_{n+1}$ be chosen such that $||x_{n+1} + v_{n+1} - z_n|| < 2^{-n}$ and let $z_{n+1} = x_{n+1} + v_{n+1}$. Thus \{z_n\} is a Cauchy sequence in $X$ and so it converges to $x \in X$. Then

$$||[x] - [x_n]|| \leq ||x - (x_n + v_n)|| = ||x - z_n||$$

and so $\lim_{n \to \infty} [x_n] = [x]$.

Next consider the claim about $\hat{A}$. This is well defined and linear because if $[x] = [x_1]$, then $x - x_1 \in \ker (A)$ and so $Ax = Ax_1$. Thus $\hat{A}([x]) = \hat{A}([x_1])$. It is linear because

$$\hat{A}(\alpha [x] + \beta [y]) = \hat{A}((\alpha x + \beta y)) = A(\alpha x + \beta y) = \alpha Ax + \beta Ay = \alpha \hat{A}([x]) + \beta \hat{A}([y])$$

Next consider the claim that $\hat{A}$ is continuous. Letting $v \in V$,

$$\left|\left|\hat{A}([x])\right|\right| \equiv \left|\left|Ax\right|\right| = \left|\left|A(x + v)\right|\right| \leq ||A|| \left|\left|x + v\right|\right|$$

and so, taking the infimum over all $v \in V$,

$$\left|\left|\hat{A}([x])\right|\right| \leq ||A|| \left|\left|[x]\right|\right|$$

and this shows $\left|\left|\hat{A}\right|\right| \leq ||A||$. ■

Now with this theorem, here is an interesting application.

**Theorem 16.7.3** Let $X_1$ and $X_2$ be Banach spaces which are either reflexive or dual spaces for a separable Banach space and let $A_i \in \mathcal{L}(X_i, Y)$ for $Y$ a reflexive Banach space. The following are equivalent.

For some $k > 0$

$$A_1 \left(\overline{B}(0, 1)\right) \subseteq A_2 \left(\overline{B}(0, k)\right)$$

(16.7.31)

$$||A_1^*y^*|| \leq k ||A_2^*y^*||$$

(16.7.32)

for all $y \in Y^*$. If either of the above hold, then

$$A_1X_1 \subseteq A_2X_2.$$  

(16.7.33)

**Proof:** Suppose (16.7.31) first. I show this implies (16.7.32). There are two cases. First suppose $A_2$ is one to one. Then in this case, $A_2^{-1}A_1 \left(\overline{B}(0, 1)\right) \subseteq \overline{B}(0, k)$. Therefore, if $x \in X_1$,

$$A_2^{-1}A_1(x/||x||) = y \in \overline{B}(0, k)$$

and so

$$A_1(x) = ||x|| A_2(y) = A_2(||x|| y) \in A_2(X_2).$$
Next suppose $A_2$ is not one to one. In this case, letting $\hat{A}_2$ be the continuous linear map given by 
\[ \hat{A}_2 ([x]) \equiv A_2 x, \]
it follows $\hat{A}_2$ is $1-1$ on $X_2 / \ker (A_2)$. Now note that if $\|x\| \leq k$, then it is also the case that $\||[x]|| \leq k$ and so 
\[ A_2 \left( B (0, k) \right) \subseteq \hat{A}_2 \left( B_2 (0, k) \right) \]
where in the second set, $B_2 (0, k)$ is the unit ball in $X / \ker (A_2)$. It follows from 16.7.31 
\[ A_1 \left( B (0, 1) \right) \subseteq \hat{A}_2 \left( B_2 (0, k) \right) \]
and so $\hat{A}_2^{-1} A_1 \left( B (0, 1) \right) \subseteq B_2 (0, k)$ which implies 
\[ A_1 \left( x / ||x|| \right) = \hat{A}_2 [y] \in \hat{A}_2 \left( B_2 (0, k) \right) \]
Therefore, letting $[y_1] = [y]$ be such that $||y_1|| < 2k$, it follows 
\[ A_1 \left( x / ||x|| \right) = \hat{A}_2 [y_1] = A_2 (y_1) \]
and so 
\[ A_1 (x) = A_2 (||x|| y_1) \in A_2 (X_2). \]

Next I show the equivalence of 16.7.32 and 16.7.31. First I want to show 
\[ A_1 \left( B (0, r) \right) \] is closed. Suppose then that for $A = A_1$ or $A_2, A (x_n) \rightarrow y$ where $x_n \in B (0, r)$. In the case the $X_i$ are reflexive, it follows from the Eberlein Smulian theorem there exists a subsequence, still denoted as $\{x_n\}$ which converges weakly to $x \in B (0, r)$. Then $A x_n \rightarrow y$ and $x_n \rightarrow x$ weakly. Thus $(x, y)$ is in the weak closure of the graph of $A$, 
\[ \{(x, A x) : x \in X_1\} \]
This set is strongly closed and convex and hence it is weakly closed by Theorem 16.3.3 so $y = A x$ and this shows $A \left( B (0, r) \right)$ is closed. In the other case where $X_i$ is the dual space of a separable Banach space, it follows from Corollary 16.5.4 there exists a subsequence still denoted as $\{x_n\}$ such that $x_n \rightarrow x$ weak * and similarly, $(x, y)$ is in the weak * closure of the graph of $A$ which shows again by Theorem 16.3.3 that $(x, y)$ is in the graph of $A$, showing again that $A \left( B (0, r) \right)$ is closed.

Suppose $16.7.31$. Then letting $y^* \in Y^*$, 
\[ ||A_1^* y^*|| = \sup_{||x_1|| \leq 1} |y^* (A_1 x_1)| \leq \sup_{||x_2|| \leq k} |y^* (A_2 x_2)| = k ||A_2^* y^*|| \]
which shows 16.7.32.
16.7. QUOTIENT SPACES

Now suppose (16.7.32). Then if (16.7.31) does not hold, it follows from the first part which gives $A_i \left(B(0,r)\right)$ a closed set, there exists

$$A_1 x_0 \in A_1 \left(B(0,1)\right) \setminus A_2 \left(B(0,k)\right)$$

Now $A_2 \left(B(0,k)\right)$ is closed and convex, hence weakly closed, and so by Theorem (16.2.5) there exists $y_0^* \in Y'$ such that

$$\Re y_0^* \left(A_2 \left(B(0,k)\right)\right) < c < \Re y_0^* \left(A_1 x_0\right)$$

and so

$$\|A_1^* y_0^*\| = \sup_{\|x_1\|_{X_1} \leq 1} |y_0^* (A x_1)| \geq \Re y_0^* (A_1 x_0)$$

$$> c > \Re y_0^* (A_2 (x_2)) = \Re A_2^* y_0^* (x_2)$$

whenever $x_2 \in B(0,k)$ and so, taking the supremum of all such $x_2$,

$$\|A_1^* y_0^*\| > c > k \|A_2^* y_0^*\|,$$

contradicting (16.7.32).
Chapter 17

Hilbert Spaces

In this chapter, Hilbert spaces, which have been alluded to earlier are given a complete discussion. These spaces, as noted earlier are just complete inner product spaces.

17.1 Basic Theory

Definition 17.1.1 Let $X$ be a vector space. An inner product is a mapping from $X \times X$ to $\mathbb{C}$ if $X$ is complex and from $X \times X$ to $\mathbb{R}$ if $X$ is real, denoted by $(x, y)$ which satisfies the following.

\begin{align*}
&(x, x) \geq 0, \quad (x, x) = 0 \text{ if and only if } x = 0, \quad (17.1.1) \\
&(x, y) = (y, x). \quad (17.1.2)
\end{align*}

For $a, b \in \mathbb{C}$ and $x, y, z \in X$,

\begin{equation}
(ax + by, z) = a(x, z) + b(y, z). \quad (17.1.3)
\end{equation}

Note that (17.1.3) and (17.1.3) imply $(x, ay + bz) = \overline{a}(x, y) + \overline{b}(x, z)$. Such a vector space is called an inner product space.

The Cauchy Schwarz inequality is fundamental for the study of inner product spaces.

Theorem 17.1.2 (Cauchy Schwarz) In any inner product space

\begin{equation}
|(x, y)| \leq ||x|| \cdot ||y||.
\end{equation}

Proof: Let $\omega \in \mathbb{C}$, $|\omega| = 1$, and $\overline{\omega}(x, y) = |(x, y)| = \text{Re}(x, y\omega)$. Let

\begin{equation}
F(t) = (x + ty\omega, x + t\omega y).
\end{equation}
If \( y = 0 \) there is nothing to prove because
\[
(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0)
\]
and so \( (x, 0) = 0 \). Thus, it can be assumed \( y \neq 0 \). Then from the axioms of the inner product,
\[
F(t) = ||x||^2 + 2t \Re(x, \omega y) + t^2 ||y||^2 \geq 0.
\]
This yields
\[
||x||^2 + 2t||(x, y)|| + t^2 ||y||^2 \geq 0.
\]
Since this inequality holds for all \( t \in \mathbb{R} \), it follows from the quadratic formula that
\[
4||(x, y)||^2 - 4||x||^2||y||^2 \leq 0.
\]
This yields the conclusion and proves the theorem.

**Proposition 17.1.3** For an inner product space, \( ||x|| = (x, x)^{1/2} \) does specify a norm.

**Proof:** All the axioms are obvious except the triangle inequality. To verify this,
\[
||x + y||^2 = (x + y, x + y) = ||x||^2 + ||y||^2 + 2 \Re(x, y)
\]
\[
\leq ||x||^2 + ||y||^2 + 2 ||(x, y)||
\]
\[
\leq ||x||^2 + ||y||^2 + 2 ||x|| ||y|| = (||x|| + ||y||)^2.
\]
The following lemma is called the parallelogram identity.

**Lemma 17.1.4** In an inner product space,
\[
||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.
\]
The proof, a straightforward application of the inner product axioms, is left to the reader.

**Lemma 17.1.5** For \( x \in H \), an inner product space,
\[
||x|| = \sup_{||y|| \leq 1} ||x, y||
\]

**Proof:** By the Cauchy Schwarz inequality, if \( x \neq 0 \),
\[
||x|| \geq \sup_{||y|| \leq 1} ||x, y|| \geq \left( x, \frac{x}{||x||} \right) = ||x||.
\]
It is obvious that this holds in the case that \( x = 0 \).

**Definition 17.1.6** A Hilbert space is an inner product space which is complete. Thus a Hilbert space is a Banach space in which the norm comes from an inner product as described above.
In Hilbert space, one can define a projection map onto closed convex nonempty sets.

**Definition 17.1.7** A set, $K$, is convex if whenever $\lambda \in [0,1]$ and $x,y \in K$, $\lambda x + (1-\lambda)y \in K$.

**Theorem 17.1.8** Let $K$ be a closed convex nonempty subset of a Hilbert space, $H$, and let $x \in H$. Then there exists a unique point $Px \in K$ such that $\|Px - x\| \leq \|y - x\|$ for all $y \in K$.

**Proof:** Consider uniqueness. Suppose that $z_1$ and $z_2$ are two elements of $K$ such that for $i = 1, 2,$

$$||z_i - x|| \leq ||y - x|| \quad (17.1.5)$$

for all $y \in K$. Also, note that since $K$ is convex, $\frac{z_1 + z_2}{2} \in K$.

Therefore, by the parallelogram identity,

$$||z_1 - x||^2 \leq ||\frac{z_1 + z_2}{2} - x||^2 = \frac{||z_1 - x||^2 + ||z_2 - x||^2}{2} - \frac{||z_1 - z_2||^2}{2}$$

$$\leq ||z_1 - x||^2 - \frac{||z_1 - z_2||^2}{2},$$

where the last inequality holds because of letting $z_1 = z_2$ and $y = z_1$. Hence $z_1 = z_2$ and this shows uniqueness.

Now let $\lambda = \inf\{||x - y|| : y \in K\}$ and let $y_n$ be a minimizing sequence. This means $\{y_n\} \subseteq K$ satisfies $\lim_{n \to \infty} ||x - y_n|| = \lambda$. Now the following follows from properties of the norm.

$$||y_n - x + y_m - x||^2 = 4(||\frac{y_n + y_m}{2} - x||^2)$$

Then by the parallelogram identity, and convexity of $K$, $\frac{y_n + y_m}{2} \in K$, and so

$$|| (y_n - x) - (y_m - x) ||^2 = 2(||y_n - x||^2 + ||y_m - x||^2) - 4(||\frac{y_n + y_m}{2} - x||^2) \leq 2(||y_n - x||^2 + ||y_m - x||^2) - 4\lambda^2.$$

Since $||x - y_n|| \to \lambda$, this shows $\{y_n - x\}$ is a Cauchy sequence. Thus also $\{y_n\}$ is a Cauchy sequence. Since $H$ is complete, $y_n \to y$ for some $y \in H$ which must be in $K$ because $K$ is closed. Therefore

$$||x - y|| = \lim_{n \to \infty} ||x - y_n|| = \lambda.$$

Let $Px = y$. 


Corollary 17.1.9 Let \( K \) be a closed, convex, nonempty subset of a Hilbert space, \( H \), and let \( x \in H \). Then for \( z \in K \), \( z = Px \) if and only if
\[
\text{Re}(x - z, y - z) \leq 0 \tag{17.1.6}
\]
for all \( y \in K \).

Before proving this, consider what it says in the case where the Hilbert space is \( \mathbb{R}^n \).

Condition (17.1.6) says the angle, \( \theta \), shown in the diagram is always obtuse. Remember from calculus, the sign of \( x \cdot y \) is the same as the sign of the cosine of the included angle between \( x \) and \( y \). Thus, in finite dimensions, the conclusion of this corollary says that \( z = Px \) exactly when the angle of the indicated angle is obtuse. Surely the picture suggests this is reasonable.

The inequality (17.1.6) is an example of a variational inequality and this corollary characterizes the projection of \( x \) onto \( K \) as the solution of this variational inequality.

Proof of Corollary: Let \( z \in K \) and let \( y \in K \) also. Since \( K \) is convex, it follows that if \( t \in [0, 1] \),
\[
z + t(y - z) = (1 - t) z + ty \in K.
\]
Furthermore, every point of \( K \) can be written in this way. (Let \( t = 1 \) and \( y \in K \).) Therefore, \( z = Px \) if and only if for all \( y \in K \) and \( t \in [0, 1] \),
\[
||x - (z + t(y - z))||^2 = ||(x - z) - t(y - z)||^2 \geq ||x - z||^2
\]
for all \( t \in [0, 1] \) and \( y \in K \) if and only if for all \( t \in [0, 1] \) and \( y \in K \)
\[
||x - z||^2 + t^2 ||y - z||^2 - 2t \text{Re} (x - z, y - z) \geq ||x - z||^2
\]
If and only if for all \( t \in [0, 1] \),
\[
t^2 ||y - z||^2 - 2t \text{Re} (x - z, y - z) \geq 0 \tag{17.1.7}
\]
Now this is equivalent to (17.1.6) holding for all \( t \in (0, 1) \). Therefore, dividing by \( t \in (0, 1) \), (17.1.6) is equivalent to
\[
t ||y - z||^2 - 2 \text{Re} (x - z, y - z) \geq 0
\]
for all \( t \in (0, 1) \) which is equivalent to (17.1.6). This proves the corollary.

Corollary 17.1.10 Let \( K \) be a nonempty convex closed subset of a Hilbert space, \( H \). Then the projection map, \( P \) is continuous. In fact,
\[
|Px - Py| \leq |x - y|.
\]
17.1. BASIC THEORY

**Proof:** Let \( x, x' \in H \). Then by Corollary 17.1.9,

\[
\Re (x' - Px', Px - Px') \leq 0, \quad \Re (x - Px, Px' - Px) \leq 0
\]

Hence

\[
0 \leq \Re (x - Px, Px - Px') - \Re (x' - Px', Px - Px') = \Re (x - x', Px - Px') - \|Px - Px'\|^2
\]

and so

\[\|Px - Px'\|^2 \leq |x - x'| \|Px - Px'\|.
\]

This proves the corollary.

The next corollary is a more general form for the Brouwer fixed point theorem.

**Corollary 17.1.11** Let \( f : K \to K \) where \( K \) is a convex compact subset of \( \mathbb{R}^n \). Then \( f \) has a fixed point.

**Proof:** Let \( K \subseteq \overline{B(0,R)} \) and let \( P \) be the projection map onto \( K \). Then consider the map \( f \circ P \) which maps \( \overline{B(0,R)} \) to \( \overline{B(0,R)} \) and is continuous. By the Brouwer fixed point theorem for balls, this map has a fixed point. Thus there exists \( x \) such that

\[f \circ P(x) = x\]

Now the equation also requires \( x \in K \) and so \( P(x) = x \). Hence \( f(x) = x \).

**Definition 17.1.12** Let \( H \) be a vector space and let \( U \) and \( V \) be subspaces. \( U \oplus V = H \) if every element of \( H \) can be written as a sum of an element of \( U \) and an element of \( V \) in a unique way.

The case where the closed convex set is a closed subspace is of special importance and in this case the above corollary implies the following.

**Corollary 17.1.13** Let \( K \) be a closed subspace of a Hilbert space, \( H \), and let \( x \in H \). Then for \( z \in K \), \( z = Px \) if and only if

\[
(x - z, y) = 0 \tag{17.1.8}
\]

for all \( y \in K \). Furthermore, \( H = K \oplus K^\perp \) where

\[
K^\perp = \{ x \in H : (x, k) = 0 \text{ for all } k \in K \}
\]

and

\[
\|x\|^2 = \|x - Px\|^2 + \|Px\|^2. \tag{17.1.9}
\]

**Proof:** Since \( K \) is a subspace, the condition \((17.1.8)\) implies \( \Re(x - z, y) \leq 0 \) for all \( y \in K \). Replacing \( y \) with \(-y\), it follows \( \Re(x - z, -y) \leq 0 \) which implies \( \Re(x - z, y) \geq 0 \) for all \( y \). Therefore, \( \Re(x - z, y) = 0 \) for all \( y \in K \). Now let
\(|\alpha| = 1\) and \(\alpha (x - z, y) = |(x - z, y)|\). Since \(K\) is a subspace, it follows \(\overline{\alpha} y \in K\) for all \(y \in K\). Therefore,

\[
0 = \Re(x - z, \overline{\alpha} y) = (x - z, \overline{\alpha} y) = \alpha (x - z, y) = |(x - z, y)|.
\]

This shows that \(z = Px\), if and only if \(\square\).

For \(x \in H\), \(x = x - Px + Px\) and from what was just shown, \(x - Px \in K^\perp\) and \(Px \in K\). This shows that \(K^\perp + K = H\). Is there only one way to write a given element of \(H\) as a sum of a vector in \(K\) with a vector in \(K^\perp\)? Suppose \(y + z = y_1 + z_1\) where \(z, z_1 \in K^\perp\) and \(y, y_1 \in K\). Then \((y - y_1) = (z_1 - z)\) and so from what was just shown, \((y - y_1, y - y_1) = (y - y_1, z_1 - z) = 0\) which shows \(y_1 = y\) and consequently \(z_1 = z\). Finally, letting \(z = Px\),

\[
|\alpha| = 1 = \frac{(x - z + z, x - z + z)}{|x - z|^2 + (x - z, z) + |z|^2} = \frac{|x|^2}{|x|^2 + |z|^2} = 1.
\]

This proves the corollary.

The following theorem is called the Riesz representation theorem for the dual of a Hilbert space. If \(z \in H\) then define an element \(f \in H'\) by the rule \((x, z) = f (x)\). It follows from the Cauchy Schwarz inequality and the properties of the inner product that \(f \in H'\). The Riesz representation theorem says that all elements of \(H'\) are of this form.

**Theorem 17.1.14** Let \(H\) be a Hilbert space and let \(f \in H'\). Then there exists a unique \(z \in H\) such that

\[
f (x) = (x, z)
\]

for all \(x \in H\).

**Proof:** Letting \(y, w \in H\) the assumption that \(f\) is linear implies

\[
f (yf (w) - f (y)w) = f (w)f (y) - f (y)f (w) = 0
\]

which shows that \(yf (w) - f (y)w \in f^{-1} (0)\), which is a closed subspace of \(H\) since \(f\) is continuous. If \(f^{-1} (0) = H\), then \(f\) is the zero map and \(z = 0\) is the unique element of \(H\) which satisfies \(\square\). If \(f^{-1} (0) \neq H\), pick \(u \notin f^{-1} (0)\) and let \(w = u - Pu \neq 0\). Thus Corollary \(\square\) implies \((y, w) = 0\) for all \(y \in f^{-1} (0)\). In particular, let \(y = xf (w) - f (x)w\) where \(x \in H\) is arbitrary. Therefore,

\[
0 = (f (w)x - f (x)w, w) = f (w)(x, w) - f (x)||w||^2.
\]

Thus, solving for \(f (x)\) and using the properties of the inner product,

\[
f (x) = \langle x, \frac{f (w)w}{||w||^2}\rangle
\]

Let \(z = \frac{f (w)w}{||w||^2}\). This proves the existence of \(z\). If \(f (x) = (x, z)\) \(i = 1, 2\), for all \(x \in H\), then for all \(x \in H\), then \((x, z_1 - z_2) = 0\) which implies, upon taking \(x = z_1 - z_2\) that \(z_1 = z_2\). This proves the theorem.

If \(R : H \rightarrow H'\) is defined by \(Rx (y) = (y, x)\), the Riesz representation theorem above states this map is onto. This map is called the Riesz map. It is routine to show \(R\) is linear and \(|Rx| = |x|\).
17.2 The Hilbert Space $L(U)$

Let $L \in L(U,H)$. Then one can consider the image of $L$, $L(U)$ as a Hilbert space. This is another interesting application of Theorem 17.1.8. First here is a definition which involves abominable and atrociously misleading notation which nevertheless seems to be well accepted.

**Definition 17.2.1** Let $L \in L(U,H)$, the bounded linear maps from $U$ to $H$ for $U,H$ Hilbert spaces. For $y \in L(U)$, let $L^{-1}y$ denote the unique vector in 

$$\{x : Lx = y\} \equiv M_y$$

which is closest in $U$ to 0.

Note this is a good definition because $\{x : Lx = y\}$ is closed thanks to the continuity of $L$ and it is obviously convex. Thus Theorem 17.1.8 applies. With this definition define an inner product on $L(U)$ as follows. For $y,z \in L(U)$,

$$(y,z)_{L(U)} \equiv (L^{-1}y,L^{-1}z)_U$$

The notation is abominable because $L^{-1}(y)$ is the normal notation for $M_y$.

In terms of linear algebra, this $L^{-1}$ is the Moore Penrose inverse. There you obtain the least squares solution $x$ to $Lx = y$ which has smallest norm. Here there is an actual solution and among those solutions you get the one which has least norm. Of course a real honest solution is also a least squares solution so this is the Moore Penrose inverse restricted to $L(U)$.

First I want to understand $L^{-1}$ better. It is actually fairly easy to understand in terms of geometry. Here is a picture of $L^{-1}(y)$ for $y \in L(U)$.
As indicated in the picture, here is a lemma which gives a description of the situation.

**Lemma 17.2.2** In the context of the above definition, \( L^{-1}(y) \) is characterized by

\[
(L^{-1}(y), x)_U = 0 \text{ for all } x \in \ker(L)
\]

\[
L(L^{-1}(y)) = y, \quad (L^{-1}(y) \in M_y)
\]

In addition to this, \( L^{-1} \) is linear and the above definition does define an inner product.

**Proof:** The point \( L^{-1}(y) \) is well defined as noted above. I claim it is characterized by the following for \( y \in L(U) \)

\[
(L^{-1}(y), x)_U = 0 \text{ for all } x \in \ker(L)
\]

\[
L(L^{-1}(y)) = y, \quad (L^{-1}(y) \in M_y)
\]

Let \( w \in M_y \) and suppose

\[
(v, x)_U = 0, \ L(v) = y
\]

Then from the above characterization,

\[
||w||^2 = \left( \left\| \underbrace{w - v + v}_{\in \ker(L)} \right\| \right)^2 = ||w - v||^2 + ||v||^2
\]

which shows that \( w = L^{-1}(y) \) if and only if \( w = v \) just described. From this characterization, it is clear that \( L^{-1} \) is linear. Then it is also obvious that

\[
(y, z)_{L(U)} = (L^{-1}y, L^{-1}z)_U
\]

also specifies an inner product. The algebraic axioms are all obvious because \( L^{-1} \) is linear. If \( (y, y)_{L(U)} = 0 \), then \( ||L^{-1}y||_U^2 = 0 \) and so \( L^{-1}y = 0 \) which requires \( y = L(L^{-1}y) = 0 \).

With the above definition, here is the main result.

**Theorem 17.2.3** Let \( U, H \) be Hilbert spaces and let \( L \in \mathcal{L}(U, H) \). Then Definition 17.2.1 makes \( L(U) \) into a Hilbert space. Also \( L : U \to L(U) \) is continuous and \( L^{-1} : L(U) \to U \) is continuous. Also,

\[
\|L\|_{\mathcal{L}(U,H)} \|Lx\|_{L(U)} \geq \|Lx\|_H \quad (17.2.11)
\]

If \( U \) is separable, so is \( L(U) \). Also \( (L^{-1}(y), x)_U = 0 \) for all \( x \in \ker(L) \), and \( L^{-1} : L(U) \to U \) is linear. Also, in case that \( L \) is one to one, both \( L \) and \( L^{-1} \) preserve norms.
Proof: First consider the claim that \( L : U \rightarrow L(U) \) is continuous and \( L^{-1} : L(U) \rightarrow U \) is also continuous. Why is \( L \) continuous? Say \( u_n \rightarrow 0 \) in \( U \). Then

\[
\|L u_n\|_{L(U)} = \|L^{-1} (L(u_n))\|_U .
\]

Now \( \|L^{-1} (L(u_n))\|_U \leq \|u_n\|_U \) and so it converges to 0. (Recall that \( L^{-1} (L u_n) \) is the smallest vector in \( U \) which maps to \( L u_n \). Since \( u_n \) is mapped by \( L \) to \( L u_n \), it follows that \( \|L^{-1} (L(u_n))\|_U \leq \|u_n\|_U \).) Hence \( L \) is continuous.

Next, why is \( L^{-1} \) continuous? Let \( \|y_n\|_{L(U)} \rightarrow 0 \). This requires \( \|L^{-1} (y_n)\|_U \rightarrow 0 \) by definition of the norm in \( L(U) \). Thus \( L^{-1} \) is continuous.

Why is \( L(U) \) a Hilbert space? Let \( \{y_n\} \) be a Cauchy sequence in \( L(U) \). Then from what was just observed, it follows that \( L^{-1} (y_n) \) is a Cauchy sequence in \( U \). Hence \( L^{-1} (y_n) \rightarrow x \in U \). It follows that \( y_n = L (L^{-1} (y_n)) \rightarrow Lx \) in \( L(U) \). This is in the norm of \( L(U) \). It was just shown that \( L \) is continuous as a map from \( U \) to \( L(U) \). This shows that \( L(U) \) is a Hilbert space. It was already shown that it is an inner product space and this has shown that it is complete.

If \( x \in U \), then \( \|Lx\|_{U} \leq \|L\|_{L(U)} \|x\|_{U} \). It follows that

\[
\|L (x)\|_{U} = \|L (L^{-1} (L(x)))\|_{U} \leq \|L\|_{L(U)} \|L^{-1} (L(x))\|_{U} \\
= \|L\|_{L(U)} \|L(x)\|_{L(U)}. 
\]

This verifies \( \|L\|_{L(U)} \leq 1 \).

If \( U \) is separable, then letting \( D \) be a countable dense subset, it follows from the continuity of the operators \( L, L^{-1} \) discussed above that \( L(D) \) is separable in \( U \). To see this, note that

\[
\|L x_n - Lx\|_{L(U)} = \|L (L^{-1} (Lx_n - Lx))\|_{U} \leq \|L\|_{L(U)} \|L^{-1} (L(x_n - x))\|_{U} \leq \|L\|_{L(U)} \|x_n - x\|_U
\]

As before, \( L^{-1} (L(x_n - x)) \) is the smallest vector which maps onto \( L(x_n - x) \) and so its norm is no larger than \( \|x_n - x\|_U \).

Consider the last claim. If \( L \) is one to one, then for \( y \in L(U) \), there is only one vector which maps to \( y \). Therefore,

\[
L^{-1} (L(x)) = x.
\]

Hence for \( y \in L(U) \),

\[
\|y\|_{L(U)} = \|L^{-1} (y)\|_U
\]

Also,

\[
\|Lu\|_{L(U)} = \|L^{-1} (L(u))\|_U = \|u\|_U \]

Now here is another argument for various continuity claims.

\[
\|Lx\|_{L(U)} = \|L^{-1} (Lx)\|_U \leq \|x\|_U
\]

because \( L^{-1} (Lx) \) is the smallest thing in \( U \) which maps to \( Lx \) and \( x \) is something which maps to \( Lx \) so it follows that the inequality holds. Hence \( L \in \mathcal{L}(U,L(U)) \) and in fact, \( \|L\|_{\mathcal{L}(U,L(U))} = 1 \). Next, let \( y \in L(U) \),

\[
\|L^{-1} y\|_U = \|y\|_{L(U)}
and so \(\|L^{-1}\|_{\mathcal{L}(L(U), U)} = 1\) and this shows that \(L \in \mathcal{L}(U, L(U))\) while \(L^{-1} \in \mathcal{L}(L(U), U)\) and both have norm equal to 1.

Now

\[ \|Lx\|_H = \|L(L^{-1}(Lx))\|_H \leq \|L\|_{\mathcal{L}(U, H)} \|L^{-1}(Lx)\|_U \equiv \|L\|_{\mathcal{L}(U, H)} \|Lx\|_{\mathcal{L}(U)} \]

Now here are some other very interesting results. I am following [98].

**Lemma 17.2.4** Let \(L \in \mathcal{L}(U, H)\). Then \(L \left( B(0, r) \right) \) is closed and convex.

**Proof:** It is clear this is convex since \(L\) is linear. Why is it closed? \(B(0, r)\) is compact in the weak topology by the Banach Alaoglu theorem, Theorem [15.5.4] on Page 447. Furthermore, \(L\) is continuous with respect to the weak topologies on \(U\) and \(H\). Here is why this is so. Suppose \(u_n \to u\) weakly in \(U\). Then if \(h \in H\),

\[ (Lu_n, h) = (u_n, L^*h) \to (u, L^*h) = (Lu, h) \]

which shows \(Lu_n \to Lu\) weakly. Therefore, \(L \left( B(0, r) \right) \) is weakly compact because it is the continuous image of a compact set. Therefore, it must also be weakly closed because the weak topology is a Hausdorff space. (See Lemma [16.3.2] on Page 479, and so you can apply the separation theorem, Theorem [16.2.5] on Page 472 to obtain a separating functional. Thus if \(x \neq y\), there exists \(f \in H'\) such that \(\text{Re } f(y) > c > \text{Re } f(x)\) and so taking

\[ 2r < \min (c - \text{Re } f(x), \text{Re } f(y) - c), \]

\[ B_f(x, r) \cap B_f(y, r) = \emptyset \]

where

\[ B_f(x, r) \equiv \{ y \in H : |f(x - y)| < r \} \]

is an example of a basic open set in the weak topology.)

Now suppose \(p \notin L \left( B(0, r) \right)\). Since the set is weakly closed and convex, it follows by Theorem [16.2.4] and the Riesz representation theorem for Hilbert space that there exists \(z \in H\) such that

\[ \text{Re } (p, z) > c > \text{Re } (Lx, z) \]

for all \(x \in B(0, r)\). Therefore, \(p\) cannot be a strong limit point because if it were, there would exist \(x_n \in B(0, r)\) such that \(Lx_n \to p\) which would require \(\text{Re } (Lx_n, z) \to \text{Re } (p, z)\) which is prevented by the above inequality. This proves the lemma.

Now here is a very interesting result about showing that \(T_1(U_1) = T_2(U_2)\) where \(U_i\) is a Hilbert space and \(T_i \in \mathcal{L}(U_i, H)\). The situation is as indicated in the diagram.

\[ \begin{array}{ccc}
H & \uparrow & \leftarrow \\
U_1 & \nearrow & T_1 \nearrow T_2 \nearrow U_2 \\
\end{array} \]

The question is whether \(T_1U_1 = T_2U_2\).
17.2. THE HILBERT SPACE $L(U)$

**Theorem 17.2.5** Let $U_i$, $i = 1, 2$ and $H$ be Hilbert spaces and let $T_i \in L(U_i, H)$. If there exists $c \geq 0$ such that for all $x \in H$

$$||T_i^* x||_1 \leq c ||T_i^* x||_2$$  \hspace{1cm} (17.2.12)

then

$$T_1 \left( \overline{B}(0,1) \right) \subseteq T_2 \left( \overline{B}(0,c) \right)$$  \hspace{1cm} (17.2.13)

and so $T_1 (U_1) \subseteq T_2 (U_2)$. If $||T_i^* x||_1 = ||T^* x||_2$ for all $x \in H$, then $T_1 (U_1) = T_2 (U_2)$ and in addition to this,

$$||T_i^{-1} x||_1 = ||T_i^{-1} x||_2$$  \hspace{1cm} (17.2.14)

for all $x \in T_1 (U_1) = T_2 (U_2)$. In this theorem, $T_i^{-1}$ refers to Definition 17.2.1.

**Proof:** Consider the first claim. If it is not so, then there exists $u_0, \|u_0\|_1 \leq 1$ but

$$T_1 (u_0) \notin T_2 \left( \overline{B}(0,c) \right)$$

the latter set being a closed convex nonempty set thanks to Lemma 17.2.4. Then by the separation theorem, Theorem 16.2.5 there exists $z \in H$ such that

$$\text{Re} \left( (T_1 (u_0), z)_H \right) > 1 > \text{Re} \left( (T_2 (v), z)_H \right)$$

for all $\|v\|_2 \leq c$. Therefore, replacing $v$ with $v\theta$ where $\theta$ is a suitable complex number having modulus 1, it follows

$$||T_2 z|| > 1 > \left| (v, T_2^* z)_{U_2} \right|$$  \hspace{1cm} (17.2.15)

for all $\|v\|_2 \leq c$. If $c = 0$, gives a contradiction immediately because of 17.2.15. Assume then that $c > 0$. From 17.2.15, if $\|v\|_2 \leq 1$, then

$$\left| (v, T_2^* z)_{U_2} \right| < \frac{1}{c} \|T_2 z\|$$

Then from 17.2.15,

$$\|T_2^* z\|_{U_2} = \sup_{\|v\| \leq 1} \left| (v, T_2^* z)_{U_2} \right| \leq \frac{1}{c} \|T_2^* z\|$$

which contradicts 17.2.15. Therefore, it is clear that $T_1 (U_1) \subseteq T_2 (U_2)$.

Now consider the second claim. The first part shows $T_1 (U_1) = T_2 (U_2)$. Denote by $u_i \in U_i$, the point $T_i^{-1} x$. Without loss of generality, it can be assumed $x \neq 0$ because if $x = 0$, then the definition of $T_i^{-1}$ gives $T_i^{-1} (x) = 0$. Thus for $x \neq 0$ neither $u_i$ can equal 0. I need to verify that $\|u_1\|_1 = \|u_2\|_2$. Suppose then that this is not so. Say $\|u_1\|_1 > \|u_2\|_2 > 0$.

$$\frac{x}{\|u_2\|_2} = T_2 \left( \frac{u_2}{\|u_2\|_2} \right) \in T_2 \left( \overline{B}(0,1) \right)$$
But from the first part of the theorem this equals $T_1 \left( B(0,1) \right)$ and so there exists $u'_1 \in B(0,1)$ such that

$$x = T_1 u'_1$$

Hence

$$T_1 \left( u'_1 - \frac{u_1}{\|u_2\|_2} \right) = \frac{x}{\|u_2\|_2} - \frac{x}{\|u_2\|_2} = 0.$$  

From Theorem 17.2.3 this implies

$$0 = (u_1, u'_1 - \frac{u_1}{\|u_2\|_2}) \leq ||u_1||_1 ||u'_1||_1 - ||u_1||_1 \frac{||u_1||_1}{\|u_2\|_2}$$

$$= ||u_1||_1 \left( ||u'_1||_1 - \frac{||u_1||_1}{\|u_2\|_2} \right) \leq ||u_1||_1 \left( 1 - \frac{||u_1||_1}{\|u_2\|_2} \right)$$

which is a contradiction because it was assumed $||u_1||_1 > \|u_2\|_2 > 1$. This proves the theorem.

## 17.3 Approximations In Hilbert Space

The Gram Schmidt process applies in any Hilbert space.

**Theorem 17.3.1** Let $\{x_1, \cdots, x_n\}$ be a basis for $M$ a subspace of $H$ a Hilbert space. Then there exists an orthonormal basis for $M$, $\{u_1, \cdots, u_n\}$ which has the property that for each $k \leq n$, span($x_1, \cdots, x_k$) = span($u_1, \cdots, u_k$). Also if $\{x_1, \cdots, x_n\} \subseteq H$, then

$$\text{span} (x_1, \cdots, x_n)$$

is a closed subspace.

**Proof:** Let $\{x_1, \cdots, x_n\}$ be a basis for $M$. Let $u_1 = \frac{x_1}{|x_1|}$. Thus for $k = 1$, span ($u_1$) = span ($x_1$) and $\{u_1\}$ is an orthonormal set. Now suppose for some $k < n$, $u_1, \cdots, u_k$ have been chosen such that $(u_j \cdot u_i) = \delta_{ji}$ and span ($x_1, \cdots, x_k$) = span ($u_1, \cdots, u_k$). Then define

$$u_{k+1} = \frac{x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j}{\|x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j\|}$$  \hspace{1cm} (17.3.16)

where the denominator is not equal to zero because the $x_j$ form a basis and so

$$x_{k+1} \notin \text{span} (x_1, \cdots, x_k) = \text{span} (u_1, \cdots, u_k)$$

Thus by induction,

$$u_{k+1} \in \text{span} (u_1, \cdots, u_k, x_{k+1}) = \text{span} (x_1, \cdots, x_k, x_{k+1})$$.
Also, \( x_{k+1} \in \text{span} (u_1, \ldots, u_k, u_{k+1}) \) which is seen easily by solving (17.3.15) for \( x_{k+1} \) and it follows
\[
\text{span} (x_1, \ldots, x_k, x_{k+1}) = \text{span} (u_1, \ldots, u_k, u_{k+1}).
\]

If \( l \leq k \),
\[
(u_{k+1} \cdot u_l) = C \left( (x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) (u_j \cdot u_l) \right) = C \left( (x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) \delta_{lj} \right) = C ((x_{k+1} \cdot u_l) - (x_{k+1} \cdot u_l)) = 0.
\]
The vectors, \( \{u_j\}_{j=1}^{n} \), generated in this way are therefore an orthonormal basis because each vector has unit length.

Consider the second claim about finite dimensional subspaces. Without loss of generality, assume \( \{x_1, \ldots, x_n\} \) is linearly independent. If it is not, delete vectors until a linearly independent set is obtained. Then by the first part, \( \text{span} (x_1, \ldots, x_n) = \text{span} (u_1, \ldots, u_n) \equiv M \) where the \( u_i \) are an orthonormal set of vectors. Suppose \( \{y_k\} \subseteq M \) and \( y_k \to y \in H \). Is \( y \in M \)? Let
\[
y_k = \sum_{j=1}^{n} c_j^k u_j
\]
Then let \( c^k = (c_1^k, \ldots, c_n^k)^T \). Then
\[
|c^k - c'|^2 = \sum_{j=1}^{n} |c_j^k - c_j'|^2 = \left( \sum_{j=1}^{n} (c_j^k - c_j') u_j \right) \left( \sum_{j=1}^{n} (c_j^k - c_j') u_j \right) = \|y_k - y_l\|^2
\]
which shows \( \{c^k\} \) is a Cauchy sequence in \( \mathbb{F}^n \) and so it converges to \( c \in \mathbb{F}^n \). Thus
\[
y = \lim_{k \to \infty} y_k = \lim_{k \to \infty} \sum_{j=1}^{n} c_j^k u_j = \sum_{j=1}^{n} c_j u_j \in M.
\]
This completes the proof.

**Theorem 17.3.2** Let \( M \) be the span of \( \{u_1, \ldots, u_n\} \) in a Hilbert space, \( H \) and let \( y \in H \). Then \( Py \) is given by
\[
Py = \sum_{k=1}^{n} \langle y, u_k \rangle u_k \tag{17.3.17}
\]
and the distance is given by

\[ \sqrt{|y|^2 - \sum_{k=1}^{n} |(y, u_k)|^2}. \]  

(17.3.18)

**Proof:**

\[
\begin{align*}
(y - \sum_{k=1}^{n} (y, u_k) u_k, y_p) &= (y, u_p) - \sum_{k=1}^{n} (y, u_k) (u_k, u_p) \\
&= (y, u_p) - (y, u_p) = 0
\end{align*}
\]

It follows that

\[
\left( y - \sum_{k=1}^{n} (y, u_k) u_k, u \right) = 0
\]

for all \( u \in M \) and so by Corollary 17.1.13 this verifies 17.3.17.

The square of the distance, \( d \) is given by

\[
d^2 = \left( y - \sum_{k=1}^{n} (y, u_k) u_k, y - \sum_{k=1}^{n} (y, u_k) u_k \right)
\]

\[
= |y|^2 - 2 \sum_{k=1}^{n} |(y, u_k)|^2 + \sum_{k=1}^{n} |(y, u_k)|^2
\]

and this shows 17.3.18.

What if the subspace is the span of vectors which are not orthonormal? There is a very interesting formula for the distance between a point of a Hilbert space and a finite dimensional subspace spanned by an arbitrary basis.

**Definition 17.3.3** Let \( \{x_1, \cdots, x_n\} \subseteq H \), a Hilbert space. Define

\[
G(x_1, \cdots, x_n) \equiv \begin{pmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & \ddots & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{pmatrix} \quad (17.3.19)
\]

Thus the \( ij \)th entry of this matrix is \((x_i, x_j)\). This is sometimes called the Gram matrix. Also define \( G(x_1, \cdots, x_n) \) as the determinant of this matrix, also called the Gram determinant.

\[
G(x_1, \cdots, x_n) \equiv \begin{vmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & \ddots & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{vmatrix} \quad (17.3.20)
\]

The theorem is the following.
Theorem 17.3.4 Let \( M = \text{span} (x_1, \cdots, x_n) \subseteq H \), a Real Hilbert space where \( \{x_1, \cdots, x_n\} \) is a basis and let \( y \in H \). Then letting \( d \) be the distance from \( y \) to \( M \),
\[
d^2 = \frac{G(x_1, \cdots, x_n, y)}{G(x_1, \cdots, x_n)}. \tag{17.3.21}
\]

**Proof:** By Theorem 17.3.1 \( M \) is a closed subspace of \( H \). Let \( \sum_{k=1}^{n} \alpha_k x_k \) be the element of \( M \) which is closest to \( y \). Then by Corollary 17.1.13,
\[
\left( y - \sum_{k=1}^{n} \alpha_k x_k, x_p \right) = 0
\]
for each \( p = 1, 2, \cdots, n \). This yields the system of equations,
\[
(y, x_p) = \sum_{k=1}^{n} (x_p, x_k) \alpha_k, p = 1, 2, \cdots, n \tag{17.3.22}
\]
Also by Corollary 17.1.13,
\[
||y||^2 = \left\| y - \sum_{k=1}^{n} \alpha_k x_k \right\|^2 + \left\| \sum_{k=1}^{n} \alpha_k x_k \right\|^2
\]
and so, using (17.3.22),
\[
||y||^2 = d^2 + \sum_{j} \left( \sum_{k} \alpha_k (x_k, x_j) \right) \alpha_j
\]
\[
= d^2 + \sum_{j} (y, x_j) \alpha_j \tag{17.3.23}
\]
\[
\equiv d^2 + y_x^T \alpha \tag{17.3.24}
\]
in which
\[
y_x^T = ((y, x_1), \cdots, (y, x_n)), \quad \alpha = (\alpha_1, \cdots, \alpha_n).
\]
Then (17.3.21) and (17.3.22) imply the following system
\[
\begin{pmatrix}
G(x_1, \cdots, x_n) & 0 \\
y_x^T & 1
\end{pmatrix}
\begin{pmatrix}
\alpha \\
d^2
\end{pmatrix}
= 
\begin{pmatrix}
y_x \\
||y||^2
\end{pmatrix}
\]
By Cramer’s rule,

\[
d^2 = \frac{\det \left( \begin{array}{ccc} \mathcal{G} (x_1, \cdots, x_n) & y_x & ||y||^2 \\ y^T_x & 0 \\ y^T_x \end{array} \right)}{\det \left( \begin{array}{ccc} \mathcal{G} (x_1, \cdots, x_n) & 0 \\ 0 & 1 \end{array} \right)}
\]

\[
= \frac{\det \left( \begin{array}{ccc} \mathcal{G} (x_1, \cdots, x_n) & y_x \\ y^T_x & ||y||^2 \end{array} \right)}{\det (\mathcal{G} (x_1, \cdots, x_n))}
\]

\[
= \frac{\det (\mathcal{G} (x_1, \cdots, x_n, y))}{\det (\mathcal{G} (x_1, \cdots, x_n))} = \frac{G (x_1, \cdots, x_n, y)}{G (x_1, \cdots, x_n)}
\]

and this proves the theorem.

17.4 The M"untz Theorem

Recall the polynomials are dense in \( C ([0, 1]) \). This is a consequence of the Weierstrass approximation theorem. Now consider finite linear combinations of the functions, \( \{p_0, p_1, p_2, \cdots \} \) is a sequence of nonnegative real numbers, \( p_0 \equiv 0 \). The M"untz theorem says this set, \( S \) of finite linear combinations is dense in \( C ([0, 1]) \) exactly when \( \sum_{k=1}^{\infty} \frac{1}{p_k} = \infty \). There are two versions of this theorem, one for density of \( S \) in \( L^2 (0, 1) \) and one for \( C ([0, 1]) \). The presentation follows Cheney [32].

Recall the Cauchy identity presented earlier, Theorem [4.9.1] on Page 85 which is stated here for convenience.

**Theorem 17.4.1** The following identity holds.

\[
\prod_{i,j} \left| \begin{array}{ccc} \frac{1}{a_i+b_j} & \cdots & \frac{1}{a_i+b_n} \\ \vdots & \ddots & \vdots \\ \frac{1}{a_n+b_j} & \cdots & \frac{1}{a_n+b_n} \end{array} \right| = \prod_{j < i} (a_i - a_j) (b_i - b_j) . \tag{17.4.25}
\]

**Lemma 17.4.2** Let \( m, p_1, \cdots, p_n \) be distinct real numbers larger than \(-1/2\). Thus the functions, \( f_m (x) \equiv x^m, f_{p_i} (x) \equiv x^{p_i} \) are all in \( L^2 (0, 1) \). Let \( M = \text{span} (f_{p_1}, \cdots, f_{p_n}) \). Then the \( L^2 \) distance, \( d \) between \( f_m \) and \( M \) is

\[
d = \frac{1}{\sqrt{2m+1}} \prod_{j=1}^{n} \frac{|m-p_j|}{m+p_j+1}
\]

**Proof:** By Theorem [17.3.4]

\[
d^2 = \frac{G (f_{p_1}, \cdots, f_{p_n}, f_m)}{G (f_{p_1}, \cdots, f_{p_n})}.
\]

\[
(f_{p_i}, f_{p_j}) = \int_0^1 x^{p_i} x^{p_j} dx = \frac{1}{1+p_i+p_j}
\]
Therefore,

\[ d^2 = \frac{\prod_{j<i} (p_i - p_j)(p_i - p_j)}{\prod_{i,j} (p_i + p_j + 1)^2} \]

Now from the Cauchy identity, letting \( a_i = p_i + \frac{1}{2} \) and \( b_j = \frac{1}{2} + p_j \) with \( p_{n+1} = m \), the numerator of the above equals

\[ \prod_{j<i} (p_i - p_j)(p_i - p_j) \]

while the denominator equals

\[ \prod_{i,j} (p_i + p_j + 1)^2 \]

Therefore,

\[ d^2 = \frac{\prod_{k=1}^n (m-p_k)^2 \prod_{i,j} (p_i - p_j)^2}{\prod_{i=1}^n (m+p_i+1)^2 \prod_{i,j} (p_i + p_j + 1)(2m+1)^2} \]

which shows

\[ d = \frac{1}{\sqrt{2m + 1}} \prod_{k=1}^n \frac{|m-p_k|}{m+p_k+1} \]

and this proves the lemma.
The following lemma relates an infinite sum to a product. First consider the graph of $\ln (1 - x)$ for $x \in [0, \frac{1}{2}]$. Here is a rough sketch with two lines, $y = -x$ which lies above the graph of $\ln (1 - x)$ and $y = -2x$ which lies below.

**Lemma 17.4.3** Let $a_n \neq 1, a_n > 0$, and $\lim_{n \to \infty} a_n = 0$. Then

$$\prod_{k=1}^{\infty} (1 - a_n) \equiv \lim_{n \to \infty} \prod_{k=1}^{n} (1 - a_n) = 0$$

if and only if

$$\sum_{n=1}^{\infty} a_n = +\infty.$$

**Proof:** Without loss of generality, you can assume $a_n < 1/2$ because the two conditions are determined by the values of $a_n$ for $n$ large. By the above sketch the following is obtained.

$$\ln \prod_{k=1}^{n} (1 - a_k) = \sum_{k=1}^{n} \ln (1 - a_k) \in \left[ -2 \sum_{k=1}^{n} a_k, -\sum_{k=1}^{n} a_k \right].$$

Therefore,

$$e^{-2 \sum_{k=1}^{n} a_k} \leq \prod_{k=1}^{n} (1 - a_k) \leq e^{-\sum_{k=1}^{n} a_k}$$

The conclusion follows.

The following is Müntz’s first theorem.

**Theorem 17.4.4** Let $\{p_n\}$ be a sequence of real numbers larger than $-1/2$ such that $\lim_{n \to \infty} p_n = \infty$. Let $S$ denote the set of finite linear combinations of the functions, $\{x^{p_1}, x^{p_2}, \ldots\}$. Then $S$ is dense in $L^2(0,1)$ if and only if

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty.$$
17.4. THE MÜNTZ THEOREM

Proof: The polynomials are dense in $L^2(0,1)$ and so $S$ is dense in $L^2(0,1)$ if and only if for every $\varepsilon > 0$ there exists a function $f$ from $S$ such that for each integer $m \geq 0$, \[
\left( \int_0^1 |f(x) - x^m|^2 \, dx \right)^{1/2} < \varepsilon.
\] This happens if and only if for all $n$ large enough, the distance in $L^2(0,1)$ between the function, $x \to x^m$ and \(\text{span}(x^{p_1}, x^{p_2}, \cdots, x^{p_n})\) is less than $\varepsilon$. However, from Lemma 17.4.2 this distance equals
\[
\frac{1}{\sqrt{2m + 1}} \prod_{k=1}^n \frac{|m - p_k|}{m + p_k + 1}.
\]
Thus $S$ is dense if and only if
\[
\prod_{k=1}^\infty \left( 1 - \left( 1 - \frac{|m - p_k|}{m + p_k + 1} \right) \right) = 0
\]
which, by Lemma 17.4.3, happens if and only if
\[
\sum_{k=1}^\infty \left( 1 - \frac{|m - p_k|}{m + p_k + 1} \right) = +\infty
\]
But this sum equals
\[
\sum_{k=1}^\infty \left( \frac{m + p_k + 1 - |m - p_k|}{m + p_k + 1} \right)
\]
which has the same convergence properties as $\sum \frac{1}{p_k}$ by the limit comparison test. This proves the theorem.

The following is Müntz’s second theorem.

Theorem 17.4.5 Let $S$ be finite linear combinations of $\{1, x^{p_1}, x^{p_2}, \cdots\}$ where $p_j \geq 1$ and $\lim_{n \to \infty} p_n = \infty$. Then $S$ is dense in $C([0,1])$ if and only if $\sum_{k=1}^\infty \frac{1}{p_k} = \infty$.

Proof: If $S$ is dense in $C([0,1])$ then $S$ must also be dense in $L^2(0,1)$ and so by Theorem 17.4.4 $\sum_{k=1}^\infty \frac{1}{p_k} = \infty$.

Suppose then that $\sum_{k=1}^\infty \frac{1}{p_k} = \infty$ so that by Theorem 17.4.4, $S$ is dense in $L^2(0,1)$. The theorem will be proved if it is shown that for all $m$ a nonnegative integer,
\[
\max \{|x^m - f(x)| : x \in [0,1]\} < \varepsilon
\]
for some $f \in S$. This is true if $m = 0$ because $1 \in S$. Suppose then that $m > 0$. Let $S'$ denote finite linear combinations of the functions
\[
\{x^{p_1-1}, x^{p_2-1}, \cdots\}.
\]
These functions are also dense in \( L^2 (0,1) \) because \( \sum_{p_i} \frac{1}{p_i} = \infty \) by the limit comparison test. Then by Theorem 17.4.4 there exists \( f \in S' \) such that
\[
\left( \int_0^1 |f(x) - mx^{m-1}|^2 \, dx \right)^{1/2} < \varepsilon.
\]
Thus \( F(x) \equiv \int_0^x f(t) \, dt \in S \) and
\[
|F(x) - x^m| = \left| \int_0^x \left( f(t) - mt^{m-1} \right) \, dt \right|
\leq \int_0^x |f(t) - mt^{m-1}| \, dt
\leq \left( \int_0^1 |f(t) - mt^{m-1}|^2 \, dt \right)^{1/2} \left( \int_0^1 \, dx \right)^{1/2}
\leq \varepsilon
\]
and this proves the theorem.

### 17.5 Orthonormal Sets

The concept of an orthonormal set of vectors is a generalization of the notion of the standard basis vectors of \( \mathbb{R}^n \) or \( \mathbb{C}^n \).

**Definition 17.5.1** Let \( H \) be a Hilbert space. \( S \subseteq H \) is called an orthonormal set if \( ||x|| = 1 \) for all \( x \in S \) and \( (x,y) = 0 \) if \( x,y \in S \) and \( x \neq y \). For any set, \( D \),
\[
D^\perp \equiv \{ x \in H : (x,d) = 0 \text{ for all } d \in D \}.
\]
If \( S \) is a set, \( \text{span} (S) \) is the set of all finite linear combinations of vectors from \( S \).

You should verify that \( D^\perp \) is always a closed subspace of \( H \).

**Theorem 17.5.2** In any separable Hilbert space, \( H \), there exists a countable orthonormal set, \( S = \{ x_i \} \) such that the span of these vectors is dense in \( H \). Furthermore, if \( \text{span} (S) \) is dense, then for \( x \in H \),
\[
x = \sum_{i=1}^{\infty} (x,x_i) x_i \equiv \lim_{n \to \infty} \sum_{i=1}^{n} (x,x_i) x_i. \quad (17.5.26)
\]

**Proof:** Let \( \mathcal{F} \) denote the collection of all orthonormal subsets of \( H \). \( \mathcal{F} \) is nonempty because \( \{ x \} \in \mathcal{F} \) where \( ||x|| = 1 \). The set, \( \mathcal{F} \), is a partially ordered set with the order given by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, \( \mathcal{C} \) in \( \mathcal{F} \). Then let \( S \equiv \cup \mathcal{C} \). It follows \( S \) must be a maximal orthonormal set of vectors. Why? It remains to verify that \( S \) is countable span (S)
is dense, and the condition holds. To see note that if \( x, y \in S \), then
\[
||x - y||^2 = ||x||^2 + ||y||^2 - 2 \Re \langle x, y \rangle = ||x||^2 + ||y||^2 = 2.
\]
Therefore, the open sets, \( B\left(x, \frac{1}{2}\right) \) for \( x \in S \) are disjoint and cover \( S \). Since \( H \) is assumed to be separable, there exists a point from a countable dense set in each of these disjoint balls showing there can only be countably many of the balls and that consequently, \( S \) is countable as claimed.

It remains to verify and that span (\( S \)) is dense. If span (\( S \)) is not dense, then span (\( S \)) is a closed proper subspace of \( H \) and letting \( y \notin \) span (\( S \)),
\[
z \equiv \frac{y - Py}{||y - Py||} \in \text{span} \left( S \right)^\perp.
\]
But then \( S \cup \{z\} \) would be a larger orthonormal set of vectors contradicting the maximality of \( S \).

It remains to verify. Let \( S = \{x_i\}_{i=1}^\infty \) and consider the problem of choosing the constants, \( c_k \) in such a way as to minimize the expression
\[
\left\| x - \sum_{k=1}^n c_k x_k \right\|^2 = ||x||^2 + \sum_{k=1}^n |c_k|^2 - \sum_{k=1}^n \overline{c_k} \langle x, x_k \rangle - \sum_{k=1}^n c_k \langle x, x_k \rangle.
\]
This equals
\[
||x||^2 + \sum_{k=1}^n |c_k - (x, x_k)|^2 - \sum_{k=1}^n |(x, x_k)|^2
\]
and therefore, this minimum is achieved when \( c_k = (x, x_k) \) and equals
\[
||x||^2 - \sum_{k=1}^n |(x, x_k)|^2
\]
Now since span (\( S \)) is dense, there exists \( n \) large enough that for some choice of constants, \( c_k \),
\[
\left\| x - \sum_{k=1}^n c_k x_k \right\|^2 < \varepsilon.
\]
However, from what was just shown,
\[
\left\| x - \sum_{i=1}^n \langle x, x_i \rangle x_i \right\|^2 \leq \left\| x - \sum_{k=1}^n c_k x_k \right\|^2 < \varepsilon
\]
showing that \( \lim_{n \to \infty} \sum_{i=1}^n \langle x, x_i \rangle x_i = x \) as claimed. This proves the theorem.

The proof of this theorem contains the following corollary.
**Corollary 17.5.3** Let $S$ be any orthonormal set of vectors and let
\[
\{x_1, \ldots, x_n\} \subseteq S.
\]
Then if $x \in H$
\[
\left\| x - \sum_{k=1}^{n} c_k x_k \right\|^2 \geq \left\| x - \sum_{i=1}^{n} (x, x_i) x_i \right\|^2
\]
for all choices of constants, $c_k$. In addition to this, Bessel’s inequality
\[
||x||^2 \geq \sum_{k=1}^{n} |(x, x_k)|^2.
\]
If $S$ is countable and span $(S)$ is dense, then letting \( \{x_i\}_{i=1}^{\infty} = S \), \[17.5.26\] follows.

### 17.6 Fourier Series, An Example

In this section consider the Hilbert space, $L^2(0, 2\pi)$ with the inner product,
\[
(f, g) \equiv \int_{0}^{2\pi} f \overline{g} dm.
\]
This is a Hilbert space because of the theorem which states the $L^p$ spaces are complete, Theorem \[13.1.10\] on Page 383. An example of an orthonormal set of functions in $L^2(0, 2\pi)$ is
\[
\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}
\]
for $n$ an integer. Is it true that the span of these functions is dense in $L^2(0, 2\pi)$?

**Theorem 17.6.1** Let $S = \{\phi_n\}_{n \in \mathbb{Z}}$. Then span $(S)$ is dense in $L^2(0, 2\pi)$.

**Proof:** By regularity of Lebesgue measure, it follows from Theorem \[13.2.4\] that $C_c(0, 2\pi)$ is dense in $L^2(0, 2\pi)$. Therefore, it suffices to show that for $g \in C_c(0, 2\pi)$, then for every $\varepsilon > 0$ there exists $h \in \text{span}(S)$ such that $\|g - h\|_{L^2(0, 2\pi)} < \varepsilon$.

Let $T$ denote the points of $\mathbb{C}$ which are of the form $e^{it}$ for $t \in \mathbb{R}$. Let $A$ denote the algebra of functions consisting of polynomials in $z$ and $1/z$ for $z \in T$. Thus a typical such function would be one of the form
\[
\sum_{k=-m}^{m} c_k z^k
\]
for $m$ chosen large enough. This algebra separates the points of $T$ because it contains the function, $p(z) = z$. It annihilates no point of $t$ because it contains the constant function $1$. Furthermore, it has the property that for $f \in A$, $\overline{f} \in A$. By the Stone Weierstrass approximation theorem, Theorem \[7.2.10\] on Page 189, $A$ is dense in
COMPACT OPERATORS

C(T). Now for \( g \in C_c \left( 0, 2\pi \right) \), extend \( g \) to all of \( \mathbb{R} \) to be \( 2\pi \) periodic. Then letting \( G \left( e^{it} \right) \equiv g \left( t \right) \), it follows \( G \) is well defined and continuous on \( T \). Therefore, there exists \( H \in \mathcal{A} \) such that for all \( t \in \mathbb{R} \),

\[
|H \left( e^{it} \right) - G \left( e^{it} \right)| < \varepsilon /2\pi.
\]

Thus \( H \left( e^{it} \right) \) is of the form

\[
H \left( e^{it} \right) = \sum_{k=-m}^{m} c_k e^{ikt} \in \text{span } (S).
\]

Let \( h \left( t \right) = \sum_{k=-m}^{m} c_k e^{ikt} \). Then

\[
\left( \int_{0}^{2\pi} \left| g - h \right|^2 \, dx \right)^{1/2} \leq \left( \int_{0}^{2\pi} \max \left\{ |g \left( t \right) - h \left( t \right)| : t \in [0, 2\pi] \right\} \, dx \right)^{1/2}
\]

\[
= \left( \int_{0}^{2\pi} \max \left\{ |G \left( e^{it} \right) - H \left( e^{it} \right)| : t \in [0, 2\pi] \right\} \, dx \right)^{1/2}
\]

\[
< \left( \int_{0}^{2\pi} \varepsilon^2 /2\pi \right)^{1/2} = \varepsilon.
\]

This proves the theorem.

**Corollary 17.6.2** For \( f \in L^2 \left( 0, 2\pi \right) \),

\[
\lim_{m \to \infty} \left\| f - \sum_{k=-m}^{m} (f, \phi_k) \phi_k \right\|_{L^2 \left( 0, 2\pi \right)} = 0.
\]

**Proof:** This follows from Theorem 17.5.2 on Page 524.

### 17.7 Compact Operators

**Definition 17.7.1** Let \( A \in \mathcal{L}(H, H) \) where \( H \) is a Hilbert space. Then \( |(Ax, y)| \leq ||A|| ||x|| ||y|| \) and so the map, \( x \to (Ax, y) \) is continuous and linear. By the Riesz representation theorem, there exists a unique element of \( H \), denoted by \( A^*y \) such that

\[
(Ax, y) = (x, A^*y).
\]

It is clear \( y \to A^*y \) is linear and continuous. \( A^* \) is called the adjoint of \( A \). \( A \) is a self adjoint operator if \( A = A^* \). Thus for a self adjoint operator, \( (Ax, y) = (x, Ay) \) for all \( x, y \in H \). \( A \) is a compact operator if whenever \( \{x_k\} \) is a bounded sequence, there exists a convergent subsequence of \( \{Ax_k\} \). Equivalently, \( A \) maps bounded sets to sets whose closures are compact.
The big result is called the Hilbert Schmidt theorem. It is a generalization to arbitrary Hilbert spaces of standard finite dimensional results having to do with diagonalizing a symmetric matrix. There is another statement and proof of this theorem around Page 649.

**Theorem 17.7.2** Let \( A \) be a compact self adjoint operator defined on a Hilbert space, \( H \). Then there exists a countable set of eigenvalues, \( \{ \lambda_i \} \) and an orthonormal set of eigenvectors, \( u_i \), satisfying

\[
\lambda_i \text{ is real, } |\lambda_n| \geq |\lambda_{n+1}|, \quad Au_i = \lambda_i u_i, \tag{17.7.27}
\]

and either

\[
\lim_{n \to \infty} \lambda_n = 0, \tag{17.7.28}
\]

or for some \( n \),

\[
\text{span} (u_1, \cdots, u_n) = H. \tag{17.7.29}
\]

In any case,

\[
\text{span} (\{u_i\}_{i=1}^\infty) \text{ is dense in } A(H). \tag{17.7.30}
\]

and for all \( x \in H \),

\[
Ax = \sum_{k=1}^\infty \lambda_k \langle x, u_k \rangle u_k. \tag{17.7.31}
\]

This sequence of eigenvectors and eigenvalues also satisfies

\[
|\lambda_n| = ||A_n||, \tag{17.7.32}
\]

and

\[
A_n : H_n \to H_n. \tag{17.7.33}
\]

where \( H \equiv H_1 \) and \( H_n \equiv \{u_1, \cdots, u_{n-1}\}^\perp \) and \( A_n \) is the restriction of \( A \) to \( H_n \).

**Proof:** If \( A = 0 \) then pick \( u \in H \) with \( ||u|| = 1 \) and let \( \lambda_1 = 0 \). Since \( A(H) = 0 \) it follows the span of \( u \) is dense in \( A(H) \) and this proves the theorem in this uninteresting case.

Assume from now on \( A \neq 0 \). Let \( \lambda_1 \) be real and \( \lambda_1^2 \equiv ||A||^2 \). From the definition of \( ||A|| \) there exists \( x_n, ||x_n|| = 1 \), and \( ||Ax_n|| \to ||A|| = |\lambda_1| \). Now it is clear that \( A^2 \) is also a compact self adjoint operator. Consider

\[
((\lambda_1^2 - A^2) x_n, x_n) = \lambda_1^2 - ||Ax_n||^2 \to 0.
\]

Since \( A \) is compact, there exists a subsequence of \( \{x_n\} \) still denoted by \( \{x_n\} \) such that \( Ax_n \) converges to some element of \( H \). Thus since \( \lambda_1^2 - A^2 \) satisfies

\[
((\lambda_1^2 - A^2) y, y) \geq 0
\]
in addition to being self adjoint, it follows \( x, y \rightarrow ((\lambda_1^2 - A^2)x, y) \) satisfies all the axioms for an inner product except for the one which says that \((z, z) = 0\) only if \(z = 0\). Therefore, the Cauchy Schwarz inequality may be used to write

\[
\left| ((\lambda_1^2 - A^2)x, y) \right| \leq \left( (\lambda_1^2 - A^2)y, y \right)^{1/2} \left( (\lambda_1^2 - A^2)x, x \right)^{1/2} \leq e_n \|y\|.
\]

where \(e_n \to 0\) as \(n \to \infty\). Therefore, taking the sup over all \(\|y\| \leq 1\),

\[
\lim_{n \to \infty} \left| (\lambda_1^2 - A^2)x_n \right| = 0.
\]

Since \(A^2x_n\) converges, it follows since \(\lambda_1 \neq 0\) that \(\{x_n\}\) is a Cauchy sequence converging to \(x\) with \(\|x\| = 1\). Therefore, \(A^2x_n \to A^2x\) and so

\[
\left| (\lambda_1^2 - A^2)x \right| = 0.
\]

Now

\[
(\lambda_1 I - A)(\lambda_1 I + A)x = (\lambda_1 I + A)(\lambda_1 I - A)x = 0.
\]

If \((\lambda_1 I - A)x = 0\), let \(u_1 \equiv x\). If \((\lambda_1 I - A)x = y \neq 0\), let \(u_1 \equiv \frac{y}{\|y\|}\).

Suppose \(\{u_1, \cdots, u_n\}\) is such that \(Au_k = \lambda_k u_k\) and \(|\lambda_k| \geq |\lambda_{k+1}|, |\lambda_k| = ||A_k||\) and \(A_k : H_k \to H_k\) for \(k \leq n\). If

\[
\text{span} \langle u_1, \cdots, u_n \rangle = H
\]

this yields the conclusion of the theorem in the situation of \(\ref{17.7.29}\). Therefore, assume the span of these vectors is always a proper subspace of \(H\). It is shown next that \(A_{n+1} : H_{n+1} \to H_{n+1}\). Let

\[
y \in H_{n+1} \equiv \{u_1, \cdots, u_n\}^\perp
\]

Then for \(k \leq n\)

\[
(Ay, u_k) = (y, Au_k) = \lambda_k (y, u_k) = 0,
\]

showing \(A_{n+1} : H_{n+1} \to H_{n+1}\) as claimed. There are two cases. Either \(\lambda_n = 0\) or it is not. In the case where \(\lambda_n = 0\) it follows \(A_n = 0\). Every element of \(H\) is the sum of one in \(\text{span} \langle u_1, \cdots, u_n \rangle\) and one in \(\text{span} \langle u_1, \cdots, u_n \rangle^\perp\). (note \(\text{span} \langle u_1, \cdots, u_n \rangle^\perp\) is a closed subspace.) Thus, if \(x \in H, x = y + z\) where \(y \in \text{span} \langle u_1, \cdots, u_n \rangle\) and \(z \in \text{span} \langle u_1, \cdots, u_n \rangle^\perp\) and \(Az = 0\). Say \(y = \sum_{j=1}^n c_j u_j\). Then

\[
Ax = Ay = \sum_{j=1}^n c_j Au_j
\]

\[
= \sum_{j=1}^n c_j \lambda_j u_j \in \text{span} \langle u_1, \cdots, u_n \rangle.
\]

The conclusion of the theorem holds in this case because the above equation holds if with \(c_i = (x, u_i)\).
Now consider the case where \( \lambda_n \neq 0 \). In this case repeat the above argument used to find \( u_{n+1} \) and \( \lambda_{n+1} \) for the operator, \( A_{n+1} \). This yields \( u_{n+1} \in H_{n+1} \equiv \{ u_1, \cdots, u_n \}^\perp \) such that

\[
\|u_{n+1}\| = 1, \|Au_{n+1}\| = |\lambda_{n+1}| = \|A_{n+1}\| \leq \|A\| = |\lambda_n|
\]

and if it is ever the case that \( \lambda_n = 0 \), it follows from the above argument that the conclusion of the theorem is obtained.

I claim \( \lim_{n \to \infty} \lambda_n = 0 \). If this were not so, then for some \( \varepsilon > 0 \), \( 0 < \varepsilon = \lim_{n \to \infty} \|\lambda_n\| \) but then

\[
\|Au_n - Au_m\|^2 = \|\lambda_n u_n - \lambda_m u_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon^2
\]

and so there would not exist a convergent subsequence of \( \{Au_k\}_{k=1}^\infty \) contrary to the assumption that \( A \) is compact. This verifies the claim that \( \lim_{n \to \infty} \lambda_n = 0 \).

It remains to verify that \( \text{span} (\{u_i\}) \) is dense in \( A(H) \). If \( w \in \text{span} (\{u_i\})^\perp \) then \( w \in H_n \) for all \( n \) and so for all \( n \),

\[
\|Aw\| \leq \|A_n\| \|w\| \leq \|\lambda_n\| \|w\|.
\]

Therefore, \( Aw = 0 \). Now every vector from \( H \) can be written as a sum of one from

\[
\text{span} (\{u_i\})^\perp = \text{span} (\{u_i\})^\perp
\]

and one from \( \overline{\text{span}} (\{u_i\}) \). Therefore, if \( x \in H \), \( x = y + w \) where \( y \in \text{span} (\{u_i\}) \) and \( w \in \overline{\text{span}} (\{u_i\})^\perp \). It follows \( Aw = 0 \). Also, since \( y \in \text{span} (\{u_i\}) \), there exist constants, \( c_k \) and \( n \) such that

\[
\left\| y - \sum_{k=1}^n c_k u_k \right\| < \varepsilon.
\]

Therefore, from Corollary \[17.5.3\]

\[
\left\| y - \sum_{k=1}^n (y, u_k) u_k \right\| = \left\| y - \sum_{k=1}^n (x, u_k) u_k \right\| < \varepsilon.
\]

Therefore,

\[
\|A\| \varepsilon \geq \left\| A \left( y - \sum_{k=1}^n (x, u_k) u_k \right) \right\| = \left\| Ax - \sum_{k=1}^n (x, u_k) \lambda_k u_k \right\|.
\]

Since \( \varepsilon \) is arbitrary, this shows \( \text{span} (\{u_i\}) \) is dense in \( A(H) \) and also implies \[17.7.31\].
This proves the theorem.
17.7. COMPACT OPERATORS

Define \( v \otimes u \in \mathcal{L}(H, H) \) by

\[
v \otimes u(x) = (x, u)v,
\]

then (17.7.31) is of the form

\[
A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k
\]

This is the content of the following corollary.

**Corollary 17.7.3** The main conclusion of the above theorem can be written as

\[
A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k
\]

where the convergence of the partial sums takes place in the operator norm.

**Proof:** Using (17.7.31)

\[
\left\| \left( A - \sum_{k=1}^{n} \lambda_k u_k \otimes u_k \right) x, y \right\|
= \left\| \left( Ax - \sum_{k=1}^{n} \lambda_k (x, u_k) u_k, y \right) \right\|
= \left\| \sum_{k=n}^{\infty} \lambda_k (x, u_k) u_k, y \right\|
= \left\| \sum_{k=n}^{\infty} \lambda_k (x, u_k) (u_k, y) \right\|
\leq |\lambda_n| \left( \sum_{k=n}^{\infty} |(x, u_k)|^2 \right)^{1/2} \left( \sum_{k=n}^{\infty} |(y, u_k)|^2 \right)^{1/2}
\leq |\lambda_n| ||x|| ||y||
\]

It follows

\[
\left\| \left( A - \sum_{k=1}^{n} \lambda_k u_k \otimes u_k \right) x \right\| \leq |\lambda_n| ||x||
\]

and this proves the corollary.

**Corollary 17.7.4** Let \( A \) be a compact self adjoint operator defined on a separable Hilbert space, \( H \). Then there exists a countable set of eigenvalues, \( \{\lambda_i\} \) and an orthonormal set of eigenvectors, \( v_i \) satisfying

\[
Av_i = \lambda_i v_i, ||v_i|| = 1, \quad (17.7.34)
\]

\[
\text{span} \{v_i\}_{i=1}^{\infty} \text{ is dense in } H. \quad (17.7.35)
\]

Furthermore, if \( \lambda_i \neq 0 \), the space, \( V_{\lambda_i} \equiv \{ x \in H : Ax = \lambda_i x \} \) is finite dimensional.
Proof: In the proof of the above theorem, let \( W \equiv \text{span} \left( \{ u_i \} \right) \). By Theorem 17.5.2, there is an orthonormal set of vectors, \( \{ w_i \}_{i=1}^{\infty} \) whose span is dense in \( W \). As shown in the proof of the above theorem, \( Aw = 0 \) for all \( w \in W \). Let \( \{ v_i \}_{i=1}^{\infty} = \{ u_i \}_{i=1}^{\infty} \cup \{ w_i \}_{i=1}^{\infty} \).

It remains to verify the space, \( V_{\lambda_i} \), is finite dimensional. First observe that \( A : V_{\lambda_i} \rightarrow V_{\lambda_i} \). Since \( A \) is continuous, it follows that \( A : V_{\lambda_i} \rightarrow V_{\lambda_i} \). Thus \( A \) is a compact self adjoint operator on \( V_{\lambda_i} \) and by Theorem 17.7.2, \( 17.7.29 \) holds because the only eigenvalue is \( \lambda_i \). This proves the corollary.

Note the last claim of this corollary holds independent of the separability of \( H \). This proves the corollary.

Suppose \( \lambda \notin \{ \lambda_n \} \) and \( \lambda \neq 0 \). Then the above formula for \( A \), \( 17.7.31 \), yields an interesting formula for \( (A - \lambda I)^{-1} \). Note first that since \( \lim_{n \to \infty} \lambda_n = 0 \), it follows that \( \lambda_n^2 / (\lambda_n - \lambda)^2 \) must be bounded, say by a positive constant, \( M \).

**Corollary 17.7.5** Let \( A \) be a compact self adjoint operator and let \( \lambda \notin \{ \lambda_n \}_{n=1}^{\infty} \) and \( \lambda \neq 0 \) where the \( \lambda_n \) are the eigenvalues of \( A \). Then

\[
(A - \lambda I)^{-1} x = -\frac{1}{\lambda} x + \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \lambda} (x, u_k) u_k. \tag{17.7.36}
\]

**Proof:** Let \( m < n \). Then since the \( \{ u_k \} \) form an orthonormal set,

\[
\left| \sum_{k=m}^{n} \frac{\lambda_k}{\lambda_k - \lambda} (x, u_k) u_k \right| = \left( \sum_{k=m}^{n} \left( \frac{\lambda_k}{\lambda_k - \lambda} \right)^2 |(x, u_k)|^2 \right)^{1/2} \leq M \left( \sum_{k=m}^{n} |(x, u_k)|^2 \right)^{1/2}. \tag{17.7.37}
\]

But from Bessel’s inequality,

\[
\sum_{k=1}^{\infty} |(x, u_k)|^2 \leq ||x||^2
\]

and so for \( m \) large enough, the first term in \( 17.7.37 \) is smaller than \( \varepsilon \). This shows the infinite series in \( 17.7.36 \) converges. It is now routine to verify that the formula in \( 17.7.36 \) is the inverse.

### 17.8 Sturm Liouville Problems

A Sturm Liouville problem involves the differential equation,

\[
(p(x) y')' + (\lambda q(x) + r(x)) y = 0, \quad x \in [a, b], \quad p(x) \geq 0 \tag{17.8.38}
\]

where we assume that \( q(x) \geq 0 \) for \( x \in [a, b] \) and is positive except for finitely many points. Also, assume it is continuous. Probably, you could generalize this to assume
less about $q$ if this is of interest. There will also be boundary conditions at $a, b$. These are typically of the form

\begin{align*}
C_1 y(a) + C_2 y'(a) &= 0 \\
C_3 y(b) + C_4 y'(b) &= 0
\end{align*}

(17.8.39)

where

\begin{align*}
C_1^2 + C_2^2 > 0, \text{ and } C_3^2 + C_4^2 > 0.
\end{align*}

(17.8.40)

Also we assume here that $a$ and $b$ are finite numbers. In the example, the constants $C_i$ are given and $\lambda$ is called the eigenvalue while a solution of the differential equation and given boundary conditions corresponding to $\lambda$ is called an eigenfunction.

There is a simple but important identity related to solutions of the above differential equation. Suppose $\lambda_i$ and $y_i$ for $i = 1, 2$ are two solutions of \[17.8.38\]. Thus from the equation, we obtain the following two equations.

\begin{align*}
(p(x)y_1')'y_2 + (\lambda_1 q(x) + r(x)) y_1 y_2 &= 0, \\
(p(x)y_2')'y_1 + (\lambda_2 q(x) + r(x)) y_1 y_2 &= 0.
\end{align*}

Subtracting the second from the first yields

\begin{align*}
(p(x)y_1')'y_2 - (p(x)y_2')'y_1 + (\lambda_1 - \lambda_2) q(x) y_1 y_2 &= 0. \quad (17.8.41)
\end{align*}

Now we note that

\begin{align*}
(p(x)y_1')'y_2 - (p(x)y_2')'y_1 &= \frac{d}{dx} \left( (p(x)y_1') y_2 - (p(x)y_2') y_1 \right)
\end{align*}

and so integrating \[17.8.41\] from $a$ to $b$, we obtain

\begin{align*}
\left( (p(x)y_1') y_2 - (p(x)y_2') y_1 \right) \bigg|_a^b + (\lambda_1 - \lambda_2) \int_a^b q(x) y_1(x) y_2(x) \, dx &= 0 \quad (17.8.42)
\end{align*}

We have been purposely vague about the nature of the boundary conditions because of a desire to not lose generality. However, we will always assume the boundary conditions are such that whenever $y_1$ and $y_2$ are two eigenfunctions, it follows that

\begin{align*}
\left( (p(x)y_1') y_2 - (p(x)y_2') y_1 \right) \bigg|_a^b &= 0 \quad (17.8.43)
\end{align*}

In the case where the boundary conditions are given by \[17.8.39\] and \[17.8.40\] we obtain \[17.8.43\]. To see why this is so, consider the top limit. This yields

\begin{align*}
p(b) \left[ y_1'(b) y_2(b) - y_2'(b) y_1(b) \right]
\end{align*}

However we know from the boundary conditions that

\begin{align*}
C_3 y_1(b) + C_4 y_1'(b) &= 0 \\
C_3 y_2(b) + C_4 y_2'(b) &= 0
\end{align*}
and that from \(17.8.40\) that not both \(C_3\) and \(C_4\) equal zero. Therefore the determinant of the matrix of coefficients must equal zero. But this implies

\[
[y_1'(b) y_2(b) - y_2'(b) y_1(b)] = 0
\]

which yields the top limit is equal to zero. A similar argument holds for the lower limit. **Note that** \(y_1, y_2\) **satisfy different differential equations because of different eigenvalues.**

From now on the boundary condition will be conditions \(L, \hat{L}, L_y(a), y'(a)) = 0, \hat{L}_y(b), y'(b)) = 0\) which imply that if \(y_i\) correspond to two different eigenvalues, \((p(x)y_1'y_2 - (p(x)y_2'y_1))|_a^b = 0\) \((*)\)

and if \(\alpha\) is a constant, if \(L_y(a), y'(a)) = 0, \hat{L}_y(b), y'(b)) = 0\), then also

\(L(\alpha y(a), \alpha y'(a)) = 0, \hat{L}(\alpha y(b), \alpha y'(b)) = 0\)

For example, maybe one wants to say that \(y\) is bounded at \(a, b\).

With the identity \(17.8.42\) here is a result on orthogonality of the eigenfunctions.

**Proposition 17.8.1** Suppose \(y_i\) solves the boundary conditions and the differential equation for \(\lambda = \lambda_i\) where \(\lambda_1 \neq \lambda_2\). Then we have the orthogonality relation

\[
\int_a^b q(x) y_1(x) y_2(x) \, dx = 0. \quad (17.8.44)
\]

In addition to this, if \(u, v\) are two solutions to the differential equation corresponding to a single \(\lambda\), not necessarily the boundary conditions, \(\text{same differential equation} \) then there exists a constant, \(C\) such that

\[
W(u, v)(x) p(x) = C \quad (17.8.45)
\]

for all \(x \in [a, b]\). In this formula, \(W(u, v)\) denotes the Wronskian given by

\[
\det\begin{pmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{pmatrix}. \quad (17.8.46)
\]

**Proof:** The orthogonality relation, \(17.8.44\) follows from the fundamental assumption, \(17.8.38\) and \(17.8.42\).

It remains to verify \(17.8.45\). We have from \(17.8.43\)

\[
0 = (\lambda - \lambda) q(x) uv + (p(x) u')' v - (p(x) v')' u
= \frac{d}{dx} (p(x) u' v - p(x) v' u) = \frac{d}{dx} (p(x) W(v, u)(x))
\]

and so \(p(x) W(u, v)(x) = -p(x) W(v, u)(x) = C\) as claimed. ■

Now consider the differential equation,

\[
(p(x)y')' + r(x)y = 0. \quad (17.8.47)
\]

This is obtained from the one of interest by letting \(\lambda = 0\).
17.8. STURM LIOUVILLE PROBLEMS

Criterion 17.8.2 Suppose we are able to find functions, $u$ and $v$ such that they solve the differential equation, \[ 17.8.47 \] and $u$ solves the boundary condition at $x = a$ while $v$ solves the boundary condition at $x = b$. Assume both are in $L^2(a,b)$ and $W(u,v) \neq 0$. It follows that both are in $L^2(a,b,q)$, the $L^2$ functions with respect to the measure $q(x) \, dx$. Thus

\[
(f,g)_{L^2(a,b,q)} = \int_a^b f(x) g(x) q(x) \, dx
\]

If $p(x) > 0$ on $[a,b]$ it is typically clear from the fundamental existence and uniqueness theorems for ordinary differential equations that such functions $u$ and $v$ exist. (See any good differential equations book or Problem 10 on Page 26.)

However, such functions might exist even if $p$ vanishes at the end points.

Lemma 17.8.3 Assume Criterion 17.8.2. A function $y$ is a solution to the boundary conditions along with the equation,

\[
(p(x)y')' + r(x)y = g
\]  \hfill (17.8.48)

if

\[
y(x) = \int_a^b G(t,x) g(t) \, dt
\]  \hfill (17.8.49)

where

\[
G(t,x) = \begin{cases} 
  c^{-1}(v(x)u(t)) & \text{if } t < x \\
  c^{-1}(v(t)u(x)) & \text{if } t > x
\end{cases}
\]  \hfill (17.8.50)

where $c$ is the constant of Proposition 17.8.1 which satisfies $p(x)W(u,v)(x) = c$.

Proof: Why does $y$ solve the equation 17.8.48 along with the boundary conditions?

\[
y(x) = \frac{1}{c} \int_a^x g(t) u(t) v(x) \, dt + \frac{1}{c} \int_x^b g(t) v(t) u(x) \, dt
\]

Differentiate

\[
y'(x) = \frac{1}{c} g(x) u(x) v(x) + \frac{1}{c} \int_a^x g(t) u(t) v'(x) \, dt
\]

\[
- \frac{1}{c} g(x) v(x) u(x) + \frac{1}{c} \int_x^b g(t) v(t) u'(x) \, dt
\]

\[
= \frac{1}{c} \int_a^x g(t) u(t) v'(x) \, dt + \frac{1}{c} \int_x^b g(t) v(t) u'(x) \, dt
\]

Then

\[
p(x)y'(x) = \frac{1}{c} \int_a^x g(t) u(t) p(x) v'(x) \, dt + \frac{1}{c} \int_x^b g(t) v(t) p(x) u'(x) \, dt
\]
Then 
\[ (p(x)y'(x))' = \]
\[ \frac{1}{c} g(x)p(x)u(x)v'(x) - \frac{1}{c} g(x)p(x)v(x)u'(x) \]
\[ + \frac{1}{c} \int_a^x g(t)u(t)(p(x)v'(x))' \, dt + \frac{1}{c} \int_x^b g(t)v(t)(p(x)u'(x))' \, dt \]

From the definition of \( c \), this equals
\[ = g(x) + \frac{1}{c} \int_a^x g(t)u(t)(p(x)v'(x))' \, dt + \frac{1}{c} \int_x^b g(t)v(t)(p(x)u'(x))' \, dt \]
\[ = g(x) + \frac{1}{c} \int_a^x g(t)u(t)(-r(x)v(x)) \, dt + \frac{1}{c} \int_x^b g(t)v(t)(-r(x)u(x)) \, dt \]
\[ = g(x) - r(x) \left( \frac{1}{c} \int_a^x g(t)u(t)v(x) \, dt + \frac{1}{c} \int_x^b g(t)v(t)u(x) \, dt \right) \]
\[ = g(x) - r(x)y(x) \]

Thus
\[ (p(x)y'(x))' + r(x)y(x) = g(x) \]

so \( y \) satisfies the equation. As to the boundary conditions, by assumption,
\[ \hat{L}(y(b), y'(b)) = \hat{L}(v(b) \frac{1}{c} \int_a^b g(t)u(t) \, dt, v'(b) \frac{1}{c} \int_a^b g(t)u(t) \, dt) = 0 \]

because \( v \) satisfies the boundary condition at \( b \). The other boundary condition is exactly similar.

Now in the case of Criterion 17.8.2, \( y \) is a solution to the Sturm Liouville eigenvalue problem, if and only if \( y \) solves the boundary conditions and the equation,
\[ (p(x)y'(x))' + r(x)y(x) = -\lambda q(x)y(x). \]

This happens if
\[ y(x) = -\frac{\lambda}{c} \int_a^x q(t)y(t)u(t)v(x) \, dt \]
\[ + \frac{\lambda}{c} \int_x^b q(t)y(t)v(t)u(x) \, dt, \quad (17.8.51) \]

Letting \( \mu = \frac{1}{\lambda} \), this is of the form
\[ \mu y(x) = \int_a^b G(t,x)q(t)y(t) \, dt \quad (17.8.52) \]

where
\[ G(t,x) = \begin{cases} 
-\mu^{-1}(v(x)u(t)) & \text{if } t < x \\
-\mu^{-1}(v(t)u(x)) & \text{if } t > x 
\end{cases} \quad (17.8.53) \]
Could \( \mu = 0 \)? If this happened, then from Lemma 17.8.3, we would have that \( y = 0 \) is a solution of 17.8.48 where the right side is \(-q(t)y(t)\) which would imply that \( q(t)y(t) = 0 \) (since the left side is 0) for all \( t \) which implies \( y(t) = 0 \) for all \( t \) thanks to assumptions on \( q(t) \). Thus we are not interested in this case. It follows from 17.8.53 that \( G : [a, b] \times [a, b] \rightarrow \mathbb{R} \) is continuous and symmetric, \( G(t, x) = G(x, t) \).

\[
G(x, t) = \begin{cases} 
-c^{-1}(v(t)u(x)) & \text{if } x < t \\
-c^{-1}(v(x)u(t)) & \text{if } x > t 
\end{cases} = G(t, x)
\]

Also we see that for \( f \in C([a, b]) \), and

\[
w(x) = \int_a^b G(t, x) q(t) f(t) \, dt,
\]

Lemma 17.8.3 implies \( w \) is a solution to the boundary conditions and the equation

\[
(p(x)y')' + r(x)y = -q(x)f(x) \quad (17.8.54)
\]

**Theorem 17.8.4** Suppose \( u, v \) are given in Criterion 17.8.2. Then there exists a sequence of functions, \( \{y_n\}_{n=1}^{\infty} \) and real numbers, \( \lambda_n \) such that

\[
(p(x)y_n')' + (\lambda_nq(x) + r(x))y_n = 0, \quad x \in [a, b],
\]

\[
L(y(a), y'(a)) = 0,
\]

\[
\hat{L}(y(b), y'(b)) = 0. \quad (17.8.56)
\]

and

\[
\lim_{n \to \infty} |\lambda_n| = \infty \quad (17.8.57)
\]

such that for all \( f \in C([a, b]) \), whenever \( w \) satisfies 17.8.54 and the boundary conditions,

\[
w(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (f, y_n) y_n. \quad (17.8.58)
\]

Also the functions, \( \{y_n\} \) form a dense set in \( L^2(a, b, q) \) which satisfy the orthogonality condition, 17.8.44.

**Proof:** Let \( Ay(x) = \int_a^b G(t, x) q(t) y(t) \, dt \) where \( G \) is defined above in 17.8.53. Then from symmetry and Fubini’s theorem,

\[
(Ay, z)_{L^2(a, b, q)} =
\]
\[
\int_a^b \int_a^b G(t, x) y(t) z(x) q(x) q(t) \, dt \, dx = \int_a^b \int_a^b G(x, t) y(x) z(t) q(t) q(x) \, dx \, dt
\]

This shows that \( A \) is self adjoint. For \( y \in L^2(a, b, q) \),

\[
Ay(x) = \int_a^x (-c^{-1}(v(t)u(x))) y(t) q(t) \, dt + \int_x^b (-c^{-1}(v(x)u(t))) q(t) y(t) \, dt
\]

If you have \( y_n \to y \) weakly in \( L^2(a, b, q) \), then it is clear that \( Ay_n \to Ay \) for each \( x \), this from the above formula. Consider now \( \|Ay_n - Ay\|_{L^2(a,b,q)} \). Look at the first term in the above. Is it true that the following converges to 0?

\[
\int_a^b \left| \int_a^x (-c^{-1}(v(t)u(x))) (y_n(t) - y(t)) q(t) \, dt \right|^2 q(x) \, dx \quad (**)
\]

We know that the integrand converges to 0 for each \( x \). Is there a dominating function? If so, then the dominated convergence theorem gives the result.

\[
\left| \int_a^x (-c^{-1}(v(t)u(x))) q(t) (y_n(t) - y(t)) \, dt \right| \leq |u(x)| C \left( \int_a^x |v(t)(y_n(t) - y(t))| \, dt \right)
\]

\[
\leq |u(x)| C \|v\|_{L^2(a,b,q)} \|y_n - y\|_{L^2(a,b,q)}
\]

The last factor is uniformly bounded due to the weak convergence of \( y_n \) to \( y \). Therefore, there is a constant \( C \) such that the integrand is bounded by \( |u(x)|^2 C \).

Hence the dominated convergence theorem applies and we can conclude that ** converges to 0. Thus \( A \) is a compact, self adjoint operator on \( L^2(a, b, q) \).

Therefore, by Theorem 17.8.5, there exist functions \( y_n \) and real constants, \( \mu_n \) such that \( \|y_n\| = 1 \) and \( Ay_n = \mu_n y_n \) and

\[
|\mu_n| \geq |\mu_{n+1}|, \quad Au_i = \mu_i u_i, \quad (17.8.59)
\]

and either

\[
\lim_{n \to \infty} \mu_n = 0, \quad (17.8.60)
\]

or for some \( n \),

\[
\text{span} (y_1, \cdots, y_n) = H \equiv L^2(a, b, q). \quad (17.8.61)
\]

Of course, \( H \) is not finite dimensional and so the second will not hold. Also from Theorem 17.8.5, \( \text{span} (\{y_i\}_{i=1}^\infty) \) is dense in \( A(H) \).
and so for all \( f \in C([a,b]) \),

\[
Af = \sum_{k=1}^{\infty} \mu_k (f, y_k) y_k. \tag{17.8.63}
\]

Thus for \( w \) a solution of \( 17.8.54 \) and suitable boundary conditions as above which cause \(*\),

\[
w \equiv Af = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (f, y_k) y_k.
\]

The last claim follows from Corollary \( 17.7.4 \) and the observation above that \( \mu \) is never equal to zero. \( \square \)

Note that if \( q(x) \neq 0 \) we can say that for a given \( g \in C([a,b]) \), one can define \( f \) by \( g(x) = -q(x) f(x) \) and so if \( w \) is a solution to the boundary conditions and the equation

\[
(p(x) w'(x))' + r(x) w(x) = g(x) = -q(x) f(x),
\]

one obtains the formula

\[
w(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (f, y_k) y_k
= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left( \frac{-g}{q}, y_k \right) y_k.
\]

More can be said about convergence of these series based on the eigenfunctions of a Sturm Liouville problem. In particular, it can be shown that for reasonable functions the pointwise convergence properties are like those of Fourier series and that the series converges to the midpoint of the jump. This is partly done for the Legendre polynomials in \([27]\). For more on these topics see the old book by Ince, written in Egypt in the 1920’s, \([65]\), \([66]\) or the 1955 book on differential equations by Coddington and Levinson \([30]\).

As an example, consider the following eigenvalue problem

\[
x^2 y'' + xy' + (\lambda x^2 - n^2) y = 0, \quad C_1 y(L) + C_2 y'(L) = 0, \quad x \in [0, L] \quad (*)
\]

not both \( C_i \) equal zero. Then you can write the equation in “self adjoint” form as

\[
(xy')' + \left( \lambda x - \frac{n^2}{x} \right) y = 0
\]

Multiply by \( y \) and integrate from 0 to \( L \). Then the boundary terms cancel and you get

\[
\int_0^L \left( \lambda x - \frac{n^2}{x} \right) y^2 dx = 0
\]

and so you must have \( \lambda > 0 \).
Now it follows that corresponding to different values of $\lambda$ the eigenfunctions are orthogonal with respect to $x$. So what are the values of $\lambda$ and how can we describe the corresponding eigenfunctions?

Let $y$ be an eigenfunction. Let $z\left(\sqrt{\lambda}x\right) = y\left(x\right)$. Then

$$0 = x^2 y'' + xy' + \left(\lambda x^2 - n^2\right) y$$

$$= \lambda x^2 z'' \left(\sqrt{\lambda}x\right) + \sqrt{\lambda}x z' \left(\sqrt{\lambda}x\right) + \left(\lambda x^2 - n^2\right) z \left(\sqrt{\lambda}x\right)$$

Now replace $\sqrt{\lambda}x$ with $u$. Then

$$u^2 z'' \left(u\right) + uz' \left(u\right) + \left(u^2 - n^2\right) z \left(u\right) = 0$$

Then we need

$$z \left(\sqrt{\lambda}L\right) = 0$$

and $z$ is bounded near 0. This happens if and only if $z \left(u\right) = J_n \left(u\right)$ because the other solution to the Bessel equation is unbounded near 0. Then $J_n \left(\sqrt{\lambda}L\right) = 0$ and so for some $\alpha$ a zero of $J_n$,

$$\sqrt{\lambda}L = \alpha, \quad \lambda = \frac{\alpha^2}{L^2}.$$ 

Thus the eigenvalues are

$$\frac{\alpha^2}{L^2}, \alpha \text{ a zero of } J_n \left(x\right)$$

and the eigenfunctions are

$$x \rightarrow J_n \left(\frac{\alpha}{L}x\right).$$

Then Theorem 17.8.4 implies that if you have any $f \in L^2 \left(a,b,x\right)$, you can obtain it as an expansion in terms of the functions $x \rightarrow J_n \left(\frac{\alpha_k}{L}x\right)$ where $\alpha_k$ are the zeros of the Bessel function. Note that this theorem and what was shown above also shows that there are countably many zeros of $J_n$ also.

17.8.1 Nuclear Operators

Definition 17.8.5 A self adjoint operator $A \in \mathcal{L} \left(H,H\right)$ for $H$ a separable Hilbert space is called a nuclear operator if for some complete orthonormal set, $\{e_k\}$,

$$\sum_{k=1}^{\infty} |\langle Ae_k, e_k\rangle| < \infty$$

To begin with here is an interesting lemma.
**Lemma 17.8.6** Suppose \( \{A_n\} \) is a sequence of compact operators in \( \mathcal{L}(X,Y) \) for two Banach spaces, \( X \) and \( Y \) and suppose \( A \in \mathcal{L}(X,Y) \) and

\[
\lim_{n \to \infty} ||A - A_n|| = 0.
\]

Then \( A \) is also compact.

**Proof:** Let \( B \) be a bounded set in \( X \) such that \( ||b|| \leq C \) for all \( b \in B \). I need to verify \( AB \) is totally bounded. Suppose then it is not. Then there exists \( \varepsilon > 0 \) and a sequence, \( \{Ab_i\} \) where \( b_i \in B \) and

\[
||Ab_i - Ab_j|| \geq \varepsilon
\]

whenever \( i \neq j \). Then let \( n \) be large enough that

\[
||A - A_n|| \leq \frac{\varepsilon}{4C}.
\]

Then

\[
||A_n b_i - A_n b_j|| = ||Ab_i - Ab_j + (A_n - A) b_i - (A_n - A) b_j||
\]

\[
\geq ||Ab_i - Ab_j|| - ||(A_n - A) b_i|| - ||(A_n - A) b_j||
\]

\[
\geq ||Ab_i - Ab_j|| - \frac{\varepsilon}{4C}C - \frac{\varepsilon}{4C}C \geq \frac{\varepsilon}{2},
\]

a contradiction to \( A_n \) being compact. This proves the lemma.

Then one can prove the following lemma. In this lemma, \( A \geq 0 \) will mean \( (Ax,x) \geq 0 \).

**Lemma 17.8.7** Let \( A \geq 0 \) be a nuclear operator defined on a separable Hilbert space, \( H \). Then \( A \) is compact and also, whenever \( \{e_k\} \) is a complete orthonormal set,

\[
A = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (A e_i, e_j) e_i \otimes e_j.
\]

**Proof:** First consider the formula. Since \( A \) is given to be continuous,

\[
Ax = A \left( \sum_{j=1}^{\infty} (x, e_j) e_j \right) = \sum_{j=1}^{\infty} (x, e_j) Ae_j,
\]

the series converging because

\[
x = \sum_{j=1}^{\infty} (x, e_j) e_j
\]
Then also since $A$ is self adjoint,

$$
\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (A e_i, e_j) e_i \otimes e_j (x) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (A e_i, e_j) e_i
$$

$$
= \sum_{j=1}^{\infty} (x, e_j) \sum_{i=1}^{\infty} (A e_i, e_j) e_i
$$

$$
= \sum_{j=1}^{\infty} (x, e_j) \sum_{i=1}^{\infty} (A e_i, e_j) e_i
$$

$$
= \sum_{j=1}^{\infty} (x, e_j) A e_j
$$

Next consider the claim that $A$ is compact. Let $C_A \equiv \left( \sum_{j=1}^{\infty} |(A e_j, e_j)| \right)^{1/2}$. Let $A_n$ be defined by

$$
A_n \equiv \sum_{j=1}^{\infty} \sum_{i=1}^{n} (A e_i, e_j) (e_i \otimes e_j).
$$

Then $A_n$ has values in span $(e_1, \cdots, e_n)$ and so it must be a compact operator because bounded sets in a finite dimensional space must be precompact. Then

$$
|(Ax - A_n x, y)| = \left| \sum_{j=1}^{\infty} \sum_{i=n+1}^{\infty} (A e_i e_j) (y, e_j) (e_i, x) \right|
$$

$$
= \left| \sum_{j=1}^{\infty} (y, e_j) \sum_{i=n+1}^{\infty} (A e_i e_j) (e_i, x) \right|
$$

$$
\leq \sum_{j=1}^{\infty} (y, e_j) |(A e_j, e_j)|^{1/2} \sum_{i=n+1}^{\infty} (A e_i e_i)^{1/2} |(e_i, x)|
$$

$$
\leq \left( \sum_{j=1}^{\infty} (y, e_j)^2 \right)^{1/2} \left( \sum_{j=1}^{\infty} |(A e_j, e_j)| \right)^{1/2}
$$

$$
\cdot \left( \sum_{i=n+1}^{\infty} |(x, e_i)|^2 \right)^{1/2} \left( \sum_{i=n+1}^{\infty} |(A e_i, e_i)| \right)^{1/2}
$$

$$
\leq |y| |x| C_A \left( \sum_{i=n+1}^{\infty} |(A e_i, e_i)| \right)^{1/2}.$$
and this shows that if \( n \) is sufficiently large,
\[
|(A - A_n)x, y| \leq \varepsilon |x||y|.
\]

Therefore,
\[
\lim_{n \to \infty} ||A - A_n|| = 0
\]
and so \( A \) is the limit in operator norm of finite rank bounded linear operators, each of which is compact. Therefore, \( A \) is also compact.

**Definition 17.8.8** The trace of a nuclear operator \( A \in \mathcal{L}(H, H) \) such that \( A \geq 0 \) is defined to equal
\[
\sum_{k=1}^{\infty} (Ae_k, e_k)
\]
where \( \{e_k\} \) is an orthonormal basis for the Hilbert space, \( H \).

**Theorem 17.8.9** Definition 17.8.8 is well defined and equals \( \sum_{j=1}^{\infty} \lambda_j \) where the \( \lambda_j \) are the eigenvalues of \( A \).

**Proof:** Suppose \( \{u_k\} \) is some other orthonormal basis. Then
\[
e_k = \sum_{j=1}^{\infty} u_j (e_k, u_j)
\]
By Lemma 17.8.7 \( A \) is compact and so
\[
A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k
\]
where the \( u_k \) are the orthonormal eigenvectors of \( A \) which form a complete orthonormal set. Then
\[
\sum_{k=1}^{\infty} (Ae_k, e_k) = \sum_{k=1}^{\infty} \left( A \left( \sum_{j=1}^{\infty} u_j (e_k, u_j) \right) , \sum_{j=1}^{\infty} u_j (e_k, u_j) \right)
\]
\[
= \sum_{k=1}^{\infty} \sum_{ij} (A_{ij}) (e_k, u_i) (u_j, e_k)
\]
\[
= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (A_{ij}) |(e_k, u_j)|^2
\]
\[
= \sum_{j=1}^{\infty} (A_{ij}) \sum_{k=1}^{\infty} |(e_k, u_j)|^2 = \sum_{j=1}^{\infty} (A_{ij}) |u_j|^2
\]
\[
= \sum_{j=1}^{\infty} (A_{ij}) = \sum_{j=1}^{\infty} \lambda_j
\]
and this proves the theorem.

This is just like it is for a matrix. Recall the trace of a matrix is the sum of the
eigenvalues.

It is also easy to see that in any separable Hilbert space, there exist nuclear
operators. Let \( \sum_{k=1}^{\infty} |\lambda_k| < \infty \). Then let \( \{e_k\} \) be a complete orthonormal set of
vectors. Let
\[
A \equiv \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k.
\]
It is not too hard to verify this works.

Much more can be said about nuclear operators.

17.8.2 Hilbert Schmidt Operators

**Definition 17.8.10** Let \( H \) and \( G \) be two separable Hilbert spaces and let \( T \) map
\( H \) to \( G \) be linear. Then \( T \) is called a Hilbert Schmidt operator if there exists some
orthonormal basis for \( H \), \( \{e_j\} \) such that
\[
\sum_j ||Te_j||^2 < \infty.
\]
The collection of all such linear maps will be denoted by \( \mathcal{L}_2(H,G) \).

**Theorem 17.8.11** \( \mathcal{L}_2(H,G) \subseteq \mathcal{L}(H,G) \) and \( \mathcal{L}_2(H,G) \) is a separable Hilbert space
with norm given by
\[
||T||_{\mathcal{L}_2} \equiv \left( \sum_k ||Te_k||^2 \right)^{1/2}
\]
where \( \{e_k\} \) is some orthonormal basis for \( H \). Also \( \mathcal{L}_2(H,G) \subseteq \mathcal{L}(H,G) \) and
\[
||T|| \leq ||T||_{\mathcal{L}_2}.
\]
(17.8.64)

All Hilbert Schmidt operators are compact. Also for \( X \in H \) and \( Y \in G \), \( X \otimes Y \in \mathcal{L}_2(H,G) \) and
\[
||X \otimes Y||_{\mathcal{L}_2} = ||X||_H ||Y||_G
\]
(17.8.65)

**Proof:** First I want to show \( \mathcal{L}_2(H,G) \subseteq \mathcal{L}(H,G) \) and \( ||T|| \leq ||T||_{\mathcal{L}_2} \). Pick an
orthonormal basis for \( H \), \( \{e_k\} \) and an orthonormal basis for \( G \), \( \{f_k\} \). Then letting
\[
x = \sum_{k=1}^{n} x_k e_k,
\]
\[
Tx = T \left( \sum_{k=1}^{n} x_k e_k \right) = \sum_{k=1}^{n} x_k T(e_k)
\]
where \( x_k \equiv (x, e_k) \). Therefore using Minkowski’s inequality,

\[
||Tx|| = \left( \sum_{k=1}^{\infty} |(Tx, f_k)|^2 \right)^{1/2}
\]

\[
= \left( \sum_{k=1}^{\infty} \left( \sum_{j=1}^{n} x_j Te_j, f_k \right)^2 \right)^{1/2}
\]

\[
= \left( \sum_{k=1}^{\infty} \sum_{j=1}^{n} (x_j Te_j, e_k)^2 \right)^{1/2}
\]

\[
\leq \sum_{j=1}^{n} \left( \sum_{k=1}^{\infty} |(x_j Te_j, e_k)|^2 \right)^{1/2}
\]

\[
\leq \sum_{j=1}^{n} |x_j| \left( \sum_{k=1}^{\infty} |(Te_j, e_k)|^2 \right)^{1/2}
\]

\[
= \sum_{j=1}^{n} |x_j| ||Te_j|| \leq \left( \sum_{j=1}^{n} |x_j|^2 \right)^{1/2} ||T||_{L^2}
\]

\[
= ||x|| ||T||_{L^2}
\]

Therefore, since finite sums of the form \( \sum_{k=1}^{n} x_k e_k \) are dense in \( H \), it follows \( T \in \mathcal{L}(H, G) \) and \( ||T|| \leq ||T||_{L^2} \).

Next consider the norm. I need to verify the norm does not depend on the choice of orthonormal basis. Let \( \{f_k\} \) be an orthonormal basis for \( G \). Then for \( \{e_k\} \) an orthonormal basis for \( H \),

\[
\sum_{k} ||Te_k||^2 = \sum_{k} \sum_{j} |(Te_k, f_j)|^2 = \sum_{k} \sum_{j} |(e_k, T^* f_j)|^2
\]

\[
= \sum_{j} \sum_{k} |(e_k, T^* f_j)|^2 = \sum_{j} ||T^* f_j||^2.
\]

The above computation makes sense because it was just shown that \( T \) is continuous. The same result would be obtained for any other orthonormal basis \( \{e'_k\} \) and this shows the norm is at least well defined. It is clear this does indeed satisfy the axioms of a norm and this proves the above claims.

It only remains to verify \( L^2(H, G) \) is a separable Hilbert space. It is clear it is an inner product space because you only have to pick an orthonormal basis, \( \{e_k\} \) and define the inner product as

\[
(S, T) \equiv \sum_{k} (Se_k, Te_k).
\]
The only remaining issue is the completeness. Suppose then that \( \{ T_n \} \) is a Cauchy sequence in \( L_2(H, G) \). Then from Theorem 17.8.64 \( \{ T_n \} \) is a Cauchy sequence in \( L(H, G) \) and so there exists a unique \( T \) such that \( \lim_{n \to \infty} ||T_n - T|| = 0 \). Then it only remains to verify \( T \in L_2(H, G) \). But by Fatou’s lemma,

\[
\sum_k ||T e_k||^2 \leq \lim \inf_{n \to \infty} \sum_k ||T_n e_k||^2 = \lim \inf_{n \to \infty} ||T_n||^2_{L_2} < \infty.
\]

All that remains is to verify \( L_2(H, G) \) is separable and these Hilbert Schmidt operators are compact. I will show an orthonormal basis for \( L_2(H, G) \) is \( \{ f_j \otimes e_k \} \) where \( \{ f_k \} \) is an orthonormal basis for \( G \) and \( \{ e_k \} \) is an orthonormal basis for \( H \). Here, for \( f \in G \) and \( e \in H \),

\[
f \otimes e(x) \equiv (x, e) f.
\]

I need to show \( f_j \otimes e_k \in L_2(H, G) \) and that it is an orthonormal basis for \( L_2(H, G) \) as claimed.

\[
\sum_k ||f_j \otimes e_i (e_k)||^2 = \sum_k ||f_j \delta_{i k}||^2 = ||f_j||^2 = 1 < \infty
\]

so each of these operators is in \( L_2(H, G) \). Next I show they are orthonormal.

\[
(f_j \otimes e_k, f_s \otimes e_r) = \sum_p (f_j \otimes e_k (e_p), f_s \otimes e_r (e_p)) = \sum_p \delta_{_{rp}} \delta_{kp} (f_j, f_s) = \sum_p \delta_{_{rp}} \delta_{kp} \delta_{js}
\]

If \( j = s \) and \( k = r \) this reduces to 1. Otherwise, this gives 0. Thus these operators are orthonormal. Now let \( T \in L_2(H, G) \). Consider

\[
T_n = \sum_{i=1}^n \sum_{j=1}^n (T e_i, f_j) f_j \otimes e_i
\]

Then

\[
T_n e_k = \sum_{i=1}^n \sum_{j=1}^n (T e_i, f_j) (e_k, e_i) f_j
\]

\[
= \sum_{j=1}^n (T e_k, f_j) f_j
\]

It follows

\[
||T_n e_k|| \leq ||T e_k||
\]
and
\[ \lim_{n \to \infty} T_n e_k = T e_k. \]
Therefore, from the dominated convergence theorem,
\[ \lim_{n \to \infty} \|T - T_n\|_{L_2}^2 = \lim_{n \to \infty} \sum_k \|(T - T_n) e_k\|^2 = 0. \]
Therefore, the linear combinations of the \( f_j \otimes e_i \) are dense in \( L_2(H,G) \) and this proves completeness of the orthonormal basis.

This also shows \( L_2(H,G) \) is separable. From 17.8.64 it also shows that every \( T \in L_2(H,G) \) is the limit in the operator norm of a sequence of compact operators. This follows because each of the \( f_j \otimes e_i \) is easily seen to be a compact operator because if \( x \) converges weakly, then
\[ f_j \otimes e_i (x_m) = (x_m, e_i) f_j \rightarrow (x, e_i) f_j = f_j \otimes e_i (x) \]
and since if \( \{x_m\} \) is any bounded sequence, there exists a subsequence, \( \{x_n_k\} \) which converges weakly and by the above, \( f_j \otimes e_i (x_n_k) \rightarrow f_j \otimes e_i (x) \) showing bounded sets are mapped to precompact sets. Therefore, each \( T \in L_2(H,G) \) must also be a compact operator. Here is why.

Let \( B \) be a bounded set in which \( ||x|| < M \) for all \( x \in B \) and consider \( TB \). I need to show \( TB \) is totally bounded. Let \( \varepsilon > 0 \) be given. Then let \( ||T_m - T|| < \frac{\varepsilon}{3M} \) where \( T_m \) is a compact operator like those described above and let \( \{T_m x_j\}_{j=1}^N \) be an \( \varepsilon/3 \) net for \( T_m (B) \). Then
\[ ||Tx_j - T_m x_j|| < \frac{\varepsilon}{3} \]
and so letting \( x \in B \), pick \( x_j \) such that \( ||T_m x - T_m x_j|| < \varepsilon/3 \). Then
\[
||Tx - Tx_j|| \leq ||Tx - T_m x|| + ||T_m x - T_m x_j|| + ||T_m x_j - Tx_j||
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]
showing \( \{Tx_j\}_{j=1}^N \) is an \( \varepsilon \) net for \( TB \).

Finally, consider 17.8.63. Let \( \{e_k\} \) be an orthonormal basis for \( H \) and consider the following computation which establishes this equation.
\[
||Y \otimes X||_{L_2}^2 = \sum_{k=1}^\infty ||Y \otimes X (e_k)||^2 = \sum_{k=1}^\infty ||(e_k, X) Y||^2 = ||Y||_G^2 \sum_{k=1}^{\infty} |(e_k, X)|^2 = ||Y||_G^2 ||X||_H^2 < \infty. \quad (17.8.66)
\]
This proves the theorem.
17.9 Compact Operators In Banach Space

In general for \( A \in \mathcal{L}(X,Y) \) the following definition holds.

**Definition 17.9.1** Let \( A \in \mathcal{L}(X,Y) \). Then \( A \) is compact if whenever \( B \subseteq X \) is a bounded set, \( AB \) is precompact. Equivalently, if \( \{x_n\} \) is a bounded sequence in \( X \), then \( \{Ax_n\} \) has a subsequence which converges in \( Y \).

An important result is the following theorem about the adjoint of a compact operator.

**Theorem 17.9.2** Let \( A \in \mathcal{L}(X,Y) \) be compact. Then the adjoint operator, \( A^* \in \mathcal{L}(Y',X') \) is also compact.

**Proof:** Let \( \{y^*_n\} \) be a bounded sequence in \( Y' \). Let \( B \) be the closure of the unit ball in \( X \). Then \( AB \) is precompact. Then it is clear that the functions \( \{y^*_n\} \) are equicontinuous and uniformly bounded on the compact set, \( A(B) \). By the Ascoli Arzela theorem, there is a subsequence \( \{y^*_{nk}\} \) which converges uniformly to a continuous function, \( f \) on \( A(B) \). Now define \( g \) on \( AX \) by

\[
g(Ax) = \|x\| f \left( A \left( \frac{x}{\|x\|} \right) \right), g(A0) = 0.
\]

Thus for \( x_1, x_2 \neq 0 \), and \( a, b \) scalars,

\[
g(aAx_1 + bAx_2) = \|ax_1 + bx_2\| f \left( A \frac{ax_1 + bx_2}{\|ax_1 + bx_2\|} \right)
\]

\[
= \lim_{k \to \infty} \|ax_1 + bx_2\| y^*_n \left( A \frac{ax_1 + bx_2}{\|ax_1 + bx_2\|} \right)
\]

\[
= \lim_{k \to \infty} ay^*_{nk}(Ax_1) + by^*_{nk}(Ax_2)
\]

\[
= a \lim_{k \to \infty} \|x_1\| y^*_{nk} \left( A \frac{x_1}{\|x_1\|} \right) + b \lim_{k \to \infty} \|x_2\| y^*_{nk} \left( A \frac{x_2}{\|x_2\|} \right)
\]

\[
= a \|x_1\| f \left( A \frac{x_1}{\|x_1\|} \right) + b \|x_2\| f \left( A \frac{x_2}{\|x_2\|} \right)
\]

showing that \( g \) is linear on \( AX \). Also

\[
|g(Ax)| = \lim_{k \to \infty} \|x\| y^*_{nk} \left( A \frac{x}{\|x\|} \right) \leq C \|x\| \left\| A \left( \frac{x}{\|x\|} \right) \right\| = C \|Ax\|
\]

and so by the Hahn Banach theorem, there exists \( y^* \) extending \( g \) to all of \( Y \) having the same operator norm.

\[
y^*(Ax) = \lim_{k \to \infty} \|x\| y^*_{nk} \left( A \frac{x}{\|x\|} \right) = \lim_{k \to \infty} y^*_{nk}(Ax)
\]
Thus \( A^* y_n^* (x) \rightarrow A^* y^* (x) \) for every \( x \). In addition to this, for \( x \in B \),
\[
\left\| A^* y^* (x) - A^* y_n^* (x) \right\| = \left\| y^* (Ax) - y_n^* (Ax) \right\|
= \left\| g (Ax) - y_n^* (Ax) \right\|
= \left\| \left\| x \right\| f \left( A \left( \frac{x}{\left\| x \right\|} \right) \right) \right\| - \left\| \left\| x \right\| y_n^* \left( \frac{Ax}{\left\| x \right\|} \right) \right\|
\leq \left\| f \left( A \left( \frac{x}{\left\| x \right\|} \right) \right) - y_n^* \left( \frac{Ax}{\left\| x \right\|} \right) \right\|
\]
and this is uniformly small for large \( k \) due to the uniform convergence of \( y_n^* \) to \( f \) on \( A(B) \). Therefore, \( \left\| A^* y^* - A^* y_n^* \right\| \rightarrow 0 \).

### 17.10 The Fredholm Alternative

Recall that if \( A \) is an \( n \times n \) matrix and if the only solution to the system, \( Ax = 0 \) is \( x = 0 \) then for any \( y \in \mathbb{R}^n \) it follows that there exists a unique solution to the system \( Ax = y \). This holds because the first condition implies \( A \) is one to one and therefore, \( A^{-1} \) exists. Of course things are much harder in a general Banach space. Here is a simple example for a Hilbert space.

**Example 17.10.1** Let \( L^2 (\mathbb{N}; \mu) = H \) where \( \mu \) is counting measure. Thus an element of \( H \) is a sequence, \( a = \{a_i\}_{i=1}^\infty \) having the property that
\[
\left\| a \right\|_H \equiv \left( \sum_{k=1}^\infty |a_k|^2 \right)^{1/2} < \infty.
\]
Define \( A : H \rightarrow H \) by
\[
Aa \equiv b \equiv \{0, a_1, a_2, \cdots \}.
\]
Thus \( A \) slides the sequence to the right and puts a zero in the first slot. Clearly \( A \) is one to one and linear but it cannot be onto because it fails to yield \( e_1 = \{1, 0, 0, \cdots \} \).

Notwithstanding the above example, there are theorems which are like the linear algebra theorem mentioned above which hold in an arbitrary Banach spaces in the case where the operator is compact. To begin with here is an interesting lemma.

**Lemma 17.10.2** Suppose \( A \in \mathcal{L} (X, X) \) is compact for \( X \) a Banach space. Then \( (I - A) (X) \) is a closed subspace of \( X \).

**Proof:** Suppose \((I - A)x_n \rightarrow y\). Let
\[
\alpha_n \equiv \text{dist} \left( x_n, \text{ker} (I - A) \right)
\]
and let \( z_n \in \text{ker} (I - A) \) be such that
\[
\alpha_n \leq \left\| x_n - z_n \right\| \leq \left( 1 + \frac{1}{n} \right) \alpha_n.
\]
Thus \((I - A) (x_n - z_n) \rightarrow y\) because \((I - A) z_n = 0\).

**Case 1:** \(\{x_n - z_n\}\) has a bounded subsequence.

If this is so, the compactness of \(A\) implies there exists a subsequence, still denoted by \(n\) such that \(\{A(x_n - z_n)\}_{n=1}^{\infty}\) is a Cauchy sequence. Since \((I - A) (x_n - z_n) \rightarrow y\), this implies \(\{(x_n - z_n)\}\) is also a Cauchy sequence converging to a point, \(x \in X\).

Then, taking the limit as \(n \rightarrow \infty\), \((I - A) x = y\) and so \(y \in (I - A) (X)\).

**Case 2:** \(\lim_{n \rightarrow \infty} ||x_n - z_n|| = \infty\). I will show this case cannot occur.

In this case, let \(w_n = \frac{x_n - z_n}{||x_n - z_n||}\). Thus \((I - A) w_n \rightarrow 0\) and \(w_n\) is bounded. Therefore, there exists a subsequence, still denoted by \(n\) such that \(\{Aw_n\}\) is a Cauchy sequence. Now it follows

\[
Aw_n - Aw_m + e_n - e_m = w_n - w_m
\]

where \(e_k \rightarrow 0\) as \(k \rightarrow \infty\). This implies \(\{w_n\}\) is a Cauchy sequence which must converge to some \(w_\infty \in X\). Therefore, \((I - A) w_\infty = 0\) and so \(w_\infty \in \ker (I - A)\).

However, this is impossible because of the following argument. If \(z \in \ker (I - A)\),

\[
||w_n - z|| = \frac{1}{||x_n - z_n||} ||x_n - z_n - ||x_n - z_n|| z|| \\
\geq \frac{1}{||x_n - z_n||} \alpha_n \geq \frac{\alpha_n}{(1 + \frac{1}{n}) \alpha_n} = \frac{n}{n + 1}.
\]

Taking the limit, \(||w_\infty - z|| \geq 1\). Since \(z \in \ker (I - A)\) is arbitrary, this shows \(\text{dist} (w_\infty, \ker (I - A)) \geq 1\).

Since Case 2 does not occur, this proves the lemma.

**Theorem 17.10.3** Let \(A \in \mathcal{L} (X, X)\) be a compact operator and let \(f \in X\). Then there exists a solution, \(x\), to

\[
x - Ax = f
\]

if and only if

\[
x^* (f) = 0
\]

for all \(x^* \in \ker (I - A^*)\).

**Proof:** Suppose \(x\) is a solution to \((17.10.67)\) and let \(x^* \in \ker (I - A^*)\). Then

\[
x^* (f) = x^* ((I - A) (x)) = ((I - A^*) x^*) (x) = 0.
\]

Next suppose \(x^* (f) = 0\) for all \(x^* \in \ker (I - A^*)\). I will show there exists \(x\) solving \((17.10.67)\). By Lemma 17.10.2, \((I - A) (X)\) is a closed subspace of \(X\). Is \(f \in (I - A) (X)\)? If not, then by the Hahn Banach theorem, there exists \(x^* \in X'\) such that \(x^* (f) \neq 0\) but \(x^* ((I - A) (x)) = 0\) for all \(x \in X\). However last statement says nothing more nor less than \((I - A^*) x^* = 0\). This is a contradiction because for such \(x^*\), it is given that \(x^* (f) = 0\). This proves the theorem.

The following corollary is called the Fredholm alternative.
Corollary 17.10.4 Let $A \in \mathcal{L}(X, X)$ be a compact operator. Then there exists a solution to the equation
\[ x - Ax = f \] (17.10.69)
for all $f \in X$ if and only if $(I - A^*)$ is one to one on $X'$.

Proof: Suppose $(I - A^*)$ is one to one first. Then if $x^* - A^*x^* = 0$ it follows $x^* = 0$ and so for any $f \in X$, $x^*(f) = 0$ for all $x^* \in \ker(I - A^*)$. By 17.10.3 there exists a solution to $(I - A)x = f$.

Now suppose there exists a solution, $x$, to $(I - A)x = f$ for every $f \in X$. If $(I - A^*)x^* = 0$, then for every $x \in X$,
\[ (I - A^*)x^*(x) = x^*((I - A)(x)) = 0 \]
Since $(I - A)$ is onto, this shows $x^* = 0$ and so $(I - A^*)$ is one to one as claimed. This proves the corollary.

The following is just an easier version of the above.

Corollary 17.10.5 In the case where $X$ is a Hilbert space, the conclusions of Corollary 17.10.4, Theorem 17.10.3, and Lemma 17.10.2 remain true if $H'$ is replaced by $H$ and the adjoint is understood in the usual manner for Hilbert space. That is
\[ (Ax, y)_H = (x, A^*y)_H \]

17.11 Square Roots

In this section, $H$ will be a Hilbert space, real or complex, and $T$ will denote an operator which satisfies the following definition. A useful theorem about the existence of square roots of certain operators is presented. This proof is very elementary. I found it in [2].

Definition 17.11.1 Let $T \in \mathcal{L}(H, H)$ satisfy $T = T^*$ (Hermitian) and for all $x \in H$,
\[ (Tx, x) \geq 0 \] (17.11.70)
Such an operator is referred to as positive and self adjoint. It is probably better to refer to such an operator as “nonnegative” since the possibility that $Tx = 0$ for some $x \neq 0$ is not being excluded. Instead of “self adjoint” you can also use the term, Hermitian. To save on notation, write
\[ T \geq 0 \]
to mean $T$ is positive, satisfying 17.11.70.

With the above definition here is a fundamental result about positive self adjoint operators.
Proposition 17.11.2 Let $S, T$ be positive and self adjoint such that $ST = TS$. Then $ST$ is also positive and self adjoint.

Proof: It is obvious that $ST$ is self adjoint. The only problem is to show that $ST$ is positive. To show this, first suppose $S \leq I$. The idea is to write

$$S = S_{n+1} + \sum_{k=0}^{n} S_k^2$$

where $S_0 = S$ and the operators $S_k$ are self adjoint. This is a useful idea because it is then obvious that the sum is positive. If we want such a representation as above, then it follows that $S_0 \equiv S$ and

$$S_{n+1} \equiv S_n - S_n^2.$$ 

Thus it is obvious that the $S_k$ are all self adjoint. Also, the following claim holds.

Claim: $I \geq S_n \geq 0$.

Proof of the claim: This is true if $n = 0$. Assume true for $n$. Then from the definition,

$$S_{n+1} = S_n^2 (I - S_n) + (I - S_n)^2 S_n$$

and it is obvious from the definition that the sum of positive operators is positive. Therefore, it suffices to show the two terms in the above are both positive. It is clear from the definition that each $S_n$ is Hermitian (self adjoint) because they are just polynomials in $S$. Also each must commute with $T$ for the same reason. Therefore,

$$(S_n^2 (I - S_n) x, x) = ((I - S_n) S_n x, S_n x) \geq 0$$

and also

$$((I - S_n)^2 S_n x, x) = (S_n (I - S_n) x, (I - S_n) x) \geq 0$$

This proves the claim.

Now each $S_k$ commutes with $T$ because this is true of $S_0$ and succeeding $S_k$ are polynomials in terms of $S_0$. Therefore,

$$(ST x, x) = \left( S_{n+1} + \sum_{k=0}^{n} S_k^2 \right) T x, x = (S_{n+1} T x, x) + \sum_{k=0}^{n} (S_k^2 T x, x) = (T x, S_{n+1} x) + \sum_{k=0}^{n} (T S_k x, S_k x)$$

(17.11.71)

Consider $S_{n+1} x$. From the claim,

$$(S x, x) = (S_{n+1} x, x) + \sum_{k=0}^{n} |S_k x|^2 \geq \sum_{k=0}^{n} |S_k x|^2$$
and so \( \lim_{n \to \infty} S_n x = 0 \). Hence from \( \text{Proposition 17.11.2} \),

\[
\lim \inf_{n \to \infty} (ST x, x) = (ST x, x) = \lim \inf_{n \to \infty} \sum_{k=0}^{n} (TS_k x, S_k x) \geq 0.
\]

All this was based on the assumption that \( S \leq I \). The next task is to remove this assumption. Let \( ST = TS \) where \( T \) and \( S \) are positive self adjoint operators. Then consider \( S/\|S\| \). This is still a positive self adjoint operator and it commutes with \( T \) just like \( S \) does. Therefore, from the first part, \( 0 \leq \left( \frac{S}{\|S\|} T x, x \right) = \frac{1}{\|S\|} (ST x, x) \).

The proposition is like the familiar statement about real numbers which says that when you multiply two nonnegative real numbers the result is a nonnegative real number. The next lemma is a generalization of the familiar fact that if you have an increasing sequence of real numbers which is bounded above, then the sequence converges.

**Lemma 17.11.3** Let \( \{T_n\} \) be a sequence of self adjoint operators on a Hilbert space, \( H \) and let \( T_n \leq T_{n+1} \) for all \( n \). Also suppose there exists \( K \), a self adjoint operator such that for all \( n, T_n \leq K \). Suppose also that each operator commutes with all the others and that \( K \) commutes with all the \( T_n \). Then there exists a self adjoint continuous operator, \( T \) such that for all \( x \in H \),

\[
T_n x \to Tx,
\]

\( T \leq K \), and \( T \) commutes with all the \( T_n \) and with \( K \).

**Proof:** Consider \( K - T_n \equiv S_n \). Then the \( \{S_n\} \) are decreasing, that is,

\[
\{(S_n x, x)\}
\]
is a decreasing sequence and from the hypotheses, \( S_n \geq 0 \) so the above sequence is bounded below by 0. Therefore, \( \lim_{n \to \infty} (S_n x, x) \) exists. By Proposition \( \text{Proposition 17.11.2} \), if \( n > m \),

\[
S_m^2 - S_m S_n = S_m (S_m - S_n) \geq 0
\]

and similarly,

\[
S_n S_m - S_n^2 = S_n (S_m - S_n) \geq 0.
\]

Therefore, since \( S_n \) is self adjoint,

\[
|T_n x - T_m x|^2 = |S_n x - S_m x|^2 = \left( (S_n - S_m)^2 x, x \right)
\]

\[
= ((S_n^2 - 2S_n S_m + S_m^2) x, x) = ((S_m^2 - S_m S_n) x, x) + ((S_n^2 - S_n S_m) x, x)
\]

\[
\leq ((S_m^2 - S_m S_n) x, x) \leq ((S_n^2 - S_m^2) x, x)
\]
\[ \begin{align*}
&= ((S_m - S_n)(S_m + S_n) x, x) \leq 2 ((S_m - S_n) K x, x) \\
&\leq 2 ((S_m - S_n) K x, K x)^{1/2} ((S_m - S_n) x, x)^{1/2}
\end{align*} \]

The last step follows from an application of the Cauchy Schwarz inequality along with the fact \( S_m - S_n \geq 0 \). The last expression converges to 0 because \( \lim_{n \to \infty} (S_n x, x) \) exists for each \( x \). It follows \{\( T_n x \)\} is a Cauchy sequence. Let \( T x \) be the thing to which it converges. \( T \) is obviously linear and

\[ (Tx, x) = \lim_{n \to \infty} (T_n x, x) \leq (K x, x). \]

Also

\[ (KTx, y) = \lim_{n \to \infty} (KT_n x, y) = \lim_{n \to \infty} (T_n K x, y) = (TK x, y) \]

and so \( TK = KT \). Similarly, \( T \) commutes with all \( T_n \).

In order to show \( T \) is continuous, apply the uniform boundedness principle, Theorem 15.1.8. The convergence of \{\( T_n x \)\} implies there exists a uniform bound on the norms, \( ||T_n|| \) and so

\[ ||(T_n x, y)|| \leq C ||x|| ||y|. \]

Now take the limit as \( n \to \infty \) to conclude

\[ ||(Tx, y)|| \leq C ||x|| ||y|| \]

which shows \( ||T|| \leq C \). This proves the lemma.

With this preparation, here is the theorem about square roots.

**Theorem 17.11.4** Let \( T \in \mathcal{L}(H, H) \) be a positive self adjoint linear operator. Then there exists a unique square root, \( A \) with the following properties. \( A^2 = T, A \) is positive and self adjoint, \( A \) commutes with every operator which commutes with \( T \).

**Proof:** First suppose \( T \leq I \). Then define

\[ A_0 \equiv 0, A_{n+1} = A_n + \frac{1}{2} (T - A_n^2). \]

From this it follows that every \( A_n \) is a polynomial in \( T \). Therefore, \( A_n \) commutes with \( T \) and with every operator which commutes with \( T \).

**Claim 1:** \( A_n \leq I \).
17.11. SQUARE ROOTS

Proof of Claim 1: This is true if $n = 0$. Suppose it is true for $n$. Then by the assumption that $T \leq I$,

$$I - A_{n+1} = I - A_n + \frac{1}{2} \left( A_n^2 - T \right)$$

$$\geq I - A_n + \frac{1}{2} \left( A_n^2 - I \right)$$

$$= I - A_n - \frac{1}{2} (I - A_n) (I + A_n)$$

$$= (I - A_n) \left( I - \frac{1}{2} (I + A_n) \right)$$

$$= (I - A_n) (I - A_n) \frac{1}{2} \geq 0.$$  

Claim 2: $A_n \leq A_{n+1}$

Proof of Claim 2: From the definition of $A_n$, this is true if $n = 0$ because $A_1 = T \geq 0 = A_0$.

Suppose true for $n$. Then from Claim 1,

$$A_{n+2} - A_{n+1} = A_{n+1} + \frac{1}{2} \left( T - A_{n+1} \right) - \left[ A_n + \frac{1}{2} \left( T - A_n \right) \right]$$

$$= A_{n+1} - A_n + \frac{1}{2} \left( A_n^2 - A_{n+1} \right)$$

$$= (A_{n+1} - A_n) \left( I - \frac{1}{2} (A_n + A_{n+1}) \right)$$

$$\geq (A_{n+1} - A_n) \left( I - \frac{1}{2} (2I) \right) = 0.$$  

Claim 3: $A_n \geq 0$

Proof of Claim 3: This is true if $n = 0$. Suppose it is true for $n$.  

$$(A_{n+1}x, x) = (A_nx, x) + \frac{1}{2} (Tx, x) - \frac{1}{2} (A_n^2x, x)$$

$$\geq (A_nx, x) + \frac{1}{2} (Tx, x) - \frac{1}{2} (A_nx, x) \geq 0$$

because $A_n - A_n^2 = A_n (I - A_n) \geq 0$ by Proposition 17.11.2.

Now \{A_n\} is a sequence of positive self adjoint operators which are bounded above by $I$ such that each of these operators commutes with every operator which commutes with $T$. By Lemma 17.11.4, there exists a bounded linear operator, $A$ such that for all $x$,

$$A_nx \rightarrow Ax$$

Then $A$ commutes with every operator which commutes with $T$ because each $A_n$ has this property. Also $A$ is a positive operator because each $A_n$ is. From passing
to the limit in the definition of \( A_n \),

\[
Ax = Ax + \frac{1}{2} (Tx - A^2 x)
\]

and so \( Tx = A^2 x \). This proves the theorem in the case that \( T \leq I \).

In the general case, consider \( T/||T|| \). Then

\[
\left( \frac{T}{||T||}, x, x \right) = \frac{1}{||T||} (Tx, x) \leq |x|^2 = (Ix, x)
\]

and so \( T/||T|| \leq I \). Therefore, it has a square root, \( B \). Let \( A = \sqrt{||T||}B \). Then \( A \) has all the right properties and \( A^2 = ||T||B^2 = ||T||(T/||T||) = T \). This proves the existence part of the theorem.

Next suppose both \( A \) and \( B \) are square roots of \( T \) having all the properties stated in the theorem. Then \( AB = BA \) because both \( A \) and \( B \) commute with every operator which commutes with \( T \).

\[
(A (A - B) x, (A - B) x), (B (A - B) x, (A - B) x) \geq 0 \quad (17.11.72)
\]

Therefore, on adding these,

\[
\left( (A^2 - AB + BA - B^2) x, (A - B) x \right) = \left( (A^2 - B^2) x, (A - B) x \right) = \left( (T - T) x, (A - B) x \right) = 0.
\]

It follows both expressions in \( 17.11.72 \) equal 0 since both are nonnegative and when they are added the result is 0. Now applying the existence part of the theorem to \( A \), there exists a positive square root of \( A \) which is self adjoint. Thus

\[
\left( \sqrt{A} (A - B) x, \sqrt{A} (A - B) x \right) = 0
\]

so \( \sqrt{A} (A - B) x = 0 \) which implies \( A (A - B) x = 0 \). Similarly, \( B (A - B) x = 0 \). Subtracting these and taking the inner product with \( x \),

\[
0 = ((A (A - B) - B (A - B)) x, x) = \left( (A - B)^2 x, x \right) = ||(A - B) x||^2
\]

and so \( Ax = Bx \) which shows \( A = B \) since \( x \) was arbitrary. This proves the theorem.

17.12 General Theory Of Continuous Semigroups

Much more on semigroups is available in Yosida [115]. This is just an introduction to the subject.
17.12. GENERAL THEORY OF CONTINUOUS SEMIGROUPS

Definition 17.12.1 A strongly continuous semigroup defined on $H$, a Banach space is a function $S : [0, \infty) \to H$ which satisfies the following for all $x_0 \in H$.

\[
S(t) \in \mathcal{L}(H, H), S(t + s) = S(t)S(s),
\]

$t \to S(t)x_0$ is continuous, \( \lim_{t \to 0^+} S(t)x_0 = x_0 \)

Sometimes such a semigroup is said to be $C_0$. It is said to have the linear operator $A$ as its generator if

\[
D(A) = \left\{ x : \lim_{h \to 0} \frac{S(h)x - x}{h} \text{ exists} \right\}
\]

and for $x \in D(A)$, $A$ is defined by

\[
\lim_{h \to 0} \frac{S(h)x - x}{h} = Ax
\]

The assertion that $t \to S(t)x_0$ is continuous and that $S(t) \in \mathcal{L}(H, H)$ is not sufficient to say there is a bound on $\|S(t)\|$ for all $t$. Also the assertion that for each $x_0$,

\[
\lim_{t \to 0^+} S(t)x_0 = x_0
\]

is not the same as saying that $S(t) \to I$ in $\mathcal{L}(H, H)$. It is a much weaker assertion. The next theorem gives information on the growth of $\|S(t)\|$. It turns out it has exponential growth.

Lemma 17.12.2 Let $M = \sup \{ \|S(t)\| : t \in [0, T] \}$. Then $M < \infty$.

**Proof:** If this is not true, then there exists $t_n \in [0, T]$ such that $\|S(t_n)\| \geq n$. That is the operators $S(t_n)$ are not uniformly bounded. By the uniform boundedness principle, Theorem 15.1.8, there exists $x \in H$ such that $\|S(t_n)x\|$ is not bounded. However, this is impossible because it is given that $t \to S(t)x$ is continuous on $[0, T]$ and so $t \to \|S(t)x\|$ must achieve its maximum on this compact set.

Now here is the main result for growth of $\|S(t)\|$.

Theorem 17.12.3 For $M$ described in Lemma 17.12.2, there exists $\alpha$ such that

\[
\|S(t)\| \leq Me^{\alpha t}.
\]

In fact, $\alpha$ can be chosen such that $M^{1/T} = e^\alpha$.

**Proof:** Let $t$ be arbitrary. Then $t = nT + r(t)$ where $0 \leq r(t) < T$. Then by the semigroup property

\[
\|S(t)\| = \|S(nT + r(t))\| = \|S(r(t)) S(T)^m\| \leq M^{m+1}
\]
Now \( mT \leq t \leq mT + r(t) \leq (m + 1)T \) and so
\[
m \leq \frac{t}{T} \leq m + 1
\]
Therefore,
\[
\|S(t)\| \leq M^{(t/T)+1} = M \left( M^{1/T} \right)^t.
\]
Let \( M^{1/T} \equiv e^\alpha \) and then
\[
\|S(t)\| \leq Me^{\alpha t}
\]
This proves the theorem.

**Definition 17.12.4** Let \( S(t) \) be a continuous semigroup as described above. It is called a contraction semigroup if for all \( t \geq 0 \)
\[
\|S(t)\| \leq 1.
\]
It is called a bounded semigroup if there exists \( M \) such that for all \( t \geq 0 \),
\[
\|S(t)\| \leq M
\]
Note that for \( S(t) \) an arbitrary continuous semigroup satisfying
\[
\|S(t)\| \leq Me^{\alpha t},
\]
It follows that the semigroup,
\[
T(t) = e^{-\alpha t} S(t)
\]
is a bounded semigroup which satisfies
\[
\|T(t)\| \leq M.
\]

**Proposition 17.12.5** Given a continuous semigroup \( S(t) \), its generator \( A \) exists and is a closed densely defined operator. Furthermore, for
\[
\|S(t)\| \leq Me^{\alpha t}
\]
and \( \lambda > \alpha \), \( \lambda I - A \) is onto and \( (\lambda I - A)^{-1} \) maps \( H \) onto \( D(A) \) and is in \( \mathcal{L}(H,H) \). Also for these values of \( \lambda \),
\[
(\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt.
\]
For \( \lambda > \alpha \), the following estimate holds.
\[
\left\| (\lambda I - A)^{-1} \right\| \leq \frac{M}{|\lambda - \alpha|}
\]
Proof: First note $D(A) \neq \emptyset$. In fact $0 \in D(A)$. It follows from Theorem that for all $\lambda$ large enough, one can define a Laplace transform,

$$R(\lambda) x = \int_0^\infty e^{-\lambda t} S(t) x dt \in H.$$  

Here the integral is the ordinary improper Riemann integral. I claim each of these is in $D(A)$.

Using the semigroup property and changing the variables in the first of the above integrals, this equals

$$= \frac{1}{h} \left( e^{\lambda h} \int_0^\infty e^{-\lambda t} S(t) x dt - \int_0^h e^{-\lambda t} S(t) x dt \right)$$

The limit as $h \to 0$ exists and equals

$$\lambda R(\lambda) x - x$$

Thus $R(\lambda) x \in D(A)$ as claimed and

$$AR(\lambda) x = \lambda R(\lambda) x - x$$

Hence

$$x = (\lambda I - A) R(\lambda) x.$$  \hspace{1cm} (17.12.73)

Since $x$ is arbitrary, this shows that for $\lambda$ large enough, $\lambda I - A$ is onto.}

Why is $D(A)$ dense? It was shown above that $R(\lambda) x$ and thererfore $\lambda R(\lambda) x \in D(A)$. Then for $\lambda > \alpha$ where $\|S(t)\| \leq Me^{\alpha t}$,

$$\|\lambda R(\lambda) x - x\| = \left\| \int_0^\infty \lambda e^{-\lambda t} S(t) x dt - \int_0^\infty \lambda e^{-\lambda t} x dt \right\|$$

$$\leq \int_0^\infty \|\lambda e^{-\lambda t} S(t) x - x\| dt$$

$$= \int_0^h \|\lambda e^{-\lambda t} (S(t) x - x)\| dt + \int_h^\infty \|\lambda e^{-\lambda t} (S(t) x - x)\| dt$$

$$\leq \int_0^h \|\lambda e^{-\lambda t} (S(t) x - x)\| dt + \int_h^\infty \lambda e^{-(\lambda - \alpha)t} dt (M + 1) \|x\|$$

Now since $S(t) x - x \to 0$, it follows that for $h$ sufficiently small

$$\leq \frac{\epsilon}{2} \int_0^h \lambda e^{-\lambda t} dt + \frac{\lambda}{\lambda - \alpha} e^{-(\lambda - \alpha)h} (M + 1) \|x\|$$

$$\leq \frac{\epsilon}{2} + \frac{\lambda}{\lambda - \alpha} e^{-(\lambda - \alpha)h} (M + 1) \|x\| < \epsilon$$
whenever $\lambda$ is large enough. Thus $D(A)$ is dense as claimed.

Let $x \in D(A)$. Then for $y^* \in H'$,
\[
y^* \left( \int_0^t S(s) Ax \, ds \right) = \int_0^t y^* \left( S(s) \lim_{h \to 0^+} \frac{S(h)x - x}{h} \right) \, ds
\]
The difference quotient is given to have a limit and so the difference quotients are bounded. Therefore, one can use the dominated convergence theorem to take the limit outside the integral and write the above equals
\[
\lim_{h \to 0^+} \int_0^t y^* \left( S(s) \frac{S(h)x - x}{h} \right) \, ds
\]
\[
= \lim_{h \to 0^+} y^* \left( \frac{1}{h} \left( \int_t^{t+h} S(s) x \, ds - \int_0^t S(s) x \, ds \right) \right)
\]
\[
= \lim_{h \to 0^+} y^* \left( \int_t^{t+h} S(s) x \, ds - \int_0^h S(s) x \, ds \right)
\]
\[
= y^* (S(t)x - x).
\]
Thus since $y^*$ is arbitrary, for $x \in D(A)$
\[
S(t)x = x + \int_0^t S(s) Ax \, ds
\]
Why is $A$ closed? Suppose $x_n \to x$ and $x_n \in D(A)$ while $Ax_n \to z$. From what was just shown
\[
S(t)x_n = x_n + \int_0^t S(s) Ax_n \, ds
\]
and so, passing to the limit this yields
\[
S(t)x = x + \int_0^t S(s) z \, ds
\]
which implies
\[
\lim_{t \to 0^+} \frac{S(t)x - x}{h} = \lim_{h \to 0^+} \frac{1}{h} \int_0^t S(s) z \, ds = z
\]
which shows $Ax = z$ and $x \in D(A)$. Thus $A$ is closed.

Because of 17.12.73 it follows $R(\lambda)x = (\lambda I - A)^{-1}x$. Also
\[
\|R(\lambda)x\| \leq \left\| \int_0^\infty e^{-\lambda t} S(t) \, x \, dt \right\|
\]
\[
\leq \int_0^\infty e^{-\lambda t} M e^{\alpha t} \|x\| \leq \frac{M}{|\lambda - \alpha|} \|x\|
\]
so $R(\lambda) = (\lambda I - A)^{-1} \in \mathcal{L}(H,H)$ and this also proves the last estimate. Also from 17.12.73 $R(\lambda)$ maps $H$ onto $D(A)$. This proves the proposition.

The linear mapping $(\lambda I - A)^{-1}$ is called the resolvent.

The above proof contains an argument which implies the following corollary.
Corollary 17.12.6 Let $S(t)$ be a continuous semigroup and let $A$ be its generator. Then for $0 < a < b$ and $x \in D(A)$

$$S(b)x - S(a)x = \int_a^b S(t)Axdt$$

and also for $t > 0$ you can take the derivative from the left,

$$\lim_{h \to 0^+} \frac{S(t)x - S(t-h)x}{h} = S(t)Ax$$

Proof: Letting $y^* \in H'$,

$$y^* \left( \int_a^b S(t)Axdt \right) = \int_a^b y^* \left( S(t) \frac{S(h)x-x}{h} \right)dt$$

The difference quotients are bounded because they converge to $Ax$. Therefore, from the dominated convergence theorem,

$$y^* \left( \int_a^b S(t)Axdt \right) = \lim_{h \to 0} \int_a^b y^* \left( S(t) \frac{S(h)x-x}{h} \right)dt$$

$$= \lim_{h \to 0} y^* \left( \int_a^b S(t) \frac{S(h)x-x}{h}dt \right)$$

$$= \lim_{h \to 0} y^* \left( \frac{1}{h} \int_{a+h}^{a+h} S(t)xdt - \frac{1}{h} \int_a^b S(t)xdt \right)$$

$$= \lim_{h \to 0} y^* \left( \frac{1}{h} \int_h^{b+h} S(t)xdt - \frac{1}{h} \int_a^a S(t)xdt \right)$$

$$= \frac{1}{h} \int_{t-h}^{t} S(s)Axds$$

which converges to $S(t)Ax$ as $h \to 0^+$. This proves the corollary.

Given a closed densely defined operator, when is it the generator of a bounded semigroup? This is answered in the following theorem which is called the Hille Yosida theorem.

Theorem 17.12.7 Suppose $A$ is a densely defined linear operator which has the property that for all $\lambda > 0$,

$$(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$$
which means that \( \lambda I - A : D(A) \to H \) is one to one and onto with continuous inverse. Suppose also that for all \( n \in \mathbb{N} \),

\[
\left\| \left( (\lambda I - A)^{-1} \right)^n \right\| \leq \frac{M}{\lambda^n}. \tag{17.12.74}
\]

Then there exists a continuous semigroup, \( S(t) \) which has \( A \) as its generator and satisfies \( \|S(t)\| \leq M \) and \( A \) is closed. In fact letting

\[
S_\lambda(t) \equiv \exp \left( -\lambda + \lambda^2 (\lambda I - A)^{-1} \right)
\]

it follows \( \lim_{\lambda \to \infty} S_\lambda(t)x = S(t)x \) uniformly on finite intervals. Conversely, if \( A \) is the generator of \( S(t) \), a bounded continuous semigroup having \( \|S(t)\| \leq M \), then \( (\lambda I - A)^{-1} \in \mathcal{L}(H,H) \) for all \( \lambda > 0 \) and \([17.12.75] \) holds.

**Proof:** Consider the operator

\[
\lambda (\lambda I - A)^{-1} A
\]

On \( D(A) \), this equals

\[
-\lambda + \lambda^2 (\lambda I - A)^{-1} \tag{17.12.75}
\]

which makes sense on all of \( H \), not just on \( D(A) \). Also this last expression equals

\[
\lambda A (\lambda I - A)^{-1}
\]

on all of \( H \) because \( \lambda I - A \) is given to be onto. Denote this as \( A_\lambda \) to save notation. Thus on \( D(A) \),

\[
\lambda A (\lambda I - A)^{-1} = \lambda (\lambda I - A)^{-1} A
\]

For \( x \in D(A) \),

\[
\left\| \lambda (\lambda I - A)^{-1} x - x \right\| = \left\| (\lambda I - A)^{-1} (\lambda x - (\lambda I - A)x) \right\| = \left\| (\lambda I - A)^{-1} Ax \right\| \leq \frac{M}{\lambda} \|Ax\|
\]

which converges to 0. Therefore, for \( x \in D(A) \),

\[
\|A_\lambda x - Ax\| = \left\| \lambda (\lambda I - A)^{-1} Ax - Ax \right\| \leq \frac{M \|Ax\|}{\lambda} \tag{17.12.76}
\]

so it also converges to 0. Because of \([17.12.76] \), the operator \( \lambda A (\lambda I - A)^{-1} \) is continuous. Now using \([17.12.76] \) define an approximate semigroup

\[
S_\lambda(t) \equiv e^{-M} \sum_{k=0}^{\infty} \frac{t^k (\lambda^2 (\lambda I - A)^{-1})^k}{k!}
\]
The sum converges in \( L(H, H) \) because it converges absolutely and \( L(H, H) \) is complete. Here is why it converges absolutely. It follows from the assumption in the lemma.

\[
\sum_{k=0}^{\infty} t^k \left( \frac{\left( \lambda^2 (\lambda I - A) \right)^{-1}}{k!} \right) \leq \sum_{k=0}^{\infty} t^k \left( \frac{\left( \lambda^2 (\lambda I - A) \right)^{-1}}{k!} \right) \leq \sum_{k=0}^{\infty} \frac{t^k \lambda^k M}{k!} = Me^{t\lambda}
\]

Thus

\[
||S_\lambda(t)|| \leq e^{-\lambda t} Me^{t\lambda} = M
\]

The series converges uniformly on any finite interval thanks to the Weierstrass M test. Thus \( t \to S_\lambda(t) \) is continuous and it is also routine to verify the semigroup identity. Clearly \( \lim_{t \to 0} S_\lambda(t) x = x \). It is also the case that \( S_\lambda(t) \) is generated by \(-\lambda + \lambda^2 (\lambda I - A)^{-1} = A_\lambda \). This is easy to show from differentiating the power series which has a continuous derivative. Thus

\[
(-\lambda) e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k \left( \lambda^2 (\lambda I - A)^{-1} \right)^{k+1} x}{k!} = (-\lambda + \lambda^2 (\lambda I - A)^{-1}) S_\lambda(t) x = S_\lambda(t) \left( -\lambda + \lambda^2 (\lambda I - A)^{-1} \right) x
\]

Now let \( t \to 0^+ \) to obtain \((-\lambda + \lambda^2 (\lambda I - A)^{-1}) x = A_\lambda x\).

**Claim:** For \( \lambda, \mu > 0 \), \((\lambda I - A)^{-1}\) and \((\mu I - A)^{-1}\) commute.

**Proof of claim:** Suppose

\[
y = (\mu I - A)^{-1} (\lambda I - A)^{-1} x \quad (17.12.77)
\]

\[
z = (\lambda I - A)^{-1} (\mu I - A)^{-1} x \quad (17.12.78)
\]

I need to show \( y = z \). First note \( z \in D(A) \) and

\[
(\lambda I - A) z = (\mu I - A)^{-1} x \in D(A).
\]

Hence

\[
(\mu I - A) z = (\mu - \lambda) z + (\lambda I - A) z \in D(A).
\]

Similarly

\[
(\lambda I - A) y, (\mu I - A) y \in D(A).
\]

From (17.12.77)

\[
(\lambda I - A) (\mu I - A) y = x
\]
and using \(17.12.78\),

\[
\begin{align*}
x &= (\mu I - A) (\lambda I - A) z \\
&= ((\mu - \lambda) I + (\lambda I - A)) (\lambda I - A) z \\
&= (\mu - \lambda) (\lambda I - A) z + (\lambda I - A)^2 z \\
&= (\lambda I - A) (\mu - \lambda) z + (\lambda I - A) (\lambda I - A) z \\
&= (\lambda I - A) (\mu - \lambda) z + (\lambda I - A) ((\lambda - \mu) I + (\lambda I - A)) z \\
&= (\lambda I - A) (\mu - \lambda) z + (\lambda I - A) ((\lambda - \mu) z + (\mu I - A) z) \\
&= (\lambda I - A) (\mu - \lambda) z + (\lambda I - A) (\lambda - \mu) z + (\lambda I - A) (\mu I - A) z \\
&= (\lambda I - A) (\mu I - A) z
\end{align*}
\]

Thus

\[
x = (\lambda I - A) (\mu I - A) z = (\lambda I - A) (\mu I - A) y
\]

and so \( z = y \). This proves the claim.

It follows from the description of \( S_\lambda (t) \) that \( S_\lambda (t) \) and \( S_\mu (s) \) commute and also \( A_\lambda \) commutes with \( S_\mu (t) \) for any \( t \).

I want to show that for each \( x \in D (A) \),

\[
\lim_{\lambda \to \infty} S_\lambda (t) x = S(t) x
\]

where \( S(t) \) is the desired semigroup. Let \( x \in D (A) \)

\[
||S_\lambda (t) x - S_\mu (t) x|| = \left|\left| \int_0^t \frac{d}{dr} (S_\lambda (t - r) S_\mu (r)) x dr \right| \right|
\]

Since \( A_\lambda \) commutes with \( S_\mu (r) \), the following formula follows from \(17.12.78\),

\[
\begin{align*}
&= \left|\left| \int_0^t (S_\lambda (t - r) S_\mu (r) A_\mu x - S_\lambda (t - r) A_\lambda S_\mu (r) x) dr \right| \right|
\\
&\leq \int_0^t ||S_\lambda (t - r) S_\mu (r) (A_\mu x - A_\lambda x)|| dr
\\
&\leq M^2 t ||A_\mu x - A_\lambda x|| \leq M^2 t (||A_\mu x - A x|| + ||A x - A_\lambda x||)
\\
&\leq \left( \frac{||A x||}{\mu} + \frac{||A x||}{\lambda} \right) t M^2
\end{align*}
\]

Hence whenever \( \mu, \lambda \) large enough, \( ||S_\lambda (t) x - S_\mu (t) x|| \) is small. Thus \( S_\lambda (t) x \) converges uniformly on finite intervals to something denoted by \( S(t) x \). Therefore, \( t \to S(t) x \) is continuous for each \( x \in D (A) \) and also

\[
||S(t) x|| = \lim_{\lambda \to \infty} ||S_\lambda (t) x|| \leq M ||x||
\]
17.12. GENERAL THEORY OF CONTINUOUS SEMIGROUPS

so that $S(t)$ can be extended to a continuous linear map, still called $S(t)$ defined on all of $H$ which also satisfies $\|S(t)\| \leq M$ since $D(A)$ is dense in $H$. If $x$ is arbitrary, let $y \in D(A)$ be close to $x$. Then

$$
\begin{align*}
\|S(t)x - S_\lambda(t)x\| & \leq \|S(t)x - S(t)y\| + \|S(t)y - S_\lambda(t)y\| \\
& \quad + \|S_\lambda(t)y - S_\lambda(t)x\| \\
& \leq 2M \|x - y\| + \|S(t)y - S_\lambda(t)y\|
\end{align*}
$$

and so $\lim_{\lambda \to \infty} S_\lambda(t)x = S(t)x$ for all $x$, uniformly on finite intervals. Thus $t \to S(t)x$ is continuous for any $x \in H$.

It remains to verify $A$ generates $S(t)$ and for all $x$, $S(t)x - x \to 0$. From the above,

$$
S_\lambda(t)x = x + \int_0^t S_\lambda(s) A \lambda x ds \quad (17.12.79)
$$

and so

$$
\lim_{t \to 0^+} \|S_\lambda(t)x - x\| = 0
$$

By the uniform convergence just shown, there exists $\lambda$ large enough that for all $t \in [0, \delta]$,

$$
\|S(t)x - S_\lambda(t)x\| < \varepsilon.
$$

Then

$$
\begin{align*}
\limsup_{t \to 0^+} \|S(t)x - x\| & \leq \limsup_{t \to 0^+} (\|S(t)x - S_\lambda(t)x\| + \|S_\lambda(t)x - x\|) \\
& \leq \limsup_{t \to 0^+} (\varepsilon + \|S_\lambda(t)x - x\|) \leq \varepsilon
\end{align*}
$$

It follows $\lim_{t \to 0^+} S(t)x = x$ because $\varepsilon$ is arbitrary.

Next, $\lim_{\lambda \to \infty} A_\lambda x = Ax$ for all $x \in D(A)$ by (17.12.76). Therefore, passing to the limit in (17.12.79) yields from the uniform convergence

$$
S(t)x = x + \int_0^t S(s) Ax ds
$$

and by continuity of $s \to S(s)Ax$, it follows

$$
\lim_{h \to 0^+} \frac{S(h)x - x}{h} = \lim_{h \to 0^+} \frac{1}{h} \int_0^h S(s) Ax ds = Ax
$$

Thus letting $B$ denote the generator of $S(t)$, $D(A) \subseteq D(B)$ and $A = B$ on $D(A)$. It only remains to verify $D(A) = D(B)$.

To do this, let $\lambda > 0$ and consider the following where $y \in H$ is arbitrary.

$$
(\lambda I - B)^{-1} y = (\lambda I - B)^{-1} \left( (\lambda I - A) (\lambda I - A)^{-1} y \right)
$$
Now \((\lambda I - A)^{-1} y \in D(A) \subseteq D(B)\) and \(A = B\) on \(D(A)\) and so

\[(\lambda I - A) (\lambda I - A)^{-1} y = (\lambda I - B) (\lambda I - A)^{-1} y\]

which implies,

\[(\lambda I - B)^{-1} y = (\lambda I - B)^{-1} ((\lambda I - B) (\lambda I - A)^{-1} y) = (\lambda I - A)^{-1} y\]

Recall from Proposition 17.12.5 that an arbitrary element of \(D(B)\) is of the form \((\lambda I - B)^{-1} y\) and this has shown every such vector is in \(D(A)\), in fact it equals \((\lambda I - A)^{-1} y\). Hence \(D(B) \subseteq D(A)\) which shows \(A\) generates \(S(t)\) and this proves the first half of the theorem.

Next suppose \(A\) is the generator of a semigroup \(S(t)\) having \(||S(t)|| \leq M\). Then by Proposition 17.12.5 for all \(\lambda > 0\), \((\lambda I - A)\) is onto and

\[(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} S(t) \, dt\]

thus

\[
\left\| \left( (\lambda I - A)^{-1} \right)^n \right\| = \left\| \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1 + \cdots + t_n)} S(t_1 + \cdots + t_n) \, dt_1 \cdots dt_n \right\| \\
\leq \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1 + \cdots + t_n)} M dt_1 \cdots dt_n = \frac{M}{\lambda^n}.
\]

This proves the theorem.

### 17.12.1 An Evolution Equation

When \(\Lambda\) generates a continuous semigroup, one can consider a very interesting theorem about evolution equations of the form

\[y' - \Lambda y = g(t)\]

provided \(t \rightarrow g(t)\) is \(C^1\).

**Theorem 17.12.8** Let \(\Lambda\) be the generator of \(S(t)\), a continuous semigroup on \(H\), a Banach space and let \(t \rightarrow g(t)\) be in \(C^1(0, \infty; H)\). Then there exists a unique solution to the initial value problem

\[y' - \Lambda y = g, \ y(0) = y_0 \in D(\Lambda)\]

and it is given by

\[y(t) = S(t) y_0 + \int_0^t S(t - s) g(s) \, ds. \quad (17.12.80)\]

This solution is continuous having continuous derivative and has values in \(D(\Lambda)\).
Proof: First I show the following claim.

Claim: \( \int_0^t S(t-s)g(s)\,ds \in D(\Lambda) \) and

\[
\Lambda \left( \int_0^t S(t-s)g(s)\,ds \right) = S(t)g(0) - g(t) + \int_0^t S(t-s)g'(s)\,ds
\]

Proof of the claim:

\[
\frac{1}{h} \left( \int_0^h S(t) g(s)\,ds - \int_0^t S(t-s)g(s)\,ds \right)
\]

\[
= \frac{1}{h} \left( \int_0^t S(t-s+h)g(s)\,ds - \int_0^t S(t-s)g(s)\,ds \right)
\]

\[
= \frac{1}{h} \left( \int_{-h}^t S(t-s)g(s+h)\,ds - \int_0^t S(t-s)g(s)\,ds \right)
\]

\[
= \frac{1}{h} \int_{-h}^0 S(t-s)g(s+h)\,ds + \int_0^t S(t-s)g(s+h) - g(s)
\]

\[
= \frac{1}{h} \int_{-h}^t S(t-s)g(s)\,ds
\]

Using the estimate in Theorem 17.12.3 on Page 557 and the dominated convergence theorem, the limit as \( h \to 0 \) of the above equals

\[
S(t)g(0) - g(t) + \int_0^t S(t-s)g'(s)\,ds
\]

which proves the claim.

Since \( y_0 \in D(\Lambda) \),

\[
S(t)\Lambda y_0 = S(t) \lim_{h \to 0} \frac{S(h)y_0 - y_0}{h}
\]

\[
= \lim_{h \to 0} \frac{S(t+h) - S(t)}{h} y_0
\]

\[
= \lim_{h \to 0} \frac{S(h)S(t)y_0 - S(t)y_0}{h}
\]

(17.12.81)

Since this limit exists, the last limit in the above exists and equals

\[
\Lambda S(t) y_0
\]

(17.12.82)

and so \( S(t) y_0 \in D(\Lambda) \). Now consider 17.12.80.

\[
y(t+h) - y(t) = \frac{S(t+h) - S(t)}{h} y_0 +
\]
\[ \frac{1}{h} \left( \int_0^{t+h} S(t-s+h)g(s)\,ds - \int_0^t S(t-s)g(s)\,ds \right) \]

\[ = \frac{S(t+h) - S(t)}{h}y_0 + \frac{1}{h} \int_t^{t+h} S(t-s)g(s)\,ds \]

\[ + \frac{1}{h} \left( S(h) \int_0^t S(t-s)g(s)\,ds - \int_0^t S(t-s)g(s)\,ds \right) \]

From the claim and 17.12.81, 17.12.82, the limit of the right side is

\[ \Lambda S(t)y_0 + g(t) + \Lambda \left( \int_0^t S(t-s)g(s)\,ds \right) \]

\[ = \Lambda \left( S(t)y_0 + \int_0^t S(t-s)g(s)\,ds \right) + g(t) \]

Hence

\[ y'(t) = \Lambda y(t) + g(t) \]

and from the formula, \( y' \) is continuous since by the claim and 17.12.82, it also equals

\[ S(t)\Lambda y_0 + g(t) + S(t)g(0) - g(t) + \int_0^t S(t-s)g'(s)\,ds \]

which is continuous. The claim and 17.12.82 also shows \( y(t) \in D(\Lambda) \). This proves the existence part of the lemma.

It remains to prove the uniqueness part. It suffices to show that if

\[ y' - \Lambda y = 0, \quad y(0) = 0 \]

and \( y \) is \( C^1 \) having values in \( D(\Lambda) \), then \( y = 0 \). Suppose then that \( y \) is this way. Letting \( 0 < s < t \),

\[ \frac{d}{ds} \left( S(t-s)y(s) \right) \]

\[ = \lim_{h \to 0} S(t-s-h) \frac{y(s+h) - y(s)}{h} - \frac{S(t-s)y(s) - S(t-s-h)y(s)}{h} \]

provided the limit exists. Since \( y' \) exists and \( y(s) \in D(\Lambda) \), this equals

\[ S(t-s)y'(s) - S(t-s)y(s) = 0. \]

Let \( y^* \in H' \). This has shown that on the open interval \((0,t)\) the function \( s \to y^*(S(t-s)y(s)) \) has a derivative equal to \( 0 \). Also from continuity of \( S \) and \( y \), this function is continuous on \([0,t] \). Therefore, it is constant on \([0,t] \) by the mean value theorem. At \( s = 0 \), this function equals \( 0 \). Therefore, it equals \( 0 \) on \([0,t] \). Thus for fixed \( s > 0 \) and letting \( t > s \), \( y^*(S(t-s)y(s)) = 0 \). Now let \( t \) decrease toward \( s \). Then \( y^*(y(s)) = 0 \) and since \( y^* \) was arbitrary, it follows \( y(s) = 0 \). This proves uniqueness.
17.12. GENERAL THEORY OF CONTINUOUS SEMIGROUPS

17.12.2 Adjoints, Hilbert Space

In Hilbert space, there are some special things which are true.

Definition 17.12.9 Let $A$ be a densely defined closed operator on $H$ a real Hilbert space. Then $A^*$ is defined as follows.

$$D(A^*) \equiv \{ y \in H : |(Ax,y)| \leq C|x| \}$$

Then since $D(A)$ is dense, there exists a unique element of $H$ denoted by $A^*y$ such that

$$(Ax,y) = (x,A^*y)$$

for all $x \in D(A)$.

Lemma 17.12.10 Let $A$ be closed and densely defined on $D(H) \subseteq H$, a Hilbert space. Then $A^*$ is also closed and densely defined. Also $(A^*)^* = A$. In addition to this, if $(\lambda I - A)^{-1} \in \mathcal{L}(H,H)$, then $(\lambda I - A^*)^{-1} \in \mathcal{L}(H,H)$ and

$$
\left( \left( (\lambda I - A)^{-1} \right)^n \right)^* = \left( (\lambda I - A^*)^{-1} \right)^n
$$

Proof: Denote by $[x,y]$ an ordered pair in $H \times H$. Define $\tau : H \times H \to H \times H$ by

$$\tau [x,y] \equiv [-y,x]$$

Then the definition of adjoint implies that for $G(B)$ equal to the graph of $B$,

$$G(A^*) = (\tau G(A))^\perp$$

In this notation the inner product on $H \times H$ with respect to which $\perp$ is defined is given by

$$( [x,y] , [a,b] ) \equiv (x,a) + (y,b) .$$

Here is why this is so. For $[x,A^*x] \in G(A^*)$ it follows that for all $y \in D(A)$

$$([x,A^*x],[Ay,y]) = -(Ay,x) + (y,A^*x) = 0$$

and so $[x,A^*x] \in (\tau G(A))^\perp$ which shows

$$G(A^*) \subseteq (\tau G(A))^\perp$$

To obtain the other inclusion, let $[a,b] \in (\tau G(A))^\perp$. This means that for all $x \in D(A)$,

$$([a,b],[Ax,x]) = 0.$$
and so $|(Ax, a)| \leq C|x|$ for all $x \in D(A)$ which shows $a \in D(A^*)$ and

$$(x, A^*a) = (x, b)$$

for all $x \in D(A)$. Therefore, since $D(A)$ is dense, it follows $b = A^*a$ and so $[a, b] \in G(A^*)$. This shows the other inclusion.

Note that if $V$ is any subspace of the Hilbert space $H \times H$,

$$(V^\perp)^\perp = V$$

and $S^\perp$ is always a closed subspace. Also $\tau$ and $\perp$ commute. The reason for this is that $[x, y] \in (\tau V)^\perp$ means that $[x, -y] + [y, a] = 0$ for all $[a, b] \in V$ and $[x, y] \in \tau (V^\perp)$ means $[-y, x] \in V^\perp$ so for all $[a, b] \in V$,

$$(-y, a) + (x, b) = 0$$

which says the same thing. It is also clear that $\tau \circ \tau$ has the effect of multiplication by $-1$.

It follows from the above description of the graph of $A^*$ that even if $G(A)$ were not closed it would still be the case that $G(A^*)$ is closed.

Why is $D(A^*)$ dense? Suppose $z \in D(A^*)^\perp$. Then for all $y \in D(A^*)$ so that $[y, Ay] \in G(A^*)$, it follows $[z, 0] \in G(A^*)^\perp = (G(A))^\perp = \tau G(A)$ but this implies

$$[0, z] \in -G(A)$$

and so $z = -A0 = 0$. Thus $D(A^*)$ must be dense since there is no nonzero vector in $D(A^*)^\perp$.

Since $A$ is a closed operator, meaning $G(A)$ is closed in $H \times H$, it follows from the above formula that

$$G((A^*)^*) = \left(\tau \left(\tau G(A)^\perp\right)\right)^\perp = (\tau G(A))^\perp = \left((-G(A))^\perp\right)^\perp = (G(A))^\perp = G(A)$$

and so $(A^*)^* = A$.

Now consider the final claim. First let $y \in D(A^*) = D(\lambda I - A^*)$. Then letting $x \in H$ be arbitrary,

$$\left(x, \left((\lambda I - A)(\lambda I - A)^{-1}\right)^* y\right) = \left((\lambda I - A)(\lambda I - A)^{-1} x, y\right) = \left(x, (\lambda I - A)^{-1}\right)^* (\lambda I - A^*) y$$

Thus

$$\left((\lambda I - A)(\lambda I - A)^{-1}\right)^* = I = \left((\lambda I - A)^{-1}\right)^* (\lambda I - A^*) \quad (17.12.84)$$
on $D(A^*)$. Next let $x \in D(A) = D(\lambda I - A)$ and $y \in H$ arbitrary.

$$(x, y) = \left( (\lambda I - A)^{-1} (\lambda I - A) x, y \right) = \left( (\lambda I - A) x, (\lambda I - A)^{-1} y \right)$$

Now it follows $\left| (\lambda I - A) x, (\lambda I - A)^{-1} y \right| \leq \|y\| \|x\|$ for any $x \in D(A)$ and so

$$(\lambda I - A)^{-1} y \in D(A^*)$$

Hence

$$(x, y) = \left( x, (\lambda I - A^*) \left( (\lambda I - A)^{-1} \right)^* y \right).$$

Since $x \in D(A)$ is arbitrary and $D(A)$ is dense, it follows

$$(\lambda I - A^*) \left( (\lambda I - A)^{-1} \right)^* = I \quad (17.12.85)$$

From (17.12.84) and (17.12.85) it follows

$$(\lambda I - A^*)^{-1} = \left( (\lambda I - A)^{-1} \right)^*$$

and $(\lambda I - A^*)$ is one to one and onto with continuous inverse. Finally, from the above,

$$\left( (\lambda I - A^*)^{-1} \right)^n = \left( \left( (\lambda I - A)^{-1} \right)^* \right)^n = \left( \left( (\lambda I - A)^{-1} \right)^n \right)^*.$$

This proves the lemma.

With this preparation, here is an interesting result about the adjoint of the generator of a continuous bounded semigroup. I found this in Balakrishnan [11].

**Theorem 17.12.11** Suppose $A$ is a densely defined closed operator which generates a continuous semigroup, $S(t)$. Then $A^*$ is also a closed densely defined operator which generates $S^*(t)$ and $S^*(t)$ is also a continuous semigroup.

**Proof:** First suppose $S(t)$ is also a bounded semigroup, $\|S(t)\| \leq M$. From Lemma (17.12.10) $A^*$ is closed and densely defined. It follows from the Hille Yosida theorem, Theorem (17.12.7) that

$$\left| (\lambda I - A)^{-1} \right| \leq \frac{M}{\lambda^n}$$

From Lemma (17.12.11) and the fact the adjoint of a bounded linear operator preserves the norm,

$$\frac{M}{\lambda^n} \geq \left| \left( (\lambda I - A)^{-1} \right)^n \right| = \left| \left( (\lambda I - A)^{-1} \right)^* \right| = \left| \left( (\lambda I - A^*)^{-1} \right)^n \right|.$$
and so by Theorem 17.12.7 again it follows \( A^* \) generates a continuous semigroup, \( T(t) \) which satisfies \( \|T(t)\| \leq M \). I need to identify \( T(t) \) with \( S^*(t) \). However, from the proof of Theorem 17.12.7 and Lemma 17.12.10, it follows that for \( x \in D(A^*) \) and a suitable sequence \( \{\lambda_n\} \),

\[
(T(t)x, y) = \left( \lim_{n \to \infty} e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left( \lambda_n^2 (\lambda_n I - A)^{-1} \right)^k}{k!} x, y \right)
\]

\[
= \lim_{n \to \infty} \left( e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left( \lambda_n^2 (\lambda_n I - A)^{-1} \right)^k}{k!} x, y \right)
\]

\[
= \lim_{n \to \infty} \left( x, e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left( \lambda_n^2 (\lambda_n I - A)^{-1} \right)^k}{k!} \right)
\]

\[
= (x, S(t)y) = (S^*(t)x, y).
\]

Therefore, since \( y \) is arbitrary, \( S^*(t) = T(t) \) on \( x \in D(A^*) \) a dense set and this shows the two are equal. This proves the proposition in the case where \( S(t) \) is also bounded.

Next only assume \( S(t) \) is a continuous semigroup. Then by Proposition 17.12.5 there exists \( \alpha > 0 \) such that

\[
\|S(t)\| \leq Me^{\alpha t}.
\]

Then consider the operator \(-\alpha I + A\) and the bounded semigroup \( e^{-\alpha t}S(t) \). For \( x \in D(A) \)

\[
\lim_{h \to 0^+} \frac{e^{-\alpha h}S(h)x - x}{h} = \lim_{h \to 0^+} \left( e^{-\alpha h}S(h)x - x + \frac{e^{-\alpha h} - 1}{h} x \right)
\]

\[
= -\alpha x + Ax.
\]

Thus \(-\alpha I + A\) generates \( e^{-\alpha t}S(t) \) and it follows from the first part that \(-\alpha I + A^*\) generates \( e^{-\alpha t}S^*(t) \). Thus

\[
-\alpha x + A^*x = \lim_{h \to 0^+} \frac{e^{-\alpha h}S^*(h)x - x}{h}
\]

\[
= \lim_{h \to 0^+} \left( e^{-\alpha h}S^*(h)x - x + \frac{e^{-\alpha h} - 1}{h} x \right)
\]

\[
= -\alpha x + \lim_{h \to 0^+} \frac{S^*(h)x - x}{h}
\]

showing that \( A^* \) generates \( S^*(t) \). It follows from Proposition 17.12.5 that \( A^* \) is closed and densely defined. It is obvious \( S^*(t) \) is a semigroup. Why is it continuous?
This also follows from the first part of the argument which establishes that
\[ e^{-\alpha t} S^* (t) \]
is continuous. This proves the theorem.

### 17.12.3 Adjoints, Reflexive Banach Space

Here the adjoint of a generator of a semigroup is considered. I will show that the
adjoint of the generator generates the adjoint of the semigroup in a reflexive Banach
space. This is about as far as you can go although a general but less satisfactory
result is given in Yosida [115].

**Definition 17.12.12** Let \( A \) be a densely defined closed operator on \( H \) a real Banach
space. Then \( A^* \) is defined as follows.

\[
D (A^*) \equiv \{ y^* \in H' : |y^* (Ax)| \leq C ||x|| \text{ for all } x \in D (A) \}
\]

Then since \( D (A) \) is dense, there exists a unique element of \( H' \) denoted by \( A^* y \) such
that
\[
A^* (y^*) (x) = y^* (Ax)
\]
for all \( x \in D (A) \).

**Lemma 17.12.13** Let \( A \) be closed and densely defined on \( D (A) \subseteq H \), a Banach
space. Then \( A^* \) is also closed and densely defined. Also \( (A^*)^* = A \). In addition to
this, if \( (\lambda I - A)^{-1} \in \mathcal{L} (H, H) \), then \( (\lambda I - A^*)^{-1} \in \mathcal{L} (H', H') \) and

\[
\left( \left( \lambda I - A \right)^{-1} \right)^n = \left( \left( \lambda I - A^\ast \right)^{-1} \right)^n
\]

**Proof:** Denote by \([x, y]\) an ordered pair in \( H \times H \). Define \( \tau : H \times H \rightarrow H \times H \)
by
\[
\tau [x, y] \equiv [-y, x]
\]
A similar notation will apply to \( H' \times H' \). Then the definition of adjoint implies
that for \( \mathcal{G} (B) \) equal to the graph of \( B \),
\[
\mathcal{G} (A^*) = (\tau \mathcal{G} (A))^\perp
\]  
(17.12.86)
For \( S \subseteq H \times H \), define \( S^\perp \) by
\[
\{ [a^*, b^*] \in H' \times H' : a^* (x) + b^* (y) = 0 \text{ for all } [x, y] \in S \}
\]
If \( S \subseteq H' \times H' \) a similar definition holds.
\[
\{ [x, y] \in H \times H : a^* (x) + b^* (y) = 0 \text{ for all } [a^*, b^*] \in S \}
\]
Here is why 17.12.86 is so. For \([a^*, A^* x^*] \in \mathcal{G} (A^*) \) it follows that for all \( y \in D (A) \)
\[
x^* (Ay) = A^* x^* (y)
\]
and so for all \([y, Ay] \in \mathcal{G}(A)\),
\[-x^*(Ay) + A^*x^*(y) = 0\]
which is what it means to say \([x^*, A^*x^*] \in (\tau \mathcal{G}(A))^\perp\). This shows
\[\mathcal{G}(A^*) \subseteq (\tau \mathcal{G}(A))^\perp\]
To obtain the other inclusion, let \([a^*, b^*] \in (\tau \mathcal{G}(A))^\perp\). This means that for all \([x, Ax] \in \mathcal{G}(A)\),
\[-a^*(Ax) + b^*(x) = 0\]
In other words, for all \(x \in D(A)\),
\[|a^*(Ax)| \le ||b^*|| ||x||\]
which means by definition, \(a^* \in D(A^*)\) and \(A^*a^* = b^*\). Thus \([a^*, b^*] \in \mathcal{G}(A^*)\). This shows the other inclusion.

Note that if \(V\) is any subspace of \(H \times H\), \((V^\perp)^\perp = V\) and \(S^\perp\) is always a closed subspace. Also \(\tau\) and \(\perp\) commute. The reason for this is that \([x^*, y^*] \in (\tau V)^\perp\) means \(-x^*(y) + y^*(x) = 0\) for all \([a, b] \in V\) and \([x^*, y^*] \in \tau (V^\perp)\) means \([-y^*, x^*] \in (-V^\perp) = V^\perp\) so for all \([a, b] \in V\),
\[-y^*(a) + x^*(b) = 0\]
which says the same thing. It is also clear that \(\tau \circ \tau\) has the effect of multiplication by \(-1\). If \(V \subseteq H' \times H'\), the argument for commuting \(\perp\) and \(\tau\) is similar.

It follows from the above description of the graph of \(A^*\) that even if \(\mathcal{G}(A)\) were not closed it would still be the case that \(\mathcal{G}(A^*)\) is closed.

Why is \(D(A^*)\) dense? If it is not dense, then by a typical application of the Hahn Banach theorem, there exists \(y^{**} \in H''\) such that \(y^{**}(D(A^*)) = 0\) but \(y^{**} \neq 0\). Since \(H\) is reflexive, there exists \(y \in H\) such that \(x^*(y) = 0\) for all \(x^* \in D(A^*)\).

Thus
\[[y, 0] \in \mathcal{G}(A^*)^\perp = (\tau \mathcal{G}(A))^\perp = \tau \mathcal{G}(A)\]
and so \([0, y] \in \mathcal{G}(A)\) which means \(y = A0 = 0\), a contradiction. Thus \(D(A^*)\) is indeed dense. Note this is where it was important to assume the space is reflexive.

If you consider \(C([0, 1])\) it is not dense in \(L^\infty([0, 1])\) but if \(f \in L^1([0, 1])\) satisfies \(\int_0^1 fg dm = 0\) for all \(g \in C([0, 1])\), then \(f = 0\). Hence there is no nonzero \(f \in C([0, 1])^\perp\).
Since $A$ is a closed operator, meaning $G(A)$ is closed in $H \times H$, it follows from the above formula that

$$G((A^*)^*) = \left(\tau \left(\tau G(A)^\perp\right)^\perp\right)^\perp = \left(\tau G(A)^\perp\right)^\perp = G(A)$$

and so $(A^*)^* = A$.

Now consider the final claim. First let $y^* \in D(A^*) = D(\lambda I - A^*)$. Then letting $x \in H$ be arbitrary,

$$y^*(x) = \left(\lambda I - A\right)\left(\lambda I - A\right)^{-1} y^*(x)$$

$$= y^* \left(\lambda I - A\right)\left(\lambda I - A\right)^{-1} x$$

Since $y^* \in D(A^*)$ and $(\lambda I - A)^{-1} x \in D(A)$, this equals

$$(\lambda I - A)^* y^* \left((\lambda I - A)^{-1} x\right)$$

Now by definition, this equals

$$\left((\lambda I - A)^{-1}\right)^* (\lambda I - A)^* y^*(x)$$

It follows that for $y^* \in D(A^*)$,

$$\left((\lambda I - A)^{-1}\right)^* (\lambda I - A)^* y^*$$

$$= \left((\lambda I - A)^{-1}\right)^* (\lambda I - A^*) y^* = y^* \quad (17.12.87)$$

Next let $y^* \in H'$ be arbitrary and $x \in D(A)$

$$y^*(x) = y^* \left((\lambda I - A)^{-1} (\lambda I - A) x\right)$$

$$= \left((\lambda I - A)^{-1}\right)^* y^* \left((\lambda I - A) x\right)$$

$$= (\lambda I - A)^* \left((\lambda I - A)^{-1}\right)^* y^* (x)$$

In going from the second to the third line, the first line shows $\left((\lambda I - A)^{-1}\right)^* y^* \in D(A^*)$ and so the third line follows. Since $D(A)$ is dense, it follows

$$(\lambda I - A^*) \left((\lambda I - A)^{-1}\right)^* = I \quad (17.12.88)$$

Then (17.12.87) and (17.12.88) show $\lambda I - A^*$ is one to one and onto from $D(A^*)$ to $H'$ and

$$(\lambda I - A^*)^{-1} = \left((\lambda I - A)^{-1}\right)^*.$$
Finally, from the above,
\[
\left( (\lambda I - A^*)^{-1} \right)^n = \left( (\lambda I - A)^{-1} \right)^n = \left( (\lambda I - A)^{-1} \right)^n \ast.
\]

This proves the lemma.

With this preparation, here is an interesting result about the adjoint of the generator of a continuous bounded semigroup.

**Theorem 17.12.14** Suppose \( A \) is a densely defined closed operator which generates a continuous semigroup, \( S(t) \). Then \( A^\ast \) is also a closed densely defined operator which generates \( S^\ast(t) \) and \( S^\ast(t) \) is also a continuous semigroup.

**Proof:** First suppose \( S(t) \) is also a bounded semigroup, \( \| S(t) \| \leq M \). From Lemma 17.12.13 \( A^\ast \) is closed and densely defined. It follows from the Hille Yosida theorem, Theorem 17.12.7 that
\[
\left\| \left( (\lambda I - A)^{-1} \right)^n \right\| \leq \frac{M}{\lambda^n}
\]

From Lemma 17.12.13 and the fact the adjoint of a bounded linear operator preserves the norm,
\[
\frac{M}{\lambda^n} \geq \left\| \left( (\lambda I - A)^{-1} \right)^n \right\| = \left\| \left( (\lambda I - A)^{-1} \right)^n \right\|
\]

and so by Theorem 17.12.7 again it follows \( A^\ast \) generates a continuous semigroup, \( T(t) \) which satisfies \( \| T(t) \| \leq M \). I need to identify \( T(t) \) with \( S^\ast(t) \). However, from the proof of Theorem 17.12.7 and Lemma 17.12.13 it follows that for \( x^\ast \in D(A^\ast) \) and a suitable sequence \( \{\lambda_n\} \),
\[
T(t) x^\ast(y) = \lim_{n \to \infty} e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \lambda_n^2 (\lambda_n I - A)^{-1} \right)^k x^\ast(y)
\]

\[
= \lim_{n \to \infty} e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \lambda_n^2 (\lambda_n I - A)^{-1} \right)^k x^\ast(y)
\]

\[
= \lim_{n \to \infty} x^\ast \left( e^{-\lambda_n t} \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \lambda_n^2 (\lambda_n I - A)^{-1} \right)^k y \right) \right)
\]

\[
= x^\ast(S(t)y) = S^\ast(t)x^\ast(y).
\]
17.12. GENERAL THEORY OF CONTINUOUS SEMIGROUPS

Therefore, since \( y \) is arbitrary, \( S^* (t) = T (t) \) on \( x \in D (A^*) \) a dense set and this shows the two are equal. In particular, \( S^* (t) \) is a semigroup because \( T (t) \) is. This proves the proposition in the case where \( S (t) \) is also bounded.

Next only assume \( S (t) \) is a continuous semigroup. Then by Proposition \ref{prop:17.12.5} there exists \( \alpha > 0 \) such that

\[
\| S (t) \| \leq M e^{\alpha t}.
\]

Then consider the operator \(-\alpha I + A\) and the bounded semigroup \( e^{-\alpha t} S (t) \). For \( x \in D (A) \)

\[
\lim_{h \to 0^+} \frac{e^{-\alpha h} S (h) x - x}{h} = \lim_{h \to 0^+} \left( e^{-\alpha h} S (h) x - x + \frac{e^{-\alpha h} - 1}{h} x \right) = -\alpha x + Ax
\]

Thus \(-\alpha I + A\) generates \( e^{-\alpha t} S (t) \) and it follows from the first part that \(-\alpha I + A^*\) generates the semigroup \( e^{-\alpha t} S^* (t) \). Thus

\[
-\alpha x + A^* x = \lim_{h \to 0^+} \frac{e^{-\alpha h} S^* (h) x - x}{h} = \lim_{h \to 0^+} \left( e^{-\alpha h} S^* (h) x - x + \frac{e^{-\alpha h} - 1}{h} x \right) = -\alpha x + \lim_{h \to 0^+} \frac{S^* (h) x - x}{h}
\]

showing that \( A^* \) generates \( S^* (t) \). It follows from Proposition \ref{prop:17.12.5} that \( A^* \) is closed and densely defined. It is obvious \( S^* (t) \) is a semigroup. Why is it continuous? This also follows from the first part of the argument which establishes that

\[
t \to e^{-\alpha t} S^* (t) x
\]

is continuous. This proves the theorem.
Chapter 18

Representation Theorems

18.1 Radon Nikodym Theorem

This chapter is on various representation theorems. The first theorem, the Radon Nikodym Theorem, is a representation theorem for one measure in terms of another. The approach given here is due to Von Neumann and depends on the Riesz representation theorem for Hilbert space, Theorem 17.1.14 on Page 510.

Definition 18.1.1 Let $\mu$ and $\lambda$ be two measures defined on a $\sigma$-algebra, $\mathcal{S}$, of subsets of a set, $\Omega$. $\lambda$ is absolutely continuous with respect to $\mu$, written as $\lambda \ll \mu$, if $\lambda(E) = 0$ whenever $\mu(E) = 0$.

It is not hard to think of examples which should be like this. For example, suppose one measure is volume and the other is mass. If the volume of something is zero, it is reasonable to expect the mass of it should also be equal to zero. In this case, there is a function called the density which is integrated over volume to obtain mass. The Radon Nikodym theorem is an abstract version of this notion. Essentially, it gives the existence of the density function.

Theorem 18.1.2 (Radon Nikodym) Let $\lambda$ and $\mu$ be finite measures defined on a $\sigma$-algebra, $\mathcal{S}$, of subsets of $\Omega$. Suppose $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega, \mu)$ such that $f(x) \geq 0$ and

$$\lambda(E) = \int_E f \, d\mu.$$ 

If it is not necessarily the case that $\lambda \ll \mu$, there are two measures, $\lambda_\perp$ and $\lambda_\parallel$ such that $\lambda = \lambda_\perp + \lambda_\parallel$, $\lambda_\parallel \ll \mu$ and there exists a set of $\mu$ measure zero, $N$ such that for all $E$ measurable, $\lambda_\perp(E) = \lambda(E \cap N) = \lambda_\parallel(E \cap N)$. In this case the two measures, $\lambda_\perp$ and $\lambda_\parallel$ are unique and the representation of $\lambda = \lambda_\perp + \lambda_\parallel$ is called the Lebesgue decomposition of $\lambda$. The measure $\lambda_\parallel$ is the absolutely continuous part of $\lambda$ and $\lambda_\perp$ is called the singular part of $\lambda$. 

579
**Proof:** Let $\Lambda : L^2(\Omega, \mu + \lambda) \to \mathbb{C}$ be defined by

$$\Lambda g = \int_{\Omega} g \, d\lambda.$$ 

By Holder’s inequality,

$$|\Lambda g| \leq \left( \int_{\Omega} |g|^2 \, d(\lambda + \mu) \right)^{1/2} = \lambda(\Omega)^{1/2} ||g||_2$$

where $||g||_2$ is the $L^2$ norm of $g$ taken with respect to $\mu + \lambda$. Therefore, since $\Lambda$ is bounded, it follows from Theorem 15.1.4 on Page 421 that $\Lambda \in (L^2(\Omega, \mu + \lambda))'$, the dual space $L^2(\Omega, \mu + \lambda)$. By the Riesz representation theorem in Hilbert space, Theorem 17.1.14, there exists a unique $h \in L^2(\Omega, \mu + \lambda)$ with

$$\Lambda g = \int_{\Omega} g \, d\lambda = \int_{\Omega} h g d(\mu + \lambda). \quad (18.1.1)$$

The plan is to show $h$ is real and nonnegative at least a.e. Therefore, consider the set where $\text{Im} h > 0$.

$$E = \{x \in \Omega : \text{Im} h(x) > 0\},$$

Now let $g = X_E$ and use (18.1.1) to get

$$\lambda(E) = \int_{E} (\text{Re} \, h + i \, \text{Im} \, h) d(\mu + \lambda). \quad (18.1.2)$$

Since the left side of (18.1.2) is real, this shows

$$0 = \int_{E} (\text{Im} \, h) \, d(\mu + \lambda)$$
$$\geq \int_{E_n} (\text{Im} \, h) \, d(\mu + \lambda)$$
$$\geq \frac{1}{n} (\mu + \lambda)(E_n)$$

where

$$E_n = \left\{ x : \text{Im} h(x) \geq \frac{1}{n} \right\}$$

Thus $(\mu + \lambda)(E_n) = 0$ and since $E = \bigcup_{n=1}^{\infty} E_n$, it follows $(\mu + \lambda)(E) = 0$. A similar argument shows that for

$$E = \{x \in \Omega : \text{Im} h(x) < 0\},$$

$(\mu + \lambda)(E) = 0$. Thus there is no loss of generality in assuming $h$ is real-valued.

The next task is to show $h$ is nonnegative. This is done in the same manner as above. Define the set where it is negative and then show this set has measure zero.
Let $E \equiv \{ x : h(x) < 0 \}$ and let $E_n \equiv \{ x : h(x) < -\frac{1}{n} \}$. Then let $g = \chi_{E_n}$. Since $E = \cup_n E_n$, it follows that if $(\mu + \lambda)(E) > 0$ then this is also true for $(\mu + \lambda)(E_n)$ for all $n$ large enough. Then from (18.1.2)

$$
\lambda(E_n) = \int_{E_n} h \, d(\mu + \lambda) \leq - \left( \frac{1}{n} \right) (\mu + \lambda)(E_n) < 0,
$$
a contradiction. Thus it can be assumed $h \geq 0$.

At this point the argument splits into two cases.

**Case Where $\lambda \ll \mu$.** In this case, $h < 1$.

Let $E = [h \geq 1]$ and let $g = \chi_E$. Then

$$
\lambda(E) = \int_E h \, d(\mu + \lambda) \geq \mu(E) + \lambda(E).
$$

Therefore $\mu(E) = 0$. Since $\lambda \ll \mu$, it follows that $\lambda(E) = 0$ also. Thus it can be assumed

$$
0 \leq h(x) < 1
$$
for all $x$.

From (18.1.1), whenever $g \in L^2(\Omega, \mu + \lambda),

$$
\int_{\Omega} g(1 - h) \, d\lambda = \int_{\Omega} h \, g \, d\mu.
$$

(18.1.3)

Now let $E$ be a measurable set and define

$$
g(x) \equiv \sum_{i=0}^{n} h^i(x) \chi_E(x)
$$
in (18.1.3). This yields

$$
\int_E (1 - h^{n+1}(x)) \, d\lambda = \int_E \sum_{i=1}^{n+1} h^i(x) \, d\mu.
$$

(18.1.4)

Let $f(x) = \sum_{i=1}^{\infty} h^i(x)$ and use the Monotone Convergence theorem in (18.1.3) to let $n \to \infty$ and conclude

$$
\lambda(E) = \int_E f \, d\mu.
$$

$f \in L^1(\Omega, \mu)$ because $\lambda$ is finite.

The function, $f$ is unique $\mu$ a.e. because, if $g$ is another function which also serves to represent $\lambda$, consider for each $n \in \mathbb{N}$ the set,

$$
E_n \equiv \left\{ f - g > \frac{1}{n} \right\}
$$

and conclude that

$$
0 = \int_{E_n} (f - g) \, d\mu \geq \frac{1}{n} \mu(E_n).
$$
Therefore, $\mu(E_n) = 0$. It follows that
\[
\mu([f - g > 0]) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0
\]
Similarly, the set where $g$ is larger than $f$ has measure zero. This proves the theorem.

Case where it is not necessarily true that $\lambda \ll \mu$.

In this case, let $N = [h \geq 1]$ and let $g = X_N$. Then
\[
\lambda(N) = \int_N h \, d(\mu + \lambda) \geq \mu(N) + \lambda(N).
\]
and so $\mu(N) = 0$. Now define a measure, $\lambda_{\perp}$ by
\[
\lambda_{\perp}(E) \equiv \lambda(E \cap N)
\]
so $\lambda_{\perp}(E \cap N) = \lambda(E \cap N \cap N) \equiv \lambda_{\perp}(E)$ and let $\lambda_{||} \equiv \lambda - \lambda_{\perp}$. Therefore,
\[
\mu(E) = \mu(E \cap N^C)
\]
Also,
\[
\lambda_{||}(E) = \lambda(E) - \lambda_{\perp}(E) \equiv \lambda(E) - \lambda(E \cap N) = \lambda(E \cap N^C).
\]
Suppose $\lambda_{||}(E) > 0$. Therefore, since $h < 1$ on $N^C$
\[
\lambda_{||}(E) = \lambda(E \cap N^C) = \int_{E \cap N^C} h \, d(\mu + \lambda) < \mu(E \cap N^C) + \lambda(E \cap N^C) = \mu(E) + \lambda_{||}(E),
\]
which is a contradiction unless $\mu(E) > 0$. Therefore, $\lambda_{||} \ll \mu$ because if $\mu(E) = 0$, the above inequality cannot hold.

It only remains to verify the two measures $\lambda_{\perp}$ and $\lambda_{||}$ are unique. Suppose then that $\nu_1$ and $\nu_2$ play the roles of $\lambda_{\perp}$ and $\lambda_{||}$ respectively. Let $N_1$ play the role of $N$ in the definition of $\nu_1$ and let $f_1$ play the role of $f$ for $\nu_2$. I will show that $f = f_1 \mu$ a.e. Let $E_k \equiv [f_1 - f > 1/k]$ for $k \in \mathbb{N}$. Then on observing that $\lambda_{\perp} - \nu_1 = \nu_2 - \lambda_{||}$
\[
0 = (\lambda_{\perp} - \nu_1)\left(E_k \cap (N_1 \cup N)^C\right) = \int_{E_k \cap (N_1 \cup N)^C} (g_1 - g) \, d\mu
\]
\[
\geq \frac{1}{k} \mu\left(E_k \cap (N_1 \cup N)^C\right) = \frac{1}{k} \mu(E_k).
\]
and so $\mu(E_k) = 0$. Therefore, $\mu([f_1 - f > 0]) = 0$ because $[f_1 - f > 0] = \bigcup_{k=1}^{\infty} E_k$. It follows $f_1 \leq f \mu$ a.e. Similarly, $f_1 \geq f \mu$ a.e. Therefore, $\nu_2 = \lambda_{||}$ and so $\lambda_{\perp} = \nu_1$ also.

The $f$ in the theorem for the absolutely continuous case is sometimes denoted by $\frac{d\lambda}{d\mu}$ and is called the Radon Nikodym derivative.

The next corollary is a useful generalization to $\sigma$ finite measure spaces.

Corollary 18.1.3 Suppose $\lambda \ll \mu$ and there exist sets $S_n \in \mathcal{S}$ with 

$$S_n \cap S_m = \emptyset, \quad \cup_{n=1}^{\infty} S_n = \Omega,$$

and $\lambda(S_n), \mu(S_n) < \infty$. Then there exists $f \geq 0$, where $f$ is $\mu$ measurable, and 

$$\lambda(E) = \int_E f \, d\mu$$

for all $E \in \mathcal{S}$. The function $f$ is $\mu + \lambda$ a.e. unique.

Proof: Define the $\sigma$ algebra of subsets of $S_n$, 

$$S_n = \{E \cap S_n : E \in \mathcal{S}\}.$$ 

Then both $\lambda$ and $\mu$ are finite measures on $S_n$, and $\lambda \ll \mu$. Thus, by Theorem

there exists a nonnegative $S_n$ measurable function $f_n$ with $\lambda(E) = \int_E f_n \, d\mu$

for all $E \in S_n$. Define $f(x) = f_n(x)$ for $x \in S_n$. Since the $S_n$ are disjoint and their union is all of $\Omega$, this defines $f$ on all of $\Omega$. The function, $f$ is measurable because 

$$f^{-1}((a, \infty]) = \cup_{n=1}^{\infty} f_n^{-1}((a, \infty]) \in \mathcal{S}.$$ 

Also, for $E \in \mathcal{S}$, 

$$\lambda(E) = \sum_{n=1}^{\infty} \lambda(E \cap S_n) = \sum_{n=1}^{\infty} \int X_{E \cap S_n}(x) f_n(x) \, d\mu$$ 

$$= \sum_{n=1}^{\infty} \int X_{E \cap S_n}(x) f(x) \, d\mu$$

By the monotone convergence theorem 

$$\sum_{n=1}^{\infty} \int X_{E \cap S_n}(x) f(x) \, d\mu = \lim_{N \to \infty} \sum_{n=1}^{N} \int X_{E \cap S_n}(x) f(x) \, d\mu$$ 

$$= \lim_{N \to \infty} \sum_{n=1}^{N} X_{E \cap S_n}(x) f(x) \, d\mu$$ 

$$= \int \sum_{n=1}^{\infty} X_{E \cap S_n}(x) f(x) \, d\mu = \int_{E} f \, d\mu.$$ 

This proves the existence part of the corollary.

To see $f$ is unique, suppose $f_1$ and $f_2$ both work and consider for $n \in \mathbb{N}$ 

$$E_k = \left[ f_1 - f_2 > \frac{1}{k} \right].$$ 

Then 

$$0 = \lambda(E_k \cap S_n) - \lambda(E_k \cap S_n) = \int_{E_k \cap S_n} f_1(x) - f_2(x) \, d\mu.$$
Hence $\mu(E_k \cap S_n) = 0$ for all $n$ so
\[ \mu(E_k) = \lim_{n \to \infty} \mu(E \cap S_n) = 0. \]
Hence $\mu([f_1 - f_2 > 0]) \leq \sum_{k=1}^{\infty} \mu(E_k) = 0$. Therefore, $\lambda([f_1 - f_2 > 0]) = 0$ also. Similarly
\[ (\mu + \lambda)([f_1 - f_2 < 0]) = 0. \]

This version of the Radon Nikodym theorem will suffice for most applications, but more general versions are available. To see one of these, one can read the treatment in Hewitt and Stromberg [RS]. This involves the notion of decomposable measure spaces, a generalization of $\sigma$ finite.

Not surprisingly, there is a simple generalization of the Lebesgue decomposition part of Theorem [RS].

**Corollary 18.1.4** Let $(\Omega, \mathcal{S})$ be a set with a $\sigma$ algebra of sets. Suppose $\lambda$ and $\mu$ are two measures defined on the sets of $\mathcal{S}$ and suppose there exists a sequence of disjoint sets of $\mathcal{S}$, $\{\Omega_i\}_{i=1}^{\infty}$ such that $\lambda(\Omega_i), \mu(\Omega_i) < \infty$. Then there is a set of $\mu$ measure zero, $N$ and measures $\lambda_|\parallel$ and $\lambda|\parallel$ such that
\[ \lambda_|\parallel + \lambda|\parallel = \lambda, \lambda_|\parallel \ll \mu, \lambda_|\parallel(E) = \lambda(E \cap N) = \lambda|\parallel(E \cap N). \]

**Proof:** Let $\mathcal{S}_i \equiv \{E \cap \Omega_i : E \in \mathcal{S}\}$ and for $E \in \mathcal{S}_i$, let $\lambda_|\parallel^i(E) = \lambda(E)$ and $\mu^i(E) = \mu(E)$. Then by Theorem [RS], there exist unique measures $\lambda_|\parallel^i$ and $\lambda|\parallel^i$ such that $\lambda_i = \lambda_|\parallel^i + \lambda|\parallel^i$, a set of $\mu^i$ measure zero, $N_i \in \mathcal{S}_i$ such that for all $E \in \mathcal{S}_i$, $\lambda_|\parallel^i(E) = \lambda^i(E \cap N_i)$ and $\lambda|\parallel^i \ll \mu^i$. Define for $E \in \mathcal{S}$
\[ \lambda_|\parallel(E) \equiv \sum_{i} \lambda_|\parallel^i(E \cap \Omega_i), \lambda|\parallel(E) \equiv \sum_{i} \lambda|\parallel^i(E \cap \Omega_i), \quad N \equiv \cup_i N_i. \]

First observe that $\lambda_|\parallel$ and $\lambda|\parallel$ are measures.
\[ \lambda_|\parallel\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{i} \lambda_|\parallel^i\left(\bigcup_{j=1}^{\infty} E_j \cap \Omega_i\right) = \sum_{i} \sum_{j} \lambda_|\parallel^i\left(E_j \cap \Omega_i\right) = \sum_{j} \sum_{i} \lambda_|\parallel^i\left(E_j \cap \Omega_i\right) = \sum_{j} \lambda_|\parallel\left(E_j\right). \]

The argument for $\lambda|\parallel$ is similar. Now
\[ \mu(N) = \sum_{i} \mu(N \cap \Omega_i) = \sum_{i} \mu^i(N_i) = 0 \]
and
\[ \lambda_|\parallel(E) \equiv \sum_{i} \lambda_|\parallel^i(E \cap \Omega_i) = \sum_{i} \lambda^i(E \cap \Omega_i \cap N_i) = \sum_{i} \lambda(E \cap \Omega_i \cap N) = \lambda(E \cap N). \]
Also if \( \mu(E) = 0 \), then \( \mu^i(E \cap \Omega_i) = 0 \) and so \( \lambda^0_i(E \cap \Omega_i) = 0 \). Therefore,

\[
\lambda_\| (E) = \sum_i \lambda^0_i(E \cap \Omega_i) = 0.
\]

The decomposition is unique because of the uniqueness of the \( \lambda^0_i \) and \( \lambda_L^i \) and the observation that some other decomposition must coincide with the given one on the \( \Omega_i \).

### 18.2 Vector Measures

The next topic will use the Radon Nikodym theorem. It is the topic of vector and complex measures. The main interest is in complex measures although a vector measure can have values in any topological vector space. Whole books have been written on this subject. See for example the book by Diestal and Uhl \([38]\) titled Vector measures.

**Definition 18.2.1** Let \((V, \|\cdot\|)\) be a normed linear space and let \((\Omega, S)\) be a measure space. A function \( \mu : S \to V \) is a vector measure if \( \mu \) is countably additive. That is, if \( \{E_i\}_{i=1}^\infty \) is a sequence of disjoint sets of \( S \),

\[
\mu(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i).
\]

Note that it makes sense to take finite sums because it is given that \( \mu \) has values in a vector space in which vectors can be summed. In the above, \( \mu(E_i) \) is a vector. It might be a point in \( \mathbb{R}^n \) or in any other vector space. In many of the most important applications, it is a vector in some sort of function space which may be infinite dimensional. The infinite sum has the usual meaning. That is

\[
\sum_{i=1}^\infty \mu(E_i) = \lim_{n \to \infty} \sum_{i=1}^n \mu(E_i)
\]

where the limit takes place relative to the norm on \( V \).

**Definition 18.2.2** Let \( (\Omega, S) \) be a measure space and let \( \mu \) be a vector measure defined on \( S \). A subset, \( \pi(E) \), of \( S \) is called a partition of \( E \) if \( \pi(E) \) consists of finitely many disjoint sets of \( S \) and \( \cup \pi(E) = E \). Let

\[
|\mu|(E) = \sup \left\{ \sum_{F \in \pi(E)} \|\mu(F)\| : \pi(E) \text{ is a partition of } E \right\}.
\]

\( |\mu| \) is called the total variation of \( \mu \).

The next theorem may seem a little surprising. It states that, if finite, the total variation is a nonnegative measure.
THEOREM 18.2.3 If \(|\mu| (\Omega) < \infty\), then \(|\mu|\) is a measure on \(S\). Even if \(|\mu| (\Omega) = \infty\), \(|\mu| (\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} |\mu| (E_i)\). That is \(|\mu|\) is subadditive and \(|\mu| (A) \leq |\mu| (B)\) whenever \(A, B \in S\) with \(A \subseteq B\).

**Proof:** Consider the last claim. Let \(a < |\mu| (A)\) and let \(\pi (A)\) be a partition of \(A\) such that

\[
a < \sum_{F \in \pi (A)} ||\mu (F)||.
\]

Then \(\pi (A) \cup \{B \setminus A\}\) is a partition of \(B\) and

\[
|\mu| (B) \geq \sum_{F \in \pi (A)} ||\mu (F)|| + ||\mu (B \setminus A)|| > a.
\]

Since this is true for all such \(a\), it follows \(|\mu| (B) \geq |\mu| (A)\) as claimed.

Let \(\{E_j\}_{j=1}^{\infty}\) be a sequence of disjoint sets of \(S\) and let \(E_\infty = \cup_{j=1}^{\infty} E_j\). Then letting \(a < |\mu| (E_\infty)\), it follows from the definition of total variation there exists a partition of \(E_\infty\), \(\pi (E_\infty) = \{A_1, \cdots, A_n\}\) such that

\[
a < \sum_{i=1}^{n} ||\mu (A_i)||.
\]

Also,

\[
A_i = \cup_{j=1}^{\infty} A_i \cap E_j
\]

and so by the triangle inequality, \(||\mu (A_i)|| \leq \sum_{j=1}^{\infty} ||\mu (A_i \cap E_j)||\). Therefore, by the above, and either Fubini’s theorem or Lemma 9.3.3 on Page 221

\[
a < \sum_{i=1}^{n} ||\mu (A_i || \geq ||\mu (A_i))
\]

\[
= \sum_{j=1}^{\infty} \sum_{i=1}^{n} ||\mu (A_i \cap E_j)||
\]

\[
\leq \sum_{j=1}^{\infty} |\mu| (E_j)
\]

because \(\{A_i \cap E_j\}_{i=1}^{n}\) is a partition of \(E_j\).

Since \(a\) is arbitrary, this shows

\[
|\mu| (\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} |\mu| (E_j).
\]

If the sets, \(E_j\) are not disjoint, let \(F_1 = E_1\) and if \(F_n\) has been chosen, let \(F_{n+1} = E_{n+1} \setminus \cup_{i=1}^{n} E_i\). Thus the sets, \(F_i\) are disjoint and \(\cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} E_i\). Therefore,

\[
|\mu| (\cup_{j=1}^{\infty} E_j) = |\mu| (\cup_{j=1}^{\infty} F_j) \leq \sum_{j=1}^{\infty} |\mu| (F_j) \leq \sum_{j=1}^{\infty} |\mu| (E_j)
\]
18.2. VECTOR MEASURES

and proves $|\mu|$ is always subadditive as claimed regardless of whether $|\mu|(\Omega) < \infty$.

Now suppose $|\mu|(\Omega) < \infty$ and let $E_1$ and $E_2$ be sets of $\mathcal{S}$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A^1_1 \cdots A^k_n\} = \pi(E_i)$, a partition of $E_i$ which is chosen such that

$$|\mu|(E_i) - \varepsilon < \sum_{j=1}^{n_i} ||\mu(A^i_j)|| \quad i = 1, 2.$$  

Such a partition exists because of the definition of the total variation. Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ denoted by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|\mu|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} ||\mu(F)|| > |\mu|(E_1) + |\mu|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|\mu|(E_1 \cup E_2) \geq |\mu|(E_1) + |\mu|(E_2). \quad (18.2.5)$$

Then (18.2.4) implies that whenever the $E_i$ are disjoint, $|\mu|(\cup_{j=1}^{n} E_j) \geq \sum_{j=1}^{n} |\mu|(E_j)$. Therefore,

$$\sum_{j=1}^{\infty} |\mu|(E_j) \geq |\mu|(\cup_{j=1}^{\infty} E_j) \geq |\mu|(\cup_{j=1}^{n} E_j) \geq \sum_{j=1}^{n} |\mu|(E_j).$$

Since $n$ is arbitrary,

$$|\mu|(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |\mu|(E_j)$$

which shows that $|\mu|$ is a measure as claimed. $\blacksquare$

The following corollary is interesting. It concerns the case that $\mu$ is only finitely additive.

**Corollary 18.2.4** Suppose $(\Omega, \mathcal{F})$ is a set with a $\sigma$ algebra of subsets $\mathcal{F}$ and suppose $\mu : \mathcal{F} \to \mathbb{C}$ is only finitely additive. That is, $\mu(\cup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mu(E_i)$ whenever the $E_i$ are disjoint. Then $|\mu|$, defined in the same way as above, is also finitely additive provided $|\mu|$ is finite.

**Proof:** Say $E \cap F = \emptyset$ for $E, F \in \mathcal{F}$. Let $\pi(E), \pi(F)$ suitable partitions for which the following holds,

$$|\mu|(E \cup F) \geq \sum_{A \in \pi(E)} |\mu(A)| + \sum_{B \in \pi(F)} |\mu(B)| \geq |\mu|(E) + |\mu|(F) - 2\varepsilon.$$  

Similar considerations apply to any finite union.
Now let $E = \bigcup_{i=1}^{n} E_i$ where the $E_i$ are disjoint. Then letting $\pi(E)$ be a partition of $E$,

$$|\mu|(E) - \varepsilon \leq \sum_{F \in \pi(E)} |\mu(F)|,$$

it follows that

$$|\mu|(E) \leq \varepsilon + \sum_{F \in \pi(E)} |\mu(F)| = \varepsilon + \sum_{F \in \pi(E)} \left| \sum_{i=1}^{n} \mu(F \cap E_i) \right|$$

$$\leq \varepsilon + \sum_{i=1}^{n} \left| \sum_{F \in \pi(E)} |\mu(F \cap E_i)| \right| \leq \varepsilon + \sum_{i=1}^{n} |\mu|(E_i)$$

which shows $|\mu|$ is finitely additive. ■

In the case that $\mu$ is a complex measure, it is always the case that $|\mu|(\Omega) < \infty$.

**Theorem 18.2.5** Suppose $\mu$ is a complex measure on $(\Omega, S)$ where $S$ is a $\sigma$-algebra of subsets of $\Omega$. That is, whenever, $\{E_i\}$ is a sequence of disjoint sets of $S$,

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Then $|\mu|(\Omega) < \infty$.

**Proof:** First here is a claim.

**Claim:** Suppose $|\mu|(E) = \infty$. Then there are disjoint subsets of $E$, $A$ and $B$ such that $E = A \cup B$, $|\mu(A)|, |\mu(B)| > 1$ and $|\mu|(B) = \infty$.

**Proof of the claim:** From the definition of $|\mu|$, there exists a partition of $E, \pi(E)$ such that

$$\sum_{F \in \pi(E)} |\mu(F)| > 20 (1 + |\mu(E)|). \quad (18.2.6)$$

Here 20 is just a nice sized number. No effort is made to be delicate in this argument. Also note that $\mu(E) \in \mathbb{C}$ because it is given that $\mu$ is a complex measure. Consider the following picture consisting of two lines in the complex plane having slopes 1 and -1 which intersect at the origin, dividing the complex plane into four closed sets, $R_1, R_2, R_3,$ and $R_4$ as shown.
Let $\pi_i$ consist of those sets, $A$ of $\pi(E)$ for which $\mu(A) \in R_i$. Thus, some sets, $A$ of $\pi(E)$ could be in two of the $\pi_i$ if $\mu(A)$ is on one of the intersecting lines. This is not important. The thing which is important is that if $\mu(A) \in R_1$ or $R_3$, then $\sqrt{2} |\mu(A)| \leq |\text{Re}(\mu(A))|$ and if $\mu(A) \in R_2$ or $R_4$ then $\sqrt{2} |\mu(A)| \leq |\text{Im}(\mu(A))|$ and $\text{Re}(z)$ has the same sign for $z$ in $R_1$ and $R_3$ while $\text{Im}(z)$ has the same sign for $z$ in $R_2$ or $R_4$. Then by (18.2.6), it follows that for some $i$,

$$\sum_{F \in \pi_i} |\mu(F)| > 5 (1 + |\mu(E)|). \quad (18.2.7)$$

Suppose $i$ equals 1 or 3. A similar argument using the imaginary part applies if $i$ equals 2 or 4. Then,

$$\left| \sum_{F \in \pi_i} \mu(F) \right| \geq \left| \sum_{F \in \pi_i} \text{Re}(\mu(F)) \right| = \sum_{F \in \pi_i} |\text{Re}(\mu(F))| \geq \frac{\sqrt{2}}{2} \sum_{F \in \pi_i} |\mu(F)| > \frac{5 \sqrt{2}}{2} (1 + |\mu(E)|).$$

Now letting $C$ be the union of the sets in $\pi_i$,

$$|\mu(C)| = \left| \sum_{F \in \pi_i} \mu(F) \right| > \frac{5}{2} (1 + |\mu(E)|) > 1. \quad (18.2.8)$$

Define $D \equiv E \setminus C$.

Then $\mu(C) + \mu(E \setminus C) = \mu(E)$ and so

$$\frac{5}{2} (1 + |\mu(E)|) < |\mu(C)| = |\mu(E) - \mu(E \setminus C)| = |\mu(E) - \mu(D)| \leq |\mu(E)| + |\mu(D)|$$

and so

$$1 < \frac{5}{2} + \frac{3}{2} |\mu(E)| < |\mu(D)|.$$
Now since \(|\mu|(E) = \infty\), it follows from Theorem 18.2.4 that \(\infty = |\mu|(E) \leq |\mu|(C) + |\mu|(D)\) and so either \(|\mu|(C) = \infty\) or \(|\mu|(D) = \infty\). If \(|\mu|(C) = \infty\), let \(B = C\) and \(A = D\). Otherwise, let \(B = D\) and \(A = C\). This proves the claim.

Now suppose \(|\mu|(\Omega) = \infty\). Then from the claim, there exist \(A_1\) and \(B_1\) such that \(|\mu|(B_1) = \infty\), \(|\mu(B_1)|, \lambda(A_1)| > 1\), and \(A_1 \cup B_1 = \Omega\). Let \(B_1 = \Omega \setminus A\) play the same role as \(\Omega\) and obtain \(B_2, B_2 \subseteq B_1\) such that \(|\mu|(B_2) = \infty, |\mu(B_2)|, |\mu(A_2)| > 1\), and \(B_2 \cup B_2 = B_1\). Continue in this way to obtain a sequence of disjoint sets, \(\{A_i\}\) such that \(|\mu(A_i)| > 1\). Then since \(\mu\) is a measure,

\[
\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)
\]

but this is impossible because \(\lim_{i \to \infty} \mu(A_i) \neq 0\). This proves the theorem.

**Theorem 18.2.6** Let \((\Omega, S)\) be a measure space and let \(\lambda : S \to \mathbb{C}\) be a complex vector measure. Thus \(|\lambda|(\Omega) < \infty\). Let \(\mu : S \to [0, \mu(\Omega)]\) be a finite measure such that \(\lambda \ll \mu\). Then there exists a unique \(f \in L^1(\Omega)\) such that for all \(E \in S\),

\[
\int_E f \, d\mu = \lambda(E).
\]

**Proof:** It is clear that \(\text{Re} \lambda\) and \(\text{Im} \lambda\) are real-valued vector measures on \(S\). Since \(|\lambda|(\Omega) < \infty\), it follows easily that \(|\text{Re} \lambda|(\Omega)\) and \(|\text{Im} \lambda|(\Omega) < \infty\). This is clear because

\[
|\lambda(E)| \geq |\text{Re} \lambda(E)|, |\text{Im} \lambda(E)|.
\]

Therefore, each of

\[
\frac{|\text{Re} \lambda| + \text{Re} \lambda}{2}, \quad \frac{|\text{Re} \lambda| - \text{Re}(\lambda)}{2}, \quad \frac{|\text{Im} \lambda| + \text{Im} \lambda}{2}, \quad \text{and} \quad \frac{|\text{Im} \lambda| - \text{Im}(\lambda)}{2}
\]

are finite measures on \(S\). It is also clear that each of these finite measures are absolutely continuous with respect to \(\mu\) and so there exist unique nonnegative functions in \(L^1(\Omega)\), \(f_1, f_2, g_1, g_2\) such that for all \(E \in S\),

\[
\begin{align*}
\frac{1}{2}(|\text{Re} \lambda| + \text{Re} \lambda)(E) &= \int_E f_1 \, d\mu, \\
\frac{1}{2}(|\text{Re} \lambda| - \text{Re}(\lambda))(E) &= \int_E f_2 \, d\mu, \\
\frac{1}{2}(|\text{Im} \lambda| + \text{Im} \lambda)(E) &= \int_E g_1 \, d\mu, \\
\frac{1}{2}(|\text{Im} \lambda| - \text{Im}(\lambda))(E) &= \int_E g_2 \, d\mu.
\end{align*}
\]

Now let \(f = f_1 - f_2 + i(g_1 - g_2)\).

The following corollary is about representing a vector measure in terms of its total variation. It is like representing a complex number in the form \(re^{i\theta}\). The proof requires the following lemma.
Lemma 18.2.7 Suppose $\Omega, S, \mu$ is a measure space and $f$ is a function in $L^1(\Omega, \mu)$ with the property that 
\[ |\int_E f \, d\mu| \leq \mu(E) \]
for all $E \in S$. Then $|f| \leq 1$ a.e.

Proof of the lemma: Consider the following picture.

where $B(p, r) \cap B(0, 1) = \emptyset$. Let $E = f^{-1}(B(p, r))$. In fact $\mu(E) = 0$. If $\mu(E) \neq 0$ then
\[ \left| \frac{1}{\mu(E)} \int_E f \, d\mu - p \right| = \left| \frac{1}{\mu(E)} \int_E (f - p) \, d\mu \right| \leq \frac{1}{\mu(E)} \int_E |f - p| \, d\mu < r \]
because on $E$, $|f(x) - p| < r$. Hence
\[ \left| \frac{1}{\mu(E)} \int_E f \, d\mu \right| > 1 \]
because it is closer to $p$ than $r$. (Refer to the picture.) However, this contradicts the assumption of the lemma. It follows $\mu(E) = 0$. Since the set of complex numbers, $z$ such that $|z| > 1$ is an open set, it equals the union of countably many balls, \( \{B_i\}_{i=1}^{\infty} \). Therefore,
\[ \mu \left( f^{-1}(\{z \in \mathbb{C} : |z| > 1\}) \right) = \mu \left( \bigcup_{k=1}^{\infty} f^{-1}(B_k) \right) \leq \sum_{k=1}^{\infty} \mu \left( f^{-1}(B_k) \right) = 0. \]

Thus $|f(x)| \leq 1$ a.e. as claimed. This proves the lemma.

Corollary 18.2.8 Let $\lambda$ be a complex vector measure with $||\lambda||(\Omega) < \infty$. Then there exists a unique $f \in L^1(\Omega)$ such that $\lambda(E) = \int_E f \, d|\lambda|$. Furthermore, $|f| = 1$ for $|\lambda|$ a.e. This is called the polar decomposition of $\lambda$.

Proof: First note that $\lambda \ll |\lambda|$ and so such an $L^1$ function exists and is unique. It is required to show $|f| = 1$ a.e. If $||\lambda||(\Omega) \neq 0$,
\[ \left| \frac{\lambda(E)}{||\lambda||(\Omega)} \right| = \left| \frac{1}{||\lambda||(\Omega)} \int_E f \, d|\lambda| \right| \leq 1. \]

\[ ^1 \text{As proved above, the assumption that } |\lambda|(\Omega) < \infty \text{ is redundant.} \]
Therefore by Lemma 18.2.7, \(|f| \leq 1, |\lambda|\) a.e. Now let

\[ E_n = \left[ |f| \leq 1 - \frac{1}{n} \right]. \]

Let \(\{F_1, \ldots, F_m\}\) be a partition of \(E_n\). Then

\[
\sum_{i=1}^{m} |\lambda(F_i)| = \sum_{i=1}^{m} \left| \int_{F_i} f d|\lambda| \right| \leq \sum_{i=1}^{m} \int_{F_i} |f| d|\lambda| \\
\leq \sum_{i=1}^{m} \int_{F_i} \left( 1 - \frac{1}{n} \right) d|\lambda| = \sum_{i=1}^{m} \left( 1 - \frac{1}{n} \right) |\lambda|(F_i) \\
= |\lambda|(E_n) \left( 1 - \frac{1}{n} \right).
\]

Then taking the supremum over all partitions,

\[
|\lambda|(E_n) \leq \left( 1 - \frac{1}{n} \right) |\lambda|(E_n)
\]

which shows \(|\lambda|(E_n) = 0\). Hence \(|\lambda|([|f| < 1]) = 0\) because \([|f| < 1] = \cup_{n=1}^{\infty} E_n\). This proves Corollary 18.2.8.

**Corollary 18.2.9** Let \(\lambda\) be a complex vector measure such that \(\lambda \ll \mu\) where \(\mu\) is \(\sigma\)-finite. Then there exists a unique \(g \in L^1(\Omega, \mu)\) such that \(\lambda(E) = \int_{E} g d\mu\).

**Proof:** By Corollary 18.2.8 and Theorem 18.2.5 which says that \(|\lambda|\) is finite, there exists a unique \(f\) such that \(|f| = 1 |\lambda|\) a.e. and

\[
\lambda(E) = \int_{E} f d|\lambda|.
\]

Now \(|\lambda| \ll \mu\) and so it follows from Corollary 18.2.8 there exists a unique nonnegative measurable function \(h\) such that for all \(E\) measurable,

\[
|\lambda|(E) = \int_{E} h d\mu
\]

where since \(|\lambda|\) is finite, \(h \in L^1(\Omega, \mu)\). It follows from approximating \(f\) with simple functions and using the above formula that

\[
\lambda(E) = \int_{E} f h d\mu.
\]

Then let \(g = L^1(\Omega, \mu)\). This proves the corollary.

**Corollary 18.2.10** Suppose \((\Omega, S)\) is a measure space and \(\mu\) is a finite nonnegative measure on \(S\). Then for \(h \in L^1(\mu)\), define a complex measure, \(\lambda\) by

\[
\lambda(E) = \int_{E} h d\mu.
\]
18.3. REPRESENTATION THEOREMS FOR THE DUAL SPACE OF $L^p$  593

Then

$$|\lambda|(E) = \int_E |h| \, d\mu.$$  

Furthermore, $|h| = \overline{g}h$ where $gd|\lambda|$ is the polar decomposition of $\lambda$,

$$\lambda(E) = \int_E gd|\lambda|$$

Proof: From Corollary 18.2.8 there exists $g$ such that $|g| = 1, |\lambda|$ a.e. and for all $E \in S$

$$\lambda(E) = \int_E gd|\lambda| = \int_E hd\mu.$$  

Let $s_n$ be a sequence of simple functions converging pointwise to $\overline{g}$. Then from the above,

$$\int_E gs_n d|\lambda| = \int_E s_n hd\mu.$$  

Passing to the limit using the dominated convergence theorem,

$$\int_E d|\lambda| = \int_E \overline{g}hd\mu.$$  

It follows $\overline{g}h \geq 0$ a.e. and $|\overline{g}| = 1$. Therefore, $|h| = |\overline{g}h| = \overline{g}h$. It follows from the above, that

$$|\lambda|(E) = \int_E d|\lambda| = \int_E \overline{g}hd\mu = \int_E d|\lambda| = \int_E |h| \, d\mu$$

and this proves the corollary.

18.3 Representation Theorems For The Dual Space Of $L^p$

Recall the concept of the dual space of a Banach space in the Chapter on Banach space starting on Page 419. The next topic deals with the dual space of $L^p$ for $p \geq 1$ in the case where the measure space is $\sigma$ finite or finite. In what follows $q = \infty$ if $p = 1$ and otherwise, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 18.3.1 (Riesz representation theorem) Let $p > 1$ and let $(\Omega, S, \mu)$ be a finite measure space. If $\Lambda \in (L^p(\Omega))'$, then there exists a unique $h \in L^q(\Omega)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that

$$\Lambda f = \int_{\Omega} hf \, d\mu.$$  

This function satisfies $||h||_q = ||\Lambda||$ where $||\Lambda||$ is the operator norm of $\Lambda$.  

Proof: (Uniqueness) If $h_1$ and $h_2$ both represent $\Lambda$, consider

$$f = |h_1 - h_2|^q - 2(\overline{h_1} - \overline{h_2}),$$

where $\overline{h}$ denotes complex conjugation. By Holder’s inequality, it is easy to see that $f \in L^p(\Omega)$. Thus

$$0 = \Lambda f - \Lambda f = \int h_1|h_1 - h_2|^q - 2(\overline{h_1} - \overline{h_2}) - h_2|h_1 - h_2|^q - 2(\overline{h_1} - \overline{h_2})d\mu = \int |h_1 - h_2|^q d\mu.$$ 

Therefore $h_1 = h_2$ and this proves uniqueness.

Now let $\lambda(E) = \Lambda(X_E)$. Since this is a finite measure space $X_E$ is an element of $L^p(\Omega)$ and so it makes sense to write $\Lambda(X_E)$. In fact $\lambda$ is a complex measure having finite total variation. Let $A_1, \cdots, A_n$ be a partition of $\Omega$.

$$|\Lambda X_{A_i}| = w_i(\Lambda X_{A_i}) = \Lambda(w_i X_{A_i})$$

for some $w_i \in \mathbb{C}$, $|w_i| = 1$. Thus

$$\sum_{i=1}^{n} |\lambda(A_i)| = \sum_{i=1}^{n} |\Lambda(X_{A_i})| = \Lambda\left(\sum_{i=1}^{n} w_i X_{A_i}\right)$$

$$\leq ||\Lambda||(\int |\sum_{i=1}^{n} w_i X_{A_i}|^p d\mu)^{\frac{1}{p}} = ||\Lambda||(\int \mu)^{\frac{1}{p}} = ||\Lambda||\mu(\Omega)^{\frac{1}{p}}.$$ 

This is because if $x \in \Omega$, $x$ is contained in exactly one of the $A_i$ and so the absolute value of the sum in the first integral above is equal to 1. Therefore $|\lambda|(\Omega) < \infty$ because this was an arbitrary partition. Also, if $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathcal{S}$, let

$$F_n = \bigcup_{i=1}^{n} E_i, \quad F = \bigcup_{i=1}^{\infty} E_i.$$ 

Then by the Dominated Convergence theorem,

$$||X_{F_n} - X_F||_p \to 0.$$ 

Therefore, by continuity of $\Lambda$,

$$\lambda(F) = \Lambda(X_F) = \lim_{n \to \infty} \Lambda(X_{F_n}) = \lim_{n \to \infty} \sum_{k=1}^{n} \Lambda(X_{E_k}) = \sum_{k=1}^{\infty} \lambda(E_k).$$

This shows $\lambda$ is a complex measure with $|\lambda|$ finite.

It is also clear from the definition of $\lambda$ that $\lambda \ll \mu$. Therefore, by the Radon Nikodym theorem, there exists $h \in L^1(\Omega)$ with

$$\lambda(E) = \int_E hd\mu = \Lambda(X_E).$$
18.3. REPRESENTATION THEOREMS FOR THE DUAL SPACE OF $L^p$

Actually $h \in L^q$ and satisfies the other conditions above. Let $s = \sum_{i=1}^{m} c_i X_{E_i}$ be a simple function. Then since $\Lambda$ is linear,

$$\Lambda(s) = \sum_{i=1}^{m} c_i \Lambda(X_{E_i}) = \sum_{i=1}^{m} c_i \int_{E_i} h \, d\mu = \int h \, s \, d\mu. \quad (18.3.9)$$

**Claim:** If $f$ is uniformly bounded and measurable, then

$$\Lambda (f) = \int h \, f \, d\mu.$$  

**Proof of claim:** Since $f$ is bounded and measurable, there exists a sequence of simple functions, $\{s_n\}$ which converges to $f$ pointwise and in $L^p(\Omega)$. This follows from Theorem 9.3.9 on Page 227 upon breaking $f$ up into positive and negative parts of real and complex parts. In fact this theorem gives uniform convergence. Then

$$\Lambda (f) = \lim_{n \to \infty} \Lambda (s_n) = \lim_{n \to \infty} \int h s_n \, d\mu = \int h f \, d\mu,$$

the first equality holding because of continuity of $\Lambda$, the second following from 18.3.9 and the third holding by the dominated convergence theorem.

This is a very nice formula but it still has not been shown that $h \in L^q(\Omega)$.

Let $E_n = \{x : |h(x)| \leq n\}$. Thus $|h X_{E_n}| \leq n$. Then

$$|h X_{E_n}|^{q - 2} (h X_{E_n}) \in L^p(\Omega).$$

By the claim, it follows that

$$\|h X_{E_n}\|_q^q = \int |h X_{E_n}|^{q - 2} (h X_{E_n}) \, d\mu = \Lambda(|h X_{E_n}|^{q - 2} (h X_{E_n}))$$

$$\leq \|\Lambda\| \|h X_{E_n}|^{q - 2} (h X_{E_n})\|_p = \|\Lambda\| \|h X_{E_n}\|_q^q,$$

the last equality holding because $q - 1 = \frac{q}{p}$ and so

$$\left( \int |h X_{E_n}|^{q - 2} (h X_{E_n}) \, d\mu \right)^{1/p} = \left( \int \left(|h X_{E_n}|^{q/p} \right)^p \, d\mu \right)^{1/p} = \|h X_{E_n}\|_q^{\frac{2}{p}}$$

Therefore, since $q - \frac{2}{p} = 1$, it follows that

$$\|h X_{E_n}\|_q \leq \|\Lambda\|.$$  

Letting $n \to \infty$, the Monotone Convergence theorem implies

$$\|h\|_q \leq \|\Lambda\|. \quad (18.3.10)$$
Now that $h$ has been shown to be in $L^q(\Omega)$, it follows from \textbf{Theorem 18.3.1 and the density of the simple functions, Theorem 18.3.4 on Page 587} that

$$\Lambda f = \int hf \, d\mu$$

for all $f \in L^p(\Omega)$.

It only remains to verify the last claim.

$||A|| = \sup \{ \int hf : ||f||_p \leq 1 \} \leq ||h||_q \leq ||A||$

by \textbf{Theorem 18.3.3} and Holder’s inequality. This proves the theorem.

To represent elements of the dual space of $L^1(\Omega)$, another Banach space is needed.

\textbf{Definition 18.3.2} Let $(\Omega, \mathcal{S}, \mu)$ be a measure space. $L^\infty(\Omega)$ is the vector space of measurable functions such that for some $M > 0$, $|f(x)| \leq M$ for all $x$ outside of some set of measure zero $(|f(x)| \leq M \text{ a.e.})$. Define $f = g$ when $f(x) = g(x)$ a.e. and $||f||_\infty \equiv \inf \{M : |f(x)| \leq M \text{ a.e.}\}$.

\textbf{Theorem 18.3.3} $L^\infty(\Omega)$ is a Banach space.

\textbf{Proof:} It is clear that $L^\infty(\Omega)$ is a vector space. Is $|| \cdot ||_\infty$ a norm?

\textbf{Claim:} If $f \in L^\infty(\Omega)$, then $|f(x)| \leq ||f||_\infty$ a.e.

\textbf{Proof of the claim:} $\{x : |f(x)| \geq ||f||_\infty + n^{-1}\} \equiv E_n$ is a set of measure zero according to the definition of $||f||_\infty$. Furthermore, $\{x : |f(x)| > ||f||_\infty\} = \bigcup_n E_n$ and so it is also a set of measure zero. This verifies the claim.

Now if $||f||_\infty = 0$ it follows that $f(x) = 0$ a.e. Also if $f, g \in L^\infty(\Omega)$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty$$

a.e. and so $||f||_\infty + ||g||_\infty$ serves as one of the constants, $M$ in the definition of $||f + g||_\infty$. Therefore,

$$||f + g||_\infty \leq ||f||_\infty + ||g||_\infty.$$  

Next let $c$ be a number. Then $|cf(x)| = |c||f(x)| \leq |c||f||_\infty$ and so $||cf||_\infty \leq |c||f||_\infty$. Therefore since $c$ is arbitrary, $||f||_\infty = ||c(1/c) f||_\infty \leq |1/c||cf||_\infty$ which implies $|c||f||_\infty \leq ||cf||_\infty$. Thus $|| \cdot ||_\infty$ is a norm as claimed.

To verify completeness, let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega)$ and use the above claim to get the existence of a set of measure zero, $E_{nm}$ such that for all $x \notin E_{nm}$,

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$$

Let $E = \bigcup_{n,m} E_{nm}$. Thus $\mu(E) = 0$ and for each $x \notin E$, $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ \lim_{n \to \infty} f_n(x) & \text{if } x \notin E \end{cases} = \lim_{n \to \infty} \lambda_{E \cap C} f_n(x).$$
18.3. REPRESENTATION THEOREMS FOR THE DUAL SPACE OF $L^p$  

Then $f$ is clearly measurable because it is the limit of measurable functions. If

$$F_n = \{ x : |f_n(x)| > ||f_n||_\infty \}$$

and $F = \cup_{n=1}^{\infty} F_n$, it follows $\mu(F) = 0$ and that for $x \notin F \cup E$,

$$|f(x)| \leq \lim \inf_{n \to \infty} |f_n(x)| \leq \lim \inf_{n \to \infty} ||f_n||_\infty < \infty$$

because $\{ ||f_n||_\infty \}$ is a Cauchy sequence. ($||f_n||_\infty - ||f_m||_\infty | \leq ||f_n - f_m||_\infty$ by the triangle inequality.) Thus $f \in L^\infty(\Omega)$. Let $n$ be large enough that whenever $m > n$,

$$||f_m - f_n||_\infty < \varepsilon.$$ 

Then, if $x \notin E$,

$$|f(x) - f_n(x)| = \lim_{m \to \infty} |f_m(x) - f_n(x)| \leq \lim_{m \to \infty} \inf_{m \to \infty} ||f_m - f_n||_\infty < \varepsilon.$$

Hence $||f - f_n||_\infty < \varepsilon$ for all $n$ large enough. This proves the theorem.

The next theorem is the Riesz representation theorem for $(L^1(\Omega))'$.

**Theorem 18.3.4 (Riesz representation theorem)** Let $(\Omega, S, \mu)$ be a finite measure space. If $\Lambda \in (L^1(\Omega))'$, then there exists a unique $h \in L^\infty(\Omega)$ such that

$$\Lambda(f) = \int \Omega hf \, d\mu$$

for all $f \in L^1(\Omega)$. If $h$ is the function in $L^\infty(\Omega)$ representing $\Lambda \in (L^1(\Omega))'$, then $||h||_\infty = ||\Lambda||$.

**Proof:** Just as in the proof of Theorem 18.3.1, there exists a unique $h \in L^1(\Omega)$ such that for all simple functions, $s$,

$$\Lambda(s) = \int s \, d\mu \quad \text{(18.3.11)}$$

To show $h \in L^\infty(\Omega)$, let $\varepsilon > 0$ be given and let

$$E = \{ x : |h(x)| \geq ||\Lambda|| + \varepsilon \}.$$ 

Let $|k| = 1$ and $hk = |h|$. Since the measure space is finite, $k \in L^1(\Omega)$. As in Theorem 18.3.1 let $\{s_n\}$ be a sequence of simple functions converging to $k$ in $L^1(\Omega)$, and pointwise. It follows from the construction in Theorem 9.3.9 on Page 227 that it can be assumed $|s_n| \leq 1$. Therefore

$$\Lambda(k\mathcal{X}_E) = \lim_{n \to \infty} \Lambda(s_n \mathcal{X}_E) = \lim_{n \to \infty} \int_E hs_n \, d\mu = \int_E hk \, d\mu$$
where the last equality holds by the Dominated Convergence theorem. Therefore,
\[
||\Lambda|| \mu(E) \geq |\Lambda(k\chi_E)| = |\int \Omega h k \chi_E d\mu| = \int_E |h|d\mu \\
\geq (||\Lambda|| + \varepsilon)\mu(E).
\]
It follows that \(\mu(E) = 0\). Since \(\varepsilon > 0\) was arbitrary, \(||\Lambda|| \geq ||h||_\infty\). It was shown that \(h \in L^\infty(\Omega)\), the density of the simple functions in \(L^1(\Omega)\) and [18.3.11] imply
\[
\Lambda f = \int_{\Omega} h f d\mu, ||\Lambda|| \geq ||h||_\infty.
\] (18.3.12)
This proves the existence part of the theorem. To verify uniqueness, suppose \(h_1\) and \(h_2\) both represent \(\Lambda\) and let \(f \in L^1(\Omega)\) be such that \(|f| \leq 1\) and \(f(h_1 - h_2) = |h_1 - h_2|\). Then
\[
0 = \Lambda f - \Lambda f = \int (h_1 - h_2) f d\mu = \int |h_1 - h_2| d\mu.
\]
Thus \(h_1 = h_2\). Finally,
\[
||\Lambda|| = \sup\{||\int h f d\mu| : ||f||_1 \leq 1\} \leq ||h||_\infty \leq ||\Lambda||
\]
by [18.3.11].

Next these results are extended to the \(\sigma\) finite case.

**Lemma 18.3.5** Let \((\Omega, S, \mu)\) be a measure space and suppose there exists a measurable function, \(r\) such that \(r(x) > 0\) for all \(x\), there exists \(M\) such that \(|r(x)| < M\) for all \(x\), and \(\int r d\mu < \infty\). Then for
\[
\Lambda \in (L^p(\Omega, \mu))', \ p \geq 1,
\]
there exists a unique \(h \in L^{p'}(\Omega, \mu), L^\infty(\Omega, \mu)\) if \(p = 1\) such that
\[
\Lambda f = \int_{\Omega} h f d\mu.
\]
Also \(||h|| = ||\Lambda||. \ (||h|| = ||h||_{p'} \ if \ p > 1, \ ||h||_\infty \ if \ p = 1)\). Here
\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

**Proof:** Define a new measure \(\tilde{\mu}\), according to the rule
\[
\tilde{\mu}(E) \equiv \int_E r d\mu.
\] (18.3.13)
Thus \(\tilde{\mu}\) is a finite measure on \(S\). Now define a mapping, \(\eta : L^p(\Omega, \mu) \to L^p(\Omega, \tilde{\mu})\) by
\[
\eta f = r^{-\frac{1}{p}} f.
\]
18.3. REPRESENTATION THEOREMS FOR THE DUAL SPACE OF $L^p$

Then

$$||\eta f||_{L^p(\tilde{\mu})}^p = \int |r^{-\frac{1}{p}}f|^p r d\mu = ||f||_{L^p(\mu)}^p$$

and so $\eta$ is one to one and in fact preserves norms. I claim that also $\eta$ is onto. To see this, let $g \in L^p(\Omega, \tilde{\mu})$ and consider the function, $r^{\frac{1}{p}}g$. Then

$$\int |r^{\frac{1}{p}}g|^p d\mu = \int |g|^p r d\mu = \int |g|^p d\tilde{\mu} < \infty$$

Thus $r^{\frac{1}{p}}g \in L^p(\Omega, \mu)$ and $\eta(r^{\frac{1}{p}}g) = g$ showing that $\eta$ is onto as claimed. Thus $\eta$ is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which $\eta^*$ must be one to one and onto.

\[
\begin{array}{ccc}
  h, L^{p'}(\tilde{\mu}) & \xrightarrow{\eta^*} & L^p(\tilde{\mu})' \\
  L^p(\tilde{\mu}) & \xrightarrow{\eta} & L^p(\mu) \\
  \Lambda, L^p(\mu)' & \xleftarrow{\Lambda} & L^p(\mu)
\end{array}
\]

Then for $\Lambda \in L^p(\mu)'$, there exists a unique $\tilde{\Lambda} \in L^p(\tilde{\mu})'$ such that $\eta^*\tilde{\Lambda} = \Lambda, ||\tilde{\Lambda}|| = ||\Lambda||$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in L^{p'}(\tilde{\mu})$ which represents $\tilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $||h||_{L^{p'}(\tilde{\mu})} = ||\tilde{\Lambda}|| = ||\Lambda||$ and for all $f \in L^p(\mu)$,

$$\Lambda(f) = \eta^*\tilde{\Lambda}(f) \equiv \tilde{\Lambda}(\eta f) = \int h(\eta f) d\tilde{\mu} = \int r^{\frac{-1}{p'}}h f d\mu.$$

Now

$$\int |r^{\frac{1}{p'}}h|^p d\mu = \int |h|^p r d\mu = ||h||_{L^{p'}(\mu)} < \infty.$$

Thus $||r^{\frac{1}{p'}}h||_{L^{p'}(\mu)} = ||h||_{L^{p'}(\tilde{\mu})} = ||\tilde{\Lambda}|| = ||\Lambda||$ and represents $\Lambda$ in the appropriate way. If $p = 1$, then $1/p' \equiv 0$. This proves the Lemma.

A situation in which the conditions of the lemma are satisfied is the case where the measure space is $\sigma$ finite. In fact, you should show this is the only case in which the conditions of the above lemma hold.

**Theorem 18.3.6** (Riesz representation theorem) Let $(\Omega, \mathcal{S}, \mu)$ be $\sigma$ finite and let $\Lambda \in (L^p(\Omega, \mu))'$, $p \geq 1$.

Then there exists a unique $h \in L^q(\Omega, \mu), L^{\infty}(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int hf d\mu.$$
Also $|h| = ||\Lambda||$. ($|h| = ||h||_q$ if $p > 1$, $||h||_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{q} = 1.$$

**Proof:** Let $\{\Omega_n\}$ be a sequence of disjoint elements of $S$ having the property that

$$0 < \mu(\Omega_n) < \infty, \cup_{n=1}^\infty \Omega_n = \Omega.$$

Define

$$r(x) = \sum_{n=1}^\infty \frac{1}{n^2} \chi_{\Omega_n}(x) \mu(\Omega_n)^{-1}, \tilde{\mu}(E) = \int_E r d\mu.$$  

Thus

$$\int_\Omega r d\mu = \tilde{\mu}(\Omega) = \sum_{n=1}^\infty \frac{1}{n^2} < \infty$$

so $\tilde{\mu}$ is a finite measure. The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

With the Riesz representation theorem, it is easy to show that

$$L^p(\Omega), \ p > 1$$

is a reflexive Banach space. Recall Definition \[\text{15.4.14}\] on Page \436\ for the definition.

**Theorem 18.3.7** For $(\Omega, S, \mu)$ a $\sigma$ finite measure space and $p > 1$, $L^p(\Omega)$ is reflexive.

**Proof:** Let $\delta_r : (L'(\Omega))' \rightarrow L'(\Omega)$ be defined for $\frac{1}{r} + \frac{1}{r'} = 1$ by

$$\int (\delta_r \Lambda)g \ d\mu = \Lambda g$$

for all $g \in L'(\Omega)$. From Theorem \[\text{15.2.10}\] $\delta_r$ is one to one, onto, continuous and linear. By the open map theorem, $\delta^{-1}_r$ is also one to one, onto, and continuous ($\delta_r \Lambda$ equals the representor of $\Lambda$). Thus $\delta^*_r$ is also one to one, onto, and continuous by Corollary \[15.2.11\]. Now observe that $J = \delta^*_r \circ \delta^{-1}_q$. To see this, let $z^* \in (L^q)', y^* \in (L^p)'$,

$$\delta^*_p \circ \delta^{-1}_q (\delta_q z^*)(y^*) = (\delta^*_p z^*)(y^*)$$

$$= z^*(\delta_p y^*)$$

$$= \int (\delta_q z^*)(\delta_p y^*) d\mu,$$

$$J(\delta_q z^*)(y^*) = y^*(\delta_q z^*)$$

$$= \int (\delta_p y^*)(\delta_q z^*) d\mu.$$

Therefore $\delta^*_p \circ \delta^{-1}_q = J$ on $\delta_q (L^q)' = L^p$. But the two $\delta$ maps are onto and so $J$ is also onto.
18.4 The Dual Space Of $L^\infty(\Omega)$

What about the dual space of $L^\infty(\Omega)$? This will involve the following Lemma. Also recall the notion of total variation defined in Definition 18.2.2.

**Lemma 18.4.1** Let $(\Omega, F)$ be a measure space. Denote by $BV(\Omega)$ the space of finitely additive complex measures $\nu$ such that $|\nu| (\Omega) < \infty$. Then defining $||\nu|| \equiv |\nu| (\Omega)$, it follows that $BV(\Omega)$ is a Banach space.

**Proof:** It is obvious that $BV(\Omega)$ is a vector space with the obvious conventions involving scalar multiplication. Why is $||\cdot||$ a norm? All the axioms are obvious except for the triangle inequality. However, this is not too hard either.

$$||\mu + \nu|| \equiv |\mu + \nu| (\Omega) = \sup_{\pi(\Omega)} \left\{ \sum_{A \in \pi(\Omega)} |\mu(A) + \nu(A)| \right\}$$

$$\leq \sup_{\pi(\Omega)} \left\{ \sum_{A \in \pi(\Omega)} |\mu(A)| \right\} + \sup_{\pi(\Omega)} \left\{ \sum_{A \in \pi(\Omega)} |\nu(A)| \right\}$$

$$\equiv |\mu| (\Omega) + |\nu| (\Omega) = ||\mu|| + ||\nu||.$$

Suppose now that $\{\nu_n\}$ is a Cauchy sequence. For each $E \in F$,

$$|\nu_n(E) - \nu_m(E)| \leq ||\nu_n - \nu_m||$$

and so the sequence of complex numbers $\nu_n(E)$ converges. That to which it converges is called $\nu(E)$. Then it is obvious that $\nu(E)$ is finitely additive. Why is $|\nu|$ finite? Since $||\cdot||$ is a norm, it follows that there exists a constant $C$ such that for all $n$,

$$|\nu_n| (\Omega) < C$$

Let $\pi(\Omega)$ be any partition. Then

$$\sum_{A \in \pi(\Omega)} |\nu(A)| = \lim_{n \to \infty} \sum_{A \in \pi(\Omega)} |\nu_n(A)| \leq C.$$ 

Hence $\nu \in BV(\Omega)$. Let $\varepsilon > 0$ be given and let $N$ be such that if $n, m > N$, then

$$||\nu_n - \nu_m|| < \varepsilon/2.$$ 

Pick any such $n$. Then choose $\pi(\Omega)$ such that

$$|\nu - \nu_n| (\Omega) - \varepsilon/2 < \sum_{A \in \pi(\Omega)} |\nu(A) - \nu_n(A)|$$

$$= \lim_{m \to \infty} \sum_{A \in \pi(\Omega)} |\nu_m(A) - \nu_n(A)| < \lim_{m \to \infty} \inf_{\pi(\Omega)} |\nu_n - \nu_m| (\Omega) \leq \varepsilon/2$$

It follows that

$$\lim_{n \to \infty} ||\nu - \nu_n|| = 0. \blacksquare$$
Corollary 18.4.2 Suppose \((\Omega, F)\) is a measure space as above and suppose \(\mu\) is a measure defined on \(F\). Denote by \(BV(\Omega; \mu)\) those finitely additive measures of \(BV(\Omega)\) \(\nu\) such that \(\nu \ll \mu\) in the usual sense that if \(\mu(E) = 0\), then \(\nu(E) = 0\). Then \(BV(\Omega; \mu)\) is a closed subspace of \(BV(\Omega)\).

Proof: It is clear that it is a subspace. Is it closed? Suppose \(\nu_n \to \nu\) and each \(\nu_n\) is in \(BV(\Omega; \mu)\). Then if \(\mu(E) = 0\), it follows that \(\nu_n(E) = 0\) and so \(\nu(E) = 0\) also, being the limit of 0.

Definition 18.4.3 For a simple function \(s(\omega) = \sum_{k=1}^{n} c_k \chi_{E_k}(\omega)\) and \(\nu \in BV(\Omega)\), define an “integral” with respect to \(\nu\) as follows.

\[
\int s d\nu \equiv \sum_{k=1}^{n} c_k \nu(E_k).
\]

For a function which is in \(L^\infty(\Omega; \mu)\), define \(\int f d\nu\) as follows. Applying Theorem 9.3.9, to the positive and negative parts of real and imaginary parts of \(f\), there exists a sequence of simple functions \(\{s_n\}\) which converges uniformly to \(f\) off a set of \(\mu\) measure zero. Then

\[
\int f d\nu \equiv \lim_{n \to \infty} \int s_n d\nu
\]

Lemma 18.4.4 The above definition of the integral with respect to a finitely additive measure in \(BV(\Omega; \mu)\) is well defined.

Proof: First consider the claim about the integral being well defined on the simple functions. This is clearly true if it is required that the \(c_k\) are disjoint and the \(E_k\) also disjoint having union equal to \(\Omega\). Thus define the integral of a simple function in this manner. First write the simple function as

\[
\sum_{k=1}^{n} c_k \chi_{E_k}
\]

where the \(c_k\) are the values of the simple function. Then use the above formula to define the integral. Next suppose the \(E_k\) are disjoint but the \(c_k\) are not necessarily distinct. Let the distinct values of the \(c_k\) be \(a_1, \cdots, a_m\).

\[
\sum_{k} c_k \chi_{E_k} = \sum_{j} a_j \left( \sum_{i : c_i = a_j} \chi_{E_i} \right) = \sum_{j} a_j \nu \left( \bigcup_{i : c_i = a_j} E_i \right) = \sum_{j} a_j \sum_{i : c_i = a_j} \nu(E_i) = \sum_{k} c_k \nu(E_k)
\]

and so the same formula for the integral of a simple function is obtained in this case also. Now consider two simple functions

\[
s = \sum_{k=1}^{n} a_k \chi_{E_k}, \quad t = \sum_{j=1}^{m} b_j \chi_{F_j}
\]
18.4. THE DUAL SPACE OF $L^\infty (\Omega)$

where the $a_k$ and $b_j$ are the distinct values of the simple functions. Then from what was just shown,

$$\int (\alpha s + \beta t) \, d\nu = \int \left( \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha a_k \chi_{E_k \cap F_j} + \sum_{j=1}^{m} \sum_{k=1}^{n} \beta b_j \chi_{E_k \cap F_j} \right) \, d\nu$$

$$= \int \left( \sum_{j,k} \alpha a_k \chi_{E_k \cap F_j} + \beta b_j \chi_{E_k \cap F_j} \right) \, d\nu$$

$$= \sum_{j,k} (\alpha a_k + \beta b_j) \nu (E_k \cap F_j)$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha a_k \nu (E_k \cap F_j) + \sum_{j=1}^{m} \sum_{k=1}^{n} \beta b_j \nu (E_k \cap F_j)$$

$$= \sum_{k=1}^{n} \alpha a_k \nu (E_k) + \sum_{j=1}^{m} \beta b_j \nu (F_j)$$

$$= \alpha \int s \, d\nu + \beta \int t \, d\nu$$

Thus the integral is linear on simple functions so, in particular, the formula given in the above definition is well defined regardless.

So what about the definition for $f \in L^\infty (\Omega; \mu)$? Since $f \in L^\infty$, there is a set of $\mu$ measure zero $N$ such that on $N^C$ there exists a sequence of simple functions which converges uniformly to $f$ on $N^C$. Consider $s_n$ and $s_m$. As in the above, they can be written as

$$\sum_{k=1}^{p} c_k^n \chi_{E_k}, \ \sum_{k=1}^{p} c_k^m \chi_{E_k}$$

respectively, where the $E_k$ are disjoint having union equal to $\Omega$. Then by uniform convergence, if $m, n$ are sufficiently large, $|c_k^n - c_k^m| < \varepsilon$ or else the corresponding $E_k$ is contained in $N^C$. Consider $s_n$ and $s_m$. As in the above, the integrals of these simple functions converge. Similar reasoning shows that the definition is not dependent on the choice of approximating sequence.

Note also that for $s$ simple,

$$\left| \int s \, d\nu - \int s_m \, d\nu \right| = \left| \sum_{k=1}^{p} (c_k^n - c_k^m) \nu (E_k) \right|$$

$$\leq \sum_{k=1}^{p} |c_k^n - c_k^m| \nu (E_k) \leq \varepsilon \|\nu\|$$

and so the integrals of these simple functions converge. Similar reasoning shows that the definition is not dependent on the choice of approximating sequence. ■

Note also that for $s$ simple,

$$\left| \int s \, d\nu \right| \leq ||s||_{L^\infty} \|\nu\| (\Omega) = ||s||_{L^\infty} \|\nu\|$$
Next the dual space of $L^\infty (\Omega; \mu)$ will be identified with $BV (\Omega; \mu)$. First here is a simple observation. Let $\nu \in BV (\Omega; \mu)$. Then define the following for $f \in L^\infty (\Omega; \mu)$.

$$T_\nu (f) \equiv \int f \, d\nu$$

**Lemma 18.4.5** For $T_\nu$ just defined,

$$|T_\nu f| \leq ||f||_{L^\infty} ||\nu||$$

**Proof:** As noted above, the conclusion true if $f$ is simple. Now if $f$ is in $L^\infty$, then it is the uniform limit of simple functions off a set of $\mu$ measure zero. Therefore, by the definition of the $T_\nu$,

$$|T_\nu f| = \lim_{n \to \infty} |T_\nu s_n| \leq \liminf_{n \to \infty} ||s_n||_{L^\infty} ||\nu|| = ||f||_{L^\infty} ||\nu||. \square$$

Thus each $T_\nu$ is in $(L^\infty (\Omega; \mu))'$. Here is the representation theorem, due to Kantorovitch, for the dual of $L^\infty (\Omega; \mu)$.

**Theorem 18.4.6** Let $\theta : BV (\Omega; \mu) \to (L^\infty (\Omega; \mu))'$ be given by $\theta (\nu) \equiv T_\nu$. Then $\theta$ is one to one, onto and preserves norms.

**Proof:** It was shown in the above lemma that $\theta$ maps into $(L^\infty (\Omega; \mu))'$. It is obvious that $\theta$ is linear. Why does it preserve norms? From the above lemma,

$$||\theta \nu|| \equiv \sup_{||f||_\infty \leq 1} |T_\nu f| \leq ||\nu||$$

It remains to turn the inequality around. Let $\pi (\Omega)$ be a partition. Then

$$\sum_{A \in \pi (\Omega)} |\nu (A)| = \sum_{A \in \pi (\Omega)} \text{sgn} (\nu (A)) \nu (A) \equiv \int f \, d\nu$$

where $\text{sgn} (\nu (A))$ is defined to be a complex number of modulus 1 such that $\text{sgn} (\nu (A)) \nu (A) = |\nu (A)|$ and

$$f (\omega) = \sum_{A \in \pi (\Omega)} \text{sgn} (\nu (A)) \chi_A (\omega).$$

Therefore, choosing $\pi (\Omega)$ suitably, since $||f||_\infty \leq 1$,

$$||\nu|| - \varepsilon = |\nu| (\Omega) - \varepsilon \leq \sum_{A \in \pi (\Omega)} |\nu (A)| = T_\nu (f)$$

$$= |T_\nu (f)| = ||\theta (\nu) (f)|| \leq ||\theta (\nu)|| \leq ||\nu||$$

Thus $\theta$ preserves norms. Hence it is one to one also. Why is $\theta$ onto? Let $\Lambda \in (L^\infty (\Omega; \mu))'$. Then define

$$\nu (E) \equiv \Lambda (\chi_E) \quad (18.4.14)$$
This is obviously finitely additive because \( \Lambda \) is linear. Also, if \( \mu (E) = 0 \), then \( X_E = 0 \) in \( L^\infty \) and so \( \Lambda (X_E) = 0 \). If \( \pi (\Omega) \) is any partition of \( \Omega \), then
\[
\sum_{A \in \pi (\Omega)} |\nu (A)| = \sum_{A \in \pi (\Omega)} |\Lambda (X_A)| = \sum_{A \in \pi (\Omega)} \text{sgn} (\Lambda (X_A)) \Lambda (X_A)
\]
\[
= \Lambda \left( \sum_{A \in \pi (\Omega)} \text{sgn} (\Lambda (X_A)) X_A \right) \leq ||\Lambda||
\]
and so \( ||\nu|| \leq ||\Lambda|| \) showing that \( \nu \in BV (\Omega; \mu) \). Also from 18.4.14, if \( s = \sum_{k=1}^n c_k X_{E_k} \) is a simple function,
\[
\int s d\nu = \sum_{k=1}^n c_k \nu (E_k) = \sum_{k=1}^n c_k \Lambda (X_{E_k}) = \Lambda \left( \sum_{k=1}^n c_k X_{E_k} \right) = \Lambda (s)
\]
Then letting \( f \in L^\infty (\Omega; \mu) \), there exists a sequence of simple functions converging to \( f \) uniformly off a set of \( \mu \) measure zero and so passing to a limit in the above with \( s \) replaced with \( s_n \) it follows that
\[
\Lambda (f) = \int f d\nu
\]
and so \( \theta \) is onto. ■

### 18.5 Non \( \sigma \) Finite Case

It is not necessary to assume \( \mu \) is either finite or \( \sigma \) finite to establish the Riesz representation theorem for \( 1 < p < \infty \). This involves the notion of uniform convexity. First we recall Clarkson’s inequalities.

**Lemma 18.5.1** Let \( 2 \leq p \). Then
\[
\left\| \frac{f + g}{2} \right\|_p^p + \left\| \frac{f - g}{2} \right\|_p^p \leq \frac{1}{2} \left( ||f||_L^p + ||g||_L^p \right)
\]
Let \( 1 < p < 2 \), then for \( 1/p + 1/q = 1 \),
\[
\left\| \frac{f + g}{2} \right\|_L^q + \left\| \frac{f - g}{2} \right\|_L^q \leq \left( \frac{1}{2} ||f||_L^p + \frac{1}{2} ||g||_L^p \right)^{q/p}
\]
Recall also the following definition of uniform convexity.

**Definition 18.5.2** A Banach space, \( X \), is said to be uniformly convex if whenever \( ||x_n|| \leq 1 \) and \( ||\frac{x_n + x_m}{2}|| \to 1 \) as \( n, m \to \infty \), then \( \{x_n\} \) is a Cauchy sequence and \( x_n \to x \) where \( ||x|| = 1 \).
Observe that Clarkson’s inequalities imply $L^p$ is uniformly convex for all $p > 1$. This was Theorem 15.3.10. Uniformly convex spaces have a very nice property which is described in the following lemma. Roughly, this property is that any element of the dual space achieves its norm at some point of the closed unit ball.

Lemma 18.5.3 Let $X$ be uniformly convex and let $\phi \in X'$. Then there exists $x \in X$ such that

$$||x|| = 1, \phi(x) = ||\phi||.$$

**Proof:** Let $||x_n|| \leq 1$ and $|\phi(x_n)| \to ||\phi||$. Let $x_n = w_n \tilde{x}_n$ where $|w_n| = 1$ and $w_n \phi \tilde{x}_n = ||\phi \tilde{x}_n||$.

Thus $\phi(x_n) = |\phi(x_n)| = |\phi(\tilde{x}_n)| \to ||\phi||$.

$\phi(x_n) \to ||\phi||, ||x_n|| \leq 1.$

We can assume, without loss of generality, that

$$\phi(x_n) = |\phi(x_n)| \geq \frac{||\phi||}{2}$$

and $\phi \neq 0$.

**Claim** $||\frac{x_n + x_m}{2}|| \to 1$ as $n, m \to \infty$.

**Proof of Claim:** Let $n, m$ be large enough that $\phi(x_n), \phi(x_m) \geq ||\phi|| - \frac{\varepsilon}{2}$ where $0 < \varepsilon$. Then $||x_n + x_m|| \neq 0$ because if it equals 0, then $x_n = -x_m$ so $-\phi(x_n) = \phi(x_m)$ but both $\phi(x_n)$ and $\phi(x_m)$ are positive. Therefore consider $\frac{x_n + x_m}{||x_n + x_m||}$, a vector of norm 1. Thus,

$$||\phi|| \geq \phi\left(\frac{x_n + x_m}{||x_n + x_m||}\right) \geq \frac{2||\phi|| - \varepsilon}{||x_n + x_m||}.$$

Hence

$$||x_n + x_m|| ||\phi|| \geq 2||\phi|| - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n,m \to \infty} ||x_n + x_m|| = 2$. This proves the claim.

By uniform convexity, $\{x_n\}$ is Cauchy and $x_n \to x$, $||x|| = 1$. Thus $\phi(x) = \lim_{n \to \infty} \phi(x_n) = ||\phi||$.

The proof of the Riesz representation theorem will be based on the following lemma which says that if you can show a directional derivative exists, then it can be used to represent a functional.

Lemma 18.5.4 (McShane) Let $X$ be a complex normed linear space and let $\phi \in X'$. Suppose there exists $x \in X$, $||x|| = 1$ with $\phi x = ||\phi|| \neq 0$. Let $y \in X$ and let $\psi_y(t) = ||x + ty||$ for $t \in \mathbb{R}$. Suppose $\psi'_y(0)$ exists for each $y \in X$. Then for all $y \in X$,

$$\psi'_y(0) + i \psi'_{-iy}(0) = ||\phi||^{-1} \phi(y).$$
**Proof:** Suppose first that $||\phi|| = 1$. Then since $\phi(x) = 1$, $\phi(y - \phi(y)x) = 0$ and so

$$\phi(x + t(y - \phi(y)x)) = \phi(x) = 1 = ||\phi||.$$ 

Therefore, $||x + t(y - \phi(y)x)|| \geq 1$ since otherwise $||x + t(y - \phi(y)x)|| = r < 1$ and so

$$\phi \left( \frac{x + t(y - \phi(y)x)}{r} \right) = \frac{1}{r} \phi(x) = \frac{1}{r}$$

which would imply that $||\phi|| > 1$.

Also for small $t$, $|\phi(y)| < 1$, and so

$$1 \leq ||x + t(y - \phi(y)x)|| = ||(1 - \phi(y)t)x + ty||$$

This implies

$$|1 + t\phi(y) + o(t)| = \frac{1}{|1 - t\phi(y)|}$$

$$\leq \left| x + \frac{t}{1 - \phi(y)}y \right| = ||x + ty + o(t)|| \quad (18.5.15)$$

where $\lim_{t \to 0} o(t) (t^{-1}) = 0$. Thus for $t > 0$,

$$\Re \phi(y) \leq |\Re \phi(y)| \leq \frac{|1 + t\phi(y)| - 1}{t} \leq \frac{||x + ty|| - ||x||}{t} + \frac{o(t)}{t}$$

and for $t < 0$,

$$\Re \phi(y) \geq \frac{|1 + t\phi(y)| - 1}{t} \geq \frac{||x + ty|| - ||x||}{t} + \frac{o(t)}{t}$$

By assumption, letting $t \to 0+$ and $t \to 0-$,

$$\Re \phi(y) = \lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} = \psi'_y(0).$$

Now

$$\phi(y) = \Re \phi(y) + i \Im \phi(y)$$

so

$$\phi(-iy) = -i(\phi(y)) = -i \Re \phi(y) + \Im \phi(y)$$

and

$$\phi(-iy) = \Re \phi(-iy) + i \Im \phi(-iy).$$

Hence

$$\Re \phi(-iy) = \Im \phi(y).$$

Consequently,

$$\phi(y) = \Re \phi(y) + i \Im \phi(y) = \Re \phi(y) + i \Re \phi(-iy)$$
= ψ′_y(0) + ivψ′_{-iy}(0).

This proves the lemma when ||ϕ|| = 1. For arbitrary ϕ ≠ 0, let ϕ(x) = ||ϕ||, ||x|| = 1. Then from above, if φ_1(y) ≡ ||ϕ||^{-1} ϕ(y), ||φ_1|| = 1 and so from what was just shown,

φ_1(y) = ϕ(y) ||ϕ|| = ψ′_y(0) + ivψ′_{-iy}(0) ■

Now here are some short observations. For t ∈ ℝ, p > 1, and x, y ∈ C, x ≠ 0

\[
\lim_{t \to 0} \frac{|x + ty|^p - |x|^p}{t} = p |x|^{p-2} (\text{Re} x \text{Re} y + \text{Im} x \text{Im} y)
\]

Also from convexity of f(r) = r^p, for |t| < 1,

\[
|x + ty|^p - |x|^p \leq ||x| + |t||y||^p - |x|^p
\]

\[
= \left[ (1 + |t|) \left( \frac{|x| + |t||y|}{1 + |t|} \right) \right]^p - |x|^p
\]

\[
\leq (1 + |t|)^p \frac{|x|^p}{1 + |t|} - |x|^p
\]

\[
\leq (1 + |t|)^{p-1} (|x|^p + |t||y|^p) - |x|^p
\]

\[
\leq (1 + |t|)^{p-1} - 1 |x|^p + 2^{p-1} |t||y|^p
\]

Now for f(t) ≡ (1 + t)^{p-1}, f'(t) is uniformly bounded, depending on p, for t ∈ [0,1]. Hence the above is dominated by an expression of the form

\[ C_p (|x|^p + |y|^p) |t| \]

(18.5.17)

The above lemma and uniform convexity of L^p can be used to prove a general version of the Riesz representation theorem next. This version makes no assumption that the measure space is σ finite. Let p > 1 and let η : L^q → (L^p)' be defined by

\[ \eta(g)(f) = \int f g d\mu. \]

(18.5.18)

**Theorem 18.5.5 (Riesz representation theorem p > 1)** The map η is 1-1, onto, continuous, and

\[ ||\eta g|| = ||g||, \ ||\eta|| = 1. \]

**Proof:** Obviously η is linear. Suppose ηg = 0. Then 0 = ∫ g f dμ for all f ∈ L^p. Let f = |g|^{q-2}g. Then f ∈ L^p and so 0 = ∫ |g|^{q-1} dμ. Hence g = 0 and η is 1-1. That ηg ∈ (L^p)' is obvious from the Holder inequality. In fact,

\[ |\eta(g)(f)| \leq ||g||_q ||f||_p, \]
and so \( ||\eta(g)|| \leq ||g||_q \). To see that equality holds, let
\[
f = |g|^{q-2} g ||g||_q^{1-q}.
\]
Then \( ||f||_p = 1 \) and
\[
\eta(g)(f) = \int_{\Omega} |g|^q d\mu ||g||_q^{1-q} = ||g||_q.
\]
Thus \( ||\eta|| = 1 \).

It remains to show \( \eta \) is onto. Let \( \phi \in (L^p)' \). Is \( \phi = \eta g \) for some \( g \in L^q \)? Without
loss of generality, assume \( \phi \neq 0 \). By uniform convexity of \( L^p \), Lemma 18.3.3, there
exists \( g \) such that
\[
\phi g = ||\phi||, \ g \in L^p, \ ||g|| = 1.
\]
For \( f \in L^p \), define \( \phi_f (t) = \int_{\Omega} |g + tf|^p d\mu \). Thus
\[
\psi_f (t) = ||g + tf||_p = \psi_f (t)^{\frac{1}{p}}.
\]
Does \( \psi_f ' (0) \) exist? Let \( [g = 0] \) denote the set \( \{ x : g(x) = 0 \} \).
\[
\frac{\phi_f (t) - \phi_f (0)}{t} = \int \frac{(|g + tf|^p - |g|^p)}{t} d\mu.
\]
From 18.3.4, the integrand is bounded by \( C_p (|f|^p + |g|^p) \). Therefore, using 18.3.4, the
dominated convergence theorem applies and it follows \( \phi_f ' (0) = \)
\[
\lim_{t \to 0} \frac{\phi_f (t) - \phi_f (0)}{t} = \lim_{t \to 0} \left[ \int_{[g = 0]} |t|^{p-1} |f|^p d\mu + \int_{[g \neq 0]} \frac{(|g + tf|^p - |g|^p)}{t} d\mu \right]
\]
\[
= p \int_{[g \neq 0]} |g|^{p-2} \text{Re}(\bar{g} f) d\mu = p \int |g|^{p-2} \text{Re}(\bar{g} f) d\mu.
\]
Hence
\[
\psi_f ' (0) = ||g||^{-\frac{1}{p}} \int |g(x)|^{p-2} \text{Re}(g(x)\bar{f}(x)) d\mu.
\]
Note \( \frac{1}{p} - 1 = -\frac{1}{q} \). Therefore,
\[
\psi_{-f} ' (0) = ||g||^{-\frac{1}{p}} \int |g(x)|^{p-2} \text{Re}(ig(x)\bar{f}(x)) d\mu.
\]
But \( \text{Re}(i g \bar{f}) = \text{Im}(-g \bar{f}) \) and so by the McShane lemma,
\[
\phi (f) = ||\phi|| ||g||^{-\frac{1}{p}} \int |g(x)|^{p-2} [\text{Re}(g(x)\bar{f}(x)) + i \text{Re}(i g(x)\bar{f}(x))] d\mu
\]
\[
= ||\phi|| ||g||^{-\frac{1}{p}} \int |g(x)|^{p-2} [\text{Re}(g(x)\bar{f}(x)) + i \text{Im}(g(x)\bar{f}(x))] d\mu
\]
\[
= ||\phi|| ||g||^{-\frac{1}{p}} \int |g(x)|^{p-2} \bar{g}(x)f(x) d\mu.
\]
This shows that
\[
\phi = \eta(||\phi|| ||g||^{-\frac{1}{p}} |g|^{p-2} \bar{g})
\]
and verifies \( \eta \) is onto. ■
18.6 The Dual Space Of $C_0(X)$

Consider the dual space of $C_0(X)$ where $X$ is a locally compact Hausdorff space. It will turn out to be a space of measures. To show this, the following lemma will be convenient. Recall this space is defined as follows.

**Definition 18.6.1** $f \in C_0(X)$ means that for every $\varepsilon > 0$ there exists a compact set $K$ such that $|f(x)| < \varepsilon$ whenever $x \notin K$. Recall the norm on this space is

$$||f||_\infty \equiv ||f|| \equiv \sup \{ |f(x)| : x \in X \}$$

**Lemma 18.6.2** Suppose $\lambda$ is a mapping which has nonnegative values which is defined on the nonnegative functions in $C_0(X)$ such that

$$\lambda(af + bg) = a\lambda(f) + b\lambda(g) \quad (18.6.19)$$

whenever $a, b \geq 0$ and $f, g \geq 0$. Then there exists a unique extension of $\lambda$ to all of $C_0(X)$, $\Lambda$ such that whenever $f, g \in C_0(X)$ and $a, b \in \mathbb{C}$, it follows

$$\Lambda(af + bg) = a\Lambda(f) + b\Lambda(g).$$

If

$$|\lambda(f)| \leq C||f||_\infty$$

then

$$|\Lambda f| \leq C||f||_\infty$$

**Proof:** Let $C_0(X; \mathbb{R})$ be the real-valued functions in $C_0(X)$ and define

$$\Lambda_R(f) = \lambda f^+ - \lambda f^-$$

for $f \in C_0(X; \mathbb{R})$. Use the identity

$$(f_1 + f_2)^+ + f_1^- + f_2^- = f_1^+ + f_2^+ + (f_1 + f_2)^-$$

and [1,9.1.14] to write

$$\lambda(f_1 + f_2)^+ - \lambda(f_1 + f_2)^- = \lambda f_1^+ - \lambda f_1^- + \lambda f_2^+ - \lambda f_2^-,$$

it follows that $\Lambda_R(f_1 + f_2) = \Lambda_R(f_1) + \Lambda_R(f_2)$. To show that $\Lambda_R$ is linear, it is necessary to verify that $\Lambda_R(cf) = c\Lambda_R(f)$ for all $c \in \mathbb{R}$. But

$$(cf)^\pm = cf^\pm,$$

if $c \geq 0$ while

$$(cf)^+ = -c(f)^-,$$

if $c < 0$ and

$$(cf)^- = (-c)f^+,$$

and

$$\Lambda_R(cf) = \lambda(cf)^+ - \lambda(cf)^- = c\lambda(f)^+ - c\lambda(f)^- = c\Lambda_R(f).$$
18.6. **THE DUAL SPACE OF** \( C_0(X) \)

if \( c < 0 \). Thus, if \( c < 0 \),

\[
\Lambda_R(cf) = \lambda(cf)^+ - \lambda(cf)^- = \lambda((-c)f^-) - \lambda((-c)f^+)
\]

\[
= -c\lambda(f^-) + c\lambda(f^+) = c(\lambda(f^+) - \lambda(f^-)) = c\Lambda_R(f).
\]

A similar formula holds more easily if \( c \geq 0 \).

Now let

\[
\Lambda f = \Lambda_R(\text{Re } f) + i\Lambda_R(\text{Im } f)
\]

for arbitrary \( f \in C_0(X) \). This is linear as desired.

Here is why. It is obvious that \( \Lambda (f + g) = \Lambda (f) + \Lambda (g) \) from the fact that taking the real and imaginary parts are linear operations. The only thing to check is whether you can factor out a complex scalar.

\[
\Lambda ((a + ib)f) = \Lambda (af) + \Lambda (ibf)
\]

\[
\equiv \Lambda_R (a \text{Re } f) + i\Lambda_R (a \text{Im } f) + \Lambda_R (-b \text{Im } f) + i\Lambda_R (b \text{Re } f)
\]

because \( ibf = ib \text{Re } f - b \text{Im } f \) and so \( \text{Re } (ibf) = -b \text{Im } f \) and \( \text{Im } (ibf) = b \text{Re } f \). Therefore, the above equals

\[
= (a + ib) \Lambda_R (\text{Re } f) + i (a + ib) \Lambda_R (\text{Im } f)
\]

\[
= (a + ib) (\Lambda_R (\text{Re } f) + i\Lambda_R (\text{Im } f)) = (a + ib) \Lambda f
\]

The extension is obviously unique because all the above is required in order for \( \Lambda \) to be linear.

It remains to verify the claim about continuity of \( \Lambda \). From the definition of \( \lambda \), if \( 0 \leq g \leq f \), then

\[
\lambda (f) = \lambda (f - g + g) = \lambda (f - g) + \lambda (g) \geq \lambda (g)
\]

\[
|\Lambda_R f| \equiv |\lambda f^+ - \lambda f^-| \leq \max (\lambda f^+, \lambda f^-) \leq \lambda (|f|) \leq C ||f||_{\infty}
\]

Then letting \( \omega \Lambda f = |\Lambda f|, |\omega| = 1 \), and using the above,

\[
|\Lambda f| = \omega \Lambda f = \Lambda (\omega f) \equiv \Lambda_R (\text{Re } (\omega f)) = |\Lambda_R (\text{Re } (\omega f))|
\]

\[
\leq C ||\text{Re } (\omega f)|| \leq C ||f||_{\infty}
\]

This proves the lemma.

Let \( L \in C_0(X)' \). Also denote by \( C_0^+(X) \) the set of nonnegative continuous functions defined on \( X \). Define for \( f \in C_0^+(X) \)

\[
\lambda(f) = \sup \{|Lg| : |g| \leq f\}.
\]

Note that \( \lambda(f) < \infty \) because \( |Lg| \leq ||L|| ||g|| \leq ||L|| ||f|| \) for \( |g| \leq f \). Then the following lemma is important.
Lemma 18.6.3 If \( c \geq 0 \), \( \lambda(cf) = c\lambda(f) \), \( f_1 \leq f_2 \) implies \( \lambda f_1 \leq \lambda f_2 \), and
\[
\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2).
\]

Also
\[
0 \leq \lambda(f) \leq ||L|| \|f\|_\infty
\]

Proof: The first two assertions are easy to see so consider the third.
For \( f_j \in C_0^+(X) \), there exists \( g_i \in C_0(X) \) such that \( |g_i| \leq f_i \) and
\[
\lambda (f_1) + \lambda (f_2) \leq |L (g_1)| + |L (g_2)| + 2\varepsilon
= L (\omega_1 g_1) + L (\omega_2 g_2) + 2\varepsilon
= |L (\omega_1 g_1 + \omega_2 g_2)| + 2\varepsilon
\]
where \( |g_i| \leq f_i \) and \( |\omega_i| = 1 \) and \( \omega_i L (g_i) = |L (g_i)| \). Now
\[
|\omega_1 g_1 + \omega_2 g_2| \leq |g_1| + |g_2| \leq f_1 + f_2
\]
and so the above shows
\[
\lambda (f_1) + \lambda (f_2) \leq \lambda (f_1 + f_2) + 2\varepsilon.
\]
Since \( \varepsilon \) is arbitrary, \( \lambda (f_1) + \lambda (f_2) \leq \lambda (f_1 + f_2) \). It remains to verify the other inequality.
Now let \( |g| \leq f_1 + f_2 \), \( |Lg| \geq \lambda (f_1 + f_2) - \varepsilon \). Let
\[
h_i (x) = \begin{cases} 
\frac{f_1(x)g(x)}{f_1(x)+f_2(x)} & \text{if } f_1 (x) + f_2 (x) > 0, \\
0 & \text{if } f_1 (x) + f_2 (x) = 0.
\end{cases}
\]
Then \( h_i \) is continuous and \( h_1 (x) + h_2 (x) = g(x) \), \( |h_i| \leq f_i \). The reason it is continuous at a point where \( f_1 (x) + f_2 (x) = 0 \) is that at every point \( y \) where \( f_1 (y) + f_2 (y) > 0 \), the top description of the function gives
\[
\left| \frac{f_1 (y) g (y)}{f_1 (y) + f_2 (y)} \right| \leq |g(y)|
\]
Therefore,
\[
-\varepsilon + \lambda (f_1 + f_2) \leq |Lg| \leq |Lh_1 + Lh_2| \leq |Lh_1| + |Lh_2|
\leq \lambda (f_1) + \lambda (f_2).
\]
Since \( \varepsilon > 0 \) is arbitrary, this shows
\[
\lambda (f_1 + f_2) \leq \lambda (f_1) + \lambda (f_2) \leq \lambda (f_1 + f_2)
\]
The last assertion follows from
\[
\lambda (f) = \sup \{|Lg| : |g| \leq f\} \leq \sup_{||g||_\infty \leq ||f||_\infty} ||L|| \|g\|_\infty \leq ||L|| \|f\|_\infty
\]
18.6. THE DUAL SPACE OF $C_0(X)$

which proves the lemma.

Let $\Lambda$ be defined in Lemma 18.6.2. Then $\Lambda$ is linear by this lemma and also satisfies

$$|\Lambda f| \leq ||L|| \cdot ||f||_{\infty}.$$  \hfill (18.6.20)

Also, if $f \geq 0$,

$$\Lambda f = \Lambda_R f = \lambda(f) \geq 0.$$

Therefore, $\Lambda$ is a positive linear functional on $C_0(X)$. In particular, it is a positive linear functional on $C_c(X)$. By Theorem 10.3.2 on Page 269, there exists a unique measure $\mu$ such that

$$\Lambda f = \int_X f d\mu$$

for all $f \in C_c(X)$. This measure is inner regular on all open sets and on all measurable sets having finite measure. In fact, it is actually a finite measure.

**Lemma 18.6.4** Let $L \in C_0(X)'$ as above. Then letting $\mu$ be the Radon measure just described, it follows $\mu$ is finite and

$$\mu(X) = ||\Lambda|| = ||L||$$

**Proof:** First of all, why is $||\Lambda|| = ||L||$? From 18.6.20 it follows $||\Lambda|| \leq ||L||$. But also

$$|Lg| \leq \lambda(|g|) = \Lambda(|g|) \leq ||\Lambda|| \cdot ||g||_{\infty}$$

and so by definition of the operator norm, $||L|| \leq ||\Lambda||$.

Now $X$ is an open set and so

$$\mu(X) = \sup \{ \mu(K) : K \subseteq X \}$$

and so letting $K \prec f \prec X$ for one of these $K$, it also follows

$$\mu(X) = \sup \{ \Lambda f : f \prec X \}$$

However, for such $f \prec X$,

$$0 \leq \Lambda f = \Lambda_R f \leq ||L|| \cdot ||f||_{\infty} = ||L||$$

and so

$$\mu(X) \leq ||L||.$$

Now since $C_c(X)$ is dense in $C_0(X)$, there exists $f \in C_c(X)$ such that $||f|| \leq 1$ and

$$|\Lambda f| + \varepsilon > ||\Lambda|| = ||L||$$

Then also $f \prec X$ and so

$$||L|| - \varepsilon < |\Lambda f| = \Lambda f \leq \mu(X)$$

Since $\varepsilon$ is arbitrary, this shows $||L|| = \mu(X)$. This proves the lemma.

What follows is the Riesz representation theorem for $C_0(X)'$. 
Theorem 18.6.5  Let $L \in (C_0(X))'$ for $X$ a locally compact Hausdorff space. Then there exists a finite Radon measure $\mu$ and a function $\sigma \in L^\infty(X, \mu)$ such that for all $f \in C_0(X)$,

$$L(f) = \int_X f \sigma d\mu.$$ 

Furthermore,

$$\mu(X) = ||L||, |\sigma| = 1 \text{ a.e.}$$

and if

$$\nu(E) \equiv \int_E \sigma d\mu$$

then $\mu = |\nu|

Proof: From the above there exists a unique Radon measure $\mu$ such that for all $f \in C_c(X)$,

$$\Lambda f = \int_X f d\mu$$

Then for $f \in C_c(X)$,

$$|Lf| \leq \Lambda(|f|) = \int_X |f| d\mu = ||f||_{L^1(\mu)}.$$ 

Since $\mu$ is both inner and outer regular thanks to it being finite, $C_c(X)$ is dense in $L^1(X, \mu)$. (See Theorem 13.2.4 for more than is needed.) Therefore $L$ extends uniquely to an element of $(L^1(X, \mu))'$, $\bar{L}$. By the Riesz representation theorem for $L^1$ for finite measure spaces, there exists a unique $\sigma \in L^\infty(X, \mu)$ such that for all $f \in L^1(X, \mu)$,

$$\bar{L}f = \int_X f \sigma d\mu$$

In particular, for all $f \in C_0(X)$,

$$Lf = \int_X f \sigma d\mu$$

and it follows from Lemma (18.6.4) $\mu(X) = ||L||$.

It remains to verify $|\sigma| = 1$ a.e. For any $f \geq 0$,

$$\Lambda f \equiv \int_X f d\mu \geq |Lf| = \left| \int_X f \sigma d\mu \right|$$

Now if $E$ is measurable, the regularity of $\mu$ implies there exists a sequence of bounded functions $f_n \in C_c(X)$ such that $f_n(x) \to \chi_E(x)$ a.e. Then using the dominated convergence theorem in the above,

$$\int_E d\mu = \lim_{n \to \infty} \int_X f_n d\mu \geq \lim_{n \to \infty} \left| \int_X f_n \sigma d\mu \right| = \left| \int_E \sigma d\mu \right|$$
and so if \( \mu(E) > 0 \),
\[
1 \geq \left| \frac{1}{\mu(E)} \int_E \sigma d\mu \right|
\]
which shows from Lemma 18.2.7 that \( |\sigma| \leq 1 \) a.e. But also, choosing \( f_1 \) appropriately, \( ||f_1||_\infty \leq 1 \), and letting \( \omega L f_1 = |L f_1| \),
\[
\mu(X) = ||L|| = \sup_{||f||_\infty \leq 1} |Lf| \leq |Lf_1| + \varepsilon
\]
\[
\leq \int_X f_1 \omega \sigma d\mu + \varepsilon = \int_X \operatorname{Re}(f_1 \omega \sigma) d\mu + \varepsilon
\]
\[
\leq \int_X |\sigma| d\mu + \varepsilon
\]
and since \( \varepsilon \) is arbitrary,
\[
\mu(X) \leq \int_X |\sigma| d\mu
\]
which requires \( |\sigma| = 1 \) a.e. since it was shown to be no larger than 1 and if it is smaller than 1 on a set of positive measure, then the above could not hold.

It only remains to verify \( \mu = |\nu| \). By Corollary 18.2.10,
\[
|\nu|(E) = \int_E |\sigma| d\mu = \int_E 1 d\mu = \mu(E)
\]
and so \( \mu = |\nu| \). This proves the Theorem.

Sometimes people write
\[
\int_X f d\nu = \int_X f \sigma d|\nu|
\]
where \( \sigma d|\nu| \) is the polar decomposition of the complex measure \( \nu \). Then with this convention, the above representation is
\[
L(f) = \int_X f d\nu, \quad |\nu|(X) = ||L||.
\]

18.7 The Dual Space Of \( C_0(X) \), Another Approach

It is possible to obtain the above theorem by a slick trick after first proving it for the special case where \( X \) is a compact Hausdorff space. For \( X \) a locally compact Hausdorff space, \( \widehat{X} \) denotes the one point compactification of \( X \). Thus, \( \widehat{X} = X \cup \{\infty\} \) and the topology of \( \widehat{X} \) consists of the usual topology of \( X \) along with all complements of compact sets which are defined as the open sets containing \( \infty \). Also \( C_0(X) \) will denote the space of continuous functions, \( f \), defined on \( X \) such that in the topology of \( \widehat{X} \), \( \lim_{x \to \infty} f(x) = 0 \). For this space of functions, \( ||f||_0 = \sup\{|f(x)| : x \in X\} \) is a norm which makes this into a Banach space. Then the generalization is the following corollary.
Corollary 18.7.1 Let $L \in (C_0(X))^\prime$ where $X$ is a locally compact Hausdorff space. Then there exists $\sigma \in L^\infty(X, \mu)$ for $\mu$ a finite Radon measure such that for all $f \in C_0(X)$,
\[ L(f) = \int_X f\sigma d\mu. \]

Proof: Let
\[ \tilde{D} \equiv \{ f \in C(\tilde{X}) : f(\infty) = 0 \}. \]

Thus $\tilde{D}$ is a closed subspace of the Banach space $C(\tilde{X})$. Let $\theta : C_0(X) \to \tilde{D}$ be defined by
\[ \theta f(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = \infty. \end{cases} \]

Then $\theta$ is an isometry of $C_0(X)$ and $\tilde{D}$. (\[ ||\theta u|| = ||u||. \]) The following diagram is obtained.

\[ C_0(X)^\prime \xleftarrow{\theta^*} (\tilde{D})^\prime \xrightarrow{i^*} C(\tilde{X})^\prime \]
\[ C_0(X) \xrightarrow{\theta} \tilde{D} \xrightarrow{i} C(\tilde{X}) \]

By the Hahn Banach theorem, there exists $L_1 \in C(\tilde{X})^\prime$ such that $\theta^* i^* L_1 = L$. Now apply Theorem 18.6.5 to get the existence of a finite Radon measure, $\mu_1$, on $\tilde{X}$ and a function $\sigma \in L^\infty(\tilde{X}, \mu_1)$, such that
\[ L_1 g = \int_{\tilde{X}} g\sigma d\mu_1. \]

Letting the $\sigma$ algebra of $\mu_1$ measurable sets be denoted by $S_1$, define
\[ S \equiv \{ E \setminus \{\infty\} : E \in S_1 \} \]

and let $\mu$ be the restriction of $\mu_1$ to $S$. If $f \in C_0(X)$,
\[ Lf = \theta^* i^* L_1 f \equiv L_1 i \theta f = L_1 \theta f = \int_{\tilde{X}} \theta f \sigma d\mu_1 = \int_{X} f\sigma d\mu. \]

This proves the corollary.

18.8 More Attractive Formulations

In this section, Corollary 18.7.1 will be refined and placed in an arguably more attractive form. The measures involved will always be complex Borel measures defined on a $\sigma$ algebra of subsets of $X$, a locally compact Hausdorff space.
Definition 18.8.1 Let $\lambda$ be a complex measure. Then $\int f d\lambda \equiv \int fhd|\lambda|$ where $hd|\lambda|$ is the polar decomposition of $\lambda$ described above. The complex measure, $\lambda$ is called regular if $|\lambda|$ is regular.

The following lemma says that the difference of regular complex measures is also regular.

Lemma 18.8.2 Suppose $\lambda_i, i = 1, 2$ is a complex Borel measure with total variation finite defined on $X$, a locally compact Hausdorff space. Then $\lambda_1 - \lambda_2$ is also a regular measure on the Borel sets.

Proof: Let $E$ be a Borel set. That way it is in the $\sigma$ algebras associated with both $\lambda_i$. Then by regularity of $\lambda_i$, there exist $K$ and $V$ compact and open respectively such that $K \subseteq E \subseteq V$ and $|\lambda_i|(V \setminus K) < \epsilon/2$. Therefore,

$$\sum_{A \in \pi(V \setminus K)} |(\lambda_1 - \lambda_2)(A)| = \sum_{A \in \pi(V \setminus K)} |\lambda_1(A) - \lambda_2(A)|$$

$$\leq \sum_{A \in \pi(V \setminus K)} |\lambda_1(A)| + |\lambda_2(A)|$$

$$\leq |\lambda_1|(V \setminus K) + |\lambda_2|(V \setminus K) < \epsilon.$$

Therefore, $|\lambda_1 - \lambda_2|(V \setminus K) \leq \epsilon$ and this shows $\lambda_1 - \lambda_2$ is regular as claimed.

Theorem 18.8.3 Let $L \in C_0(X)'$ Then there exists a unique complex measure, $\lambda$ with $|\lambda|$ regular and Borel, such that for all $f \in C_0(X)$,

$$L(f) = \int_X f d\lambda.$$ 

Furthermore, $||L|| = |\lambda|(X)$.

Proof: By Corollary 18.7.1 there exists $\sigma \in L^\infty(X, \mu)$ where $\mu$ is a Radon measure such that for all $f \in C_0(X)$,

$$L(f) = \int_X f d\mu.$$

Let a complex Borel measure, $\lambda$ be given by

$$\lambda(E) \equiv \int_E \sigma d\mu.$$

This is a well defined complex measure because $\mu$ is a finite measure. By Corollary 18.7.1

$$|\lambda|(E) = \int_E |\sigma| d\mu \quad (18.8.21)$$

\footnote{Recall this is automatic for a complex measure.}
and \( \sigma = g|\sigma| \) where \( gd|\lambda| \) is the polar decomposition for \( \lambda \). Therefore, for \( f \in C_0(X) \),

\[
L(f) = \int_X f\sigma d\mu = \int_X f g|\sigma| d\mu = \int_X f g|\lambda| \equiv \int_X f d\lambda. \tag{18.8.22}
\]

From \( \sigma \) and the regularity of \( \mu \), it follows that \( |\lambda| \) is also regular.

What of the claim about \( ||L|| \)? By the regularity of \( |\lambda| \), it follows that \( C_0(X) \) (in fact, \( C_c(X) \)) is dense in \( L^1(X,|\lambda|) \). Since \( |\lambda| \) is finite, \( g \in L^1(X,|\lambda|) \). Therefore, there exists a sequence of functions in \( C_0(X) \), \( \{f_n\} \) such that \( f_n \to \overline{g} \) in \( L^1(X,|\lambda|) \). Therefore, there exists a subsequence, still denoted by \( \{f_n\} \) such that \( f_n(x) \to \overline{g}(x) \) \( |\lambda| \) a.e. also. But since \( |\overline{g}(x)| = 1 \) a.e. it follows that \( h_n(x) \equiv \frac{f_n(x)}{|f_n(x)| + \overline{g}} \) also converges pointwise \( |\lambda| \) a.e. Then from the dominated convergence theorem and \( \|L\| \geq \lim_{n \to \infty} \int_X h_n g d|\lambda| = |\lambda|(X) \).

Also, if \( ||f||_{C_0(X)} \leq 1 \), then

\[
|L(f)| = \left| \int_X f g d|\lambda| \right| \leq \int_X |f| d|\lambda| \leq |\lambda|(X) ||f||_{C_0(X)}
\]

and so \( ||L|| \leq |\lambda|(X) \). This proves everything but uniqueness.

Suppose \( \lambda \) and \( \lambda_1 \) both work. Then for all \( f \in C_0(X) \),

\[
0 = \int_X f d(\lambda - \lambda_1) = \int_X fhd|\lambda - \lambda_1|
\]

where \( hd|\lambda - \lambda_1| \) is the polar decomposition for \( \lambda - \lambda_1 \). By Lemma \( \sigma \), \( \lambda - \lambda_1 \) is regular and so, as above, there exists \( \{f_n\} \) such that \( |f_n| \leq 1 \) and \( f_n \to h \) pointwise. Therefore, \( \int_X d|\lambda - \lambda_1| = 0 \) so \( \lambda = \lambda_1 \). This proves the theorem.

18.9 Sequential Compactness In \( L^1 \)

**Lemma 18.9.1** Let \( C \equiv \{E_i\}_{i=1}^\infty \) be a countable collection of sets and let \( \Omega_1 \equiv \bigcup_{i=1}^\infty E_i \). Then there exists an algebra of sets, \( A \), such that \( A \supseteq C \) and \( A \) is countable.

**Proof:** Let \( C_1 \) denote all finite unions of sets of \( C \) and also include \( \Omega_1 \) and \( \emptyset \). Thus \( C_1 \) is countable. Next let \( B_1 \) denote all sets of the form \( \Omega_1 \setminus A \) such that \( A \in C_1 \). Next let \( C_2 \) denote all finite unions of sets of \( B_1 \cup C_1 \). Then let \( B_2 \) denote all sets of the form \( \Omega_1 \setminus A \) such that \( A \in C_2 \) and let \( C_3 = B_2 \cup C_2 \). Continuing this way yields an increasing sequence, \( \{C_n\} \) each of which is countable. Let

\[
A \equiv \bigcup_{i=1}^\infty C_i.
\]

Then \( A \) is countable. Also \( A \) is an algebra. Here is why. Suppose \( A, B \in A \). Then there exists \( n \) such that both \( A, B \in C_{n-1} \). It follows \( A \cup B \in C_n \subseteq A \) from the
construction. It only remains to show that \( A \setminus B \in \mathcal{A} \). Taking complements with respect to \( \Omega_1 \), it follows from the construction that \( A^C, B^C \) are both in \( \mathcal{B}_{n-1} \subseteq \mathcal{C}_n \). Thus,
\[
A^C \cup B \in \mathcal{C}_n
\]
and so
\[
A \setminus B = (A^C \cup B)^C \in \mathcal{B}_n \subseteq \mathcal{C}_{n+1} \subseteq \mathcal{A}.
\]
This shows \( \mathcal{A} \) is an algebra of sets of \( \Omega_1 \) which is also countable and contains \( \mathcal{C} \).

**Lemma 18.9.2** Let \( \{ f_n \} \) be a sequence of functions in \( L^1 (\Omega, S, \mu) \). Then there exists a \( \sigma \) finite set of \( S \), \( \Omega_1 \), and a \( \sigma \) algebra of subsets of \( \Omega_1, S_1 \), such that \( S_1 \subseteq S \), \( f_n = 0 \) off \( \Omega_1 \), \( f_n \in L^1 (\Omega_1, S_1, \mu) \), and \( S_1 = \sigma (A) \), the \( \sigma \) algebra generated by \( A \), for some \( A \) a countable algebra.

**Proof:** Let \( \mathcal{E}_n \) denote the sets which are of the form
\[
\left\{ f_n^{-1} (B(z, r)) : z \in \mathbb{Q} + i\mathbb{Q}, r > 0, r \in \mathbb{Q}, \text{ and } 0 \notin \overline{B(z, r)} \right\}
\]
Since each \( \mathcal{E}_n \) is countable, so is
\[
\mathcal{E} \equiv \cup_{n=1}^{\infty} \mathcal{E}_n
\]
Now let \( \Omega_1 \equiv \cup \mathcal{E} \). I claim \( \Omega_1 \) is \( \sigma \) finite. To see this, let
\[
W_n = \left\{ \omega \in \Omega : |f_k (\omega)| > \frac{1}{n} \text{ for some } k = 1, 2, \ldots, n \right\}
\]
Thus if \( \omega \in W_n \), it follows that for some \( r \in \mathbb{Q}, z \in \mathbb{Q} + i\mathbb{Q} \) sufficiently close to \( f_k (\omega) \)
\[
\omega \in f_k^{-1} (B(z, r)) \in \mathcal{E}_k
\]
and so \( \omega \in \cup_{k=1}^{n} \mathcal{E}_k \) and consequently, \( W_n \in \cup_{k=1}^{n} \mathcal{E}_k \). Also
\[
\mu (W_n) \frac{1}{n} \leq \int_{W_n} \sum_{k=1}^{n} |f_k (\omega)| \, d\mu < \infty.
\]
Now if \( \omega \in \Omega_1 \), then for some \( k, \omega \) is contained in a set of \( \mathcal{E}_k \). Therefore, for that \( k \),
\[
f_k (\omega) \in B(z, r)
\]
where \( r \) is a positive rational number and \( z \in \mathbb{Q} + i\mathbb{Q} \) and \( \overline{B(z, r)} \) does not contain 0. Therefore, \( f_k (\omega) \) is at a positive distance from 0 and so for large enough \( n, \omega \in W_n \). Take \( n \) so large that \( 1/n \) is less than the distance from \( \overline{B(z, r)} \) to 0 and also larger than \( k \).

By Lemma 18.9.1 there exists a countable algebra of sets \( \mathcal{A} \) which contains \( \mathcal{E} \). Let \( S_1 \equiv \sigma (A) \). It remains to show \( f_n (\omega) = 0 \) off \( \Omega_1 \) for all \( n \). Let \( \omega \notin \Omega_1 \). Then \( \omega \) is not contained in any set of \( \mathcal{E}_k \) and so \( f_k (\omega) \) cannot be nonzero. Hence \( f_k (\omega) = 0 \). This proves the lemma.

The following Theorem is the main result on sequential compactness in \( L^1 (\Omega, S, \mu) \).
Theorem 18.9.3 Let $K \subseteq L^1(\Omega, S, \mu)$ be such that for some $C > 0$ and all $f \in K$,
\[ ||f||_{L^1} \leq C \]  
(18.9.23)
and $K$ also satisfies the property that if $\{E_n\}$ is a decreasing sequence of measurable sets such that $\cap_{n=1}^{\infty} E_n = \emptyset$, then for all $\varepsilon > 0$ there exists $n_\varepsilon$ such that if $n \geq n_\varepsilon$, then
\[ \left| \int_{E_n} f \, d\mu \right| < \varepsilon \]  
(18.9.24)
for all $f \in K$. Then every sequence of functions of $K$ has an $L^1(\Omega, \mu)$ weakly convergent subsequence.

Proof: Take $\{f_n\}$ a sequence in $K$ and let $\mathcal{A}, S_1, \Omega_1$ be as in Lemma 18.9.2. Thus $\mathcal{A}$ is a countable algebra and by assumption, for each $E \in \mathcal{A}$,
\[ \left\{ \int_E f_n \, d\mu \right\} \]
is a bounded sequence and so there exists a convergent subsequence. Therefore, from a Cantor diagonalization argument, there exists a subsequence, denoted by $\{g_n\}$ such that
\[ \left\{ \int_E g_n \, d\mu \right\} \]
converges for every $E \in \mathcal{A}$.

Let
\[ \mathcal{M} \equiv \left\{ E \in S_1 = \sigma(\mathcal{A}) \text{ such that } \lim_{n \to \infty} \int_E g_n \, d\mu \text{ exists} \right\}. \]
Then it has been shown that $\mathcal{A} \subseteq \mathcal{M}$. Suppose $E_k \uparrow E$ where $E_k \in \mathcal{M}$. Then letting $\varepsilon > 0$ be given, the assumption shows that for $k$ large enough,
\[ \left| \int_{E \setminus E_k} g_n \, d\mu \right| < \varepsilon \]
for all $g_n$. Therefore, picking such a $k$,
\[ \left| \int_{E_k} g_n \, d\mu - \int_{E_k} g_m \, d\mu \right| \leq 2\varepsilon + \left| \int_{E_k} g_n \, d\mu - \int_{E_k} g_m \, d\mu \right| < 3\varepsilon \]
provided $m, n$ are large enough. Therefore, $\{\int_E g_n \, d\mu\}$ is a Cauchy sequence and so it converges.

In the case that $E_k \downarrow E$ use the assumption to conclude there exists a $k$ large enough that
\[ \left| \int_{E_k \setminus E} g_n \, d\mu \right| < \varepsilon \]
for all \( g_n \). Then

\[
\left| \int_E g_n d\mu - \int_E g_m d\mu \right| = \left| \int_{E_k} g_n d\mu - \int_{E_k} g_m d\mu \right|
+ \left| \int_{E_k \setminus E} g_n d\mu \right| + \left| \int_{E_k \setminus E} g_m d\mu \right|
\leq \left| \int_{E_k} g_n d\mu - \int_{E_k} g_m d\mu \right| + 2\varepsilon < 3\varepsilon
\]

provided \( m, n \) large enough. Again \( \{ \int_E g_n d\mu \} \) is a Cauchy sequence. This shows \( \mathcal{M} \) is a monotone class and so by the monotone class theorem, Theorem 10.10.5 on Page 302 it follows \( \mathcal{M} = S_1 \equiv \sigma (A) \).

Therefore, picking \( E \in S_1 \), you can define a complex measure,

\[ \lambda (E) = \lim_{n \to \infty} \int_E g_n d\mu \]

Then \( \lambda \ll \mu \) and so by Corollary 18.2.9 on Page 592 and the fact shown above that \( \Omega_1 \) is \( \sigma \) finite there exists a unique \( S_1 \) measurable \( g \in L^1 (\Omega_1, \mu) \) such that

\[ \lambda (E) = \int_E g d\mu = \lim_{n \to \infty} \int_E g_n d\mu. \]

Extend \( g \) to equal 0 outside \( \Omega_1 \).

It remains to show \( \{ g_n \} \) converges weakly. It has just been shown that for every \( s \) a simple function measurable with respect to \( S_1 \)

\[ \int_{\Omega_1} g_n sd\mu = \int_{\Omega_1} g_n sd\mu \to \int_{\Omega_1} gs \mu = \int_{\Omega} gs \mu \]

Now let \( f \in L^\infty (\Omega_1, S_1, \mu) \) and pick a uniformly bounded representative of this function. Then by Theorem 18.3.1 on Page 224 there exists a sequence of simple functions converging uniformly to \( f \) and so

\[ \left\{ \int_{\Omega} g_n f d\mu \right\} \]

converges because

\[
\left| \int_{\Omega} g_n f d\mu - \int_{\Omega} g f d\mu \right| \leq \left| \int_{\Omega} g_n s d\mu - \int_{\Omega} gs d\mu \right|
+ \int_{\Omega} |g_n| \varepsilon d\mu + \int_{\Omega} |g| \varepsilon d\mu
\leq C\varepsilon + ||g||_1 \varepsilon + \left| \int_{\Omega} g_n s d\mu - \int_{\Omega} gs d\mu \right|
\]

for suitable simple \( s \) satisfying \( \sup_{\omega \in \Omega} |s(\omega) - f(\omega)| < \varepsilon \) and the last term converges to 0 as \( n \to \infty \).
"(L^1(\Omega, S, \mu))' is a space I don’t know much about due to a possible lack of \(\sigma\) finiteness of \(\Omega\). However, it does follow that for \(i\) the inclusion map of \(L^1(\Omega_1, S_1, \mu)\) into \(L^1(\Omega, S, \mu)\) which merely extends the function as 0 off \(\Omega_1\) and \(f \in (L^1(\Omega, S, \mu))'\), there exists \(h \in L^\infty(\Omega_1)\) such that for all \(g \in L^1(\Omega_1, S_1, \mu)\)

\[ i^*f(g) = \int_{\Omega_1} hgd\mu. \]

This is because \(i^*f \in (L^1(\Omega_1, S_1, \mu))'\) and \(\Omega_1\) is \(\sigma\) finite and so the Riesz representation theorem applies to get a unique such \(h \in L^\infty(\Omega_1)\). Then since all the \(g_n\) equal 0 off \(\Omega_1\),

\[ f(g_n) = i^*f(g_n) = \int_{\Omega_1} h g_n d\mu \]

for a unique \(h \in L^\infty(\Omega_1, S_1, \mu)\) due to the Riesz representation theorem which holds here because \(\Omega_1\) was shown to be \(\sigma\) finite. Therefore,

\[ \lim_{n \to \infty} f(g_n) = \lim_{n \to \infty} \int_{\Omega_1} h g_n d\mu = \int_{\Omega_1} hgd\mu = i^*f(g) = f(g). \]

This proves the theorem.

For more on this theorem see [42]. I have only discussed the sufficiency of the conditions to give sequential compactness. They also discuss the necessity of these conditions.

There is another nice condition which implies the above results which is seen in books on probability. It is the concept of equi integrability.

**Definition 18.9.4** Let \((\Omega, S, \mu)\) be a measure space in which \(\mu(\Omega) < \infty\). Then \(K \subseteq L^1(\Omega, S, \mu)\) is said to be equi integrable if

\[ \lim_{\lambda \to \infty} \sup_{f \in K} \int_{\{|f| \geq \lambda\}} |f| \, d\mu = 0 \]

**Lemma 18.9.5** Let \(K\) be an equi integrable set. Then there exists \(C > 0\) such that for all \(f \in K\),

\[ ||f||_{L^1} \leq C \]  \hspace{1cm} (18.9.25)

and \(K\) also satisfies the property that if \(\{E_n\}\) is a decreasing sequence of measurable sets such that \(\cap_{n=1}^\infty E_n = \emptyset\), then for all \(\varepsilon > 0\) there exists \(n_\varepsilon\) such that if \(n \geq n_\varepsilon\), then

\[ \left| \int_{E_n} f \, d\mu \right| < \varepsilon \]  \hspace{1cm} (18.9.26)

for all \(f \in K\).

**Proof:** Choose \(\lambda_0\) such that

\[ \sup_{f \in K} \int_{\{|f| \geq \lambda_0\}} |f| \, d\mu \leq 1. \]
18.9. SEQUENTIAL COMPACTNESS IN $L^1$  

Then for $f \in K$,

$$\int_{\Omega} |f| \, d\mu = \int_{\{|f| \geq \lambda_0\}} |f| \, d\mu + \int_{\{|f| < \lambda_0\}} |f| \, d\mu \leq 1 + \lambda_0 \mu(\Omega) \equiv C$$

and this proves 18.9.25.

Next suppose $\{E_n\}$ is a decreasing sequence which has empty intersection and let $\varepsilon > 0$ and choose $\lambda_\varepsilon$ such that

$$\sup_{f \in K} \int_{\{|f| \geq \lambda_\varepsilon\}} |f| \, d\mu \leq \frac{\varepsilon}{2}.$$  

Then since $\mu$ is finite, there exists $n_\varepsilon$ such that if $n \geq n_\varepsilon$, then $\mu(E_n) \leq \varepsilon/(1 + \lambda_\varepsilon)$.

Then letting $f \in K$,

$$\int_{E_n} |f| \, d\mu = \int_{E_n \cap \{|f| \geq \lambda_\varepsilon\}} |f| \, d\mu + \int_{E_n \cap \{|f| < \lambda_\varepsilon\}} |f| \, d\mu \leq \frac{\varepsilon}{2} + \int_{E_n} \lambda_\varepsilon \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This proves 18.9.26 and proves the lemma.

**Corollary 18.9.6** Let $(\Omega, \mathcal{S}, \mu)$ be a measure space in which $\mu(\Omega) < \infty$ and let $K \subseteq L^1(\Omega, \mathcal{S}, \mu)$ be equi integrable. Then every sequence from $K$ has a weakly convergent subsequence.

**Proof:** From Lemma 18.9.5 the hypotheses of Theorem 18.9.3 are satisfied.

It is also convenient to consider the following proposition.

**Proposition 18.9.7** Let $(\Omega, \mathcal{S}, \mu)$ be a measure space in which $\mu(\Omega) < \infty$. Then $K \subseteq L^1(\Omega, \mathcal{S}, \mu)$ is equi integrable if and only if $K$ is uniformly integrable and there exists a constant, $M$ such that for all $f \in K$, $||f||_{L^1} \leq M$.

**Proof:** First suppose $K$ is equi integrable. Then pick $\lambda$ such that for all $f \in K$,

$$\int_{\{|f| \geq \lambda\}} |f| \, d\mu < 1.$$  

Then for $f \in K$

$$\int_{\Omega} |f| \, d\mu = \int_{\{|f| \geq \lambda\}} |f| \, d\mu + \int_{\{|f| < \lambda\}} |f| \, d\mu \leq 1 + \lambda \mu(\Omega) \equiv M.$$  

Also, if $\varepsilon > 0$, pick $\lambda$ so large that for all $f \in K$

$$\int_{\{|f| \geq \lambda\}} |f| \, d\mu < \frac{\varepsilon}{2}.$$
Then letting $A \in \mathcal{S}$,
\[
\int_A |f| \, d\mu = \int_{A \cap \{|f| \geq \lambda\}} |f| \, d\mu + \int_{A \cap \{|f| < \lambda\}} |f| \, d\mu < \frac{\varepsilon}{2} + \lambda P(A)
\]
and so if $P(A)$ is sufficiently small, this is less than $\varepsilon$. Thus $K$ is uniformly integrable.

Now suppose $||f||_1 \leq M$ for all $f \in K$ and $K$ is uniformly integrable. Then
\[
\int_{\{|f| \geq \lambda\}} |f| \, d\mu \geq \lambda P(\{|f| \geq \lambda\})
\]
and so
\[
P(\{|f| \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\{|f| \geq \lambda\}} |f| \, d\mu \leq \frac{1}{\lambda} \int_{\Omega} |f| \, d\mu \leq \frac{M}{\lambda}
\]
and so, by the assumption of uniform integrability,
\[
\int_{\{|f| \geq \lambda\}} |f| \, d\mu < \varepsilon
\]
for all $f \in K$ provided $\lambda$ is large enough. This proves the proposition.

18.10 Exercises

1. Suppose $\mu$ is a vector measure having values in $\mathbb{R}^n$ or $\mathbb{C}^n$. Can you show that $|\mu|$ must be finite? Hint: You might define for each $e_i$, one of the standard basis vectors, the real or complex measure, $\mu_{e_i}$, given by $\mu_{e_i}(E) \equiv e_i \cdot \mu(E)$. Why would this approach not yield anything for an infinite dimensional normed linear space in place of $\mathbb{R}^n$?

2. The Riesz representation theorem of the $L^p$ spaces can be used to prove a very interesting inequality. Let $r, p, q \in (1, \infty)$ satisfy
\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.
\]
Then
\[
\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{p} > \frac{1}{r}
\]
and so $r > q$. Let $\theta \in (0, 1)$ be chosen so that $\theta r = q$. Then also we have
\[
\frac{1}{r} = \left(1 - \frac{1}{p'}\right) + \frac{1}{q} - 1 = \frac{1}{q} - \frac{1}{p'}.
\]
and so
\[ \frac{\theta}{q} = \frac{1}{q} - \frac{1}{p'} \]
which implies \( p' (1 - \theta) = q \). Now let \( f \in L^p(\mathbb{R}^n) \), \( g \in L^q(\mathbb{R}^n) \), \( f, g \geq 0 \).

Justify the steps in the following argument using what was just shown that \( \theta r = q \) and \( p' (1 - \theta) = q \). Let

\[ h \in L^{r'}(\mathbb{R}^n) \cdot \left( \frac{1}{r} + \frac{1}{r'} = 1 \right) \]

\[ \int f \ast g(x) \, |h(x)| \, dx = \int \int f(y) \, g(x - y) \, |h(x)| \, dy \, dx. \]

\[ \leq \int \int |f(y)| \, |g(x - y)|^\theta \, |g(x - y)|^{1-\theta} \, |h(x)| \, dy \, dx \]

\[ \leq \int \left( \int \left( |g(x - y)|^{1-\theta} \, |h(x)| \right)^{r'} \, dx \right)^{1/r'} \cdot \left( \int \left( |f(y)| \, |g(x - y)|^\theta \right)^r \, dx \right)^{1/r}. \]

\[ \leq \left[ \int \left( \int \left( |g(x - y)|^{1-\theta} \, |h(x)| \right)^{r'} \, dx \right)^{1/r'} \, dy \right]^1. \]

\[ \leq \left[ \int \left( \int \left( |g(x - y)|^{1-\theta} \, |h(x)| \right)^{p/r} \, dx \right)^{r'/p'} \, dy \right]^1. \]

\[ \leq \left[ \int |f(y)|^p \left( \int |g(x - y)|^\theta \, dx \right)^{p/r} \, dy \right]^{1/p} \]

\[ = \left[ \int |h(x)|^{r'} \left( \int |g(x - y)|^{(1-\theta)p'} \, dy \right)^{r'/p'} \, dx \right]^{1/r'} \cdot ||g||_q^{q/r} ||f||_p. \]

\[ = ||g||_q^{q/r} ||f||_p ||h||_r \cdot (18.10.27) \]

Young’s inequality says that

\[ ||f \ast g||_r \leq ||g||_q ||f||_p \cdot (18.10.28) \]

Therefore \( ||f \ast g||_r \leq ||g||_q ||f||_p \cdot (18.10.28) \). How does this inequality follow from the above computation? Does (18.10.27) continue to hold if \( r, p, q \) are only assumed to be in \([1, \infty] \)? Explain. Does (18.10.28) hold even if \( r, p, \) and \( q \) are only assumed to lie in \([1, \infty] \)?
3. Show that in a reflexive Banach space, weak and weak $\ast$ convergence are the same.

4. Suppose $(\Omega, \mu, \mathcal{S})$ is a finite measure space and that $\{f_n\}$ is a sequence of functions which converge weakly to 0 in $L^p(\Omega)$. Suppose also that $f_n(x) \to 0$ a.e. Show that then $f_n \to 0$ in $L^{p-\varepsilon}(\Omega)$ for every $p > \varepsilon > 0$.

5. Give an example of a sequence of functions in $L^\infty(-\pi, \pi)$ which converges weak $\ast$ to zero but which does not converge pointwise a.e. to zero.
Chapter 19

The Bochner Integral

19.1 Strong And Weak Measurability

In this chapter \((\Omega, \mathcal{S}, \mu)\) will be a \(\sigma\) finite measure space and \(X\) will be a Banach space which contains the values of either a function or a measure. The Banach space will be either a real or a complex Banach space but the field of scalars does not matter and so it is denoted by \(F\) with the understanding that \(F = \mathbb{C}\) unless otherwise stated. The theory presented here includes the case where \(X = \mathbb{R}^n\) or \(\mathbb{C}^n\) but it does not include the situation where \(f\) could have values in a space like \([0, \infty]\). To begin with here is a definition.

**Definition 19.1.1** A function, \(x : \Omega \rightarrow X\), for \(X\) a Banach space, is a simple function if it is of the form

\[
x(s) = \sum_{i=1}^{n} a_i \chi_{B_i}(s)
\]

where \(B_i \in \mathcal{S}\) and \(\mu(B_i) < \infty\) for each \(i\). A function \(x\) from \(\Omega\) to \(X\) is said to be strongly measurable if there exists a sequence of simple functions \(\{x_n\}\) converging pointwise to \(x\). The function \(x\) is said to be weakly measurable if, for each \(f \in X',\)

\[
f \circ x
\]

is a scalar valued measurable function.

Earlier, a function was measurable if inverse images of open sets were measurable. Something similar holds here. The difference is that another condition needs to hold.

**Theorem 19.1.2** \(x\) is strongly measurable if and only if \(x^{-1}(U)\) is measurable for all \(U\) open in \(X\) and \(x(\Omega)\) is separable.
Proof: Suppose first $x^{-1}(U)$ is measurable for all $U$ open in $X$ and $x(\Omega)$ is separable. Let $\{a_n\}_{n=1}^{\infty}$ be the dense subset of $x(\Omega)$. It follows $x^{-1}(B)$ is measurable for all $B$ Borel because

$$\{B : x^{-1}(B) \text{ is measurable}\}$$

is a $\sigma$ algebra containing the open sets. Let

$$U^n_k \equiv \{z \in X : ||z - a_k|| \leq \min\{\{||z - a_l||\}_{l=1}^{n}\}\}.$$

In words, $U^n_k$ is the set of points of $X$ which are as close to $a_k$ as they are to any of the $a_l$ for $l \leq n$.

$$B^n_k \equiv x^{-1}(U^n_k), \ D^n_k \equiv B^n_k \setminus (\cup_{i=1}^{n-1} B^n_i), \ D^n_1 \equiv B^n_1,$$

and

$$x_n(s) = \sum_{k=1}^{n} a_k \chi_{D^n_k}(s).$$

Thus $x_n(s)$ is a closest approximation to $x(s)$ from $\{a_k\}_{k=1}^{n}$ and so $x_n(s) \to x(s)$ because $\{a_n\}_{n=1}^{\infty}$ is dense in $x(\Omega)$. Furthermore, $x_n$ is measurable because each $D^n_k$ is measurable.

Since $(\Omega, S, \mu)$ is $\sigma$ finite, there exists $\Omega_n \uparrow \Omega$ with $\mu(\Omega_n) < \infty$. Let

$$y_n(s) = \chi_{\Omega_n}(s) x_n(s).$$

Then $y_n(s) \to x(s)$ for each $s$ because for any $s, n \in \Omega_n$ if $n$ is large enough. Also $y_n$ is a simple function because it equals 0 off a set of finite measure.

Now suppose that $x$ is strongly measurable. Then some sequence of simple functions, $\{x_n\}$, converges pointwise to $x$. Then $x_n^{-1}(W)$ is measurable for every open set $W$ because it is just a finite union of measurable sets. Thus, $x_n^{-1}(W)$ is measurable for every Borel set $W$. This follows by considering

$$\{W : x_n^{-1}(W) \text{ is measurable}\}$$

and observing this is a $\sigma$ algebra which contains the open sets. Since $X$ is a metric space, it follows that if $U$ is an open set in $X$, there exists a sequence of open sets, $\{V_n\}$ which satisfies

$$V_n \subseteq U, \ V_n \subseteq V_{n+1}, \ U = \bigcup_{n=1}^{\infty} V_n.$$

Then

$$x^{-1}(V_m) \subseteq \bigcup_{n<\infty} \bigcap_{k\geq n} x^{-1}(V_m) \subseteq x^{-1}(V_m).$$

This implies

$$x^{-1}(U) = \bigcup_{m<\infty} x^{-1}(V_m)$$
19.1. STRONG AND WEAK MEASURABILITY

\[ \subseteq \bigcup_{m<\infty} \bigcap_{n<\infty} x_k^{-1} (V_m) \subseteq \bigcup_{m<\infty} x^{-1} (V_m) \subseteq x^{-1} (U). \]

Since

\[ x^{-1} (U) = \bigcup_{m<\infty} \bigcap_{n<\infty} x_k^{-1} (V_m), \]

it follows that \( x^{-1} (U) \) is measurable for every open \( U \). It remains to show \( x (\Omega) \) is separable. Let

\[ D \equiv \text{all values of the simple functions } x_n \]

which converge to \( x \) pointwise. Then \( D \) is clearly countable and dense in \( D \), a set which contains \( x (\Omega) \).

Claim: \( x (\Omega) \) is separable.

Proof of claim: For \( n \in \mathbb{N} \), let \( B_n \equiv \{ B (d, r) : 0 < r < \frac{1}{n}, \ r \text{ rational}, \ d \in D \} \). Thus \( B_n \) is countable. Let \( z \in D \). Consider \( B (z, \frac{1}{n}) \). Then there exists \( d \in D \cap B (z, \frac{1}{3n}) \). Now \( r \in \mathbb{Q} \cap \left( \frac{1}{3n}, \frac{1}{n} \right) \) so that \( B (d, r) \in B_n \). Now \( z \in B (d, r) \) and so this shows that \( x (\Omega) \subseteq D \subseteq \bigcup B_n \) for each \( n \). Now let \( B'_n \) denote those sets of \( B_n \) which have nonempty intersection with \( x (\Omega) \). Say \( B'_n = \{ B_k^n \}_{n,k=1}^\infty \). By the axiom of choice, there exists \( x_k^n \in B_k^n \cap x (\Omega) \). Then if \( z \in x (\Omega) \), \( z \) is contained in some set of \( B'_n \) which also contains a point of \( \{ x_k^n \}_{n,k=1}^\infty \). Therefore, \( z \) is at least as close as \( 2/n \) to some point of \( \{ x_k^n \}_{n,k=1}^\infty \) which shows \( \{ x_k^n \}_{n,k=1}^\infty \) is a countable dense subset of \( x (\Omega) \). Therefore \( x (\Omega) \) is separable. \( \blacksquare \)

The last part also shows that a subset of a separable metric space is also separable. Therefore, the following simple corollary is obtained.

Corollary 19.1.3 If \( X \) is a separable Banach space then \( x \) is strongly measurable if and only if \( x^{-1} (U) \) is measurable for all \( U \) open in \( X \).

The next lemma is interesting for its own sake. Roughly it says that if a Banach space is separable, then the unit ball in the dual space is weak \( \ast \) separable. This will be used to prove Pettis's theorem, one of the major theorems in this subject which relates weak measurability to strong measurability.

Lemma 19.1.4 If \( X \) is a separable Banach space with \( B' \) the closed unit ball in \( X' \), then there exists a sequence \( \{ f_n \}_{n=1}^\infty \equiv D' \subseteq B' \) with the property that for every \( x \in X \),

\[ ||x|| = \sup_{f \in D'} |f (x)| \]

If \( H \) is a dense subset of \( X' \) then \( D' \) may be chosen to be contained in \( H \).

Proof: Let \( \{ a_k \} \) be a countable dense set in \( X \), and consider the mapping

\[ \phi_n : B' \to \mathbb{F}^n \]

given by

\[ \phi_n (f) \equiv (f (a_1), \cdots, f (a_n)). \]
Then \( \phi_n(\mathcal{B}') \) is contained in a compact subset of \( \mathbb{F}^n \) because \( |f(a_k)| \leq ||a_k|| \). Therefore, there exists a countable dense subset of \( \phi_n(\mathcal{B}'), \{ \phi_n(f_k) \}_{k=1}^{\infty} \). Then pick \( h^k \in H \cap \mathcal{B}' \) such that \( \lim_{j \to \infty} \||f_k - h^k|| = 0 \). Then \( \{ \phi_n(h^k), k, j \} \) must also be dense in \( \phi_n(\mathcal{B}') \). Let \( D'_n = \{ h^k_{j,k}, k, j \} \). Define

\[
D' = \bigcup_{k=1}^{\infty} D'_k.
\]

Note that for each \( x \in X \), there exists \( f_x \in \mathcal{B}' \) such that \( f_x(x) = ||x|| \). From the construction,

\[
||a_m|| = \sup \{|f(a_m)| : f \in D'\}
\]

because \( f_{a_m}(a_m) \) is the limit of numbers \( f(a_m) \) for \( f \in D'_m \subseteq D' \). Therefore, for \( x \) arbitrary,

\[
||x|| \leq ||x - a_m|| + ||a_m|| = \sup \{|f(a_m)| : f \in D'\} + ||x - a_m||
\]

\[
\leq \sup \{|f(a_m - x) + f(x)| : f \in D'\} + ||x - a_m||
\]

\[
\leq \sup \{|f(x)| : f \in D'\} + 2||x - a_m|| \leq ||x|| + 2||x - a_m||.
\]

Since \( a_m \) is arbitrary and the \( \{ a_m \}_{m=1}^{\infty} \) are dense, this establishes the claim of the lemma.

The next theorem is one of the most important results in the subject. It is due to Pettis and appeared in 1938.

**Theorem 19.1.5** If \( x \) has values in a separable Banach space \( X \). Then \( x \) is weakly measurable if and only if \( x \) is strongly measurable.

**Proof:** It is necessary to show \( x^{-1}(U) \) is measurable whenever \( U \) is open. Since every open set is a countable union of balls, it suffices to show \( x^{-1}(B(a,r)) \) is measurable for any ball, \( B(a,r) \). Since every open ball is the countable union of closed balls, it suffices to verify \( x^{-1}(\overline{B(a,r)}) \) is measurable. From Lemma [M.J.]

\[
x^{-1}(\overline{B(a,r)}) = \{ s : ||x(s) - a|| \leq r \}
\]

\[
= \left\{ s : \sup_{f \in D'} |f(x(s) - a)| \leq r \right\}
\]

\[
= \bigcap_{f \in D'} \{ s : |f(x(s) - a)| \leq r \}
\]

\[
= \bigcap_{f \in D'} \{ s : |f(x(s)) - f(a)| \leq r \}
\]

\[
= \bigcap_{f \in D'} (f \circ x)^{-1} \overline{B(f(a),r)}
\]

which equals a countable union of measurable sets because it is assumed that \( f \circ x \) is measurable for all \( f \in X' \).

Next suppose \( x \) is strongly measurable. Then there exists a sequence of simple functions \( x_n \) which converges to \( x \) pointwise. Hence for all \( f \in X' \), \( f \circ x_n \) is measurable and \( f \circ x_n \to f \circ x \) pointwise. Thus \( x \) is weakly measurable. ■

The same method of proof yields the following interesting corollary.
Corollary 19.1.6 Let $X$ be a separable Banach space and let $\mathcal{B}(X)$ denote the $\sigma$ algebra of Borel sets. Let $H$ be a dense subset of $X'$. Then $\mathcal{B}(X) = \sigma(H) \equiv F$, the smallest $\sigma$ algebra of subsets of $X$ which has the property that every function, $x^* \in H$ is measurable.

Proof: First I need to show $F$ contains open balls because then $F$ will contain the open sets and hence the Borel sets. As noted above, it suffices to show $F$ contains closed balls. Let $D'$ be those functionals in $B'(X)$ defined in Lemma 19.1.4.

Then

$$\{ x : ||x - a|| \leq r \} = \{ x : \sup_{x^* \in D'} |x^*(x - a)| \leq r \}$$

$$= \bigcap_{x^* \in D'} \{ x : |x^*(x - a)| \leq r \}$$

$$= \bigcap_{x^* \in D'} \{ x : |x^*(x) - x^*(a)| \leq r \}$$

$$= \bigcap_{x^* \in D'} x^*-1(B(x^*(a), r)) \in \sigma(H)$$

which is measurable because this is a countable intersection of measurable sets. Thus $F$ contains open sets so $\sigma(H) \equiv F \supseteq \mathcal{B}(X)$.

To show the other direction for the inclusion, note that each $x^*$ is $B(X)$ measurable because $x^*-1(\text{open set}) = \text{open set}$. Therefore, $\mathcal{B}(X) \supseteq \sigma(H)$.

It is important to verify the limit of strongly measurable functions is itself strongly measurable. This happens under very general conditions. Suppose $X$ is any separable metric space and let $\tau$ denote the open sets of $X$. Then it is routine to see that

$$\tau \text{ has a countable basis, } \mathcal{B}. \quad (19.1.1)$$

Whenever $U \in \mathcal{B}$, there exists a sequence of open sets, $\{V_m\}_{m=1}^{\infty}$, such that

$$\cdots V_m \subseteq \overline{V}_m \subseteq V_{m+1} \subseteq \cdots, U = \bigcup_{m=1}^{\infty} V_m. \quad (19.1.2)$$

Theorem 19.1.7 Let $f_n$ and $f$ be functions mapping $\Omega$ to $X$ where $\mathcal{F}$ is a $\sigma$ algebra of measurable sets of $\Omega$ and $(X, \tau)$ is a topological space satisfying [19.1.7]. Then if $f_n$ is measurable, and $f(\omega) = \lim_{n \to \infty} f_n(\omega)$, it follows that $f$ is also measurable. (Pointwise limits of measurable functions are measurable.)

Proof: Let $\mathcal{B}$ be the countable basis of $\mathcal{B}$ and let $U \in \mathcal{B}$. Let $\{V_m\}$ be the sequence of $\mathcal{U}$. Since $f$ is the pointwise limit of $f_n$, $f^{-1}(V_m) \subseteq \{ \omega : f_k(\omega) \in V_m \text{ for all } k \text{ large enough} \} \subseteq f^{-1}(\overline{V}_m)$. Therefore,

$$f^{-1}(U) = \bigcup_{m=1}^{\infty} f^{-1}(V_m) \subseteq \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{m} f_k^{-1}(V_m) \subseteq \bigcup_{m=1}^{\infty} f^{-1}(\overline{V}_m) = f^{-1}(U).$$
CHAPTER 19. THE BOCHNER INTEGRAL

It follows $f^{-1}(U) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. Now let $W \in \tau$. Since $\mathcal{B}$ is countable, $W = \bigcup_{n=1}^{\infty} U_n$ for some sets $U_n \in \mathcal{B}$. Hence

$$f^{-1}(W) = \bigcup_{n=1}^{\infty} f^{-1}(U_n) \in \mathcal{F}.$$  

Note that the same conclusion would hold for any topological space with the property that for any open set $U$, it has such a sequence of $V_k$ attached to it as in 19.1.2.

**Corollary 19.1.8** $x$ is strongly measurable if and only if $x(\Omega)$ is separable and $x$ is weakly measurable.

**Proof:** Strong measurability clearly implies weak measurability. If $x_n(s) \to x(s)$ where $x_n$ is simple, then $f(x_n(s)) \to f(x(s))$ for all $f \in X'$. Hence $f \circ x$ is measurable by Theorem 19.1.7 because it is the limit of a sequence of measurable functions. Let $D$ denote the set of all values of $x_n$. Then $\overline{D}$ is a separable set containing $x(\Omega)$. Thus $\overline{D}$ is a separable metric space. Therefore $x(\Omega)$ is separable also by the last part of the proof of Theorem 19.1.5.

Now suppose $D$ is a countable dense subset of $x(\Omega)$ and $x$ is weakly measurable. Let $Z$ be the subset consisting of all finite linear combinations of $D$ with the scalars coming from the set of rational points of $\mathbb{F}$. Thus, $Z$ is countable. Letting $Y = Z$, $Y$ is a separable Banach space containing $x(\Omega)$. If $f \in Y'$, $f$ can be extended to an element of $X'$ by the Hahn Banach theorem. Therefore, $x$ is a weakly measurable $Y$ valued function. Now use Theorem 19.1.5 to conclude $x$ is strongly measurable.

Weakly measurable as defined above means $s \to x^*(x(s))$ is measurable for every $x^* \in X'$. The next lemma ties this weak measurability to the usual version of measurability in which a function is measurable when inverse images of open sets are measurable.

**Lemma 19.1.9** Let $X$ be a Banach space and let $x : (\Omega, \mathcal{F}) \to K \subseteq X$ where $K$ is weakly compact and $X'$ is separable. Then $x$ is weakly measurable if and only if $x^{-1}(U) \in \mathcal{F}$ whenever $U$ is a weakly open set.

**Proof:** By Corollary 15.5.9 on Page 449, there exists a metric $d$, such that the metric space topology with respect to $d$ coincides with the weak topology. Since $K$ is compact, it follows that $K$ is also separable. Hence it is completely separable. Therefore any weakly open set is measurable in the sense of the weak topology. Since $K$ is separable, it follows that any Borel set is measurable. Hence, any measurable Borel set is weakly measurable. Let $A$ be a countable basis of open sets $\mathcal{B}$ for the weak topology on $K$. Suppose now that $x$ is weakly measurable. To show $x^{-1}(U) \in \mathcal{F}$ whenever $U$ is weakly open, it suffices to verify $x^{-1}(B_A(z,r)) \in \mathcal{F}$ for any set, $B_A(z,r)$. Let
19.1. STRONG AND WEAK MEASURABILITY

\[ A = \{ x_1^*, \ldots, x_m^* \} \]. Then

\[ x^{-1}(B_A(z,r)) = \{ s \in \Omega : \rho_A(x(s) - z) < r \} \]

\[ \equiv \left\{ s \in \Omega : \max_{x^* \in A} |x^*(x(s) - z)| < r \right\} \]

\[ = \bigcup_{i=1}^m \{ s \in \Omega : |x_i^*(x(s)) - z| < r \} \]

\[ = \bigcup_{i=1}^m \{ s \in \Omega : |x_i^*(x(s)) - x_i^*(z)| < r \} \]

which is measurable because each \( x_i^* \circ x \) is given to be measurable.

Next suppose \( x^{-1}(U) \in \mathcal{F} \) whenever \( U \) is weakly open. Then in particular this holds when \( U = B_{x^*}(z, r) \) for arbitrary \( x^* \). Hence

\[ \{ s \in \Omega : x(s) \in B_{x^*}(z, r) \} \in \mathcal{F} \]

But this says the same as

\[ \{ s \in \Omega : |x^*(x(s)) - x^*(z)| < r \} \in \mathcal{F} \]

Since \( x^*(z) \) can be a completely arbitrary element of \( \mathcal{F} \), it follows \( x^* \circ x \) is an \( \mathcal{F} \) valued measurable function. In other words, \( x \) is weakly measurable according to the former definition. \( \blacksquare \)

One can also define weak * measurability and prove a theorem just like the Pettis theorem above. The next lemma is the analogue of Lemma 19.1.4.

Lemma 19.1.10 Let \( B \) be the closed unit ball in \( X \). If \( X' \) is separable, there exists a sequence \( \{ x_m \}_{m=1}^\infty \equiv D \subseteq B \) with the property that for all \( y^* \in X' \),

\[ ||y^*|| = \sup_{x \in D} |y^*(x)|. \]

**Proof:** Let

\[ \{ x_k^* \}_{k=1}^\infty \]

be the dense subspace of \( X' \). Define \( \phi_n : B \to \mathbb{F}^n \) by

\[ \phi_n(x) \equiv (x_1^*(x), \ldots, x_n^*(x)). \]

Then \( |x_k^*(x)| \leq ||x_k^*|| \) and so \( \phi_n(B) \) is contained in a compact subset of \( \mathbb{F}^n \). Therefore, there exists a countable set, \( D_n \subseteq B \) such that \( \phi_n(D_n) \) is dense in \( \phi_n(B) \). Let

\[ D \equiv \bigcup_{n=1}^\infty D_n. \]

It remains to verify this works. Let \( y^* \in X' \). Then there exists \( y \) such that

\[ |y^*(y)| > ||y^*|| - \varepsilon. \]

By density, there exists one of the \( x_k^* \) from the countable dense subset of \( X' \) such that also

\[ |x_k^*(y)| > ||y^*|| - \varepsilon, ||x_k^* - y^*|| < \varepsilon. \]
Now $x_k^*(y) \in \phi_k(B)$ and so there exists $x \in D_k \subseteq D$ such that
\[ |x_k^*(x)| > ||y^*|| - \varepsilon. \]
Then since $||x_k^* - y^*|| < \varepsilon$, this implies
\[ |y^*(x)| \geq ||y^*|| - 2\varepsilon. \]
Since $\varepsilon > 0$ is arbitrary,
\[ ||y^*|| \leq \sup_{x \in D} |y^*(x)| \leq ||y^*||. \]

The next theorem is another version of the Pettis theorem. First here is a definition.

**Definition 19.1.11** A function $y$ having values in $X'$ is weak $\ast$ measurable, when for each $x \in X$, $y(\cdot)(x)$ is a measurable scalar valued function.

**Theorem 19.1.12** If $X'$ is separable and $y : \Omega \rightarrow X'$ is weak $\ast$ measurable, then $y$ is strongly measurable.

**Proof:** It is necessary to show $y^{-1}(B(a^*,r))$ is measurable. This will suffice because the separability of $X'$ implies every open set is the countable union of such balls of the form $B(a^*,r)$. It also suffices to verify inverse images of closed balls are measurable because every open ball is the countable union of closed balls. From Lemma 19.1.10,
\[
y^{-1}\left(B(a^*,r)\right) = \{s : ||y(s) - a^*|| \leq r\} = \left\{s : \sup_{x \in D} |(y(s) - a^*)(x)| \leq r\right\} = \left\{s : \sup_{x \in D} |y(s)(x) - a^*(x)| \leq r\right\} = \cap_{x \in D} y(\cdot)(x)^{-1}B(a^*(x),r)\]
which is a countable intersection of measurable sets by hypothesis. ■

The following are interesting consequences of the theory developed so far and are of interest independent of the theory of integration of vector valued functions.

**Theorem 19.1.13** If $X'$ is separable, then so is $X$.

**Proof:** Let $D = \{x_m\} \subseteq B$, the unit ball of $X$, be the sequence promised by Lemma 19.1.10. Let $V$ be all finite linear combinations of elements of $\{x_m\}$ with rational scalars. Thus $V$ is a separable subspace of $X$. The claim is that $\overline{V} = X$. If not, there exists $x_0 \in X \setminus \overline{V}$. 
But by the Hahn Banach theorem there exists $x'_0 \in X'$ satisfying $x'_0 (x_0) \neq 0$, but $x'_0 (v) = 0$ for every $v \in V$. Hence

$$||x'_0|| = \sup_{x \in D} |x'_0 (x)| = 0,$$

a contradiction. ■

**Corollary 19.1.14** If $X$ is reflexive, then $X$ is separable if and only if $X'$ is separable.

**Proof:** From the above theorem, if $X'$ is separable, then so is $X$. Now suppose $X$ is separable with a dense subset equal to $D$. Then since $X$ is reflexive, $J(D)$ is dense in $X''$ where $J$ is the James map satisfying $Jx (x^*) \equiv x^* (x)$. Then since $X''$ is separable, it follows from the above theorem that $X'$ is also separable. ■

### 19.2 The Bochner Integral

#### 19.2.1 Definition And Basic Properties

**Definition 19.2.1** Let $a_k \in X$, a Banach space and let

$$x (s) = \sum_{k=1}^{n} a_k \chi_{E_k} (s) \quad (19.2.3)$$

where for each $k$, $E_k$ is measurable and $\mu (E_k) < \infty$. Then define

$$\int_{\Omega} x (s) d\mu \equiv \sum_{k=1}^{n} a_k \mu (E_k).$$

**Proposition 19.2.2** Definition (19.2.3) is well defined.

**Proof:** It suffices to verify that if

$$\sum_{k=1}^{n} a_k \chi_{E_k} (s) = 0,$$

then

$$\sum_{k=1}^{n} a_k \mu (E_k) = 0.$$

Let $f \in X'$. Then

$$f \left( \sum_{k=1}^{n} a_k \chi_{E_k} (s) \right) = \sum_{k=1}^{n} f (a_k \chi_{E_k} (s) = 0.$$
and, therefore,

\[ 0 = \int_{\Omega} \left( \sum_{k=1}^{n} f(a_k) \mathcal{X}_{E_k}(s) \right) d\mu = \sum_{k=1}^{n} f(a_k) \mu(E_k) = f \left( \sum_{k=1}^{n} a_k \mu(E_k) \right). \]

Since \( f \in X' \) is arbitrary, and \( X' \) separates the points of \( X \), it follows that

\[ \sum_{k=1}^{n} a_k \mu(E_k) = 0 \]

as claimed. \( \blacksquare \)

It follows easily from this proposition that \( \int_{\Omega} d\mu \) is well defined and linear on simple functions.

**Definition 19.2.3** A strongly measurable function \( x \) is Bochner integrable if there exists a sequence of simple functions \( x_n \) converging to \( x \) pointwise and satisfying

\[ \int_{\Omega} ||x_n(s) - x_m(s)|| d\mu \to 0 \quad \text{as} \quad m, n \to \infty. \]  

(19.2.4)

If \( x \) is Bochner integrable, define

\[ \int_{\Omega} x(s) d\mu \equiv \lim_{n \to \infty} \int_{\Omega} x_n(s) d\mu. \]  

(19.2.5)

**Theorem 19.2.4** The Bochner integral is well defined and if \( x \) is Bochner integrable and \( f \in X' \),

\[ f \left( \int_{\Omega} x(s) d\mu \right) = \int_{\Omega} f(x(s)) d\mu \]  

(19.2.6)

and

\[ \left\| \int_{\Omega} x(s) d\mu \right\| \leq \int_{\Omega} ||x(s)|| d\mu. \]  

(19.2.7)

Also, the Bochner integral is linear. That is, if \( a, b \) are scalars and \( x, y \) are two Bochner integrable functions, then

\[ \int_{\Omega} (ax(s) + by(s)) d\mu = a \int_{\Omega} x(s) d\mu + b \int_{\Omega} y(s) d\mu \]  

(19.2.8)

**Proof:** First it is shown that the triangle inequality holds on simple functions and that the limit in (19.2.4) exists. Thus, if \( x \) is given by (19.2.3) with the \( E_k \) disjoint,

\[ \left\| \int_{\Omega} x(s) d\mu \right\| \]

\[ = \left\| \int_{\Omega} \sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(s) d\mu \right\| = \left\| \sum_{k=1}^{n} a_k \mu(E_k) \right\| \]

\[ \leq \sum_{k=1}^{n} ||a_k|| \mu(E_k) = \sum_{k=1}^{n} ||a_k|| \int_{\Omega} \mathcal{X}_{E_k}(s) d\mu = \int_{\Omega} ||x(s)|| d\mu \]
19.2. THE BOCHNER INTEGRAL

which shows the triangle inequality holds on simple functions. This implies

\[
\left| \int_{\Omega} x_n(s) \, d\mu - \int_{\Omega} x_m(s) \, d\mu \right| = \left| \int_{\Omega} (x_n(s) - x_m(s)) \, d\mu \right| 
\leq \int_{\Omega} \|x_n(s) - x_m(s)\| \, d\mu
\]

which verifies the existence of the limit in 19.2.4. This completes the first part of the argument.

Next it is shown the integral does not depend on the choice of the sequence satisfying 19.2.4 so that the integral is well defined. Suppose \(y_n, x_n\) both satisfy 19.2.4 and converge to \(x\) pointwise. By Fatou’s lemma,

\[
\left| \int_{\Omega} y_n \, d\mu - \int_{\Omega} x_m \, d\mu \right| \leq \int_{\Omega} \|y_n - x\| \, d\mu + \int_{\Omega} \|x - x_m\| \, d\mu 
\leq \lim \inf_{k \to \infty} \int_{\Omega} \|y_n - y_k\| \, d\mu + \lim \inf_{k \to \infty} \int_{\Omega} \|x_k - x_m\| 
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}
\]

if \(m\) and \(n\) are chosen large enough. Since \(\varepsilon\) is arbitrary, this shows the limit is the same for both sequences and demonstrates the Bochner integral is well defined.

It remains to verify the triangle inequality on Bochner integral functions and the claim about passing a continuous linear functional inside the integral. Let \(x\) be Bochner integrable and let \(x_n\) be a sequence which satisfies the conditions of the definition. Define

\[
y_n(s) = \begin{cases} 
  x_n(s) & \text{if } \|x_n(s)\| \leq 2\|x(s)\|, \\
  0 & \text{if } \|x_n(s)\| > 2\|x(s)\|.
\end{cases}
\]

(19.2.9)

Thus

\[
y_n(s) = x_n(s) \chi_{\|x_n\| \leq 2\|x\|}(s).
\]

If \(x(s) = 0\) then \(y_n(s) = 0\) for all \(n\). If \(\|x(s)\| > 0\) then for all \(n\) large enough,

\[
y_n(s) = x_n(s).
\]

Thus, \(y_n(s) \to x(s)\) and

\[
\|y_n(s)\| \leq 2\|x(s)\|. \quad (19.2.10)
\]

By Fatou’s lemma,

\[
\int_{\Omega} \|x\| \, d\mu \leq \lim \inf_{n \to \infty} \int_{\Omega} \|x_n\| \, d\mu. \quad (19.2.11)
\]

Also from 19.2.4 and the triangle inequality on simple functions, \(\{\int_{\Omega} \|x_n\| \, d\mu\}_{n=1}^{\infty}\) is a Cauchy sequence and so it must be bounded. Therefore, by 19.2.10, 19.2.11, and the dominated convergence theorem,

\[
0 = \lim_{n,m \to \infty} \int_{\Omega} \|y_n - y_m\| \, d\mu \quad (19.2.12)
\]
and it follows $x_n$ can be replaced with $y_n$ in Definition 19.2.3.

From Definition 19.2.1,

$$f \left( \int \Omega y_n \, d\mu \right) = \int \Omega f (y_n) \, d\mu.$$ 

Thus,

$$f \left( \int \Omega x \, d\mu \right) = \lim_{n \to \infty} f \left( \int \Omega y_n \, d\mu \right) = \lim_{n \to \infty} \int \Omega f (y_n) \, d\mu = \int \Omega f (x) \, d\mu,$$

the last equation holding from the dominated convergence theorem and 19.2.10 and 19.2.11. This shows 19.2.6. To verify 19.2.7,

$$\int \Omega x (s) \, d\mu = \lim_{n \to \infty} \int \Omega y_n (s) \, d\mu \leq \lim_{n \to \infty} \int \Omega ||y_n (s)|| \, d\mu = \int \Omega ||x (s)|| \, d\mu$$

where the last equation follows from the dominated convergence theorem and 19.2.10, 19.2.11.

It remains to verify 19.2.8. Let $f \in X'$. Then from 19.2.6

$$f \left( \int \Omega (ax (s) + by (s)) \, d\mu \right) = \int \Omega \left( af (x (s)) + bf (y (s)) \right) \, d\mu \quad = \quad a \int \Omega f (x (s)) \, d\mu + b \int \Omega f (y (s)) \, d\mu \quad = \quad f \left( a \int \Omega x (s) \, d\mu + b \int \Omega y (s) \, d\mu \right).$$

Since $X'$ separates the points of $X$, it follows

$$\int \Omega (ax (s) + by (s)) \, d\mu = a \int \Omega x (s) \, d\mu + b \int \Omega y (s) \, d\mu$$

and this proves 19.2.8. \hfill \blacksquare

**Theorem 19.2.5** An $X$ valued function $x$, is Bochner integrable if and only if $x$ is strongly measurable and

$$\int \Omega ||x (s)|| \, d\mu < \infty. \quad (19.2.13)$$

In this case there exists a sequence of simple functions $\{y_n\}$ satisfying 19.2.4, $y_n (s)$ converging pointwise to $x (s)$,

$$||y_n (s)|| \leq 2 ||x (s)|| \quad (19.2.14)$$

and

$$\lim_{n \to \infty} \int \Omega ||x (s) - y_n (s)|| \, d\mu = 0. \quad (19.2.15)$$
**Proof:** Suppose $x$ is strongly measurable and condition 19.2.13 holds. Since $x$ is strongly measurable, there exists a sequence of simple functions, $\{x_n\}$ converging pointwise to $x$. As before, let

$$y_n(s) = \begin{cases} 
x_n(s) & \text{if } ||x_n(s)|| \leq 2||x(s)||, \\
0 & \text{if } ||x_n(s)|| > 2||x(s)||.
\end{cases}$$  \hfill (19.2.16)

Then 19.2.14 holds for $y_n$ and $y_n(s) \to x(s)$. Then by the dominated convergence theorem,

$$\lim_{n \to \infty} \int_{\Omega} ||x(s) - y_n(s)|| \, d\mu = 0$$

and so

$$0 = \lim_{m,n \to \infty} \int_{\Omega} ||y_n(s) - y_m(s)|| \, d\mu.$$

Now suppose $x$ is Bochner integrable. Then it is strongly measurable and there exists a sequence of simple functions $\{x_n\}$ such that $x_n(s)$ converges pointwise to $x$ and

$$\lim_{m,n \to \infty} \int_{\Omega} ||x_n(s) - x_m(s)|| \, d\mu = 0.$$  

Therefore, as before, since $\left\{\int_{\Omega} x_n d\mu \right\}_{n=1}^{\infty}$ is a Cauchy sequence, it follows

$$\left\{ \int_{\Omega} ||x_n|| \, d\mu \right\}_{n=1}^{\infty}$$

is also a Cauchy sequence because

$$\left| \int_{\Omega} ||x_n|| \, d\mu - \int_{\Omega} ||x_m|| \, d\mu \right| \leq \int_{\Omega} \left( ||x_n|| - ||x_m|| \right) \, d\mu \leq \int_{\Omega} ||x_n - x_m|| \, d\mu.$$  

Thus

$$\int_{\Omega} ||x|| \, d\mu \leq \lim_{n \to \infty} \int_{\Omega} ||x_n|| \, d\mu < \infty.$$  

Using 19.2.15 it follows $y_n$ satisfies 19.2.14, converges pointwise to $x$ and then from the dominated convergence theorem 19.2.15 holds. \hfill $\blacksquare$

Here is a simple corollary.

**Corollary 19.2.6** Let an $X$ valued function $x$ be Bochner integrable and let $L \in \mathcal{L}(X,Y)$ where $Y$ is another Banach space. Then $Lx$ is a $Y$ valued Bochner integrable function and

$$L \left( \int_{\Omega} x(s) \, d\mu \right) = \int_{\Omega} Lx(s) \, d\mu.$$
Proof: From Theorem 19.2.5 there is a sequence of simple functions \( \{y_n\} \) having the properties listed in that theorem. Then consider \( \{Ly_n\} \) which converges pointwise to \( Lx \). Since \( L \) is continuous and linear,
\[
\int_{\Omega} ||Ly_n - Lx||_Y d\mu \leq ||L|| \int_{\Omega} ||y_n - x||_X d\mu
\]
which converges to 0. This implies
\[
\lim_{m,n \to \infty} \int_{\Omega} ||Ly_n - Ly_m|| d\mu = 0
\]
and so by definition \( Lx \) is Bochner integrable. Also
\[
\int_{\Omega} x(s) d\mu = \lim_{n \to \infty} \int_{\Omega} y_n(s) d\mu
\]
\[
\int_{\Omega} Lx(s) d\mu = \lim_{n \to \infty} \int_{\Omega} Ly_n(s) d\mu = \lim_{n \to \infty} L \int_{\Omega} y_n(s) d\mu
\]
\[
||L \left( \int_{\Omega} x(s) d\mu \right) - \int_{\Omega} Lx(s) d\mu ||_Y 
\]
\[
\leq ||L \left( \int_{\Omega} x(s) d\mu \right) - L \int_{\Omega} y_n(s) d\mu ||_Y 
\]
\[
+ \int_{\Omega} \left| \int_{\Omega} Ly_n(s) d\mu - \int_{\Omega} Lx(s) d\mu \right|_Y < \varepsilon/2 + \varepsilon/2 = \varepsilon
\]
whenever \( n \) large enough. 

19.2.2 Taking A Closed Operator Out Of The Integral

Now let \( X \) and \( Y \) be separable Banach spaces and suppose \( A : D(A) \subseteq X \to Y \) be a closed operator. Recall this means that the graph of \( A \),
\[
G(A) \equiv \{(x, Ax) : x \in D(A)\}
\]
is a closed subset of \( X \times Y \) with respect to the product topology obtained from the norm
\[
||(x, y)|| = \max (||x||, ||y||).
\]
Thus also \( G(A) \) is a separable Banach space with the above norm. You can also consider \( D(A) \) as a separable Banach space having the graph norm
\[
||x||_{D(A)} \equiv \max (||x||, ||Ax||) \quad (19.2.17)
\]
which is isometric to \( G(A) \) with the mapping, \( \theta x \equiv (x, Ax) \).
Lemma 19.2.7 A closed subspace of a reflexive Banach space is reflexive.

Proof: Consider the following diagram in which $Y$ is a closed subspace of the reflexive space, $X$.

\[
\begin{array}{ccc}
Y'' & \overset{i^{**} \ 1-1}{\rightarrow} & X'' \\
Y' & \overset{i^{*} \text{onto}}{\rightarrow} & X' \\
Y & \overset{i}{\rightarrow} & X
\end{array}
\]

This diagram follows from theorems on adjoints presented earlier.

Now let $y^{**} \in Y''$. Is $y^{**} = J(y)$ for some $y \in Y$? From the above, there exists $y \in X$ such that for all $x^*$, $i^{**}y^{**}(x^*) = x^*(y)$. Is $y \in Y$? If it is not in $Y$ then there exists $x^* \in X'$ such that $x^*(y) \neq 0$ but $x^*(Y) = 0$. Then $i^*x^* = 0$. Hence

\[0 = y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = x^*(y) \neq 0,
\]
a contradiction. Hence $y \in Y$. Thus

\[y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = x^*(y) = x^*(iy) = i^*x^*(y), \quad y \in Y
\]

Now $i^*$ is onto, and so this says $y^{**}(y^*) = y^*(y)$ for all $y^* \in Y'$. In other words, $y^{**} = J(y)$.

Corollary 19.2.8 Suppose $Y$ is a reflexive Banach space and $X$ is a Banach space such that there exists a continuous one to one mapping, $g : X \rightarrow Y$ such that $g(X)$ is a closed subset of $Y$. Then $X$ is reflexive.

Proof: By the open mapping theorem, $g(X)$ and $X$ are homeomorphic since $g^{-1}$ must also be continuous. Therefore, since $g(X)$ is reflexive, it follows $X$ is also.

Lemma 19.2.9 Suppose $V$ and $W$ are reflexive Banach spaces and that $V$ is a dense subset of $W$ in the topology of $W$. Then $i^*W'$ is a dense subset of $V'$ where here $i$ is the inclusion map of $V$ into $W$.

Proof: First note that $i^*$ is one to one. If $i^*w^* = 0$ for $w^* \in W'$, then this means that for all $v \in V$,

\[i^*w(v) = w^*(v) = 0
\]

and since $V$ is dense in $W$, this shows $w^* = 0$.

Consider the following diagram

\[
\begin{array}{ccc}
V'' & \overset{i^{**}}{\rightarrow} & W'' \\
V' & \overset{i^{*}}{\rightarrow} & W' \\
V & \overset{i}{\rightarrow} & W
\end{array}
\]

in which $i$ is the inclusion map. Next suppose $i^*W'$ is not dense in $V'$. Then there exists $v^{**} \in V''$ such that $v^{**} \neq 0$ but $v^{**}(i^*W') = 0$. It follows from $V$ being
reflexive, that \( v^{**} = Jv_0 \) where \( J \) is the James map from \( V \) to \( V'' \) for some \( v_0 \in V \).

Thus for every \( w^* \in W' \),

\[
0 = v^{**} (i^* w^*) = i^{**} v^{**} (w^*) \\
= i^{**} Jv_0 (w^*) = Jv_0 (i^* w^*) \\
= i^* w^* (v_0) = w^* (v_0)
\]

and since \( W' \) separates the points of \( W \), it follows \( v_0 = 0 \) which contradicts \( v^{**} \neq 0 \).

Note that in the proof, only \( V \text{ reflexive} \) was used.

This lemma implies an easy corollary.

**Corollary 19.2.10**

Let \( E \) and \( F \) be reflexive Banach spaces and let \( A : D(A) \subseteq E \to F \). Suppose also that \( D(A) \) is dense in \( E \). Then making \( D(A) \) into a Banach space by using the graph norm given in \( 19.2.17 \), it follows that \( D(A) \) is a Banach space and \( i^* E' \) is a dense subspace of \( D(A)' \).

**Proof:**

First note that \( E \times F \) is a reflexive Banach space and \( G(A) \) is a closed subspace of \( E \times F \) so it is also a reflexive Banach space. Now \( D(A) \) is isometric to \( G(A) \) and so it follows \( D(A) \) is a dense subspace of \( E \) which is reflexive. Therefore, from Lemma \( 19.2.9 \) the conclusion follows.

With this preparation, here is another interesting theorem. This one is about taking outside the integral a closed linear operator as opposed to a continuous linear operator.

**Theorem 19.2.11**

Let \( X,Y \) be separable Banach spaces and let \( A : D(A) \subseteq X \to Y \) be a closed operator where \( D(A) \) is a dense separable subset of \( X \) with respect to the graph norm on \( D(A) \) described above. Suppose also that \( i^* X' \) is a dense subspace of \( D(A)' \) where \( D(A) \) is a Banach space having the graph norm described in \( 19.2.17 \). Suppose that \( (\Omega, \mathcal{F}, \mu) \) is a \( \sigma \) finite measure space and \( x : \Omega \to X \) is strongly measurable and it happens that \( x(s) \in D(A) \) for all \( s \in \Omega \). Then \( x \) is strongly measurable as a mapping into \( D(A) \). Also \( Ax \) is strongly measurable as a map into \( Y \) and if

\[
\int_{\Omega} ||x(s)|| \, d\mu, \int_{\Omega} ||Ax(s)|| \, d\mu < \infty, \quad (19.2.18)
\]

then

\[
\int_{\Omega} x(s) \, d\mu \in D(A) \quad (19.2.19)
\]

and

\[
A \int_{\Omega} x(s) \, d\mu = \int_{\Omega} Ax(s) \, d\mu. \quad (19.2.20)
\]

\(^1\text{Note that this follows from the assumed separability of } X,Y \text{ because the graph is a subset of the separable space } X \times Y.\)
Proof: First of all, consider the assertion that \( x \) is strongly measurable into \( D(A) \). Letting \( f \in D(A)' \) be given, there exists a sequence, \( \{g_n\} \subseteq i^*X' \) such that \( g_n \to f \) in \( D(A)' \). Therefore,

\[
s \to g_n(x(s))
\]

is measurable by assumption and

\[
g_n(x(s)) \to f(x(s))
\]

which shows that \( s \to f(x(s)) \) is measurable. By the Pettis theorem, it follows

\[
s \to x(s)
\]

is strongly measurable as a map into \( D(A) \).

It follows from Theorem 19.2.5 there exists a sequence of simple functions, \( \{x_n\} \) of the form

\[
x_n(s) = \sum_{k=1}^{m_n} a_k^n \mathcal{X}_{E_k^n}(s), x_n(s) \in D(A),
\]

which converges strongly and pointwise to \( x(s) \) in \( D(A) \). Thus

\[
x_n(s) \to x(s), Ax_n(s) \to Ax(s),
\]

which shows \( s \to Ax(s) \) is strongly measurable in \( Y \) as claimed.

It remains to verify the assertions about the integral. \[\text{(UP/A)}\] implies \( x \) is Bochner integrable as a function having values in \( D(A) \) with the norm on \( D(A) \) described above. Therefore, by Theorem 19.2.5 there exists a sequence of simple functions \( \{y_n\} \) having values in \( D(A) \),

\[
\lim_{m,n \to \infty} \int_{\Omega} ||y_n - y_m||_{D(A)} \, d\mu = 0,
\]

\( y_n(s) \) converging pointwise to \( x(s) \),

\[
||y_n(s)||_{D(A)} \leq 2 ||x(s)||_{D(A)}
\]

and

\[
\lim_{n \to \infty} \int_{\Omega} ||x(s) - y_n(s)||_{D(A)} \, ds = 0.
\]

Therefore,

\[
\int_{\Omega} y_n(s) \, d\mu \in D(A), \quad \int_{\Omega} y_n(s) \, d\mu \to \int_{\Omega} x(s) \, d\mu \text{ in } X,
\]

and since \( y_n \) is a simple function and \( A \) is linear,

\[
A \int_{\Omega} y_n(s) \, d\mu = \int_{\Omega} Ay_n(s) \, d\mu \to \int_{\Omega} Ax(s) \, d\mu \text{ in } Y.
\]
It follows, since \( A \) is a closed operator, that
\[
\int_{\Omega} x(s) \, d\mu \in D(A)
\]
and
\[
A \int_{\Omega} x(s) \, d\mu = \int_{\Omega} Ax(s) \, d\mu.
\]

Here is another version of this theorem which has different hypotheses.

**Theorem 19.2.12** Let \( X \) and \( Y \) be separable Banach spaces and let \( A : D(A) \subseteq X \rightarrow Y \) be a closed operator. Also let \( (\Omega, \mathcal{F}, \mu) \) be a \( \sigma \)-finite measure space and let \( x : \Omega \rightarrow X \) be Bochner integrable such that \( x(s) \in D(A) \) for all \( s \). Also suppose \( Ax \) is Bochner integrable. Then
\[
\int Ax \, d\mu = A \int x \, d\mu
\]
and \( \int x \, d\mu \in D(A) \).

**Proof:** Consider the graph of \( A \),
\[
G(A) \equiv \{(x, Ax) : x \in D(A)\} \subseteq X \times Y.
\]
Then since \( A \) is closed, \( G(A) \) is a closed separable Banach space with the norm \( ||(x, y)|| \equiv \max(||x||, ||y||) \). Therefore, for \( g^* \in G(A)' \), one can apply the Hahn-Banach theorem and obtain \((x^*, y^*) \in (X \times Y)'\) such that \( g^*(x, Ax) = (x^*(x), y^*(Ax)) \).

Now it follows from the assumptions that \( s \rightarrow (x^*(x(s)), y^*(Ax(s))) \) is measurable with values in \( G(A) \). It is also separably valued because this is true of \( G(A) \). By the Pettis theorem, \( s \rightarrow (x(s), A(x(s))) \) must be strongly measurable.

Also \( \int ||x(s)|| + ||A(x(s))|| \, d\mu < \infty \) by assumption and so there exists a sequence of simple functions having values in \( G(A) \), \( \{(x_n(s), Ax_n(s))\} \) which converges to \((x(s), A(s))\) pointwise such that \( \int ||(x_n, Ax_n) - (x, Ax)|| \, d\mu \rightarrow 0 \) in \( G(A) \). Now for simple functions it is routine to verify that
\[
\int (x_n, Ax_n) \, d\mu = \left( \int x_n \, d\mu, \int Ax_n \, d\mu \right)
\]
\[
= \left( \int x_n \, d\mu, A \int x_n \, d\mu \right)
\]

Also
\[
\left| \int x_n \, d\mu - \int x \, d\mu \right| \leq \int ||x_n - x|| \, d\mu
\]
\[
\leq \int ||(x_n, Ax_n) - (x, Ax)|| \, d\mu
\]
which converges to 0. Also
\[
\left\| \int Ax_n d\mu - \int A x d\mu \right\| \leq \int \left\| Ax_n - Ax \right\| d\mu \\
\leq \int \left\| (x_n, Ax_n) - (x, Ax) \right\| d\mu
\]
and this converges to 0. Therefore, \( \int x_n d\mu \to \int x d\mu \) and \( A \int x_n d\mu \to A \int x d\mu \).

19.3 Operator Valued Functions

Consider the case where \( A(s) \in \mathcal{L}(X,Y) \) for \( X \) and \( Y \) separable Banach spaces. With the operator norm \( \mathcal{L}(X,Y) \) is a Banach space and so if \( A \) is strongly measurable, the Bochner integral can be defined as before. However, it is also possible to define the Bochner integral of such operator valued functions for more general situations. In this section, \( (\Omega, \mathcal{F}, \mu) \) will be a \( \sigma \)-finite measure space as usual.

**Lemma 19.3.1** Let \( x \in X \) and suppose \( A \) is strongly measurable. Then
\[
s \to A(s) x
\]
is strongly measurable as a map into \( Y \).

**Proof:** Since \( A \) is assumed to be strongly measurable, it is the pointwise limit of simple functions of the form
\[
A_n(s) \equiv \sum_{k=1}^{m_n} A_k x_{E_k}(s)
\]
where \( A_k \in \mathcal{L}(X,Y) \). It follows \( A_n(s)x \to A(s)x \) for each \( s \) and so, since \( s \to A_n(s)x \) is a simple \( Y \)-valued function, \( s \to A(s)x \) must be strongly measurable.

**Definition 19.3.2** Suppose \( A(s) \in \mathcal{L}(X,Y) \) for each \( s \in \Omega \) where \( X,Y \) are separable Banach spaces. Suppose also that for each \( x \in X \),
\[
s \to A(s)x \text{ is strongly measurable } \quad (19.3.21)
\]
and there exists \( C \) such that for each \( x \in X \),
\[
\int_{\Omega} \| A(s)x \| d\mu < C \| x \|
\]
(19.3.22)
Then \( \int_{\Omega} A(s) d\mu \in \mathcal{L}(X,Y) \) is defined by the following formula.
\[
\left( \int_{\Omega} A(s) d\mu \right)(x) \equiv \int_{\Omega} A(s) x d\mu \quad (19.3.23)
\]
Lemma 19.3.3 The above definition is well defined. Furthermore, if holds, then is measurable and if holds, then
\[ \left\| \int_{\Omega} A(s) \, d\mu \right\| \leq \int_{\Omega} \| A(s) \| \, d\mu. \]

Proof: It is clear that in case \( s \to A(s) x \) is measurable for all \( x \in X \) there exists a unique \( \Psi \in L(X,Y) \) such that
\[ \Psi(x) = \int_{\Omega} A(s) \, x \, d\mu. \]
This is because \( x \to \int_{\Omega} A(s) \, x \, d\mu \) is linear and continuous. Thus \( \Psi = \int_{\Omega} A(s) \, d\mu \) and the definition is well defined.

Now consider the assertion about \( s \to \| A(s) \| \). Let \( D' \subseteq B' \) the closed unit ball in \( Y' \) be such that \( D' \) is countable and
\[ \| y \| = \sup_{y^* \in D'} | y^*(y) |. \]
Also let \( D \) be a countable dense subset of \( B \), the unit ball of \( X \). Then
\[ \{ s : \| A(s) \| > \alpha \} = \bigcup_{x \in D} \{ s : \| A(s) x \| > \alpha \} \]
and this is measurable because \( s \to A(s) x \) is strongly, hence weakly measurable.

Now suppose holds. Then for all \( x \),
\[ \int_{\Omega} \| A(s) x \| \, d\mu < C \| x \|. \]
It follows that for \( \| x \| \leq 1 \),
\[ \left\| \left( \int_{\Omega} A(s) \, d\mu \right)(x) \right\| = \left\| \int_{\Omega} A(s) \, x \, d\mu \right\| \leq \int_{\Omega} \| A(s) x \| \, d\mu \leq \int_{\Omega} \| A(s) \| \, d\mu \]
and so
\[ \left\| \int_{\Omega} A(s) \, d\mu \right\| \leq \int_{\Omega} \| A(s) \| \, d\mu. \]
This proves the lemma.

Now it is interesting to consider the case where \( A(s) \in L(H,H) \) where \( s \to A(s) x \) is strongly measurable and \( A(s) \) is compact and self adjoint. Recall the Kuratowski measurable selection theorem, Theorem 9.1.11 on page 212 listed here for convenience.
Theorem 19.3.4 Let $E$ be a compact metric space and let $(\Omega, \mathcal{F})$ be a measure space. Suppose $\psi : E \times \Omega \to \mathbb{R}$ has the property that $x \to \psi(x, \omega)$ is continuous and $\omega \to \psi(x, \omega)$ is measurable. Then there exists a measurable function, $f$ having values in $E$ such that
\[ \psi(f(\omega), \omega) = \sup_{x \in E} \psi(x, \omega). \]
Furthermore, $\omega \to \psi(f(\omega), \omega)$ is measurable.

19.3.1 Review Of Hilbert Schmidt Theorem
Here I will give a proof of the Hilbert Schmidt theorem which will generalize to a result about measurable operators. Recall the following.

Definition 19.3.5 Define $v \otimes u \in \mathcal{L}(H, H)$ by
\[ v \otimes u(x) = (x, u)v. \]

$A \in \mathcal{L}(H, H)$ is a compact operator if whenever $\{x_k\}$ is a bounded sequence, there exists a convergent subsequence of $\{Ax_k\}$. Equivalently, $A$ maps bounded sets to sets whose closures are compact or to use other terminology, $A$ maps bounded sets to sets which are precompact.

Lemma 19.3.6 Let $H$ be a separable Hilbert space and suppose $A \in \mathcal{L}(H, H)$ is a compact operator. Let $B$ denote the closed unit ball in $H$. Then $A$ is continuous as a map from $B$ with the weak topology into $H$ with the strong topology. For $u, v \in H$, $v \otimes u : H \to H$ is a compact operator. If $A$ is self adjoint and compact, the function
\[ x \to (Ax, x) \]
is continuous on $B$ with respect to the weak topology on $B$. The function,
\[ x \to (v \otimes u(x), x) \]
is continuous and the operator $u \otimes u$ is self adjoint.

Proof: Since $H$ is separable, it follows from Corollary on Page 449 that $B$ can be considered as a metric space. Therefore, showing continuity reduces to showing convergent sequences are taken to convergent sequences. Let $x_n \to x$ weakly in $B$. Suppose $Ax_n$ does not converge to $Ax$. Then there exists a subsequence, still denoted by $\{x_n\}$ such that
\[ ||Ax_n - Ax|| \geq \varepsilon > 0 \quad (19.3.24) \]
for all $n$. Then since $A$ maps bounded sets to compact sets, there is a further subsequence, still denoted by $\{x_n\}$ such that $Ax_n$ converges to some $y \in H$. Therefore,
\[ (y, w) = \lim_{n \to \infty} (Ax_n, w) = \lim_{n \to \infty} (x_n, A^*w) \]
\[ = (x, A^*w) = (Ax, w) \]
which shows $Ax = y$ since $w$ is arbitrary. However, this contradicts (19.3.24).

Next consider the claim about $v \otimes u$. Letting $\{x_n\}$ be a bounded sequence,

$$v \otimes u (x_n) = (x_n, u) v.$$

There exists a weakly convergent subsequence of $\{x_n\}$ say $\{x_{n_k}\}$ converging weakly to $x \in H$. Therefore,

$$||(v \otimes u (x_{n_k}) - v \otimes u (x)|| = ||(x_{n_k}, u) - (x, u)|| ||v||$$

which converges to 0. Thus $v \otimes u$ is compact as claimed. It takes bounded sets to precompact sets.

To verify the assertion about $x \rightarrow (Ax, x)$, let $x_n \rightarrow x$ weakly. Then

$$|(Ax_n, x_n) - (Ax, x)|$$

$$\leq ||(Ax_n, x_n) - (Ax, x_n)|| + ||(Ax, x_n) - (Ax, x)||$$

$$\leq ||Ax_n - Ax|| ||x_n|| + ||Ax_n - Ax|| ||x|| \leq 2 ||Ax_n - Ax||$$

which converges to 0.

$$|(v \otimes u (x_n), x_n) - (v \otimes u (x), x)|$$

$$= ||(x_n, u) (v, x_n) - (x, u) (v, x)||$$

and this converges to 0 by weak convergence. It follows from the definition that $u \otimes u$ is self adjoint. This proves the lemma.

**Observation 19.3.7** Note that if $A$ is any self adjoint operator,

$$(Ax, x) = (x, Ax) = (Ax, x).$$

so $(Ax, x)$ is real valued.

**Lemma 19.3.8** Let $A \in \mathcal{L}(H, H)$ and suppose it is self adjoint and compact. Let $B$ denote the closed unit ball in $H$. Let $e \in B$ be such that

$$|(Ae, e)| = \max_{x \in B} |(Ax, x)|.$$ 

Then letting $\lambda = (Ae, e)$, it follows $Ae = \lambda e$. If $\lambda \neq 0$, then $||e|| = 1$ and if $\lambda = 0$, it can be assumed $e = 0$ so it is still the case $Ae = \lambda e$.

**Proof:** From the above observation, $(Ax, x)$ is always real and since $A$ is compact, $|(Ax, x)|$ achieves a maximum at $e$. It remains to verify $e$ is an eigenvector. Note that $||e|| = 1$ whenever $\lambda \neq 0$ since otherwise $|(Ae, e)|$ could be made larger by replacing $e$ with $e/||e||$. 


19.3. OPERATOR VALUED FUNCTIONS

Suppose $\lambda = (Ae,e) > 0$. Then it is easy to verify that $\lambda I - A$ is a nonnegative $((\lambda I - A)x,x) \geq 0$ for all $x$) and self adjoint operator. Therefore, the Cauchy Schwarz inequality can be applied to write

$$(\lambda I - A)e,x) \leq ((\lambda I - A)e,e)^{1/2} ((\lambda I - A)x,x)^{1/2} = 0$$

Since this is true for all $x$ it follows $Ae = \lambda e$.

Next suppose $\lambda = (Ae,e) < 0$. Then $-\lambda = (-Ae,e)$ and the previous result can be applied to $-A$ and $-\lambda$. Thus $-\lambda e = -Ae$ and so $Ae = \lambda e$.

Finally consider the case where $\lambda = 0$. Then $0 = (Ae,e)$ and so it suffices to take $e = 0$ as claimed. This proves the lemma.

With these lemmas here is a major theorem, the Hilbert Schmidt theorem.

**Theorem 19.3.9** Let $A \in L(H,H)$ be a compact self adjoint operator on a Hilbert space. Then there exist real numbers $\{\lambda_k\}_{k=1}^\infty$ and vectors $\{e_k\}_{k=1}^\infty$ such that

$$||e_k|| = 1 \text{ if } \lambda_k \neq 0,$$
$$||e_k|| = 0 \text{ if } \lambda_k = 0,$$
$$(e_k,e_j)_H = 0 \text{ if } k \neq j,$$
$$A e_k = \lambda_k e_k,$$
$$|\lambda_n| \geq |\lambda_{n+1}| \text{ for all } n,$$
$$\lim_{n \to \infty} \lambda_n = 0,$$

$$\lim_{n \to \infty} \left\| A - \sum_{k=1}^{n} \lambda_k (e_k \otimes e_k) \right\| = 0. \quad (19.3.25)$$

**Proof:** This is done by considering a sequence of compact self adjoint operators, $A, A_1, A_2, \cdots$. Here is how these are defined. Using Lemma 19.3.8 let $e_1, \lambda_1$ be given by that lemma such that

$$||(A e_1, e_1) = \max_{x \in B} |(Ax,x)|, \; \lambda_1 = (A e_1, e_1).$$

Then by that lemma, $A e_1 = \lambda_1 e_1$ and $||e_1|| = 1$ if $\lambda_1 \neq 0$ while $e_1 = 0$ if $\lambda_1 = 0$.

If $A_n$ has been obtained, use Lemma 19.3.8 to obtain $e_{n+1}$ and $\lambda_{n+1}$ such that

$$||(A_n e_{n+1}, e_{n+1})) = \max_{x \in B} |(A_n x,x)|, \; \lambda_{n+1} = (A_n e_{n+1}, e_{n+1}).$$

By that lemma again, $A_n e_{n+1} = \lambda_{n+1} e_{n+1}$ and $||e_{n+1}|| = 1$ if $\lambda_{n+1} \neq 0$ while $e_{n+1} = 0$ if $\lambda_{n+1} = 0$. Then

$$A_{n+1} e_{n+1} = A - \sum_{k=1}^{n} \lambda_k e_k \otimes e_{n+1}$$

Thus

$$A_n = A - \sum_{k=1}^{n} \lambda_k e_k \otimes e_k. \quad (19.3.26)$$
Claim 1: If $k < n + 1$ then $(e_{n+1}, e_k) = 0$. Also $A e_k = \lambda_k e_k$ for all $k$.

Proof of claim: From the above,

$$\lambda_{n+1} e_{n+1} = A_n e_{n+1} = A e_{n+1} - \sum_{k=1}^{n} \lambda_k (e_{n+1}, e_k) e_k.$$  

If $\lambda_{n+1} = 0$, then $(e_{n+1}, e_k) = 0$ because $e_{n+1} = 0$. If $\lambda_{n+1} \neq 0$, then from the above and an induction hypothesis

$$\lambda_{n+1} (e_{n+1}, e_j) = (A e_{n+1}, e_j) - \sum_{k=1}^{n} \lambda_k (e_{n+1}, e_k) (e_k, e_j)$$

$$= (e_{n+1}, A e_j) - \sum_{k=1}^{n} \lambda_k (e_{n+1}, e_k) (e_k, e_j)$$

$$= \lambda_j (e_{n+1}, e_j) - \lambda_j (e_{n+1}, e_j) = 0.$$  

To verify the second part of this claim,

$$\lambda_{n+1} e_{n+1} = A_n e_{n+1} = A e_{n+1} - \sum_{k=1}^{n} \lambda_k e_k (e_{n+1}, e_k) = A e_{n+1}$$

This proves the claim.

Claim 2: $|\lambda_n| \geq |\lambda_{n+1}|$.

Proof of claim: From and the definition of $A_n$ and $e_k \otimes e_k$,

$$\lambda_{n+1} = (A_n e_{n+1}, e_{n+1})$$

$$= (A_{n-1} e_{n+1}, e_{n+1}) - \lambda_n (e_n, e_{n+1})^2$$

By the previous claim. Therefore,

$$|\lambda_{n+1}| = |(A_{n-1} e_{n+1}, e_{n+1})| \leq |(A_{n-1} e_n, e_n)| = |\lambda_n|$$

by the definition of $|\lambda_n|$. ($e_n$ makes $|(A_{n-1} x, x)|$ as large as possible, not necessarily $e_{n+1}$.)

Claim 3: $\lim_{n \to \infty} \lambda_n = 0$.

Proof of claim: If for some $n$, $\lambda_n = 0$, then $\lambda_k = 0$ for all $k > n$ by claim 2. Assume then that $\lambda_k \neq 0$ for any $k$. Then if $\lim_{k \to \infty} |\lambda_k| = \varepsilon > 0$, contrary to the claim, $||e_k|| = 1$ for all $k$ and

$$||A e_n - A e_m||^2 = ||\lambda_n e_n - \lambda_m e_m||^2$$

$$= \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon^2$$

which shows there is no Cauchy subsequence of $\{A e_n\}_{n=1}^\infty$, which contradicts the compactness of $A$. This proves the claim.

Claim 4: $||A_n|| \to 0$
Proof of claim: Let \( x, y \in B \)

\[
|\lambda_{n+1}| \geq \left| \left( A_n \frac{x + y}{2}, \frac{x + y}{2} \right) \right|
\]

\[
= \left| \frac{1}{4} (A_n x, x) + \frac{1}{4} (A_n y, y) + \frac{1}{2} (A_n x, y) \right|
\]

\[
\geq \frac{1}{2} |(A_n x, y)| - \frac{1}{4} |(A_n x, x) + (A_n y, y)|
\]

\[
\geq \frac{1}{2} |(A_n x, y)| - \frac{1}{4} (|(A_n x, x)| + |(A_n y, y)|)
\]

\[
\geq \frac{1}{2} |(A_n x, y)| - \frac{1}{2} |\lambda_{n+1}|
\]

and so

\[
3 |\lambda_{n+1}| \geq |(A_n x, y)|.
\]

It follows \( ||A_n|| \leq 3 |\lambda_{n+1}| \). This proves the claim.

By \[19.3.26\] this proves \[19.3.25\] and completes the proof.

19.3.2 Measurable Compact Operators

Here the operators will be of the form \( A(s) \) where \( s \in \Omega \) and \( s \to A(s) x \) is strongly measurable and \( A(s) \) is a compact operator in \( \mathcal{L}(H, H) \).

**Theorem 19.3.10** Let \( A(s) \in \mathcal{L}(H, H) \) be a compact self-adjoint operator and \( H \) is a separable Hilbert space such that \( s \to A(s) x \) is strongly measurable. Then there exist real numbers \( \{\lambda_k(s)\}_{k=1}^\infty \) and vectors \( \{e_k(s)\}_{k=1}^\infty \) such that

- \( ||e_k(s)|| = 1 \) if \( \lambda_k \neq 0 \),
- \( ||e_k(s)|| = 0 \) if \( \lambda_k = 0 \),
- \( (e_k(s), e_j(s))_H = 0 \) if \( k \neq j \),
- \( A(s) e_k(s) = \lambda_k(s) e_k(s) \),
- \( |\lambda_n(s)| \geq |\lambda_{n+1}(s)| \) for all \( n \),
- \( \lim_{n \to \infty} \lambda_n(s) = 0 \),
- \( \lim_{n \to \infty} \left| A(s) - \sum_{k=1}^n \lambda_k(s) (e_k(s) \otimes e_k(s)) \right|_{\mathcal{L}(H, H)} = 0 \).

The function \( s \to \lambda_j(s) \) is measurable and \( s \to e_j(s) \) is strongly measurable.

**Proof:** It is simply a repeat of the above proof of the Hilbert Schmidt theorem except at every step when the \( e_k \) and \( \lambda_k \) are defined, you use the Kuratowski measurable selection theorem, Theorem \[19.3.4\] on Page 647 to obtain \( \lambda_k(s) \) is measurable and that \( s \to e_k(s) \) is also measurable.
When you consider \( \max_{x \in B} |(A_n(s)x,x)| \), let \( \psi(x,s) = |(A_n(s)x,x)| \). Then \( \psi \) is continuous in \( x \) by Lemma 19.3.6 on Page 647 and it is measurable in \( s \) by assumption. Therefore, by the Kuratowski theorem, \( e_k(s) \) is measurable in the sense that inverse images of weakly open sets in \( B \) are measurable. However, by Lemma 19.1.9 on Page 632 this is the same as weakly measurable. Since \( H \) is separable, this implies \( s \to e_k(s) \) is also strongly measurable. The measurability of \( \lambda_k \) and \( e_k \) is the only new thing here and so this completes the proof.

### 19.4 Fubini’s Theorem For Bochner Integrals

Now suppose \((\Omega_1,F,\mu)\) and \((\Omega_2,S,\lambda)\) are two \( \sigma \)-finite measure spaces. Recall the notion of product measure. There was a \( \sigma \)-algebra, denoted by \( F \times S \) which is the smallest \( \sigma \)-algebra containing the elementary sets, (finite disjoint unions of measurable rectangles) and a measure, denoted by \( \mu \times \lambda \) defined on this \( \sigma \)-algebra such that for \( E \in F \times S \),

\[
s_1 \to \lambda(E_{s_1}), \quad (E_{s_1} \equiv \{s_2 : (s_1, s_2) \in E\})
\]

is \( \mu \) measurable and

\[
s_2 \to \mu(E_{s_2}), \quad (E_{s_2} \equiv \{s_1 : (s_1, s_2) \in E\})
\]

is \( \lambda \) measurable. In terms of nonnegative functions which are \( F \times S \) measurable,

\[
s_1 \to f(s_1, s_2) \text{ is } \mu \text{ measurable},
\]

\[
s_2 \to f(s_1, s_2) \text{ is } \lambda \text{ measurable},
\]

\[
s_1 \to \int_{\Omega_2} f(s_1, s_2) \, d\lambda \text{ is } \mu \text{ measurable},
\]

\[
s_2 \to \int_{\Omega_1} f(s_1, s_2) \, d\mu \text{ is } \lambda \text{ measurable},
\]

and the conclusion of Fubini’s theorem holds.

\[
\int_{\Omega_1 \times \Omega_2} f \, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \, d\lambda \, d\mu = \int_{\Omega_2} \int_{\Omega_1} f(s_1, s_2) \, d\mu \, d\lambda.
\]

The following theorem is the version of Fubini’s theorem valid for Bochner integrable functions.

**Theorem 19.4.1** Let \( f : \Omega_1 \times \Omega_2 \to X \) be strongly measurable with respect to \( \mu \times \lambda \) and suppose

\[
\int_{\Omega_1 \times \Omega_2} \|f(s_1, s_2)\| \, d(\mu \times \lambda) < \infty. \tag{19.4.27}
\]
Then there exist a set of \( \mu \) measure zero, \( N \) and a set of \( \lambda \) measure zero, \( M \) such that the following formula holds with all integrals making sense. 

\[
\int_{\Omega_1 \times \Omega_2} f(s_1, s_2) \, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \, d\lambda \, d\mu \\
= \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \, d\mu \, d\lambda.
\]

**Proof:** First note that from 19.4.27 and the usual Fubini theorem for nonnegative valued functions,

\[
\int_{\Omega_1 \times \Omega_2} ||f(s_1, s_2)|| \, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} ||f(s_1, s_2)|| \, d\lambda \, d\mu
\]

and so

\[
\int_{\Omega_2} ||f(s_1, s_2)|| \, d\lambda < \infty \tag{19.4.28}
\]

for \( \mu \) a.e. \( s_1 \). Say for all \( s_1 \not\in N \) where \( \mu(N) = 0 \).

Let \( \phi \in X' \). Then \( \phi \circ f \) is \( \mathcal{F} \times \mathcal{S} \) measurable and

\[
\int_{\Omega_1 \times \Omega_2} |\phi \circ f(s_1, s_2)| \, d(\mu \times \lambda) \\
\leq \int_{\Omega_1 \times \Omega_2} ||\phi|| ||f(s_1, s_2)|| \, d(\mu \times \lambda) < \infty
\]

and so from the usual Fubini theorem for complex valued functions,

\[
\int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) \, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) \, d\lambda \, d\mu. \tag{19.4.29}
\]

Now also if you fix \( s_2 \), it follows from the definition of strongly measurable and the properties of product measure mentioned above that

\[
s_1 \rightarrow f(s_1, s_2)
\]

is strongly measurable. Also, by 19.4.28

\[
\int_{\Omega_2} ||f(s_1, s_2)|| \, d\lambda < \infty
\]

for \( s_1 \not\in N \). Therefore, by Theorem 19.2.5 \( s_2 \rightarrow f(s_1, s_2) \mathcal{X}_{N_c}(s_1) \) is Bochner integrable. By 19.4.29 and 19.4.26

\[
\int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) \, d(\mu \times \lambda) \\
= \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) \, d\lambda \, d\mu \\
= \int_{\Omega_1} \int_{\Omega_2} \phi(f(s_1, s_2) \mathcal{X}_{N_c}(s_1)) \, d\lambda \, d\mu \\
= \int_{\Omega_1} \phi \left( \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N_c}(s_1) \, d\lambda \right) \, d\mu. \tag{19.4.30}
\]
Each iterated integral makes sense and
\[
\begin{align*}
\int_{\Omega_2} \phi \left( f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) \right) d\lambda \\
= \phi \left( \int_{\Omega_2} f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) d\lambda \right)
\end{align*}
\]
(19.4.31)
is \(\mu\) measurable because
\[
\begin{align*}
(s_1, s_2) \rightarrow \phi \left( f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) \right) \\
= \phi \left( f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) \right)
\end{align*}
\]
is product measurable. Now consider the function,
\[
\begin{align*}
\int_{\Omega_2} f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) d\lambda.
\end{align*}
\]
(19.4.32)
I want to show this is also Bochner integrable with respect to \(\mu\) so I can factor out \(\phi\) once again. It's measurability follows from the Pettis theorem and the above observation [19.4.31]. Also,
\[
\begin{align*}
\int_{\Omega_1} \left\| \int_{\Omega_2} f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) d\lambda \right\| d\mu \\
\leq \int_{\Omega_1} \int_{\Omega_2} |f(s_1, s_2)| d\lambda d\mu \\
= \int_{\Omega_1 \times \Omega_2} |f(s_1, s_2)| d(\mu \times \lambda) < \infty.
\end{align*}
\]
Therefore, the function in [19.4.32] is indeed Bochner integrable and so in [19.4.31] the \(\phi\) can be taken outside the last integral. Thus,
\[
\begin{align*}
\phi \left( \int_{\Omega_1 \times \Omega_2} f(s_1, s_2) d(\mu \times \lambda) \right) \\
= \int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) d(\mu \times \lambda) \\
= \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) d\lambda d\mu \\
= \int_{\Omega_1} \phi \left( \int_{\Omega_2} f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) d\lambda \right) d\mu \\
= \phi \left( \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) d\lambda d\mu \right).
\end{align*}
\]
Since \(X'\) separates the points,
\[
\int_{\Omega_1 \times \Omega_2} f(s_1, s_2) d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \chi_{\mathcal{N}C}(s_1) d\lambda d\mu.
\]
The other formula follows from similar reasoning. This proves the theorem.
19.5 The Spaces $L^p(\Omega; X)$

**Definition 19.5.1** $x \in L^p(\Omega; X)$ for $p \in [1, \infty)$ if $x$ is strongly measurable and 

$$
\int_{\Omega} ||x(s)||^p \, d\mu < \infty
$$

Also

$$
||x||_{L^p(\Omega; X)} \equiv ||x||_p \equiv \left( \int_{\Omega} ||x(s)||^p \, d\mu \right)^{1/p}. \tag{19.5.33}
$$

As in the case of scalar valued functions, two functions in $L^p(\Omega; X)$ are considered equal if they are equal a.e. With this convention, and using the same arguments found in the presentation of scalar valued functions it is clear that $L^p(\Omega; X)$ is a normed linear space with the norm given by $\| \cdot \|_{L^p(\Omega; X)}$. In fact, $L^p(\Omega; X)$ is a Banach space. This is the main contribution of the next theorem.

**Lemma 19.5.2** If $x_n$ is a Cauchy sequence in $L^p(\Omega; X)$ satisfying

$$
\sum_{n=1}^{\infty} ||x_{n+1} - x_n||_p < \infty,
$$

then there exists $x \in L^p(\Omega; X)$ such that $x_n(s) \to x(s)$ a.e. and

$$
||x - x_n||_p \to 0.
$$

**Proof:** Let 

$$
g_N(s) = \sum_{n=1}^{N} ||x_{n+1}(s) - x_n(s)||_X.
$$

Then by the triangle inequality,

$$
\left( \int_{\Omega} g_N(s)^p \, d\mu \right)^{1/p} \leq \sum_{n=1}^{N} \left( \int_{\Omega} ||x_{n+1}(s) - x_n(s)||^p \, d\mu \right)^{1/p} \\
\leq \sum_{n=1}^{\infty} ||x_{n+1} - x_n||_p < \infty.
$$

Let

$$
g(s) = \lim_{N \to \infty} g_N(s) = \sum_{n=1}^{\infty} ||x_{n+1}(s) - x_n(s)||_X.
$$

By the monotone convergence theorem,

$$
\left( \int_{\Omega} g(s)^p \, d\mu \right)^{1/p} = \lim_{N \to \infty} \left( \int_{\Omega} g_N(s)^p \, d\mu \right)^{1/p} < \infty.
$$
Therefore, there exists a set of measure 0, \( E \), such that for \( s \not\in E \), \( g(s) < \infty \). Hence, for \( s \not\in E \),

\[
\lim_{N \to \infty} x_{N+1}(s)
\]

exists because

\[
x_{N+1}(s) = x_{N+1}(s) - x_1(s) + x_1(s) = \sum_{n=1}^{N} (x_{n+1}(s) - x_n(s)) + x_1(s).
\]

Thus, if \( N > M \), and \( s \) is a point where \( g(s) < \infty \),

\[
|\|x_{N+1}(s) - x_{M+1}(s)\|_X| \leq \sum_{n=M+1}^{N} |\|x_{n+1}(s) - x_n(s)\|_X|
\]

which shows that \( \{x_{N+1}(s)\}_{N=1}^{\infty} \) is a Cauchy sequence. Now let

\[
x(s) = \begin{cases} 
\lim_{N \to \infty} x_N(s) & \text{if } s \notin E, \\
0 & \text{if } s \in E.
\end{cases}
\]

By Theorem 19.1.2, \( x_n(\Omega) \) is separable for each \( n \). Therefore, \( x(\Omega) \) is also separable. Also, if \( f \in X' \), then

\[
f(x(s)) = \lim_{N \to \infty} f(x_N(s))
\]

if \( s \not\in E \) and \( f(x(s)) = 0 \) if \( s \in E \). Therefore, \( f \circ x \) is measurable because it is the limit of the measurable functions,

\[
f \circ x \in X_{EC}'.
\]

Since \( x \) is weakly measurable and \( x(\Omega) \) is separable, Corollary 19.1.8 shows that \( x \) is strongly measurable. By Fatou’s lemma,

\[
\int_{\Omega} |\|x(s) - x_N(s)\|_X|^p \, d\mu \leq \liminf_{M \to \infty} \int_{\Omega} |\|x_M(s) - x_N(s)\|_X|^p \, d\mu.
\]

But if \( N \) and \( M \) are large enough with \( M > N \),

\[
\left( \int_{\Omega} |\|x_M(s) - x_N(s)\|_X|^p \, d\mu \right)^{1/p} \leq \sum_{n=1}^{M} |\|x_{n+1} - x_n\|_p|
\]

\[
\leq \sum_{n=N}^{\infty} |\|x_{n+1} - x_n\|_p < \varepsilon
\]

and this shows, since \( \varepsilon \) is arbitrary, that

\[
\lim_{N \to \infty} \int_{\Omega} |\|x(s) - x_N(s)\|_X|^p \, d\mu = 0.
\]
19.5. THE SPACES $L^p(\Omega; X)$

It remains to show $x \in L^p(\Omega; X)$. This follows from the above and the triangle inequality. Thus, for $N$ large enough,

$$
\left( \int_\Omega \|x(s)\|^p d\mu \right)^{1/p} \leq \left( \int_\Omega \|x_N(s)\|^p d\mu \right)^{1/p} + \left( \int_\Omega \|x(s) - x_N(s)\|^p d\mu \right)^{1/p} \leq \left( \int_\Omega \|x_N(s)\|^p d\mu \right)^{1/p} + \varepsilon < \infty.
$$

This proves the lemma.

**Theorem 19.5.3** $L^p(\Omega; X)$ is complete. Also every Cauchy sequence has a subsequence which converges pointwise.

**Proof:** If $\{x_n\}$ is Cauchy in $L^p(\Omega; X)$, extract a subsequence $\{x_{n_k}\}$ satisfying

$$
\|x_{n_{k+1}} - x_{n_k}\|_p \leq 2^{-k}
$$

and apply Lemma 19.5.2. The pointwise convergence of this subsequence was established in the proof of this lemma. This proves the theorem because if a subsequence of a Cauchy sequence converges, then the Cauchy sequence must also converge.

**Observation 19.5.4** If the measure space is Lebesgue measure then you have continuity of translation in $L^p(\mathbb{R}^n; X)$ in the usual way. More generally, for $\mu$ a Radon measure on $\Omega$ a locally compact Hausdorff space, $C_c(\Omega; X)$ is dense in $L^p(\Omega; X)$. Here $C_c(\Omega; X)$ is the space of continuous $X$ valued functions which have compact support in $\Omega$. The proof of this little observation follows immediately from approximating with simple functions and then applying the appropriate considerations to the simple functions.

Clearly Fatou’s lemma and the monotone convergence theorem make no sense for functions with values in a Banach space but the dominated convergence theorem holds in this setting.

**Theorem 19.5.5** If $x$ is strongly measurable and $x_n(s) \to x(s)$ a.e. with

$$
\|x_n(s)\| \leq g(s) \text{ a.e.}
$$

where $g \in L^1(\Omega)$, then $x$ is Bochner integrable and

$$
\int_\Omega x(s) d\mu = \lim_{n \to \infty} \int_\Omega x_n(s) d\mu.
$$
Proof: \( \|x_n(s) - x(s)\| \leq 2g(s) \) a.e. so by the usual dominated convergence theorem,
\[
0 = \lim_{n \to \infty} \int_{\Omega} \|x_n(s) - x(s)\| \, d\mu.
\]
Also,
\[
\int_{\Omega} \|x_n(s) - x_m(s)\| \, d\mu \\
\leq \int_{\Omega} \|x_n(s) - x(s)\| \, d\mu + \int_{\Omega} \|x_m(s) - x(s)\| \, d\mu,
\]
and so \( \{x_n\} \) is a Cauchy sequence in \( L^1(\Omega; X) \). Therefore, by Theorem 19.5.3, there exists \( y \in L^1(\Omega; X) \) and a subsequence \( x_{n'} \) satisfying
\[
x_{n'}(s) \to y(s) \quad \text{a.e. and in} \quad L^1(\Omega; X).
\]
But \( x(s) = \lim_{n' \to \infty} x_{n'}(s) \) a.e. and so \( x(s) = y(s) \) a.e. Hence
\[
\int_{\Omega} \|x(s)\| \, d\mu = \int_{\Omega} \|y(s)\| \, d\mu < \infty
\]
which shows that \( x \) is Bochner integrable. Finally, since the integral is linear,
\[
\left\| \int_{\Omega} x(s) \, d\mu - \int_{\Omega} x_n(s) \, d\mu \right\| = \left\| \int_{\Omega} (x(s) - x_n(s)) \, d\mu \right\| \\
\leq \int_{\Omega} \|x_n(s) - x(s)\| \, d\mu,
\]
and this last integral converges to 0. This proves the theorem.

One can also give a version of the Vitali convergence theorem.

Definition 19.5.6 Let \( A \subseteq L^1(\Omega; X) \). Then \( A \) is said to be uniformly integrable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( \mu(E) < \delta \), it follows
\[
\int_E \|f\|_X \, d\mu < \varepsilon
\]
for all \( f \in A \). It is bounded if
\[
\sup_{f \in A} \int_{\Omega} \|f\|_X \, d\mu < \infty.
\]

Theorem 19.5.7 Let \( (\Omega, \mathcal{F}, \mu) \) be a finite measure space and let \( X \) be a separable Banach space. Let \( \{f_n\} \subseteq L^1(\Omega; X) \) be uniformly integrable and bounded such that \( f_n(\omega) \to f(\omega) \) for each \( \omega \in \Omega \). Then \( f \in L^1(\Omega; X) \) and
\[
\lim_{n \to \infty} \int_{\Omega} \|f_n - f\|_X \, d\mu = 0.
\]
Proof: Let \( \varepsilon > 0 \) be given. Then by uniform integrability there exists \( \delta > 0 \) such that if \( \mu(E) < \delta \) then

\[
\int_E ||f_n|| \, d\mu < \varepsilon / 3.
\]

By Fatou’s lemma the same inequality holds for \( f \). Also Fatou’s lemma shows \( f \in L^1(\Omega; X) \), \( f \) being measurable because of Theorem 19.5.8.

By Egoroff’s theorem, Theorem 19.5.8, there exists a set of measure less than \( \delta \), \( E \) such that the convergence of \( \{f_n\} \) to \( f \) is uniform off \( E \). Therefore,

\[
\int_{\Omega} ||f - f_n|| \, d\mu \leq \int_{E} (||f||_X + ||f_n||_X) \, d\mu + \int_{E^c} ||f - f_n||_X \, d\mu < \frac{2\varepsilon}{3} + \int_{E^c} \frac{\varepsilon}{\mu(\Omega) + 1} \, d\mu < \varepsilon
\]

if \( n \) is large enough. This proves the theorem.

Note that a convenient way to achieve uniform integrability is to simply say \( \{f_n\} \) is bounded in \( L^p(\Omega; X) \) for some \( p > 1 \). This follows from Hölder’s inequality.

\[
\int_{E} ||f_n|| \, d\mu \leq \left( \int_{E} \, d\mu \right)^{1/p'} \left( \int_{\Omega} ||f_n||^p \, d\mu \right)^{1/p}.
\]

The following theorem is interesting.

Theorem 19.5.8 Let \( 1 \leq p < \infty \) and let \( p < r \leq \infty \). Then \( L^r([0, T]; X) \) is a Borel subset of \( L^p([0, T]; X) \). Letting \( C([0, T]; X) \) denote the functions having values in \( X \) which are continuous, \( C([0, T]; X) \) is also a Borel subset of \( L^p([0, T]; X) \). Here the measure is ordinary one dimensional Lebesque measure on \([0, T]\).

Proof: First consider the claim about \( L^r([0, T]; X) \). Let

\[
B_M \equiv \left\{ x \in L^p([0, T]; X) : ||x||_{L^r([0, T]; X)} \leq M \right\}.
\]

Then \( B_M \) is a closed subset of \( L^p([0, T]; X) \). Here is why. If \( \{x_n\} \) is a sequence of elements of \( B_M \) and \( x_n \to x \) in \( L^p([0, T]; X) \), then passing to a subsequence, still denoted by \( x_n \), it can be assumed \( x_n(s) \to x(s) \) a.e. Hence Fatou’s lemma can be applied to conclude

\[
\int_0^T ||x(s)||^r \, ds \leq \liminf_{n \to \infty} \int_0^T ||x_n(s)||^r \, ds \leq M^r < \infty.
\]

Now \( \bigcup_{n=1}^{\infty} B_M = L^r([0, T]; X) \). Note this did not depend on the measure space used. It would have been equally valid on any measure space.

Consider now \( C([0, T]; X) \). The norm on this space is the usual norm, \( ||\cdot||_\infty \). The argument above shows \( ||\cdot||_\infty \) is a Borel measurable function on \( L^p([0, T]; X) \). This is because \( B_M \equiv \left\{ x \in L^p([0, T]; X) : ||x||_\infty \leq M \right\} \) is a closed, hence Borel subset of \( L^p([0, T]; X) \). Now let \( \theta \in \mathcal{L}(L^p([0, T]; X), L^p(\mathbb{R}; X)) \) such that \( \theta(x(t)) = \ldots \)
$x(t)$ for all $t \in [0, T]$ and also $\theta \in \mathcal{L} (C([0, T] ; X), BC(\mathbb{R}; X))$ where $BC(\mathbb{R}; X)$ denotes the bounded continuous functions with a norm given by

$$||x|| \equiv \sup_{t \in \mathbb{R}} ||x(t)||,$$

and $\theta x$ has compact support.

For example, you could define $e_\theta x(t) \equiv \begin{cases} x(t) & \text{if } t \in [0, T] \\ x(2T - t) & \text{if } t \in [T, 2T] \\ x(-t) & \text{if } t \in [-T, 0] \\ 0 & \text{if } t \notin [-T, 2T] \end{cases}$

and let $\Phi \in C^\infty_c(-T, 2T)$ such that $\Phi(t) = 1$ for $t \in [0, T]$. Then you could let $\theta x(t) \equiv \Phi(t) e_\theta x(t)$.

Then let $\{\phi_n\}$ be a mollifier and define $\psi_n x(t) \equiv \phi_n * \theta x(t)$.

It follows $\psi_n x$ is uniformly continuous because

$$||\psi_n x(t) - \psi_n x(t')||_X \leq \int_\mathbb{R} |\phi_n (t' - s) - \phi_n (t - s)||\theta x(s)||_X ds \leq C ||x||_p \left( \int_\mathbb{R} |\phi_n (t' - s) - \phi_n (t - s)|^{p'} ds \right)^{1/p'}$$

Also for $x \in C([0, T]; X)$, it follows from usual mollifier arguments that

$$||\psi_n x - x||_{L^\infty([0, T]; X)} \to 0.$$

Here is why. For $t \in [0, T]$,

$$||\psi_n x(t) - x(t)||_X \leq \int_\mathbb{R} \phi_n(s) ||\theta x(t) - \theta x(t)||_X ds \leq C_\theta \int_{-1/n}^{1/n} \phi_n(s) ds \varepsilon = C_\theta \varepsilon$$

provided $n$ is large enough due to the compact support and consequent uniform continuity of $\theta x$.

If $||\psi_n x - x||_{L^\infty([0, T]; X)} \to 0$, then $\{\psi_n x\}$ must be a Cauchy sequence in $C([0, T]; X)$ and this requires that $x$ equals a continuous function a.e. Thus $C([0, T]; X)$ consists exactly of those functions, $x$ of $L^p([0, T]; X)$ such that $||\psi_n x - x||_\infty \to 0$. It follows

$$C([0, T]; X) = \text{...}$$
19.5. THE SPACES $L^p(\Omega; X)$

$$\cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty \left\{ x \in L^p([0, T]; X) : \| \psi_k x - x \|_\infty \leq \frac{1}{n} \right\}.$$  (19.5.34)

It only remains to show

$$S \equiv \{ x \in L^p([0, T]; X) : \| \psi_k x - x \|_\infty \leq \alpha \}$$

is a Borel set. Suppose then that $x_n \in S$ and $x_n \to x$ in $L^p([0, T]; X)$. There exists a set of measure 0 such that for all $n$, and $t$ not in this set,

$$\int_{\Omega} \| \psi_k x_n(t) - x_n(t) \| \, d\mu \to 0$$

which converges to 0 as $n \to \infty$. It follows that for a.e. $t$,

$$\| \psi_k x(t) - x(t) \| \leq \alpha.$$

Thus $S$ is closed and so the set in $L^p([0, T]; X)$ is a Borel set. This proves the theorem.

As in the scalar case, the following lemma holds in this more general context.

**Lemma 19.5.9** Let $(\Omega, \mu)$ be a regular measure space where $\Omega$ is a locally compact Hausdorff space. Then $C_c(\Omega; X)$ the space of continuous functions having compact support and values in $X$ is dense in $L^p(0, T; X)$ for all $p \in [0, \infty)$. For any $\sigma$ finite measure space, the simple functions are dense in $L^p(0, T; X)$.

**Proof:** First it is shown the simple functions are dense in $L^p(0, T; X)$. Let $f \in L^p(0, T; X)$ and let $\{x_n\}$ denote a sequence of simple functions which converge to $f$ pointwise which also have the property that

$$\| x_n(s) \| \leq 2 \| f(s) \|.$$

Then

$$\int_{\Omega} \| x_n(s) - f(s) \|^p \, d\mu \to 0$$

from the dominated convergence theorem. Therefore, the simple functions are indeed dense in $L^p(0, T; X)$.

Next suppose $(\Omega, \mu)$ is a regular measure space. If $x(s) \equiv \sum a_i \chi_{E_i}(s)$ is a simple function, then by regularity, there exist compact sets, $K_i$ and open sets, $V_i$
such that \( K_i \subseteq E_i \subseteq V_i \) and \( \mu(V_i \setminus K_i)^{1/p} < \varepsilon / \sum_i ||a_i|| \). Let \( K_i \prec h_i \prec V_i \). Then consider
\[
\sum_i a_i h_i \in C_c(\Omega).
\]

By the triangle inequality,
\[
\left( \int_\Omega \left\| \sum_i a_i h_i (s) - a_i X_{E_i}(s) \right\|^p d\mu \right)^{1/p} \\
\leq \sum_i \left( \int_\Omega ||a_i h_i (s) - X_{E_i}(s)||^p d\mu \right)^{1/p} \\
\leq \sum_i \left( \int_\Omega ||a_i||^p |h_i (s) - X_{E_i}(s)|^p d\mu \right)^{1/p} \\
\leq \sum_i ||a_i|| \left( \int_{V_i \setminus K_i} d\mu \right)^{1/p} \\
\leq \sum_i ||a_i|| \mu(V_i \setminus K_i)^{1/p} < \varepsilon
\]

Since \( \varepsilon \) is arbitrary, this and the first part of the lemma shows \( C_c(\Omega; X) \) is dense in \( L^p(\Omega; X) \).

### 19.6 Measurable Representatives

In this section consider the special case where \( X = L^1(B, \nu) \) where \((B, \mathcal{F}, \nu)\) is a \( \sigma \)-finite measure space and \( x \in L^1(\Omega; X) \). Thus for each \( s \in \Omega \), \( x(s) \in L^1(B, \nu) \). In general, the map
\[
(s, t) \rightarrow x(s)(t)
\]
will not be product measurable, but one can obtain a measurable representative. This is important because it allows the use of Fubini's theorem on the measurable representative.

By Theorem 19.2.5, there exists a sequence of simple functions, \( \{x_n\} \), of the form
\[
x_n(s) = \sum_{k=1}^m a_k X_{E_k}(s)
\]  \( (19.6.35) \)

where \( a_k \in L^1(B, \nu) \) which satisfy the conditions of Definition 19.2.3 and
\[
||x_n - x_m||_{L^1(\Omega; L^1(B))} \to 0 \text{ as } m, n \to \infty
\]  \( (19.6.36) \)
For such a simple function, you can assume the $E_k$ are disjoint and then

$$
\|x_n\|_{L^1(\Omega,L^1(B))} = \sum_{k=1}^m \|a_k\|_{L^1(B)} \mu(E_k) = \sum_{k=1}^m \int_B |a_k| \, d\nu(E_k)
$$

$$= \int_\Omega \int_B |a_k(t)| \, d\nu(t) \chi_{E_k}(s) \, d\mu(s)
$$

$$= \int_\Omega \int_B |x_n| \, d\nu d\mu
$$

Also, each $x_n$ is product measurable. Thus from \[19.6.36\],

$$\|x_n - x_m\|_{L^1(\Omega,L^1(B))} = \int_\Omega \int_B |x_n - x_m| \, d\nu d\mu
$$

which shows that $\{x_n\}$ is a Cauchy sequence in $L^1(\Omega \times B, \mu \times \lambda)$. Then there exists $y \in L^1(\Omega \times B, \mu \times \lambda)$ and a subsequence still called $\{x_n\}$ such that

$$\lim_{n \to \infty} \int_\Omega \int_B |x_n - y| \, d\nu d\mu = \lim_{n \to \infty} \|x_n - y\|_{L^1(B)} \, d\mu = \|x_n - y\|_{L^1(\Omega,L^1(B))} = 0.
$$

Now consider \[19.6.36\]. Since $\lim_{m \to \infty} x_m(s) = x(s)$ in $L^1(B)$, it follows from Fatou’s lemma that

$$\|x_n - x\|_{L^1(\Omega,L^1(B))} \leq \liminf_{m \to \infty} \|x_n - x_m\|_{L^1(\Omega,L^1(B))} < \varepsilon$$

for all $n$ large enough. Hence

$$\lim_{n \to \infty} \|x_n - x\|_{L^1(\Omega,L^1(B))} = 0$$

and so

$$x(s) = y(s) \text{ in } L^1(B) \mu \text{ a.e. } s
$$

In particular, for a.e. $s$, it follows that

$$x(s)(t) = y(s,t) \text{ for a.e. } t.
$$

Now $\int_\Omega x(s) \, d\mu \in X = L^1(B,\nu)$ so it makes sense to ask for $(\int_\Omega x(s) \, d\mu)(t)$, at least $\mu$ a.e. $t$. To find what this is, note

$$\left| \int_\Omega x_n(s) \, d\mu - \int_\Omega x(s) \, d\mu \right|_X \leq \int_\Omega \|x_n(s) - x(s)\|_X \, d\mu.
$$

Therefore, since the right side converges to 0,

$$\lim_{n \to \infty} \left| \int_\Omega x_n(s) \, d\mu - \int_\Omega x(s) \, d\mu \right|_X =
$$

$$\lim_{n \to \infty} \int_B \left| \left( \int_\Omega x_n(s) \, d\mu \right)(t) - \left( \int_\Omega x(s) \, d\mu \right)(t) \right| \, d\nu = 0.$$
CHAPTER 19. THE BOCHNER INTEGRAL

But
\[
\left( \int_{\Omega} x_n(s) \, d\mu \right)(t) = \int_{\Omega} x_n(s,t) \, d\mu \text{ a.e. } t.
\]

Therefore
\[
\lim_{n \to \infty} \int_{B} \left| \int_{\Omega} x_n(s,t) \, d\mu - \left( \int_{\Omega} x(s) \, d\mu \right)(t) \right| \, d\nu = 0. \tag{19.6.37}
\]

Also, since \( x_n \to y \) in \( L^1(\Omega \times B) \),
\[
0 = \lim_{n \to \infty} \int_{B} \int_{\Omega} |x_n(s,t) - y(s,t)| \, d\mu d\nu \geq
\]
\[
\lim_{n \to \infty} \int_{B} \left| \int_{\Omega} x_n(s,t) \, d\mu - \int_{\Omega} y(s,t) \, d\mu \right| \, d\nu. \tag{19.6.38}
\]

From (19.6.37) and (19.6.38)
\[
\int_{\Omega} y(s,t) \, d\mu = \left( \int_{\Omega} x(s) \, d\mu \right)(t) \text{ a.e. } t.
\]

This proves the following theorem.

**Theorem 19.6.1** Let \( X = L^1(B) \) where \( (B, F, \nu) \) is a \( \sigma \)-finite measure space and let \( x \in L^1(\Omega; X) \). Then there exists a measurable representative, \( y \in L^1(\Omega \times B) \), such that
\[
x(s) = y(s, \cdot) \text{ a.e. } s \text{ in } \Omega, \text{ the equation in } L^1(B),
\]
and
\[
\int_{\Omega} y(s,t) \, d\mu = \left( \int_{\Omega} x(s) \, d\mu \right)(t) \text{ a.e. } t.
\]

**19.7 Vector Measures**

There is also a concept of vector measures.

**Definition 19.7.1** Let \( (\Omega, S) \) be a set and a \( \sigma \) algebra of subsets of \( \Omega \). A mapping
\[
F : S \to X
\]
is said to be a vector measure if
\[
F(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} F(E_i)
\]
whenever \( \{E_i\}_{i=1}^{\infty} \) is a sequence of disjoint elements of \( S \). For \( F \) a vector measure,
\[
|F|(A) = \sup \left\{ \sum_{F \in \pi(A)} ||\mu(F)|| : \pi(A) \text{ is a partition of } A \right\}.
\]
This is the same definition that was given in the case where $F$ would have values in $\mathbb{C}$, the only difference being the fact that now $F$ has values in a general Banach space $X$ as the vector space of values of the vector measure. Recall that a partition of $A$ is a finite set, $\{F_1, \ldots, F_m\} \subseteq S$ such that $\bigcup_{i=1}^{m} F_i = A$. The same theorem about $|F|$ proved in the case of complex valued measures holds in this context with the same proof. For completeness, it is included here.

**Theorem 19.7.2** If $|F|(\Omega) < \infty$, then $|F|$ is a measure on $S$.

**Proof:** Let $E_1$ and $E_2$ be sets of $S$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A_1^i \cdots A_{n_i}^i\} = \pi(E_i)$, a partition of $E_i$ which is chosen such that

$$|F|(E_i) - \varepsilon < \sum_{j=1}^{n_i} ||F(A_j^i)||, \quad i = 1, 2.$$ 

Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ which is denoted here by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|F|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} ||F(F)|| > |F|(E_1) + |F|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|F|(E_1 \cup E_2) \geq |F|(E_1) + |F|(E_2). \quad (19.7.39)$$

Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $S$ and let $E_{\infty} = \bigcup_{j=1}^{\infty} E_j$. Then by the definition of total variation there exists a partition of $E_{\infty}$, $\pi(E_{\infty}) = \{A_1, \cdots, A_n\}$ such that

$$|F|(E_{\infty}) - \varepsilon < \sum_{i=1}^{n} ||F(A_i)||.$$

Also, $A_i = \bigcup_{j=1}^{\infty} A_i \cap E_j$

and so by the triangle inequality, $||F(A_i)|| \leq \sum_{j=1}^{\infty} ||F(A_i \cap E_j)||$. Therefore, by the above,

$$|F|(E_{\infty}) - \varepsilon \leq \sum_{i=1}^{n} \sum_{j=1}^{\infty} ||F(A_i \cap E_j)||$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{n} ||F(A_i \cap E_j)||$$

$$\leq \sum_{j=1}^{\infty} |F|(E_j)$$
CHAPTER 19. THE BOCHNER INTEGRAL

because \( \{ A_i \cap E_j \}_{i=1}^{n} \) is a partition of \( E_j \).

Since \( \varepsilon > 0 \) is arbitrary, this shows

\[
|F|(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} |F|(E_j).
\]

Also, \[19.7.39\] implies that whenever the \( E_i \) are disjoint,
\[
|F|(\bigcup_{j=1}^{\infty} E_j) \geq \sum_{j=1}^{n} |F|(E_j).
\]

Therefore,

\[
\sum_{j=1}^{\infty} |F|(E_j) \geq |F|(\bigcup_{j=1}^{\infty} E_j) \geq \sum_{j=1}^{n} |F|(E_j).
\]

Since \( n \) is arbitrary,

\[
|F|(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |F|(E_j)
\]

which shows that \( |F| \) is a measure as claimed. This proves the theorem.

**Definition 19.7.3** A Banach space is said to have the Radon Nikodym property if whenever

\( (\Omega, S, \mu) \) is a finite measure space

\( F : S \to X \) is a vector measure with \( |F|(\Omega) < \infty \)

\( F \ll \mu \)

then one may conclude there exists \( g \in L^1(\Omega; X) \) such that

\[
F(E) = \int_E g(s) \, d\mu
\]

for all \( E \in S \).

Some Banach spaces have the Radon Nikodym property and some don’t. No attempt is made to give a complete answer to the question of which Banach spaces have this property but the next theorem gives examples of many spaces which do.

**Theorem 19.7.4** Suppose \( X' \) is a separable dual space. Then \( X' \) has the Radon Nikodym property.

**Proof:** Let \( F \ll \mu \) and let \( |F|(\Omega) < \infty \) for \( F : S \to X' \), a vector measure. Pick \( x \in X \) and consider the map

\[
E \to F(E)(x)
\]

for \( E \in S \). This defines a complex measure which is absolutely continuous with respect to \( |F| \). Therefore, by the Radon Nikodym theorem, there exists \( f_x \in L^1(\Omega, |F|) \) such that

\[
F(E)(x) = \int_E f_x(s) \, d|F|.
\]

(19.7.40)
Claim: \(|f_x (s)| \leq \|x\|\) for \(|F|\) a.e. \(s\).

Proof of claim: Consider the closed ball in \(F\), \(\overline{B (0, ||x||)}\) and let \(F = B(p, r)\) be an open ball contained in its complement. Let \(f^{-1}_x (B) \equiv E \in \mathcal{S}\). I want to argue that \(|F| (E) = 0\) so suppose \(|F| (E) > 0\). then

\(|F| (E) \|x\| \geq \|F (E)\| \|x\| \geq |F (E) (x)|\)

and so from 19.7.40,

\[
\frac{1}{|F| (E)} \left| \int_E f_x (s) \, d|F| \right| \leq \|x\|. \tag{19.7.41}
\]

But on \(E\), \(|f_x (s) - p| < r\) and so

\[
\left| \frac{1}{|F| (E)} \int_E f_x (s) \, d|F| - p \right| < r
\]

which contradicts 19.7.41 because \(B (p, r)\) was given to have empty intersection with \(\overline{B (0, ||x||)}\). Therefore, \(|F| (E) = 0\) as hoped. Now \(F \setminus \overline{B (0, ||x||)}\) can be covered by countably many such balls and so \(|F| (F \setminus \overline{B (0, ||x||)}) = 0\).

Denote the exceptional set of measure zero by \(N_x\). By Theorem 19.1.13, \(X\) is separable. Letting \(D\) be a dense, countable subset of \(X\), define

\[N_1 \equiv \bigcup_{x \in D} N_x.\]

Thus

\(|F| (N_1) = 0.\]

For any \(E \in \mathcal{S}\), \(x, y \in D\), and \(a, b \in F\),

\[
\int_E f_{ax + by} (s) \, d|F| = F (E) (ax + by) = aF (E) (x) + bF (E) (y)
\]

\[
= \int_E (af_x (s) + bf_y (s)) \, d|F|. \tag{19.7.42}
\]

Since 19.7.42 holds for all \(E \in \mathcal{S}\), it follows

\[f_{ax + by} (s) = af_x (s) + bf_y (s)\]

for \(|F|\) a.e. \(s\) and \(x, y \in D\). Let \(\hat{D}\) consist of all finite linear combinations of the form \(\sum_{i=1}^m a_i x_i\) where \(a_i\) is a rational point of \(F\) and \(x_i \in D\). If

\[
\sum_{i=1}^m a_i x_i \in \hat{D},
\]

the above argument implies

\[f_{\sum_{i=1}^m a_i x_i} (s) = \sum_{i=1}^m a_i f_{x_i} (s)\] a.e.
Since $\tilde{D}$ is countable, there exists a set, $N_2$, with
\[ |F|(N_2) = 0 \]
such that for $s \notin N_2$,
\[ f \sum_{i=1}^{m} a_i x_i (s) = \sum_{i=1}^{m} a_i f x_i (s) \quad (19.7.43) \]
whenever $\sum_{i=1}^{m} a_i x_i \in \tilde{D}$. Let
\[ N = N_1 \cup N_2 \]
and let
\[ \tilde{h}_x (s) \equiv X_{N^c} (s) f x (s) \]
for all $x \in \tilde{D}$. Now for $x \in X$ define
\[ h_x (s) \equiv \lim_{x' \to x} \{ \tilde{h}_x (s) : x' \in \tilde{D} \}. \]
This is well defined because if $x'$ and $y'$ are elements of $\tilde{D}$, the above claim and
\[ |h_x (s) - \tilde{h}_{y'} (s)| = |\tilde{h}_{(x' - y')} (s)| \leq ||x' - y'||. \]
Using the dominated convergence theorem may be applied to conclude that for $x_n \to x$, with $x_n \in \tilde{D}$,
\[ \int_E h_x (s) \, d|F| = \lim_{n \to \infty} \int_E \tilde{h}_{x_n} (s) \, d|F| = \lim_{n \to \infty} F (E) (x_n) = F (E) (x). \quad (19.7.44) \]
It follows from the density of $\tilde{D}$ that for all $x, y \in X$ and $a, b \in F$,
\[ |h_x (s)| \leq ||x|| , \ h_{ax + by} (s) = ah_x (s) + bh_y (s), \quad (19.7.45) \]
for all $s$ because if $s \in N$, both sides of the equation in equal 0.
Let $\theta (s)$ be given by
\[ \theta (s) (x) = h_x (s). \]
By it follows that $\theta (s) \in X'$ for each $s$. Also
\[ \theta (s) (x) = h_x (s) \in L^1 (\Omega) \]
so $\theta (\cdot)$ is weak * measurable. Since $X'$ is separable, Theorem implies that $\theta$ is strongly measurable. Furthermore, by
\[ ||\theta (s)|| = \sup_{||x|| \leq 1} |\theta (s) (x)| \leq \sup_{||x|| \leq 1} |h_x (s)| \leq 1. \]
Therefore,
\[ \int_\Omega ||\theta (s)|| \, d|F| < \infty \]
so \( \theta \in L^1(\Omega; X') \). By Equation 19.2.6, if \( E \in \mathcal{S} \),

\[
\int_E h_x(s) d|F| = \int_E \theta(s)(x) d|F| = \left( \int_E \theta(s) d|F| \right)(x).
\]

(19.7.46)

From Equation 19.7.44 and 19.7.46,

\[
\left( \int_E \theta(s) d|F| \right)(x) = F(E)(x)
\]

for all \( x \in X \) and therefore,

\[
\int_E \theta(s) d|F| = F(E).
\]

Finally, since \( F \ll \mu, |F| \ll \mu \) also and so there exists \( k \in L^1(\Omega) \) such that

\[
|F|(E) = \int_E k(s) d\mu
\]

for all \( E \in \mathcal{S} \), by the Radon Nikodym Theorem. It follows

\[
F(E) = \int_E \theta(s) d|F| = \int_E \theta(s) k(s) d\mu.
\]

Letting \( g(s) = \theta(s) k(s) \), this has proved the theorem.

**Corollary 19.7.5** Any separable reflexive Banach space has the Radon Nikodym property.

It is not necessary to assume separability in the above corollary. For the proof of a more general result, consult *Vector Measures* by Diestal and Uhl, [38].

### 19.8 The Riesz Representation Theorem

The Riesz representation theorem for the spaces \( L^p(\Omega; X) \) holds under certain conditions. The proof follows the proofs given earlier for scalar valued functions.

**Definition 19.8.1** If \( X \) and \( Y \) are two Banach spaces, \( X \) is isometric to \( Y \) if there exists \( \theta \in \mathcal{L}(X,Y) \) such that

\[ ||\theta x||_Y = ||x||_X. \]

This will be written as \( X \cong Y \). The map \( \theta \) is called an isometry.

The next theorem says that \( L^p(\Omega; X') \) is always isometric to a subspace of \( (L^p(\Omega; X))' \) for any Banach space, \( X \).
Theorem 19.8.2 Let $X$ be any Banach space and let $(\Omega, S, \mu)$ be a finite measure space. Let $p \geq 1$ and let $1/p + 1/p' = 1$. (If $p = 1$, $p' \equiv \infty$.) Then $L^{p'}(\Omega; X')$ is isometric to a subspace of $(L^p(\Omega; X))'$. Also, for $g \in L^{p'}(\Omega; X')$,

$$\sup_{||f||_p \leq 1} \left| \int_{\Omega} g(s)(f(s)) \, d\mu \right| = ||g||_{p'}.$$ 

Proof: First observe that for $f \in L^p(\Omega; X)$ and $g \in L^{p'}(\Omega; X')$,

$$s \rightarrow g(s)(f(s))$$

is a function in $L^1(\Omega)$. (To obtain measurability, write $f$ as a limit of simple functions. Holder’s inequality then yields the function is in $L^1(\Omega)$.) Define

$$\theta : L^{p'}(\Omega; X') \rightarrow (L^p(\Omega; X))'$$

by

$$\theta g(f) \equiv \int_{\Omega} g(s)(f(s)) \, d\mu.$$ 

Holder’s inequality implies

$$||\theta g|| \leq ||g||_{p'}$$

and it is also clear that $\theta$ is linear. Next it is required to show

$$||\theta g|| = ||g||.$$ 

This will first be verified for simple functions. Let

$$g(s) = \sum_{i=1}^{m} c_i X_{E_i}(s)$$

where $c_i \in X'$, the $E_i$ are disjoint and

$$\cup_{i=1}^{m} E_i = \Omega.$$ 

Then $||g|| \in L^{p'}(\Omega)$. Let $\varepsilon > 0$ be given. By the scalar Riesz representation theorem, there exists $h \in L^p(\Omega)$ such that $||h||_p = 1$ and

$$\int_{\Omega} ||g(s)||_{X'} h(s) \, d\mu \geq ||g||_{L^{p'}(\Omega; X')} - \varepsilon.$$ 

Now let $d_i$ be chosen such that

$$c_i(d_i) \geq ||c_i||_{X'} - \varepsilon / ||h||_{L^1(\Omega)}$$

and $||d_i||_X \leq 1$. Let

$$f(s) \equiv \sum_{i=1}^{m} d_i h(s) X_{E_i}(s).$$
Thus $f \in L^p(\Omega; X)$ and $\|f\|_{L^p(\Omega; X)} \leq 1$. This follows from

$$\|f\|_p^p = \int_{\Omega} \sum_{i=1}^{m} |d_i|^p_X |h(s)|^p X_{E_i}(s) \, d\mu$$

$$= \sum_{i=1}^{m} \left( \int_{E_i} |h(s)|^p \, d\mu \right) \|d_i\|^p_X \leq \int_{\Omega} |h|^p \, d\mu = 1.$$ 

Also

$$|\theta g| \geq |\theta g(f)| = \left| \int_{\Omega} g(s) (f(s)) \, d\mu \right| \geq$$

$$\left| \int_{\Omega} \sum_{i=1}^{m} \left( |c_i| X' - \varepsilon / \|h\|_{L^1(\Omega)} \right) h(s) X_{E_i}(s) \, d\mu \right|$$

$$\geq \left| \int_{\Omega} |g(s)| X' h(s) \, d\mu \right| - \varepsilon \left| \int_{\Omega} h(s) / \|h\|_{L^1(\Omega)} \, d\mu \right|$$

$$\geq \|g\|_{L^{p'}(\Omega; X')} - 2\varepsilon.$$ 

Since $\varepsilon$ was arbitrary,

$$\|\theta g\| \geq \|g\| \quad (19.8.48)$$

and from (19.8.47) this shows equality holds in (19.8.48) whenever $g$ is a simple function.

In general, let $g \in L^{p'}(\Omega; X')$ and let $g_n$ be a sequence of simple functions converging to $g$ in $L^{p'}(\Omega; X')$. Then

$$\|\theta g\| = \lim_{n \to \infty} \|\theta g_n\| = \lim_{n \to \infty} \|g_n\| = \|g\|.$$ 

This proves the theorem and shows $\theta$ is the desired isometry.

**Theorem 19.8.3** If $X$ is a Banach space and $X'$ has the Radon Nikodym property, then if $(\Omega, \mathcal{S}, \mu)$ is a finite measure space,

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X')$$

and in fact the mapping $\theta$ of Theorem 19.8.2 is onto.

**Proof:** Let $l \in (L^p(\Omega; X))'$ and define $F(E) \in X'$ by

$$F(E)(x) \equiv l(X_E(\cdot) x).$$

**Lemma 19.8.4** $F$ defined above is a vector measure with values in $X'$ and $\|F\|(\Omega) < \infty$.

**Proof of the lemma:** Clearly $F(E)$ is linear. Also

$$\|F(E)\| = \sup_{\|x\| \leq 1} \|F(E)(x)\|$$
\[ \leq \|l\| \sup_{\|x\| \leq 1} \|X_E (\cdot) x\|_{L^p(\Omega; X)} \leq \|l\| \|\mu(E)\|^{1/p}. \]

Let \( \{E_i\}_{i=1}^\infty \) be a sequence of disjoint elements of \( S \) and let \( E = \bigcup_{n<\infty} E_n \).

\[
\left| F(E) (x) - \sum_{k=1}^n F(E_k) (x) \right| = \left| l(X_E (\cdot) x) - \sum_{i=1}^n l(X_{E_i} (\cdot) x) \right| \tag{19.8.49}
\leq \|l\| \left| X_E (\cdot) x - \sum_{i=1}^n X_{E_i} (\cdot) x \right|_{L^p(\Omega; X)} \\
\leq \|l\| \left( \bigcup_{k>n} E_k \right)^{1/p} \|x\|. 
\]

Since \( \mu(\Omega) < \infty \),

\[
\lim_{n \to \infty} \mu \left( \bigcup_{k>n} E_k \right)^{1/p} = 0 
\]

and so inequality \( 19.8.49 \) shows that

\[
\lim_{n \to \infty} \left| F(E) - \sum_{k=1}^n F(E_k) \right|_{X'} = 0. 
\]

To show \( |F| (\Omega) < \infty \), let \( \varepsilon > 0 \) be given, let \( \{H_1, \ldots, H_n\} \) be a partition of \( \Omega \), and let \( \|x_i\| \leq 1 \) be chosen in such a way that

\[
F(H_i) (x_i) > |F(H_i)| - \varepsilon/n. 
\]

Thus

\[
-\varepsilon + \sum_{i=1}^n |F(H_i)| < \sum_{i=1}^n l(X_{H_i} (\cdot) x_i) \leq \|l\| \left| \sum_{i=1}^n X_{H_i} (\cdot) x_i \right|_{L^p(\Omega; X)} \\
\leq \|l\| \left( \int_{\Omega} \sum_{i=1}^n X_{H_i} (s) d\mu \right)^{1/p} = \|l\| \|\mu(\Omega)\|^{1/p}. 
\]

Since \( \varepsilon > 0 \) was arbitrary,

\[
\sum_{i=1}^n |F(H_i)| < \|l\| \|\mu(\Omega)\|^{1/p}. 
\]

Since the partition was arbitrary, this shows \( |F| (\Omega) \leq \|l\| \|\mu(\Omega)\|^{1/p} \) and this proves the lemma.

Continuing with the proof of Theorem \( 19.8.3 \), note that

\[
F \ll \mu. 
\]
Since \( X' \) has the Radon Nikodym property, there exists \( g \in L^1(\Omega; X') \) such that
\[
F(E) = \int_E g(s) \, d\mu.
\]
Also, from the definition of \( F(E) \),
\[
l\left( \sum_{i=1}^n x_i \mathcal{X}_{E_i}(\cdot) \right) = \sum_{i=1}^n l(\mathcal{X}_{E_i}(\cdot) x_i)
\]
\[
= \sum_{i=1}^n F(E_i)(x_i) = \sum_{i=1}^n \int_{E_i} g(s)(x_i) \, d\mu.
\]
(19.8.50)

It follows from (19.8.50) that whenever \( h \) is a simple function,
\[
l(h) = \int_{\Omega} g(s)(h(s)) \, d\mu.
\]
(19.8.51)

Let
\[
G_n = \{ s : \|g(s)\|_{X'} \leq n \}
\]
and let
\[
j : L^p(G_n; X) \to L^p(\Omega; X)
\]
be given by
\[
jh(s) = \begin{cases} h(s) & \text{if } s \in G_n, \\ 0 & \text{if } s \notin G_n. \end{cases}
\]

Letting \( h \) be a simple function in \( L^p(G_n; X) \),
\[
j^*l(h) = l(jh) = \int_{G_n} g(s)(h(s)) \, d\mu.
\]
(19.8.52)

Since the simple functions are dense in \( L^p(G_n; X) \), and \( g \in L^{p'}(G_n; X') \), it follows (19.8.50) holds for all \( h \in L^p(G_n; X) \). By Theorem (19.8.2),
\[
\|g\|_{L^{p'}(G_n; X')} = \|j^*l\|_{(L^p(G_n; X))'} \leq \|l\|_{(L^p(\Omega; X))'}.
\]

By the monotone convergence theorem,
\[
\lim_{n \to \infty} \|g\|_{L^{p'}(G_n; X')} = \lim_{n \to \infty} \|g\|_{L^{p'}(\Omega; X')} \leq \|l\|_{(L^p(\Omega; X))'}.
\]

Therefore \( g \in L^{p'}(\Omega; X') \) and since simple functions are dense in \( L^p(\Omega; X) \), (19.8.50) holds for all \( h \in L^p(\Omega; X) \). Thus \( l = \theta g \) and the theorem is proved because, by Theorem (19.8.2), \( \|l\| = \|g\| \) and the mapping \( \theta \) is onto because \( l \) was arbitrary.

As in the scalar case, everything generalizes to the case of \( \sigma \) finite measure spaces. The proof is almost identical.
Lemma 19.8.5 Let \((\Omega, \mathcal{S}, \mu)\) be a \(\sigma\) finite measure space and let \(X\) be a Banach space such that \(X'\) has the Radon Nikodym property. Then there exists a measurable function, \(r\) such that \(r(x) > 0\) for all \(x\), such that \(|r(x)| < M\) for all \(x\), and \(\int r \, d\mu < \infty\). For
\[
\Lambda \in (L^p(\Omega; X))', \quad p \geq 1,
\]
there exists a unique \(h \in L^p(\Omega; X')\), \(L^\infty(\Omega; X')\) if \(p = 1\) such that
\[
\Lambda f = \int h(f) \, d\mu.
\]
Also \(||h|| = ||\Lambda||. (||h|| = ||h||_{p'} if \(p > 1, ||h||_\infty\) if \(p = 1\)). Here
\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

**Proof:** First suppose \(r\) exists as described. Also, to save on notation and to emphasize the similarity with the scalar case, denote the norm in the various spaces by \(|·|\). Define a new measure \(\tilde{\mu}\), according to the rule
\[
\tilde{\mu}(E) \equiv \int_E r \, d\mu. \quad (19.8.53)
\]
Thus \(\tilde{\mu}\) is a finite measure on \(\mathcal{S}\). Now define a mapping, \(\eta : L^p(\Omega; X, \mu) \rightarrow L^p(\Omega; X, \tilde{\mu})\) by
\[
\eta f = r^{-\frac{1}{p}} f.
\]
Then
\[
||\eta f||_{L^p(\tilde{\mu})}^p = \int \left| r^{-\frac{1}{p}} f \right|^p r \, d\mu = ||f||_{L^p(\mu)}^p
\]
and so \(\eta\) is one to one and in fact preserves norms. I claim that also \(\eta\) is onto. To see this, let \(g \in L^p(\Omega; X, \tilde{\mu})\) and consider the function, \(r^{\frac{1}{p}} g\). Then
\[
\int |r^{\frac{1}{p}} g|^p d\mu = \int |g|^p r \, d\mu = \int |g|^p d\tilde{\mu} < \infty
\]
Thus \(r^{\frac{1}{p}} g \in L^p(\Omega; X, \mu)\) and \(\eta \left( r^{\frac{1}{p}} g \right) = g\) showing that \(\eta\) is onto as claimed. Thus \(\eta\) is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which \(\eta^*\) must be one to one and onto.

\[
\begin{array}{ccc}
h, L^p(\tilde{\mu}) & \xrightarrow{\eta^*} & L^p(\mu) \\
L^p(\tilde{\mu}'), \tilde{\Lambda} & \xrightarrow{\eta} & L^p(\mu'), \Lambda \\
L^p(\tilde{\mu}) & \leftarrow & L^p(\mu)
\end{array}
\]

Then for \(\Lambda \in L^p(\mu)\), there exists a unique \(\tilde{\Lambda} \in L^p(\tilde{\mu})\) such that \(\eta^* \tilde{\Lambda} = \Lambda, ||\tilde{\Lambda}|| = ||\Lambda||.\) By the Riesz representation theorem for finite measure spaces, there exists
19.8. THE RIESZ REPRESENTATION THEOREM

a unique \( h \in L^p(\mu) \equiv L^p(\Omega; X', \mu) \) which represents \( \Lambda \) in the manner described in the Riesz representation theorem. Thus \( ||h||_{L^p(\mu)} = ||\Lambda|| = ||\Lambda|| \) and for all \( f \in L^p(\mu) \),

\[
\Lambda(f) = \eta^*\Lambda(f) \equiv \Lambda(\eta f) = \int h(\eta f) \, d\mu = \int rh(\eta f) \, d\mu = \int r^{\frac{1}{p'}} h f \, d\mu.
\]

Now

\[
\int |r^{\frac{1}{p'}} h|^p' \, d\mu = \int |h|^p' \, d\mu = ||h||_{L^p(\mu)} < \infty.
\]

Thus \( ||r^{\frac{1}{p'}} h||_{L^p(\mu)} = ||h||_{L^p(\mu)} = ||\Lambda|| = ||\Lambda|| \) and represents \( \Lambda \) in the appropriate way. If \( p = 1 \), then \( 1/p' \equiv 0 \). Now consider the existence of \( r \). Since the measure space is \( \sigma \) finite, there exist \( \{\Omega_n\} \) disjoint, each having positive measure and their union equals \( \Omega \). Then define

\[
r(\omega) = \sum_{n=1}^{\infty} \frac{1}{n^2} \mu(\Omega_n)^{-1} \chi_{\Omega_n}(\omega)
\]

This proves the Lemma.

**Theorem 19.8.6 (Riesz representation theorem)** Let \( (\Omega, S, \mu) \) be \( \sigma \) finite and let \( X' \) have the Radon Nikodym property. Then for

\[
\Lambda \in (L^p(\Omega; X, \mu))', \ p \geq 1
\]

there exists a unique \( h \in L^q(\Omega, X', \mu) \), \( L^\infty(\Omega, X', \mu) \) if \( p = 1 \) such that

\[
\Lambda f = \int h(f) \, d\mu.
\]

Also \( ||h|| = ||\Lambda|| \). (\( ||h|| = ||h||_q \) if \( p > 1 \), \( ||h||_\infty \) if \( p = 1 \)). Here

\[
\frac{1}{p} + \frac{1}{q} = 1.
\]

**Proof:** The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

**Corollary 19.8.7** If \( X' \) is separable, then for \( (\Omega, S, \mu) \) a \( \sigma \) finite measure space,

\[
(L^p(\Omega; X))' \cong L^{p'}(\Omega; X').
\]

**Corollary 19.8.8** If \( X \) is separable and reflexive, then for \( (\Omega, S, \mu) \) a \( \sigma \) finite measure space,

\[
(L^p(\Omega; X))' \cong L^{p'}(\Omega; X').
\]
Corollary 19.8.9 If $X$ is separable and reflexive and $(\Omega, S, \mu)$ a $\sigma$ finite measure space, then if $p \in (1, \infty)$, then $L^p(\Omega; X)$ is reflexive.

Proof: This is just like the scalar valued case.

Here is an interesting example. Obviously $L^\infty(0, T, H)$ is not separable with the normed topology. However, bounded sets turn out to be metric spaces which are complete and separable. This is the next lemma. A Polish space is a complete separable metric space. In this example, $H$ is a separable real Hilbert space or more generally a separable real Banach space.

Lemma 19.8.10 Let $B = B(0, L)$ be a closed ball in $L^\infty(0, T, H)$. Then $B$ is a Polish space with respect to the weak $*$ topology. The closure is taken with respect to the usual topology.

Proof: Let $\{z_k\}_{k=1}^\infty = X$ be a dense countable subspace in $L^1(0, T, H)$, $B$. You start with a dense countable set and then consider all finite linear combinations having coefficients in $Q$. Then the metric on $B$ is

$$d(f, g) = \sum_{k=1}^\infty 2^{-k} \frac{|\langle f - g, z_k \rangle_{L^\infty, L^1}|}{1 + |\langle f - g, z_k \rangle_{L^\infty, L^1}|}$$

is $B$ complete? Suppose you have a Cauchy sequence $\{f_n\}$. This happens if and only if $\{\langle f_n, z_k \rangle\}_{n=1}^\infty$ is a Cauchy sequence for each $k$. Therefore, there exists $\xi(z_k) = \lim_{n \to \infty} \langle f_n, z_k \rangle$. Then for $a, b \in Q$, and $z, w \in X$

$$\xi(az + bw) = \lim_{n \to \infty} \langle f_n, az + bw \rangle = \lim_{n \to \infty} a \langle f_n, z \rangle + b \langle f_n, w \rangle = a\xi(z) + b\xi(w)$$

showing that $\xi$ is linear on $X$ a dense subspace of $L^1(0, T, H)$. Is $\xi$ bounded on this dense subspace with bound $L$? For $z \in X$,

$$|\xi(z)| \equiv \lim_{n \to \infty} |\langle f_n, z \rangle| \leq \lim_{n \to \infty} \|f_n\|_{L^\infty} \|z\|_{L^1} \leq L \|z\|_{L^1}$$

Hence $\xi$ is also bounded on this dense subset of $L^1(0, T, H)$. Therefore, there is a unique bounded linear extension of $\xi$ to all of $L^1(0, T, H)'$ denoted as $\xi$ such that its norm in $L^1(0, T, H)'$ is no larger than $L$. It follows from the Riesz representation theorem that there exists a unique $f \in L^\infty(0, T, H)$ such that for all $w \in L^1(0, T, H)$, $\xi(w) = \langle f, w \rangle$ and $\|f\| \leq L$. This $f$ is the limit of the Cauchy sequence $\{f_n\}$ in $B$. Thus $B$ is complete.

Is $B$ separable? Let $f \in B$. Let $\varepsilon > 0$ be given. Choose $M$ such that

$$\sum_{k=M+1}^\infty 2^{-k} < \frac{\varepsilon}{4}$$

Then the finite set $\{z_1, \ldots, z_M\}$ is uniformly integrable. There exists $\delta > 0$ such that if $m(S) < \delta$, then

$$\int_S |z_k|_H dm \leq \left( \frac{\varepsilon}{4(1 + \|f\|_{L^\infty})} \right)$$
Then there is a sequence of simple functions \( \{ s_n \} \) which converge uniformly to \( f \) off a set of measure zero, \( N, \| s_n \|_{L^\infty} \leq \| f \|_{L^\infty} \). By regularity of the measure, there exists a continuous function with compact support \( h_n \) such that \( s_n = h_n \) off a set of measure no more than \( \frac{\delta}{2^n} \) and also \( \| h_n \|_{L^\infty} \leq \| f \|_{L^\infty} \). Then off a set of measure no more than \( \frac{1}{3} \delta \), \( h_n (r) \to f (r) \). Now by Egorov's theorem and outer regularity, one can enlarge this exceptional set to obtain an open set \( S \) of measure no more than \( \delta/2 \) such that the convergence is uniform off this exceptional set. Thus \( f \) equals the uniform limit of continuous functions on \( S^C \). Define

\[
h (r) = \begin{cases} 
\lim_{n \to \infty} h_n (r) = f (r) & \text{on } S^C \\
0 & \text{on } S \setminus N \\
0 & \text{on } N
\end{cases}
\]

Then \( \| h \|_{L^\infty} \leq \| f \|_{L^\infty} \). Now consider \( \tilde{h} \ast \psi_m (r) \) where \( \psi_r \) is approximate identity.

\[
\psi_m (t) = \frac{1}{2} m \chi_{[-1/m, 1/m]} (t), \quad \tilde{h} \ast \psi_m (t) = \frac{1}{2} m \int_{-1/m}^{1/m} \tilde{h} (t-s) \, ds = \frac{1}{2} m \int_{-1/m}^{t+1/m} \tilde{h} (s) \, ds
\]

where we define \( \tilde{h} \) to be the 0 extension of \( h \) off \( [0, T] \). This is a continuous function of \( t \). Also \( a.e.t \) is a Lebesgue point and so for \( a.e.t \),

\[
\left| \frac{1}{2} m \int_{-1/m}^{t+1/m} \tilde{h} (s) \, ds - \tilde{h} (t) \right| \to 0
\]

\[
| \tilde{h} \ast \psi_m (r) | = \left| \int_{\mathbb{R}} \tilde{h} (r-s) \psi_m (s) \, ds \right| \leq \| h \|_{L^\infty} \leq \| f \|_{L^\infty}
\]

Thus this continuous function is in \( L^\infty (0, T, H) \). Letting \( z = z_k \in L^1 (0, T, H) \) be one of those defined above,

\[
\left| \int_0^T \langle \tilde{h} \ast \psi_m (t) - f (t), z (t) \rangle \, dt \right| \leq \int_0^T \left| \langle \tilde{h} \ast \psi_m (t) - h (t), z (t) \rangle \right| \, dt
\]

\[
+ \int_0^T \| (h (t) - f (t), z (t)) \| \, dt \quad (19.8.54)
\]

for \( a.e. \ t, \tilde{h} \ast \psi_m (t) - h (t) \to 0 \) and the integrand in the first integral is bounded by \( 2 \| f \|_{L^\infty} \| z (t) \|_H \) so by the dominated convergence theorem, as \( m \to \infty \), the first integral converges to 0. As to the second, it is dominated by

\[
\int_S \| (h (t) - f (t), z (t)) \| \, dt \leq 2 \| f \|_{L^\infty} \int_S \| z (t) \| \, dt < \frac{2 \| f \|_{L^\infty} \varepsilon}{4 (1 + \| f \|_{L^\infty})} \leq \frac{\varepsilon}{2}
\]
Therefore, choosing $m$ large enough so that the first integral on the right in is less than $\frac{\varepsilon}{4}$ for each $z_k$ for $k \leq M$, then for each of these,

$$ d(f, h_{\psi_m}) \leq \frac{\varepsilon}{4} + \sum_{k=1}^{M} 2^{-k} \left( \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \right) = \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{4} $$

which appears to show that $C ([0, T], H)$ is weak * dense in $L^\infty (0, T, H)$. However, this last space is obviously separable in terms of the norm topology. Let $D$ be a countable dense subset of $C ([0, T], H)$. For $f \in L^\infty (0, T, H)$ let $g \in C ([0, T], H)$ such that $d(f, g) < \frac{\varepsilon}{4}$. Then let $h \in D$ be so close to $g$ in $C ([0, T], H)$ that

$$ \frac{1}{2} \sum_{k=1}^{M} 2^{-k} \left| \langle h - g, z_k \rangle_{L^\infty, L^1} \right| < \frac{\varepsilon}{2} $$

Then

$$ d(f, h) \leq d(f, g) + d(g, h) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon $$

It appears that $D$ is dense in $B$ in the weak * topology. $lacksquare$

19.9 Pointwise Behavior Of Weakly Convergent Sequences

There is an interesting little result which relates to weak limits in $L^2 (\Gamma, E)$ for $E$ a Banach space. I am not sure where to put this thing but think that this would be a good place for it. It obviously generalizes to $L^p$ spaces.

Proposition 19.9.1 Let $E$ be a Banach space and let $\{u_n\}$ be a sequence in $L^2 (\Gamma, E)$ and let $G(x)$ be a weakly compact set in $E$, and $u_n (x) \in G(x)$ a.e. for each $n$. Let $\limsup \{u_n (x)\}$ denote the set of all weak limits of subsequences of $\{u_n (x)\}$ and let $H(x)$ be the closure of the convex hull of $\limsup \{u_n (x)\}$. Then if $u_n \rightarrow u$ weakly in $L^2 (\Gamma, E)$, then $u (x) \in H(x)$ for a.e. $x$.

Proof: Let $H = \{w \in L^2 (\Gamma, E) : w (x) \in H(x)$ a.e.}. Then $H$ is convex. If you have $w_i \in H$, then since each $H(x)$ is convex, it follows that $\lambda w_1 (x) + (1 - \lambda) w_2 (x) \in H$ for a.e. $x$ and $\lambda \in [0, 1]$. Is $H$ closed? Suppose you have $w_n \in H$ and $w_n \rightarrow w$ in $L^2 (\Gamma, E)$. Then there is a subsequence such that pointwise convergence happens a.e. and so since $H$ is closed, you have $w (x) \in H$ for a.e. $x$. Hence $H$ is also weakly closed in $L^2 (\Gamma, H)$. Thus if $u$ is the weak limit of $\{u_n\}$ in $L^2 (\Gamma, E)$, it must be the case that $u (x) \in H (x)$ a.e. $lacksquare$

As a case of this which might be pretty interesting, suppose $G(x)$ is not just weakly compact but also convex. Then $H(x) = G(x)$ and you can say that $u (x) \in H (x)$ a.e. whenever it is a weak limit in $L^2 (\Gamma, E)$ of functions $u_n$ for which $u_n (x) \in G(x)$. 


19.10. Exercises

1. Show $L^1(\mathbb{R})$ is not reflexive. **Hint:** $L^1(\mathbb{R})$ is separable. What about $L^\infty(\mathbb{R})$?

2. If $f \in L^1(\mathbb{R}^n; X)$ for $X$ a Banach space, does the usual fundamental theorem of calculus work? That is, can you say $\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(t) \, dm = f(x)$ a.e.? If so, give a statement of the appropriate theorem and a proof.

3. Does the Vitali convergence theorem hold for Bochner integrable functions? If so, give a statement of the appropriate theorem and a proof.

4. Suppose $g \in L^1([a,b]; X)$ where $X$ is a Banach space. Then if $\int_a^b g(t) \, dt = 0$ for all $\phi \in C_c^\infty(a,b)$, then $g(t) = 0$ a.e. Show that this is the case. **Hint:** It will likely depend on the regularity properties of Lebesgue measure.

5. Suppose $f \in L^1(a,b; X)$ and for all $\phi \in C_c^\infty(a,b)$, $\int_a^b f(t) \phi'(t) \, dt = 0$. Then there exists a constant, $a \in X$ such that $f(t) = a$ a.e. **Hint:** Let

$$
\psi_\phi(x) = \int_a^x \phi(t) - \left( \int_a^b \phi(y) \, dy \right) \phi_0(t) \, dt, \quad \phi_0 \in C_c^\infty(a,b), \quad \int_a^b \phi_0(x) \, dx = 1
$$

Then explain why $\psi_\phi \in C_c^\infty(a,b)$, $\psi_\phi' = \phi - \left( \int_a^b \phi(y) \, dy \right) \phi_0$. Then use the assumption on $\psi_\phi$. Next use the above problem. Verify that

$$
f(y) = \int_a^b f(t) \phi_0(t) \, dt \quad \text{a.e.} \quad y
$$

6. Let $f \in L^1([a,b], X)$. Then we say that the weak derivative of $f$ is in $L^1([a,b], X)$ if there is a function denoted as $f' \in L^1([a,b], X)$ such that for all $\phi \in C_c^\infty(a,b)$,

$$
- \int_a^b f(t) \phi'(t) \, dt = \int_a^b f'(t) \phi(t) \, dt
$$

Show that this definition is well defined. Next, using the above problems, show that if $f, f' \in L^1([a,b], X)$, it follows that there is a continuous function, denoted by $t \to \hat{f}(t)$ such that $\hat{f}(t) = f(t)$ a.e. $t$ and

$$
\hat{f}(t) = \hat{f}(a) + \int_a^t f'(s) \, ds
$$

Thus, unlike the classical definition of the derivative, when a function and its derivative are both in $L^1$, it has a representative $\hat{f}$ which equals the function a.e. such that $\hat{f}$ can be recovered from its derivative. Recall the well known example of this not working out which is based on the Cantor function which you should see in a real analysis course. This function had zero derivative a.e. and yet it climbed from 0 to 1 on the unit interval. Thus one could not recover it from integrating its classical derivative.
Chapter 20

The Derivative

20.1 Limits Of A Function

As in the case of scalar valued functions of one variable, a concept closely related to continuity is that of the limit of a function. The notion of limit of a function makes sense at points \( x \), which are limit points of \( D(f) \) and this concept is defined next. In all that follows \((V, \|\cdot\|)\) and \((W, \|\cdot\|)\) are two normed linear spaces. Recall the definition of limit point first.

**Definition 20.1.1** Let \( A \subseteq W \) be a set. A point \( x \), is a limit point of \( A \) if \( B(x, r) \) contains infinitely many points of \( A \) for every \( r > 0 \).

**Definition 20.1.2** Let \( f : D(f) \subseteq V \rightarrow W \) be a function and let \( x \) be a limit point of \( D(f) \). Then

\[
\lim_{y \to x} f(y) = L
\]

if and only if the following condition holds. For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if

\[
0 < \|y - x\| < \delta, \text{ and } y \in D(f)
\]

then,

\[
\|L - f(y)\| < \varepsilon.
\]

**Theorem 20.1.3** If \( \lim_{y \to x} f(y) = L \) and \( \lim_{y \to x} f(y) = L_1 \), then \( L = L_1 \).

**Proof:** Let \( \varepsilon > 0 \) be given. There exists \( \delta > 0 \) such that if \( 0 < |y - x| < \delta \) and \( y \in D(f) \), then

\[
\|f(y) - L\| < \varepsilon, \|f(y) - L_1\| < \varepsilon.
\]

Pick such a \( y \). There exists one because \( x \) is a limit point of \( D(f) \). Then

\[
\|L - L_1\| \leq \|L - f(y)\| + \|f(y) - L_1\| < \varepsilon + \varepsilon = 2\varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, this shows \( L = L_1 \). 

As in the case of functions of one variable, one can define what it means for \( \lim_{y \to x} f(x) = \pm\infty \).
Definition 20.1.4 If $f(x) \in \mathbb{R}$, \( \lim_{y \to x} f(x) = \infty \) if for every number \( l \), there exists \( \delta > 0 \) such that whenever \( \|y - x\| < \delta \) and \( y \in D(f) \), then \( f(x) > l \).

\( \lim_{y \to x} f(x) = -\infty \) if for every number \( l \), there exists \( \delta > 0 \) such that whenever \( \|y - x\| < \delta \) and \( y \in D(f) \), then \( f(x) < l \).

The following theorem is just like the one variable version of calculus.

Theorem 20.1.5 Suppose \( f : D(f) \subseteq V \to \mathbb{F}^m \). Then for \( x \) a limit point of \( D(f) \),

\[
\lim_{y \to x} f(y) = L \tag{20.1.1}
\]

if and only if

\[
\lim_{y \to x} f_k(y) = L_k \tag{20.1.2}
\]

where \( f(y) = (f_1(y), \cdots, f_p(y)) \) and \( L = (L_1, \cdots, L_p) \).

Suppose here that \( f \) has values in \( W \), a normed linear space and

\[
\lim_{y \to x} f(y) = L, \quad \lim_{y \to x} g(y) = K
\]

where \( K, L \in W \). Then if \( a, b \in \mathbb{F} \),

\[
\lim_{y \to x} (af(y) + bg(y)) = aL + bK, \tag{20.1.3}
\]

If \( W \) is an inner product space,

\[
\lim_{y \to x} (f, g)(y) = (L, K) \tag{20.1.4}
\]

If \( g \) is scalar valued with \( \lim_{y \to x} g(y) = K \),

\[
\lim_{y \to x} f(y)g(y) = LK. \tag{20.1.5}
\]

Also, if \( h \) is a continuous function defined near \( L \), then

\[
\lim_{y \to x} h \circ f(y) = h(L). \tag{20.1.6}
\]

Suppose \( \lim_{y \to x} f(y) = L \). If \( \|f(y) - b\| \leq r \) for all \( y \) sufficiently close to \( x \), then \( |L - b| \leq r \) also.

Proof: Suppose 20.1.1. Then letting \( \varepsilon > 0 \) be given there exists \( \delta > 0 \) such that if \( 0 < \|y - x\| < \delta \), it follows

\[
|f_k(y) - L_k| \leq \|f(y) - L\| < \varepsilon
\]

which verifies 20.1.2.

Now suppose 20.1.2 holds. Then letting \( \varepsilon > 0 \) be given, there exists \( \delta_k \) such that if \( 0 < \|y - x\| < \delta_k \), then

\[
|f_k(y) - L_k| < \varepsilon.
\]
Let $0 < \delta < \min (\delta_1, \cdots, \delta_p)$. Then if $0 < \|y-x\| < \delta$, it follows
\[
\| f(y) - L \|_\infty < \varepsilon
\]
Any other norm on $\mathbb{F}^m$ would work out the same way because the norms are all equivalent.

Each of the remaining assertions follows immediately from the coordinate descriptions of the various expressions and the first part. However, I will give a different argument for these.

The proof of 20.1.3 is left for you. Now 20.1.4 is to be verified. Let $\varepsilon > 0$ be given. Then by the triangle inequality,
\[
| (f,g)(y) - (L,K) | \leq | (f,g)(y) - (f(y),K) | + | (f(y),K) - (L,K) | \\
\leq \| f(y) \| \| g(y) - K \| + \| K \| \| f(y) - L \|.
\]
There exists $\delta_1$ such that if $0 < \|y-x\| < \delta_1$ and $y \in D(f)$, then
\[
\| f(y) - L \| < 1,
\]
and so for such $y$, the triangle inequality implies, $\| f(y) \| < 1 + \| L \|$. Therefore, for $0 < \|y-x\| < \delta_1$,
\[
| (f,g)(y) - (L,K) | \leq (1 + \| K \| + \| L \|) \| g(y) - K \| + \| f(y) - L \|.
\]
Now let $0 < \delta_2$ be such that if $y \in D(f)$ and $0 < \|x-y\| < \delta_2$,
\[
\| f(y) - L \| < \frac{\varepsilon}{2 (1 + \| K \| + \| L \|)}, \quad \| g(y) - K \| < \frac{\varepsilon}{2 (1 + \| K \| + \| L \|)}.
\]
Then letting $0 < \delta \leq \min (\delta_1, \delta_2)$, it follows from 20.1.6 that
\[
| (f,g)(y) - (L,K) | < \varepsilon
\]
and this proves 20.1.4.

The proof of 20.1.5 is left to you.

Consider 20.1.6. Since $h$ is continuous near $L$, it follows that for $\varepsilon > 0$ given, there exists $\eta > 0$ such that if $\|y-L\| < \eta$, then
\[
\| h(y) - h(L) \| < \varepsilon
\]
Now since $\lim_{y \to x} f(y) = L$, there exists $\delta > 0$ such that if $0 < \|y-x\| < \delta$, then
\[
\| f(y) - L \| < \eta.
\]
Therefore, if $0 < \|y-x\| < \delta$,
\[
\| h(f(y)) - h(L) \| < \varepsilon.
\]

It only remains to verify the last assertion. Assume $\| f(y) - b \| \leq r$. It is required to show that $\| L - b \| \leq r$. If this is not true, then $\| L - b \| > r$. Consider
For $f$ is continuous at $x = 0$ and so whenever $x$ is close enough to $x$. Thus, by the triangle inequality,

$$\|f(y) - L\| < \|L - b\| - r$$

and so

$$r < \|L - b\| - \|f(y) - L\| \leq \|b - L\| - \|f(y) - L\|$$

a contradiction to the assumption that $\|b - f(y)\| \leq r$. $\blacksquare$

The relation between continuity and limits is as follows.

**Theorem 20.1.6** For $f : D(f) \to W$ and $x \in D(f)$ a limit point of $D(f)$, $f$ is continuous at $x$ if and only if

$$\lim_{y \to x} f(y) = f(x).$$

**Proof:** First suppose $f$ is continuous at $x$ a limit point of $D(f)$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x - y\| < \delta$ and $y \in D(f)$, then $|f(x) - f(y)| < \varepsilon$. In particular, this holds if $0 < \|x - y\| < \delta$ and this is just the definition of the limit. Hence $f(x) = \lim_{y \to x} f(y)$.

Next suppose $x$ is a limit point of $D(f)$ and $\lim_{y \to x} f(y) = f(x)$. This means that if $\varepsilon > 0$ there exists $\delta > 0$ such that for $0 < \|x - y\| < \delta$ and $y \in D(f)$, it follows $|f(y) - f(x)| < \varepsilon$. However, if $y = x$, then $|f(y) - f(x)| = |f(x) - f(x)| = 0$ and so whenever $y \in D(f)$ and $\|x - y\| < \delta$, it follows $|f(x) - f(y)| < \varepsilon$, showing $f$ is continuous at $x$. $\blacksquare$

**Example 20.1.7** Find $\lim_{(x,y) \to (3,1)} \left( \frac{x^2 - 9}{x - 3}, y \right)$.

It is clear that $\lim_{(x,y) \to (3,1)} \frac{x^2 - 9}{x - 3} = 6$ and $\lim_{(x,y) \to (3,1)} y = 1$. Therefore, this limit equals $(6, 1)$.

**Example 20.1.8** Find $\lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2}$.

First of all, observe the domain of the function is $\mathbb{R}^2 \setminus \{(0,0)\}$, every point in $\mathbb{R}^2$ except the origin. Therefore, $(0,0)$ is a limit point of the domain of the function so it might make sense to take a limit. However, just as in the case of a function of one variable, the limit may not exist. In fact, this is the case here. To see this, take points on the line $y = 0$. At these points, the value of the function equals 0. Now consider points on the line $y = x$ where the value of the function equals 1/2. Since, arbitrarily close to $(0,0)$, there are points where the function equals 1/2 and points where the function has the value 0, it follows there can be no limit. Just take $\varepsilon = 1/10$ for example. You cannot be within 1/10 of 1/2 and also within 1/10 of 0 at the same time.

Note it is necessary to rely on the definition of the limit much more than in the case of a function of one variable and there are no easy ways to do limit problems for functions of more than one variable. It is what it is and you will not deal with these concepts without suffering and anguish.
20.2 Basic Definitions

The concept of derivative generalizes right away to functions defined on a normed linear space. However, no attempt will be made to consider derivatives from one side or another. This is because there isn’t a well defined side. However, it is certainly the case that there are more general notions which include such things. I will present a fairly general notion of the derivative of a function which is defined on a normed vector space which has values in a normed vector space.

In what follows, $X, Y$ will denote normed vector spaces. Recall that $\mathcal{L}(X,Y)$ will denote the bounded linear transformations from $X$ to $Y$.

Let $U$ be an open set in $X$, and let $f : U \to Y$ be a function.

**Definition 20.2.1** A function $g$ is $o(v)$ if

$$\lim_{||v||\to 0} \frac{g(v)}{||v||} = 0$$

(20.2.8)

A function $f : U \to Y$ is differentiable at $x \in U$ if there exists a linear transformation $L \in \mathcal{L}(X,Y)$ such that

$$f(x + v) = f(x) + Lv + o(v)$$

This linear transformation $L$ is the definition of $Df(x)$. This derivative is often called the Frechet derivative.

In finite dimensions, the question whether a given function is differentiable is independent of the norm used on the finite dimensional vector space. That is, a function is differentiable with one norm if and only if it is differentiable with another norm. This is because all norms are equivalent on a finite dimensional space.

The definition 20.2.8 means the error,

$$f(x + v) - f(x) - Lv$$

converges to 0 faster than $||v||$. Thus the above definition is equivalent to saying

$$\lim_{||v||\to 0} \frac{||f(x + v) - f(x) - Lv||}{||v||} = 0$$

(20.2.9)

or equivalently,

$$\lim_{y \to x} \frac{||f(y) - f(x) - Df(x)(y - x)||}{||y - x||} = 0.$$  

(20.2.10)

The symbol $o(v)$ should be thought of as an adjective. Thus, if $t$ and $k$ are constants,

$$o(v) = o(v) + o(v), \quad o(tv) = o(v), \quad ko(v) = o(v)$$

and other similar observations hold.

**Theorem 20.2.2** The derivative is well defined.
Proof: First note that for a fixed vector \( v \), \( o(tv) = o(t) \). This is because

\[
\lim_{t \to 0} \frac{o(tv)}{|t|} = \lim_{t \to 0} \frac{||tv||}{|t|} = 0
\]

Now suppose both \( L_1 \) and \( L_2 \) work in the above definition. Then let \( v \) be any vector and let \( t \) be a real scalar which is chosen small enough that \( tv + x \in U \). Then

\[
f(x + tv) = f(x) + L_1 tv + o(tv), \quad f(x + tv) = f(x) + L_2 tv + o(tv).
\]

Therefore, subtracting these two yields \( (L_2 - L_1)(tv) = o(tv) = o(t) \). Therefore, dividing by \( t \) yields \( (L_2 - L_1)(v) = \frac{o(t)}{t} \). Now let \( t \to 0 \) to conclude that \( (L_2 - L_1)(v) = 0 \). Since this is true for all \( v \), it follows \( L_2 = L_1 \). This proves the theorem. ■

Lemma 20.2.3 Let \( f \) be differentiable at \( x \). Then \( f \) is continuous at \( x \) and in fact, there exists \( K > 0 \) such that whenever \( ||v|| \) is small enough,

\[
||f(x + v) - f(x)|| \leq K ||v||
\]

Also if \( f \) is differentiable at \( x \), then

\[
o(||f(x + v) - f(x)||) = o(||v||)
\]

Proof: From the definition of the derivative,

\[
f(x + v) - f(x) = Df(x)v + o(v).
\]

Let \( ||v|| \) be small enough that \( \frac{o(||v||)}{||v||} < 1 \) so that \( ||o(v)|| \leq ||v|| \). Then for such \( v \),

\[
||f(x + v) - f(x)|| \leq ||Df(x)v|| + ||v||
\]

\[
\leq (||Df(x)|| + 1) ||v||
\]

This proves the lemma with \( K = ||Df(x)|| + 1 \). Recall the operator norm discussed in Definition 6.1.13.

The last assertion is implied by the first as follows. Define

\[
h(v) \equiv \begin{cases} 
  \frac{o(||f(x + v) - f(x)||)}{||f(x + v) - f(x)||} & \text{if} \ ||f(x + v) - f(x)|| \neq 0 \\
  0 & \text{if} \ ||f(x + v) - f(x)|| = 0
\end{cases}
\]

Then \( \lim_{||v|| \to 0} h(v) = 0 \) from continuity of \( f \) at \( x \) which is implied by the first part. Also from the above estimate,

\[
\frac{o(||f(x + v) - f(x)||)}{||v||} = \frac{h(v)}{||v||} \frac{||f(x + v) - f(x)||}{||v||} \leq h(v) (||Df(x)|| + 1)
\]

This establishes the second claim. ■

Here \( ||Df(x)|| \) is the operator norm of the linear transformation \( Df(x) \).
20.3 The Chain Rule

With the above lemma, it is easy to prove the chain rule.

**Theorem 20.3.1** (The chain rule) Let \( U \) and \( V \) be open sets \( U \subseteq X \) and \( V \subseteq Y \). Suppose \( f : U \to V \) is differentiable at \( x \in U \) and suppose \( g : V \to \mathbb{F}^q \) is differentiable at \( f(x) \in V \). Then \( g \circ f \) is differentiable at \( x \) and

\[
D(g \circ f)(x) = Dg(f(x))Df(x).
\]

**Proof:** This follows from a computation. Let \( B(x,r) \subseteq U \) and let \( r \) also be small enough that for \( ||v|| \leq r \), it follows that \( f(x + v) \in V \). Such an \( r \) exists because \( f \) is continuous at \( x \). For \( ||v|| < r \), the definition of differentiability of \( g \) and \( f \) implies

\[
g(f(x + v)) - g(f(x)) = Dg(f(x))(f(x + v) - f(x)) + o(f(x + v) - f(x))
\]

\[
= Dg(f(x))[Df(x)v + o(v)] + o(f(x + v) - f(x))
\]

\[
= Dg(f(x))Df(x)v + o(v) + o(f(x + v) - f(x))
\]

(20.3.11)

By Lemma From the definition of the derivative \( D(g \circ f)(x) \) exists and equals \( Dg(f(x))Df(x) \). \( \blacksquare \)

20.4 The Derivative Of A Compact Mapping

Here is a little definition about compact mappings. It turns out that if you have a differentiable mapping which is also compact, then the derivative must also be compact.

**Definition 20.4.1** Let \( C \in \mathcal{L}(X,Y) \). It is said to be compact if it takes bounded sets to precompact sets. If \( f \) is a function defined on an open subset \( U \) of \( X \), then \( f \) is called compact if \( f \) (bounded set) = (precompact).

**Theorem 20.4.2** Let \( f : U \subseteq X \to Y \) where \( f \) takes bounded sets to precompact sets. Then \( Df(x) \) also takes bounded sets in \( X \) to precompact sets in \( Y \).

**Proof:** If this is not so, then there exists a bounded set \( B \) in \( X \) and for some \( \varepsilon > 0 \) a sequence of points \( Df(x)h_n \) such that all these points are further apart than \( \varepsilon \). Without loss of generality, one can assume \( B = B(0,r) \), a ball. In fact, one can assume that \( r > 0 \) is as small as desired because if \( Df(x)B(0,r) \) is precompact, then so is \( Df(x)B(0,R) \), \( R > r \). Just get an \( \varepsilon \frac{R}{r} \) net \( \{Df(x)x_n\}_{n=1}^N \) for \( Df(x)B(0,r) \) and consider \( \{\frac{R}{r}Df(x)x_n\}_{n=1}^N \cup_n B(Df(x)x_n,\varepsilon \frac{R}{r}) \) covers \( Df(x)B(0, R) \) and so \( \cup_n B(\frac{R}{r}Df(x)x_n,\varepsilon) \) covers \( Df(x)B(0, R) \).
Choose \( r \) very small so that \( r < \varepsilon / 4 \) and
\[
f(x + x_n) - f(x) = Df(x)x_n + o(x_n), \quad \|o(x_n)\| < \|x_n\|
\]
and there are infinitely many \( Df(x)x_n \) further apart than \( \varepsilon, x_n \in B(0, r) \). Then consider \( B(x, r) \) and \( \{ f(x + x_n) \}_{n=1}^{\infty} \).
\[
\|f(x + x_n) - f(x + x_m)\| \geq \|Df(x)x_n - Df(x)x_m\| - \|o(x_n) - o(x_m)\|
\]
\[
\geq \varepsilon - 2\frac{\varepsilon}{4} = \frac{\varepsilon}{2}
\]
contradicting the assertion that \( f \) takes bounded sets to precompact sets. \( \blacksquare \)

### 20.5 The Matrix Of The Derivative

The case of most interest here is the only one I will discuss. It is the case where \( X = \mathbb{R}^n \) and \( Y = \mathbb{R}^m \), the function being defined on an open subset of \( \mathbb{R}^n \). Of course this all generalizes to arbitrary vector spaces and one considers the matrix taken with respect to various bases. As above, \( f \) will be defined and differentiable on an open set \( U \subseteq \mathbb{R}^n \).

The matrix of \( Df(x) \) is the matrix having the \( i \)th column equal to \( Df(x)e_i \) and so it is only necessary to compute this. Let \( t \) be a small real number such that both
\[
\frac{f(x + te_i) - f(x) - Df(x)(te_i)}{t} = o(t)
\]
Therefore,
\[
\frac{f(x + te_i) - f(x)}{t} = Df(x)(e_i) + \frac{o(t)}{t}
\]
The limit exists on the right and so it exists on the left also. Thus
\[
\frac{\partial f(x)}{\partial x_i} \equiv \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t} = Df(x)(e_i)
\]
and so the matrix of the derivative is just the matrix which has the \( i \)th column equal to the \( i \)th partial derivative of \( f \). Note that this shows that whenever \( f \) is differentiable, it follows that the partial derivatives all exist. It does not go the other way however as discussed later.

**Theorem 20.5.1** Let \( f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \) and suppose \( f \) is differentiable at \( x \). Then all the partial derivatives \( \frac{\partial f_i(x)}{\partial x_j} \) exist and if \( Jf(x) \) is the matrix of the linear transformation, \( Df(x) \) with respect to the standard basis vectors, then the \( ij \)th entry is given by \( \frac{\partial f_i(x)}{\partial x_j} \) also denoted as \( f_{i,j} \) or \( f_{i,x_j} \). It is the matrix whose \( i \)th column is
\[
\frac{\partial f(x)}{\partial x_i} \equiv \lim_{t \to 0} \frac{f(x + te_i) - f(x)}{t}.
\]
Of course there is a generalization of this idea called the directional derivative.

**Definition 20.5.2** In general, the symbol

\[ D_v f(x) \]

is defined by

\[
\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}
\]

where \( t \in \mathbb{F} \). In case \( |v| = 1 \) and the norm is the standard Euclidean norm, this is called the directional derivative. More generally, with no restriction on the size of \( v \) and in any linear space, it is called the Gateaux derivative. \( f \) is said to be Gateaux differentiable at \( x \) if there exists \( D_v f(x) \) such that

\[
\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = D_v f(x)
\]

where \( v \to D_v f(x) \) is linear. Thus we say it is Gateaux differentiable if the Gateaux derivative exists for each \( v \) and \( v \to D_v f(x) \) is linear.

What if all the partial derivatives of \( f \) exist? Does it follow that \( f \) is differentiable? Consider the following function, \( f : \mathbb{R}^2 \to \mathbb{R}, \)

\[
f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}
\]

Then from the definition of partial derivatives,

\[
\lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

and

\[
\lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0
\]

However \( f \) is not even continuous at \((0, 0)\) which may be seen by considering the behavior of the function along the line \( y = x \) and along the line \( x = 0 \). By Lemma 20.2.3 this implies \( f \) is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

\footnote{René Gateaux was one of the many young French men killed in world war I. This derivative is named after him, but it developed naturally from ideas used in the calculus of variations which were due to Euler and Lagrange back in the 1700’s.}
20.6 A Mean Value Inequality

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem as is the next lemma.

**Lemma 20.6.1** Let \( Y \) be a normed vector space and suppose \( h : [0, 1] \to Y \) is differentiable and satisfies
\[
||h'(t)|| \leq M.
\]
Then
\[
||h(1) - h(0)|| \leq M.
\]

**Proof:** Let \( \varepsilon > 0 \) be given and let
\[
S \equiv \{ t \in [0, 1] : \text{ for all } s \in [0, t], ||h(s) - h(0)|| \leq (M + \varepsilon) s \}
\]
Then \( 0 \in S \). Let \( t = \sup S \). Then by continuity of \( h \) it follows
\[
||h(t) - h(0)|| = (M + \varepsilon) t \tag{20.6.12}
\]
Suppose \( t < 1 \). Then there exist positive numbers, \( h_k \) decreasing to 0 such that
\[
||h(t + h_k) - h(0)|| > (M + \varepsilon)(t + h_k)
\]
and now it follows from (20.6.12) and the triangle inequality that
\[
||h(t + h_k) - h(t)|| + ||h(t) - h(0)|| = ||h(t + h_k) - h(t)|| + (M + \varepsilon)t > (M + \varepsilon)(t + h_k)
\]
and so
\[
||h(t + h_k) - h(t)|| > (M + \varepsilon)h_k
\]
Now dividing by \( h_k \) and letting \( k \to \infty \)
\[
||h'(t)|| \geq M + \varepsilon,
\]
a contradiction. Thus \( t = 1 \). \( \blacksquare \)

**Theorem 20.6.2** Suppose \( U \) is an open subset of \( X \) and \( f : U \to Y \) has the property that \( Df(\mathbf{x}) \) exists for all \( \mathbf{x} \) in \( U \) and that, \( \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in U \) for all \( t \in [0, 1] \). (The line segment joining the two points lies in \( U \).) Suppose also that for all points on this line segment,
\[
||Df(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))|| \leq M.
\]
Then
\[
||f(\mathbf{y}) - f(\mathbf{x})|| \leq M ||\mathbf{y} - \mathbf{x}||.
\]
20.6. A MEAN VALUE INEQUALITY

Proof: Let
\[ h(t) = f(x + t(y - x)). \]
Then by the chain rule,
\[ h'(t) = Df(x + t(y - x))(y - x) \]
and so
\[ ||h'(t)|| = ||Df(x + t(y - x))(y - x)|| \leq M||y - x|| \]
by Lemma 20.6.1

\[ ||h(1) - h(0)|| = ||f(y) - f(x)|| \leq M||y - x||. \]

Here is a little result which will help to tie the case of \( \mathbb{R}^n \) in to the abstract theory presented for arbitrary spaces.

**Theorem 20.6.3** Let \( X \) be a normed vector space having basis \( \{v_1, \cdots, v_n\} \) and let \( Y \) be another normed vector space having basis \( \{w_1, \cdots, w_m\} \). Let \( U \) be an open set in \( X \) and let \( f: U \rightarrow Y \) have the property that the Gateaux derivatives,
\[ D_{v_k}f(x) = \lim_{t \to 0} \frac{f(x + tv_k) - f(x)}{t} \]
exist and are continuous functions of \( x \). Then \( Df(x) \) exists and
\[ Df(x)v = \sum_{k=1}^{n} D_{v_k}f(x)a_k \]
where
\[ v = \sum_{k=1}^{n} a_kv_k. \]
Furthermore, \( x \rightarrow Df(x) \) is continuous; that is
\[ \lim_{y \to x} ||Df(y) - Df(x)|| = 0. \]

Proof: Let \( v = \sum_{k=1}^{n} a_kv_k \). Then
\[ f(x + v) - f(x) = f\left(x + \sum_{k=1}^{n} a_kv_k\right) - f(x). \]
Then letting \( \sum_{k=1}^{0} \equiv 0, f(x + v) - f(x) \) is given by
\[ \sum_{k=1}^{n} \left[ f\left(x + \sum_{j=1}^{k} a_jv_j\right) - f\left(x + \sum_{j=1}^{k-1} a_jv_j\right) \right] \]
CHAPTER 20. THE DERIVATIVE

\[ = \sum_{k=1}^{n} [f(x + a_kv_k) - f(x)] + \]

\[ \sum_{k=1}^{n} \left[ \left( f(x + \sum_{j=1}^{k} a_jv_j) - f(x + a_kv_k) \right) - \left( f(x + \sum_{j=1}^{k-1} a_jv_j) - f(x) \right) \right] \]

Consider the \( k \)th term in (20.6.13). Let

\[ h(t) = f \left( x + \sum_{j=1}^{k-1} a_jv_j + ta_kv_k \right) - f(x + ta_kv_k) \]

for \( t \in [0, 1] \). Then

\[ h'(t) = a_k \lim_{h \to 0} \frac{1}{ah} \left( f \left( x + \sum_{j=1}^{k-1} a_jv_j + (t + h)a_kv_k \right) - f(x + (t + h)a_kv_k) \right) \]

\[ - \left( f \left( x + \sum_{j=1}^{k-1} a_jv_j + ta_kv_k \right) - f(x + ta_kv_k) \right) \]

and this equals

\[ \left( D_{v_k} f \left( x + \sum_{j=1}^{k-1} a_jv_j + ta_kv_k \right) - D_{v_k} f(x + ta_kv_k) \right) a_k \] (20.6.14)

Now without loss of generality, it can be assumed that the norm on \( X \) is given by

\[ ||v|| \equiv \max \left\{ |a_k| : v = \sum_{j=1}^{n} a_kv_k \right\} \]

because this is a finite dimensional space, all norms on \( X \) are equivalent. Therefore, from (20.6.14) and the assumption that the Gateaux derivatives are continuous,

\[ ||h'(t)|| = \left\| \left( D_{v_k} f \left( x + \sum_{j=1}^{k-1} a_jv_j + ta_kv_k \right) - D_{v_k} f(x + ta_kv_k) \right) a_k \right\| \]

\[ \leq \varepsilon |a_k| \leq \varepsilon ||v|| \]

provided \( ||v|| \) is sufficiently small. Since \( \varepsilon \) is arbitrary, it follows from Lemma (20.6.1) the expression in (20.6.13) is \( o(v) \) because this expression equals a finite sum of terms of the form \( h(1) - h(0) \) where \( ||h'(t)|| \leq \varepsilon ||v|| \) whenever \( ||v|| \) is small enough. Thus

\[ f(x + v) - f(x) = \sum_{k=1}^{n} [f(x + a_kv_k) - f(x)] + o(v) \]
20.6. A MEAN VALUE INEQUALITY

\[ \sum_{k=1}^{n} D_{v_k} f(x) a_k + \sum_{k=1}^{n} \left[ f(x + a_k v_k) - f(x) - D_{v_k} f(x) a_k \right] + o(v). \]

Consider the \( k \)th term in the second sum.

\[ f(x + a_k v_k) - f(x) - D_{v_k} f(x) a_k = a_k \left( \frac{f(x + a_k v_k) - f(x)}{a_k} - D_{v_k} f(x) \right) \]

where the expression in the parentheses converges to 0 as \( a_k \to 0 \). Thus whenever \( ||v|| \) is sufficiently small,

\[ ||f(x + a_k v_k) - f(x) - D_{v_k} f(x) a_k|| \leq \varepsilon |a_k| \leq \varepsilon ||v|| \]

which shows the second sum is also \( o(v) \). Therefore,

\[ f(x + v) - f(x) = \sum_{k=1}^{n} D_{v_k} f(x) a_k + o(v). \]

Defining

\[ Df(x) v = \sum_{k=1}^{n} D_{v_k} f(x) a_k \]

where \( v = \sum_k a_k v_k \), it follows \( Df(x) \in \mathcal{L}(X,Y) \) and is given by the above formula.

It remains to verify \( x \to Df(x) \) is continuous.

\[ \|(Df(x) - Df(y)) v\| \leq \sum_{k=1}^{n} \|(D_{v_k} f(x) - D_{v_k} f(y)) a_k\| \leq \max \{|a_k|, k = 1, \ldots, n\} \sum_{k=1}^{n} \|D_{v_k} f(x) - D_{v_k} f(y)\| \]

(Note that \( ||v|| \equiv \max \{|a_k|, k = 1, \ldots, n\} \) where \( v = \sum_k a_k v_k \) and so

\[ \|Df(x) - Df(y)\| \leq \sum_{k=1}^{n} \|D_{v_k} f(x) - D_{v_k} f(y)\| \]

which proves the continuity of \( Df \) because of the assumption the Gateaux derivatives are continuous.

In particular, if \( D_{v_k} f(x) \) exist and are continuous functions of \( x \), this shows that \( f \) is Gateaux differentiable and in fact the Gateaux derivatives are continuous. The following gives the corresponding result for functions defined on infinite dimensional spaces.
CHAPTER 20. THE DERIVATIVE

Theorem 20.6.4 Suppose \( f : U \to Y \) where \( U \) is an open set in \( X \), a normed linear space. Suppose that \( f \) is Gateaux differentiable on \( U \) and that the Gateaux derivative is continuous on an open set containing \( x \). Then \( f \) is Frechet differentiable at \( x \).

**Proof:** Denote by \( G(x) \in L(X,Y) \) the Gateaux derivative. Thus

\[
G(x) v \equiv \lim_{\lambda \to 0} \frac{f(x + \lambda v) - f(x)}{\lambda}
\]

It is desired to show that \( G(x) = Df(x) \). Since \( G \) is continuous, one can obtain

\[
f(x + v) - f(x) = \int_0^1 G(x + tv) v dt
\]

where this is the ordinary Riemann integral.

\[
\left\| \frac{f(x + v) - f(x) - G(x)v}{\|v\|} \right\| = \left\| \int_0^1 G(x + tv)v dt - G(x)v \right\| \leq \frac{1}{\|v\|} \int_0^1 \|G(x + tv) - G(x)\| \|v\| dt
\]

which is small provided \( \|v\| \) is sufficiently small. Thus \( G(x) = Df(x) \) as hoped.

Recall the following.

**Lemma 20.6.5** Let \( \|x\| = \sup_{\|y^*\|_{X^*} \leq 1} |\langle y^*, x \rangle| \).

**Proof:** Let \( f(kx) = k \|x\| \). Then

\[
\sup_{\|kx\| \leq 1} |\langle f, x \rangle| = \sup_{|k| \leq \|x\|} |k| \|x\| = 1
\]

Then by Hahn Banach theorem, there is \( y^* \in X^* \) which extends \( f \) and \( \|y^*\| \leq 1 \). Then

\[
\|x\| \geq \sup_{\|z^*\|_{X^*} \leq 1} |\langle z^*, x \rangle| \geq |\langle y^*, x \rangle| = \|x\| \]

One does not need continuity of \( G \) near \( x \). It suffices to have continuity at \( x \). Let \( y^* \in Y^* \). Then by the mean value theorem,

\[
\langle y^*, f(x + v) - f(x) \rangle = \langle y^*, G(x + tv) v \rangle, \ t \in [0, 1]
\]

Then

\[
\frac{1}{\|v\|} \|f(x + v) - f(x) - G(x)v\| = \frac{1}{\|v\|} \sup_{\|y^*\| \leq 1} |\langle y^*, f(x + v) - f(x) - G(x)v \rangle|
\]

\[
= \frac{1}{\|v\|} \sup_{\|y^*\| \leq 1} |\langle y^*, G(x + tv) v - G(x)v \rangle| \leq \sup_{|t| \leq 1} \|G(x + tv) - G(x)v\|_{L(X,Y)}
\]

which converges to 0 as \( \|v\| \to 0 \) thanks to continuity of \( G \) at \( x \). This proves the following.
Theorem 20.6.6 Suppose \( f: U \to Y \) where \( U \) is an open set in \( X \), a normed linear space. Suppose that \( f \) is Gateaux differentiable on \( U \) and that the Gateaux derivative is continuous at \( x \). Then \( f \) is Frechet differentiable at \( x \) and 
\[
Df(x)v = Dv f(x).
\]

Example 20.6.7 Let \( X \) be \( C^2_0(\bar{\Omega}) \) where \( \Omega \) is a bounded open set in \( \mathbb{R}^n \) consisting of those functions which are twice continuously differentiable and vanish near \( \partial \Omega \). The norm will be 
\[
\|u\|_X \equiv \|u\|_\infty + \max\{\|u_i\|_\infty, i\} + \max\{\|u_{ij}\|_\infty, i,j\}
\]
Then let \( f: X \to \mathbb{R} \) be defined by 
\[
f(u) \equiv \frac{1}{2} \int_\Omega \nabla u \cdot \nabla u dx
\]
Show \( f \) is differentiable at \( u \in X \).

Consider the Gateaux differentiability.
\[
\lim_{t \to 0} \frac{f(u + tv) - f(u)}{t} = \lim_{t \to 0} \frac{t \int_\Omega \nabla u \cdot \nabla v dx}{t} = \frac{1}{2} \int_\Omega \nabla v \cdot \nabla v
\]
so it converges to 
\[
\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega \Delta uv dx
\]
the last step comes from the divergence theorem. Clearly \( v \to -\int_\Omega \Delta uv dx \) is linear and \( \mathbb{R} \) valued.
\[
\left| -\int_\Omega \Delta uv dx \right| \leq \|v\|_X \int_\Omega |\Delta u| dx \leq \|v\|_X m(\Omega) \|u\|_X
\]
Thus this appears to be in \( L(X, \mathbb{R}) \). This also shows that,
\[
\sup_{\|v\| \leq 1} |D_v f(u) - D_v f(\hat{u})| \leq m(\Omega) \|u - \hat{u}\|_X
\]
and so \( u \to D(\cdot)(u) \) is continuous as a map from \( X \) to \( L(X, \mathbb{R}) \) so it seems that this is a differentiable function and 
\[
Df(u)(v) = -\int_\Omega \Delta uv dx
\]

Definition 20.6.8 Let \( f: U \to Y \) where \( U \) is an open set in \( X \). Then \( f \) is called \( C^1(U) \) if it Gateaux differentiable and the Gateaux derivative is continuous on \( U \).

As shown, this implies \( f \) is differentiable and the Gateaux derivative is the Frechet derivative. It is good to keep in mind the following simple example or variations of it.

Example 20.6.9 Define 
\[
f(x) \equiv \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}
\]
This function has the property that it is differentiable everywhere but is not \( C^1(\mathbb{R}) \). In fact the derivative fails to be continuous at 0.
20.7 Higher Order Derivatives

If \( f : U \subseteq X \to Y \) for \( U \) an open set, then

\[ x \to Df(x) \]

is a mapping from \( U \) to \( \mathcal{L}(X,Y) \), a normed vector space. Therefore, it makes perfect sense to ask whether this function is also differentiable.

**Definition 20.7.1** The following is the definition of the second derivative.

\[ D^2f(x) \equiv D(Df(x)). \]

Thus,

\[ Df(x + v) - Df(x) = D^2f(x) v + o(v). \]

This implies

\[ D^2f(x) \in \mathcal{L}(X,\mathcal{L}(X,Y)) , \quad D^2f(x)(u)(v) \in Y, \]

and the map

\[ (u,v) \to D^2f(x)(u)(v) \]

is a bilinear map having values in \( Y \). In other words, the two functions,

\[ u \to D^2f(x)(u)(v), \quad v \to D^2f(x)(u)(v) \]

are both linear.

The same pattern applies to taking higher order derivatives. Thus,

\[ D^3f(x) \equiv D(D^2f(x)) \]

and \( D^3f(x) \) may be considered as a trilinear map having values in \( Y \). In general \( D^k f(x) \) may be considered a \( k \) linear map. This means the function

\[ (u_1, \ldots, u_k) \to D^k f(x)(u_1) \cdots (u_k) \]

has the property

\[ u_j \to D^k f(x)(u_1) \cdots (u_j) \cdots (u_k) \]

is linear.

Also, instead of writing

\[ D^2f(x)(u)(v), \quad D^3f(x)(u)(v)(w) \]

the following notation is often used.

\[ D^2f(x)(u,v) \quad \text{or} \quad D^3f(x)(u,v,w) \]

with similar conventions for higher derivatives than 3. Another convention which is often used is the notation

\[ D^k f(x)v^k \]
instead of

\[ D^k f(x)(v, \cdots, v). \]

Note that for every \( k \), \( D^k f \) maps \( U \) to a normed vector space. As mentioned above, \( Df(x) \) has values in \( \mathcal{L}(X,Y) \), \( D^2 f(x) \) has values in \( \mathcal{L}(X, \mathcal{L}(X,Y)) \), etc. Thus it makes sense to consider whether \( D^k f \) is continuous. This is described in the following definition.

**Definition 20.7.2** Let \( U \) be an open subset of \( X \), a normed vector space, and let \( f : U \to Y \). Then \( f \) is \( C^k(U) \) if \( f \) and its first \( k \) derivatives are all continuous. Also, \( D^k f(x) \) when it exists can be considered a \( Y \)-valued multi-linear function. Sometimes these are called tensors in case \( f \) has scalar values.

### 20.8 The Derivative And The Cartesian Product

There are theorems which can be used to get differentiability of a function based on existence and continuity of the partial derivatives. A generalization of this was given above. Here a function defined on a product space is considered. It is very much like what was presented above and could be obtained as a special case but to reinforce the ideas, I will do it from scratch because certain aspects of it are important in the statement of the implicit function theorem.

The following is an important abstract generalization of the concept of partial derivative presented above. Instead of taking the derivative with respect to one variable, it is taken with respect to several but not with respect to others. This vague notion is made precise in the following definition. First here is a lemma.

**Lemma 20.8.1** Suppose \( U \) is an open set in \( X \times Y \). Then the set, \( U_y \) defined by

\[ U_y \equiv \{ x \in X : (x, y) \in U \} \]

is an open set in \( X \). Here \( X \times Y \) is a finite dimensional vector space in which the vector space operations are defined componentwise. Thus for \( a, b \in \mathbb{F} \),

\[ a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2) \]

and the norm can be taken to be

\[ \|(x, y)\| \equiv \max (\|x\|, \|y\|) \]

**Proof:** In finite dimensions it doesn’t matter how this norm is defined because all are equivalent. It obviously satisfies most axioms of a norm. The only one which is not obvious is the triangle inequality. I will show this now.

\[
\begin{align*}
\|(x, y) + (x_1, y_1)\| & \equiv \|(x + x_1, y + y_1)\| \equiv \max (\|x + x_1\|, \|y + y_1\|) \\
& \leq \max (\|x\| + \|x_1\|, \|y\| + \|y_1\|) \\
& \leq \max (\|x\|, \|y\|) + \max (\|x_1\|, \|y_1\|) \\
& \equiv \|(x, y)\| + \|(x_1, y_1)\|
\end{align*}
\]
Let $x \in U_y$. Then $(x, y) \in U$ and so there exists $r > 0$ such that
\[ B((x, y), r) \subseteq U. \]
This says that if $(u, v) \in X \times Y$ such that $\|(u, v) - (x, y)\| < r$, then $(u, v) \in U$. Thus if
\[ \|(u, y) - (x, y)\| = \|u - x\| < r, \]
then $(u, y) \in U$. This has just said that $B(x, r)$, the ball taken in $X$ is contained in $U_y$. This proves the lemma.

Or course one could also consider
\[ U_x \equiv \{ y : (x, y) \in U \} \]
in the same way and conclude this set is open in $Y$. Also, the generalization to many factors yields the same conclusion. In this case, for $x \in \prod_{i=1}^n X_i$, let
\[ \|x\| \equiv \max \left( \|x_i\|_{X_i} : x = (x_1, \ldots, x_n) \right) \]
Then a similar argument to the above shows this is a norm on $\prod_{i=1}^n X_i$. Consider the triangle inequality.
\[ \|(x_1, \ldots, x_n) + (y_1, \ldots, y_n)\| \leq \max_i \left( \|x_i + y_i\|_{X_i} \right) \leq \max_i \left( \|x_i\|_{X_i} + \|y_i\|_{X_i} \right) \]
\[ \leq \max_i \left( \|x_i\|_{X_i} \right) + \max_i \left( \|y_i\|_{X_i} \right) \]

**Corollary 20.8.2** Let $U \subseteq \prod_{i=1}^n X_i$ be an open set and let
\[ U(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \equiv \{ x \in \mathbb{R}^r : (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in U \} . \]
Then $U(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is an open set in $\mathbb{R}^r$.

**Proof:** Let $z \in U(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Then $(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \equiv x \in U$ by definition. Therefore, since $U$ is open, there exists $r > 0$ such that $B(x, r) \subseteq U$. It follows that for $B(z, r)_{X_i}$ denoting the ball in $X_i$, it follows that $B(z, r)_{X_i} \subseteq U(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ because to say that $\|z - w\|_{X_i} < r$ is to say that
\[ \|(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) - (x_1, \ldots, x_{i-1}, w, x_{i+1}, \ldots, x_n)\| < r \]
and so $w \in U(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. ■

Next is a generalization of the partial derivative.

**Definition 20.8.3** Let $g : U \subseteq \prod_{i=1}^n X_i \to Y$, where $U$ is an open set. Then the map
\[ z \to g(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \]
is a function from the open set in $X_i$,
\[ \{ z : x = (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) \in U \} \]
20.8. THE DERIVATIVE AND THE CARTESIAN PRODUCT

When this map is differentiable, its derivative is denoted by $D_i \mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v} \in X_i$, let $\theta_i \mathbf{v} \in \prod_{i=1}^n X_i$ be the vector $(0, \cdots, \mathbf{v}, \cdots, 0)$ where the $\mathbf{v}$ is in the $i^{th}$ slot and for $\mathbf{v} \in \prod_{i=1}^n X_i$, let $v_i$ denote the entry in the $i^{th}$ slot of $\mathbf{v}$. Thus, by saying $z \to \mathbf{g}(x_1, \cdots, x_{i-1}, z, x_{i+1}, \cdots, x_n)$
is differentiable is meant that for $\mathbf{v} \in X_i$ sufficiently small,

$$\mathbf{g}(\mathbf{x} + \theta_i \mathbf{v}) - \mathbf{g}(\mathbf{x}) = D_i \mathbf{g}(\mathbf{x}) \mathbf{v} + o(\mathbf{v}).$$

Note $D_i \mathbf{g}(\mathbf{x}) \in L(X_i, Y)$.

**Definition 20.8.4** Let $U \subseteq X$ be an open set. Then $\mathbf{f}: U \to Y$ is $C^1(U)$ if $\mathbf{f}$ is differentiable and the mapping

$$\mathbf{x} \to D\mathbf{f}(\mathbf{x}),$$

is continuous as a function from $U$ to $L(X,Y)$.

With this definition of partial derivatives, here is the major theorem. Note the resemblance with the matrix of the derivative of a function having values in $\mathbb{R}^m$ in terms of the partial derivatives.

**Theorem 20.8.5** Let $g, U, \prod_{i=1}^n X_i$, be given as in Definition 20.8.3. Then $g$ is $C^1(U)$ if and only if $D_i g$ exists and is continuous on $U$ for each $i$. In this case, $g$
is differentiable and

$$D\mathbf{g}(\mathbf{x})(\mathbf{v}) = \sum_k D_k \mathbf{g}(\mathbf{x}) v_k$$

(20.8.15)

where $\mathbf{v} = (v_1, \cdots, v_n)$.

**Proof:** Suppose then that $D_i \mathbf{g}$ exists and is continuous for each $i$. Note that

$$\sum_{j=1}^k \theta_j v_j = (v_1, \cdots, v_k, 0, \cdots, 0).$$

Thus $\sum_{j=1}^n \theta_j v_j = \mathbf{v}$ and define $\sum_{j=1}^0 \theta_j v_j \equiv 0$. Therefore,

$$g(\mathbf{x} + \mathbf{v}) - g(\mathbf{x}) = \sum_{k=1}^n \left[ g \left( \mathbf{x} + \sum_{j=1}^k \theta_j v_j \right) - g \left( \mathbf{x} + \sum_{j=1}^{k-1} \theta_j v_j \right) \right]$$

(20.8.16)

Consider the terms in this sum.

$$g \left( \mathbf{x} + \sum_{j=1}^k \theta_j v_j \right) - g \left( \mathbf{x} + \sum_{j=1}^{k-1} \theta_j v_j \right) = g(\mathbf{x} + \theta_k v_k) - g(\mathbf{x}) +$$

(20.8.17)
\[
\left( g \left( x + \sum_{j=1}^{k} \theta_j v_j \right) - g(x + \theta_k v_k) \right) - \left( g \left( x + \sum_{j=1}^{k-1} \theta_j v_j \right) - g(x) \right)
\]  (20.8.18)

and the expression in (20.8.18) is of the form \( h(v_k) - h(0) \) where for small \( w \in X_k \),

\[
h(w) \equiv g \left( x + \sum_{j=1}^{k-1} \theta_j v_j + \theta_k w \right) - g(x + \theta_k w).
\]

Therefore,

\[
Dh(w) = D_k g \left( x + \sum_{j=1}^{k-1} \theta_j v_j + \theta_k w \right) - D_k g (x + \theta_k w)
\]

and by continuity, \( ||Dh(w)|| < \varepsilon \) provided \( ||v|| \) is small enough. Therefore, by Theorem 20.6.2, the mean value inequality, whenever \( ||v|| \) is small enough,

\[
||h(v_k) - h(0)|| \leq \varepsilon ||v||
\]

which shows that since \( \varepsilon \) is arbitrary, the expression in (20.8.24) is \( o(v) \). Now in (20.8.17)

\[
g(x+\theta_k v_k) - g(x) = D_k g(x) v_k + o(v_k) = D_k g(x) v_k + o(v).
\]

Therefore, referring to (20.8.10)

\[
g(x + v) - g(x) = \sum_{k=1}^{n} D_k g(x) v_k + o(v)
\]

which shows \( Dg(x) \) exists and equals the formula given in (20.8.14). Also \( x \to Dg(x) \) is continuous since each of the \( D_k g(x) \) are.

Next suppose \( g \) is \( C^1 \). I need to verify that \( D_k g(x) \) exists and is continuous. Let \( v \in X_k \) sufficiently small. Then

\[
g(x + \theta_k v) - g(x) = Dg(x) \theta_k v + o(\theta_k v) = Dg(x) \theta_k v + o(v)
\]

since \( ||\theta_k v|| = ||v|| \). Then \( D_k g(x) \) exists and equals

\[Dg(x) \circ \theta_k \]

Now \( x \to Dg(x) \) is continuous. It is clear that \( \theta_k : X_k \to \prod_{i=1}^{n} X_i \) is also continuous because \( \theta_k v \) places \( v \) in the \( k^{th} \) position and \( 0 \) in every other position. 

Note that the above argument also works at a single point \( x \). That is, continuity at \( x \) of the partials implies \( Dg(x) \) exists and is continuous at \( x \).
20.9 Mixed Partial Derivatives

Let $U$ be an open set in $\prod_{i=1}^{n} X_i$ where the norm is the one described above and let $f : U \to Y$ be a function for which the higher order partial derivatives of the sort described above exist. As in the case of functions defined on open sets of $\mathbb{R}^n$ one can ask whether the mixed partials are equal.

Results of this sort were known to Euler in around 1734. The theorem was proved by Clairaut some time later. It turns out that the mixed partial derivatives, if continuous will end up being equal. It will also work in the more general situation just described.

**Theorem 20.9.1** Let $U$ be an open subset of $\prod_{i=1}^{n} X_i$ where each $X_i$ is a normed linear space and $\|x\| = \max_i \|x_i\|$. Let $f : U \to Y$ have mixed partial derivatives $D_i D_j f$ and $D_j D_i f$. Then if these are continuous at $x \in U$, it follows they will be equal in the sense that $D_i D_j f (x) (u, v) = D_j D_i f (x) (v, u)$.

**Proof:** It suffices to assume that there are only two spaces and $U$ is an open subset of $X_1 \times X_2$ because one simply specializes to two of the variables in the general case. We denote the variable for $X_1$ as $x$ and the one from $X_2$ as $y$. Also, to simplify this, first assume $f$ has values in $\mathbb{R}$. Thus it will be denoted as $f$ rather than $f$. Since $U$ is open, there exists $r > 0$ such that $B((x,y),r) \subseteq U$. Now let $t,s$ be small real numbers and consider

$$
\Delta (s,t) = \frac{1}{st} \{ f(x + tu, y + sv) - f(x + tu, y) - (f(x, y + sv) - f(x, y)) \}
$$

Then $h'(t) = D_1 f (x + tu, y + sv) (u) - D_1 f (x + tu, y) (u)$. By the mean value theorem,

$$
\Delta (s,t) = \frac{1}{s} h'(\theta t) = \frac{1}{s} D_1 f (x + \theta tu, y + sv) (u) - D_1 f (x + \theta tu, y) (u)
$$

where $\theta \in (0,1)$. Now use the mean value theorem again to obtain

$$
\Delta (s,t) = D_2 D_1 f (x + \theta tu, y + \alpha sv) (u) (\alpha) , \alpha \in (0,1).
$$

Similarly doing things in the other order writing

$$
\Delta (s,t) = \frac{1}{st} \{ (f(x + tu, y + sv) - f(x, y + sv)) - (f(x + tu, y) - f(x, y)) \}
$$

and taking the derivative first with respect to $s$ and next with respect to $t$, one can obtain

$$
\Delta (s,t) = D_1 D_2 f (x + \hat{t}u, y + \hat{\alpha} sv) (u) (\alpha)
$$

where $\hat{t}, \hat{\alpha}$ are also in $(0,1)$. Then letting $(s,t) \to (0,0)$ and using continuity of the mixed partial derivatives, one obtains that

$$
D_2 D_1 f (x,y) (u) (v) = D_1 D_2 f (x,y) (v) (u)
$$
Letting \( v = u \) yields the desired result.

The general case follows right away by applying this result to \( \langle y^*, f \rangle \). Thus one obtains

\[
\langle y^*, D_2 D_1 f (x, y) (u) \rangle = \langle y^*, D_1 D_2 f (x, y) (v) (u) \rangle
\]

for every \( y^* \in Y' \). Hence, since \( Y' \) separates the points, it follows that the mixed partials are equal. ■

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example \([3]\).

**Example 20.9.2**

Let

\[
f(x, y) = \begin{cases} 
xy(x^2 - y^2) & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

From the definition of partial derivatives it follows immediately that \( f_x(0, 0) = f_y(0, 0) = 0 \). Using the standard rules of differentiation, for \((x, y) \neq (0, 0)\),

\[
f_x = y \frac{x^4 - y^4 + 4x^2y^2}{(x^2 + y^2)^2}, \quad f_y = x \frac{x^4 - y^4 - 4x^2y^2}{(x^2 + y^2)^2}
\]

Now

\[
f_{xy}(0, 0) = \lim_{y \to 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = \lim_{y \to 0} \frac{-y^4}{(y^2)^2} = -1
\]

while

\[
f_{yx}(0, 0) = \lim_{x \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{x} = \lim_{x \to 0} \frac{x^4}{(x^2)^2} = 1
\]

showing that although the mixed partial derivatives do exist at \((0, 0)\), they are not equal there.

Incidentally, the graph of this function appears very innocent. Its fundamental sickness is not apparent. It is like one of those whitened sepulchers mentioned in the Bible.
20.10 Implicit Function Theorem

Recall the following notation. \( \mathcal{L}(X,Y) \) is the space of bounded linear mappings from \( X \) to \( Y \) where here \( (X,\|\cdot\|_X) \) and \( (Y,\|\cdot\|_Y) \) are normed linear spaces. Recall that this means that for each \( L \in \mathcal{L}(X,Y) \)
\[
\|L\| \equiv \sup_{\|x\| \leq 1} \|Lx\| < \infty
\]

As shown earlier, this makes \( \mathcal{L}(X,Y) \) into a normed linear space. In case \( X \) is finite dimensional, \( \mathcal{L}(X,Y) \) is the same as the collection of linear maps from \( X \) to \( Y \). In what follows \( X,Y \) will be Banach spaces, complete normed linear spaces. Thus these are complete normed linear space and \( \mathcal{L}(X,Y) \) is the space of bounded linear maps. I will also cease trying to write the vectors in bold face partly to emphasize that these are not in \( \mathbb{R}^n \).

**Definition 20.10.1** Let \( (X,\|\cdot\|_X) \) and \( (Y,\|\cdot\|_Y) \) be two normed linear spaces. Then \( \mathcal{L}(X,Y) \) denotes the set of linear maps from \( X \) to \( Y \) which also satisfy the following condition. For \( L \in \mathcal{L}(X,Y) \),
\[
\lim_{\|x\| \leq 1} \|Lx\|_Y \equiv \|L\| < \infty
\]

Recall that this operator norm is less than infinity is always the case where \( X \) is finite dimensional. However, if you wish to consider infinite dimensional situations, you assume the operator norm is finite as a qualification for being in \( \mathcal{L}(X,Y) \). Then here is an important theorem.

**Theorem 20.10.2** If \( Y \) is a Banach space, then \( \mathcal{L}(X,Y) \) is also a Banach space.

**Proof:** Let \( \{L_n\} \) be a Cauchy sequence in \( \mathcal{L}(X,Y) \) and let \( x \in X \).
\[
\|L_n x - L_m x\| \leq \|x\| \|L_n - L_m\|.
\]
Thus \( \{L_n x\} \) is a Cauchy sequence. Let
\[
L x = \lim_{n \to \infty} L_n x.
\]
Then, clearly, \( L \) is linear because if \( x_1, x_2 \) are in \( X \), and \( a,b \) are scalars, then
\[
L(ax_1 + bx_2) = \lim_{n \to \infty} L_n(ax_1 + bx_2) = \lim_{n \to \infty} (aL_n x_1 + bL_n x_2) = aL x_1 + bL x_2.
\]
Also \( L \) is bounded. To see this, note that \( \{|\|L_n\||\} \) is a Cauchy sequence of real numbers because \( |\|L_n\|| - |\|L_m\|| \leq |\|L_n - L_m\|| \). Hence there exists \( K > \sup\{|\|L_n\|| : n \in \mathbb{N}\} \). Thus, if \( x \in X \),
\[
\|L x\| = \lim_{n \to \infty} |\|L_n x\|| \leq K \|x\| .
\]

The following theorem is really nice. The series in this theorem is called the Neuman series.
Lemma 20.10.3 Let \((X, \| \cdot \|)\) is a Banach space, and if \(A \in \mathcal{L}(X, X)\) and \(\|A\| = r < 1\), then

\[(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \in \mathcal{L}(X, X)\]

where the series converges in the Banach space \(\mathcal{L}(X, X)\). If \(O\) consists of the invertible maps in \(\mathcal{L}(X, X)\), then \(O\) is open and if \(I\) is the mapping which takes \(A\) to \(A^{-1}\), then \(I\) is continuous.

Proof: First of all, why does the series make sense?

\[
\sum_{k=p}^{q} A^k \leq \sum_{k=p}^{q} \|A\|^k \leq \sum_{k=p}^{\infty} r^k \leq \frac{r^p}{1 - r}
\]

and so the partial sums are Cauchy in \(\mathcal{L}(X, X)\). Therefore, the series converges to something in \(\mathcal{L}(X, X)\) by completeness of this normed linear space. Now why is it the inverse?

\[
\sum_{k=0}^{\infty} A^k (I - A) = \lim_{n \to \infty} \sum_{k=0}^{n} A^k (I - A) = \lim_{n \to \infty} \left( \sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k \right) = \lim_{n \to \infty} (I - A^{n+1}) = I
\]

because \(\|A^{n+1}\| \leq \|A\|^{n+1} \leq r^{n+1}\). Similarly,

\[
(I - A) \sum_{k=0}^{\infty} A^k = \lim_{n \to \infty} (I - A^{n+1}) = I
\]

and so this shows that this series is indeed the desired inverse.

Next suppose \(A \in O\) so \(A^{-1} \in \mathcal{L}(X, X)\). Then suppose \(\|A - B\| < \frac{r}{1 + \|A^{-1}\|}, r < 1\). Does it follow that \(B\) is also invertible?

\[
B = A - (A - B) = A \left[ I - A^{-1} (A - B) \right]
\]

Then \(\|A^{-1} (A - B)\| \leq \|A^{-1}\| \|A - B\| < r\) and so \([I - A^{-1} (A - B)]^{-1}\) exists. Hence

\[
B^{-1} = \left[ I - A^{-1} (A - B) \right]^{-1} A^{-1}
\]

Thus \(O\) is open as claimed. As to continuity, let \(A, B\) be as just described. Then using the Neumann series,

\[
\| \mathcal{I}A - \mathcal{I}B \| = \left\| A^{-1} - \left[ I - A^{-1} (A - B) \right]^{-1} A^{-1} \right\|
\]

\[
= \left\| A^{-1} - \sum_{k=0}^{\infty} (A^{-1} (A - B))^k A^{-1} \right\| = \left\| \sum_{k=1}^{\infty} (A^{-1} (A - B))^k A^{-1} \right\|
\]

\[
\leq \sum_{k=1}^{\infty} \|A^{-1}\|^{k+1} \|A - B\| = \|A - B\| \|A^{-1}\|^2 \sum_{k=0}^{\infty} \|A^{-1}\|^k \left( \frac{r}{1 + \|A^{-1}\|} \right)^k
\]

\[
\leq \|B - A\| \|A^{-1}\|^2 \frac{1}{1 - r}.
\]
Thus \( \mathcal{J} \) is continuous at \( A \in \mathcal{O} \).

**Lemma 20.10.4** Let
\[
\mathcal{O} \equiv \{ A \in \mathcal{L}(X,Y) : A^{-1} \in \mathcal{L}(Y,X) \}
\]
and let
\[
\mathcal{J} : \mathcal{O} \to \mathcal{L}(Y,X), \quad \mathcal{J} A \equiv A^{-1}.
\]
Then \( \mathcal{O} \) is open and \( \mathcal{J} \) is in \( C^m(\mathcal{O}) \) for all \( m = 1, 2, \cdots \). Also
\[
D\mathcal{J}(A)(B) = -\mathcal{J}(A)(B) \mathcal{J}(A). \tag{20.10.19}
\]
In particular, \( \mathcal{J} \) is continuous.

**Proof:** Let \( A \in \mathcal{O} \) and let \( B \in \mathcal{L}(X,Y) \) with
\[
||B|| \leq \frac{1}{2} ||A^{-1}||^{-1}.
\]
Then
\[
||A^{-1}B|| \leq ||A^{-1}|| ||B|| \leq \frac{1}{2}
\]
and so by Lemma 20.10.3,
\[
(A^2B)^{-1} \in \mathcal{L}(X,X).
\]
It follows that
\[
(A + A^{-1}B)^{-1} = (I + A^{-1}B)^{-1} = (A + A^{-1}B)^{-1} \in \mathcal{L}(Y,X).
\]
Thus \( \mathcal{O} \) is an open set.

Thus
\[
(A + B)^{-1} = (I + A^{-1}B)^{-1} A^{-1} = \sum_{n=0}^{\infty} (-1)^n (A^{-1}B)^n A^{-1}
\]
\[
= [I - A^{-1}B + o(B)] A^{-1}
\]
which shows that \( \mathcal{O} \) is open and, also,
\[
\mathcal{J}(A + B) - \mathcal{J}(A) = \sum_{n=0}^{\infty} (-1)^n (A^{-1}B)^n A^{-1} - A^{-1}
\]
\[
= -A^{-1}BA^{-1} + o(B)
\]
\[
= -\mathcal{J}(A)(B) \mathcal{J}(A) + o(B)
\]
which demonstrates \( \mathcal{J} \) is continuous. It follows from this that we can continue taking derivatives of \( \mathcal{J} \). For \( ||B_1|| \) small,
\[
- [D\mathcal{J}(A + B_1)(B) - D\mathcal{J}(A)(B)] = 
\]
\[ \mathcal{J}(A + B_1)(B) \mathcal{J}(A + B_1) - \mathcal{J}(A)(B)\mathcal{J}(A) = \mathcal{J}(A + B_1)(B) \mathcal{J}(A + B_1) - \mathcal{J}(A)(B)\mathcal{J}(A) + \mathcal{J}(A)(B)\mathcal{J}(A) + o(B_1) \]

\[ = [\mathcal{J}(A)(B_1)\mathcal{J}(A) + o(B_1)](B) [A^{-1} - A^{-1}B_1A^{-1} + o(B_1)] + \mathcal{J}(A)(B)[\mathcal{J}(A)(B_1)\mathcal{J}(A) + o(B_1)] \]

\[ = \mathcal{J}(A)(B_1)\mathcal{J}(A)(B)\mathcal{J}(A) + \mathcal{J}(A)(B)\mathcal{J}(A)\mathcal{J}(A) + o(B_1) \]

and so

\[ D^2\mathcal{J}(A)(B_1)(B) = \mathcal{J}(A)(B_1)\mathcal{J}(A)(B)\mathcal{J}(A) + \mathcal{J}(A)(B)\mathcal{J}(A)\mathcal{J}(A) + o(B_1) \]

which shows \( \mathcal{J} \) is \( C^2(\mathcal{O}) \). Clearly we can continue in this way which shows \( \mathcal{J} \) is in \( C^m(\mathcal{O}) \) for all \( m = 1, 2, \ldots \).

Here are the two fundamental results presented earlier which will make it easy to prove the implicit function theorem. First is the fundamental mean value inequality.

**Theorem 20.10.5** Suppose \( U \) is an open subset of \( X \) and \( f : U \to Y \) has the property that \( Df(x) \) exists for all \( x \) in \( U \) and that, \( x + t(y - x) \in U \) for all \( t \in [0, 1] \). (The line segment joining the two points lies in \( U \).) Suppose also that for all points on this line segment,

\[ ||Df(x + t(y - x))|| \leq M. \]

Then

\[ ||f(y) - f(x)|| \leq M|y - x|. \]

Next recall the following theorem about fixed points of a contraction map. It was Corollary 10.1.6.

**Corollary 20.10.6** Let \( B \) be a closed subset of the complete metric space \( (X, d) \) and let \( f : B \to X \) be a contraction map

\[ d(f(x), f(\hat{x})) \leq rd(x, \hat{x}), \quad r < 1. \]

Also suppose there exists \( x_0 \in B \) such that the sequence of iterates \( \{f^n(x_0)\}_{n=1}^{\infty} \) remains in \( B \). Then \( f \) has a unique fixed point in \( B \) which is the limit of the sequence of iterates. This is a point \( x \in B \) such that \( f(x) = x \). In the case that \( B = \overline{B(x_0, \delta)} \), the sequence of iterates satisfies the inequality

\[ d(f^n(x_0), x_0) \leq \frac{d(x_0, f(x_0))}{1 - r} \]

and so it will remain in \( B \) if

\[ \frac{d(x_0, f(x_0))}{1 - r} < \delta. \]
The implicit function theorem deals with the question of solving, \( f(x, y) = 0 \) for \( x \) in terms of \( y \) and how smooth the solution is. It is one of the most important theorems in mathematics. The proof I will give holds with no change in the context of infinite dimensional complete normed vector spaces when suitable modifications are made on what is meant by \( \mathcal{L}(X, Y) \). There are also even more general versions of this theorem than to normed vector spaces.

Recall that for \( X, Y \) normed vector spaces, the norm on \( X \times Y \) is of the form
\[
\|(x, y)\| = \max (\|x\|, \|y\|).
\]

**Theorem 20.10.7 (implicit function theorem)** Let \( X, Y, Z \) be Banach spaces and suppose \( U \) is an open set in \( X \times Y \). Let \( f : U \to Z \) be in \( C^1(U) \) and suppose \( f(x_0, y_0) = 0 \), \( D_1 f(x_0, y_0)^{-1} \in \mathcal{L}(Z, X) \).

Then there exist positive constants, \( \delta, \eta \), such that for every \( y \in B(y_0, \eta) \) there exists a unique \( x(y) \in B(x_0, \delta) \) such that
\[
f(x(y), y) = 0.
\]

Furthermore, the mapping, \( y \to x(y) \) is in \( C^1(B(y_0, \eta)) \).

**Proof:** Let \( T(x, y) \equiv x - D_1 f(x_0, y_0)^{-1} f(x, y) \). Therefore,
\[
D_1 T(x, y) = I - D_1 f(x_0, y_0)^{-1} D_1 f(x, y).
\]

by continuity of the derivative which implies continuity of \( D_1 T \), it follows there exists \( \delta > 0 \) such that if \( \|x-x_0\| < \delta \) and \( \|y-y_0\| < \delta \), then
\[
\|D_1 T(x, y)\| < \frac{1}{2} \text{ and } D_1 f(x, y)^{-1} \text{ exists}
\]

The second claim follows from Lemma 20.10.4. By the mean value inequality, Theorem 20.10.5, whenever \( x, x' \in B(x_0, \delta) \) and \( y \in B(y_0, \delta) \),
\[
\|T(x, y) - T(x', y)\| \leq \frac{1}{2} \|x - x'\|.
\]

Also, it can be assumed \( \delta \) is small enough that for some \( M \) and all such \( (x, y) \),
\[
\|D_1 f(x_0, y_0)^{-1}\| \|D_2 f(x, y)\| < M
\]

Next, consider only \( y \) such that \( \|y-y_0\| < \eta \) where \( \eta \) is so small that
\[
\|T(x_0, y) - x_0\| < \delta \frac{3}{2}.
\]

Then for such \( y \), consider the mapping \( T_y(x) = T(x, y) \). Thus by Corollary 20.10.6, for each \( n \in \mathbb{N} \),
\[
\delta > \frac{2}{3} \delta \geq \frac{\|T_y(x_0) - x_0\|}{1 - (1/2)} \geq \|T_y^n(x_0) - x_0\|
\]
Then by the sequence of iterations of this map $T_y$ converges to a unique fixed point $x(y)$ in the ball $B(x_0, \delta)$. Thus, from the definition of $T$, $f(x(y), y) = 0$. This is the implicitly defined function.

Next we show that this function is Lipschitz continuous. For $y, \hat{y}$ in $B(y_0, \eta)$,

$$\|T(x,y) - T(x,\hat{y})\| = \left\|D_1 f(x_0, y_0)^{-1} f(x, y) - D_1 f(x_0, y_0)^{-1} f(x, \hat{y})\right\| \leq M \|y - \hat{y}\|$$

thanks to the above estimate and the mean value inequality, Theorem 20.10.25. Note how convexity of $B(y_0, \eta)$ which says that the line segment joining $y, \hat{y}$ is contained in $B(y_0, \eta)$ is important to use this theorem. Then from this,

$$\|x(y) - x(\hat{y})\| = \|T(x(y), y) - T(x(\hat{y}), \hat{y})\| \leq \|T(x(y), y) - T(x(y), \hat{y})\| + \|T(x(y), \hat{y}) - T(x(\hat{y}), \hat{y})\|$$

$$\leq M \|y - \hat{y}\| + \frac{1}{2} \|x(y) - x(\hat{y})\|$$

Hence,

$$\|x(y) - x(\hat{y})\| \leq 2M \|y - \hat{y}\| \quad (20.10.26)$$

Finally consider the claim that this implicitly defined function is $C^1$.

$$0 = f(x(y+u), y+u) - f(x(y), y)$$

$$= D_1 f(x(y), y)(x(y+u) - x(y)) + D_2 f(x(y), y) u$$

$$+ o(x(y+u) - x(y), u) \quad (20.10.27)$$

Consider the last term. $o(x(y+u) - x(y), u)/\|u\|$ equals

$$\begin{cases} 
\frac{o(x(y+u) - x(y), u)}{\|x(y+u) - x(y), u\|} \max_{\|u\|} \|x(y+u) - x(y), u\| & \text{if} \ \|x(y+u) - x(y), u\|_{X \times Y} \neq 0 \\
0 & \text{if} \ \|x(y+u) - x(y), u\|_{X \times Y} = 0 
\end{cases}$$

Now the Lipschitz condition just established shows that

$$\max \left( \|x(y+u) - x(y), u\| \right)$$

is bounded for nonzero $u$ sufficiently small that $y, y + u \in B(y_0, \eta)$. Therefore,

$$\lim_{u \to 0} \frac{o(x(y+u) - x(y), u)}{\|u\|} = 0$$

Then shows that

$$0 = D_1 f(x(y), y)(x(y+u) - x(y)) + D_2 f(x(y), y) u + o(u)$$

Therefore, solving for $x(y+u) - x(y)$, it follows that

$$x(y+u) - x(y) = -D_1 f(x(y), y)^{-1} D_2 f(x(y), y) u + D_1 f(x(y), y)^{-1} o(u)$$

$$= -D_1 f(x(y), y)^{-1} D_2 f(x(y), y) u + o(u)$$
20.10. IMPLICIT FUNCTION THEOREM

and now, the continuity of the partial derivatives $D_1f, D_2f$, continuity of the map $A \to A^{-1}$, along with the continuity of $y \to x(y)$ shows that $y \to x(y)$ is $C^1$ with derivative equal to $-D_1f (x(y), y)^{-1} D_2 f (x(y), y)$. 

It is easy to give a version of this theorem in which the function $f$ also depends on a parameter $\lambda \in \Lambda$, a metric space.

**Corollary 20.10.8** Let $X,Y,Z$ be Banach spaces and suppose $U$ is an open set in $X \times Y$. Let $f : U \times \Lambda \to Z$ satisfy $f (\cdot, \cdot, \lambda)$ is in $C^1 (U)$ and suppose for each $\lambda$,

$$f (x_0, y_0, \lambda) = 0, \quad D_1 f (x_0, y_0, \lambda)^{-1} \in \mathcal{L} (Z, X). \quad (20.10.28)$$

Also suppose $(x, y) \to D_1 f (x, y, \lambda)$ is continuous uniformly in $\lambda$ and $D_2 (x, y, \lambda)$ is uniformly bounded in $\lambda$ for $(x, y)$ sufficiently close to $(x_0, y_0)$. Then there exist positive constants, $\delta, \eta$, such that for every $y \in B (y_0, \eta)$ there exists a unique $x(y, \lambda) \in B (x_0, \delta)$ such that

$$f (x (y, \lambda), y, \lambda) = 0. \quad (20.10.29)$$

Furthermore, the mapping, $y \to x (y, \lambda)$ is in $C^1 (B (y_0, \eta))$ and $\lambda \to x (y, \lambda)$ is continuous.

**Proof:** It is just a repeat of the above proof except you use the uniform contraction principle, Corollary [6.7.4] to get the fixed point. 

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [56], [57] and in [6].

**Theorem 20.10.9** (inverse function theorem) Let $x_0 \in U$, an open set in $X$, and let $f : U \to Y$ where $X,Y$ are finite dimensional normed vector spaces. Suppose

$$f \text{ is } C^1 (U), \quad \text{and } \quad Df(x_0)^{-1} \in \mathcal{L}(Y, X). \quad (20.10.30)$$

Then there exist open sets $W$, and $V$ such that

$$x_0 \in W \subseteq U, \quad (20.10.31)$$

$$f : W \to V \text{ is one to one and onto,} \quad (20.10.32)$$

$$f^{-1} \text{ is } C^1. \quad (20.10.33)$$

**Proof:** Apply the implicit function theorem to the function

$$F (x, y) \equiv f (x) - y$$

where $y_0 \equiv f (x_0)$. Thus the function $y \to x(y)$ defined in that theorem is $f^{-1}$.

Now let

$$W \equiv B (x_0, \delta) \cap f^{-1} (B (y_0, \eta))$$

and

$$V \equiv B (y_0, \eta). \blacksquare$$
20.11 More Derivatives

When you consider a $C^k$ function $f$ defined on an open set $U$, you obtain the following

$$Df(x) \in \mathcal{L}(X,Y), D^2f(x) \in \mathcal{L}(X,\mathcal{L}(X,Y)), D^3f(x) \in \mathcal{L}(X,\mathcal{L}(X,\mathcal{L}(X,Y)))$$

and so forth. Thus they can each be considered as a linear transformation with values in some vector space. When you consider the vector spaces, you see that these can also be considered as multilinear functions on $X$ with values in $Y$. Now consider the product of two linear transformations $A(y) B(y) w$, where everything is given to make sense and here $w$ is an appropriate vector. Then if each of these linear transformations can be differentiated, you would do the following simple computation.

$$(A(y + u) B(y + u) - A(y) B(y))(w)$$

$$= (A(y + u) B(y + u) - A(y) B(y + u) + A(y) B(y + u) - A(y) B(y))(w)$$

$$= ((DA(y)u + o(u)) B(y + u) + A(y) (DB(y)u + o(u)))(w)$$

$$= (DA(y)(u) B(y + u) + A(y) DB(y)(u) + o(u))(w)$$

$$= (DA(y)(u) B(y) + A(y) DB(y)(u) + o(u))(w)$$

Then

$$u \to (DA(y)(u) B(y) + A(y) DB(y)(u))(w)$$

is clearly linear and

$$(u,w) \to (DA(y)(u) B(y) + A(y) DB(y)(u))(w)$$

is bilinear and continuous as a function of $y$. By this we mean that for a fixed choice of $(u,w)$ the resulting $Y$ valued function just described is continuous. Now if each of $A,B,DA,DB$ can be differentiated, you could replace $y$ with $y + \hat{u}$ and do a similar computation to obtain as many differentiations as desired, the $k^{th}$ differentiation yielding a $k$ linear function. You can do this as long as $A$ and $B$ have derivatives.

Now in the case of the implicit function theorem, you have

$$Dx(y) = -D_1 f(x(y),y)^{-1} D_2 f(x(y),y). \quad (20.11.34)$$

By Lemma 20.10.4 and the implicit function theorem and the chain rule, this is the situation just discussed. Thus $D^2x(y)$ can be obtained. Then the formula for it will only involve $Dx$ which is known to be continuous. Thus one can continue in this way finding derivatives till $f$ fails to have them. The inverse map never creates difficulties because it is differentiable of order $m$ for any $m$ thanks to Lemma 20.10.4. Thus one can conclude the following corollary.

**Corollary 20.11.1** In the implicit and inverse function theorems, you can replace $C^1$ with $C^k$ in the statements of the theorems for any $k \in \mathbb{N}$.
20.12 Lyapunov Schmidt Procedure

You have \( f : X \times \Lambda \to Y \) where here \( X, \Lambda \) are Banach spaces. Suppose \( (0, 0) \in X \times \Lambda \) and \( f(0, 0) = 0 \). Then if \( D_1 f(0, 0)^{-1} \) is in \( \mathcal{L}(Y, X) \), the implicit function theorem says that there exists \( x(\lambda) \) a \( C^p \) function such that locally \( f(x(\lambda), \lambda) = 0 \). So what if \( D_1 f(0, 0) \) fails to be one to one? Sometimes this case is also considered. It may be that \( D_1 f(0, 0) \) is one to one on some subspace and other nice things happen. In particular, suppose the following.

Letting \( X_2 \equiv \ker D_1 f(0, 0) \) assume

\[
X = X_1 \oplus X_2, \quad \dim(X_2) < \infty
\]

where \( X_1 \) is a closed subspace. Thus \( D_1 f(0, 0) \) is one to one on \( X_1 \). We let

\[
Y_1 = D_1 f(0, 0)(X_1)
\]

and suppose that \( Y = Y_1 \oplus Y_2 \) where \( \dim(Y_2) < \infty, Y_1 \) also a closed subspace.

\[
\begin{align*}
X_1 &\xrightarrow{D_1 f(0,0)} Y_1 = D_1 f(0, 0)(X_1), \quad Y_1 \text{ closed} \quad < \infty \\
Y &= Y_1 \oplus Y_2, \quad \dim(Y_2) < \infty
\end{align*}
\]

By the open mapping theorem, \( D_1 f(0, 0)^{-1} \) is also continuous.

Let \( Q \) be a continuous projection onto \( Y_1 \) which is assumed to exist\(^2\) so that \((I - Q)\) is a projection onto \( Y_2 \). Then the equation \( f(x(\lambda), \lambda) = 0 \) can be written as the pair

\[
\begin{align*}
Qf(x, \lambda) &= 0 \\
(I - Q)f(x, \lambda) &= 0
\end{align*}
\]

Consider the top. For \( x = x_1 + x_2 \) where \( x_i \in X_i \), this is

\[
Qf(x_1 + x_2, \lambda) = 0
\]

Then if \( g(x_1, x_2, \lambda) = Qf(x_1 + x_2, \lambda) \), one has \( g : X_1 \times X_2 \times \Lambda \to Y_1 \)

\[
D_1 g(x_1, x_2, \lambda) h = D_1 Qf(x_1 + x_2, \lambda) h, \quad h \in X_1.
\]

Thus \( D_1 g(0, 0, 0)^{-1} \) is continuous by the open mapping theorem \((D_1 f(0, 0)\) is one to one on \( X_1 \)), and by the implicit function theorem, there is a solution to

\[
Qf(x_1 + x_2, \lambda) = 0
\]

for \( x_1 = x_1(x_2, \lambda) \). (Note how it is important that \( X_1 \) and \( Y_1 \) be Banach spaces.) Then the other equation yields

\[
(I - Q)f(x_1(x_2, \lambda) + x_2, \lambda) = 0
\]

\(^2\)In Hilbert space, the existence of this projection map is obvious and it is assumed that it exists here.
and so for fixed \( \lambda \), this is a finite set of equations of a variable in a finite dimensional space.

This depends on being able to write \( X = X_1 \oplus X_2 \) where \( X_1 \) is closed, \( X_2 = \ker D_1 f (0, 0) \), a similar situation for \( Y = Y_1 \oplus Y_2 \). So when does this happen? Are there conditions on \( D_1 f (0, 0) \) which will cause it to occur?

There are such conditions. For example, \( D_1 f (0, 0) \) could be a Fredholm operator defined in Definition 5.6.7. The following are some easy examples in which all that nonsense about things being finite dimensional and part of a direct sum does not need to be considered.

**Example 20.12.1** Say \( X = \mathbb{R}^2 \) and \( \Lambda = \mathbb{R} \). Let \( f (x, y, \lambda) = x + xy + y^2 + \lambda \). Then

\[
D_1 f (0, 0, 0) = (1, 0)
\]

this \( 1 \times 2 \) matrix mapping \( \mathbb{R}^2 \) to \( \mathbb{R} \). Thus \( X_2 = (0, \alpha)^T : \alpha \in \mathbb{R} \) and \( X_1 = (\alpha, 0)^T : \alpha \in \mathbb{R} \). In this case, \( Y_1 = \mathbb{R} \) and so \( Q = I \). Thus the above reduces to the single equation

\[
f ((\alpha, 0) + (0, \beta), \lambda) = 0
\]

and so since \( D_1 f (0, 0, 0) \) is one to one, \( x_1 = (\alpha, 0) = x_1 ((0, \beta), \lambda) \). Of course this is completely obvious because if you consider \( f \) in the natural way as a function of three variables, then the implicit function theorem immediately gives \( x = x(y, \lambda) \) which is essentially the same result. We just write \( (\alpha, 0) \) in place of \( \alpha \). The first independent variable is a function of the other two.

**Example 20.12.2** Here is another easy example. \( f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \)

\[
f (x, y, \lambda) = \begin{pmatrix} x + xy + y^2 + \sin (\lambda) \\ x + y^2 - x^2 + \lambda \end{pmatrix}
\]

Then

\[
D_1 f (x, y, \lambda) = \begin{pmatrix} 1 + y & x + 2y \\ 1 - 2x & 2y \end{pmatrix}
\]

So

\[
D_1 f ((0, 0), 0) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}
\]

Then

\[
X_2 = \ker D_1 f ((0, 0), 0) = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\}
\]

and \( X_1 = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \) and clearly \( D_1 f ((0, 0), 0) \) is indeed one to one on \( X_1 \).

\[
D_1 f (0, 0) (X_1) = \left\{ \begin{pmatrix} y \\ y \end{pmatrix} : y \in \mathbb{R} \right\} = Y_1.
\]

In this case, let

\[
Q \left( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) = \begin{pmatrix} \frac{\alpha + \beta}{2} \\ \frac{\alpha + \beta}{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]
so \((I - Q) = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}\). Thus the equations are

\[
\begin{align*}
Q f(x, \lambda) &= 0 \\
(I - Q) f(x, \lambda) &= 0
\end{align*}
\]

This reduces to

\[
\begin{pmatrix}
-\frac{1}{2} x^2 + \frac{1}{2} x y + x + y^2 + \frac{1}{2} \lambda + \frac{1}{2} \sin \lambda \\
-\frac{1}{2} x^2 + \frac{1}{2} x y + x + y^2 + \frac{1}{2} \lambda + \frac{1}{2} \sin \lambda \\
\frac{1}{2} x^2 + \frac{1}{2} x y - \frac{1}{2} \lambda + \frac{1}{2} \sin \lambda \\
-\frac{1}{2} x^2 - \frac{1}{2} x y + \frac{1}{2} \lambda - \frac{1}{2} \sin \lambda
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

Note how in both the top and the bottom, there is only one equation and one can solve for \(x\) in terms of \(y, \lambda\) near \((0, 0, 0)\) which is what the above general argument shows. Of course you can see this directly using the implicit function theorem. Then can you solve for \(y = y(\lambda)\)? This would involve trying to solve for \(y\) as a function of \(\lambda\) in the following where \(x(y, \lambda)\) comes from the first equations.

\[
\frac{1}{2} x^2 (y, \lambda) + \frac{1}{2} y x (y, \lambda) - \frac{1}{2} \lambda + \frac{1}{2} \sin \lambda = 0
\]

If you can do this, then you would have found \((x, y)\) as a function of \(\lambda\) for small \(\lambda\).

In this example, in the top equation, at \((0, 0, 0), x, y = 0\). Also \(x, y = -1\) so \(x(y, \lambda) \approx -\lambda\) other than higher order terms for small \(y, \lambda\). Then in the bottom equation, for all variables very small, you would have \(\lambda^2 + y(-\lambda) - \lambda + \sin \lambda = 0\), \(y(\lambda) = -1 + \frac{\sin(\lambda)}{\lambda} + \lambda\) at least approximately. Thus it seems there is a nonzero solution to the equation \(f(x, y, \lambda) = 0\) which is valid for small \(\lambda, x, y\), this in addition to the zero solution. Note that for small nonzero \(\lambda, -1 + \frac{\sin(\lambda)}{\lambda} + \lambda \neq 0\). It equals approximately \(\lambda - \frac{\lambda^3}{3!}\) for small \(\lambda\) from the power series for \(\sin\).

In the next example, the same procedure gives a solution to a problem \(f((x, y), \lambda) = 0\) such that for small \(\lambda, (x, y)\) is a function of \(\lambda\) which is nonzero and \(f((0, 0), \lambda) = 0\). Thus for small \(\lambda\), there are two solutions to the nonlinear system of equations.

**Example 20.12.3** Let

\[
f((x, y), \lambda) = \begin{pmatrix} x + xy + y^2 + x \sin(\lambda) \\ x + y^2 - x^2 + x \lambda \end{pmatrix}
\]

In this case \(f((0, 0), \lambda) = 0\) even though \(\lambda\) might not be 0. The Lyapunov Schmidt procedure will be used to show that there are nonzero solutions \(x(\lambda), y(\lambda)\) such that \(f((x(\lambda), y(\lambda)), \lambda) = 0\).

At origin,

\[
D_1 f((0, 0), \lambda) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}
\]
Thus $X_1 = \text{span}(e_1)$ and $X_2 = \text{span}(e_2)$. Then $Y_1 = \text{span}(e_1 + e_2)$ and $Y_2 = \text{span}(e_1 - e_2)$. Also $D_1f((0,0),0)$ is one to one on $X_1$ and its range is $Y_1$. Then let

$$Q \begin{pmatrix} \alpha \\
\beta \end{pmatrix} = \begin{pmatrix} \frac{\alpha+\beta}{2} \\
\frac{\alpha-\beta}{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\
1/2 & -1/2 \end{pmatrix} \begin{pmatrix} \alpha \\
\beta \end{pmatrix}$$

$$(I-Q) = \begin{pmatrix} 1/2 & -1/2 \\
-1/2 & 1/2 \end{pmatrix}$$

Then $Qf = 0$ is yields the equation

$$x + \frac{1}{2}x\lambda + \frac{1}{2}x\sin \lambda + \frac{1}{2}xy - \frac{1}{2}x^2 + y^2 = 0$$

Also $(I - Q)f = 0$ yields the equation

$$\frac{1}{2}x\sin \lambda - \frac{1}{2}x\lambda + \frac{1}{2}xy + \frac{1}{2}x^2 = 0$$

Now consider $x_y$ and $x_\lambda$ at $(0,0)$ from the first equation. Both of these are easily seen to be $0$. Now consider $x_{yy}$. After some computations, this is seen to be $x_{yy} = -2$. Similarly, $x_{y\lambda}(0,0) = 0, x_{\lambda\lambda}(0,0) = 0$ also. Thus up to terms of degree $3$,

$$x(\lambda) = -2 = \frac{1}{2}(-2)y^2$$

Place this in the bottom equation.

$$\frac{1}{2}y^2\lambda - \frac{1}{2}y^2\sin \lambda - \frac{1}{2}y^3 + \frac{1}{2}y^4 = 0$$

Now the idea is to find $y = y(\lambda)$, hopefully nonzero. Divide by $y^2$ and multiply by $2$.

$$y^2 - y + \lambda - \sin \lambda = 0$$

Then for small $\lambda$ this is approximately equal to

$$y^2 - y + \frac{\lambda^3}{6} = 0$$

Then a solution for $y$ for small $\lambda$ is

$$y = 1 + \sqrt{1 - \frac{2}{3}\lambda^3}$$

Of course there is another solution as well, when you replace the $+$ with a minus sign. This is the one we want because when $\lambda = 0$ it reduces to $y = 0$. This shows that there exist solutions to the equations $f((x,y),\lambda) = 0$ which for small $\lambda$ are approximately

$$(x(\lambda), y(\lambda)) = \begin{pmatrix} -y^2, 1 - \sqrt{1 - \frac{2}{3}\lambda^3} \end{pmatrix}$$
In terms of \( \lambda \) very small,

\[
(x(\lambda), y(\lambda)) = \left( \frac{1}{6} \lambda^3 + \frac{1}{6} \sqrt{3} \sqrt{3 - 2\lambda^3} - \frac{1}{2}, \frac{1}{2} \sqrt{3 \lambda^3} - 2 \right)
\]

Using a power series in \( \lambda \) to approximate these functions, this reduces to

\[
(x(\lambda), y(\lambda)) = \left( -\frac{1}{36} \lambda^6, \frac{1}{6} \lambda^3 + \frac{1}{36} \lambda^6 + \frac{1}{108} \lambda^9 \right)
\]

where higher order terms are neglected. Thus there exist other solutions than the zero solution even though \( \lambda \) may be nonzero. Note that in this example, \( f((0,0), \lambda) = 0 \).

20.13 Analytic Functions

In calculus, there was a difference between functions of a real variable and functions of a complex variable. In the latter case the existence of a single derivative implied the existence of all derivatives and in fact the Taylor series converged to the function. It is reasonable to ask if a similar phenomenon occurs in the case of complex Banach spaces versus real Banach spaces. This section presents a quick introduction to this topic based on the assumption that the reader has had some exposure to complex analysis. Some of the details involving questions of convergence and term by term differentiation are left to the reader. Also if \( h \) maps an open subset of \( \mathbb{C} \) to a complex Banach space \( X \), and has a first derivative, then the usual Cauchy integral formula,

\[
h(z) = \frac{1}{2\pi i} \int_C \frac{h(w)}{w - z} dw,
\]

holds if \( C \) is a circle contained, together with its interior, in the open set on which \( h \) has a derivative. The integral can be defined as the ordinary Riemann integral using Riemann sums or it can be defined in terms of a Bochner integral. These details are routine and are left to the reader. There are several equivalent definitions of an analytic function defined on a complex Banach space. The following is the one we will use since it resembles the familiar definition encountered in undergraduate complex variable courses.

**Definition 20.13.1** Let \( X \) and \( Y \) be complex Banach spaces and let \( U \subseteq X \) be an open set. We say \( f : U \to Y \) is analytic and bounded on \( U \) if

\[
z \to f(x + zh) \text{ is analytic for } x \in U, h \in X \text{ and } |z| \text{ small enough}
\]

exists for all \( x \in U \) and also \( ||f(x)|| \leq M < \infty \) for all \( x \in U \). Here \( z \in \mathbb{C} \) and \( x, h \in X \).
Let $h \in X^l$ and consider all $z \in \mathbb{C}^l$ with $\|z\|_{\mathbb{C}^l} \equiv \max(\|z_m\|, m = 1, \ldots, l)$ sufficiently small. Let $C_1$ be a sufficiently small circle centered at 0. Then consider  

$$z_m \to f \left( x + \sum_{m=1}^l z_m h_m \right)$$

which is analytic on and inside $C_1$. Thus using the Cauchy integral formula,

$$f \left( x + z_1 h_1 + \sum_{m=2}^l z_m h_m \right)$$

$$= \frac{1}{2\pi i} \int_{C_1} f \left( x + w_1 h_1 + \sum_{m=2}^l z_m h_m \right) \frac{1}{(w_1 - z_1)} \, dw_1$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{1}{w_1 - z_1} \cdot$$

$$\int_{C_1} f \left( x + w_1 h_1 + w_2 h_2 + \sum_{m=3}^l z_m h_m \right) \frac{1}{w_2 - z_2} \, dw_2 \, dw_1 =$$

$$\left( \frac{1}{2\pi i} \right)^l \int_{C_1} \cdots \int_{C_1} f \left( x + w_1 h_1 + w_2 h_2 + \cdots + w_l h_l \right) \frac{1}{\prod_{m=1}^l (w_m - z_m)} \, dw_l \cdots dw_1.$$ 

Consider the case when $l = 2$.

$$\left( \frac{1}{2\pi i} \right)^2 \int_{C_1} \int_{C_1} f \left( x + w_1 h_1 + w_2 h_2 \right) \frac{1}{(w_1 - z_1)(w_2 - z_2)} \, dw_2 \, dw_1 =$$

$$\left( \frac{1}{2\pi i} \right)^2 \int_{C_1} \int_{C_1} f \left( x + w_1 h_1 + w_2 h_2 \right) \cdot$$

$$\sum_{k_2=0}^{\infty} \frac{z_{k_2}}{w_2^{k_2+1}} \sum_{k_1=0}^{\infty} \frac{z_{k_1}}{w_1^{k_1+1}} \, dw_2 \, dw_1 =$$

$$\left( \frac{1}{2\pi i} \right)^2 \sum_{k_2=0}^{\infty} \sum_{k_1=0}^{\infty} \left( \int_{C_1} \int_{C_1} f \left( x + w_1 h_1 + w_2 h_2 \right) \frac{1}{w_2^{k_2+1} w_1^{k_1+1}} \, dw_2 \, dw_1 \right) z_{k_2} z_{k_1}.$$

Similarly, for arbitrary $l$, and letting $C$ be any circle centered at 0 with radius smaller than $\frac{\delta}{l}$,

$$f \left( x + \sum_{m=1}^l z_m h_m \right) = \sum_{k_l=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} a_{k_1 \cdots k_l} (x, h_l, \ldots, h_1) z_{k_1}^{z_1} \cdots z_{k_l}^{z_l} \quad (20.13.35)$$

where

$$a_{k_1 \cdots k_l} (x, h_l, \ldots, h_1)$$

$$= \left( \frac{1}{2\pi i} \right)^l \int_C \cdots \int_C f \left( x + \sum_{m=1}^l w_m h_m \right) \frac{1}{\prod_{m=1}^l w_m^{k_m+1}} \, dw_l \cdots dw_1. \quad (20.13.36)$$
Lemma 20.13.2 Let \( l \geq 1 \) and let \( t_m \in \mathbb{C} \). Then if \( h \in X^l \), then whenever \(|z|\) is small enough, \(20.13.35\) holds. Also the coefficients satisfy

\[
a_{k_1 \ldots k_l} (x, t_l h_l, \ldots, t_1 h_1) = \left( \prod_{m=1}^{l} t_m^{k_m} \right) a_{k_1 \ldots k_l} (x, h_l, \ldots, h_1) \tag{20.13.37}
\]

and

\[
\|a_{k_1 \ldots k_l} (x, h_l, \ldots, h_1)\| \leq C \prod_{m=1}^{l} \|h_m\| \tag{20.13.38}
\]

for some constant \( C \).

**Proof:** Let \( C \) be small enough that the circles \( t_mC \) for all \( m = 1, \ldots, l \) and \( C \) have radius less than \( \delta \). First assume \( t_m \neq 0 \) for all \( m \). Then

\[
a_{k_1 \ldots k_l} (x, t_l h_l, \ldots, t_1 h_1)
\]

\[
= \left( \frac{1}{2\pi i} \right)^l \int_{t_l C} \cdots \int_{t_1 C} f \left( x + \sum_{m=1}^{l} w_m t_m h_m \right) \prod_{m=1}^{l} \left( w_m t_m \right)^{k_m+1} \cdot \prod_{m=1}^{l} t_m^{k_m+1} \, dw_1 \cdots dw_l
\]

Here we just multiplied and divided by \( \prod_{m=1}^{l} t_m^{k_m+1} \).

\[
= \left( \frac{1}{2\pi i} \right)^l \int_{t_l C} \cdots \int_{t_1 C} f \left( x + \sum_{m=1}^{l} u_m h_m \right) \prod_{m=1}^{l} \left( u_m \right)^{k_m+1} \, du_1 \cdots du_l \prod_{m=1}^{l} t_m^{k_m} 
\]

\[
= a_{k_1 \ldots k_l} (x, h_l, \ldots, h_1) \prod_{m=1}^{l} t_m^{k_m} 
\]

Formally, \( w_i \in C \) and so \( t_i w_i \equiv u_i \in t_i C \). Then \( t_i dw_i = du_i \) and so \( dw_i = (1/t_i) \, du_i \). This is why \( \prod_{m=1}^{l} t_m^{k_m+1} \) gets changed to \( \prod_{m=1}^{l} t_m^{k_m} \).

If \( t_m = 0 \) for any \( m \), the result of both sides in the above equals zero due to the fact that

\[
\int_{C} \frac{1}{w_m^{k_m+1}} \, dw_m = 0
\]

whenever \( k_m \geq 1 \).

To verify \(20.13.35\), use \(20.13.37\) to conclude

\[
\|a_{k_1 \ldots k_l} (x, h_l \cdots h_1)\| \leq \left\| a_{k_1 \ldots k_l} \left( x, \frac{h_l}{\|h_l\|}, \ldots, \frac{h_1}{\|h_1\|} \right) \right\| \prod_{m=1}^{l} \|h_m\|^{k_m}
\]
and \(|a_{k_1 \cdots k_l}(x, h_1, \ldots, h_l)|\) is bounded by

\[
\frac{M}{(2\pi)^l} \int_C \cdots \int_C \frac{1}{\prod_{m=1}^l |w_m|^{k_{m+1}}} d|w_1| \cdots d|w_l| = C. \blacktriangleleft
\]

**Lemma 20.13.3** Suppose

\[
g(x + zh) = g(x) + \sum_{m=1}^{\infty} b_m(x, h) z^m
\]

for all \(z\) small enough. Then

\[
b_1(x, h_1 + h_2) = b_1(x, h_1) + b_1(x, h_2).
\]

**Proof:** Recall that

\[
f \left( x + \sum_{m=1}^{l} z_m h_m \right) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_l=0}^{\infty} a_{k_1 \cdots k_l}(x, h_1, \ldots, h_l) z_1^{k_1} \cdots z_l^{k_l}
\]

and so one can write the following where \(g_{nm}\) is defined in the following expression.

\[
g(x + z_1 h_1 + z_2 h_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{mn}(x, h_1, h_2) z_1^m z_2^n.
\]

Thus,

\[
g(x + z_1 h_1) = \sum_{m=0}^{\infty} g_{m0}(x, h_1, h_2) z_1^m = g(x) + \sum_{m=1}^{\infty} b_m(x, h) z_1^m,
\]

\[
g(x + z_2 h_2) = \sum_{n=0}^{\infty} g_{0n}(x, h_1, h_2) z_2^n = g(x) + \sum_{n=1}^{\infty} b_n(x, h) z_2^n
\]

which implies

\[
g_{m0}(x, h_1, h_2) = b_m(x, h_1), \quad g_{0n}(x, h_1, h_2) = b_n(x, h_2).
\]

Now let \(z_1 = z_2 = z\). Then

\[
g(x + z(h_1 + h_2)) = g(x) + \sum_{n=0}^{\infty} b_n(x, h_1 + h_2) z^n
\]

\[
= g(x) + z (g_{10}(x, h_1, h_2) + g_{01}(x, h_1, h_2)) + \text{higher order terms in } z.
\]

Therefore,

\[
b_1(x, h_1 + h_2) = g_{10}(x, h_1, h_2) + g_{01}(x, h_1 h_2)
\]

\[
= b_1(x, h_1) + b_1(x, h_2) \blacktriangleleft
\]
Lemma 20.13.4 Suppose \( a(x, h, \cdots, h_1) \) is multilinear, \((h_i \to a(x, h, \cdots, h_1)\) is linear),

\[
||a(x, h, \cdots, h_1)|| \leq C \prod_{m=1}^{l} ||h_m||,
\]

and

\[
D^{l-1} f(x + h_{l-1}) \cdots (h_1) - D^{l-1} f(x) (h_{l-1}) \cdots (h_1) - a(x, h, \cdots, h_1) = o(||h||).
\]

Then \( D^l f(x) \) exists and

\[
D^{l} f(x) (h_{l-1}) \cdots (h_1) = a(x, h, \cdots, h_1).
\]

Proof: If \( l = 1 \), the conclusion is obvious and is nothing more than the definition of the derivative.

\[
f(x + h) - f(x) - a(x, h) = o(||h||)
\]

and so from the definition of the derivative, \( a(x, h) = Df(x) h \).

Next let \( n = 2 \). By assumption,

\[
Df(x + h)(h_1) - Df(x)(h_1) - a(x, h, h_1) = o(||h||).
\]

Let \( L(x) \) be defined by

\[
L(x)(h_1) \equiv a(x, h, h_1).
\]

Then \( L(x) \in \mathcal{L}(U, \mathcal{L}(X, Y)) \) because

\[
||L(x)|| \equiv \sup_{||h|| \leq 1} ||L(x)(h)|| \equiv \sup_{||h|| \leq 1} \sup_{||h_1|| \leq 1} ||L(x)(h)(h_1)|| \leq C.
\]

Also

\[
||Df(x + h) - Df(x) - L(x) h||
\]

\[
\equiv \sup_{||h_1|| \leq 1} ||Df(x + h)(h_1) - Df(x)(h_1) - L(x)(h)(h_1)||
\]

\[
= \sup_{||h_1|| \leq 1} ||Df(x + h)(h_1) - Df(x)(h_1) - a(x, h, h_1)|| = o(||h||)
\]

and so \( L(x) = D^2 f(x) \). Continuing in this way, we verify the conclusion of the lemma.

Lemma 20.13.5 If \( f \) is analytic on \( U \), then \( f \in C^\infty(U) \). Also

**Proof:** By Lemma 20.13.3 applied to \( g = f \) and Lemma 20.13.2, \( Df(x) \) exists and

\[
Df(x)(h) = a_1(x, h).
\]
These lemmas implied that \( h \to a_1(x, h) \) was linear. Suppose \( D^{l-1}f(x) \) exists for \( l \geq 2 \).

\[
f(x + \sum_{m=1}^{l} z_m h_m) = \sum_{n_l=0}^{\infty} \ldots \sum_{n_1=0}^{\infty} a_{n_1 \ldots n_l} (x, h_l, \ldots, h_1) z_1^{n_1} \ldots z_l^{n_l}.
\]

Differentiate with respect to \( z_1, \ldots, z_{l-1} \) to obtain

\[
D^{l-1}f \left( x + \sum_{m=1}^{l} z_m h_m + z_l h_l \right) (h_{l-1}) \ldots (h_1) =
\sum_{n_l=0}^{\infty} \ldots \sum_{n_1=1}^{\infty} a_{n_1 n_{l-1} \ldots n_l} (x, h_{l-1} \ldots h_1) \left( \prod_{m=1}^{l-1} n_m \right) z_1^{n_1-1} \ldots z_{l-1}^{n_{l-1}-1} z_l^{n_l}.
\]

Take \( z_i = 0 \) for \( i = 1, \ldots, l-1 \). Then

\[
D^{l-1}f(x + z_l h_l)(h_{l-1}) \ldots (h_1) = \sum_{n_l=0}^{\infty} a_{n_1 \ldots n_l} (x, h_l, \ldots, h_1) z_l^{n_l}.
\]

Now we apply Lemma 20.13.3 to the function

\[
z_l \to D^{l-1}f(x + z_l h_l)(h_{l-1}) \ldots (h_1)
\]

and conclude

\[
h_l \to a_{1 \ldots 1}(x, h_l, \ldots, h_1)
\]

is linear. This involved taking \( n_l = 1 \) to get \( a_{1 \ldots 1}(x, h_l, \ldots, h_1) \). Thus from

\[
D^{l-1}f(x + z_l h_l)(h_{l-1}) \ldots (h_1) - D^{l-1}f(x)(h_{l-1}) \ldots (h_1)
= a_{1 \ldots 1}(x, h_l, \ldots, h_1) z_l + o(z_l h_l).
\]

From this equation, it follows that

\[
a_{1 \ldots 1} \left( x, h_l \ldots h_l + \hat{h}_l \ldots h_1 \right) z_l - a_{1 \ldots 1} \left( x, h_l \ldots h_l \right) z_l
= a_{1 \ldots 1} \left( x, h_l \ldots h_l \right) z_l = o(z_l h_l)
\]

because for each \( z_l \), the left side of 20.13.34 is linear in \( h_i \) for each \( i \leq l-1 \). Dividing both sides of the above by \( z_l \) and then letting \( z_l \to 0 \), we see that \( a_{n_1 \ldots n_l} \) is linear in each of \( h_i \). Denoting \( h_l \) by \( h_l \),

\[
D^{l-1}f(x + h_l)(h_{l-1}) \ldots (h_1) - D^{l-1}f(x)(h_{l-1}) \ldots (h_1)
= a_{1 \ldots 1} (x, h_l, \ldots, h_1) + o(\|h_l\|)
\]

and so by Lemma 20.13.3, \( Df(x) \) exists and

\[
Df(x)(h_l) \ldots (h_1) = a_{1 \ldots 1}(x, h_l, \ldots, h_1).
\]

With these lemmas, the main result can be established. This is the generalization of the well known result for analytic functions.
**Theorem 20.13.6** Let $X$ and $Y$ be two complex Banach spaces and let $U$ be an open set in $X$. Then $f : U \to Y$ is analytic on $U$ if and only if $Df(x)$ exists for each $x \in U$ and in this case, $f \in C^\infty(U)$, and if $h \in X$, then whenever $z$ is small enough,

$$f(x + zh) = f(x) + \sum_{n=1}^{\infty} \frac{D^n f(x) h^n z^n}{n!}.$$

**Proof:** We know

$$f(x + zh) = f(x) + \sum_{n=1}^{\infty} a_n(x, h) z^n.$$ Differentiating, we obtain

$$D^k f(x + zh) h^k = k! a_k(x, h) + \sum_{n=k+1}^{\infty} n(n-1) \cdots (n-k+1) z^{n-k}.$$ Letting $z = 0$ this shows

$$D^k f(x) h^k = k! a_k(x, h)$$ and this proves half the theorem.

Conversely, if $Df(x)$ exists on $U$, it is clear that $f$ is analytic on some ball, $B(x, r) \subseteq U$, $z \to f(y + zh)$ is analytic for $y \in B(x, r)$ and small enough $z$. Therefore the formula involving the series follows. ■

### 20.14 Ordinary Differential Equations

In this section we give an application to ordinary differential equations. To begin with, here are two Banach spaces which will be of use. Let $Z$ be a complex Banach space and let $X$ be the space of functions mapping $D_1 \equiv B(0, 1)$ to $Z$ such that the functions are continuous on $D_1$ and analytic on $B_1 \equiv B(0, 1)$, the derivative is the restriction to $B_1$ of a continuous function defined on $D_1$, and the function equals 0 at 0.

$$X \equiv \{ \phi \in C(D_1, X) : \phi(0) = 0 \}$$

The norm on $X$ will be

$$||\phi||_X \equiv ||\phi||_\infty + ||\phi'||_\infty$$

where

$$||\phi||_\infty \equiv \sup \{ ||\phi(t)||_Z : t \in B_1 \}.$$ (Note that for a function continuous on $D_1$ it does not matter in the above definition of $||\cdot||_\infty$ whether we use $B_1$ or $D_1$ in the definition.) We define $Y$ to be the space of continuous functions which are defined on $D_1$ having values in $Z$ which are also analytic on $B_1$. The norm on $Y$ is defined as

$$||\phi||_\infty \equiv ||\phi||_Y.$$ Note that $B_1$ is in $\mathbb{C}$. 

Lemma 20.14.1 The spaces $X$ and $Y$ with the given norms are Banach spaces and if $L : X \to Y$ is defined as $L \phi (t) = \phi'(t)$ for all $t \in B_1$, then $L$ is one to one, onto and continuous.

**Proof:** It is clear that $X$ and $Y$ are both normed linear spaces. It remains to show they are Banach spaces. Suppose $\{ \phi_n \}$ is a Cauchy sequence in $X$. Then $\phi_n \to \phi$ uniformly and $\phi_n' \to \psi$ uniformly where $\psi$ and $\phi$ are continuous on $D_1$. We need to verify that $\psi = \phi'$ on $B_1$. Letting $C_1$ be the unit circle, the Cauchy integral formula implies for $t \in B_1$,

$$\phi(t) = \lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{C_1} \frac{\phi_n(w)}{w-t} \, dw = \frac{1}{2\pi i} \int_{C_1} \frac{\phi(w)}{w-t} \, dw$$

which shows $\phi'(t)$ exists on $B_1$. Also for $t \in B_1$,

$$\psi(t) = \lim_{n \to \infty} \phi_n'(t) = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{C_1} \frac{\phi_n'(w)}{(w-t)^2} \, dw = \frac{1}{2\pi i} \int_{C_1} \frac{\phi(w)}{(w-t)^2} \, dw = \phi'(t).$$

This shows $X$ is a Banach space. A similar argument using the Cauchy integral theorem shows $Y$ is a Banach space also. It is obvious that $L$ is continuous. It remains to show $L$ is one to one and onto.

Let $\phi \in Y$. We need to show $\phi = L\psi$ for some $\psi \in X$. Let

$$\psi(t) \equiv \int_{\Gamma} \phi(w) \, dw$$

where $\Gamma$ is any piecewise smooth curve from 0 to $t$. By the Cauchy integral theorem, this definition is well defined and it is clear that $\psi(0) = 0$, $\psi'(t) = \phi(t)$, and $\psi$ is continuous on $D_1$. This shows $L$ is onto.

It only remains to show $L$ is one to one. Suppose $L\phi = 0$. Since $\phi(0) = 0$,

$$\phi(t) = \int_0^1 \phi'(ts) \, ds = 0$$

if $t \neq 0$. But $\phi(0)$ is given to equal zero. Thus $L$ is one to one as claimed. ■

Theorem 20.14.2 Let $\Lambda$ and $Z$ be complex Banach spaces and let $W$ be an open subset of $\mathbb{C} \times Z \times \Lambda$ containing $(0, y_0, \lambda)$. Also let $f : W \to Z$ be analytic. Then there exists a unique $y = y(y_0, \lambda)$ solving

$$y' = f(t, y, \lambda), \quad y(0) = y_0 \quad (20.14.41)$$

valid for $t \in D_\alpha \equiv \overline{B(0, |\alpha|)}$ where $\alpha = \alpha(y_0, \lambda)$. Furthermore, the map

$$(t, y_0, \lambda) \to y(y_0, \lambda)(t)$$

is analytic.
20.14. ORDINARY DIFFERENTIAL EQUATIONS

Proof: Let \( \alpha = t \) and define \( \phi(s) \equiv y(t) - y_0 \). Then \( y \) is a solution to (20.14.41) for \( t \in D_\alpha \) if and only if \( \phi \) is a solution for \( s \in D_1 \equiv B(0,1) \) to the equations

\[
\phi'(s) = \alpha f(\alpha s, \phi(s) + y_0, \lambda), \quad \phi(0) = 0.
\]

Let \( X, Y, \) and \( L \) be given above and define

\[
\widetilde{W} \equiv \{(\alpha, \tilde{y}_0, \mu, \phi) \in \mathbb{C} \times Z \times \Lambda \times X : \text{for } s \in D_1, (s\alpha, \tilde{y}_0 + \phi(s), \mu) \in W\}.
\]

For a given \((\alpha, \tilde{y}_0, \mu, \phi) \in \widetilde{W}\),

\[
\{(s\alpha, \tilde{y}_0 + \phi(s), \mu) : s \in D_1\}
\]

is a compact subset of \( W \). This is because you have \( s \to (\alpha, \tilde{y}_0 + \phi(s), \mu) \) is the continuous image of a compact set which is assumed to be in \( W \). Consequently, the distance from this set to \( W^C \) is positive and so if \((\beta, y_0, \lambda, \psi) \) is sufficiently close to \((\alpha, \tilde{y}_0, \mu, \phi) \) in \( \mathbb{C} \times Z \times \Lambda \times X \) it follows \((\beta, y_0, \lambda, \psi) \) is also in \( \tilde{W} \). This shows \( \tilde{W} \) is an open subset of \( \mathbb{C} \times Z \times \Lambda \times X \).

Now define \( F: \tilde{W} \to Y \) (Recall that \( Y \) was a space of functions.) by

\[
F(\alpha, \tilde{y}_0, \mu, \phi)(s) \equiv L\phi(s) - \alpha f(\alpha s, \phi(s) + \tilde{y}_0, \mu).
\]

Then

\[
F(0, y_0, \lambda, 0) = L\phi = 0,
\]

and \( F \) is analytic in \( \tilde{W} \). Also

\[
D_4F(0, y_0, \lambda, 0) \psi = L\psi = \psi'
\]

and so \( D_4F(0, y_0, \lambda, 0) \in \mathcal{L}(X,Y) \), is one to one, onto and continuous by Lemma 20.14.1.

By the open mapping theorem, its inverse is also continuous. Therefore, the conditions of the implicit function theorem are satisfied and so there exists \( r > 0 \) such that if

\[
|\alpha| + \|\mu - \lambda\| + \|\tilde{y}_0 - y_0\| < r,
\]

then there exists a unique \( \phi \in X \) such that

\[
F(\alpha, \tilde{y}_0, \mu, \phi) = 0,
\]

and \( \phi \) is an analytic function of \((\alpha, \tilde{y}_0, \mu)\). Fixing \( 0 < \alpha < r \), it follows

\[
(\tilde{y}_0, \mu) \to y(\tilde{y}_0, \mu)
\]

is analytic on an open subset of \( Z \times \Lambda \). Also \( t \to y(\tilde{y}_0, \mu)(t) \) is an analytic function because of the definition of \( y \) in terms of \( \phi, \phi(s) \equiv y(t) - y_0 \). It follows that for \( t \in B(0,|\alpha|) \),

\[
(\tilde{y}_0, \mu) \to y(\tilde{y}_0, \mu)(t) \quad \text{and} \quad t \to y(\tilde{y}_0, \mu)(t)
\]

are both analytic. \( \blacksquare \)
20.15 Exercises

1. Suppose \( L \in \mathcal{L}(X,Y) \) where \( X \) and \( Y \) are two finite dimensional normed vector spaces and suppose \( L \) is one to one. Show there exists \( r > 0 \) such that for all \( x \in X \),

\[
\|Lx\| \geq r \|x\|.
\]

**Hint:** Show that \( \|x\| \equiv \|Lx\| \) is a norm. Now suppose \( L \in \mathcal{L}(X,Y) \) is one to one and onto for \( X,Y \) Banach spaces. Explain why the same result holds.

**Hint:** Recall open mapping theorem.

2. Suppose \( B \) is an open ball in \( X \), a Banach space, and \( f : B \to Y \) is differentiable. Suppose also there exists \( L \in \mathcal{L}(X,Y) \) such that

\[
\|Df(x) - L\| < k
\]

for all \( x \in B \). Show that if \( x_1, x_2 \in B \),

\[
\|f(x_1) - f(x_2) - L(x_1 - x_2)\| \leq k\|x_1 - x_2\|.
\]

**Hint:** Consider \( Tx = f(x) - Lx \) and argue \( \|DT(x)\| < k \).

3. \( \uparrow \) Let \( U \) be an open subset of \( X \), \( f : U \to Y \) where \( X,Y \) are finite dimensional normed linear spaces and suppose \( f \in C^1(U) \) and \( Df(x_0) \) is one to one. Then show \( f \) is one to one near \( x_0 \). **Hint:** Show using the assumption that \( f \) is \( C^1 \) that there exists \( \delta > 0 \) such that if

\[
x_1, x_2 \in B(x_0, \delta),
\]

then

\[
|f(x_1) - f(x_2) - Df(x_0)(x_1 - x_2)| \leq \frac{r}{2}\|x_1 - x_2\| \quad (20.15.42)
\]

then use Problem \( \boxed{\square} \). In case \( X,Y \) are Banach spaces, assume \( Df(x_0) \) is one to one and onto.

4. Suppose \( U \subseteq X \) is an open subset of \( X \) a Banach space and that \( f : U \to Y \) is differentiable at \( x_0 \in U \) such that \( Df(x_0) \) is one to one and onto from \( X \) to \( Y \). \( (Df(x_0)^{-1} \in \mathcal{L}(Y,X)) \) Then show that \( f(x) \neq f(x_0) \) for all \( x \) sufficiently near but not equal to \( x_0 \). In this case, you only know the derivative exists at \( x_0 \).

5. Suppose \( M \in \mathcal{L}(X,Y) \) where \( X \) and \( Y \) are finite dimensional linear spaces and suppose \( M \) is onto. Show there exists \( L \in \mathcal{L}(Y,X) \) such that

\[
LMx = Px
\]

where \( P \in \mathcal{L}(X,X) \), and \( P^2 = P \). Also show \( L \) is one to one and onto from \( X_1 \) to \( Y \). **Hint:** Let \( \{y_1 \cdots y_n\} \) be a basis of \( Y \) and let \( Mx_i = y_i \). Then define

\[
Ly = \sum_{i=1}^{n} \alpha_i x_i \text{ where } y = \sum_{i=1}^{n} \alpha_i y_i.
\]
Suppose \( \{x_1, \cdots, x_n\} \) is a linearly independent set and show you can obtain \( \{x_1, \cdots, x_n, \cdots, x_m\} \), a basis for \( X \) in which \( Mx_j = 0 \) for \( j > n \). Then let
\[
P(x) = \sum_{i=1}^{n} \alpha_i x_i \]
where
\[
x = \sum_{i=1}^{m} \alpha_i x_i.
\]

6. ↑ Let \( f : U \subseteq X \to Y \), \( f \) is \( C^1 \), and \( Df(x) \) is onto for each \( x \in U \). Then show \( f \) maps open subsets of \( U \) onto open sets in \( Y \). **Hint:** Let \( P = LDF(x) \) as in Problem 3. Argue \( L \) maps open sets from \( Y \) to open sets of \( X \) \( \equiv \) \( PX \) and \( L^{-1} \) maps open sets from \( X \) to open sets of \( Y \). Then \( Lf(x + v) = Lf(x) + LDF(x)v + o(v) \). Now for \( \varepsilon \in X_1 \), let \( h(\varepsilon) = Lf(x + \varepsilon) - Lf(x) \). Then \( h \) is \( C^1 \) on some small open subset of \( X_1 \) containing \( 0 \) and \( Dh(0) = LDF(x) \) which is seen to be one to one and onto and in \( L(X_1, X_1) \). Therefore, if \( r \) is small enough, \( h(B(0,r)) \) equals an open set in \( X_1, V \). This is by the inverse function theorem. Hence \( L(f(x + B(0,r)) - f(x)) = V \) and so \( f(x + B(0,r)) - f(x) = L^{-1}(V) \), an open set in \( Y \).

7. Suppose \( U \subseteq \mathbb{R}^2 \) is an open set and \( f : U \to \mathbb{R}^3 \) is \( C^1 \). Suppose \( Df(s_0, t_0) \) has rank two and
\[
f(s_0, t_0) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.
\]
Show that for \( (s,t) \) near \( (s_0, t_0) \), the points \( f(s,t) \) may be realized in one of the following forms.
\[
\{(x, y, \phi(x, y)) : (x, y) \text{ near } (x_0, y_0)\},
\]
or
\[
\{(\phi(y, z), y, z) : (y, z) \text{ near } (y_0, z_0)\},
\]
or
\[
\{(x, \phi(x, z), z) : (x, z) \text{ near } (x_0, z_0)\}.
\]
This shows that parametrically defined surfaces can be obtained locally in a particularly simple form.

8. Let \( f : U \to Y \), \( Df(x) \) exists for all \( x \in U \), \( B(x_0, \delta) \subseteq U \), and there exists \( L \in L(X,Y) \), such that \( L^{-1} \in L(Y,X) \), and for all \( x \in B(x_0, \delta) \)
\[
||Df(x) - L|| < \frac{r}{||L^{-1}||}, \quad r < 1.
\]
Show that there exists \( \varepsilon > 0 \) and an open subset of \( B(x_0, \delta), V \), such that \( f : V \to B(f(x_0), \varepsilon) \) is one to one and onto. Also \( Df^{-1}(y) \) exists for each \( y \in B(f(x_0), \varepsilon) \) and is given by the formula
\[
Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}.
\]
CHAPTER 20. THE DERIVATIVE

**Hint:** Let
\[ T_y(x) \equiv T(x, y) \equiv x - L^{-1}(f(x) - y) \]
for \(|y - f(x)| < \frac{(1-r)s}{2||L||} \). Consider \( \{T^n_y(x_0)\} \). This is a version of the inverse function theorem for \( f \) only differentiable, not \( C^1 \).

9. Denote by \( C([0,T], X) \) the space of functions which are continuous having values in \( X \) and define a norm on this linear space as follows.
\[ ||f||_\lambda = \max \{|f(t)| e^{\lambda t} : t \in [0,T]\}. \]
Show for each \( \lambda \in \mathbb{R} \), this is a norm and that \( C([0,T]; X) \) is a complete normed linear space with this norm.

10. Let \( f: [0,T] \times X \to X \) be continuous and suppose \( f \) satisfies a Lipschitz condition,
\[ |f(t,x) - f(t,y)| \leq K |x-y| \]
and let \( x_0 \in X \). Show there exists a unique solution to the Cauchy problem,
\[ x' = f(t,x), \quad x(0) = x_0, \]
for \( t \in [0,T] \). **Hint:** Consider the map \( G: C([0,T]; X) \to C([0,T]; X) \)
defined by
\[ Gx(t) \equiv x_0 + \int_0^t f(s, x(s)) \, ds, \]
where the integral is defined componentwise. Show \( G \) is a contraction map for \( ||\cdot||_\lambda \) given in Problem 9 for a suitable choice of \( \lambda \) and that therefore, it has a unique fixed point in \( C([0,T]; X) \). Next argue, using the fundamental theorem of calculus, that this fixed point is the unique solution to the Cauchy problem.

11. Use Theorem 6.7.5 to give another proof of the above theorem. **Hint:** Use the same mapping and show that a large power is a contraction map.

12. Suppose you know that \( u(t) \leq a + \int_0^t k(s) u(s) \, ds \) where \( k(s) \geq 0 \) and \( k \) is in \( L^1([0,T]) \). Show that then \( u(t) \leq a \exp \left( \int_0^t k(s) \, ds \right) \). This is a version of Gronwall’s inequality. **Hint:** Let \( W(t) = \int_0^t k(s) u(s) \, ds \). Then explain why \( W'(t) - k(t) W(t) \leq ak(t) \). Now use the usual technique of an integrating factor you saw in beginning differential equations.

13. Use the above Gronwall’s inequality to establish a result of continuous dependence on the initial condition and \( f \) in the ordinary differential equation of Problem 13.
14. The existence of partial derivatives does not imply continuity as was shown in an example. However, much more can be said than this. Consider

$$f(x, y) = \begin{cases} \frac{(x^2 - y^4)^2}{(x^2 + y^4)^2} & \text{if } (x, y) \neq (0, 0), \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show the directional derivative of $f$ at $(0, 0)$ exists and equals 0 for every direction. The directional derivative in the direction $(v_1, v_2)$ is defined as

$$\lim_{t \to 0} \frac{f(x + tv_1, y + tv_2) - f(x, y)}{t}.$$

Now consider the curve $x^2 = y^4$ and the curve $y = 0$ to verify the function fails to be continuous at $(0, 0)$.

15. Let

$$f(x, y) = \begin{cases} \frac{x^2 y^4}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that this function is not continuous at $(0, 0)$ but that it has all directional derivatives at $(0, 0)$ and they all equal 0.

16. Let $X_i$ be a normed linear space having norm $||\cdot||_i$. Then we can make $\prod_{i=1}^n X_i$ into a normed linear space by defining a norm on $x \in \prod_{i=1}^n X_i$ by

$$||x|| \equiv \max \{||x_i||_i : i = 1, \cdots, n\}.$$

Show this is a norm on $\prod_{i=1}^n X_i$ as claimed.

17. Suppose $f : U \subseteq X \times Y \to Z$ and $D_2 f (x_0, y_0)^{-1} \in \mathcal{L}(X, Y)$ exists and $f$ is $C^1$ so the conditions of the implicit function theorem are satisfied. Also suppose that all these are complex Banach spaces. Show that then the implicitly defined function $y = y(x)$ is analytic. Thus it has infinitely many derivatives and can be given as a power series as described above.
CHAPTER 20. THE DERIVATIVE
Chapter 21

Degree Theory, An Introduction

This chapter is on the Brouwer degree, a very useful concept with numerous and important applications. The degree can be used to prove some difficult theorems in topology such as the Brouwer fixed point theorem, the Jordan separation theorem, and the invariance of domain theorem. It also is used in bifurcation theory and many other areas in which it is an essential tool. The degree will be developed for $\mathbb{R}^n$ first. When this is understood, it is not too difficult to extend to versions of the degree which hold in Banach space. There is more on degree theory in the book by Deimling \cite{35} and much of the presentation here follows this reference. Another more recent book which is really good is \cite{40}. This is a whole book on degree theory.

To give you an idea what the degree is about, consider a real valued $C^1$ function defined on an interval, $I$, and let $y \in f(I)$ be such that $f'(x) \neq 0$ for all $x \in f^{-1}(y)$. In this case the degree is the sum of the signs of $f'(x)$ for $x \in f^{-1}(y)$, written as $d(f, I, y)$.

In the above picture, $d(f, I, y)$ is 0 because there are two places where the sign is 1 and two where it is $-1$.

The amazing thing about this is the number you obtain in this simple manner
is a specialization of something which is defined for continuous functions and which has nothing to do with differentiability.

There are many ways to obtain the Brouwer degree. The method I will use here is due to Heinz [56] and appeared in 1959. It involves first studying the degree for functions in $C^\infty$ and establishing all its most important topological properties with the aid of an integral. Then when this is done, it is extended to general continuous functions.

When you have the topological degree, you can get all sorts of amazing theorems like the invariance of domain theorem and others. The first section contains fundamental theorems from measure theory.

### 21.1 Sard’s Lemma

The following lemma is a wonderful application of the Vitali covering theorem.

**Lemma 21.1.1** Let $h$ be differentiable on $U$. If $T \subseteq U$ and $m_p(T) = 0$, then $m_p(h(T)) = 0$.

**Proof:** For $k \in \mathbb{N}$

$$T_k \equiv \{x \in T : ||Dh(x)|| < k\}$$

and let $\varepsilon > 0$ be given. Since $T_k$ is a subset of a set of measure zero, it is measurable, but we don’t need to pay much attention to this fact. Now by outer regularity, there exists an open set $V$, containing $T_k$ which is contained in $U$ such that $m_p(V) < \varepsilon$.

Let $x \in T_k$. Then by differentiability,

$$h(x + v) = h(x) + Dh(x)v + o(v)$$

and so there exist arbitrarily small $r_x < 1$ such that $B(x, 5r_x) \subseteq V$ and whenever $||v|| \leq 5r_x, ||o(v)|| < \frac{1}{k} ||v||$. Thus

$$h(B(x, 5r_x)) \subseteq Dh(x) (B(0, 5r_x)) + h(x) + B(0, r_x) \subseteq B(0, k5r_x) + B(0, r_x) + h(x) \subseteq B(h(x), (5k + 1) r_x) \subseteq B(h(x), 6kr_x)$$

From the Vitali covering theorem, there exists a countable disjoint sequence of these balls, $\{B(x_i, r_i)\}_{i=1}^\infty$ such that $\{B(x_i, 5r_i)\}_{i=1}^\infty = \{\hat{B}_i\}_{i=1}^\infty$ covers $T_k$. Then letting $\overline{m_p}$ denote the outer measure determined by $m_p$,

$$\overline{m_p}(h(T_k)) \leq \overline{m_p}(h\left(\bigcup_{i=1}^\infty \hat{B}_i\right)) \leq \overline{m_p}\left(\bigcup_{i=1}^\infty h\left(\hat{B}_i\right)\right)$$

$$\leq \sum_{i=1}^\infty \overline{m_p}(h\left(\hat{B}_i\right)) \leq \sum_{i=1}^\infty m_p(B(h(x_i), 6kr_x))$$

$$= \sum_{i=1}^\infty m_p(B(x_i, 6kr_x)) = (6k)^p \sum_{i=1}^\infty m_p(B(x_i, r_x)) \leq (6k)^p m_p(V) \leq (6k)^p \varepsilon$$
21.1. SARD’S LEMMA

Since \( \varepsilon > 0 \) is arbitrary, this shows \( m_p(h(T_k)) = \overline{m_p}(h(T_k)) = 0 \). Now

\[
m_p(h(T)) = \lim_{k \to \infty} m_p(h(T_k)) = 0. \]

The following is Sard’s lemma. In the proof, it does not matter which norm you use in defining balls but it may be easiest to consider the norm

\[
||x|| \equiv \max\{|x_i|, i = 1, \ldots, p\}
\]

It is a very simple proof because it uses the Vitali covering theorem to remove technical considerations.

**Lemma 21.1.2 (Sard)** Let \( U \) be an open set in \( \mathbb{R}^p \) and let \( h: U \to \mathbb{R}^p \) be differentiable. Let

\[
Z \equiv \{x \in U : \det Dh(x) = 0\}.
\]

Then \( m_p(h(Z)) = 0 \).

**Proof:** For convenience, assume the balls in the following argument come from \( ||\cdot||_\infty \). First note that \( Z \) is a Borel set because \( h \) is continuous and so the component functions of the Jacobian matrix are each Borel measurable. Hence the determinant is also Borel measurable.

Suppose that \( U \) is a bounded open set. Let \( \varepsilon > 0 \) be given. Also let \( V \supseteq Z \) with \( V \subseteq U \) open, and

\[
m_p(Z) + \varepsilon > m_p(V).
\]

Now let \( x \in Z \). Then since \( h \) is differentiable at \( x \), there exists \( \delta_x > 0 \) such that if \( r < \delta_x \), then \( B(x,r) \subseteq V \) and also \( o(v) < \eta \|v\| \) for \( \|v\| < r \). Thus

\[
h(x + B(0,r)) = h(B(x,r)) \subseteq h(x) + Dh(x) \cdot (B(0,r)) + B(0,\eta r), \eta < 1.
\]

Regard \( Dh(x) \) as an \( n \times n \) matrix, the matrix of the linear transformation \( Dh(x) \) with respect to the usual coordinates. Since \( x \in Z \), it follows that there exists an invertible matrix \( A \) such that \( ADh(x) \) is in row reduced echelon form with a row of zeros on the bottom. Therefore,

\[
m_p(A(h(B(x,r)))) \leq m_p(ADh(x)(B(0,r)) + AB(0,\eta r)) \quad (21.1.1)
\]

The diameter of \( ADh(x)(B(0,r)) \) is no larger than \( ||A|| \|Dh(x)||2r \) and it lies in \( \mathbb{R}^{p-1} \times \{0\} \). The diameter of \( AB(0,\eta r) \) is no more than \( ||A|| \|2\eta \| \). Therefore, the measure of the right side in (21.1.1) is no more than

\[
[||A|| \|Dh(x)||2r + ||A|| (2\eta)) r^{p-1} \| \eta
\]

\[
\leq C (||A|| \|Dh(x)||) (2r)^p \eta
\]

That is,

\[
m_p(A(h(B(x,r)))) \leq C (||A|| \|Dh(x)||) (2r)^p \eta
\]
Hence from the change of variables formula for linear maps,
\[ m_p(h(B(x,r))) \leq \eta C \left( \frac{||A||, ||Dh(x)||}{|\det(A)|} \right) m_p(B(x,r)) \]
Then letting \( \delta_x \) be still smaller if necessary, corresponding to sufficiently small \( \eta \),
\[ m_p(h(B(x,r))) \leq \varepsilon m_p(B(x,r)) \]
The balls of this form constitute a Vitali cover of \( Z \). Hence, by the Vitali covering theorem, Corollary 11.4.6, there exists \( \{B_i\}_{i=1}^\infty, B_i = B_i(x_i, r_i) \), a collection of disjoint balls, each of which is contained in \( V \), such that \( m_p(h(B_i)) \leq \varepsilon m_p(B_i) \) and \( m_p(Z \setminus \bigcup_i B_i) = 0 \). Hence from Lemma 21.1.1,
\[ m_p(h(Z) \setminus \bigcup_i h(B_i)) \leq m_p(h(Z \setminus \bigcup_i B_i)) = 0 \]
Therefore,
\[ m_p(h(Z)) \leq \sum_i m_p(h(B_i)) \leq \varepsilon \sum_i m_p(B_i) \leq \varepsilon (m_p(V)) \leq \varepsilon (m_p(Z) + \varepsilon) \]
Since \( \varepsilon \) is arbitrary, this shows \( m_p(h(Z)) = 0 \). What if \( U \) is not bounded? Then consider \( Z_n = Z \cap B(0, n) \). From what was just shown, \( h(Z_n) \) has measure 0 and so it follows that \( h(Z) \) also does, being the countable union of sets of measure zero.

### 21.2 Preliminary Results

In this chapter \( \Omega \) will refer to a bounded open set.

**Definition 21.2.1** For \( \Omega \) a bounded open set, denote by \( C(\Omega) \) the set of functions which are restrictions of functions in \( C_c(\mathbb{R}^n) \) to \( \Omega \) and by \( C^m(\Omega) \), \( m \leq \infty \) the space of restrictions of functions in \( C^m_c(\mathbb{R}^n) \) to \( \Omega \). If \( f \in C(\Omega) \) the symbol \( f \) will also be used to denote a function defined on \( \mathbb{R}^n \) equaling \( f \) on \( \Omega \) when convenient. This saves the trouble of having to extend to all of \( \mathbb{R}^n \) using something like Theorem 14.2.3. The subscript \( c \) indicates that the functions have compact support. The norm on \( C(\Omega) \) is defined as follows.

\[ ||f||_\infty \equiv \sup \{ |f(\mathbf{x})| : \mathbf{x} \in \overline{\Omega} \} \]

If the functions take values in \( \mathbb{R}^n \) write \( C^m(\Omega; \mathbb{R}^n) \) or \( C(\Omega; \mathbb{R}^n) \) for these functions if there is no differentiability assumed. The norm on \( C(\Omega; \mathbb{R}^n) \) is defined in the same way as above,

\[ ||f||_\infty \equiv \sup \{ |f(\mathbf{x})| : \mathbf{x} \in \overline{\Omega} \} \]

Of course if \( m = \infty \), the notation means that there are infinitely many derivatives. Also, \( C(\Omega; \mathbb{R}^n) \) consists of functions which are continuous on \( \Omega \) that have values in \( \mathbb{R}^n \) and \( C^m(\Omega; \mathbb{R}^n) \) denotes the functions which have \( m \) continuous derivatives defined on \( \Omega \).
21.2. PRELIMINARY RESULTS

**Theorem 21.2.2** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and let $f \in C_c(\mathbb{R}^n)$. Then there exists $g \in C^\infty_c(\mathbb{R}^n)$ with $\|g - f\|_{\infty, \mathbb{R}^n} < \varepsilon$.

**Proof:** Form $g \equiv f \ast \psi_n$ for a mollifier $\psi_n$. This will approximate $f$ uniformly on $\Omega$, and will be in $C^\infty_c(\mathbb{R}^n)$. $\blacksquare$

Using the Weierstrass approximation theorem, you could also get $g$ to equal a polynomial for all $x \in \Omega$.

Applying this result to the components of a vector valued function yields the following corollary.

**Corollary 21.2.3** If $f \in C(\overline{\Omega}; \mathbb{R}^n)$ for $\Omega$ a bounded subset of $\mathbb{R}^n$, then for all $\varepsilon > 0$, there exists $g \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ such that

$$\|g - f\|_{\infty} < \varepsilon.$$ 

Lemma [14.3.1] on Page 413 will also play an important role in the definition of the Brouwer degree. Earlier it made possible an easy proof of the Brouwer fixed point theorem. Later in this chapter, it is used to show the definition of the degree is well defined. For convenience, here it is stated again.

**Lemma 21.2.4** Let $g : U \rightarrow \mathbb{R}^n$ be $C^2$ where $U$ is an open subset of $\mathbb{R}^n$. Then

$$\sum_{j=1}^{n} \text{cof}((Dg)_{ij,j}) = 0,$$

where here $(Dg)_{ij} \equiv b_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\text{cof}((Dg)_{ij}) = \frac{\partial \text{det}(Dg)}{\partial g_{i,j}}$.

Another simple result which will be used whenever convenient is the following lemma.

**Lemma 21.2.5** Let $K$ be a compact set and $C$ a closed set in a complete normed vector space such that $K \cap C = \emptyset$. Then

$$\text{dist}(K, C) > 0.$$ 

**Proof:** Let

$$d \equiv \inf \{||k - c|| : k \in K, c \in C\}$$

Let $\{k_n\}, \{c_n\}$ be such that

$$d + \frac{1}{n} > ||k_n - c_n||.$$ 

Since $K$ is compact, there is a subsequence still denoted by $\{k_n\}$ such that $k_n \rightarrow k \in K$. Then also

$$||c_n - c_m|| \leq ||c_n - k_n|| + ||k_n - k_m|| + ||c_m - k_m||$$
If \( d = 0 \), then as \( m, n \to \infty \) it follows \( ||c_n - c_m|| \to 0 \) and so \( \{c_n\} \) is a Cauchy sequence which must converge to some \( c \in C \). But then \( ||c - k|| = \lim_{n \to \infty} ||c_n - k_n|| = 0 \) and so \( c = k \in C \cap K \), a contradiction to these sets being disjoint.

In particular the distance between a point and a closed set is always positive if the point is not in the closed set. Of course this is obvious even without the above lemma. The above lemmas will be used now to prove technical lemmas which are the basis for everything.

**Definition 21.2.6** Let \( g \in C^\infty (\overline{\Omega}; \mathbb{R}^n) \) where \( \Omega \) is a bounded open set. Also let \( \phi_\varepsilon \) be a mollifier.

\[
\phi_\varepsilon \in C^\infty_c (B(0, \varepsilon)), \phi_\varepsilon \geq 0, \quad \int \phi_\varepsilon dx = 1.
\]

First, here is a technical lemma which is the reason it all works out. It is a result on homotopy invariance for functions which are \( C^\infty \).

**Lemma 21.2.7** If \( h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is in \( C^\infty_c (\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n) \), and \( 0 \notin h (\partial \Omega \times [\alpha, \beta]) \) then for \( 0 < \varepsilon < \text{dist} (0, h (\partial \Omega \times [\alpha, \beta])) \),

\[
t \to \int_\Omega \phi_\varepsilon (h (x, t)) \det D_1 h (x, t) \, dx
\]

is constant for \( t \in (a, b) \), an open set which contains \( [\alpha, \beta] \).

**Proof:** By continuity, we can get such an open interval, \( (a, b) \) such that it contains \( [\alpha, \beta] \) and \( 0 \notin h (\partial \Omega \times [a, b]) \). Let \( \varepsilon > 0 \) be such that for all \( t \in [a, b] \),

\[
B(0, \varepsilon) \cap h (\partial \Omega \times [a, b]) = \emptyset \quad (21.2.2)
\]

Define for \( t \in (a, b) \),

\[
H (t) \equiv \int_\Omega \phi_\varepsilon (h (x, t)) \det D_1 h (x, t) \, dx
\]

Then if \( t \in (a, b) \),

\[
H' (t) = \int_\Omega \sum_\alpha \phi_\varepsilon (h (x, t)) h_{\alpha, t} (x, t) \det D_1 h (x, t) \, dx
\]

\[
+ \int_\Omega \phi_\varepsilon (h (x, t)) \sum_{\alpha, j} \det D_1 (h (x, t))_{\alpha j} h_{\alpha, j} \, dx
\]

\[
\equiv A + B.
\]

In this formula, the function \( \det \) is considered as a function of the \( n^2 \) entries in the \( n \times n \) matrix and the \( \alpha j \) represents the derivative with respect to the \( \alpha j \)th entry \( h_{\alpha, j} \). Now as in the proof of Lemma 14.3.7 on Page 413:

\[
\det D_1 (h (x, t))_{\alpha j} = (\text{cof} D_1 (h (x, t)))_{\alpha j}
\]
and so
\[ B = \int_{\Omega} \sum_{\alpha} \sum_{j} \phi_{\epsilon} (h(x,t)) (\text{cof } D_1 (h(x,t)))_{\alpha j} h_{\alpha,j} dx. \]

By hypothesis
\[ x \rightarrow \phi_{\epsilon} (h(x,t)) (\text{cof } D_1 (h(x,t)))_{\alpha j} \]
is in \( C_\epsilon^\infty (\Omega) \) because if \( x \in \partial \Omega \), it follows that for all \( t \in [a,b] \)
\[ h(x,t) \notin B(0,\epsilon) \]
and so \( \phi_{\epsilon} (h(x,t)) = 0 \). Thus it equals 0 on \( \partial \Omega \). Therefore, integrate by parts and write
\[ B = - \int_{\Omega} \sum_{\alpha} \sum_{j} \frac{\partial}{\partial x_j} (\phi_{\epsilon} (h(x,t))) (\text{cof } D_1 (h(x,t)))_{\alpha j} h_{\alpha,t} dx + \]
\[ - \int_{\Omega} \sum_{\alpha} \sum_{j} \phi_{\epsilon} (h(x,t)) (\text{cof } D (h(x,t)))_{\alpha j,j} h_{\alpha,t} dx \]
The second term equals zero by Lemma 21.2.4. Simplifying the first term yields
\[ B = - \int_{\Omega} \sum_{\alpha} \sum_{j} \phi_{\epsilon,\beta} (h(x,t)) h_{\beta,j} h_{\alpha,t} (\text{cof } D_1 (h(x,t)))_{\alpha j} dx \]
Now the sum on \( j \) is the dot product of the \( \beta^{th} \) row with the \( \alpha^{th} \) row of the cofactor matrix which equals zero unless \( \beta = \alpha \) because it would be a cofactor expansion of a matrix with two equal rows. When \( \beta = \alpha \), the sum on \( j \) reduces to
\[ \det (D_1 (h(x,t))) \]
Thus \( B \) reduces to
\[ - \int_{\Omega} \sum_{\alpha} \phi_{\epsilon,\alpha} (h(x,t)) h_{\alpha,j} \sum_{j} h_{\alpha,j} (x,t) (\text{cof } D_1 (h(x,t)))_{\alpha j} dx \]
\[ = - \int_{\Omega} \sum_{\alpha} \phi_{\epsilon,\alpha} (h(x,t)) h_{\alpha,t} \det (D_1 (h(x,t))) dx \]
Now \( A \) equals
\[ \int_{\Omega} \sum_{\alpha} \phi_{\epsilon,\alpha} (h(x,t)) h_{\alpha,t} (x,t) \det D_1 h(x,t) dx \]
which is the same thing with opposite sign. Hence these sum to 0. Therefore, \( H'(t) = 0 \) and so \( H \) is a constant on \( (a,b) \supseteq [\alpha,\beta] \).

The following is a situation in which one only has continuity in \( t \). Of course the difficulty is that it is not yet clear whether the constant depends on \( \epsilon \).
Corollary 21.2.8 If \( h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is in \( C_c (\mathbb{R}^n \times \mathbb{R},\mathbb{R}^n) \), and for each \( t \in [0,1] \), \( h(\cdot, t) \in C_c^\infty (\mathbb{R},\mathbb{R}^n) \) and for all \( 0 \notin h(\partial \Omega \times [0,1]) \) then for \( \varepsilon \) small enough, 

\[
\phi (h(x),t) \rightarrow \int_{\Omega} \phi (h(x),t) \det D_1 h(x,t) \, dx
\]

is constant for \( t \in [0,1] \).

**Proof:** Let \( 0 < 6\delta < \text{dist} (0, h(\partial \Omega \times [0,1])) \). Now let \( 0 = t_0 < t_1 < \ldots < t_m = 1 \). Also let \( \hat{h}(x,t_k) = h(x,t_k) \) and for \( t \in (t_{k-1}, t_k) \), 

\[
\hat{h}(x,t) \equiv h(x,t_{k-1}) + \frac{t-t_{k-1}}{t_k-t_{k-1}} (h(x,t_k) - h(x,t_{k-1}))
\]

Thus for fixed \( x \), this gives a piecewise linear function approximating \( t \rightarrow h(x,t) \). Let these \( t_k \) be close enough together that on each \( [t_{k-1}, t_k] \), 

\[
\max_{t \in [t_{k-1}, t_k]} \left\| \hat{h}(\cdot,t) - h(\cdot,t) \right\|_\infty < \delta
\]

It follows that on \( [t_{k-1}, t_k] \), for \( x \in \partial \Omega \),

\[
\min_{t \in [t_{k-1}, t_k]} \left| \hat{h}(x,t) - 0 \right| > \delta
\]

This function \( (x,t) \rightarrow \hat{h}(x,t) \) can be considered the restriction of a function in \( C_c^\infty (\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n) \) and \( 0 \notin \hat{h}(\partial \Omega, t) \) for all \( t \in [t_{k-1}, t_k] \). By Lemma 21.2.7, for all \( \varepsilon \) small enough, say \( \varepsilon < \delta \), 

\[
\int_{\Omega} \phi (\hat{h}(x,t)) \det (D_1 \hat{h}(x,t)) \, dx
\]

is constant on this interval \( [t_{k-1}, t_k] \).

This has shown that for \( k, j \in \{ 0, 2, \cdots, m \} \), 

\[
\int_{\Omega} \phi (h(x,t_k)) \det (D_1 h(x,t_k)) \, dx = \int_{\Omega} \phi (h(x,t_j)) \det (D_1 h(x,t_j)) \, dx
\]

You could include \( 1/2 \) in the partition. Then this would show that whatever the partition including this point,

\[
\int_{\Omega} \phi (h(x,t_k)) \det (D_1 h(x,t_k)) \, dx = \int_{\Omega} \phi (h(x,1/2)) \det (D_1 h(x,1/2)) \, dx
\]

Thus this integral is constant for \( t \in [0,1] \) as claimed. \( \blacksquare \)

This corollary is applied later to the situation where \( h(x,t) \) is of the form \( h(x,t) - y(t) \) where \( t \rightarrow y(t) \) is continuous and \( y(t) \notin h(\partial \Omega, t) \).
21.3 Definitions And Elementary Properties

First is what is meant by two functions being homotopic.

**Definition 21.3.1** \( \mathcal{U}_y \equiv \{ f \in C(\overline{\Omega};\mathbb{R}^n) : y \notin f(\partial\Omega) \} \).

(Recall that \( \partial\Omega = \overline{\Omega} \setminus \Omega \)).

For two functions, \( f, g \in \mathcal{U}_y \),

\[ f \sim g \text{ if there exists a continuous function,} \]

\[ h : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \]

such that \( h(x,1) = g(x) \) and \( h(x,0) = f(x) \) for \( x \in \overline{\Omega} \) and \( x \to h(x,t) \in \mathcal{U}_y \) for all \( t \in [0,1] \) \( y \notin h(\partial\Omega, t) \)). This function \( h \), is called a homotopy and \( f \) and \( g \) are homotopic.

**Definition 21.3.2** For \( W \) an open set in \( \mathbb{R}^n \) and \( g \in C^1(W;\mathbb{R}^n) \) \( y \) is called a regular value of \( g \) if whenever \( x \in g^{-1}(y) \), \( \det(Dg(x)) \neq 0 \). Note that if \( g^{-1}(y) = \emptyset \), it follows that \( y \) is a regular value from this definition. Denote by \( S_g \) the set of singular values of \( g \), those \( y \) such that \( \det(Dg(x)) = 0 \) for some \( x \in g^{-1}(y) \).

**Lemma 21.3.3** The relation \( \sim \) is an equivalence relation and, denoting by \([f]\) the equivalence class determined by \( f \), it follows that \([f]\) is an open subset of

\[ \mathcal{U}_y \equiv \{ f \in C(\overline{\Omega};\mathbb{R}^n) : y \notin f(\partial\Omega) \} . \]

Furthermore, \( \mathcal{U}_y \) is an open set in \( C(\overline{\Omega};\mathbb{R}^n) \). Let \( g \in C^\infty(\overline{\Omega};\mathbb{R}^n) \) and \( y \notin g(\partial\Omega) \) so \( \text{dist}(y, g(\partial\Omega)) > 5\delta \) for some positive \( \delta \). Then there exists \( g_1 \) such that \( \| g - g_1 \|_\infty < \delta \), \( y \) is a regular value of \( g_1 \), and

\[ \int_{\Omega} \phi(1-t) g(x) + t g_1(x) - y \det(D(1-t) g + t g_1))(x) dx \]

is constant for \( t \in [0,1] \).

**Proof:** In showing that \( \sim \) is an equivalence relation, it is easy to verify that \( f \sim f \) and that if \( f \sim g \), then \( g \sim f \). To verify the transitive property for an equivalence relation, suppose \( f \sim g \) and \( g \sim k \), with the homotopy for \( f \) and \( g \), the function, \( h_1 \) and the homotopy for \( g \) and \( k \), the function \( h_2 \). Thus \( h_1(x,0) = f(x) \), \( h_1(x,1) = g(x) \) and \( h_2(x,0) = g(x) \), \( h_2(x,1) = k(x) \). Then define a homotopy of \( f \) and \( k \) as follows.

\[ h(x,t) = \begin{cases} h_1(x,2t) & \text{if } t \in [0,\frac{1}{2}] \\ h_2(x,2t-1) & \text{if } t \in [\frac{1}{2},1] \end{cases} . \]

It is obvious that \( \mathcal{U}_y \) is an open subset of \( C(\overline{\Omega};\mathbb{R}^n) \). If \( g \in \mathcal{U}_y \) then \( y \notin g(\partial\Omega) \) a compact set. Hence if \( f \) is close enough to \( g \), the same is true of \( f \).
Next consider the claim that \([f]\) is also an open set. If \(f \in U_y\), there exists \(\delta > 0\) such that \(B(y, 2\delta) \cap f(\partial\Omega) = \emptyset\). Let \(f_1 \in C(\Omega; \mathbb{R}^n)\) with \(||f_1 - f||_{\infty} < \delta\). Then if \(t \in [0, 1]\), and \(x \in \partial\Omega\)

\[
|f(x) + t(f_1(x) - f(x)) - y| \geq |f(x) - y| - t||f - f_1||_{\infty} > 2\delta - t\delta > 0.
\]

Therefore, \(B(f, \delta) \subseteq [f]\) because if \(f_1 \in B(f, \delta)\), this shows that, letting \(h(x, t) \equiv f(x) + t(f_1(x) - f(x)), f_1 \sim f\).

Consider the last claim. There exists \(y_1 \in B(y, \delta)\) such that \(y_1\) is a regular value of \(g\) thanks to Sard's theorem. Let \(g_1(x) \equiv g(x) + y - y_1\) for all \(x \in \Omega\). Then \(||g - g_1||_{\infty} < \delta\) and so it follows that for \(t \in [0, 1]\), and \(x \in \partial\Omega\),

\[
|((1-t)g(x) + tg_1(x)) - y| = |g(x) - y + t(y - y_1)| \geq 5\delta - \delta > 0.
\]

Therefore, from Lemma 21.3.3, the integral is constant. Also \(y\) is a regular value of \(g_1\) because \(g_1(x) = y\) if and only if \(g(x) - y_1\) and \(y_1\) is a regular value for \(g\).

21.3.1 The Degree For \(C^\infty(\Omega; \mathbb{R}^n)\)

Here I will give a definition of the degree which works for all functions in \(C^\infty(\Omega; \mathbb{R}^n)\). These are the restrictions to \(\Omega\) of functions in \(C^\infty_c(\mathbb{R}^n; \mathbb{R}^n)\).

**Definition 21.3.4** For \(g \in C^\infty(\Omega; \mathbb{R}^n), y \notin g(\partial\Omega),\)

\[
d(g, \Omega, y) = \lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon(g(x) - y) \det Dg(x) \, dx
\]

The next lemma has to do with the existence of the limit in the definition of the degree and the fact that it is always an integer.

**Lemma 21.3.5** The above definition is well defined. In particular the limit exists. In fact

\[
\int_{\Omega} \phi_\varepsilon(g(x) - y) \det Dg(x) \, dx
\]

does not depend on \(\varepsilon\) whenever \(\varepsilon\) is small enough. If \(y\) is a regular value for \(g\) then for all \(\varepsilon\) small enough,

\[
\int_{\Omega} \phi_\varepsilon(g(x) - y) \det Dg(x) \, dx = \sum \{\text{sgn} (\det Dg(x)) : x \in g^{-1}(y)\}
\]

(21.3.3)

If \(g, f \in U_y \cap C^\infty(\Omega; \mathbb{R}^n)\), and

\[
y \notin (tf + (1-t)g)(\partial\Omega), \text{ all } t \in [0, 1]
\]

(21.3.4)
then
\[ d(f, \Omega, y) = d(g, \Omega, y) \]

If \( \text{dist} (y, g(\partial \Omega)) > 5\delta \) and \( y_1 \in B(y, \delta) \), then \( d(g, \Omega, y) = d(g, \Omega, y_1) \). Also, the appropriate integrals are equal. See the following picture for an illustration of this last claim. The degree has integer values.

**Proof:** If \( y \) is not a value of \( g \) then there is not much to show. For small enough \( \varepsilon \), you will get 0 in the integral.

The case where \( y \) is a regular value

First consider the case where \( y \) is a regular value of \( g \). I will show that in this case, the integral expression is eventually constant for small \( \varepsilon > 0 \) and equals the right side of 21.3.3. I claim the right side of this equation is actually a finite sum. This follows from the inverse function theorem because \( g^{-1}(y) \) is a closed, hence compact subset of \( \Omega \) due to the assumption that \( y \notin g(\partial \Omega) \). If \( g^{-1}(y) \) had infinitely many points in it, there would exist a sequence of distinct points \( \{x_k\} \subseteq g^{-1}(y) \). Since \( \Omega \) is bounded, some subsequence \( \{x_{k_l}\} \) would converge to a limit point \( x_\infty \). By continuity of \( g \), it follows \( x_\infty \in g^{-1}(y) \) also and so \( x_\infty \in \Omega \). Therefore, since \( y \) is a regular value, there is an open set, \( U_{x_\infty} \), containing \( x_\infty \) such that \( g(U_{x_\infty}) \) is an open set containing \( y \). Therefore, this set is finite and so the sum is well defined.

Thus the right side of 21.3.3 is finite when \( y \) is a regular value. Next I need to show the left side of this equation is eventually constant and equals the right side. By what was just shown, there are finitely many points, \( \{x_i\}_{i=1}^m = g^{-1}(y) \). By the inverse function theorem, there exist disjoint open sets \( U_i \) with \( x_i \in U_i \), such that \( g \) is one to one on \( U_i \) with \( \det(Dg(x)) \) having constant sign on \( U_i \) and \( g(U_i) \) is an open set containing \( y \). Then let \( \varepsilon \) be small enough that \( B(y, \varepsilon) \subseteq \bigcap_{i=1}^m g(U_i) \) and let \( V_i = g^{-1}(B(y, \varepsilon)) \cap U_i \).
Therefore, for any \( \varepsilon \) this small,

\[
\int_{\Omega} \phi_\varepsilon (g(x) - y) \det Dg(x) \, dx = \sum_{i=1}^{m} \int_{V_i} \phi_\varepsilon (g(x) - y) \det Dg(x) \, dx
\]

The reason for this is as follows. The integrand on the left is nonzero only if 

\[ g(x) - y \in B(0, \varepsilon) \]

which occurs only if 

\[ g(x) \in B(y, \varepsilon) \]

which is the same as 

\[ x \in g^{-1}(B(y, \varepsilon)) \].

Therefore, the integrand is nonzero only if \( x \) is contained in exactly one of the disjoint sets, \( V_i \). Now using the change of variables theorem,

\[
(z = g(x) - y, g^{-1}(y + z) = x)
\]

By the chain rule,

\[
I = Dg(g^{-1}(y + z)) Dg^{-1}(y + z)
\]

and so

\[
\det Dg(g^{-1}(y + z)) |\det Dg^{-1}(y + z)|
\]

\[
= \text{sgn} (\det Dg(g^{-1}(y + z))) |\det Dg(g^{-1}(y + z))| |\det Dg^{-1}(y + z)|
\]

\[
= \text{sgn} (\det Dg(x)) = \text{sgn} (\det Dg(x_i)).
\]

Therefore, this reduces to

\[
\sum_{i=1}^{m} \text{sgn} (\det Dg(x_i)) \int_{g(V_i) - y} \phi_\varepsilon(z) \, dz
\]

\[
= \sum_{i=1}^{m} \text{sgn} (\det Dg(x_i)) \int_{B(0, \varepsilon)} \phi_\varepsilon(z) \, dz = \sum_{i=1}^{m} \text{sgn} (\det Dg(x_i)).
\]

In case \( g^{-1}(y) = \emptyset \), there exists \( \varepsilon > 0 \) such that \( g(\Omega) \cap B(y, \varepsilon) = \emptyset \) and so for \( \varepsilon \) this small,

\[
\int_{\Omega} \phi_\varepsilon (g(x) - y) \det Dg(x) \, dx = 0.
\]
21.3. Definitions and Elementary Properties

Showing the integral is constant for small \( \varepsilon \)

Let \( y \notin g(\partial \Omega) \). By Lemma 21.3.3, \( y \) is a regular value of \( g \) where

\[
\int_{\Omega} \phi_{\varepsilon} (g(x) + t(g_1(x) - g(x)) - y) \det(D(g + t(g_1 - g))(x)) \, dx
\]

is constant for \( t \in [0, 1] \). Letting \( \varepsilon \) be small enough, and \( t = 0 \) and then \( 1 \),

\[
\int_{\Omega} \phi_{\varepsilon} (g(x) - y) \det(Dg(x)) \, dx = \int_{\Omega} \phi_{\varepsilon} (g_1(x) - y) \det(Dg_1(x)) \, dx = d(g_1, \Omega, y)
\]

Therefore, \( \int_{\Omega} \phi_{\varepsilon} (g(x) - y) \det(Dg(x)) \, dx \) does not change for sufficiently small \( \varepsilon \).
Thus the limit exists and equals an integer. This shows the degree is an integer.

The next claim follows right away from the above. Suppose \( 0 \notin (tf + (1 - t)g)(\partial \Omega) - y \) for all \( t \in [0, 1] \). Then choosing \( \varepsilon \) small enough, it follows \( d(f, \Omega, y) = d(g, \Omega, y) \) because the two integrals defining the degree for small \( \varepsilon \) are equal, this by Lemma 21.3.3.

It also follows from the above argument and Lemma 21.3.3 that if \( \text{dist}(y, g(\partial \Omega)) > 5\delta \) and if \( y_1 \in B(y, \delta) \), then \( d(g, \Omega, y_1) = d(g, \Omega, y) \) because the corresponding integrals are equal for all \( \varepsilon \) small enough. \( \blacksquare \)

21.3.2 Definition Of The Degree For Continuous Functions

With the above results, it is now possible to extend the definition of the degree to continuous functions which have no differentiability. It is desired to preserve the homotopy invariance. This requires the following definition.

Definition 21.3.6 Let \( y \in \mathbb{R}^n \setminus f(\partial \Omega) \) where \( f \in C(\overline{\Omega}; \mathbb{R}^n) \) Then

\[
d(f, \Omega, y) \equiv d(g, \Omega, y)
\]

where \( y \notin g(\partial \Omega), \ g \in C^\infty(\overline{\Omega}; \mathbb{R}^n) \) and \( f \sim g \).

Theorem 21.3.7 The definition of the degree given in Definition 21.3.6 is well defined, equals an integer, and satisfies the following properties. In what follows, \( I(x) = x \).

1. \( d(I, \Omega, y) = 1 \) if \( y \in \Omega \).
2. If \( \Omega_1 \subseteq \Omega, \Omega_2 \) open, and \( \Omega_1 \cap \Omega_2 = \emptyset \) and if \( y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)) \), then
   \[d(f, \Omega_1, y) + d(f, \Omega_2, y) = d(f, \Omega, y)\]
3. For \( y \in \mathbb{R}^n \setminus f(\partial \Omega) \), if \( d(f, \Omega, y) \neq 0 \) then \( f^{-1}(y) \cap \Omega \neq \emptyset \).
4. If \( t \to y(t) \) is continuous \( h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) is continuous and if \( y(t) \notin h(\partial \Omega, t) \) for all \( t \), then \( t \to d(h(-, t), \Omega, y(t)) \) is constant.
5. \( d(f, \Omega, \cdot) \) is continuous and constant on every connected component of \( \mathbb{R}^n \setminus f(\partial \Omega) \).

6. \( d(g, \Omega, y) = d(f, \Omega, y) \) if \( g|_{\partial \Omega} = f|_{\partial \Omega} \).

**Proof:** First it is necessary to show the definition is well defined. There are two parts to this. First I need to show there exists \( g \) with the desired properties and then I need to show that it doesn’t matter which \( g \) I happen to pick. The first part is easy. Let \( \delta \) be small enough that

\[
B(y, \delta) \cap f(\partial \Omega) = \emptyset.
\]

Then by Lemma 21.3.3 there exists \( g \in C^\infty(\overline{\Omega}; \mathbb{R}^n) \) such that \( \|g - f\|_\infty < \delta \). It follows that for \( t \in [0, 1] \),

\[
y \notin (tg + (1 - t)f)(\partial \Omega)
\]

and so \( g \sim f \). The reason is that if \( x \in \partial \Omega \),

\[
|tg(x) + (1 - t)f(x) - y| \geq |f(x) - y| - t|g(x) - f(x)| > \delta - \delta = 0
\]

Now consider the second part. Suppose \( g \sim f \) and \( g_1 \sim f \). Then by Lemma 21.3.3 again

\[
g \sim g_1
\]

Thus there is a function \( h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) such that \( h(x, 0) = g(x) \) and \( h(x, 1) = g_1(x) \). The difficulty is that it is only known that this function is continuous. It is not known that \( h(\cdot, t) \) is \( C^\infty(\overline{\Omega}; \mathbb{R}^n) \). Let \( \psi \) be a mollifier. Thus it is infinitely differentiable, has support in \( B(0, \varepsilon) \) and \( \int_{\mathbb{R}^n} \psi(x) \, dx = 1 \). Then define

\[
h_\varepsilon(x, t) \equiv h(\cdot, t) * \psi_\varepsilon(x) \equiv \int_{\mathbb{R}^n} h(x - u, t) \psi_\varepsilon(u) \, du.
\]

Then as \( \varepsilon \to 0 \), the convergence is uniform on \( \overline{\Omega} \times [0, 1] \). Now there exists \( \delta > 0 \) such that \( B(y, 6\delta) \cap h(\partial \Omega \times [0, 1]) = \emptyset \) for all \( t \). Hence, by uniform convergence, for \( \varepsilon \) small enough, \( B(y, 5\delta) \cap h_\varepsilon(\partial \Omega, t) = \emptyset \) for all \( t \) and in fact, \( \max_{t \in [0, 1]} \|h_\varepsilon(\cdot, t) - h(\cdot, t)\|_\infty < \delta \). Then by Lemma 21.2.4, it follows that the following integral is constant for \( t \in [0, 1] \).

\[
\int_{\Omega} \phi_\varepsilon(h_\varepsilon(x, t) - y) \det(D_1 h_\varepsilon(\cdot, t)(x)) \, dx
\]

Thus, letting \( \varepsilon \) be smaller if necessary, \( d(h_\varepsilon(\cdot, 0), \Omega, y) = d(h_\varepsilon(\cdot, 1), \Omega, y) \). Since \( \|h_\varepsilon(\cdot, 1) - g_1\|_\infty < \delta \), it follows that \( y \notin t h_\varepsilon(x, 1) + (1 - t) g_1(x) \) for all \( x \in \partial \Omega \). Then by Lemma 21.3.3, formula 21.2.4,

\[
d(h_\varepsilon(\cdot, 1), \Omega, y) = d(g_1, \Omega, y)
\]

similarly,

\[
d(h_\varepsilon(\cdot, 0), \Omega, y) = d(g, \Omega, y)
\]
Therefore,
\[ d(g, \Omega, y) = d(h_\varepsilon (\cdot, 0), \Omega, y) = d(h_\varepsilon (\cdot, 1), \Omega, y) = d(g_1, \Omega, y) \]
which implies \( d(g_1, \Omega, y) = d(g, \Omega, y) \). Thus the definition is well defined.

Now consider the properties. The first, \( \text{(i)} \), is obvious since \( y \) is a regular value of \( I \).

Consider \( \text{(ii)} \) about \( y \notin f(\Omega \setminus (\Omega_1 \cup \Omega_2)) \).

The assumption implies
\[ y \notin f(\partial \Omega) \cup f(\partial \Omega_1) \cup f(\partial \Omega_2) \]
Recall that \( y \notin f(\Omega \setminus (\Omega_1 \cup \Omega_2)) \). Say
\[ \text{dist} (y, f(\Omega \setminus (\Omega_1 \cup \Omega_2))) > 5\delta, \delta > 0 \]
Then let \( g \in C^\infty (\Omega; \mathbb{R}^n) \) such that \( ||f - g||_\infty < \delta \). It follows that \( f \sim g \) and so by definition,
\[ d(f, \Omega, y) = d(g, \Omega, y), \quad d(f, \Omega_1, y) = d(g, \Omega_1, y), \quad d(f, \Omega_2, y) = d(g, \Omega_2, y) \]
since \( f(\Omega \setminus (\Omega_1 \cup \Omega_2)) \) includes \( f(\partial \Omega) \cup f(\partial \Omega_1) \cup f(\partial \Omega_2) \). One can use for a homotopy \( tf + (1 - t)g \) in every case.

Hence
\[ d(f, \Omega, y) = \int_{\Omega_1} \phi_\varepsilon (g(x) - y) Dg(x) \, dx \]
for all \( \varepsilon \) sufficiently small. However, \( \phi_\varepsilon (g(x) - y) = 0 \) if \( x \notin \Omega_1 \cup \Omega_2 \) whenever \( \varepsilon \) is sufficiently small, \( (\varepsilon < \delta \text{ will do}) \). Hence the above integral is
\[ \int_{\Omega_1} \phi_\varepsilon (g(x) - y) Dg(x) \, dx + \int_{\Omega_2} \phi_\varepsilon (g(x) - y) Dg(x) \, dx \]
for all \( \varepsilon \) small enough, and hence
\[ d(f, \Omega, y) = d(g, \Omega, y) = d(g, \Omega_1, y) + d(g, \Omega_2, y) = d(f, \Omega_1, y) + d(f, \Omega_1, y) \].

Property \( \text{(ii)} \) is very important because it can be used to deduce the existence of solutions to a nonlinear equation. Suppose \( f^{-1}(y) \cap \Omega = \emptyset \). I will show this
requires \( d(f, \Omega, y) = 0 \). It is assumed \( y \notin f(\partial \Omega) \) and so if \( f^{-1}(y) \cap \Omega = \emptyset \), then \( y \notin f(\bar{\Omega}) \). Choosing \( g \in C^\infty (\bar{\Omega}; \mathbb{R}^n) \) such that \( ||f - g||_\infty \) is sufficiently small, it can be assumed

\[
y \notin g(\bar{\Omega}), \ y \notin ((1 - t)f + tg)(\partial \Omega) \text{ for all } t \in [0, 1].
\]

Then it follows from the definition of the degree

\[
d(f, \Omega, y) = d(g, \Omega, y) \equiv \lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon (g(x) - y) Dg(x) \, dx = 0
\]

because eventually \( \varepsilon \) is smaller than the distance from \( y \) to \( g(\bar{\Omega}) \) and so

\[
\phi_\varepsilon (g(x) - y) = 0
\]

for all \( x \in \Omega \).

Consider \( \Box \). There is a \( \delta > 0 \) such that \( B(y(t), 6\delta) \cap h(\partial \Omega, t) \) for all \( t \). If this were not so, there would be a sequence \( \{t_k\} \subseteq [0, 1] \), \( \{x_k\} \subseteq \partial \Omega \) such that \( \text{dist}(y(t_k), h(x_k, t_k)) \leq 1/k \). Now taking a convergent subsequence \( t_k \to t \in [0, 1] \), \( x_k \to x \in \partial \Omega \), it follows from continuity that \( y(t) = h(x, t) \) which is a contradiction. Let \( \psi_\varepsilon \) be a mollifier and let \( h_\varepsilon(x, t) \equiv h(\cdot, t) \ast \psi_\varepsilon(x) \). Then by the uniform convergence, whenever \( \varepsilon \) is sufficiently small, \( B(y(t), 5\delta) \cap h_\varepsilon(\partial \Omega, t) = \emptyset \) because for all \( t \),

\[
\|h(\cdot, t) - h_\varepsilon(\cdot, t)\|_\infty < \delta.
\]

Therefore, \( h(\cdot, t) \sim h_\varepsilon(\cdot, t) \) for all \( t \) since \( y(t) \notin (1 - \lambda)h(\partial \Omega, t) + \lambda h_\varepsilon(\partial \Omega, t) \), \( \lambda \in [0, 1] \). To see this, let \( x \in \partial \Omega \)

\[
|\lambda h(x, t) + \lambda h_\varepsilon(x, t) - y(t)| \geq |h(x, t) - y(t)| - \lambda |h(x, t) - h_\varepsilon(x, t)|
\]

\[
\geq 6\delta - \lambda \delta \geq 5\delta > 0
\]

Then from the definition of the degree above, which was shown above to be well defined, it follows that for all \( t \),

\[
d(h(\cdot, t) \cap \Omega, y(t)) = d(h_\varepsilon(\cdot, t) \cap \Omega, y(t))
\]

and the expression on the right is constant in \( t \) thanks to Corollary \( \Box \).

Property \( \Box \) about being constant on connected components is done by showing \( y \to d(f, \Omega, y) \) is continuous. Then, since it is integer valued, it must be constant on every connected component of \( f(\partial \Omega)^C \). Suppose \( \text{dist}(y, f(\partial \Omega)) = 5\delta > 0 \) and suppose \( \hat{y} \in B(y, \delta) \). Let \( y(t) = ty + (1 - t)\hat{y}, t \in [0, 1] \). Then \( y(t) \notin f(\partial \Omega) \) and so by \( \Box \)

\[
d(f, \Omega, y(t))
\]

is a constant. Letting \( t = 0 \) and then letting \( t = 1 \), it follows that \( d(f, \Omega, y) = d(f, \Omega, \hat{y}) \) showing that in fact \( y \to d(f, \Omega, y) \) is locally constant. Hence it is
21.4. BORSUK’S THEOREM

continuous and so it is also constant on every connected component of \( f(\partial \Omega) \) by Corollary 21.3.11.

Consider property \( \Box \) about the degree in which \( f = g \) on \( \partial \Omega \). This one is easy because for \( y \in \mathbb{R}^n \setminus f(\partial \Omega) = \mathbb{R}^n \setminus g(\partial \Omega) \), and \( x \in \partial \Omega \),

\[
t f(\mathbf{x}) + (1-t)g(\mathbf{x}) - y = f(\mathbf{x}) - y \neq 0
\]

for all \( t \in [0, 1] \) and so by \( \Box \), \( d(f, \Omega, y) = d(g, \Omega, y) \). \( \blacksquare \)

From the above, there is an easy corollary which gives related properties of the degree.

**Corollary 21.3.8** The following additional properties of the degree are also valid.

1. If \( y \notin f(\overline{\Omega} \setminus \Omega_1) \) and \( \Omega_1 \) is an open subset of \( \Omega \), then \( d(f, \Omega, y) = d(f, \Omega_1, y) \).

2. \( d(\cdot, \Omega, y) \) is defined and constant on

\[
\{ g \in C(\overline{\Omega}; \mathbb{R}^n) : ||g - f||_{\infty} < r \}
\]

where \( r = \text{dist}(y, f(\partial \Omega)) \).

3. If \( \text{dist}(y, f(\partial \Omega)) \geq \delta \) and \( |z - y| < \delta \), then \( d(f, \Omega, y) = d(f, \Omega, z) \).

**Proof:** Consider \( \Box \). This really follows from \( \Box \) of previous theorem. You can take \( \Omega_2 = \emptyset \). I leave the details to you. To be more careful, you can modify the proof of \( \Box \) of the previous theorem slightly. Consider \( \Box \). To verify, let \( h(x, t) = tg(x) + (1-t)f(x) \). Then note that \( y \notin h(\partial \Omega, t) \) and use Property \( \Box \) of the previous theorem. Finally, consider \( \Box \). Let \( y(t) \equiv (1-t)y + tz \). Then for \( x \in \partial \Omega \)

\[
|(1-t)y + tz - f(x)| \geq |y - f(x) + t(z - y)| \geq \delta - t|z - y| > \delta - \delta = 0
\]

Then by \( \Box \) of the previous theorem, \( d(f, \Omega, (1-t)y + tz) \) is constant. When \( t = 0 \) you get \( d(f, \Omega, y) \) and when \( t = 1 \) you get \( d(f, \Omega, z) \). \( \blacksquare \)

**21.4 Borsuk’s Theorem**

In this section is an important theorem which can be used to verify that \( d(f, \Omega, y) \neq 0 \). This is significant because when this is known, it follows from Theorem 21.3.7 that \( f^{-1}(y) \neq \emptyset \). In other words there exists \( x \in \Omega \) such that \( f(x) = y \).

**Definition 21.4.1** A bounded open set, \( \Omega \) is symmetric if \( -\Omega = \Omega \). A continuous function, \( f : \Omega \to \mathbb{R}^n \) is odd if \( f(-x) = -f(x) \).
Suppose $\Omega$ is symmetric and $g \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ is an odd map for which $0$ is a regular value. Then the chain rule implies $Dg(-x) = Dg(x)$ and so $d(g, \Omega, 0)$ must equal an odd integer because if $x \in g^{-1}(0)$, it follows that $-x \in g^{-1}(0)$ also and since $Dg(-x) = Dg(x)$, it follows the overall contribution to the degree from $x$ and $-x$ must be an even integer. Also $0 \in g^{-1}(0)$ and so the degree equals an even integer added to $\text{sgn} \det(Dg(0))$, an odd integer, either $-1$ or $1$. It seems reasonable to expect that something like this would hold for an arbitrary continuous odd function defined on symmetric $\Omega$. In fact this is the case and this is next. The following lemma is the key result used. This approach is due to Gromes \cite{gromes}. See also Deimling \cite{deimling} which is where I found this argument.

The idea is to start with a smooth odd map and approximate it with a smooth odd map which also has $0$ a regular value.

**Lemma 21.4.2** Let $g \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ be an odd map. Then for every $\varepsilon > 0$, there exists $h \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ such that $h$ is also an odd map, $\|h - g\|_\infty < \varepsilon$, and $0$ is a regular value of $h$. Here $\Omega$ is a symmetric bounded open set. In addition, $d(g, \Omega, 0)$ is an odd integer.

**Proof:** In this argument $\eta > 0$ will be a small positive number and $C$ will be a constant which depends only on the diameter of $\Omega$. Let $h_0(x) = g(x) + \eta x$ where $\eta$ is chosen such that $\det Dh_0(0) \neq 0$. Note that $h_0$ is odd, close to $g$ and so $0$ is a value of $h_0$ thanks to $h_0(0) = 0$. The idea is to modify $h_0$ such that for all $x \in h_0^{-1}(0)$, $\det Dh_0(x) \neq 0$.

Let $\Omega_i = \{x \in \Omega : x_i \neq 0\}$. In other words, leave out the plane $x_i = 0$ from $\Omega$ in order to obtain $\Omega_i$. A succession of modifications is about to take place on $\Omega_i, \Omega_1 \cup \Omega_2$, etc. Finally a function will be obtained on $\bigcup_{i=1}^n \Omega_i$ which is everything except $0$.

Define $h_1(x) = h_0(x) - y^1 x_1^3$ where $|y^1| < \eta$ and $y^1 = (y_1^1, \cdots, y_n^1)$ is a regular value of the function, $x \rightarrow \frac{h_0(x)}{x_1^3}$ for $x \in \Omega_1$. The existence of $y^1$ follows from Sard’s lemma because this function is in $C^\infty(\Omega_1; \mathbb{R}^n)$. Thus $h_1(x) = 0$ if and only if $y^1 = \frac{h_0(x)}{x_1^3}$. Since $y^1$ is a regular value, it follows that for such $x$ satisfying $h_1(x) = 0$,

$$
\det \left( h_{0i,j}(x) x_1^3 - \frac{\partial}{\partial x_j} (x_1^3) h_{0i}(x) \right) = 0
$$

implying that

$$
\det \left( h_{0i,j}(x) x_1^3 - \frac{\partial}{\partial x_j} (x_1^3) y_i^1 x_1^3 \right) \neq 0
$$

This shows $0$ is a regular value of $h_1$ on the set $\Omega_1$ and it is clear $h_1$ is an odd map in $C^\infty(\Omega; \mathbb{R}^n)$ and $\|h_1 - g\|_\infty \leq C\eta$ where $C$ depends only on the diameter of $\Omega$. 


Now suppose for some $k$ such that $1 \leq k < n$ there exists an odd mapping $h_k$ in $C^\infty (\bar{\Omega}; \mathbb{R}^n)$ such that 0 is a regular value of $h_k$ on $\cup_{i=1}^k \Omega_i$ and $||h_k - g||_\infty \leq C\eta$. Sard’s theorem implies there exists $y^{k+1}$ a regular value of the function $x \to h_k(x)/x_{k+1}^3$ defined on $\Omega_{k+1}$ such that $||y^{k+1}|| < \eta$ and let $h_{k+1}(x) \equiv h_k(x) - y^{k+1}x_{k+1}^3$. As before, $h_{k+1}(x) = 0$ if and only if $h_k(x)/x_{k+1}^3 = y^{k+1}$, a regular value of $x \to h_k(x)/x_{k+1}^3$. Consider such $x$ for which $h_{k+1}(x) = 0$. First suppose $x \in \Omega_{k+1}$. Then

$$\det \left( \frac{h_{k,i,j}(x) x_{k+1}^3 - \frac{\partial}{\partial x_j} \left( x_{k+1}^3 \right) \frac{\partial}{\partial x_i} \right) \neq 0$$

which implies that whenever $h_{k+1}(x) = 0$ and $x \in \Omega_{k+1}$,

$$\det \left( h_{k,i,j}(x) - \frac{\partial}{\partial x_j} \left( x_{k+1}^3 \right) \frac{\partial}{\partial x_i} \right) = \det (Dh_{k+1}(x)) \neq 0. \quad (21.4.5)$$

However, if $x \in \cup_{i=1}^k \Omega_k$ but $x \notin \Omega_{k+1}$, then $x_{k+1} = 0$ since $\Omega_{k+1}$ consists of those $x$ such that $x_{k+1} \neq 0$, and so the left side of (21.4.5) reduces to $det(h_{k,i,j}(x))$ which is not zero because 0 is assumed a regular value of $h_k$ on $\cup_{i=1}^k \Omega_i$. Therefore, 0 is a regular value for $h_{k+1}$ on $\cup_{i=1}^k \Omega_k$. (For $x \in \cup_{i=1}^k \Omega_k$, either $x \in \Omega_{k+1}$ or $x \notin \Omega_{k+1}$. If $x \in \Omega_{k+1}$, 0 is a regular value by the construction above. In the other case, 0 is a regular value by the induction hypothesis.) Also $h_{k+1}$ is odd and in $C^\infty (\bar{\Omega}; \mathbb{R}^n)$, and $||h_{k+1} - g||_\infty \leq C\eta$.

Let $h \equiv h_n$. Then 0 is a regular value of $h$ for $x \in \cup_{j=1}^n \Omega_j$. The point of $\Omega$ which is not in $\cup_{j=1}^n \Omega_j$ is 0. If $x = 0$, then from the construction, $Dh(0) = Dh_0(0)$ and so 0 is a regular value of $h$ for $x \in \Omega$. By choosing $\eta$ small enough, it follows $||h - g||_\infty < \varepsilon$.

For the last part, let $3\delta = dist (g (\partial \Omega), 0)$ and let $h$ be as described above with $||h - g||_\infty < \delta$. Then $0 \notin \{th + (1 - t)g \mid t \in [0, 1] \}$ and so by the homotopy invariance of the degree, $t \to d (th + (1 - t)g, \Omega, 0)$ is constant for $t \in [0, 1]$. Therefore,

$$d (g, \Omega, 0) = d (h, \Omega, 0)$$

So what is $d (h, \Omega, 0)$? Since 0 is a regular value and $h$ is odd,

$$h^{-1} (0) = \{x_1, \ldots, x_r, -x_1, \ldots, -x_r, 0\}.$$ 

So consider $Dh(x)$ and $Dh(-x)$.

$$Dh(-x) u + o(u) = h(-x + u) - h(-x) = -h(x+(-u)) + h(x) = - (Dh(x)(-u)) + o(-u) = Dh(x)(u) + o(u)$$
Hence $D_h(x) = D_h(-x)$ and so the determinants of these two are the same. It follows that
\[
d(h, \Omega, 0) = \sum_{i=1}^{r} \text{sgn} \left( \det (D_h(x_i)) \right) + \sum_{i=1}^{r} \text{sgn} \left( \det (D_h(-x_i)) \right) + \text{sgn} \left( \det (D_h(0)) \right) = 2m \pm 1 \text{ some integer } m
\]
an odd integer. ■

Theorem 21.4.3 (Borsuk) Let $f \in C(\overline{\Omega}; \mathbb{R}^n)$ be odd and let $\Omega$ be symmetric with $0 \notin f(\partial \Omega)$. Then $d(f, \Omega, 0)$ equals an odd integer.

Proof: Let $\psi_n$ be a mollifier which is symmetric, $\psi(-x) = \psi(x)$. Also recall that $f$ is the restriction to $\overline{\Omega}$ of a continuous function, still denoted as $f$ which is defined on all of $\mathbb{R}^n$. Let $g$ be the odd part of this function. That is,
\[
g(x) \equiv \frac{1}{2} (f(x) - f(-x))
\]
Since $f$ is odd, $g = f$ on $\overline{\Omega}$. Then
\[
g_n(-x) \equiv g * \psi_n(-x) = \int_{\mathbb{R}^n} g(-x - y) \psi_n(y) \, dy = -\int_{\mathbb{R}^n} g(x + y) \psi_n(y) \, dy = -\int_{\mathbb{R}^n} g(x - (y)) \psi_n(-y) \, dy = -g_n(x)
\]
Thus $g_n$ is odd and is infinitely differentiable. Let $3\delta = \text{dist}(f(\partial \Omega), 0)$ and let $n$ be large enough that $\|g_n - f\|_{\infty} < \delta$. Then $0 \notin (tg_n + (1-t)f)(\partial \Omega)$ for $t \in [0,1]$ and so by homotopy invariance,
\[
d(f, \Omega, 0) = d(g, \Omega, 0) = d(g_n, \Omega, 0)
\]
and by Lemma 21.4.2 this is an odd integer. ■

21.5 Applications

With these theorems it is possible to give easy proofs of some very important and difficult theorems.

Definition 21.5.1 If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ where $U$ is an open set. Then $f$ is locally one to one if for every $x \in U$, there exists $\delta > 0$ such that $f$ is one to one on $B(x, \delta)$.

As a first application, consider the invariance of domain theorem. This result says that a one to one continuous map takes open sets to open sets. It is an amazing result which is essential to understand if you wish to study manifolds. In fact, the following theorem only requires $f$ to be locally one to one. First here is a lemma which has the main idea.
Lemma 21.5.2 Let \( g : B(0,r) \to \mathbb{R}^n \) be one to one and continuous where here \( B(0,r) \) is the ball centered at \( 0 \) of radius \( r \) in \( \mathbb{R}^n \). Then there exists \( \delta > 0 \) such that 
\[
g(0) + B(0,\delta) \subseteq g(B(0,r)).
\]
The symbol on the left means: \( \{ g(0) + x : x \in B(0,\delta) \} \).

**Proof:** For \( t \in [0,1] \), let 
\[
h(x,t) \equiv g \left( \frac{x}{1+t} \right) - g \left( \frac{-tx}{1+t} \right).
\]
Then for \( x \in \partial B(0,r) \), \( h(x,t) \neq 0 \) because if this were so, the fact \( g \) is one to one implies
\[
\frac{x}{1+t} = \frac{-tx}{1+t}
\]
and this requires \( x = 0 \) which is not the case since \( \|x\| = r \). Then \( d(h(\cdot,t),B(0,r),0) \) is constant. Hence it is an odd integer for all \( t \) thanks to Borsuk’s theorem, because \( h(\cdot,1) \) is odd. Now let \( B(0,\delta) \) be such that \( B(0,\delta) \setminus h(\partial \Omega,0) = 0 \). Then
\[
d(h(\cdot,0),B(0,r),0) = d(h(\cdot,0),B(0,r),z)
\]
for \( z \in B(0,\delta) \) because the degree is constant on connected components of \( \mathbb{R}^n \setminus h(\partial \Omega,0) \). Hence \( z = h(x,0) = g(x) - g(0) \) for some \( x \in B(0,r) \). Thus
\[
g(B(0,r)) \supseteq g(0) + B(0,\delta) \quad \blacksquare
\]

Now with this lemma, it is easy to prove the very important invariance of domain theorem.

A function \( f \) is locally one to one on an open set \( \Omega \) if for every \( x_0 \in \Omega \), there exists \( B(x_0,r) \subseteq \Omega \) such that \( f \) is one to one on \( B(x_0,r) \).

**Theorem 21.5.3 (invariance of domain)** Let \( \Omega \) be any open subset of \( \mathbb{R}^n \) and let \( f : \Omega \to \mathbb{R}^n \) be continuous and locally one to one. Then \( f \) maps open subsets of \( \Omega \) to open sets in \( \mathbb{R}^n \).

**Proof:** Let \( B(x_0,r) \subseteq \Omega \) where \( f \) is one to one on \( B(x_0,r) \). Let \( g \) be defined on \( B(0,r) \) given by
\[
g(x) \equiv f(x + x_0)
\]
Then \( g \) satisfies the conditions of Lemma 21.5.2, being one to one and continuous. It follows from that lemma there exists \( \delta > 0 \) such that
\[
f(\Omega) \supseteq f(B(x_0,r)) = f(x_0 + B(0,r)) = g(B(0,r)) \supseteq g(0) + B(0,\delta) = f(x_0) + B(0,\delta) = B(f(x_0),\delta)
\]
This shows that for any \( x_0 \in \Omega \), \( f(x_0) \) is an interior point of \( f(\Omega) \) which shows \( f(\Omega) \) is open. \( \blacksquare \)

With the above, one gets easily the following amazing result. It is something which is clear for linear maps but this is a statement about continuous maps.
Corollary 21.5.4 If \( n > m \) there does not exist a continuous one to one map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).

**Proof:** Suppose not and let \( f \) be such a continuous map,
\[
f(x) \equiv (f_1(x), \cdots, f_m(x))^T.
\]
Then let \( g(x) \equiv (f_1(x), \cdots, f_m(x), 0, \cdots, 0)^T \) where there are \( n - m \) zeros added in. Then \( g \) is a one to one continuous map from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) and so \( g(\mathbb{R}^n) \) would have to be open from the invariance of domain theorem and this is not the case.

Corollary 21.5.5 If \( f \) is locally one to one and continuous, \( f : \mathbb{R}^n \to \mathbb{R}^n \), and
\[
\lim_{|x| \to \infty} |f(x)| = \infty,
\]
then \( f \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \).

**Proof:** By the invariance of domain theorem, \( f(\mathbb{R}^n) \) is an open set. It is also true that \( f(\mathbb{R}^n) \) is a closed set. Here is why. If \( f(x_k) \to y \), the growth condition ensures that \( \{x_k\} \) is a bounded sequence. Taking a subsequence which converges to \( x \in \mathbb{R}^n \) and using the continuity of \( f \), it follows \( f(x) = y \). Thus \( f(\mathbb{R}^n) \) is both open and closed which implies \( f \) must be an onto map since otherwise, \( \mathbb{R}^n \) would not be connected.

The next theorem is the famous Brouwer fixed point theorem.

**Theorem 21.5.6 (Brouwer fixed point)** Let \( B = \overline{B(0,r)} \subseteq \mathbb{R}^n \) and let \( f : B \to B \) be continuous. Then there exists a point \( x \in B \), such that \( f(x) = x \).

**Proof:** Assume there is no fixed point. Consider \( h(x,t) \equiv t f(x) - x \) for \( t \in [0,1] \). Then for \( \|x\| = r \),
\[
0 \notin tf(x) - x, t \in [0,1]
\]
By homotopy invariance,
\[
t \to d(tf - I, B, 0)
\]
is constant. But when \( t = 0 \), this is \( d(-I, B, 0) = (-1)^n \neq 0 \). Hence \( d(f - I, B, 0) \neq 0 \) so there exists \( x \) such that \( f(x) - x = 0 \). ■

It is easy to generalize this to an arbitrary closed bounded convex set \( K \) in \( \mathbb{R}^n \) as follows. You use the fact that \( K \) is a retract, Theorem 14.2.5. Thus there exists a mapping \( P \) which is continuous and maps all of \( \mathbb{R}^n \) to \( K \) and fixed points of \( K \). In particular, \( P \) maps \( \overline{B(0,r)} \) to \( K \). Thus there exists \( x \in \overline{B(0,r)} \) such that \( f(P(x)) = x \). But then \( x \in K \) and so \( P(x) = x \). Thus \( f(x) = x \).

You can also use standard stuff from Hilbert space to get this. Let \( K \) be a closed bounded convex set and let \( f : K \to K \) be continuous. Let \( P \) be the projection map onto \( K \). Then \( P \) is continuous because \( |P(x) - P(y)| \leq |x - y| \). Recall why this
is. From the material on Hilbert space, \((x - Px, y - Py) \leq 0\) for all \(y \in K\). Indeed, this characterizes \(Px\). Therefore,

\[(x - Px, Py - Px) \leq 0, \quad (y - Py, Px - Py) \leq 0\] 

so \((y - Py, Py - Px) \geq 0\). Hence, subtracting the first from the last,

\[(y - Py - (x - Px), Py - Px) \geq 0\]

consequently,

\[|x - y| |Py - Px| \geq (y - x, Py - Px) \geq |Py - Px|^2\]

and so \(|Py - Px| \leq |y - x|\) as claimed.

Now let \(r\) be so large that \(K \subseteq B(0, r)\). Then consider \(f \circ P\). This map takes \(B(0, r)\) to \(K\). Therefore, being the composition of continuous functions, it is continuous and so has a fixed point in \(B(0, r)\) denoted as \(x\). Hence \(f(P(x)) = x\). Now, since \(f\) maps into \(K\), it follows that \(x \in K\). Hence \(Px = x\) and so \(f(x) = x\). This has proved the following general Brouwer fixed point theorem.

**Theorem 21.5.7** Let \(f : K \to K\) be continuous where \(K\) is compact and convex and nonempty, \(K \subseteq \mathbb{R}^n\). Then \(f\) has a fixed point.

**Definition 21.5.8** \(f\) is a retract of \(B(0, r)\) onto \(\partial B(0, r)\) if \(f\) is continuous,

\[f\left(B(0, r)\right) \subseteq \partial B(0, r)\]

and \(f(x) = x\) for all \(x \in \partial B(0, r)\).

**Theorem 21.5.9** There does not exist a retraction of \(B(0, r)\) onto its boundary, \(\partial B(0, r)\).

**Proof:** Suppose \(f\) were such a retraction. Then for all \(x \in \partial B(0, r)\), \(f(x) = x\) and so from the properties of the degree, the one which says if two functions agree on \(\partial \Omega\), then they have the same degree,

\[1 = d(I, B(0, r), 0) = d(f, B(0, r), 0)\]

which is clearly impossible because \(f^{-1}(0) = \emptyset\) which implies \(d(f, B(0, r), 0) = 0\).

You should now use this theorem to give another proof of the Brouwer fixed point theorem.

The proofs of the next two theorems make use of the Tietze extension theorem, Theorem \[6.6.7\].

**Theorem 21.5.10** Let \(\Omega\) be a symmetric open set in \(\mathbb{R}^n\) such that \(0 \in \Omega\) and let \(f : \partial \Omega \to V\) be continuous where \(V\) is an \(m\) dimensional subspace of \(\mathbb{R}^n, m < n\). Then \(f(-x) = f(x)\) for some \(x \in \partial \Omega\).
Proof: Suppose not. Using the Tietze extension theorem or Theorem 14.2.3, extend $f$ to all of $\mathbb{R}^n$, $f(\bar{\Omega}) \subseteq V$. (Here the extended function is also denoted by $f$.) Let $g(x) = f(x) - f(-x)$. Then $0 \notin g(\partial\Omega)$ and so for some $r > 0$, $B(0,r) \subseteq \mathbb{R}^n \setminus g(\partial\Omega)$. For $z \in B(0,r)$,

$$d(g,\Omega,z) = d(g,\Omega,0) \neq 0$$

because $B(0,r)$ is contained in a component of $\mathbb{R}^n \setminus g(\partial\Omega)$ and Borsuk’s theorem implies that $d(g,\Omega,0) \neq 0$ since $g$ is odd. Hence

$$V \supseteq g(\Omega) \supseteq B(0,r)$$

and this is a contradiction because $V$ is $m$ dimensional. ■

This theorem is called the Borsuk Ulam theorem. Note that it implies there exist two points on opposite sides of the surface of the earth which have the same atmospheric pressure and temperature, assuming the earth is symmetric and that pressure and temperature are continuous functions. The next theorem is an amusing result which is like combing hair. It gives the existence of a “cowlick”.

**Theorem 21.5.11** Let $n$ be odd and let $\Omega$ be an open bounded set in $\mathbb{R}^n$ with $0 \in \Omega$. Suppose $f : \partial\Omega \to \mathbb{R}^n \setminus \{0\}$ is continuous. Then for some $x \in \partial\Omega$ and $\lambda \neq 0$, $f(x) = \lambda x$.

Proof: Using the Tietze extension theorem or Theorem 14.2.3, extend $f$ to all of $\mathbb{R}^n$. Also denote the extended function by $f$. Suppose for all $x \in \partial\Omega$, $f(x) \neq \lambda x$ for all $\lambda \in \mathbb{R}$. Then

$$0 \notin tf(x) + (1-t)x, \ (x,t) \in \partial\Omega \times [0,1]$$

Thus there exists a homotopy of $f$ and $I$ and a homotopy of $f$ and $-I$. Then by the homotopy invariance of degree,

$$d(f,\Omega,0) = d(I,\Omega,0), \ d(f,\Omega,0) = d(-I,\Omega,0).$$

But this is impossible because $d(I,\Omega,0) = 1$ but $d(-I,\Omega,0) = (-1)^n = -1$. ■

### 21.6 The Product Formula

This section is on the product formula for the degree which is used to prove the Jordan separation theorem. To begin with here is a lemma which is similar to an earlier result except here there are $r$ points.

**Lemma 21.6.1** Let $y_1, \cdots, y_r$ be points not in $f(\partial\Omega)$ and let $\delta > 0$. Then there exists $\tilde{f} \in C^2(\bar{\Omega};\mathbb{R}^n)$ such that $\|\tilde{f} - f\|_{\infty} < \delta$ and $y_i$ is a regular value for $\tilde{f}$ for each $i$. 

Proof: Let \( f_0 \in C^2(\mathbb{R}; \mathbb{R}^n) \), \( ||f_0 - f||_\infty < \frac{\delta}{2} \). Let \( \bar{y}_1 \) be a regular value for \( f_0 \) and \( |\bar{y}_1 - y_1| < \frac{\delta}{3r} \). Let \( f_1(x) \equiv f_0(x) + y_1 - \bar{y}_1 \). Thus \( y_1 \) is a regular value of \( f_1 \) because \( Df_1(x) = Df_0(x) \) and if \( f_1(x) = y_1 \), this is the same as having \( f_0(x) = \bar{y}_1 \) where \( \bar{y}_1 \) is a regular value of \( f_0 \). Then also

\[
||f - f_1||_\infty \leq ||f - f_0||_\infty + ||f_0 - f_1||_\infty \\
= ||f - f_0||_\infty + |\bar{y}_1 - y_1| \\
< \frac{\delta}{3r} + \frac{\delta}{2}.
\]

Suppose now there exists \( f_k \in C^2(\mathbb{R}; \mathbb{R}^n) \) with each of the \( y_i \) for \( i = 1, \ldots, k \) a regular value of \( f_k \) and

\[
||f - f_k||_\infty < \frac{\delta}{2} + \frac{k}{r} \left( \frac{\delta}{3} \right).
\]

Then letting \( S_k \) denote the singular values of \( f_k \), Sard’s theorem implies there exists \( \bar{y}_{k+1} \) such that

\[
|\bar{y}_{k+1} - y_{k+1}| < \frac{\delta}{3r}
\]

and

\[
\bar{y}_{k+1} \notin S_k \cup \bigcup_{i=1}^k (S_k + y_{k+1} - y_i).
\]

Let

\[
f_{k+1}(x) \equiv f_k(x) + y_{k+1} - \bar{y}_{k+1}.
\]

If \( f_{k+1}(x) = y_i \) for some \( i \leq k \), then

\[
f_k(x) + y_{k+1} - y_i = \bar{y}_{k+1}
\]

and so \( f_k(x) \) is a regular value for \( f_k \) since by 21.6.6 \( \bar{y}_{k+1} \notin S_k + y_{k+1} - y_i \) and so \( f_k(x) \notin S_k \). Therefore, for \( i \leq k \), \( y_i \) is a regular value of \( f_{k+1} \) since by 21.6.7, \( Df_{k+1} = Df_k \). Now suppose \( f_{k+1}(x) = y_{k+1} \). Then

\[
y_{k+1} = f_k(x) + y_{k+1} - \bar{y}_{k+1}
\]

so \( f_k(x) = \bar{y}_{k+1} \) implying that \( f_k(x) = \bar{y}_{k+1} \notin S_k \). Hence \( \det Df_{k+1}(x) = \det Df_k(x) \neq 0 \). Thus \( y_{k+1} \) is also a regular value of \( f_{k+1} \). Also,

\[
||f_{k+1} - f|| \leq ||f_{k+1} - f_k|| + ||f_k - f|| \\
\leq \frac{\delta}{3r} + \frac{\delta}{2} + \frac{k}{r} \left( \frac{\delta}{3} \right) = \frac{\delta}{2} + \frac{k+1}{r} \left( \frac{\delta}{3} \right).
\]

Let \( \bar{f} = f_k \). Then

\[
||f - \bar{f}||_\infty < \frac{\delta}{2} + \left( \frac{\delta}{3} \right) < \delta
\]

and each of the \( y_i \) is a regular value of \( \bar{f} \). ■
\textbf{Definition 21.6.2} Let the connected components of $\mathbb{R}^n \setminus f(\partial \Omega)$ be denoted by $K_i$. From the properties of the degree listed in Theorem 21.3.7, $d(f, \Omega, \cdot)$ is constant on each of these components. Denote by $d(f, \Omega, K_i)$ the constant value on the component, $K_i$.

The product formula considers the situation depicted in the following diagram in which $y \notin g(f(\partial \Omega))$ and the $K_i$ are the connected components of $\mathbb{R}^n \setminus f(\partial \Omega)$.

\[
\begin{array}{ccc}
\Omega & \xrightarrow{f} & f(\Omega) \\
& \mathbb{R}^n \setminus f(\partial \Omega) = \bigcup_i K_i & \xrightarrow{g} \mathbb{R}^n \\
\end{array}
\]

The following diagram may be helpful in remembering what it says.

\[
\begin{array}{ccc}
\Omega & \xrightarrow{f} & K_1 \xrightarrow{g} y \\
& \xrightarrow{K_2} & \xrightarrow{K_3} \end{array}
\]

\textbf{Lemma 21.6.3} Let $f \in C(\overline{\Omega}; \mathbb{R}^n)$, $g \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, and $y \notin g(f(\partial \Omega))$. Suppose also that $y$ is a regular value of $g$. Then the following product formula holds where $K_i$ are the bounded components of $\mathbb{R}^n \setminus f(\partial \Omega)$.

\[d(g \circ f, \Omega, y) = \sum_{i=1}^{\infty} d(f, \Omega, K_i) d(g, K_i, y).\]

All but finitely many terms in the sum are zero.

\textbf{Proof:} First note that if $K_i$ is unbounded, $d(f, \Omega, K_i) = 0$ because there exists a point, $z \in K_i$ such that $f^{-1}(z) = \emptyset$ due to the fact that $f(\overline{\Omega})$ is compact and is consequently bounded. Thus it makes no difference in the above formula whether the $K_i$ are arbitrary components or only bounded components. Let $\{x^i_j\}_{j=1}^{m_i}$ denote the points of $g^{-1}(y)$ which are contained in $K_i$, the $i^{th}$ bounded component of $\mathbb{R}^n \setminus f(\partial \Omega)$. Then $m_i < \infty$ because if not, there would exist a limit point $x$ for this sequence. Then $g(x) = y$ and so $x \notin f(\partial \Omega)$. Thus $\det(Dg(x)) \neq 0$ and so by the inverse function theorem, $g$ would be one to one on an open ball containing $x$ which contradicts having $x$ a limit point.

Note also that $g^{-1}(y) \cap f(\overline{\Omega})$ is a compact set covered by the components of $\mathbb{R}^n \setminus f(\partial \Omega)$ because by assumption, $g^{-1}(y) \cap f(\partial \Omega) = \emptyset$. It follows $g^{-1}(y) \cap f(\overline{\Omega})$ is covered by finitely many of these components. It is not in $f(\partial \Omega)$. 
21.6. THE PRODUCT FORMULA

The only terms in the above sum which are nonzero are those corresponding to \( K_i \) having nonempty intersection with \( g^{-1}(y) \cap f(\bar{\Omega}) \). The other components contribute 0 to the above sum because if \( K_i \cap g^{-1}(y) = \emptyset \), it follows from Theorem 21.3.7 that \( d(g, K_i, y) = 0 \). If \( K_i \) does not intersect \( f(\Omega) \), then \( d(f, \Omega, K_i) = 0 \). Therefore, the above sum is actually a finite sum since \( g^{-1}(y) \cap f(\Omega) \), being a compact set, is covered by finitely many of the \( K_i \). Thus there are no convergence problems.

Let \( d(f, \Omega, K_i) = d(f, \Omega, u^i_j) \) where the \( \{u^i_j\}_{j=1}^{m_i} \) are the points in \( g^{-1}(y) \cap K_i \). By Lemma 21.6.1, there exists \( \tilde{f} \) such that \( \|\tilde{f} - f\|_\infty \) is very small and each of the \( u^i_j \) are regular values for \( \tilde{f} \). If \( \|\tilde{f} - f\|_\infty \) is small enough, then \( \left(f + t \left(\tilde{f} - f\right)\right)(\partial\Omega) \) does not contain any of the \( u^i_j \). This is so because by the definition of \( u^i_j \) they are in some \( K_i \) and these are connected components of \( \mathbb{R}^n \setminus f(\partial\Omega) \). Thus

\[
d(f, \Omega, K_i) = d(f, \Omega, u^i_j) = d(\tilde{f}, \Omega, u^i_j)
\]

by the homotopy invariance of the degree, this for each \( j = 1, 2, \ldots, m_i \). Also if \( \|\tilde{f} - f\|_\infty \) is small enough, one can have \( \left(g \circ f + t \left(g \circ \tilde{f} - g \circ f\right)\right)(\partial\Omega) \) does not contain \( y \) for all \( t \in [0, 1] \). Hence by homotopy invariance again,

\[
d(g \circ f, \Omega, y) = d(g \circ \tilde{f}, \Omega, y) .
\]

Now \( \tilde{f}^{-1}(u^i_j) \) is a finite set because \( \bar{\Omega} \) is a bounded open set and \( u^i_j \) is a regular value. It follows from 21.6.8

\[
d(g \circ f, \Omega, y) = d(g \circ \tilde{f}, \Omega, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \sum_{z \in f^{-1}(u^i_j)} \text{sgn det } Dg \left( \frac{u^i_j}{\tilde{f}(z)} \right) \text{sgn det } D\tilde{f}(z)
\]

\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \text{sgn det } Dg \left(u^i_j\right) d(\tilde{f}, \Omega, x^i_j)
\]

\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} d(g, K_i, y) d(\tilde{f}, \Omega, x^i_j)
\]

With this lemma, the following is the product formula.
Theorem 21.6.4 \textit{(product formula)} Let \( \{K_i\}_{i=1}^{\infty} \) be the bounded components of \( \mathbb{R}^n \setminus f(\partial\Omega) \) for \( f \in C(\overline{\Omega}, \mathbb{R}^n) \), let \( g \in C(\mathbb{R}^n, \mathbb{R}^n) \), and suppose that \( y \notin g(f(\partial\Omega)) \). Then
\[
\sum_{i=1}^{\infty} d(g, K_i, y) d(f, \Omega, K_i) = d(g \circ f, \Omega, y). \tag{21.6.9}
\]

All but finitely many terms in the sum are zero.

\textbf{Proof:} Let \( \sup \{ |\tilde{g}(z) - g(z)| : z \in f(\overline{\Omega}) \} \) be sufficiently small that
\[
y \notin (g \circ f + t(\tilde{g} \circ f - g \circ f))(\partial\Omega), \quad t \in [0, 1]
\]
\( \tilde{g} \) being \( C^2(\mathbb{R}^n, \mathbb{R}^n) \) with \( y \) a regular value of \( \tilde{g} \). It follows that
\[
d(g \circ f, \Omega, y) = d(\tilde{g} \circ f, \Omega, y). \tag{21.6.10}
\]

Now also, the \( K_i \) are the open components of \( \mathbb{R}^n \setminus f(\partial\Omega) \) and so \( \partial K_i \subseteq f(\partial\Omega) \) (if \( x \in \partial K_i \), then if \( x \notin f(\partial\Omega) \), it would be in a ball contained in one of the \( K_j \) and so could not be in \( \partial K_i \)) and so if \( z \in \partial K_i \), then \( g(z) \in g(f(\partial\Omega)) \). Consequently, for \( t \in [0, 1], \)
\[
y \notin (g + t(\tilde{g} - g))(\partial K_i)
\]
\( y \) is not in the larger set \( (g \circ f + t(\tilde{g} \circ f - g \circ f))(\partial\Omega) \) which shows that, by homotopy invariance,
\[
d(g, K_i, y) = d(\tilde{g}, K_i, y). \tag{21.6.11}
\]
Therefore, by Lemma 21.6.3,
\[
d(g \circ f, \Omega, y) = d(\tilde{g} \circ f, \Omega, y) = \sum_{i=1}^{\infty} d(\tilde{g}, K_i, y) d(f, \Omega, K_i)
\]
\[
= \sum_{i=1}^{\infty} d(g, K_i, y) d(f, \Omega, K_i)
\]
and the sum has only finitely many non zero terms. \( \blacksquare \)

Note there are no convergence problems because these sums are actually finite sums because, as in the previous lemma, \( g^{-1}(y) \cap f(\overline{\Omega}) \) is a compact set covered by the components of \( \mathbb{R}^n \setminus f(\partial\Omega) \) and so it is covered by finitely many of these components. For the other components, \( d(f, \Omega, K_i) = 0 \) or else \( d(g, K_i, y) = 0 \).

The following theorem is the Jordan separation theorem, a major result. A homeomorphism is a function which is one to one onto and continuous having continuous inverse. Before the theorem, here is a helpful lemma.

\textbf{Lemma 21.6.5} Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), \( f \in C(\overline{\Omega} \setminus f(\partial\Omega)) \), and suppose \( \{\Omega_i\}_{i=1}^{\infty} \) are disjoint open sets contained in \( \Omega \) such that
\[
y \notin f(\overline{\Omega} \setminus \cup_{j=1}^{\infty} \Omega_j)
\]
Then
\[ d(f, \Omega, y) = \sum_{j=1}^{\infty} d(f, \Omega_j, y) \]
where the sum has all but finitely many terms equal to 0.

**Proof:** By assumption, the compact set \( f^{-1}(y) \equiv \{ x \in \bar{\Omega} : f(x) = y \} \) has empty intersection with
\[ \Pi \setminus \bigcup_{j=1}^{\infty} \Omega_j \]
and so this compact set is covered by finitely many of the \( \Omega_j \), say \( \{ \Omega_1, \cdots, \Omega_n \} \) and
\[ y \notin f \left( \bigcup_{j=n}^{\infty} \Omega_j \right). \]
By Theorem 21.3.7 and letting \( O = \bigcup_{j=n}^{\infty} \Omega_j \),
\[ d(f, \Omega, y) = \sum_{j=1}^{n-1} d(f, \Omega_j, y) + d(f, O, y) = \sum_{j=1}^{\infty} d(f, \Omega_j, y) \]
because \( d(f, O, y) = 0 \) as is \( d(f, \Omega_j, y) \) for every \( j \geq n \).

**Lemma 21.6.6** Define \( \partial U \) to be those points \( x \) with the property that for every \( r > 0 \), \( B(x, r) \) contains points of \( U \) and points of \( U^C \). Then for \( U \) an open set,
\[ \partial U = U \setminus \overline{U} \]
Let \( C \) be a closed subset of \( \mathbb{R}^n \) and let \( K \) denote the set of components of \( \mathbb{R}^n \setminus C \). Then if \( K \) is one of these components, it is open and
\[ \partial K \subseteq C \]

**Proof:** Let \( x \in U \setminus \overline{U} \). If \( B(x, r) \) contains no points of \( U \), then \( x \notin \overline{U} \). If \( B(x, r) \) contains no points of \( U^C \), then \( x \notin \overline{U} \). Therefore, \( U \setminus \overline{U} \subseteq \partial U \). Now let \( x \in \partial U \). If \( x \in U \), then since \( U \) is open there is a ball containing \( x \) which is contained in \( U \) contrary to \( x \in \partial U \). Therefore, \( x \notin U \). If \( x \) is not a limit point of \( U \), then some ball containing \( x \) contains \( U \) contrary to \( x \in \partial U \). Therefore, \( x \in U \setminus \overline{U} \) which shows the two sets are equal.

Why is \( K \) open for \( K \) a component of \( \mathbb{R}^n \setminus C \)? This is obvious because in \( \mathbb{R}^n \) an open ball is connected. Thus if \( k \in K \), letting \( B(k, r) \subseteq C^C \), it follows \( K \cup B(k, r) \) is connected and contained in \( C^C \). Thus \( K \cup B(k, r) \) is connected, contained in \( C^C \), and therefore is contained in \( K \) because \( K \) is maximal with respect to being connected and contained in \( C^C \).

Now for \( K \) a component of \( \mathbb{R}^n \setminus C \), why is \( \partial K \subseteq C \)? Let \( x \in \partial K \). If \( x \notin C \), then \( x \in K_1 \), some component of \( \mathbb{R}^n \setminus C \). If \( K_1 \neq K \) then \( x \) cannot be a limit point of \( K \) and so it cannot be in \( \partial K \). Therefore, \( K = K_1 \) but this also is a contradiction because if \( x \in \partial K \) then \( x \notin K \).

I will give a shorter version of the proof and a longer version. First is the shorter version which leaves out a few details which may or may not be clear. Sometimes, it seems to me that when you put in too many details, one loses the forest by stumbling around hitting trees. It may still have too many details.
Theorem 21.6.7 (Jordan separation theorem) Let $f$ be a homeomorphism of $C$ and $f(C)$ where $C$ is a compact set in $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus C$ and $\mathbb{R}^n \setminus f(C)$ have the same number of connected components.

Proof: Denote by $K$ the bounded components of $\mathbb{R}^n \setminus C$ and denote by $L$, the bounded components of $\mathbb{R}^n \setminus f(C)$. Also, using the Tietze extension theorem, there exists $\bar{f}$ an extension of $f$ to all of $\mathbb{R}^n$ which maps into a bounded set and let $\bar{f}^{-1}$ be an extension of $f^{-1}$ to all of $\mathbb{R}^n$ which also maps into a bounded set. Pick $K \in K$ and take $y \in K$. Then $\partial K \subseteq C$ and so

$$y \notin \bar{f}^{-1}(\bar{f}(\partial K))$$

Since $\bar{f}^{-1} \circ \bar{f}$ equals the identity $I$ on $\partial K$, it follows from the properties of the degree that

$$1 = d(I, K, y) = d(\bar{f}^{-1} \circ \bar{f}, K, y).$$

Recall that if two functions agree on the boundary, then they have the same degree. Let $H$ denote the set of bounded components of $\mathbb{R}^n \setminus f(\partial K)$. These will be as large as those in $L$ and if a set in $L$ intersects one of these larger $H \in H$ then $H$ contains the component in $L$. By the product formula,

$$1 = d(\bar{f}^{-1} \circ \bar{f}, K, y) = \sum_{H \in H} d(\bar{f}, K, H) d(\bar{f}^{-1}, H, y),$$

the sum being a finite sum from the product formula. That is, there are finitely many $H$ involved in the sum, the other terms being zero.

What about those sets of $H$ which contain no set of $L$? These sets also have empty intersection with all sets of $L$. Therefore, for $H$ one of these, $H \subseteq f(C)$. Therefore,

$$d(\bar{f}^{-1}, H, y) = d(\bar{f}^{-1}, H, y) = 0$$

because $y \in K$ a component of $\mathbb{R}^n \setminus C$, but for $u \in H \subseteq f(C), f^{-1}(u) \in C$ so $f^{-1}(u) \neq y$ implying that $d(\bar{f}^{-1}, H, y) = 0$. Thus in sum all such terms are zero. Then letting $H_1$ be those sets of $H$ which contain (intersect) some sets of $L$, the above sum reduces to

$$\sum_{H \in H_1} d(\bar{f}, K, H) d(\bar{f}^{-1}, H, y) = \sum_{H \in H_1} d(\bar{f}, K, H) \sum_{L \in L_H} d(\bar{f}^{-1}, L, y)$$

where $L_H$ are those sets of $L$ contained in $H$. If $L_H = \emptyset$, the above shows that the second sum is 0 with the convention that $\sum_{\emptyset} = 0$. Now $d(\bar{f}, K, H) = d(\bar{f}, K, L)$ where $L \in L_H$. Therefore,

$$\sum_{H \in H_1} \sum_{L \in L_H} d(\bar{f}, K, H) d(\bar{f}^{-1}, L, y) = \sum_{H \in H_1} \sum_{L \in L_H} d(\bar{f}, K, L) d(\bar{f}^{-1}, L, y)$$
As noted above, there are finitely many $H \in \mathcal{H}$ which are involved. $\mathbb{R}^n \setminus \mathbf{f} (C) \subseteq \mathbb{R}^n \setminus \mathbf{f} (\partial K)$ and so every $L$ must be contained in some $H \in \mathcal{H}_1$. It follows that the above reduces to

$$\sum_{L \in \mathcal{L}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, y \right)$$

Thus from 21.6.12,

$$1 = \sum_{L \in \mathcal{L}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, y \right) = \sum_{L \in \mathcal{L}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, K \right) \quad (21.6.13)$$

Let $|\mathcal{K}|$ denote the number of components in $\mathcal{K}$ and similarly, $|\mathcal{L}|$ denotes the number of components in $\mathcal{L}$. Thus

$$|\mathcal{K}| = \sum_{K \in \mathcal{K}} 1 = \sum_{K \in \mathcal{K}} \sum_{L \in \mathcal{L}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, K \right)$$

Similarly,

$$|\mathcal{L}| = \sum_{L \in \mathcal{L}} 1 = \sum_{L \in \mathcal{L}} \sum_{K \in \mathcal{K}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, K \right)$$

If $|\mathcal{K}| < \infty$, then $\sum_{K \in \mathcal{K}} \sum_{L \in \mathcal{L}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, K \right) < \infty$. The summation which equals 1 is a finite sum and so is the outside sum. Hence we can switch the order of summation and get

$$|\mathcal{K}| = \sum_{L \in \mathcal{L}} \sum_{K \in \mathcal{K}} d (\bar{f}, K, L) d \left( \bar{f}^{-1}, L, K \right) = |\mathcal{L}|$$

A similar argument applies if $|\mathcal{L}| < \infty$. Thus if one of these numbers is finite, so is the other and they are equal. It follows that $|\mathcal{L}| = |\mathcal{K}|$. ■

Now is the same proof with more details included.

**Theorem 21.6.8 (Jordan separation theorem)** Let $\mathbf{f}$ be a homeomorphism of $C$ and $\mathbf{f} (C)$ where $C$ is a compact set in $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus C$ and $\mathbb{R}^n \setminus \mathbf{f} (C)$ have the same number of connected components.

**Proof:** Denote by $\mathcal{K}$ the bounded components of $\mathbb{R}^n \setminus C$ and denote by $\mathcal{L}$, the bounded components of $\mathbb{R}^n \setminus \mathbf{f} (C)$. Also, using the Tietze extension theorem, there exists $\bar{f}$ an extension of $\mathbf{f}$ to all of $\mathbb{R}^n$ which maps into a bounded set and let $\bar{f}^{-1}$ be an extension of $\mathbf{f}^{-1}$ to all of $\mathbb{R}^n$ which also maps into a bounded set. Pick $K \in \mathcal{K}$ and take $y \in K$. Then

$$y \notin \bar{f}^{-1} (\bar{f} (\partial K))$$

because by Lemma 21.6.6, $\partial K \subseteq C$ and on $C, \bar{f} = \mathbf{f}$. Thus the right side is of the form

$$\bar{f}^{-1} \left( \bar{f} (\partial K) \right) = \mathbf{f}^{-1} (\mathbf{f} (\partial K)) \subseteq C$$
and \( y \notin C \). Since \( f^{-1} \circ \bar{f} \) equals the identity \( I \) on \( \partial K \), it follows from the properties of the degree that

\[
1 = d(I, K, y) = d\left(f^{-1} \circ \bar{f}, K, y\right).
\]

Recall that if two functions agree on the boundary, then they have the same degree. Let \( \mathcal{H} \) denote the set of bounded components of \( \mathbb{R}^n \setminus f(\partial K) \). (These will be as large as those in \( \mathcal{L} \)) By the product formula,

\[
1 = d\left(f^{-1} \circ \bar{f}, K, y\right) = \sum_{H \in \mathcal{H}} d(H, K, H) d\left(f^{-1}, H, y\right),
\]

the sum being a finite sum from the product formula. It might help to consult the following diagram.

Now letting \( x \in L \in \mathcal{L} \), if \( S \) is a connected set containing \( x \) and contained in \( \mathbb{R}^n \setminus f(C) \), then it follows \( S \) is contained in \( \mathbb{R}^n \setminus f(\partial K) \) because \( \partial K \subseteq C \). Therefore, every set of \( \mathcal{L} \) is contained in some set of \( \mathcal{H} \). Furthermore, if any \( L \in \mathcal{L} \) has nonempty intersection with \( H \in \mathcal{H} \) then it must be contained in \( H \). This is because

\[
L = (L \cap H) \cup (L \cap \partial H) \cup \left(L \cap \overline{H}^C\right).
\]

Now by Lemma 21.6.1,

\[
L \cap \partial H \subseteq L \cap f(\partial K) \subseteq L \cap f(C) = \emptyset.
\]

Since \( L \) is connected, \( L \cap \overline{H}^C = \emptyset \). Letting \( \mathcal{L}_H \) denote those sets of \( \mathcal{L} \) which are contained in \( H \) equivalently having nonempty intersection with \( H \), if \( p \in H \setminus \cup \mathcal{L}_H = H \setminus \cup \mathcal{L} \), then \( p \in H \cap f(C) \) and so

\[
H = (\cup \mathcal{L}_H) \cup (H \cap f(C))
\]

Claim 1:

\[
\overline{H} \setminus \cup \mathcal{L}_H \subseteq f(C).
\]

**Proof of the claim:** Suppose \( p \in \overline{H} \setminus \cup \mathcal{L}_H \) but \( p \notin f(C) \). Then \( p \in L \in \mathcal{L} \). It must be the case that \( L \) has nonempty intersection with \( H \) since otherwise \( p \) could not be in \( \overline{H} \). However, as shown above, this requires \( L \subseteq H \) and now by \( p \notin \cup \mathcal{L}_H \), it follows \( p \in f(C) \) after all. This proves the claim.

**Claim 2:** \( y \notin f^{-1}(\overline{H} \setminus \cup \mathcal{L}_H) \). Recall \( y \in K \in \mathcal{K} \) the bounded components of \( \mathbb{R}^n \setminus C \).
21.6. THE PRODUCT FORMULA

Proof of the claim: If not, then \( f^{-1}(z) = y \) where \( z \in \overline{H} \cup L \subseteq f(C) \) and so \( z = f(w) \) for some \( w \in C \) and so \( y = f^{-1}(f(w)) = w \in C \) contrary to \( y \in K \), a component of \( \mathbb{R}^n \setminus C \).

Now every set of \( L \) is contained in some set of \( H \). What about those sets of \( H \) which contain no set of \( L \) so that \( L_H = \emptyset \)? From (21.6.15) it follows \( H \subseteq f(C) \). Therefore,

\[
d \left( f^{-1}, H, y \right) = d \left( f^{-1}, H, y \right) = 0
\]

because \( y \in K \), a component of \( \mathbb{R}^n \setminus C \). Therefore, letting \( H_1 \) denote those sets of \( H \) which contain some set of \( L \), (21.6.14) is of the form

\[
1 = \sum_{H \in H_1} d \left( f, K, H \right) d \left( f^{-1}, H, y \right)
\]

and it is still a finite sum because the terms in the sum are 0 for all but finitely many \( H \in H_1 \). I want to expand \( d \left( f^{-1}, H, y \right) \) as a sum of the form

\[
\sum_{L \in \mathcal{L}_H} d \left( f^{-1}, L, y \right)
\]

using Lemma (21.6.5). Therefore, I must verify

\[
y \notin f^{-1}(\overline{H} \cup L_H)
\]

but this is just Claim 2. By Lemma (21.6.5), I can write the above sum in place of \( d \left( f^{-1}, H, y \right) \). Therefore,

\[
1 = \sum_{H \in H_1} d \left( f, K, H \right) d \left( f^{-1}, H, y \right) = \sum_{H \in H_1} d \left( f, K, H \right) \sum_{L \in \mathcal{L}_H} d \left( f^{-1}, L, y \right)
\]

where there are only finitely many \( H \) which give a nonzero term and for each of these, there are only finitely many \( L \) in \( \mathcal{L}_H \) which yield \( d \left( f^{-1}, L, y \right) \neq 0 \). Now the above equals

\[
= \sum_{H \in H_1} \sum_{L \in \mathcal{L}_H} d \left( f, K, H \right) d \left( f^{-1}, L, y \right)
\]

(21.6.16)

By definition,

\[
d \left( f, K, H \right) = d \left( f, K, x \right)
\]

where \( x \) is any point of \( H \). In particular \( d \left( f, K, H \right) = d \left( f, K, L \right) \) for any \( L \in \mathcal{L}_H \). Therefore, the above reduces to

\[
= \sum_{L \in \mathcal{L}} d \left( f, K, L \right) d \left( f^{-1}, L, y \right)
\]

(21.6.17)
Here is why. There are finitely many $H \in \mathcal{H}_1$ for which the term in the double sum of 21.6.16 is not zero, say $H_1, \ldots, H_m$. Then the above sum in 21.6.17 equals

$$\sum_{k=1}^{m} \sum_{L \in \mathcal{L}_{H_k}} d(\bar{f}, K, L) d(\bar{f}, L, y) + \sum_{L' \in \mathcal{L} \setminus \bigcup_{k=1}^{m} \mathcal{L}_{H_k}} d(\bar{f}, K, L') d(\bar{f}, L, y).$$

The second sum equals 0 because those $L$ are contained in some $H \in \mathcal{H}$ for which

$$0 = d(\bar{f}, K, H) d(\bar{f}, L, y) = d(\bar{f}, K, H) \sum_{L \in \mathcal{L}_H} d(\bar{f}, L, y).$$

Therefore, the sum in 21.6.17 reduces to

$$\sum_{k=1}^{m} \sum_{L \in \mathcal{L}_{H_k}} d(\bar{f}, K, L) d(\bar{f}, L, y)$$

which is the same as the sum in 21.6.16. Therefore, 21.6.17 does follow. Then the sum in 21.6.17 reduces to

$$= \sum_{L \in \mathcal{L}} d(\bar{f}, K, L) d(\bar{f}, L, y).$$

and all but finitely many terms in the sum are 0.

By the same argument,

$$1 = \sum_{K \in \mathcal{K}} d(\bar{f}, K, L) d(\bar{f}, L, K)$$

and all but finitely many terms in the sum are 0. Letting $|\mathcal{K}|$ denote the number of elements in $\mathcal{K}$, similar for $\mathcal{L}$,

$$|\mathcal{K}| = \sum_{K \in \mathcal{K}} 1 = \sum_{K \in \mathcal{K}} \left( \sum_{L \in \mathcal{L}} d(\bar{f}, K, L) d(\bar{f}, L, K) \right)$$

$$|\mathcal{L}| = \sum_{L \in \mathcal{L}} 1 = \sum_{L \in \mathcal{L}} \left( \sum_{K \in \mathcal{K}} d(\bar{f}, K, L) d(\bar{f}, L, K) \right)$$

Suppose $|\mathcal{K}| < \infty$. Then you can switch the order of summation in the double sum for $|\mathcal{K}|$ and so

$$|\mathcal{K}| = \sum_{K \in \mathcal{K}} \left( \sum_{L \in \mathcal{L}} d(\bar{f}, K, L) d(\bar{f}, L, K) \right)$$

$$= \sum_{L \in \mathcal{L}} \left( \sum_{K \in \mathcal{K}} d(\bar{f}, K, L) d(\bar{f}, L, K) \right) = |\mathcal{L}|.$$
It follows that if either $|K|$ or $|L|$ is finite, then they are equal. Thus if one is infinite, so is the other. This proves the theorem because if $n > 1$ there is exactly one unbounded component to both $\mathbb{R}^n \setminus C$ and $\mathbb{R}^n \setminus f(C)$ and if $n = 1$ there are exactly two unbounded components. ■

As an application, here is a very interesting little result. It has to do with $d(f, \Omega, f(x))$ in the case where $f$ is one to one and $\Omega$ is connected. You might imagine this should equal 1 or $-1$ based on one dimensional analogies. In fact this is the case and it is a nice application of the Jordan separation theorem and the product formula.

**Proposition 21.6.9** Let $\Omega$ be an open connected bounded set in $\mathbb{R}^n$, $n \geq 1$ such that $\mathbb{R}^n \setminus \partial \Omega$ consists of two, three if $n = 1$, connected components. Let $f \in C(\overline{\Omega} ; \mathbb{R}^n)$ be continuous and one to one. Then $f(\Omega)$ is the bounded component of $\mathbb{R}^n \setminus f(\partial \Omega)$ and for $y \in f(\Omega)$, $d(f, \Omega, y)$ either equals 1 or $-1$.

**Proof:** First suppose $n \geq 2$. By the Jordan separation theorem, $\mathbb{R}^n \setminus f(\partial \Omega)$ consists of two components, a bounded component $B$ and an unbounded component $U$. Using the Tietze extension theorem, there exists $g$ defined on $\mathbb{R}^n$ such that $g = f^{-1}$ on $f(\overline{\Omega})$. Thus on $\partial \Omega$, $g \circ f = \text{id}$. It follows from this and the product formula that

$$
1 = d(\text{id}, \Omega, g(y)) = d(g \circ f, \Omega, g(y))
= d(g, B, g(y)) d(f, \Omega, B) + d(f, \Omega, U) d(g, U, g(y))
= d(g, B, g(y)) d(f, \Omega, B)
$$

The reduction happens because $d(f, \Omega, U) = 0$ as explained above. Since $U$ is unbounded, there are points in $U$ which cannot be in the compact set $f(\overline{\Omega})$. For such, the degree is 0 but the degree is constant on $U$, one of the components of $f(\partial \Omega)$. Therefore, $d(f, \Omega, B) \neq 0$ and so for every $z \in B$, it follows $z \in f(\Omega)$. Thus $B \subseteq f(\Omega)$. On the other hand, $f(\Omega)$ cannot have points in both $U$ and $B$ because it is a connected set. Therefore $f(\Omega) \subseteq B$ and this shows $B = f(\Omega)$. Thus $d(f, \Omega, B) = d(f, \Omega, y)$ for each $y \in B$ and the above formula shows this equals either 1 or $-1$ because the degree is an integer. In the case where $n = 1$, the argument is similar but here you have 3 components in $\mathbb{R}^1 \setminus f(\partial \Omega)$ so there are more terms in the above sum although two of them give 0. ■

### 21.7 A Function With Values In Smaller Dimensions

Recall that we have the degree defined $d(f, \Omega, y)$ for continuous functions on $\overline{\Omega}$ and $y \notin f(\partial \Omega)$. It had properties as follows.

1. $d(\text{id}, \Omega, y) = 1$ if $y \in \Omega$.
2. If $\Omega_1 \subseteq \Omega$, $\Omega_1$ open, and $\Omega_1 \cap \Omega_2 = \emptyset$ and if $y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then $d(f, \Omega_1, y) + d(f, \Omega_2, y) = d(f, \Omega, y)$. 
3. If \( y \notin f(\overline{\Omega} \setminus \Omega) \) and \( \Omega_1 \) is an open subset of \( \Omega \), then
\[
d(f, \Omega, y) = d(f, \Omega_1, y).
\]

4. For \( y \in \mathbb{R}^n \setminus f(\partial \Omega) \), if \( d(f, \Omega, y) \neq 0 \) then \( f^{-1}(y) \cap \Omega \neq \emptyset \).

5. If \( t \to y(t) \) is continuous \( h : \Omega \times [0,1] \to \mathbb{R}^n \) is continuous and if \( y(t) \notin h(\partial \Omega, t) \) for all \( t \), then \( t \to d(h(\cdot, t), \Omega, y(t)) \) is constant.

6. \( d(\cdot, \Omega, y) \) is defined and constant on
\[
\{ g \in C(\overline{\Omega}; \mathbb{R}^n) : \| g - f \|_{\infty} < r \}
\]
where \( r = \text{dist}(y, f(\partial \Omega)) \).

7. \( d(f, \Omega, \cdot) \) is constant on every connected component of \( \mathbb{R}^n \setminus f(\partial \Omega) \).

8. \( d(g, \Omega, y) = d(f, \Omega, y) \) if \( g|_{\partial \Omega} = f|_{\partial \Omega} \).

**Theorem 21.7.1** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \) and let \( f \in C(\overline{\Omega}; \mathbb{R}^m) \) where \( \mathbb{R}^m = \{ x \in \mathbb{R}^n : x_k = 0 \text{ for } k > m \} \). Thus \( \mathbf{x} \) concludes with a column of \( n-m \) zeros. Let \( y \in \mathbb{R}^n \setminus (\text{id} - f)(\partial \Omega) \). Then \( d(\text{id} - f, \Omega, y) = d(\text{id} - f)(\overline{\Omega \cap \mathbb{R}^m}, y) \).

**Proof:** To save space, let \( g = \text{id} - f \). Then there is no loss of generality in assuming at the outset that \( y \) is a regular value for \( g \). Indeed, everything above was reduced to this case. Then for \( x \in g^{-1}(y) \) and letting \( x_m \) be the first \( m \) variables for \( x \),
\[
Dg(x) = \begin{pmatrix}
D_{x_m}g(x) & * \\
0 & I_{n-m}
\end{pmatrix}
\]
Then it follows that
\[
0 \neq \det(Dg(x)) = \det\left(\begin{pmatrix}
D_{x_m}g(x) & * \\
0 & I_{n-m}
\end{pmatrix}\right) = \det\left(\begin{pmatrix}
D_{x_m}g(x) \\
0
\end{pmatrix}\right) = \det(D_{x_m}g(x))
\]
This last is just the determinant of the derivative of the function which results from restricting \( g \) to the first \( m \) variables. Now \( y \in \mathbb{R}^n \), and \( f \) also is given to have values in \( \mathbb{R}^m \), so if \( g(x) = y \), then you have \( x - f(x) = y \) which requires \( x \in \mathbb{R}^m \), also. Therefore, \( g^{-1}(y) \) consists of points in \( \mathbb{R}^m \) only. Thus, \( y \) is also a regular value of the function which results from restricting \( g \) to \( \mathbb{R}^m \cap \Omega \).
\[
d(\text{id} - f, \Omega, y) = d(g, \Omega, y)
\]
\[
= \sum_{x \in g^{-1}(y)} \text{sign}(\det(Dg(x)))
\]
\[
= \sum_{x \in g^{-1}(y)} \text{sign}(\det(D_{x_m}g(x))) = d(\text{id} - f)(\overline{\Omega \cap \mathbb{R}^m}, \Omega \cap \mathbb{R}^m, y) \quad \blacksquare
\]
21.7. A FUNCTION WITH VALUES IN SMALLER DIMENSIONS

Recall that for \( g \in C^2 (\bar{\Omega} ; \mathbb{R}^n) \),

\[
d (g, \Omega, y) = \lim_{\varepsilon \to 0} \int_{\Omega} \varphi_\varepsilon (g (x) - y) \det Dg (x) \, dx
\]

In fact, it can be shown that the degree is unique based on its Properties, 1,2,5 above. It involves reducing to linear maps and then some complicated arguments involving linear algebra. It is done in \[39\]. Here we will be a little less ambitious. The following lemma will be useful when extending the degree to finite dimensional normed linear spaces and from there to Banach spaces. It is motivated by the following diagram.

\[
\begin{array}{c c c c}
\theta^{-1} (y) & \theta^{-1} (\Omega) \\
\uparrow \theta^{-1} \circ g \circ \theta & \uparrow g \\
\theta^{-1} (\bar{\Omega}) & \rightarrow \Omega
\end{array}
\]

Lemma 21.7.2 Let \( y \notin g (\partial \Omega) \) and let \( \theta \) be an isomorphism of \( \mathbb{R}^n \). That is, \( \theta \) is one to one onto and linear. Then

\[
d (\theta^{-1} \circ g \circ \theta, \theta^{-1} (\bar{\Omega}), \theta^{-1} (y)) = d (g, \Omega, y)
\]

Proof: It suffices to consider \( g \in C^2 (\bar{\Omega} ; \mathbb{R}^n) \) for which \( y \) is a regular value because you can get such a \( \hat{g} \) with \( \| \hat{g} - g \| \infty < \delta \) where

\[
B (y, 2\delta) \cap g (\partial \Omega) = \emptyset.
\]

Thus \( B (y, \delta) \cap \hat{g} (\partial \Omega) = \emptyset \) and so \( d (g, \Omega, y) = d (\hat{g}, \Omega, y) \). One can assume similarly that \( \| \hat{g} - g \| \infty \) is sufficiently small that

\[
d (\theta^{-1} \circ g \circ \theta, \theta^{-1} (\bar{\Omega}), \theta^{-1} (y)) = d (\theta^{-1} \circ \hat{g} \circ \theta, \theta^{-1} (\bar{\Omega}), \theta^{-1} (y))
\]

because both \( \theta \) and \( \theta^{-1} \) are continuous. Thus it suffices to consider at the outset \( g \in C^2 (\bar{\Omega} ; \mathbb{R}^n) \). Then from the definition of degree for \( C^2 \) maps,

\[
d (\theta^{-1} \circ g \circ \theta, \theta^{-1} (\bar{\Omega}), \theta^{-1} (y))
\]

\[
= \lim_{\varepsilon \to 0} \int_{\theta^{-1} \Omega} \varphi_\varepsilon ((\theta^{-1} \circ g \circ \theta) (z) - \theta^{-1} (y)) \det D (\theta^{-1} \circ g \circ \theta) (z) \, dz
\]

Now \( D (\theta^{-1} \circ g \circ \theta) (z) = \theta^{-1} D (g \circ \theta) (z) = \theta^{-1} Dg (\theta (z)) \theta z. \) Changing the variables \( x = \theta z, z = \theta^{-1} x, \) this last integral equals

\[
\int_{\Omega} \varphi_\varepsilon ((\theta^{-1} g \circ \theta) (\theta^{-1} (x)) - \theta^{-1} (y)) \det Dg (x) \, dx
\]

\[
= \int_{\Omega} \varphi_\varepsilon (\theta^{-1} g (x) - \theta^{-1} (y)) \det Dg (x) \, dx
\]

Recall that \( \varphi_\varepsilon \) is a mollifier which is nonzero only in \( B (0, \varepsilon) \). Now

\[
g^{-1} (y) = \{ x_1, \cdots, x_m \} = (\theta^{-1} g)^{-1} (\theta^{-1} (y))
\]
and so \( \mathbf{g}(x_i) = y \) and \( \theta^{-1}\mathbf{g}(x_i) = \theta^{-1} y \). By the inverse function theorem, there exist disjoint open sets \( U_i \) with \( x_i \in U_i \), such that \( \theta^{-1}\mathbf{g} \) is one to one on \( U_i \) with \( \det \left( D \left( \theta^{-1}\mathbf{g} \right)(x) \right) = \det \left( \theta^{-1} \right) \det D\mathbf{g}(x) \) having constant sign on \( U_i \) and \( \theta^{-1}\mathbf{g}(U_i) \) is an open set containing \( \theta^{-1}y \). Then let \( \varepsilon \) be small enough that \( \cap_{i=1}^m \theta^{-1}\mathbf{g}(U_i) \) and let \( V_i \equiv \left( \theta^{-1}\mathbf{g} \right)^{-1} \left( B \left( \theta^{-1}y, \varepsilon \right) \right) \cap U_i \). Thus for small \( \varepsilon \), the \( V_i \) are disjoint open sets in \( \Omega \) and

\[
\int_{\Omega} \phi_{\varepsilon} \left( \theta^{-1}\mathbf{g}(x) - \theta^{-1}y \right) \det D\mathbf{g}(x) \left| \det \theta^{-1} \right| dx
= \sum_{i=1}^m \int_{V_i} \phi_{\varepsilon} \left( \theta^{-1}\mathbf{g}(x) - y \right) \det D\mathbf{g}(x) \left| \det \theta^{-1} \right| dx
\]

Now just let \( z = \mathbf{g}(x) - y \) and change the variables.

\[
= \sum_{i=1}^m \left| \det \theta^{-1} \right| \int_{g(V_i) - y} \phi_{\varepsilon} \left( \theta^{-1}z \right) \det D\mathbf{g} \left( g^{-1}(y + z) \right) \left| \det Dg^{-1}(y + z) \right| dz
\]

By the chain rule, \( I = D\mathbf{g} \left( g^{-1}(y + z) \right) Dg^{-1}(y + z) \) and so

\[
\det D\mathbf{g} \left( g^{-1}(y + z) \right) \left| \det Dg^{-1}(y + z) \right|
= \sgn \left( \det D\mathbf{g} \left( g^{-1}(y + z) \right) \right) \cdot \left| \det Dg^{-1}(y + z) \right|
\]

\[
= \sgn \left( \det D\mathbf{g} \left( g^{-1}(y + z) \right) \right) \left| \det Dg^{-1}(y + z) \right|
\]

\[
= \sgn \left( \det D\mathbf{g}(x) \right) = \sgn \left( \det D\mathbf{g}(x_i) \right).
\]

and so it all reduces to

\[
\sum_{i=1}^m \sgn \left( \det D\mathbf{g}(x_i) \right) \int_{g(V_i) - y} \phi_{\varepsilon} \left( \theta^{-1}z \right) dz
= \sum_{i=1}^m \sgn \left( \det D\mathbf{g}(x_i) \right) \int_{\theta B(0,\varepsilon)} \left| \det \theta^{-1} \right| \phi_{\varepsilon} \left( \theta^{-1}z \right) dz
\]

\[
= \sum_{i=1}^m \sgn \left( \det D\mathbf{g}(x_i) \right) \int_{B(0,\varepsilon)} \phi_{\varepsilon} \left( w \right) \left| \det \theta^{-1} \right| \left| \det \theta \right| dw
= \sum_{i=1}^m \sgn \left( \det D\mathbf{g}(x_i) \right) = d \left( \mathbf{g}, \Omega, y \right). \quad \blacksquare
\]

What about functions which have values in finite dimensional vector spaces?
21.7. A FUNCTION WITH VALUES IN SMALLER DIMENSIONS

**Theorem 21.7.3** Let \( \Omega \) be an open bounded set in \( V \) a real normed \( n \) dimensional vector space. Then there exists a topological degree \( d(f, \Omega, y) \) for \( f \in C(\Omega, V) \), \( y \notin f(\partial \Omega) \) which satisfies all the properties of the degree for functions having values in \( \mathbb{R}^n \) described above,

1. \( d(id, \Omega, y) = 1 \) if \( y \in \Omega \).
2. If \( \Omega_1 \subseteq \Omega, \Omega_2 \) open, and \( \Omega_1 \cap \Omega_2 = \emptyset \) and if \( y \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)) \), then \( d(f, \Omega_1, y) + d(f, \Omega_2, y) = d(f, \Omega, y) \).
3. If \( y \notin f(\overline{\Omega} \setminus \Omega_1) \) and \( \Omega_1 \) is an open subset of \( \Omega \), then
   \[ d(f, \Omega, y) = d(f, \Omega_1, y). \]
4. For \( y \in \mathbb{R}^n \setminus f(\partial \Omega) \), if \( d(f, \Omega, y) \neq 0 \) then \( f^{-1}(y) \cap \Omega \neq \emptyset \).
5. If \( t \to y(t) \) is continuous \( h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n \) is continuous and if \( y(t) \notin h(\partial \Omega, t) \) for all \( t \), then \( t \to d(h(\cdot, t), \Omega, y(t)) \) is constant.
6. \( d(\cdot, \Omega, y) \) is defined and constant on
   \[ \{ g \in C(\overline{\Omega}; \mathbb{R}^n) : \| g - f \|_\infty < r \} \]
   where \( r = \text{dist}(y, f(\partial \Omega)) \).
7. \( d(f, \Omega, \cdot) \) is constant on every connected component of \( \mathbb{R}^n \setminus f(\partial \Omega) \).
8. \( d(g, \Omega, y) = d(f, \Omega, y) \) if \( g|_{\partial \Omega} = f|_{\partial \Omega} \).

**Proof:** There is an isomorphism \( \theta : \mathbb{R}^n \to V \) which also preserves all topological properties. This follows from the properties of finite dimensional vector spaces. In fact, every algebraic isomorphism is automatically a homeomorphism preserving all topological properties. Then it is pretty easy to see what the degree should be.

\[ d(f, \Omega, y) \equiv d(\theta^{-1} \circ f \circ \theta, \theta^{-1}(\Omega), \theta^{-1}y) \]

Then by standard material on finite dimensional vector spaces, the norm on \( V \) is equivalent to the norm defined by \( \|v\| \equiv \|\theta^{-1}v\|_{\mathbb{R}^n} \). Hence all of those properties hold.

By Lemma 21.7.2 this definition does not depend on the particular isomorphism used. If \( \theta \) is another one, then one would need to verify that

\[ d(\theta^{-1} \circ f \circ \theta, \theta^{-1}(\Omega), \theta^{-1}y) = d(\hat{\theta}^{-1} \circ f \circ \hat{\theta}, \hat{\theta}^{-1}(\Omega), \hat{\theta}^{-1}y) \]

However, you could use that lemma to conclude that

\[ d(\hat{\theta}^{-1} \circ f \circ \hat{\theta}, \hat{\theta}^{-1}(\Omega), \hat{\theta}^{-1}y) = d(\alpha^{-1} \circ \hat{\theta}^{-1} \circ f \circ \alpha \circ \alpha^{-1} \theta^{-1}(\Omega), \alpha^{-1} \hat{\theta}^{-1}y) \]

where \( \alpha \) is such that \( \hat{\theta} \circ \alpha = \theta \). Then this verifies the appropriate equation. ■

Next one considers what happens when the function \( f \) has values in a smaller dimensional subspace.
Theorem 21.7.4 Let $\Omega$ be a bounded open set in $V$ an $n$ dimensional normed linear space and let $f \in C(\overline{\Omega}; V_m)$ where $V_m$ is an $m$ dimensional subspace. Let $y \in V_m \setminus (I - f)(\partial \Omega)$. Then $d(I - f, \Omega, y) = d((I - f)|_{\overline{\Omega} \cap V_m}, \Omega \cap V_m, y)$.

Proof: Letting $\{v_1, \ldots, v_m\}$ be a basis for $V_m$, let a basis for $V$ be

$$\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_n\}$$

Let $\theta$ be the isomorphism which satisfies $\theta e_i = v_i$ where the $e_i$ denotes the standard basis vectors for $\mathbb{R}^n$. Then from the above,

$$d(I - f, \Omega, y) = d(\theta^{-1} \circ (I - f) \circ \theta, \theta^{-1}\Omega, \theta^{-1}y)$$

$$= d(\theta^{-1} \circ (I - f) \circ \theta|_{\overline{\Omega} \cap \mathbb{R}^n_m}, \theta^{-1}\Omega \cap \mathbb{R}^n_m, \theta^{-1}y)$$

$$\equiv d((I - f)|_{\overline{\Omega} \cap V_m}, \Omega \cap V_m, y). \qed$$

### 21.8 The Leray Schauder Degree

This is a very important generalization to Banach spaces. It turns out you can define the degree of $I - F$ where $F$ is a compact mapping. To recall what one of these is, here is the definition.

**Definition 21.8.1** Let $\Omega$ be a bounded open set in $X$ a Banach space and let $F : \overline{\Omega} \to X$ be continuous. Then $F$ is called compact if $F(B)$ is precompact whenever $B$ is bounded. That is, if $\{x_n\}$ is a bounded sequence, then there is a subsequence $\{x_{n_k}\}$ such that $\{F(x_{n_k})\}$ converges.

**Theorem 21.8.2** Let $F : \overline{\Omega} \to X$ as above be compact. Then for each $\varepsilon > 0$, there exists $F_\varepsilon : \overline{\Omega} \to X$ such that $F$ has values in a finite dimensional subspace of $X$ and $\sup_{x \in \Omega} \|F_\varepsilon(x) - F(x)\| < \varepsilon$. In addition to this, $(I - F)^{-1}$ (compact set) = compact set. (This is called “proper”.)

**Proof:** It is known that $\overline{F(\Omega)}$ is compact. Therefore, there is an $\varepsilon$ net for $F(\Omega)$, $\{F_{x_k}\}_{k=1}^n$ satisfying

$$\overline{F(\Omega)} \subseteq \bigcup_k B(F_{x_k}, \varepsilon)$$

Now let

$$\phi_k(Fx) \equiv (\varepsilon - \|Fx - F_{x_k}\|)^+$$

Thus this is equal to 0 if $\|Fx_k - Fx\| \geq \varepsilon$ and is positive if $\|Fx_k - Fx\| < \varepsilon$. Then consider

$$F_{\varepsilon}(x) \equiv \sum_{k=1}^n F(x_k) \frac{\phi_k(Fx)}{\sum_i \phi_i(Fx)}$$
21.8. THE LERAY SCHAUDEL DEGREE

It clearly has values in \( \text{span } \{ Fx_k \}_{k=1}^{n} \). How close is it to \( F(x) \)? Say \( Fx \in B(Fx_k, \varepsilon) \). Then for such \( x \), \( \| F(x) - F(x_k) \| < \varepsilon \) by definition. Hence

\[
\| F(x) - F_{\varepsilon}(x) \| = \sum_{k: \| F(x) - Fx_k \| < \varepsilon} \| F(x_k) - F(x) \| \frac{\phi_k(Fx)}{\sum_i \phi_i(Fx)} < \varepsilon \sum_k \sum_i \phi_i(Fx) = \varepsilon
\]

Of course \( x \) is arbitrary and so

\[
\sup_{x \in \Omega} \| F_{\varepsilon}(x) - F(x) \| < \varepsilon.
\]

Next consider the second claim. Let \( K \) be compact. Consider \( \{ x_k \} \subseteq (I - F)^{-1}(K) \).

It is necessary to show that it has a convergent subsequence. Then \( \{ (I - F)(x_k) \} \) is a sequence in \( K \) and so it has a convergent subsequence still denoted with subscript \( k \) such that \( (I - F)(x_k) \to y \). The \( x_k \) are in a bounded set \( \Omega \) and so, from compactness of \( F \), there is a further subsequence, still denoted with subscript \( k \) such that \( Fx_k \to z \). It follows that \( x_k \to y - z \) and hence every sequence in \( (I - F)^{-1}(K) \) has a convergent subsequence.

**Corollary 21.8.3** Let \( F : \Omega \to X \) as above be compact. Then for each \( \varepsilon > 0 \), there exists \( F_{\varepsilon} : \Omega \to X \) such that \( F \) has values in a finite dimensional subspace of \( X \) and \( \sup_{x \in \Omega} \| F_{\varepsilon}(x) - F(x) \| < \varepsilon \). In addition to this, \( (I - F)^{-1}(K) = \text{compact set} \). (This is called “proper”.) If \( \Omega \) is symmetric and \( F \) is odd \( (F(-x) = -F(x)) \) then one can also assume \( F_{\varepsilon} \) is also odd.

**Proof:** Suppose \( \Omega \) is symmetric in that \( x \in \Omega \) iff \( -x \in \Omega \). Suppose also that \( F \) is odd. Thus \( F(\Omega) \) is also symmetric. Thus \( F(\Omega) \) is compact and symmetric. If \( y \in F(\Omega) \), then \( y = Fx \) and so \( -y = F(-x) = F(x) \in F(\Omega) \). Choose the \( \varepsilon \) net to be symmetric. That is, you have \( (Fx)_k \) in the net if and only if \( -(Fx)_k \) is in the net. Just add them in if needed. Therefore, there is an \( \varepsilon \) net for \( F(\Omega) \), \( \{(Fx)_k\}_{k=-m_{\varepsilon}}^{m_{\varepsilon}} \) satisfying

\[
F(\Omega) \subseteq \bigcup_k B(Fx_k, \varepsilon), \{ Fx_k \} \text{ is symmetric}.
\]

Number these so that

\[
Fx_{-k} = -Fx_k = F(-x_k), |k| \leq m_{\varepsilon}.
\]

Now let

\[
\phi_k(Fx) \equiv (\varepsilon - \|Fx - Fx_k\|)^+
\]
\(\phi_{-k}(Fx) = (\varepsilon - \|Fx - Fx_{-k}\|)^+\)

\(= (\varepsilon - \|Fx + Fx_{k}\|)^+\)

\(= (\varepsilon - \|Fx - (-Fx_{k})\|)^+\)

\(= \phi_{k}(-Fx)\)

that is, \(\phi_{-k}\) is centered at \(-Fx_{k}\) while \(\phi_{k}\) is centered at \(Fx_{k}\), each function equal to 0 off \(B(Fx_{k}, \varepsilon)\) and is positive on \(B(Fx_{k}, \varepsilon)\). Then consider

\[F_\varepsilon(x) = \sum_{k=-m_\varepsilon}^{m_\varepsilon} F(x_k) \frac{\phi_k(Fx)}{\sum_{i} \phi_i(Fx)}\]

\[F_\varepsilon(-x) = \sum_{k=-m_\varepsilon}^{m_\varepsilon} F(x_k) \frac{\phi_{-k}(F(-x))}{\sum_{i} \phi_i(F(-x))} = \sum_{k=-m_\varepsilon}^{m_\varepsilon} F(x_k) \frac{\phi_{-k}(F(x))}{\sum_{i} \phi_{-i}(F(x))}\]

\[= - \sum_{k=-m_\varepsilon}^{m_\varepsilon} F(x_k) \frac{\phi_{-k}(F(x))}{\sum_{i} \phi_{-i}(F(x))} = - \sum_{k=-m_\varepsilon}^{m_\varepsilon} F(x_{-k}) \frac{\phi_{-k}(F(x))}{\sum_{i} \phi_{-i}(F(x))}\]

The rest of the argument is the same.

Now let \(F : \overline{\Omega} \to X\) be compact and consider \(I - F\). Is \((I - F)(\partial \Omega)\) closed? Suppose \((I - F)x_k \to y\). Then \(K \equiv y \cup \{(I - F)x_k\}_{k=1}^\infty\) is a compact set because if you have any open cover, one of the open sets contains \(y\) and hence it contains all \((I - F)x_k\) except for finitely many which can then be covered by finitely many open sets in the open cover. Hence, since \((I - F)\) is proper, \((I - F)^{-1}(K)\) is compact. It follows that there is a subsequence, still called \(x_k\) such that \(x_k \to x \in (I - F)^{-1}(K)\). Then by continuity of \(F\),

\((I - F)(x_k) \to (I - F)(x)\)

\((I - F)(x_k) \to y\)

It follows \(y = (I - F)x\) and so in fact \((I - F)(\partial \Omega)\) is closed.

Lemma 21.8.4 If \(F : \overline{\Omega} \to X\) is compact and \(\Omega\) is a bounded open set in \(X\), then \((I - F)(\partial \Omega)\) is closed.

**Justification for definition of Leray Schauder Degree**

Now let \(y \notin (I - F)(\partial \Omega)\), a closed set. Hence dist \((y, (I - F)(\partial \Omega)) > 4\delta > 0\). Now let \(F_k\) be a sequence of approximations to \(F\) which has values in an increasing sequence of finite dimensional subsets \(V_k\) each of which contains \(y\). Thus \(\lim_{k \to \infty} \sup_{x \in \Omega} \|F(x) - F_k(x)\| = 0\). Consider

\[d(I - F_k|_{V_k}, \Omega \cap V_k, y)\]
21.8. THE LERAY SCHAUDER DEGREE

Each of these is a well defined integer according to Theorem 21.7.3. For all $k$ large enough,

$$\sup_{x \in \Omega} \| (I - F^k)(x) - (I - F)(x) \| < \delta$$

Hence, for all such $k$,

$$B(y, 3\delta) \cap (I - F^k)(\partial \Omega) = \emptyset,$$
that is $\text{dist}(y, (I - F^k)(\partial \Omega)) > 3\delta$ \hspace{1cm} (21.8.18)

Note that this implies

$$\text{dist}(y, (I - F^k)(\partial (\Omega \cap V))) > 3\delta$$

for any subspace $V$. If $k < l$ are two such indices, then consider

$$d(I - F^k|_{V_k}, \Omega \cap V_k, y), d(I - F^l|_{V_l}, \Omega \cap V_l, y)$$

Are they equal? Let $V = V_k + V_l$. Then by Theorem 21.7.4,

$$d(I - F^l|_{V_l}, \Omega \cap V_l, y) = d(I - F^l|_{V_l}, \Omega \cap V_l, y)$$

$$d(I - F^k|_{V_k}, \Omega \cap V_k, y) = d(I - F^k|_{V_k}, \Omega \cap V_k, y)$$

So what about $d(I - F^l|_{V_l}, \Omega \cap V_l, y), d(I - F^k|_{V_k}, \Omega \cap V_k, y)$? Are these equal?

$$\sup_{x \in \Omega \cap V} \| F^l(x) - F^k(x) \| \leq \sup_{x \in \Omega \cap V} \| F^l(x) - F(x) \| + \sup_{x \in \Omega \cap V} \| F(x) - F^k(x) \| < 2\delta$$

This implies for

$$h(x, t) = t(I - F^l)(x) + (1 - t)(I - F^k)(x),$$

and $x \in \Omega \cap V, y \notin h(\partial (\Omega \cap V), t)$ for all $t \in [0, 1]$. To see this, let $x \in \partial \Omega$

$$\| t(I - F^l)(x) + (1 - t)(I - F^k)(x) - y \| = \| t(I - F^l)(x) + t(F^k x - F^l x) + (1 - t)(I - F^k)(x) - y \|$$

$$\| (I - F^k)(x) + t(F^k x - F^l x) \| \geq 3\delta - t2\delta \geq \delta$$

Hence

$$d(I - F^l|_{V_l}, \Omega \cap V_l, y) = d(I - F^k|_{V_k}, \Omega \cap V_k, y)$$

and so

$$\lim_{k \to \infty} d(I - F^k|_{V_k}, \Omega \cap V_k, y)$$
exists. A similar argument shows that this limit is independent of the sequence $\{F_k\}$ of approximating functions having values in a finite dimensional space. Thus we have the following definition of the Leray Schauder degree.
Definition 21.8.5 Let $X$ be a Banach space and let $F : X \to X$ be compact. That is, $F(\Omega)$ is precompact whenever $\Omega$ is bounded. Let $\Omega$ be a bounded open set in $X$ and let $y \notin (I - F)(\partial \Omega)$. Let $F_k$ be a sequence of operators which have values in finite dimensional spaces $V_k$ such that $V_k \subseteq V_{k+1} \cdots, y \in V_k$, and $\lim_{k \to \infty} \sup_{x \in \Omega} \|F(x) - F_k(x)\| = 0$. Then

$$D(I - F, \Omega, y) = \lim_{k \to \infty} d(I - F_k|_{V_k}, \Omega \cap V_k, y)$$

In fact, the sequence on the right is eventually constant. So

$$D(I - F, \Omega, y) \equiv d(I - F_k|_{V_k}, \Omega \cap V_k, y)$$

for all $k$ sufficiently large.

The main properties of the Leray Schauder degree follow from the corresponding properties of Brouwer degree.

Theorem 21.8.6 Let $D$ be the Leray Schauder degree just defined and let $\Omega$ be a bounded open set $y \notin (I - F)(\partial \Omega)$ where $F$ is always a compact mapping. Then the following properties hold:

1. $D(I, \Omega, y) = 1$

2. If $\Omega_i \subseteq \Omega$ where $\Omega_i$ is open, $\Omega_1 \cap \Omega_2 = \emptyset$, and $y \notin \overline{\Omega \setminus (\Omega_1 \cup \Omega_2)}$ then

$$D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$$

3. If $t \to y(t)$ is continuous $h : \overline{\Omega} \times [0, 1] \to X$ is continuous, $(x, t) \to h(x, t)$ is compact, (It takes bounded subsets of $\overline{\Omega} \times [0, 1]$ to precompact sets in $X$) and if $y(t) \notin (I - h)(\partial \Omega, t)$ for all $t$, then $t \to D((I - h)(\cdot, t), \Omega, y(t))$ is constant.

Proof: The mapping $x \to 0$ is clearly compact. Then an approximating sequence is $F_k, F_kx = 0$ for all $k$. Then

$$D(I, \Omega, y) = \lim_{k \to \infty} d(I|_{V_k}, \Omega \cap V_k, y) = 1$$

For the second part, let $k$ be large enough that for $U = \Omega, \Omega_1, \Omega_2$,

$$D(I - F, U, y) = d(I - F_k|_{V_k}, U \cap V_k, y)$$

where $F_k$ is the sequence of approximating functions having finite dimensional range. Then the result follows from the Brouwer degree. In fact,

$$D(I - F, \Omega, y) = d(I - F_k|_{V_k}, \Omega \cap V_k, y) \equiv d(I - F_k|_{V_k}, \Omega_1 \cap V_k, y) + d(I - F_k|_{V_k}, \Omega_2 \cap V_k, y) \equiv D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$$
21.8. THE LERAY SCHAUDER DEGREE

this does the second claim of the theorem. Now consider the third one about homotopy invariance.

Claim: If \( \text{dist} (y, (I - F) \partial \Omega) \geq 6 \delta \), and if \( \| y - z \| < \delta \), then

\[
D (I - F, \Omega, y) = D (I - F, \Omega, z)
\]

Proof of claim: Let \( F_k \) be the approximations and include both \( y, z \) in all the finite dimensional subspaces \( V_k \). Then for \( k \) large enough, \( \sup_{x \in \Omega} \| F (x) - F_k (x) \| < \delta \) and also,

\[
D (I - F, \Omega, y) = d ((I - F_k) |_{V_k}, \Omega \cap V_k, y)
\]

\[
D (I - F, \Omega, z) = d ((I - F_k) |_{V_k}, \Omega \cap V_k, z)
\]

Now for \( x \in \partial (\Omega \cap V_k) \),

\[
\|(I - F_k) (x) - y\| \geq \|(I - F) (x) + (F (x) - F_k (x)) - y\|
\]

\[
\geq \|(I - F) (x) - y\| - \|F (x) - F_k (x)\|
\]

\[
> 6 \delta - \delta = 5 \delta
\]

Hence \( \text{dist} (y, (I - F_k) \partial \Omega) \geq 5 \delta \) while \( \| y - z \| < \delta \). Hence

\[
d ((I - F_k) |_{V_k}, \Omega \cap V_k, y) = d ((I - F_k) |_{V_k}, \Omega \cap V_k, z)
\]

by Theorem \ref{thm:main}.

From compactness of \( h \), there is an \( \varepsilon \) net for \( h (\bar{\Omega} \times [0, 1]), \{ h (x_k, t_k) \} \) such that

\[
h (\bar{\Omega} \times [0, T]) \subseteq \bigcup_{k=1}^{n} B (h (x_k, t_k), \varepsilon).
\]

Say the \( t_k \) are ordered. Then, as before,

\[
\phi_k (x) \equiv (\varepsilon - \| h (x, t) - h (x_k, t_k) \|)^+
\]

\[
h_n (x, t) \equiv \sum_{k=1}^{n} h (x_k, t_k) \frac{\phi_k (h (x, t))}{\sum_{i} \phi_i (h (x, t))}
\]

Then this is clearly continuous and has values in \( \text{span} (\{h (x_k, t_k)\}) \). How well does it approximate? Say \( h (x, t) \in h (\bar{\Omega} \times [0, T]) \). Then it is in some

\[
B (h (x, t), \varepsilon),
\]

maybe several. Thus letting \( \mathcal{K} (x, t) \) be those indices \( k \) such that

\[
h (x, t) \in B (h (x, t), \varepsilon)
\]

\[
\| h_n (x, t) - h (x, t) \| \leq \sum_{k \in \mathcal{K} (x, t)} \| h (x_k, t_k) - h (x_k, t_k) \| \frac{\phi_k (h (x, t))}{\sum_{i} \phi_i (h (x, t))}
\]

\[
\leq \varepsilon \sum_{k=1}^{n} \frac{\phi_k (h (x, t))}{\sum_{i} \phi_i (h (x, t))} = \varepsilon
\]
Now here is a claim.

Claim: There exists $\delta > 0$ such that for all $t \in [0, 1]$,

$$\operatorname{dist}(y(t), (I - h)(\partial \Omega, t)) > 6\delta$$

Proof of claim: If not, there is $(x_n, t_n) \in \partial \Omega \times [0, 1]$ such that

$$\|y(t_n) - (I - h)(x_n, t_n)\| < 1/n$$

Then $h(x_n, t_n)$ is in a compact set because of compactness of $h$. Also, the $y(t_n)$ are in a compact set because $y$ is continuous and $y([0, T])$ must therefore be compact. It follows that $(x_n, t_n)$ must be in a compact subset of $\partial \Omega \times [0, 1]$. It follows there is a subsequence, still denoted as $(x_n, t_n)$ which converges to $(x, t)$ in $\partial \Omega \times [0, 1]$. Then by continuity, $\|y(t) - (I - h)(x, t)\| = 0$ contrary to assumption. This proves the claim.

As with $h$ there exists a sequence $\{y_k(t)\}$ such that $y_k(t) \to y(t)$ uniformly in $t \in [0, 1]$ but $y_k$ has values in a finite dimensional subspace of $X, Y_k$. Choose $k_0$ large enough that for all $t \in [0, 1], \|y(t) - y_{k_0}(t)\| < \delta$. Thus by the first claim,

$$D(h(\cdot, t), \Omega, y(t)) = D(h(\cdot, t), \Omega, y_{k_0}(t))$$

for all $t$. Also,

$$\operatorname{dist}(y_0(t), (I - h)(\partial \Omega, t)) > 5\delta$$

From the above, let $h_k \to h$ uniformly on $\overline{\Omega} \times [0, 1]$ but $h_k$ has values in a finite dimensional subspace $V_k$. Let all the $V_k$ contain the values of $y_{k_0}$ and so, for all $k$ large enough,

$$\sup_{\overline{\Omega} \times [0, T]} \|h(t, x) - h_k(t, x)\| < \delta$$

so for such $k$,

$$\operatorname{dist}(y_0(t), (I - h_k)(\partial(\Omega \cap V_k), t)) \\
\geq \operatorname{dist}(y_0(t), (I - h_k)(\partial \Omega, t)) > 4\delta$$

Then

$$D(h(\cdot, t), \Omega, y(t)) = D(h(\cdot, t), \Omega, y_{k_0}(t)) \\
= \lim_{k \to \infty} d(h_k(\cdot, t)\mid_{\overline{\Omega \cap V_k}}, \Omega \cap V_k, y_{k_0}(t))$$

and $d(h_k(\cdot, t)\mid_{\overline{\Omega \cap V_k}}, \Omega \cap V_k, y_{k_0}(t))$ is constant in $t$ for all large enough $k$. Thus

$$D(h(\cdot, t), \Omega, y(t)) = \lim_{k \to \infty} a_k, \ a_k \text{ independent of } t. \blacksquare$$

One of the nice results which follows right away from this is the Schauder fixed point theorem.

Theorem 21.8.7 Let $B = \overline{B(0, r)}$ and let $F : B \to B$ be compact. Then $F$ has a fixed point.
21.8. THE LERAY SCHAUDEL DEGREE

**Proof:** Suppose it does not. Then consider $D(I - tF, B(0, r), 0)$. If $t = 1$, $0 \notin (I - tF)(\partial B)$ since otherwise, there would be a fixed point. If $t < 1$ there is no point of $\partial B$ which $I - tF$ sends to 0 because if so, $x - tFx = 0$, $\|x\| = 1$, $\|Fx\| \leq 1$.

Therefore, by homotopy invariance, $t \to D(I - tF, B(0, r), 0)$ is constant for $t \in [0, 1]$. It must equal $D(I, B(0, r), 0) = D(I - F, B(0, r), 0) = 1$.

Therefore, there exists $x \in B(0, r)$ such that $(I - F)(x) = 0$ so $F$ which means $F$ has a fixed point after all.

One can get an improved version of this easily.

**Theorem 21.8.8** Let $K$ be a closed bounded convex subset of a Banach space $X$ and suppose $F : K \to K$ is compact. Then $F$ has a fixed point.

**Proof:** By Theorem [14.2.5](#), $K$ is a retract. Thus there is a continuous function $R : X \to K$ which leaves points of $K$ unchanged. Then you consider $F \circ R$. It is still a compact mapping obviously. Let $B(0, r)$ be so large that it contains $K$. Then from the above theorem, it has a fixed point in $B(0, r)$ denoted as $x$. Then $F(R(x)) = x$. But $F(R(x)) \in K$ and so $x \in K$. Hence $Rx = x$ and so $Fx = x$.

There is an easy modification of the above which is often useful. If $F : X \to F(X)$ where $F(X)$ is bounded and in a compact set, and $F$ is a compact map, then you could consider $F : \text{conv } F(X) \to F(X) \subseteq \text{conv } F(X)$ where here $\text{conv } F(X)$ is a closed bounded convex subset of $X$. Then by the Schauder theorem, there is a fixed point for $F$.

Here is an easy application of this theorem to ordinary differential equations.

**Theorem 21.8.9** Let $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let $F : C([0, T] ; \mathbb{R}^n) \to C([0, T] ; \mathbb{R}^n)$ be given by

$$F(y)(t) = y_0 + \int_0^t g(s, y(s)) \, ds$$

Suppose that whenever

$$y(s) = F(y)(s), \text{ for } s \leq t,$$

it follows that $\max_{s \in [0, t]} |y(s)| < M, |y_0| < M$. Then there exists a solution to the integral equation

$$y(t) = y_0 + \int_0^t g(s, y(s)) \, ds$$

for $t \in [0, T]$.
CHAPTER 21. DEGREE THEORY, AN INTRODUCTION

Proof: Let $r_M$ be the radial projection in $\mathbb{R}^n$ onto $\overline{B(0,M)}$. Then $F \circ r_M$ is compact because $|g(s, r_M y)|$ is bounded. It also maps into a compact subset of $C([0, T]; \mathbb{R}^n)$ thanks to the Arzela Ascoli theorem. Then by the Schauder fixed point theorem, there exists a solution $y = F \circ r_M$ to

$$y(t) = y_0 + \int_0^t g(s, r_M y(s)) \, ds$$

Then for $s \in [0, \hat{T}]$ where $\hat{T}$ is the largest such that $\|y(s)\| \leq M$ for $s \in [0, \hat{T}]$. Thus on $[0, \hat{T}], r_M$ has no effect. If $\hat{T} < T$, then by the estimate, $|y(\hat{T})| < M$. Hence $\hat{T}$ is not really the last. Thus $\hat{T} = T$. ■

The Schauder alternative or Schaefer fixed point theorem is as follows [33].

Theorem 21.8.10 Let $f : X \to X$ be a compact map. Then either

1. There is a fixed point for $tf$ for all $t \in [0, 1]$ or
2. For every $r > 0$, there exists a solution to $x = tf(x)$ for $t \in (0, 1)$ such that $\|x\| > r$.

Proof: Suppose there is $t_0 \in [0, 1]$ such that $t_0f$ has no fixed point. Then $t_0 \neq 0, t_0f$ obviously has a fixed point if $t_0 = 0$. Thus $t_0 \in (0, 1]$. Then let $r_M$ be the radial retraction onto $\overline{B(0, M)}$. By Schauder’s theorem there exists $x \in \overline{B(0, M)}$ such that $t_0r_M f(x) = x$. Then if $\|f(x)\| \leq M$, $r_M$ has no effect and so $t_0f(x) = x$ which is assumed not to take place. Hence $\|f(x)\| > M$ and so $\|r_M f(x)\| = M$ so $\|x\| = t_0 M$. Also $t_0r_M f(x) = t_0 M \frac{f(x)}{\|f(x)\|} = x$ and so $x = tf(x), t = t_0 M \frac{M}{\|f(x)\|} < 1$. Since $M$ is arbitrary, it follows that the solutions to $x = tf(x)$ for $t \in (0, 1)$ are unbounded. It was just shown that there is a solution to $x = tf(x), t < 1$ such that $\|x\| = t_0 M$ where $M$ is arbitrary. Thus the second of the two alternatives holds. ■

There is a lot more on degree theory in [31]. Here is a very interesting theorem from this reference which pertains specifically to infinite dimensional spaces.

Theorem 21.8.11 Let $X$ be an infinite dimensional Banach space and let $0 \notin \partial \Omega$ where $\Omega$ is an open bounded subset of $X$. Let $F : \overline{\Omega} \to X$ be compact. Suppose that $F x \neq \lambda x$ for all $x \in \partial \Omega$ and that $0 \notin F(\partial \Omega)$. Then $D(I - F, \Omega, 0) = 0$.

Proof: Recall that $D(I - F, \Omega, 0) \equiv \lim_{k \to \infty} d(I - F_k, \Omega \cap V_k, 0)$ where $F_k$ has values in a finite dimensional subspace $V_k$,

$$\sup_{x \in \Omega} \|F_k(x) - F(x)\| < 1/k$$

Since the dimension of $X$ is infinite, it can always be assumed that span $(-F_k(\Omega))$ is a proper subspace of $V_k$ and this will be assumed. This is where it is significant that the dimension of $X$ is infinite. Also recall that in the limit, eventually $d(I - F_k, \Omega \cap V_k, 0)$ is a constant. Then the fact that $F x \neq \lambda x$ for all $x \in \partial \Omega$ will persist for $F_k$ for all $k$ large enough.
21.8. THE LERAY SCHAUDER DEGREE

If not, then there exists $x_k \in \partial \Omega$, $F_k x_k = \lambda_k x_k$ for some $x_k \in \partial \Omega$ and $\lambda_k \in [0, 1]$. Then there are subsequence $\lambda_k \to \lambda_0 \in [0, 1]$. Then

$$Fx_k - \lambda_k x_k \to 0$$

because it is uniformly close to $F_k x_k - \lambda_k x_k$. Now by assumption $0 \notin \overline{F(\partial \Omega)}$. If $\lambda_0 = 0$, then you would have $Fx_k \to 0$ which does not happen because 0 is at a positive distance from $\overline{F(\partial \Omega)}$. Hence for all $k$ large enough,

$$F_k x \neq \lambda x$$

for all $\lambda \in [0, 1]$. Pick $k$ sufficiently large that in the limit for the Leray Schauder degree $d(I - F_k, \Omega \cap V_k, 0)$ remains constant. Then for $\lambda \in [0, 1]$,

$$d(\lambda I - F_k, \Omega \cap V_k, 0) = d(F_k, \Omega \cap V_k, 0) = d(F_k, \Omega \cap V_k, p)$$

for all $p \notin \text{span}(-F_k(\Omega))$ which is also close enough to 0. Hence, since the degree of this last equals 0 for such $p$, it follows that

$$d(I - F_k, \Omega \cap V_k, 0) = 0$$

Hence $D(I - F, \Omega, 0) = 0$ as claimed. ■

This theorem implies a very strange fixed point theorem. It is strange because it only applies to infinite dimensions.

**Corollary 21.8.12** Let $X$ be an infinite dimensional Banach space. Let $0 \in \Omega_0 \subseteq \Omega$ be two open sets. Let $F : \Omega \to X$ be a compact mapping which satisfies

1. $\|Fx\| \leq \|x\|$ for $x \in \partial \Omega_0$
2. $\|Fx\| \geq \|x\|$ for $x \in \partial \Omega$

Then $F$ has a fixed point in $\overline{\Omega \setminus \Omega_0}$.

**Proof:** First note that $\overline{\Omega \setminus \Omega_0}$ is like an annulus with both edges included. Suppose $F$ does not have a fixed point in $\overline{\Omega \setminus \Omega_0}$. What if $t = 1$ and $x \in \partial \Omega$? Could $0 = (I - F)(x)$? If so, the fixed point is obtained so assume this is not so. Then for $t < 1$ and $x \in \partial \Omega$, if you have $x = t^{-1}Fx$, this would mean that $\|Fx\| = t \|x\|$ and $\|Fx\| < \|x\|$ which is assumed not to happen. Hence for $t \in (0, 1)$ we can assume that $0 \notin (I - tF)(\partial \Omega)$. If $(I - F)(x) = 0$ for $x \notin \partial \Omega_0$ then the fixed point has been found. For $t \in [0, 1)$, you can’t have $(I - tF)(x) = 0$ for $x \in \partial \Omega_0$ because then you would have $\|x\| = t \|Fx\|$ and so $\|Fx\| > \|x\|$ which is assumed not to happen. Therefore, we can assume that for $x \in \partial \Omega_0$, $(I - tF)(x) \neq 0$. Therefore, $D(I - F, \Omega_0, 0) = D(I, \Omega_0, 0) = 1$ by homotopy invariance. Also from properties of the degree,

$$D(I - F, \Omega_0, 0) + D(I - F, \Omega \setminus \overline{\Omega_0}, 0) = D(I - F, \Omega, 0)$$
Recall that this is true if \( 0 \not\in (I - F) (\overline{\Omega} - \Omega_0) \) which is assumed to take place when we assume there is no fixed point. It is desired to use the above theorem so we need to consider \( F (\partial \Omega) \) and whether \( 0 \) is in this set. Condition 2 implies \( 0 \) is not in this set. Then Theorem 21.8.11 implies that \( D (I - F, \Omega, 0) = 0 \) and so \( D (I - F, \Omega \setminus \overline{\Omega_0}, 0) = -1 \). Hence there is a fixed point in \( \Omega \setminus \Omega_0 \) after all contrary to the assumption that there was no such thing. 

This only works in infinite dimensions. Consider an annulus in \( \mathbb{R}^2 \) and let \( F \) be a rotation through an angle of 30 degrees. It clearly has no fixed point but the above conditions are satisfied. This seems very interesting, something which happens in infinite dimensions but not in finite dimensions.

### 21.9 Exercises

1. Show the Brouwer fixed point theorem is equivalent to the nonexistence of a continuous retraction onto the boundary of \( B (0, r) \).

2. Using the Jordan separation theorem, prove the invariance of domain theorem \( n \geq 2 \). Thus an open ball goes to some open. **Hint:** You might consider \( B (x, r) \) and show \( f \) maps the inside to one of two components of \( \mathbb{R}^n \setminus f (\partial B (x, r)) \). etc.

3. Give a version of Proposition 21.6.9 which is valid for the case where \( n = 1 \).

4. It was shown that if \( f \) is locally one to one and continuous, \( f : \mathbb{R}^n \to \mathbb{R}^n \), and

   \[
   \lim_{|x| \to \infty} |f (x)| = \infty,
   \]

then \( f \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \). Suppose you have \( f : \mathbb{R}^m \to \mathbb{R}^n \) where \( f \) is one to one and \( \lim_{|x| \to \infty} |f (x)| = \infty \). Show that \( f \) cannot be onto.

5. Can there exists a one to one onto continuous map, \( f \) which takes the unit interval \([0, 1]\) to the unit disk \( B (0, 1) \)? **Hint:** Think in terms of invariance of domain.

6. Let \( m < n \) and let \( B_m (0, r) \) be the ball in \( \mathbb{R}^m \) and \( B_n (0, r) \) be the ball in \( \mathbb{R}^n \). Show that there is no one to one continuous map from \( B_m (0, r) \) to \( B_n (0, r) \). **Hint:** It is like the above problem.

7. Consider the unit disk,

   \[
   \{ (x, y) : x^2 + y^2 \leq 1 \} \equiv D
   \]

and the annulus

   \[
   \{ (x, y) : \frac{1}{2} \leq x^2 + y^2 \leq 1 \} \equiv A
   \]

Is it possible there exists a one to one onto continuous map \( f \) such that \( f (D) = A \)? Thus \( D \) has no holes and \( A \) is really like \( D \) but with one hole punched.
out. Can you generalize to different numbers of holes? **Hint:** Consider the invariance of domain theorem. The interior of $D$ would need to be mapped to the interior of $A$. Where do the points of the boundary of $A$ come from? Consider Theorem 6.9.3.

8. Suppose $C$ is a compact set in $\mathbb{R}^n$ which has empty interior and $f : C \rightarrow \Gamma \subseteq \mathbb{R}^n$ is one to one onto and continuous with continuous inverse. Could $\Gamma$ have nonempty interior? Show also that if $f$ is one to one and onto $\Gamma$ then if it is continuous, so is $f^{-1}$.

9. Let $K$ be a nonempty closed and convex subset of $\mathbb{R}^n$. Recall $K$ is convex means that if $x, y \in K$, then for all $t \in [0, 1]$, $tx + (1 - t)y \in K$. Show that if $x \in \mathbb{R}^n$ there exists a unique $z \in K$ such that

$$|x - z| = \min \{|x - y| : y \in K\}.$$ 

This $z$ will be denoted as $Px$. **Hint:** First note you do not know $K$ is compact. Establish the parallelogram identity if you have not already done so,

$$|u - v|^2 + |u + v|^2 = 2|u|^2 + 2|v|^2.$$ 

Then let \{z_k\} be a minimizing sequence,

$$\lim_{k \to \infty} |z_k - x|^2 = \inf \{|x - y| : y \in K\} = \lambda.$$ 

Now using convexity, explain why

$$\left|\frac{z_k - z_m}{2}\right|^2 + \left|\frac{x - z_k + z_m}{2}\right|^2 = 2\left|\frac{x - z_k}{2}\right|^2 + 2\left|\frac{x - z_m}{2}\right|^2$$ 

and then use this to argue \{z_k\} is a Cauchy sequence. Then if $z_i$ works for $i = 1, 2$, consider $(z_1 + z_2)/2$ to get a contradiction.

10. In Problem 8 show that $Px$ satisfies the following variational inequality.

$$(x - Px) \cdot (y - Px) \leq 0$$ 

for all $y \in K$. Then show that $|Px_1 - Px_2| \leq |x_1 - x_2|$. **Hint:** For the first part note that if $y \in K$, the function $t \to |x - (Px + t(y - Px))|^2$ achieves its minimum on $[0, 1]$ at $t = 0$. For the second part,

$$(x_1 - Px_1) \cdot (Px_2 - Px_1) \leq 0, \ (x_2 - Px_2) \cdot (Px_1 - Px_2) \leq 0.$$ 

Explain why

$$(x_2 - Px_2 - (x_1 - Px_1)) \cdot (Px_2 - Px_1) \geq 0$$ 

and then use a some manipulations and the Cauchy Schwarz inequality to get the desired inequality. Thus $P$ is called a retraction onto $K$. 

11. Establish the Brouwer fixed point theorem for any convex compact set in $\mathbb{R}^n$. 

**Hint:** If $K$ is a compact and convex set, let $R$ be large enough that the closed ball, $D(0,R) \supseteq K$. Let $P$ be the projection onto $K$ as in Problem 10 above. If $f$ is a continuous map from $K$ to $K$, consider $f \circ P$. You want to show $f$ has a fixed point in $K$.

12. Suppose $D$ is a set which is homeomorphic to $B(0,1)$. This means there exists a continuous one to one map, $h$ such that $h(B(0,1)) = D$ such that $h^{-1}$ is also one to one. Show that if $f$ is a continuous function which maps $D$ to $D$ then $f$ has a fixed point. Now show that it suffices to say that $h$ is one to one and continuous. In this case the continuity of $h^{-1}$ is automatic. Sets which have the property that continuous functions taking the set to itself have at least one fixed point are said to have the fixed point property. Work Problem 7 using this notion of fixed point property. What about a solid ball and a donut? Could these be homeomorphic?

13. Suppose $\Omega$ is any open bounded subset of $\mathbb{R}^n$ which contains 0 and that $f : \overline{\Omega} \rightarrow \mathbb{R}^n$ is continuous with the property that $f(x) \cdot x \geq 0$ for all $x \in \partial \Omega$. Show that then there exists $x \in \overline{\Omega}$ such that $f(x) = 0$. Give a similar result in the case where the above inequality is replaced with $\leq$.

**Hint:** You might consider the function $h(t,x) \equiv tf(x) + (1 - t)x$.

14. Suppose $\Omega$ is an open set in $\mathbb{R}^n$ containing 0 and suppose that $f : \overline{\Omega} \rightarrow \mathbb{R}^n$ is continuous and $|f(x)| \leq |x|$ for all $x \in \partial \Omega$. Show $f$ has a fixed point in $\overline{\Omega}$.

**Hint:** Consider $h(t,x) \equiv t(x - f(x)) + (1 - t)x$ for $t \in [0,1]$. If $t = 1$ and some $x \in \partial \Omega$ is sent to 0, then you are done. Suppose therefore, that no fixed point exists on $\partial \Omega$. Consider $t < 1$ and use the given inequality.

15. Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and let $f, g : \overline{\Omega} \rightarrow \mathbb{R}^n$ both be continuous such that $|f(x)| - |g(x)| > 0$ for all $x \in \partial \Omega$. Show that then $d(f - g, \Omega, 0) = d(f, \Omega, 0)$.

**Hint:** You might consider $h(t,x) \equiv (1 - t)f(x) + t(f(x) - g(x))$ and argue $0 \notin h(t, \partial \Omega)$ for $t \in [0,1]$.

16. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous and satisfies $|f(x) - f(y)| \geq \alpha |x - y|$, $\alpha > 0$,
Show that $f$ must map $\mathbb{R}^n$ onto $\mathbb{R}^n$. **Hint:** First show $f$ is one to one. Then use invariance of domain. Next show, using the inequality, that the points not in $f(\mathbb{R}^n)$ must form an open set because if $y$ is such a point, then there can be no sequence $\{f(x_n)\}$ converging to it. Finally recall that $\mathbb{R}^n$ is connected. It is obvious that $f$ is one to one. This follows from the inequality. If $U$ are the points not in the image of $f$, then $U$ must be open because if not, then for $y$ one of these points, there would be a sequence $f(x_n) \to y$. Then by the inequality, $\{x_n\}$ is a Cauchy sequence and so it converges to $x$. Thus $f(x) = \lim_{n \to \infty} f(x_n) = y$. Now by invariance of domain, $f(\mathbb{R}^n)$ is open. However, $\mathbb{R}^n$ is connected and so in fact, $U$ is empty.

17. Let $f : \mathbb{C} \to \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers. Thus $f$ has a real and imaginary part. Letting $z = x + iy$,

$$f(z) = u(x, y) + iv(x, y)$$

Recall that the norm in $\mathbb{C}$ is given by $|x + iy| = \sqrt{x^2 + y^2}$ and this is the usual norm in $\mathbb{R}^2$ for the ordered pair $(x, y)$. Thus complex valued functions defined on $\mathbb{C}$ can be considered as $\mathbb{R}^2$ valued functions defined on some subset of $\mathbb{R}^2$. Such a complex function is said to be analytic if the usual definition holds. That is

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}.$$  \hspace{1cm} (21.9.19)

In other words,

$$f(z + h) = f(z) + f'(z)h + o(h)$$

at a point $z$ where the derivative exists. Let $f(z) = z^n$ where $n$ is a positive integer. Thus $z^n = p(x, y) + iq(x, y)$ for $p, q$ suitable polynomials in $x$ and $y$. Show this function is analytic. Next show that for an analytic function and $u$ and $v$ the real and imaginary parts, the Cauchy Riemann equations hold.

$$u_x = v_y, \ u_y = -v_x.$$  

In terms of mappings show 21.9.18 has the form

$$\begin{pmatrix} u(x + h_1, y + h_2) \\ v(x + h_1, y + h_2) \end{pmatrix} = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + \begin{pmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + o(h)$$

$$= \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} + \begin{pmatrix} u_x(x, y) & -v_x(x, y) \\ v_x(x, y) & u_x(x, y) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + o(h)$$

where $h = (h_1, h_2)^T$ and $h$ is given by $h_1 + ih_2$. Thus the determinant of the above matrix is always nonnegative. Letting $B_r$ denote the ball $B(0, r) = B((0, 0), r)$ show

$$d(f, B_r, 0) = n.$$
where \( f(z) = z^n \). In terms of mappings on \( \mathbb{R}^2 \),

\[
f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.
\]

Thus show \( d(f, B_r, 0) = n \).

**Hint:** You might consider

\[
g(z) = \prod_{j=1}^{n} (z - a_j)
\]

where the \( a_j \) are small real distinct numbers and argue that both this function and \( f \) are analytic but that \( 0 \) is a regular value for \( g \) although it is not so for \( f \). However, for each \( a_j \) small but distinct \( d(f, B_r, 0) = d(g, B_r, 0) \).

18. Using Problem 17, prove the fundamental theorem of algebra as follows. Let \( p(z) \) be a nonconstant polynomial of degree \( n \),

\[
p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots
\]

Show that for large enough \( r, |p(z)| > |p(z) - a_n z^n| \) for all \( z \in \partial B(0, r) \).

Now from Problem 19 you can conclude \( d(p, B_r, 0) = d(f, B_r, 0) = n \) where \( f(z) = a_n z^n \).

19. The proof of Sard’s lemma made use of the hard Vitali covering theorem. Here is another way to do something similar. Let \( U \) be a bounded open set and let \( f : U \rightarrow \mathbb{R}^n \) be in \( C^1(U) \). Let \( S \) denote the set of \( x \in U \) such that \( Df(x) \) has rank less than \( n \). Thus it is a closed set. Let \( U_m = \{ x \in U : \|Df(x)\| \leq m \} \), a closed set. It suffices to show that for \( S_m \equiv U_m \cap S, f(S_m) \) has measure zero because \( f(S) = \cup_m f(S_m) \) these sets increasing in \( m \). By definition of differentiability,

\[
\lim_{k \to \infty} \sup_{\|v\| \leq 1/k} \|f(x + v) - f(x) - Df(x)v\| = 0
\]

for each \( x \in U \). Explain why the above function of \( x \) is measurable. Now by Eggoroff’s theorem, there is measurable set \( A \) of measure less than \( \frac{\varepsilon}{m^{n+1}} \) such that off \( A \), the convergence is uniform. Let \( C_k \) be a countable union of non overlapping half open rectangles one of which is of the form \( \prod_{i=1}^{n} (a_i, b_i] \) such that each has diameter less than \( 2^{-k} \). Consider the half open rectangles which have nonempty intersection with \( S_m \setminus A, I_k \). Then repeat the argument given in the first section of this chapter. Show that for \( k \) large enough, the rank condition and uniform convergence above implies that \( m_n(\cup \{ f(I) : I \in I_k \}) \) is less than \( \varepsilon \). Now show that \( f(A) \) is contained in a set of measure no more than \( m^n 10^n \frac{\varepsilon}{m^{n+1}} = 2\varepsilon \). Thus \( f(S_m) \) has measure no more than \( 3\varepsilon \). Since \( \varepsilon \) is arbitrary, this establishes the desired conclusion.
21.9. EXERCISES

20. Let $X$ be a Banach space and let $\Omega$ be a symmetric and bounded open set. Let $F : \Omega \to X$ be odd and compact $0 \notin (I - F)(\partial \Omega)$. Show using Corollary 21.8.3 that $D(I - F, \Omega, 0)$ is an odd integer.

21. Let $F$ be compact. Suppose $I - F$ is one to one on $B(0, r)$. Then using similar reasoning to the finite dimensional case, show that there is a $\delta > 0$ such that

$$(I - F)(0) + B(0, \delta) \subseteq (I - F)(B(0, r))$$

22. Let $F$ be compact. Suppose $I - F$ is locally one to one on an open set $\Omega$. Show that $(I - F)$ maps open sets to open sets. This is a version of invariance of domain.

23. Suppose $(I - F)$ is locally one to one and $F$ is compact. Suppose also that $\lim_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} = \infty$. Show that $(I - F)$ is onto.

24. As a variation of the above problem, suppose $F : X \to X$ is compact and

$$\lim_{\|x\| \to \infty} \frac{\|F(x)\|}{\|x\|} = 0$$

Then $I - F$ is onto. Note that $I - F$ is not one to one.

25. Suppose $F$ is compact and $\|(I - F)x - (I - F)y\| \geq \alpha \|x - y\|$. Show that then $(I - F)$ is onto.

26. The Jordan curve theorem is: Let $C$ denote the unit circle,

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$ 

Suppose $\gamma : C \to \Gamma \subseteq \mathbb{R}^2$ is one to one onto and continuous. Then $\mathbb{R}^2 \setminus \Gamma$ consists of two components, a bounded component (called the inside) $U_1$ and an unbounded component (called the outside), $U_2$. Also the boundary of each of these two components of $\mathbb{R}^2 \setminus \Gamma$ is $\Gamma$ and $\Gamma$ has empty interior. Using the Jordan separation theorem, prove this important result.

27. This problem is from [40] Recall Theorem 21.8.11. It allowed you to say that $D(I - F, \Omega, 0) = 0$ provided $0 \notin F(\partial \Omega)$ and $\lambda x \neq Fx$ for all $x \in \partial \Omega$, $\lambda \in [0, 1]$. This was for $F$ compact and defined on an infinite dimensional space $X$. Suppose now that $F$ is compact and $F : \Omega \to X$ where $0 \in \Omega$ an open set in $X$. Suppose also that $F(0) = 0$ and that

$$\lim \inf_{x \to 0} \frac{\|F(x)\|}{\|x\|} \equiv \lim_{r \to 0^+} \inf \left\{ \frac{\|F(x)\|}{\|x\|} : \|x\| \leq r \right\} = \infty$$

Show that there is a sequence $\alpha_n \to 0$ each $\alpha_n \neq 0$, and for some $x_n \neq 0$, $x_n - \alpha_n F(x_n) = 0$. Note that when $\alpha = 0$, there is only one solution to $(I - \alpha F)(x) = 0$, but this says that there are many small $\alpha_n \neq 0$ for which there is a nonzero solution to $(I - \alpha_n F)(x) = 0$. That is there exist arbitrarily
small $\alpha_n$ such that $(I - \alpha_n) F (x_n) = 0$. This says that 0 is a bifurcation point for $I - \alpha F$. \textbf{Hint:} Let $\alpha_n \downarrow 0$ and pick $r_n$ such that for all $\|x\| = r_n,$

$$\|\alpha_n F (x)\| > \|x\|$$

Thus $0 \notin \overline{\alpha_n F (\partial B (0, r_n))}$ and also $\alpha_n F (x) \neq \lambda x$ for all $x \in \partial B (0, r_n)$. Use the theorem to conclude that

$$D (I - \alpha_n F, B (0, r_n), 0) = 0$$

and then consider the homotopy $I - \alpha_t F$. If it sends no point of $\partial B (0, r_n)$ to 0 then you would have

$$D (I - \alpha_n t F, B (0, r_n), 0) = D (I, B (0, r_n), 0) = 1$$
Chapter 22

Critical Points

22.1 Mountain Pass Theorem In Hilbert Space

This is from Evans \[126\]. It is an interesting theorem. See also \[52\] for more general versions. It has to do with differentiable functions defined on a Hilbert space $H$. Thus $I : H \to \mathbb{R}$ will be differentiable. Then the following is the Palais Smale condition.

**Definition 22.1.1** A functional $I$ satisfies the Palais Smale conditions means that if $\{I(u_k)\}$ is a bounded sequence and $I'(u_k) \to 0$, then $\{u_k\}$ is precompact. That is, it has a subsequence which converges.

It will be assumed that $I$ is $C^1(H; \mathbb{R})$ and also that $I'$ is Lipschitz on bounded sets. By $I'(u)$ is meant the element of $H$ such that

$$I(u + v) = I(u) + (I'(u), v)_H + o(v)$$

Such exists because of the Riesz representation theorem. Note that, from the assumption that $I'$ is Lipschitz continuous, it follows that $I'$ is bounded on every bounded set.

First is a deformation theorem. The notation $[I(u) \in S]$ means $\{u : I(u) \in S\}$.

**Theorem 22.1.2** Let $I$ be $C^1$, $I$ is non constant, satisfy the Palais Smale condition, and $I'$ is Lipschitz continuous on bounded sets. Also suppose that $c \in \mathbb{R}$ is such that either $[I(u) \in [c - \delta, c + \delta]] = \emptyset$ for some $\delta > 0$ or $[I(u) \in [c - \delta, c + \delta]] \neq \emptyset$ for all $\delta > 0$ and $I(u) = c$, then $I'(u) \neq 0$. Then for each sufficiently small $\varepsilon > 0$, there is a constant $\delta \in (0, \varepsilon)$ and a function $\eta : [0,1] \times H \to H$ such that

1. $\eta(0, u) = u$
2. $\eta(1, u) = u$ on $[I(u) \notin (c - \varepsilon, c + \varepsilon)]$
3. $I(\eta(t, u)) \leq I(u)$
4. $\eta(1, [I(u) \leq c + \delta]) \subseteq [I(u) \leq c - \delta]$

The main part of this conclusion is the statement about $u \to \eta(1, u)$ contained in parts 2. and 4. The other two parts are there to facilitate these two although they are certainly interesting for their own sake.

**Proof:** Suppose $[I(u) \in [c - \delta, c + \delta]] = \emptyset$ for some $\delta > 0$. Then $[I(u) \leq c + \delta/2] \subseteq [I(u) \leq c - \delta/2]$ and you could take $\varepsilon = \delta$ and let $\eta(t, u) = u$. Therefore, assume $[I(u) \in [c - \delta, c + \delta]] \neq \emptyset$ for all $\delta > 0$. Since $I$ is nonconstant, $\varepsilon > 0$ can be chosen small enough that $[I(u) \notin (c - \varepsilon, c + \varepsilon)] \neq \emptyset$. Always let $\varepsilon$ be this small.

**Claim 1:** For all small enough $\varepsilon > 0$, if $u \in [I(u) \in [c - \varepsilon, c + \varepsilon]]$, $I'(u) \neq 0$ and in fact, for such $\varepsilon$, there exists $\sigma(\varepsilon) > 0$ such that $\sigma(\varepsilon) < \varepsilon$, $\|I'(u)\| > \sigma(\varepsilon)$ for all $u \in [I(u) \in [c - \varepsilon, c + \varepsilon]]$.

**Proof of Claim 1:** If claim is not so, then there is $\{u_k\}, \varepsilon_k, \sigma_k \to 0, \|I'(u_k)\| < \sigma_k$, and $I(u_k) \in [c - \varepsilon_k, c + \varepsilon_k]$ but $\|I'(u_k)\| \leq \sigma_k$. However, from the Palais Smale condition, there is a subsequence, still denoted as $u_k$ which converges to some $u$. Now $I(u_k) \in [c - \varepsilon_k, c + \varepsilon_k]$ and so $I(u) = c$ while $I'(u) = 0$ contrary to the hypothesis. This proves Claim 1. From now on, $\varepsilon$ will be sufficiently small.

Now define for $\delta < \varepsilon$ (The description of small $\delta$ will be described later.)

$$A \equiv [I(u) \notin (c - \varepsilon, c + \varepsilon)]$$
$$B \equiv [I(u) \in [c - \delta, c + \delta]]$$

Thus $A$ and $B$ are disjoint closed sets. Recall that it is assumed that $B \neq \emptyset$ since otherwise, there is nothing to prove. Also it is assumed throughout that $\varepsilon > 0$ is such that $A \neq \emptyset$ thanks to $I$ not being constant. Thus these are nonempty sets and we do not have to fuss with worrying about meaning when one is empty.

**Claim 2:** For any $u$, $\text{dist} (u, A) + \text{dist} (u, B) > 0$.

This is so because if not, then both would be zero and this requires that $u \in A \cap B$ since these sets are closed. But $A \cap B = \emptyset$.

Now define a function

$$g(u) \equiv \frac{\text{dist} (u, A)}{\text{dist} (u, A) + \text{dist} (u, B)}$$

It is a continuous function of $u$ which has values in $[0, 1]$. Consider the ordinary differential initial value problem

$$\eta'(t, u) + g(u) h (\|I'(\eta(t, u))\|) I'(\eta(t, u)) = 0 \quad (22.1.1)$$
$$\eta(0, u) = u \quad (22.1.2)$$

where $r \to h(r)$ is a decreasing function which has values in $(0, 1]$ and equals $1$ for $r \in [0, 1]$ and equals $1/r$ for $r > 1$. Here $u$ is given and the $\eta'$ is the time derivative is with respect to $t$. Thus, by assumption, the function

$$\eta \to g(u) h (\|I'(\eta)\|) I'(\eta)$$
is Lipschitz continuous on bounded sets and so there exists a solution to the above initial value problem valid for all $t \in [0, 1]$. To see this, you can let $P$ be the projection map onto the closed ball of radius $M > \|u\|$ and the system

$$
\eta'(t, u) + g(u) h(\|I'(P(\eta(t, u)))\|) I'(P(\eta(t, u))) = 0
$$

$$
\eta(0, u) = u
$$

Then by Lipschitz continuity, there is a global solution for all $t \geq 0$. Hence there is a local solution to \ref{22.1.1}, \ref{22.1.2}. Note that

$$
\|g(u) h(\|I'(P(\eta(t, u)))\|) I'(P(\eta(t, u)))\| \leq 1
$$

Taking inner products with $\eta(t, u)$ and integrating $\int_0^t$ for this local solution,

$$
\frac{1}{2} \|\eta(t, u)\|^2 - \frac{1}{2} \|u\|^2 + \int_0^t g(u) h(\|I'(P\eta(s, u))\|) (I'(P\eta(s, u)), \eta(s, u)) \, ds = 0
$$

$$
\frac{1}{2} \|\eta(t, u)\|^2 \leq \frac{1}{2} \|u\|^2 + \int_0^t g(u) h(\|I'(P\eta(t, u))\|) \|I'(P\eta(t, u))\| \|\eta(s, u)\| \, ds
$$

It follows that for $t \leq 1$,

$$
\|\eta(t, u)\|^2 \leq \|u\|^2 + 2 \int_0^t \|\eta(s, u)\| \, ds
$$

$$
\leq \|u\|^2 + 1 + \int_0^t \|\eta(s, u)\|^2 \, ds
$$

and so from Gronwall’s inequality, for $t \leq 1$,

$$
\|\eta(t, u)\|^2 \leq \left(\|u\|^2 + 1\right) e^t
$$

Thus we pick $M > e \left(\|u\|^2 + 1\right)$ and then we obtain that for $t \in [0, 1]$, the projection map does not change anything. Hence there exists a solution to \ref{22.1.1}, \ref{22.1.2} on $[0, 1]$ as desired.

Then for this solution, $\eta(0, u) = u$ because of the above initial condition. If $u \in [I(u) \notin [c - \varepsilon, c + \varepsilon]]$, then $u \in A$ and so $g(u) = 0$ so $\eta(t, u) = u$ for all $t \in [0, 1]$. This gives the first two conditions. Consider the third.

$$
\frac{d}{dt} (I(\eta(t, u))) = (I'(\eta), \eta') = - (I'(\eta), g(u) h(\|I'(\eta)\|) I'(\eta))
$$

$$
= -g(u) h(\|I'(\eta)\|) \|I'(\eta)\|^2
$$

and so this implies the third condition since it says that the function $t \rightarrow I(\eta(t, u))$ is decreasing.
It remains to consider the last condition. This involves choosing $\delta$ still smaller if necessary. It is desired to verify that

$$\eta(1, [I(u) \leq c + \delta]) \subseteq [I(u) \leq c - \delta]$$

Suppose it is not so. Then there exists $u \in [I(u) \leq c + \delta]$ but $I(\eta(1,u)) > c - \delta$.

$$c - \delta < I(\eta(1,u)) = I(u) - \int_0^1 (I'(\eta) \cdot g(u) h(\|I'(\eta)\|) I'(\eta)) \, dt$$

$$= I(u) - g(u) \int_0^1 h(\|I'(\eta)\|) \|I'(\eta(t,u))\|^2 \, dt$$

$$< c + \delta - g(u) \int_0^1 h(\|I'(\eta)\|) \|I'(\eta(t,u))\|^2 \, dt$$

Then

$$c - 2\delta + g(u) \int_0^1 h(\|I'(\eta(t,u))\|) \|I'(\eta(t,u))\|^2 \, dt < c$$

If $I(u) \leq c - \delta$, there is nothing to show because in this case $I(\eta(1,u)) \leq I(u) \leq c - \delta$. Hence we can assume that $I(u) > c - \delta$ and also that $I(u) \leq c + \delta$. Thus $u \in B$ and so $g(u) = 1$. Thus

$$c - 2\delta + \int_0^1 h(\|I'(\eta(t,u))\|) \|I'(\eta(t,u))\|^2 \, dt < c$$

Also, it is being assumed that $I(\eta(1,u)) > c - \delta$ and so by the third conclusion shown above, $\eta(t,u) \in B$ for $t \in [0,1]$. We also know that for such values of $\eta(t,u), \|I'(\eta(t,u))\| \geq \sigma(\varepsilon)$ from Claim 1. If $\|I'(\eta(t,u))\| > 1$, the integrand equals $\|I'(\eta(t,u))\| \geq \sigma(\varepsilon)$. If $\|I'(\eta(t,u))\| \leq 1$, the integrand is $\|I'(\eta(t,u))\|^2 \geq \sigma(\varepsilon)^2$. Thus

$$c - 2\delta + \int_0^1 \min(\sigma(\varepsilon), \sigma(\varepsilon)^2) \, dt < c$$

and the only restriction on $\delta$ was that it should be smaller than $\varepsilon$. Although it was not mentioned above, $\delta$ was chosen so small that $-2\delta + \min(\sigma(\varepsilon), \sigma(\varepsilon)^2) > 0$. Hence this yields a contradiction. Thus the last conclusion is verified.

Imagine a valley surrounded by a ring of mountains. On the other side of this ring of mountains, there is another low place. Then there must be some path from the valley to the exterior low place which goes through a point where the gradient equals 0, the gradient being the gradient of a function $f$ which gives the altitude of the land. This is the idea of the mountain pass theorem. The critical point where $\nabla f = 0$ is the mountain pass.

**Theorem 22.1.3** Let $H$ be a Hilbert space and let $I : H \to \mathbb{R}$ be a $C^1$ functional having $I'$ Lipschitz continuous and such that $I$ satisfies the Palais Smale condition.
Suppose $I(0) = 0$ and $I(u) \geq a > 0$ for all $\|u\| = r$. Suppose also that there exists $v$, $\|v\| > r$ such that $I(v) \leq 0$. Then define

$$\Gamma \equiv \{ g \in C([0, 1]; H) : g(0) = 0, g(1) = v \}$$

Let

$$c \equiv \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t))$$

Then $c$ is a critical value of $I$ meaning that there exists $u$ such that $I(u) = c$ and $I'(u) = 0$. In particular, there is $u \neq 0$ such that $I'(u) = 0$.

**Proof:** First note that $c \geq a > 0$. Suppose $c$ is not a critical value. Then by the deformation theorem, for $\varepsilon > 0$, sufficiently small, there is $\eta : H \to H$ and a $\delta < \varepsilon$ small enough that $\eta([I(u) \leq c + \delta]) \subseteq [I(u) \leq c - \delta]$ and $\eta$ leaves unchanged $[I(u) \notin (c - \varepsilon, c + \varepsilon)]$. Then there is $g \in \Gamma$ such that

$$\max_{t \in [0, 1]} I(g(t)) < c + \delta$$

Then in particular, $I(g(t)) < c + \delta$ for every $t$. Hence you look at $\eta \circ g$. We know that $g(0), g(1)$ are both in the set $[I(u) \notin (c - \varepsilon, c + \varepsilon)]$ because they are both 0 and so $\eta$ leaves these unchanged. Hence $\eta \circ g \in \Gamma$ and

$$I(\eta \circ g(t)) \leq c - \delta$$

for all $t \in [0, 1]$. Thus

$$c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t)) \leq \max_{t \in [0, 1]} I(\eta \circ g(t)) \leq c - \delta$$

which is clearly a contradiction. ■

The Palais Smale conditions are pretty restrictive. For example, let $I(x) = \cos x$. Thus $I : \mathbb{R} \to \mathbb{R}$. Then let $u_k = k \pi$. Clearly $I(u_k)$ is bounded and $\lim_{k \to \infty} I(u_k) = 0$ but $\{u_k\}$ is not precompact. However, here is a simple case which does satisfy the Palais Smale conditions.

**Example 22.1.4** Let $I : \mathbb{R}^d \to \mathbb{R}$ satisfy $\lim_{|x| \to \infty} I(x) = \infty$. Then $I$ satisfies the Palais Smale conditions.

The growth condition implies that if $I(x_k)$ is bounded, then so is $\{x_k\}$ and so this sequence is precompact. Nothing needs to be said about $I'(x_k)$.

### 22.1.1 A Locally Lipschitz Selection, Pseudogradients

When you have a functional $\phi$ defined on a Banach space $X$, $\phi'(u)$ is in $X'$ and it isn’t obvious how you can understand it in terms of an element in $X$ like what is done with Hilbert space using the Riesz representation theorem. However, there is something called a pseudogradient which is defined next.
Definition 22.1.5 Let $\phi : X \to \mathbb{R}$ be $C^1$. Then $v$ is a pseudogradients for $\phi$ at $x$ if the following hold.

1. $\|v\|_X \leq 2\|\phi' (x)\|_{X'}$
2. $\|\phi' (x)\|^2_{X'} \leq \langle \phi' (x), v \rangle$

A pseudogradients field $\mathcal{V}$ is a locally Lipschitz selection of $G (x)$ where $G (x)$ is defined to be the set of pseudogradients of $\phi$ at $x$. Thus $\mathcal{V} (x) \in G (x)$ and $\mathcal{V} (x)$ is a pseudogradient for $\phi$ at each regular point of $\phi$.

Note how this generalizes the case of Hilbert space. In the Hilbert space case, you have $\phi' (x)$ which technically is in $H'$ and you have the gradient, written here as $\nabla \phi$ which is in $H$ such that

\[(\nabla \phi (x), v)_{H} \equiv \langle \phi' (x), v \rangle_{H', H},\]

the existence of $\nabla \phi (x)$ coming from the Riesz representation theorem which also gives that $\nabla \phi (x) = R^{-1} \phi' (x)$ and so $\|\nabla \phi (x)\|_H = \|\phi' (x)\|_{H'}$, so the above two conditions hold for the gradient field except for one thing. Why is $x \to \nabla \phi (x)$ locally Lipschitz. We don’t know this, but with a pseudogradient field, we do. Also, the pseudogradient field is only required at regular points of $\phi$ where $\phi' (x) \neq 0$. If you had strict inequalities holding in the above definition, then they would continue to hold for $x$ near $x$. Thus if you had

\[\|v\|_X < 2\|\phi' (x)\|_{X'} \text{, } \|\phi' (x)\|^2_{X'} < \langle \phi' (x), v \rangle\]

and $\Gamma (x)$ were the set of such $v$, then there would be an open set $U$ containing $x$ such that $\cap_{\hat{y} \in U} \Gamma (\hat{y}) = \emptyset$. In fact, the intersection would contain $v$.

This very nice lemma is from Gasinski L. and Papageorgiou N. [52]. It is a lovely application of Stone’s theorem and partitions of unity for a metric space.

Lemma 22.1.6 Let $Y$ be a metric space and let $X$ be a normed linear space. (We will want to add in $X$.) Let $\Gamma : Y \to \mathcal{P} (X)$ such that $\Gamma (y)$ is a nonempty convex set. Suppose that for each $y \in Y$, there exists an open set $U$ containing $y$ such that

\[\emptyset \neq \cap_{\hat{y} \in U} \Gamma (\hat{y})\]

Then there exists a locally Lipschitz map $\gamma : Y \to X$ such that $\gamma (y) \in \Gamma (y)$ for all $y$.

Proof: Let $\mathcal{U}$ denote the collection of all open sets $U$ such that the nonempty intersection described above holds. Let $\mathcal{V}$ be a locally finite open refinement which also covers. Thus for any $V \in \mathcal{V}$

\[\emptyset \neq \cap_{\hat{y} \in V} \Gamma (\hat{y})\]

because it is a smaller intersection. Let $\{\phi_{V}\}_{V \in \mathcal{V}}$ be a partition of unity subordinate to the open covering $\mathcal{V}$. In fact, we can have $\phi_{V}$ locally Lipschitz. This follows from
22.1. MOUNTAIN PASS THEOREM IN HILBERT SPACE

the above construction of the partition of unity in Theorem 22.1.6. Pick \( x_V \in \cap_{\hat{y} \in V} \Gamma (\hat{y}) \). Then consider

\[
\gamma (y) = \sum_{V \in V} x_V \phi_V (y)
\]

It is clearly locally Lipschitz because near any point \( y \), it is a finite sum of Lipschitz functions. Pick \( y \in Y \). Then it is in some \( V \in V \). In fact, it is finitely many, \( V_1, \cdots , V_n \) and for other \( V \in V, \phi_V (y) = 0 \). Therefore,

\[
\gamma (y) = \sum_{i=1}^{n} x_{V_i} \phi_{V_i} (y)
\]

which is a convex combination of the \( x_{V_i} \). Now \( x_{V_i} \in \cap_{\hat{y} \in V_i} \Gamma (\hat{y}) \subseteq \Gamma (y) \), this for each \( i \). Hence this is a convex combination of points in a nonempty convex set \( \Gamma (y) \).

Thus \( \gamma (y) \in \Gamma (y) \). \( \blacksquare \)

The following lemma says that if \( \phi \) is \( C^1 \) on \( X \), then it has a pseudogradient field on \( \{ x : \phi' (x) \neq 0 \} \), the set of regular points.

**Lemma 22.1.7** Let \( \phi \) be a \( C^1 \) function defined on \( X \) a Banach space. Then there exists a pseudogradient field for \( \phi \) on the set of regular points. \( (V (x) \in G (x) \) and \( x \rightarrow V (x) \) is locally Lipschitz on the set of regular points.\)

**Proof:** First consider whether \( G (x) \), the set of pseudogradients of \( \phi \) at \( x \) is nonempty for \( \phi' (x) \neq 0 \). From the definition of the operator norm, there exists \( u \) such that \( ||u||_X = 1 \) and \( \langle \phi' (x) , u \rangle \geq \delta ||\phi' (x)||_X \), where \( \delta \in (0,1) \). Then let \( v = ru \langle \phi' (x) , u \rangle \), where \( r \in (1,2) \).

\[
\langle \phi' (x) , v \rangle = \langle \phi' (x) , ru \langle \phi' (x) , u \rangle \rangle = r \langle \phi' (x) , u \rangle \langle \phi' (x) , u \rangle \geq r\delta ||\phi' (x)||^2
\]

Then choose \( r, \delta \) such that \( r\delta > 1 \) and \( r < 2 \). Then if these were chosen this way in the above reasoning, it follows that

\[
||v|| < 2 ||\phi' (x)|| \quad \text{and} \quad \langle \phi' (x) , v \rangle > ||\phi' (x)||^2
\]

That \( \phi' (x) \neq 0 \) is needed to insure that the above strict inequalities hold.

Thus, letting \( Y \) be the metric space consisting of the regular points of \( \phi \), the continuity of \( \phi' \) implies that the above inequalities persist for all \( y \) close enough to \( x \). Thus there is an open set \( U \) containing \( x \) such that \( v \) satisfies the above inequalities for \( x \) replaced with arbitrary \( y \in U \). Thus

\[
v \in \cap_{y \in U} G (y)
\]

Since it is clear that each \( G (y) \) is convex, Lemma 22.1.6 implies the existence of a locally Lipschitz selection from \( G \). That is \( x \rightarrow V (x) \) is locally Lipschitz and \( V (x) \in G (x) \) for all regular \( x \). \( \blacksquare \)

It will be important to consider \( y' = f (y) \) where \( f \) is locally Lipschitz and \( y \) is just in a Banach space. This is more complicated than in Hilbert space because of the lack of a convenient projection map.
Theorem 22.1.8 Let \( f : U \rightarrow X \) be locally Lipschitz where \( X \) is a Banach space and \( U \) is an open set. Then there exists a unique local solution to the IVP

\[
y' = f(y), \quad y(0) = y_0 \in U
\]

**Proof:** Let \( B \) be a closed ball of radius \( R \) centered at \( y_0 \) such that \( f \) has Lipschitz constant \( K \) on \( B \). Then

\[
y_1(t) = y_0 + \int_0^t f(y_0) \, ds
\]

and if \( y_n(t) \) has been obtained,

\[
y_{n+1}(t) = y_0 + \int_0^t f(y_n(s)) \, ds \quad (22.1.3)
\]

Now \( t < T \) where \( T \) is so small that \( \|f(y_0)\| Te^{KT} < R \).

**Claim:** \( \|y_n(t) - y_{n-1}(t)\| \leq \|f(y_0)\| t^n K^{n-1} \frac{1}{(n-1)!} \)

**Proof of claim:** First

\[
\|y_1(t) - y_0\| \leq \int_0^t \|f(y_0)\| \, ds \leq \|f(y_0)\| t
\]

Now suppose it is so for \( n \). Then

\[
\|y_{n+1}(t) - y_n(t)\| \leq \int_0^t \|f(y_n(s)) - f(y_{n-1}(s))\| \, ds
\]

By induction, \( y_n(s), y_{n-1}(s) \) are still in \( B \). This is because

\[
\|y_n(t) - y_0\| \leq \sum_{k=1}^n \|y_k(t) - y_{k-1}(t)\| \\
\leq \sum_{k=1}^n \|f(y_0)\| \frac{1}{(k-1)!} t^k K^{k-1} \\
\leq \|f(y_0)\| t e^{Kt} < R \quad (22.1.4)
\]

showing that \( y_n(t) \) stays in \( B \). Then since all values of the iterates remain in \( B \), induction gives

\[
\|y_{n+1}(t) - y_n(t)\| \leq \int_0^t K \|y_n(s) - y_{n-1}(s)\| \, ds \\
\leq K \int_0^t \|f(y_0)\| \frac{1}{(n-1)!} s^n K^{n-1} \, ds = K^n \frac{1}{(n-1)!} \|f(y_0)\| \int_0^t s^n \, ds \\
= K^n \frac{1}{n!} \|f(y_0)\| t^{n+1}
\]
which proves the claim. Since the inequality of the claim shows that \(|y_n - y_{n-1}|\) is summable, it follows that \(\{y_n\}\) is a Cauchy sequence in \(C([0,T], X)\). It satisfies \(|y_n - y_0| < R\) and so \(y_n\) converges uniformly to some \(y \in C([0,T], X)\). Hence one can pass to a limit in \(22.1.3\) and obtain
\[
y(t) = y_0 + \int_0^t f(y(s))
\]
for \(t \in [0,T]\). Also \(|y(t) - y_0| \leq R\) and on \(B(y_0, R)\), \(f\) is Lipschitz continuous so Gronwall’s inequality gives uniqueness of solutions which remain in \(B\). □

Here is an alternate proof which other than the ugly lemma, seems more elegant to me. However, it is a useful lemma.

**Lemma 22.1.9** Define
\[
\gamma(x) = \begin{cases} 
x & \text{if } \|x - y_0\| \leq R \\
y_0 + \frac{x - y_0}{\|x - y_0\|} R & \text{if } \|x - y_0\| > R
\end{cases}
\]
Then \(|\gamma(x) - \gamma(y)| \leq 3\|x - y\|\) for all \(x, y \in X\). Thus
\[
\|\gamma(x) - y_0\| \leq R.
\]

**Proof:** In case both of \(x, y\) are in \(B = \overline{B(y_0, R)}\), there is nothing to show.
Suppose then that \(|y - y_0| \leq R\) but \(|x - y_0| > R\). Then, assuming \(y - y_0 \neq 0\),
\[
\|\gamma(x) - \gamma(y)\| = \left\| y_0 + \frac{x - y_0}{\|x - y_0\|} R - y \right\| = \left\| \frac{x - y_0}{\|x - y_0\|} R - (y - y_0) \right\|
\]
\[
= \left\| \frac{x - y_0}{\|x - y_0\|} R - \frac{y - y_0}{\|y - y_0\|} (y - y_0) \right\|
\]
\[
\leq \left\| \frac{x - y_0}{\|x - y_0\|} R - \frac{(y - y_0)}{\|y - y_0\|} R \right\| +
\]
\[
+ \left\| \frac{R}{\|y - y_0\|} (y - y_0) - \frac{y - y_0}{\|y - y_0\|} (y - y_0) \right\| = A + B
\]
Now
\[
B = (R - \|y - y_0\|) < \|x - y_0\| - \|y - y_0\| \leq \|x - y\|
\]
\[
A \leq \left\| \frac{x - y_0}{\|x - y_0\|} R - \frac{(y - y_0)}{\|y - y_0\|} R \right\| \leq R \left\| \frac{(x - y_0)}{\|x - y_0\|} \|y - y_0\| - \frac{(y - y_0)}{\|y - y_0\|} \|y - y_0\| \|x - y_0\| \right\|
\]
\[
\leq \frac{R}{\|x - y_0\| \|y - y_0\|} \left( \|x - y_0\| \|y - y_0\| - \|y - y_0\| \|y - y_0\| \right) + \|y - y_0\| \|y - y_0\| - \|y - y_0\| \|x - y_0\|)
\]
\[
\leq \frac{R}{\|x - y_0\| \|y - y_0\|} (\|y - y_0\| \|x - y\| + \|y - y_0\| \|y - x\|)
\]
\[
\leq \frac{R}{\|x - y_0\|} (\|x - y\| + \|y - x\|) < 2 \|y - x\|
\]
CHAPTER 22. CRITICAL POINTS

In case $y = y_0$, you have

$$\|\gamma(x) - \gamma(y)\| = \left\| \frac{x - y_0}{\|x - y_0\|} R \right\| = \left\| \frac{x - y}{\|x - y_0\|} R \right\| < \|x - y\|$$

The only other case is where both $x, y$ are in $X \setminus B$. In this case, you get

$$\|\gamma(x) - \gamma(y)\| = \left\| y_0 + \frac{x - y_0}{\|x - y_0\|} R - \left( y_0 + \frac{y - y_0}{\|y - y_0\|} R \right) \right\|
= \left\| \frac{x - y_0}{\|x - y_0\|} R - \frac{y - y_0}{\|y - y_0\|} R \right\| \leq 2 \|x - y\|$$

by the same reasoning used above to estimate $A$. ■

**Alternate Proof of Theorem 22.1.8:** Let $B$ be a closed ball of radius $R$ centered at $y_0$ such that $f$ has Lipschitz constant $K$ on $B$. Let $\gamma$ be as in Lemma 22.1.9. Consider $g(x) = f(\gamma(x))$. Then

$$\|g(x) - g(y)\| = \|f(\gamma(x)) - f(\gamma(y))\| \leq K \|\gamma(x) - \gamma(y)\| \leq 3K \|x - y\|.$$

Now consider for $y \in C([0, T], X)$

$$Fy(t) = y_0 + \int_0^t g(y(s)) \, ds$$

Then

$$\|Fy(t) - Fz(t)\| \leq \int_0^t K \|y(s) - z(s)\| \, ds$$

Thus, iterating this inequality, it follows that a large enough power of $F$ is a contraction map. Therefore, there is a unique fixed point. Now letting $y$ be this fixed point,

$$\|y(t) - y_0\| \leq \int_0^t 3K \|y(s) - y_0\| \, ds + \|f(y_0)\| T$$

It follows that

$$\|y(t) - y_0\| \leq \|f(y_0)\| T e^{3KT}$$

Choosing $T$ small enough, it follows that $\|y(t) - y_0\| < R$ on $[0, T]$ and so $\gamma$ has no effect. Thus this yields a local solution to the initial value problem. ■

In the case that $U = X$, the above argument shows that there exists a solution on some $[0, T)$ where $T$ is maximal.

$$y(t) = y_0 + \int_0^t f(y(s)) \, ds, \ t < T$$

Suppose $T < \infty$. Suppose $\int_0^T \|f(y(s))\| \, ds < \infty$. Then you can consider $y_0 + \int_0^T f(y(s)) \, ds$ as an initial condition for the equation and obtain a unique solution...
22.1. MOUNTAIN PASS THEOREM IN HILBERT SPACE

z valid on \([T, T + \delta]\). Then one could consider \(\hat{y}(t) = y(t)\) for \(t < T\) and for \(t \geq T\), \(\hat{y}(t) = z(t)\). Then for \(t \in [T, T + \delta]\),

\[
\hat{y}(t) = z(t) = y_0 + \int_0^T f(y(s)) \, ds + \int_T^t f(\hat{y}(s)) \, ds
\]

and so in fact, for all \(t \in [0, T + \delta]\),

\[
\hat{y}(t) = y_0 + \int_0^t f(\hat{y}(s)) \, ds
\]

counter to the maximality of \(T\). Hence it cannot be the case that \(T < \infty\). Thus it must be the case that \(\int_0^T \|f(y(s))\| \, ds = \infty\) if the solution is not global.

From the above observation, here is a corollary.

Corollary 22.1.10 Let \(f : X \to X\) be locally Lipschitz where \(X\) is a Banach space. Then there exists a unique local solution to the IVP

\[ y' = f(y), \quad y(0) = y_0 \]

If \(f\) is bounded, then in fact the solutions exists on \([0, T]\) for any \(T > 0\).

**Proof:** Say \(\|f(x)\| \leq M\) for all \(M\). Then letting \([0, \hat{T})\) be the maximal interval, it must be the case that \(\int_0^\hat{T} \|f(y(t))\| \, dt = \infty\), but this does not happen if \(f\) is bounded.  

Note that this conclusion holds just as well if \(f\) has linear growth, \(\|f(u)\| \leq a + b\|u\|\) for \(a, b \geq 0\). One just uses an application of Gronwall’s inequality to verify a similar conclusion.

One can also give a simple modification of these theorems as follows.

Corollary 22.1.11 Suppose \(f : X \to X\) is continuous and \(f\) is locally Lipschitz on \(U\), an open subset of \(X\), a Banach space. Suppose also that \(f(x) = 0\) for all \(x \notin U\) and that \(\|f(x)\| < M\) for all \(x \in X\). Then there exists a solution to the IVP

\[ y' = f(y), \quad y(0) = y_0 \]

for \(t \in [0, T]\) for any \(T > 0\).

**Proof:** Let \(T\) be given. If \(y_0 \notin U\), there is nothing to show. The solution is \(y(t) \equiv y_0\). Suppose then that \(y_0 \in U\). Then by Theorem 22.1.8, there exists a unique solution to the initial value problem on an interval \([0, \hat{T})\) of maximal length. If \(\hat{T} = T\), then as \(t_n \to T\), \(\{y(t_n)\}\) must converge. This is because for \(t_m < t_n\),

\[
\|y(t_n) - y(t_m)\| \leq M|t_n - t_m|
\]
showing that this is a Cauchy sequence. Since all such sequences lead to a Cauchy sequence, it must be the case that \( \lim_{t \to T} y(t) \) exists. Thus it equals

\[
y_0 + \int_0^T f(y(t)) \, dt
\]

We let \( y(T) \) equal the above and it follows from Gronwall’s inequality that there is a unique solution to the IVP on \([0, T]\) so the claim is true in this case.

Otherwise, if \( \hat{T} < T \), then one can define

\[
y(\hat{T}) \equiv y_0 + \int_0^{\hat{T}} f(y(s)) \, ds
\]

If \( y(\hat{T}) \in U \), then by the assumption that \( f \) is bounded, one could consider a new initial condition and extend the solution further violating the maximality of the length of \([0, \hat{T}]\). Therefore, it must be the case that \( y(\hat{T}) \in U^C \). Then the solution is

\[
y'(t) = \begin{cases} y(t), & t < \hat{T} \\ y(\hat{T}), & t > \hat{T} \end{cases}
\]

because \( f(y(\hat{T})) = 0 \) by assumption. ■

One could also change the above argument for Corollary 22.1.11 to include the case that \( f \) has linear growth.

### 22.1.2 Mountain Pass Theorem In Banach Space

In this section, is a more general version of the mountain pass theorem. It is generalized in two ways. First, the space is not a Hilbert space and second, the derivative of the functional is not assumed to be Lipschitz. Instead of using \( I' \) one uses the pseudogradient in an appropriate differential equation. This is a significant generalization because there is no convenient projection map from \( X' \) to \( X \) like there is in Hilbert space. This is why the use of the pseudogradient is so interesting. For many more considerations of this sort of thing, see [52]. First is a deformation theorem. Here \( I \) will be defined on a Banach space \( X \) and \( I'(x) \in X' \). First recall the Palais Smale conditions.

**Definition 22.1.12** A functional \( I \) satisfies the Palais Smale conditions means that if \( \{ I(u_k) \} \) is a bounded sequence and \( I'(u_k) \to 0 \), then \( \{ u_k \} \) is precompact. That is, it has a subsequence which converges.

Here is a picture which illustrates the main conclusion of the following theorem. The idea is that you modify the functional on some set making it smaller and leaving it unchanged off that set.
Theorem 22.1.13 Let $I$ be $C^1$, $I$ is non-constant, satisfy the Palais Smale condition, and $I'$ is bounded on bounded sets. Also suppose that $c \in \mathbb{R}$ is such that either $I^{-1}([c-\delta, c+\delta]) = \emptyset$ for some $\delta > 0$ or $I^{-1}([c-\delta, c+\delta]) \neq \emptyset$ for all $\delta > 0$ and $\textbf{IF} I(u) = c$, then $I'(u) \neq 0$. Then for each sufficiently small $\varepsilon > 0$, there is a constant $\delta \in (0, \varepsilon)$ and a function $\eta : [0, 1] \times X \to X$ such that

1. $\eta(0, u) = u$
2. $\eta(1, u) = u$ on $I^{-1}(X \setminus (c-\varepsilon, c+\varepsilon))$
3. $I(\eta(t, u)) \leq I(u)$
4. $\eta(1, I^{-1}(-\infty, c+\delta)) \subseteq I^{-1}(-\infty, c-\delta]$, so $I(\eta(1, u)) \leq c-\delta$ if $I(u) \leq c+\delta$.

The main part of this conclusion is the statement about $u \to \eta(1, u)$ contained in parts 2. and 4. The other two parts are there to facilitate these two although they are certainly interesting for their own sake.

**Proof:** Suppose $I^{-1}\left([-\delta, c+\delta]\right) = \emptyset$ for some $\delta > 0$. Then

$$I^{-1}\left((-\infty, c+\frac{\delta}{2}\right) \subseteq I^{-1}\left((-\infty, c-\frac{\delta}{2}\right)$$

and you could take $\varepsilon = \delta$ and let $\eta(t, u) = u$. The conclusion holds with $\delta = \delta/2$.

Therefore, assume $I^{-1}([c-\delta, c+\delta]) \neq \emptyset$ for all $\delta > 0$. Since $I$ is nonconstant, $\varepsilon > 0$ can be chosen small enough that

$$I^{-1}(X \setminus (c-\varepsilon, c+\varepsilon)) \neq \emptyset.$$

Always let $\varepsilon$ be this small. Note that $I$ nonconstant is part of the assumptions.

**Claim 1:** For all small enough $\varepsilon > 0$, if $u \in I^{-1}([c-\varepsilon, c+\varepsilon])$, then $I'(u) \neq 0$ and in fact, for such $\varepsilon$, there exists $\sigma(\varepsilon) > 0$, such that $\sigma(\varepsilon) < \min(\varepsilon, 1)$, $\|I'(u)\| > \sigma(\varepsilon)$ for all $u \in I^{-1}([c-\varepsilon, c+\varepsilon])$. 
Proof of Claim 1: If the claim is not so, then there is \( \{ u_k \}, \varepsilon_k, \sigma_k \to 0, \| I' (u_k) \|_{X'} < \sigma_k, \) and \( I (u_k) \in [c - \varepsilon_k, c + \varepsilon_k] \) but \( \| I' (u_k) \|_{X'} \leq \sigma_k. \) However, from the Palais Smale condition, there is a subsequence, still denoted as \( u_k \) which converges to some \( u. \) Now \( I (u_k) \in [c - \varepsilon_k, c + \varepsilon_k] \) and so \( I (u) = c \) while \( I' (u) = 0 \) contrary to the hypothesis. This proves Claim 1. From now on, \( \varepsilon \) will be sufficiently small that this holds.

Now define for \( \delta < \varepsilon \) (The precise description of small \( \delta \) will be described later. However, it will be \( \delta < \sigma (\varepsilon) / 2 \), but this exact description is only used at the end.)

\[
A \equiv \; I^{-1} (X \setminus (c - \varepsilon, c + \varepsilon)) \\
B \equiv \; I^{-1} ([c - \delta, c + \delta])
\]

Thus \( A \) and \( B \) are disjoint closed sets. Recall that it is assumed that \( B \neq \emptyset \) since otherwise, there is nothing to prove. Also it is assumed throughout that \( \varepsilon > 0 \) is such that \( A \neq \emptyset \) thanks to \( I \) not being constant. Thus these are nonempty sets and we do not have to fuss with worrying about meaning when one is empty.

Claim 2: For any \( u, \) \( \text{dist}(u, A) + \text{dist}(u, B) > 0. \)

This is so because if not, then both summands would be zero and this requires that \( u \in A \cap B \) since these sets are closed. But \( A \cap B = \emptyset. \)

Now define a function

\[
g (u) = \frac{\text{dist}(u, A)}{\text{dist}(u, A) + \text{dist}(u, B)}
\]

It is a continuous function of \( u \) which has values in \([0, 1]\). It is 1 on \( B \) and 0 on \( A. \) Also define \( V (x) \) as a pseudogradient field for \( I \) on the regular points of \( I. \) At points \( I' (x) = 0, \) let \( V (x) = 0. \) Recall what this means:

\[
\| I' (x) \|_{X'}^2 \leq \langle I' (x), V (x) \rangle, \quad \| V (x) \|_X \leq 2 \| I' (x) \|_{X'}, \tag{22.1.5}
\]

and also \( V \) is locally Lipschitz on the regular points of \( I. \) Thus \( x \to V (x) \) is continuous on \( X, \) thanks to continuity of \( I', \) satisfies the above inequalities, and is locally Lipschitz on \( U = \{ x : I' (x) \neq 0 \}. \) It exists because of Lemma \([22.1.4].\)

Consider the ordinary differential initial value problem

\[
\eta' (t, u) + g (u) h (\| V (\eta (t, u)) \|) V (\eta (t, u)) = 0 \quad \tag{22.1.6}
\]

\[
\eta (0, u) = u \quad \tag{22.1.7}
\]

where \( r \to h (r) \) is a decreasing function which has values in \((0, 1]\) and equals 1 for \( r \in [0, 1] \) and equals \( 1/r \) for \( r > 1. \)

Here \( u \) is given and the \( \eta' \) is the time derivative is with respect to \( t. \) By Corollary \([22.1.8],\) there exists a solution to this for \( t \in [0, 1]. \)
Then for this solution, \( \eta (0, u) = u \) because of the above initial condition. If \( u \in I^{-1} (X \setminus \{c - \varepsilon, c + \varepsilon\}) \), then \( u \in A \) and so \( g(u) = 0 \) so \( \eta(t, u) = u \) for all \( t \in [0, 1] \). This gives the first two conditions. Consider the third.

\[
\frac{d}{dt} (I(\eta(t, u))) = \langle I'(\eta), \eta' \rangle = -\langle I'(\eta), g(u) h(||V(\eta(t, u))||) V(\eta(t, u)) \rangle
\]

\[
= -g(u) h(||V(\eta(t, u))||) \langle I'(\eta), V(\eta(t, u)) \rangle
\]

\[
\leq -g(u) h(||V(\eta(t, u))||) ||I'(\eta)||_{X^*}^2 \leq 0
\]

this last inequality from the inequalities of Claim 1, and so this implies the third condition since it says that the function \( t \rightarrow I(\eta(t, u)) \) is decreasing.

It remains to consider the last condition. This involves an appropriate choice of small \( \delta \). It was chosen small and now it will be seen how small. It is desired to verify that

\[
\eta (1, I^{-1} ((-\infty, c + \delta])) \subseteq I^{-1} ((-\infty, c - \delta])
\]

Suppose it is not so. Then there exists \( u \) such that \( I(u) \in (c - \delta, c + \delta] \) but \( I(\eta(1, u)) > c - \delta \). We can assume that \( I(u) = (c - \delta, c + \delta] \) because if \( I(u) \leq c - \delta \), then so is \( I(\eta(1, u)) \) from what was just shown. Hence \( g(u) = 1 \). Then using the fact that \( g(u) = 1 \),

\[
c - \delta < I(\eta(1, u)) = I(u) + \int_0^1 \frac{d}{dt} (I(\eta)) dt
\]

\[
= I(u) - \int_0^1 \langle I'(\eta), h(||V(\eta)||) V(\eta) \rangle dt
\]

\[
= I(u) + \int_0^1 -h(||V(\eta)||) \langle I'(\eta), V(\eta) \rangle dt
\]

\[
\leq I(u) + \int_0^1 \left( -h(||V(\eta)||) ||I'(\eta)||^2 \right) dt
\]

Then

\[
c - \delta + \int_0^1 h(||V(\eta)||) ||I'(\eta)||^2 dt < I(u) \leq c + \delta
\]

Thus

\[
c - 2\delta + \int_0^1 h(||V(\eta)||) ||I'(\eta)||^2 dt < c
\]

Also, it is being assumed that \( I(\eta(1, u)) > c - \delta \) and so by the third conclusion shown above, \( \eta(t, u) \in B \) for \( t \in [0, 1] \). We also know that for such values of \( \eta(t, u) \), \( ||I'(\eta(t, u))|| \geq \sigma(\varepsilon) \) from Claim 1. Now

\[
||I'(x)||_{X^*}^2 \leq \langle I'(x), V(x) \rangle \leq ||I'(x)|| ||V(x)||
\]

and so

\[
||V(\eta(t, u))||_X \geq ||I'(\eta(t, u))||_{X^*} \geq \sigma(\varepsilon) \cdot (22.1.8)
\]
Thus the above inequality yields
\[ c - 2\delta + \int_{0}^{1} h(\|V(\eta)\|) \sigma(\varepsilon)^2 \, dt < c \]
Now what is the value of \( h(\|V(\eta)\|) \)? From 22.1.8
\[ h(\|V(\eta(t,u))\|_X) \leq h(\|I'(\eta(t,u))\|_{X'}) \leq h(\sigma(\varepsilon)) \leq \frac{1}{\sigma(\varepsilon)} \]
In fact, \( \sigma(\varepsilon) < 1 \) so \( h(\sigma(\varepsilon)) = 1 \) so the above estimate, while correct is sloppy.

Hence
\[ c - 2\delta + \int_{0}^{1} \frac{1}{\sigma(\varepsilon)} \sigma(\varepsilon)^2 \, dt < c \]
So far it was only assumed \( \delta < \varepsilon \). As indicated above, \( \delta \) was chosen small enough that \(-2\delta + \sigma(\varepsilon) > 0\). Hence this yields a contradiction. Thus the last conclusion is verified.

Imagine a valley surrounded by a ring of mountains. On the other side of this ring of mountains, there is another low place. Then there must be some path from the valley to the exterior low place which goes through a point where the gradient equals 0, the gradient being the gradient of a function \( f \) which gives the altitude of the land. This is the idea of the mountain pass theorem. The critical point where \( \nabla f = 0 \) is the mountain pass.

**Theorem 22.1.14** Let \( X \) be a Banach space and let \( I: X \rightarrow \mathbb{R} \) be a \( C^1 \) functional having \( I' \) bounded on bounded sets and such that \( I \) satisfies the Palais Smale condition. Suppose \( I(0) = 0 \) and \( I(u) \geq a > 0 \) for all \( \|u\| = r \). Suppose also that there exists \( v, \|v\| > r \) such that \( I(v) = 0 \). Then define
\[ \Gamma \equiv \{ g \in C([0,1] ; X) : g(0) = 0, g(1) = v \} \]
Let
\[ c \equiv \inf \max_{g \in \Gamma} I(\gamma(t)) \]
Then \( c \) is a critical value of \( I \) meaning that there exists \( u \) such that \( I(u) = c \) and \( I'(u) = 0 \). In particular, there is \( u \neq 0 \) such that \( I'(u) = 0 \).

**Proof:** First note that \( c \geq a > 0 \). Suppose \( c \) is not a critical value. Then either \( I^{-1}((c - \delta, c + \delta)) = \emptyset \) for some \( \delta > 0 \) in which case the conclusion of the deformation theorem, (Theorem 22.1.13) holds, or for all \( \delta > 0 \), \( I^{-1}((c - \delta, c + \delta)) \neq \emptyset \) and if \( I(u) = c \), then \( I'(u) \neq 0 \) in which case the deformation theorem also holds. Then by this theorem, for \( \varepsilon > 0, \varepsilon \) sufficiently small, \( \varepsilon < c \), there is \( \eta: X \rightarrow X \) and a \( \delta < \varepsilon \) small enough that
\[ \eta(I^{-1}((-\infty, c + \delta))) \subseteq I^{-1}((-\infty, c - \delta)) \]
and \( \eta \) leaves unchanged \( I^{-1}(X \setminus (c - \varepsilon, c + \varepsilon)) \). Then there is \( g \in \Gamma \) such that
\[ \max_{t \in [0,1]} I(g(t)) < c + \delta \]
Then in particular, \( I(g(t)) < c + \delta \) for every \( t \). Hence you look at \( \eta \circ g \). We know that \( g(0), g(1) \) are both in the set \( I(u) \notin (c - \epsilon, c + \epsilon) \) because they are both 0 or less than 0 and so \( \eta \) leaves these unchanged. Hence \( \eta \circ g \in \Gamma \) and

\[
I(\eta \circ g(t)) \leq c - \delta
\]

for all \( t \in [0, 1] \). Thus

\[
c = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t)) \leq \max_{t \in [0,1]} I(\eta \circ g(t)) \leq c - \delta
\]

which is clearly a contradiction. \( \blacksquare \)
Chapter 23

Nonlinear Operators

In this chapter, is a discussion of various kinds of nonlinear operators. Some standard references on these operators are [36, 37, 21, 23, 12, 82, 105, 24] and references listed there. The most important examples of these operators seem to be due to Brezis in the 1960’s and these things have been generalized and used by many others since this time. I am following many of these, but the stuff about maximal monotone operators is mainly from Barbu [12]. I am trying to include all the necessary basic results such as fixed point theorems which are needed to prove the main theorems and also to re write in a manner understandable to me.

It seems like the main issue is the following. When does \( \langle f_n, x_n \rangle \) converge to \( \langle f, x \rangle \) given that \( f_n \) and \( x_n \) both converge weakly to \( f \) and \( x \) respectively? There is no problem in finite dimensions because in finite dimensions, there is only one meaning for convergence. However, in infinite dimensions, there certainly is a problem as can be instantly realized by consideration of the Riemann-Lebesgue lemma, for example. You know that \( \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \to 0 \) so \( \sin(nx) \) converges weakly to 0 but \( \int_{-\pi}^{\pi} \sin^2(nx) \, dx \) certainly does not converge to 0.

The idea behind all of these considerations is that \( f_n \) is to come from some nonlinear operator which has properties which will allow one to successfully pass to a limit. When the operator is linear, there usually is no problem because the graph is a subspace and so if it is closed, it will also be weakly closed. Thus, if \( x_n \to x \) weakly and \( Lx_n \to f \) weakly, then \( f = Lx \). However, nothing like this happens with nonlinear operators. Consideration of when this happens is the purpose of this catalogue of nonlinear operators, and also to generalize to set valued operators. First is a section on single valued nonlinear operators and then the case of set valued nonlinear operators is discussed.

23.1 Some Nonlinear Single Valued Operators

Here is an assortment of nonlinear operators which are useful in applications to nonlinear partial differential equations. Generalizations of the notion of a pseudomonono-
tone map will be presented later to include the case of set valued pseudomonotone maps. This is on the single valued version of some of these and these ideas originate with Brezis in the 1960’s. A good description is given in Lions [82].

**Definition 23.1.1** For a real Banach space, $A : V \to V'$ is a pseudomonotone map if whenever

$$u_n \rightharpoonup u$$

and

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0$$

it follows that for all $v \in V$,

$$\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle.$$

The half arrows denote weak convergence.

If $V$ is finite dimensional, then pseudomonotone maps are continuous. Also the property of being pseudomonotone is preserved when restriction is made to finite dimensional spaces. The notation is explained in the following diagram.

\[ W' \xleftarrow{i} V' \xrightarrow{i^*} W \]

The map $i$ is just the inclusion map. $iw \equiv w$ and $i^*$ is the usual adjoint map. $\langle i^* f, w \rangle_{W', W} = \langle f, iw \rangle_{V', V} = \langle f, w \rangle_{V', V}$. Thus $i^* A_i(w) \in W'$ and it is defined by

$$\langle i^* A_i(w), z \rangle_{W', W} = \langle Aw, z \rangle_{V', V}$$

in other words, you restrict $A$ to $W$ and only consider what the resulting functional does to things in $W$.

**Proposition 23.1.2** Let $V$ be finite dimensional and let $A : V \to V'$ be pseudomonotone and bounded (meaning $A$ maps bounded sets to bounded sets). Then $A$ is continuous. Also, if $A : V \to V'$ is pseudomonotone and bounded, and if $W \subseteq V$ is a finite dimensional subspace, then $i^* A_i$ is pseudomonotone as a map from $W$ to $W'$.

**Proof:** Say $u_n \to u$. Does it follow that $Au_n \to Au$? If not, then there is a subsequence such that $Au_n \to \xi \neq Au$ thanks to $\{Au_n\}$ being bounded. Then the lim sup condition holds obviously. In fact the limit of $\langle Au_n, u_n - u \rangle$ exists and equals 0. Hence for all $v$,

$$\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle$$

Therefore,

$$\langle \xi, u - v \rangle \geq \langle Au, u - v \rangle$$
23.1. SOME NONLINEAR SINGLE VALUED OPERATORS

for all \( v \) and so in fact \( \xi = Au \) after all. Thus \( A \) must be continuous.

As to the second part of this proposition, if you have \( w_n \rightharpoonup w \) in \( W \), then in fact convergence takes place strongly because weak and strong convergence are the same in finite dimensions. Hence the same argument given above holds to show that \( i^* A \) is continuous.

**Definition 23.1.3** \( A : V \to V' \) is monotone if for all \( v, u \in V \),

\[
\langle Au - Av, u - v \rangle \geq 0,
\]

and \( A \) is Hemicontinuous if for all \( v, u \in V \),

\[
\lim_{t \to 0^+} \langle A(u + t(v - u)), u - v \rangle = \langle Au, u - v \rangle.
\]

**Theorem 23.1.4** Let \( V \) be a Banach space and let \( A : V \to V' \) be monotone and hemi-continuous. Then \( A \) is pseudomonotone.

**Proof:** Let \( A \) be monotone and Hemicontinuous. First here is a claim.

**Claim:** If 23.1.1 and 23.1.2 hold, then \( \lim_{n \to \infty} \langle Au_n, u_n - u \rangle = 0 \).

**Proof of the claim:** Since \( A \) is monotone,

\[
\langle Au_n - Au, u_n - u \rangle \geq 0
\]

so

\[
\langle Au_n, u_n - u \rangle \geq \langle Au, u_n - u \rangle.
\]

Therefore,

\[
0 = \liminf_{n \to \infty} \langle Au, u_n - u \rangle \leq \liminf_{n \to \infty} \langle Au_n, u_n - u \rangle \leq \limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0.
\]

Now using that \( A \) is monotone again, then letting \( t > 0 \),

\[
\langle Au_n - A(u + t(v - u)), u_n - u + t(u - v) \rangle \geq 0
\]

and so

\[
\langle Au_n, u_n - u + t(u - v) \rangle \geq \langle A(u + t(v - u)), u_n - u + t(u - v) \rangle.
\]

Taking the \( \liminf \) on both sides and using the claim and \( t > 0 \),

\[
t \liminf_{n \to \infty} \langle Au_n, u - v \rangle \geq t \langle A(u + t(v - u)), (u - v) \rangle.
\]

Next divide by \( t \) and use the Hemicontinuity of \( A \) to conclude that

\[
\liminf_{n \to \infty} \langle Au_n, u - v \rangle \geq \langle Au, u - v \rangle.
\]

From the claim,

\[
\liminf_{n \to \infty} \langle Au_n, u - v \rangle = \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle.
\]

Monotonicity is very important in the above proof. The next example shows that even if the operator is linear and bounded, it is not necessarily pseudomonotone.
Example 23.1.5 Let $H$ be any Hilbert space (complete inner product space, more on these later) and let $A : H \to H'$ be given by
\[
\langle Ax, y \rangle \equiv (-x, y)_H.
\]
Then $A$ fails to be pseudomonotone.

**Proof:** Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set of vectors in $H$. Then Parsevall’s inequality implies
\[
||x||^2 \geq \sum_{n=1}^{\infty} |(x_n, x)|^2
\]
and so for any $x \in H$, $\lim_{n \to \infty} (x_n, x) = 0$. Thus $x_n \rightharpoonup 0 \equiv x$. Also
\[
\lim \sup_{n \to \infty} \langle Ax_n, x_n - x \rangle =
\lim \sup_{n \to \infty} \langle Ax_n, x_n - 0 \rangle = \lim \sup_{n \to \infty} (-||x_n||^2) = -1 \leq 0.
\]
If $A$ were pseudomonotone, we would need to be able to conclude that for all $y \in H$,
\[
\lim \inf_{n \to \infty} \langle Ax_n, x_n - y \rangle \geq \langle Ax, x - y \rangle = 0.
\]
However,
\[
\lim \inf_{n \to \infty} \langle Ax_n, x_n - 0 \rangle = -1 < 0 = \langle A0, 0 - 0 \rangle.
\]
The following proposition is useful.

**Proposition 23.1.6** Suppose $A : V \to V'$ is pseudomonotone and bounded where $V$ is separable. Then it must be demicontinuous. This means that if $u_n \to u$, then $Au_n \rightharpoonup Au$. In case that $V$ is reflexive, you don’t need the assumption that $V$ is separable.

**Proof:** Since $u_n \to u$ is strong convergence and since $Au_n$ is bounded, it follows
\[
\lim \sup_{n \to \infty} \langle Au_n, u_n - u \rangle = \lim_{n \to \infty} \langle Au_n, u_n - u \rangle = 0.
\]
Suppose this is not so that $Au_n$ converges weakly to $Au$. Since $A$ is bounded, there exists a subsequence, still denoted by $n$ such that $Au_n \rightharpoonup \xi$ weak *. I need to verify $\xi = Au$. From the above, it follows that for all $v \in V$
\[
\langle Au, u - v \rangle \leq \lim \inf_{n \to \infty} \langle Au_n, u_n - v \rangle
\]
\[
= \lim \inf_{n \to \infty} \langle Au_n, u - v \rangle = \langle \xi, u - v \rangle
\]
Hence $\xi = Au$. ■

There is another type of operator which is more general than pseudomonotone.
Definition 23.1.7 Let $A : V \to V'$ be an operator. Then $A$ is called type M if whenever $u_n \rightharpoonup u$ and $Au_n \rightharpoonup \xi$, and
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle \]
it follows that $Au = \xi$.

Proposition 23.1.8 If $A$ is pseudomonotone, then $A$ is type M.

Proof: Suppose $A$ is pseudomonotone and $u_n \rightharpoonup u$ and $Au_n \rightharpoonup \xi$, and
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle \]
Then
\[ \limsup_{n \to \infty} \langle Au_n, u_n - u \rangle = \limsup_{n \to \infty} \langle Au_n, u_n \rangle - \langle \xi, u \rangle \leq 0 \]
Hence
\[ \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle \]
for all $v \in V$. Consequently, for all $v \in V$,
\[ \langle Au, u - v \rangle \leq \limsup_{n \to \infty} \langle Au_n, u_n - v \rangle \]
\[ = \limsup_{n \to \infty} (\langle Au_n, u - v \rangle) \]
\[ = \langle \xi, u - v \rangle + \liminf_{n \to \infty} \langle Au_n, u_n - u \rangle \]
and so $Au = \xi$. ■

An interesting result is the following which states that a monotone linear function added to a type M is also type M.

Proposition 23.1.9 Suppose $A : V \to V'$ is type M and suppose $L : V \to V'$ is monotone, bounded and linear. Then $L + A$ is type M. Let $V$ be separable or reflexive so that the weak convergences in the following argument are valid.

Proof: Suppose $u_n \rightharpoonup u$ and $Au_n + Lu_n \rightharpoonup \xi$ and also that
\[ \limsup_{n \to \infty} \langle Au_n + Lu_n, u_n \rangle \leq \langle \xi, u \rangle \]
Does it follow that $\xi = Au + Lu$? Suppose not. There exists a further subsequence, still called $n$ such that $Lu_n \rightharpoonup Lu$. This follows because $L$ is linear and bounded. Then from monotonicity,
\[ \langle Lu_n, u_n \rangle \geq \langle Lu_n, u \rangle + \langle L(u), u_n - u \rangle \]
Hence with this further subsequence, the lim sup is no larger and so
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle + \lim_{n \to \infty} ((Lu_n, u) + (L(u), u_n - u)) \leq \langle \xi, u \rangle \]
and so
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi - Lu, u \rangle \]
It follows since \( A \) is type \( M \) that \( Au = \xi - Lu \), which contradicts the assumption that \( \xi \neq Au + Lu \).

There is also the following useful generalization of the above proposition.

**Corollary 23.1.10** Suppose \( A : V \to V' \) is type \( M \) and suppose \( L : W \to W' \) is monotone, bounded and linear where \( V \subseteq W \) and \( V \) is dense in \( W \) so that \( W' \subseteq V' \).

Then for \( u_0 \in W \) define \( M(u) = L(u - u_0) \). Then \( M + A \) is type \( M \). Let \( V \) be separable or reflexive so that the weak convergences in the following argument are valid.

**Proof:** Suppose \( u_n \rightharpoonup u \) and \( Au_n + Mu_n \rightharpoonup \xi \) and also that
\[ \limsup_{n \to \infty} \langle Au_n + Mu_n, u_n \rangle \leq \langle \xi, u \rangle \]

Does it follow that \( \xi = Au + Mu \)? Suppose not. By assumption, \( u_n \rightharpoonup u \) and so,
\[ u_n - u_0 \rightharpoonup u - u_0 \]
weak convergence in \( W \)

since \( L \) is bounded, there is a further subsequence, still called \( n \) such that
\[ Mu_n = L(u_n - u_0) \rightharpoonup L(u - u_0) = Mu. \]

Since \( M \) is monotone,
\[ \langle Mu_n - Mu, u_n - u \rangle \geq 0 \]
Thus
\[ \langle Mu_n, u_n \rangle - \langle Mu_n, u \rangle - \langle Mu, u_n \rangle + \langle Mu, u \rangle \geq 0 \]
and so
\[ \langle Mu_n, u_n \rangle \geq \langle Mu_n, u \rangle + \langle Mu, u_n - u \rangle \]
Hence with this further subsequence, the lim sup is no larger and so
\[ \langle \xi, u \rangle \geq \limsup_{n \to \infty} \langle Au_n + Mu_n, u_n \rangle \]
\[ \geq \limsup_{n \to \infty} (\langle Au_n, u_n \rangle + \langle Mu_n, u \rangle + \langle Mu, u_n - u \rangle) \]
\[ = \limsup_{n \to \infty} \langle Au_n, u_n \rangle + \lim_{n \to \infty} (\langle Mu_n, u \rangle + \langle M(u), u_n - u \rangle) \leq \langle \xi, u \rangle \]
and so
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi - Mu, u \rangle \]
It follows since \( A \) is type \( M \) that \( Au = \xi - Mu \), which contradicts the assumption that \( \xi \neq Au + Mu \).

The following is Browder’s lemma. It is a very interesting application of the Brouwer fixed point theorem.
23.1. SOME NONLINEAR SINGLE VALUED OPERATORS

Lemma 23.1.11 (Browder) Let $K$ be a convex closed and bounded set in $\mathbb{R}^n$ and let $A : K \to \mathbb{R}^n$ be continuous and $f \in \mathbb{R}^n$. Then there exists $x \in K$ such that for all $y \in K$,

$$(f - Ax, y - x)_{\mathbb{R}^n} \leq 0$$

If $K$ is convex, closed, bounded subset of $V$ a finite dimensional vector space, then the same conclusion holds. If $f \in V'$, there exists $x \in K$ such that for all $y \in K$,

$$\langle f - Ax, y - x \rangle_{V', V} \leq 0$$

Proof: Let $P_K$ denote the projection onto $K$. Thus $P_K$ is Lipschitz continuous.

$$x \to P_K(f - Ax + x)$$

is a continuous map from $K$ to $K$. By the Brouwer fixed point theorem, it has a fixed point $x \in K$. Therefore, for all $y \in K$,

$$(f - Ax + x - x, y - x) = (f - Ax, y - x) \leq 0$$

As to the second claim. Consider the following diagram.

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\theta^*} & V' \\
\mathbb{R}^n & \xrightarrow{\theta} & V
\end{array}$$

where

$$\theta(x) = \sum_{i=1}^{n} x_i v_i$$

Thus $\theta$ and $\theta^*$ are both continuous linear and one to one and onto. Hence there is $x \in \theta^{-1}K$ a closed convex and bounded subset of $\mathbb{R}^n$ such that $x = \theta^{-1}u, u \in K$, and

$$\langle \theta^* f - \theta^* A \theta (\theta^{-1} u), \theta^{-1} y - \theta^{-1} u \rangle_{\mathbb{R}^n} = \langle f - Au, y - u \rangle_{V', V} \leq 0$$

for all $y \in K$. $\blacksquare$

From this lemma, there is an interesting theorem on surjectivity.

Proposition 23.1.12 Let $A : V \to V'$ be continuous and coercive,

$$\lim_{\|v\| \to \infty} \frac{\langle A(v + v_0), v \rangle}{\|v\|_V} = \infty$$

for some $v_0$. Then for all $f \in V'$, there exists $v \in V$ such that $Av = f$.

Proof: Define the closed convex sets $B_n \equiv \overline{B(v_0, n)}$. By Browder’s lemma, there exists $x_n$ such that

$$(f - Av_n, y - v_n) \leq 0$$

for all $y \in B_n$. Then taking $y = v_0$,

$$\langle Av_n, v_n - v_0 \rangle \leq \langle f, v_n - v_0 \rangle$$
letting \( w_n = v_n - v_0 \),

\[
\langle A (w_n + v_0), w_n \rangle \leq \langle f, w_n \rangle
\]

and so

\[
\frac{\langle A (w_n + v_0), w_n \rangle}{\|w_n\|} \leq \|f\|
\]

which implies that the \( \|w_n\| \) and hence the \( \|v_n\| \) are bounded. It follows that for large \( n \), \( v_n \) is an interior point of \( B_n \). Therefore,

\[
(f - Av_n, z)_{V', V} \leq 0
\]

for all \( z \) in some open ball centered at \( v_0 \). Hence \( f - Av_n = 0 \).

**Lemma 23.1.13** Let \( A : V \to V' \) be type \( M \) and bounded and suppose \( V \) is reflexive or \( V \) is separable. Then \( A \) is demicontinuous.

**Proof:** Suppose \( u_n \to u \) and \( Au_n \) fails to converge weakly to \( Au \). Then there is a further subsequence, still denoted as \( u_n \) such that \( Au_n \rightharpoonup \zeta \neq Au \). Then thanks to the strong convergence, you have

\[
\limsup_{n \to \infty} \langle Au_n, u_n \rangle = \langle \zeta, u \rangle
\]

which implies \( \zeta = Au \) after all.

With these lemmas and the above proposition, there is a very interesting surjectivity result.

**Theorem 23.1.14** Let \( A : V \to V' \) be type \( M \), bounded, and coercive

\[
\lim_{\|u\| \to \infty} \frac{\langle A (u + u_0), u \rangle}{\|u\|} = \infty, \quad (23.1.4)
\]

for some \( u_0 \), where \( V \) is a separable reflexive Banach space. Then \( A \) is surjective.

**Proof:** Since \( V \) is separable, there exists an increasing sequence of finite dimensional subspaces \( \{V_n\} \) such that \( \bigcup_n V_n = V \) and each \( V_n \) contains \( u_0 \). Say \( \text{span}(v_1, \ldots, v_n) = V_n \). Then consider the following diagram.

\[
\begin{array}{ccc}
V_n' & \xrightarrow{i^*} & V' \\
\downarrow & & \downarrow \\
V_n & \xrightarrow{i} & V
\end{array}
\]

The map \( i \) is the inclusion map. Consider the map \( i^*Ai \). By Lemma 23.1.13, this map is continuous.

\[
\frac{\langle (i^*Ai (v + u_0), v)_{V_n' V_n} \rangle}{\|v\|} = \frac{\langle A (v + u_0), v \rangle_{V' V}}{\|v\|}
\]

Hence \( i^*Ai \) is coercive. Let \( f \in V' \). Then from Proposition 23.1.12, there exists \( x_n \) such that

\[
i^*Av_n = i^*f
\]
23.1. SOME NONLINEAR SINGLE VALUED OPERATORS

In other words,

\[
\langle Av_n, y \rangle_{V^*V} = \langle f, y \rangle_{V^*V} \tag{23.1.5}
\]

for all \( y \in V_n \). Letting \( y \equiv v_n - u_0 \equiv w_n \),

\[
\langle A(w_n + u_0), w_n \rangle = \langle f, w_n \rangle
\]

Then from the coercivity condition \(23.1.4\), the \( w_n \) are bounded independent of \( n \). Hence this is also true of the \( v_n \). Since \( V \) is reflexive, there is a subsequence, still called \( \{v_n\} \) which converges weakly to \( v \in V \). Since \( A \) is bounded, it can also be assumed that \( Av_n \to \zeta \in V' \). Then

\[
\limsup_{n \to \infty} \langle Av_n, v_n \rangle = \limsup_{n \to \infty} \langle f, v_n \rangle = \langle f, v \rangle
\]

Also, passing to the limit in \(23.1.5\),

\[
\langle \zeta, y \rangle = \langle f, y \rangle
\]

for any \( y \in V_n \), this for any \( n \). Since the union of these \( V_n \) is dense, it follows that the above equation holds for all \( y \in V \). Therefore, \( f = \zeta \) and so

\[
\limsup_{n \to \infty} \langle Av_n, v_n \rangle = \limsup_{n \to \infty} \langle f, v_n \rangle = \langle f, \zeta \rangle = \langle f, v \rangle
\]

Since \( A \) is type \( M \), \( Av = \zeta = f \). \( \blacksquare \)

You can generalize pseudomonotone slightly without any trouble.

**Definition 23.1.15** Let \( V \) be a Banach space and let \( K \) be a closed convex nonempty subset of \( V \). Then \( A : K \to V' \) is pseudomonotone if similar conditions hold as above. That is, if

\[
u_n \rightharpoonup u \tag{23.1.6}
\]

and

\[
\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0 \tag{23.1.7}
\]

it follows that for all \( v \in K \),

\[
\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle. \tag{23.1.8}
\]

Then it is easy to give a nice result on variational inequalities.

**Proposition 23.1.16** Let \( K \) be a closed convex nonempty subset of \( V \) a separable reflexive Banach space. Let \( A : K \to V' \) be pseudomonotone and bounded. Also assume that either \( K \) is bounded or there is a coercivity condition

\[
\lim_{\|u\| \to \infty} \frac{\langle Au, u - u_0 \rangle}{\|u\|} = \infty, \ u_0 \in K
\]

then for \( f \in V' \), there exists \( u \in K \) such that for all \( v \in K \),

\[
\langle Au, u - v \rangle \leq \langle f, u - v \rangle
\]
Proof: Let $V_n$ be finite dimensional spaces whose union is dense in $V$, each containing $u_0, n > \|u_0\|$. By a repeat of the proof of Proposition 23.1.2, $i^* A_i$ will be continuous on $K$. Therefore, by Browder’s lemma, there exists $u_n \in K_n \equiv K \cap B(0, n) \cap V_n$ such that for all $v \in K_n$,

$$\langle i^* f - i^* A_i u_n, v - u_n \rangle_{V_n^*, V_n} = \langle f - A u_n, v - u_n \rangle_{V^*, V} \leq 0$$

Now assume we don’t know that $K$ is bounded. In case it is bounded, the argument simplifies. In the harder case, the coercivity condition implies that the $u_n$ are bounded in $V$. This follows from letting $v = u_0$ in the above inequality. Thus

$$\langle f, u_n - u_0 \rangle \geq \langle A u_n, u_n - u_0 \rangle$$

Hence

$$\frac{\langle A u_n, u_n - u_0 \rangle}{\|u_n\|} \leq \frac{\|f\|\|u_n - u_0\|}{\|u_n\|}$$

The right side is bounded and so it follows that the left side is also bounded. Therefore, $\|u_n\|$ must be bounded. Taking a subsequence and using the assumption that $V$ is reflexive, we can obtain

$$u_n \to u \text{ weakly in } V$$

By the fact that convex closed sets are weakly closed also, it follows that $u \in K$. Also, given $M$, eventually all $\|u_n\|$ and $\|u\|$ are less than $M$. Now from the inequality,

$$\langle A u_n, u_n - v \rangle \leq \langle f, u_n - v \rangle$$

Thus

$$\langle A u_n, u_n - u \rangle + \langle A u_n, u - v \rangle \leq \langle f, u_n - u \rangle + \langle f, u - v \rangle$$

Then taking $\limsup_{n \to \infty}$ one gets

$$\limsup_{n \to \infty} \langle A u_n, u_n - u \rangle + \langle \xi, u - v \rangle \leq \langle f, u - v \rangle$$

This holds for $v \in K_m$ where $m$ is arbitrary. Hence one could let $v_m \to u$. Thus eventually $\|v_m\| < M$ and so for large $m, v_m \in K_m$. Then it follows that

$$\limsup_{n \to \infty} \langle A u_n, u_n - u \rangle \leq 0.$$ 

Consequently, by the assumption that $A$ is pseudomonotone on $K$, for every $v \in K$,

$$\langle A u, u - v \rangle \leq \liminf_{n \to \infty} \langle A u_n, u_n - v \rangle$$

for all $v \in K$. Then from the inequality obtained from Browder’s lemma,

$$\langle A u_n, u_n - v \rangle_{V^*, V} \leq \langle f, u_n - v \rangle_{V^*, V}$$

and so * implies on taking $\liminf$ that for all $v \in K$,

$$\langle A u, u - v \rangle_{V^*, V} \leq \langle f, u - v \rangle_{V^*, V}$$
23.2 Duality Maps

The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space.

Definition 23.2.1 A Banach space is said to be strictly convex if whenever \( \|x\| = \|y\| \) and \( x \neq y \), then
\[
\left\| \frac{x + y}{2} \right\| < \|x\|.
\]

\( F : X \to X' \) is said to be a duality map if it satisfies the following: a.) \( \|F(x)\| = \|x\|^{p-1} \). b.) \( F(x)(x) = \|x\|^p \), where \( p > 1 \).

Duality maps exist. Here is why. Let
\[
F(x) \equiv \left\{ x^* : \|x^*\| \leq \|x\|^{p-1} \text{ and } x^*(x) = \|x\|^p \right\}
\]

Then \( F(x) \) is not empty because you can let \( f(\alpha x) = \alpha \|x\|^p \). Then \( f \) is linear and defined on a subspace of \( X \). Also
\[
\sup_{\|\alpha x\| \leq 1} |f(\alpha x)| = \sup_{\|\alpha x\| \leq 1} |\alpha| \|x\|^p \leq \|x\|^{p-1}
\]

Also from the definition,
\[
f(x) = \|x\|^p
\]

and so, letting \( x^* \) be a Hahn Banach extension, it follows \( x^* \in F(x) \). Also, \( F(x) \) is closed and convex. It is clearly closed because if \( x^*_n \to x^* \), the condition on the norm clearly holds and also the other one does too. It is convex because
\[
\|x^* \lambda + (1 - \lambda) y^*\| \leq \lambda \|x^*\| + (1 - \lambda) \|y^*\| \leq \lambda \|x\|^{p-1} + (1 - \lambda) \|y\|^{p-1}
\]

If the conditions hold for \( x^* \), then we can show that in fact \( \|x^*\| = \|x\|^{p-1} \).

This is because
\[
\|x^*\| \geq \left( \frac{x}{\|x\|} \right) = \frac{1}{\|x\|} \|x^*(x)\| = \|x\|^{p-1}.
\]

Now how many things are in \( F(x) \) assuming the norm on \( X' \) is strictly convex? Suppose \( x^*_1 \) and \( x^*_2 \) are two things in \( F(x) \). Then by convexity, so is \( (x^*_1 + x^*_2)/2 \). Hence by strict convexity, if the two are different, then
\[
\left\| \frac{x^*_1 + x^*_2}{2} \right\| = \|x\|^{p-1} < \frac{1}{2} \|x^*_1\| + \frac{1}{2} \|x^*_2\| = \|x\|^{p-1}
\]

which is a contradiction. Therefore, \( F \) is an actual mapping.

What are some of its properties? First is one which is similar to the Cauchy Schwarz inequality. Since \( p - 1 = p'/p' \),
\[
\sup_{\|y\| \leq 1} |\langle Fx, y \rangle| = \|x\|^{p/p'}
\]
and so for arbitrary \( y \neq 0 \),
\[
\left| \langle Fx, y \rangle \right| = \|y\| \left| \frac{\langle Fx, \frac{y}{\|y\|} \rangle}{\|y\|} \right| \leq \|y\| \|x\|^{p/p'} = \|\langle Fy, y \rangle \|^{1/p} |\langle Fx, y \rangle|^{1/p'}
\]
Next we can show that \( F \) is monotone.
\[
\langle Fx - Fy, x - y \rangle = \langle Fx, x \rangle - \langle Fx, y \rangle - \langle Fy, x \rangle + \langle Fy, y \rangle \\
\geq \|x\|^p + \|y\|^p - \|x\|^{p/p'} - \|y\|^{p/p'} \|x\| \\
\geq \|x\|^p + \|y\|^p - \left( \|y\|^{p/p} + \|x\|^{p/p'} \right) - \left( \|y\|^{p/p'} + \|x\|^p \right) = 0
\]
Next it can be shown that \( F \) is hemicontinuous. By the construction, \( F(x + ty) \) is bounded as \( t \to 0 \). Let \( t \to 0 \) be a subsequence such that \( F(x + ty) \to \xi \) weak *
Then we ask: Does \( \xi \) do what it needs to do in order to be \( F(x) \)? The answer is yes. First of all \( \|F(x + ty)\| = \|x + ty\|^{p-1} \to \|x\|^{p-1} \). The set
\[
\left\{ x^* : \|x^*\| \leq \|x\|^{p-1} + \varepsilon \right\}
\]
is closed and convex and so it is weak * closed as well. For all small enough \( t \), it follows \( F(x + ty) \) is in this set. Therefore, the weak limit is also in this set and it follows \( \|\xi\| \leq \|x\|^{p-1} + \varepsilon \). Since \( \varepsilon \) is arbitrary, it follows \( \|\xi\| \leq \|x\|^{p-1} \). Is \( \xi(x) = \|x\|^p \)? We have
\[
\|x\|^p = \lim_{t \to 0} \|x + ty\|^p = \lim_{t \to 0} \langle F(x + ty), x + ty \rangle = \lim_{t \to 0} \langle F(x + ty), x \rangle = \langle \xi, x \rangle
\]
and so, \( \xi \) does what it needs to do to be \( F(x) \). This would be clear if \( \|\xi\| = \|x\|^{p-1} \). However, \( |\langle \xi, x \rangle| = \|x\|^p \) and so \( \|\xi\| \geq \left| \langle \xi, \frac{x}{\|x\|} \rangle \right| = \|x\|^{p-1} \). Thus \( \|\xi\| = \|x\|^{p-1} \)
which shows \( \xi \) does everything it needs to do to equal \( F(x) \) and so it is \( F(x) \) weakly. Since this conclusion follows for any convergent sequence, it follows that \( F(x + ty) \) converges to \( F(x) \) weakly as \( t \to 0 \). This is what it means to be hemicontinuous. This proves the following theorem. One can show also that \( F \) is demicontinuous which means strongly convergent sequences go to weakly convergent sequences. Here is a proof for the case where \( p = 2 \). You can clearly do the same thing for arbitrary \( p \).

**Lemma 23.2.2** Let \( F \) be a duality map for \( p = 2 \) where \( X, X' \) are reflexive and have strictly convex norms. (If \( X \) is reflexive, there is always an equivalent strictly convex norm \( [72] \).) Then \( F \) is demicontinuous.
23.2. DUALITY MAPS

Proof: Say \( x_n \to x \). Then does it follow that \( Fx_n \to Fx \)? Suppose not. Then there is a subsequence, still denoted as \( x_n \) such that \( x_n \to x \) but \( Fx_n \to y \neq Fx \) where here \( \to \) denotes weak convergence. This follows from the Eberlein Smulian theorem. Then

\[
\langle y, x \rangle = \lim_{n \to \infty} \langle Fx_n, x_n \rangle = \lim_{n \to \infty} \| x_n \|^2 = \| x \|^2
\]

Also, there exists \( z, \| z \| = 1 \) and \( \langle y, z \rangle \geq \| y \| - \varepsilon \). Then

\[
\| y \| - \varepsilon \leq \langle y, z \rangle = \lim_{n \to \infty} \langle Fx_n, z \rangle \leq \liminf_{n \to \infty} \| Fx_n \| = \liminf_{n \to \infty} \| x_n \| = \| x \|
\]

and since \( \varepsilon \) is arbitrary, \( \| y \| \leq \| x \| \). It follows from the above construction of \( Fx \), that \( y = Fx \) after all, a contradiction.

Theorem 23.2.3 Let \( X \) be a reflexive Banach space with \( X' \) having strictly convex norm. Then for \( p > 1 \), there exists a mapping \( F : X \to X' \) which is bounded, monotone, hemicontinuous, coercive in the sense that \( \lim_{|x| \to \infty} \langle Fx, x \rangle / |x| = \infty \), which also satisfies the inequalities

\[
|\langle Fx, y \rangle| \leq |\langle Fx, x \rangle|^{1/p'} |\langle Fy, y \rangle|^{1/p}
\]

Note that these conclusions about duality maps show that they map onto the dual space.

The duality map was onto and it was monotone. This was shown above. Consider the form of a duality map for the \( L^p \) spaces. Let \( F : L^p \to (L^p)' \) be the one which satisfies

\[
\| Ff \| = \| f \|^{p-1}, \quad \langle Ff, f \rangle = \| f \|^p
\]

Then in this case,

\[
Ff = |f|^{p-2} \overline{f}
\]

This is because it does what it needs to do.

\[
\| Ff \|_{L^{p'}} = \left( \int_{\Omega} \left( |f|^{p-1} \right)^{p'} d\mu \right)^{1/p'} = \left( \int_{\Omega} \left( |f|^{p/p'} \right)^{p'} d\mu \right)^{1/p'} = \left( \int_{\Omega} |f|^{p} d\mu \right)^{1-(1/p)} = \left( \int_{\Omega} |f|^{p} d\mu \right)^{1/p} = \| f \|_{L^p}^{p-1}
\]

while it is obvious that

\[
\langle Ff, f \rangle = \int_{\Omega} |f|^{p} d\mu = \| f \|_{L^p(\Omega)}^{p}.
\]

Now here is an interesting inequality which I will only consider in the case where the quantities are real valued.

\footnote{It is known that if the space is reflexive, then there is an equivalent norm which is strictly convex. However, in most examples, this strict convexity is obvious.}
Lemma 23.2.4 Let \( p \geq 2 \). Then for \( a, b \) real numbers,

\[
\left( |a|^{p-2} a - |b|^{p-2} b \right) (a - b) \geq C \abs{a - b}^p
\]

for some constant \( C \) independent of \( a, b \).

Proof: There is nothing to show if \( a = b \). Without loss of generality, assume \( a > b \). Also assume \( p > 2 \). There is nothing to show if \( p = 2 \). I want to show that there exists a constant \( C \) such that for \( a > b \),

\[
\frac{|a|^{p-2} a - |b|^{p-2} b}{|a - b|^{p-1}} \geq C
\]

(23.2.9)

First assume also that \( b \geq 0 \). Now it is clear that as \( a \to \infty \), the quotient above converges to 1. Take the derivative of this quotient. This yields

\[
(p - 1) |a - b|^{p-2} \frac{|a|^{p-2} (a - b) - \left( |a|^{p-2} a - |b|^{p-2} b \right)}{|a - b|^{2p-2}}
\]

Now remember \( a > b \). Then the above reduces to

\[
(p - 1) |a - b|^{p-2} b |b|^{p-2} - |a|^{p-2} \frac{|a|^{p-2} a - |b|^{p-2} b}{|a - b|^{2p-2}}
\]

Since \( b \geq 0 \), this is negative and so 1 would be a lower bound. Now suppose \( b < 0 \). Then the above derivative is negative for \( b < a \leq -b \) and then it is positive for \( a > -b \). It equals 0 when \( a = -b \). Therefore the quotient in (23.2.9) achieves its minimum value when \( a = -b \). This value is

\[
\frac{|-b|^{p-2} (-b) - |b|^{p-2} b}{|-b - b|^{p-1}} = \frac{|b|^{p-2} - 2b}{|2b|^{p-1}} = \frac{|b|^{p-2}}{|2b|^{p-2}} = \frac{1}{2^{p-2}}.
\]

Therefore, the conclusion holds whenever \( p \geq 2 \). That is

\[
\left( |a|^{p-2} a - |b|^{p-2} b \right) (a - b) \geq \frac{1}{2^{p-2}} \abs{a - b}^p
\]

This proves the lemma.

However, in the context of strictly convex norms on the reflexive Banach space \( X \), the following important result holds. I will give it first for the case where \( p = 2 \) since this is the case of most interest.

Theorem 23.2.5 Let \( X \) be a reflexive Banach space and \( X, X' \) have strictly convex norms as discussed above. Let \( F \) be the duality map with \( p = 2 \). Then \( F \) is strictly monotone. This means

\[
\langle Fu - Fv, u - v \rangle \geq 0
\]

and it equals 0 if and only if \( u = v \).
Proof: First why is it monotone? By definition of $F$, $\langle F(u), u \rangle = ||u||^2$ and $||F(u)|| = ||u||$. Then

$$\langle Fu, v \rangle = \left( \frac{v}{||v||} \right) ||v|| \leq ||F|| \cdot ||v|| = ||u|| \cdot ||v||$$

Hence

$$\langle Fu - Fv, u - v \rangle \geq ||u||^2 + ||v||^2 - \langle Fu, v \rangle - \langle Fv, u \rangle \geq 0$$

Now suppose $||x|| = ||y|| = 1$ but $x \neq y$. Then

$$\left\langle Fx, \frac{x + y}{2} \right\rangle \leq \left\| \frac{x + y}{2} \right\| < \frac{||x|| + ||y||}{2} = 1$$

It follows that

$$\left\{ \frac{1}{2} \langle Fx, x \rangle + \frac{1}{2} \langle Fx, y \rangle = \frac{1}{2} + \frac{1}{2} \langle Fx, y \rangle < 1 \right. \right\}$$

and so

$$\langle Fx, y \rangle < 1$$

For arbitrary $x, y$, $x/||x|| \neq y/||y||$

$$\langle Fx, y \rangle = ||x|| \cdot ||y|| \left\langle F \left( \frac{x}{||x||} \right), \left( \frac{y}{||y||} \right) \right\rangle$$

It is easy to check that $F(\alpha x) = \alpha F(x)$. Therefore,

$$||\langle Fx, y \rangle|| = ||x|| \cdot ||y|| \left\langle F \left( \frac{x}{||x||} \right), \left( \frac{y}{||y||} \right) \right\rangle < ||x|| \cdot ||y||$$

Now say that $x \neq y$ and consider

$$\langle Fx - Fy, x - y \rangle$$

First suppose $x = \alpha y$. Then the above is

$$\langle F(\alpha y) - Fy, (\alpha - 1) y \rangle = (\alpha - 1) \left\langle (F(\alpha y), y) - ||y||^2 \right\rangle$$

$$= (\alpha - 1) \left\langle (\alpha F(y), y) - ||y||^2 \right\rangle$$

$$= (\alpha - 1)^2 ||y||^2 > 0$$

The other case is that $x/||x|| \neq y/||y||$ and in this case,

$$\langle Fx - Fy, x - y \rangle = ||x||^2 + ||y||^2 - \langle Fx, y \rangle - \langle Fy, x \rangle$$

$$> ||x||^2 + ||y||^2 - 2 \cdot ||x|| \cdot ||y|| \geq 0$$

Thus $F$ is strictly monotone as claimed. ■

As mentioned, this will hold for any $p > 1$. Here is a proof in the case that the Banach space is real which is the usual case of interest. First here is a simple observation.
**Observation 23.2.6** Let \( p > 1 \). Then \( x \rightarrow |x|^{p-2} x \) is strictly monotone. Here \( x \in \mathbb{R} \).

To verify this observation,

\[
\frac{d}{dx} \left( (x^2)^{\frac{p-2}{2}} x \right) = \frac{1}{x^2} (p-1) (x^2)^{\frac{1}{2}p} > 0
\]

**Theorem 23.2.7** Let \( X \) be a real reflexive Banach space and \( X, X' \) have strictly convex norms as discussed above. Let \( F \) be the duality map for \( p > 1 \). Then \( F \) is strictly monotone. This means

\[
\langle Fu - Fv, u - v \rangle \geq 0
\]

and it equals 0 if and only if \( u - v \).

**Proof:** First why is it monotone? By definition of \( F \),

\[
\langle Fu, \frac{v}{\|v\|} \rangle \leq \|Fu\| \|v\| = \|u\|^{p-1} \|v\|
\]

Hence

\[
\langle Fu - Fv, u - v \rangle = \|u\|^p + \|v\|^p - \langle Fu, v \rangle - \langle Fv, u \rangle 
\geq \|u\|^p + \|v\|^p - \|u\|^{p-1} \|v\| - \|u\| \|v\|^{p-1} 
\geq \|u\|^p + \|v\|^p - \left( \frac{\|u\|^p}{p} + \frac{\|v\|^p}{p} \right) - \left( \frac{\|u\|^p}{p} + \frac{\|v\|^p}{p} \right) = 0
\]

Now suppose \( \|x\| = \|y\| = 1 \) but \( x \neq y \). Then

\[
\left\langle Fx, \frac{x + y}{2} \right\rangle \leq \|x\|^{p-1} \left\| \frac{x + y}{2} \right\| < \frac{\|x\| + \|y\|}{2} = 1
\]

It follows that

\[
\frac{1}{2} \langle Fx, x \rangle + \frac{1}{2} \langle Fx, y \rangle = \frac{1}{2} + \frac{1}{2} \langle Fx, y \rangle < 1
\]

and so

\[
\langle Fx, y \rangle < 1
\]

It is easy to check that for nonzero \( \alpha \), \( F(\alpha x) = |\alpha|^{p-2} \alpha F(x) \). This is because

\[
\|\alpha|^{p-2} \alpha F(x)\| = |\alpha|^{p-1} \|x\|^{p-1} = \|\alpha x\|^{p-1} 
\]

\[
\langle |\alpha|^{p-2} \alpha F(x), \alpha x \rangle = |\alpha|^p \|x\|^p = \|\alpha x\|^p
\]
23.2. DUALITY MAPS

and so, since $|α|^{p-2} αF(x)$ acts like $F(αx)$, it is $F(αx)$. It follows that for arbitrary $x, y$, such that $x/∥x∥ ≠ y/∥y∥$

$$\langle Fx, y \rangle = ∥x∥^{p-1} ∥y∥ \left( F\left( \frac{x}{∥x∥} \right), \left( \frac{y}{∥y∥} \right) \right)$$

Therefore,

$$\langle Fx, y \rangle = ∥x∥^{p-1} ∥y∥ \left( F\left( \frac{x}{∥x∥} \right), \left( \frac{y}{∥y∥} \right) \right) < ∥x∥^{p-1} ∥y∥ \quad (23.2.10)$$

Now say that $x ≠ y$ and consider

$$\langle Fx - Fy, x - y \rangle$$

First suppose $x = αy$. This is the case where $x$ is a multiple of $y$. Then the above is

$$\langle F(αy) - Fy, (α - 1) y \rangle = (α - 1) (∥F(αy)∥ - ∥y∥^p)$$

$$= (α - 1) (|α|^{p-2} α ∥y∥^p - ∥y∥^p) = (α - 1) (|α|^{p-2} α - 1) ∥y∥^p > 0$$

by the above observation that $x → |x|^{p-2} x$ is strictly monotone. Similarly, $⟨Fx - Fy, x - y⟩ > 0$ if $y = αx$ for $α ≠ 1$.

Thus the desired result holds in the case that one vector is a multiple of the other. The other case is that neither vector is a multiple of the other. Thus, in particular, $x/∥x∥ ≠ y/∥y∥$, and in this case, it follows from 23.2.10

$$\langle Fx - Fy, x - y \rangle = ∥x∥^p + ∥y∥^p - ⟨Fx, y⟩ - ⟨Fy, x⟩$$

$$> ∥x∥^p + ∥y∥^p - ∥x∥^{p-1} ∥y∥ - ∥y∥^{p-1} ∥x∥$$

$$≥ ∥x∥^p + ∥y∥^p - \left( ∥x∥^p - ∥y∥^p \right) - \left( ∥y∥^p - ∥x∥^p \right) = 0$$

Thus $F$ is strictly monotone as claimed.

Another useful observation about duality maps for $p = 2$ is that $∥F^{-1}y^*∥_{V'} = ∥y^*∥_{V'}$. This is because

$$∥y^*∥_{V'} = ∥FF^{-1}y^*∥_{V'} = ∥F^{-1}y^*∥_{V'}$$

also from similar reasoning,

$$\langle y^*, F^{-1}y^* \rangle = ∥FF^{-1}y^*, F^{-1}y^*∥ = ∥F^{-1}y^*∥_{V'}^2 = ∥y^*∥_{V'}^2$$

You can give specific inequalities in certain cases. Here is a nice little inequality which will allow this.

**Theorem 23.2.8** Let $p ≥ 2$ then for $x, y ∈ \mathbb{R}^n$,

$$\left( |x|^{p-2} x - |y|^{p-2} y, x - y \right) ≥ \frac{1}{2^{p-1}} |x - y|^p \quad (*)$$
Proof: We have \((x, y) = \frac{1}{2} \left( |x|^2 + |y|^2 - |x - y|^2 \right)\). Consider the following.

\[
\frac{1}{2} \left( \frac{|x|^{p-2} + |y|^{p-2}}{|x|^{p-2} - |y|^{p-2}} \right) |x - y|^p + \frac{1}{2} \left( |x|^{p-2} - |y|^{p-2} \right) \left( |x|^2 - |y|^2 \right)
\]

multiplying this out gives

\[
\frac{1}{2} \left( |x|^{p-2} + |y|^{p-2} \right) \left( |x|^2 + |y|^2 - 2 (x, y) \right) + \frac{1}{2} \left( |x|^p - |x|^2 |y|^{p-2} + |y|^p - |x|^{p-2} |y|^2 \right)
\]

thus this yields

\[
\frac{1}{2} \left( |x|^2 + |y|^2 \right) + \frac{1}{2} \left( |x|^p + |y|^p - \left( |x|^2 |y|^{p-2} + |x|^{p-2} |y|^2 \right) \right)
\]

It simplifies to

\[
|x|^p + |y|^p - 2 (x, y) \left( |x|^{p-2} + |y|^{p-2} \right)
\]

On the left side of *, when you multiply it out, you get

\[
|x|^p - |x|^{p-2} (x, y) - |y|^{p-2} (x, y) + |y|^p
\]

which is exactly the same thing. Therefore,

\[
\left( |x|^{p-2} x - |y|^{p-2} y, x - y \right) = \frac{1}{2} \left( \frac{|x|^{p-2} + |y|^{p-2}}{|x|^{p-2} - |y|^{p-2}} \right) |x - y|^p \quad (**)
\]

\[
\frac{1}{2} \left( |x|^{p-2} - |y|^{p-2} \right) \left( |x|^2 - |y|^2 \right) \geq 0
\]

Suppose first that \(p \geq 3\). Now \(p \geq 3\) and so \(|x|^{p-2}\) is convex. Hence

\[
\left| \frac{x + (-y)}{2} \right|^{p-2} \leq \frac{1}{2} \left( |x|^{p-2} + |y|^{p-2} \right)
\]

and so

\[
\left( |x|^{p-2} x - |y|^{p-2} y, x - y \right) \geq \left| \frac{x - y}{2} \right|^{p-2} \frac{1}{|x - y|^{p-2}} \left| x - y \right|^p = \frac{1}{2^{p-2}} |x - y|^p
\]

Next suppose \(p > 2\). There is nothing to show if \(p = 2\). Then for a positive integer \(m\), you can get \(m (p - 2) > 1\). Then

\[
\left( |x|^{p-2} + |y|^{p-2} \right)^m \geq |x|^{m(p-2)} + |y|^{m(p-2)} \geq 2^{1-m(p-2)} |x - y|^{m(p-2)}
\]
Thus we can raise both sides of the above to \(1/m\) and conclude
\[
|x|^{p-2} + |y|^{p-2} \geq 2^{1/(p-2)} |x - y|^{p-2}
\]
Then we use this in ** to obtain
\[
\left( |x|^{p-2} x - |y|^{p-2} y, x - y \right) \geq \frac{1}{2} \left( |x|^{p-2} + |y|^{p-2} \right) |x - y|^p
\]
\[
\geq \frac{1}{2} \frac{1}{2^{(p-2)-(1/m)}} |x - y|^p \geq \frac{1}{2^{p-1}} |x - y|^p \quad \blacksquare
\]
Thus, if you have the duality map \(F\) for \(p \geq 2\) for real valued \(L^p(\Omega)\) to \(L^{p'}(\Omega)\), it is clear that \(Ff = |f|^{p-2} f\) and so
\[
\langle Ff - Fg, f - g \rangle = \int_\Omega \left( |f|^{p-2} f - |g|^{p-2} g \right) (f - g) \, d\mu \geq \frac{1}{2^{p-1}} \int_\Omega |f - g|^p \, d\mu
\]
\[
\langle Ff - Fg, f - g \rangle \geq \frac{1}{2^{p-1}} \|f - g\|_{L^p(\Omega)}^p
\]
A similar result would hold for the duality map from \((L^p(\Omega))^n\) to \(\left(L^{p'}(\Omega)\right)^n\).

### 23.3 Penalizaton And Projection Operators

In this section, \(X\) will be a reflexive Banach space such that \(X, X'\) has a strictly convex norm. Let \(K\) be a closed convex set in \(X\). Then the following lemma is obtained.

**Lemma 23.3.1** Let \(K\) be closed and convex nonempty subset of \(X\) a reflexive Banach space which has strictly convex norm. Then there exists a projection map \(P\) such that \(Px \in K\) and for all \(y \in K\),
\[
\|y - x\| \geq \|x - Px\|
\]

**Proof:** Let \(\{y_n\}\) be a minimizing sequence for \(y \to \|y - x\|\) for \(y \in K\). Thus
\[
d \equiv \inf \{\|y - x\| : y \in K\} = \lim_{n \to \infty} \|y_n - x\|
\]
Then obviously \(\{y_n\}\) is bounded. Hence there is a subsequence, still denoted by \(n\) such that \(y_n \to w \in K\). Then
\[
\|w - x\| \leq \liminf_{n \to \infty} \|y_n - x\| = d
\]
How many closest points to \(x\) are there? Suppose \(w_1\) is another one. Then
\[
\frac{\|w_1 + w - x\|}{2} = \frac{\|w_1 - x + w - x\|}{2} < \frac{\|w_1 - x\|}{2} + \frac{\|w - x\|}{2} = d
\]
contradicting the assumption that both \( w, w_1 \) are closest points to \( x \). Therefore, \( Px \) consists of a single point. □

Denote by \( F \) the duality map such that \( \langle Fx, x \rangle = \|x\|^2 \). This is described earlier but there is also a very nice treatment which is somewhat different in [12]. Everything can be generalized and is in [82] but here I will only consider this case. First here is a useful result.

**Proposition 23.3.2** Let \( F \) be the duality map just described. Let \( \phi(x) \equiv \|x\|^2 \). Then \( F(x) = \partial \phi(x) \).

**Proof:** This follows from
\[
\langle Fx, y - x \rangle \leq \langle Fx, y \rangle - \langle Fx, x \rangle \leq \langle Fx, x \rangle^{1/2} \langle Fy, x \rangle^{1/2} - \langle Fx, x \rangle^{1/2} = \frac{\|y\|^2 - \|x\|^2}{2} \leq \frac{\|y\|^2 - \|x\|^2}{2}.
\]

Next is a really nice result about the characterization of \( Px \) in terms of \( F \).

**Proposition 23.3.3** Let \( K \) be a nonempty closed convex set in \( X \) a reflexive Banach space in which both \( X, X' \) have strictly convex norms. Then \( w \in K \) is equal to \( Px \) if and only if \( \langle F(x - w), y - w \rangle \leq 0 \) for every \( y \in K \).

**Proof:** First suppose the condition. Then for \( y \in K \), it follows from the above proposition about the subgradient,
\[
\frac{1}{2} \|x - y\|^2 - \frac{1}{2} \|x - w\|^2 \geq \langle F(x - w), w - y \rangle \geq 0
\]
and so since this holds for all \( y \) it follows that
\[
\|x - y\| \geq \|x - w\|
\]
for all \( y \) which says that \( w = Px \).

Next, using the subgradient idea again, for \( \theta \in [0, 1] \), suppose \( w = Px \) then for \( y \in K \) arbitrary,
\[
0 \geq \frac{1}{2} \|x - w\|^2 - \frac{1}{2} \|x - (w + \theta(y - w))\|^2 \geq \langle F(x - (w + \theta(y - w))), \theta(y - x) \rangle
\]
Now divide by \( \theta \) and let \( \theta \downarrow 0 \) and use the hemicontinuity of \( F \) given above. Then
\[
0 \geq \langle F(x - w), y - x \rangle \quad \Box
\]

**Definition 23.3.4** An operator of penalization is an operator \( f : X \to X' \) such that \( f = 0 \) on \( K \), \( f \) is monotone and nonzero off \( K \) as well as demicontinuous. (Strong convergence goes to weak convergence.) Actually, in applications, it is usually easy to give an ad hoc description of an appropriate penalization operator.
Proposition 23.3.5 Let $K$ be a closed convex nonempty subset of $X$ a reflexive Banach space such that $X, X'$ have strictly convex norms. Then

$$ f(x) \equiv F(x - Px) $$

is an operator of penalization. Here $P$ is the projection onto $K$. This operator of penalization is demicontinuous.

**Proof:** First, observe that $f(x)$ is 0 on $K$ and nonzero off $K$. Why is it monotone?

$$ \langle F(x - Px) - F(x_1 - Px_1), x - x_1 \rangle $$

The first term is $\geq 0$ because $F$ is monotone. As to the second, it equals

$$ \langle F(x - Px) - F(x_1 - Px_1), x - Px - Px_1 \rangle $$

and both of these are $\geq 0$ because of Proposition 23.3.3 which characterizes the projection map.

Now why is this hemicontinuous? Let $x_n \to x$. Then $Px_n$ is clearly bounded. Taking a subsequence, it can be assumed that $Px_n \to \xi$ weakly. Is $\xi = Px$?

$$ \|x - Px\| \leq \|x - Px_n\| \leq \|x - x_n\| + \|x_n - Px_n\| $$

$$ \|x_n - Px_n\| \leq \|x_n - Px\| \leq \|x_n - x\| + \|x - Px\| $$

It follows that

$$ \|x - Px\| - \|x_n - Px_n\| \leq \|x - x_n\| $$

$$ \|x_n - Px_n\| - \|x - Px\| \leq \|x - x_n\| $$

Hence $\|x_n - Px_n\| \to \|x - Px\|$. However, from convexity and strong lower semicontinuity implying weak lower semicontinuity,

$$ \|x - \xi\| \leq \lim_{n \to \infty} \inf \|x_n - Px_n\| = \|x - Px\| $$

and so $\xi = Px$ because there is only one value in $Px$. This has shown that, thanks to uniqueness of $Px$, $x_n \to x$ implies $Px_n \to Px$ weakly.

Next we show that $f$ is demicontinuous. Suppose $x_n \to x$. Then from what was just shown, $Px_n \to Px$ weakly. Thus $x_n - Px_n \to x - Px$ weakly. Then

$$ \lim_{n \to \infty} \sup \langle F(x_n - Px_n), x_n - Px_n - (x - Px) \rangle $$

$$ = \lim_{n \to \infty} \sup \langle F(x_n - Px_n), Px - Px_n \rangle \leq 0 $$
from Proposition 23.3.3 which characterizes the projection map. It follows that, since $F$ is monotone hemicontinuous and bounded, it is also pseudomonotone and so for all $v$

$$
\lim_{n \to \infty} \langle F(x_n - Px_n), (x_n - Px_n) - v \rangle \\
\geq \langle F(x - Px), (x - Px) - v \rangle
$$

Now $F(x_n - Px_n)$ is bounded. If it converges to $\xi$, then

$$
\lim_{n \to \infty} \langle F(x_n - Px_n), (x_n - Px_n) - v \rangle \\
\leq \limsup_{n \to \infty} \left[ \langle F(x_n - Px_n), (x_n - Px_n) - (x - Px) \rangle + \langle F(x_n - Px_n), (x - Px) - v \rangle \right] \\
\leq \langle \xi, (x - Px) - v \rangle
$$

It follows that

$$
\langle \xi, (x - Px) - v \rangle \geq \lim_{n \to \infty} \inf \langle F(x_n - Px_n), (x_n - Px_n) - v \rangle \\
\geq \langle F(x - Px), (x - Px) - v \rangle
$$

Since $v$ is arbitrary, it follows that $\xi = F(x - Px)$. Hence $F(x_n - Px_n) \to F(x - Px)$ weakly. Thus this is demicontinuous.

### 23.4 Set-Valued Maps, Pseudomonotone Operators

In the abstract theory of partial differential equations and variational inequalities, it is important to consider set-valued maps from a Banach space to the power set of its dual. In this section we give an introduction to this theory by proving a general result on surjectivity for a class of such operators.

To begin with, if $A : X \to P(Y)$ is a set-valued map, define the graph of $A$ by

$$
G(A) \equiv \{(x, y) : y \in Ax\}.
$$

First consider a map $A$ which maps $C^n$ to $P(C^n)$ which satisfies

$$
Ax \text{ is compact and convex.} \quad (23.4.11)
$$

and also the condition that if $O$ is open and $O \supseteq Ax$, then there exists $\delta > 0$ such that if

$$
y \in B(x, \delta), \text{ then } Ay \subseteq O. \quad (23.4.12)
$$

This last condition is sometimes referred to as upper semicontinuity. In words, $A$ is upper semicontinuous and has values which are compact and convex. As to the last condition of upper semi continuity, here is the formal definition.
Definition 23.4.1 Let \( F : X \to \mathcal{P}(Y) \) be a set valued function. Then \( F \) is upper semicontinuous at \( x \) if for every open set \( V \supseteq F(x) \) there exists an open set \( U \) containing \( x \) such that whenever \( \hat{x} \in U \), it follows that \( F(\hat{x}) \subseteq V \).

Lemma 23.4.2 Let \( A \) satisfy Definition 23.4.1. Then \( AK \) is a subset of a compact set whenever \( K \) is compact. Also the graph of \( A \) is closed if \( Ax \) is closed.

Proof: Let \( x \in K \). Then \( Ax \) is compact and contained in some open set whose closure is compact, \( U_x \). By assumption, there exists an open set \( V_x \) containing \( x \) such that if \( y \in V_x \), then \( Ay \subseteq U_x \). Let \( V_{x_1}, \ldots, V_{x_m} \) cover \( K \). Then \( AK \subseteq \bigcup_{i=1}^{m} V_{x_i} \), a compact set.

To see the graph of \( A \) is closed when \( Ax \) is closed, let \( x_k \to x, y_k \to y \) where \( y_k \in Ax_k \). Then letting \( O = Ax + B(0, r) \) it follows from Lemma 23.4.2 that \( y_k \in Ax_k \subseteq O \) for all \( k \) large enough. Therefore, \( y \in Ax + B(0, 2r) \) and since \( r \) is arbitrary and \( Ax \) is closed it follows \( y \in Ax \). \( \blacksquare \)

Also, there is a general consideration relative to upper semicontinuous functions.

Lemma 23.4.3 If \( f \) is upper semicontinuous on some set \( K \) and \( g \) is continuous and defined on \( f(K) \), then \( g \circ f \) is also upper semicontinuous.

Proof: Let \( x_n \to x \) in \( K \). Let \( U \supseteq g \circ f(x) \). Is \( g \circ f(x_n) \subseteq U \) for all \( n \) large enough? We have \( f(x) \in g^{-1}(U) \), an open set. Therefore, if \( n \) is large enough, \( f(x_n) \in g^{-1}(U) \). It follows that for large enough \( n \), \( g \circ f(x_n) \subseteq U \) and so \( g \circ f \) is upper semicontinuous on \( K \). \( \blacksquare \)

The next theorem is an application of the Brouwer fixed point theorem. First define an \( n \) simplex, denoted by \([x_0, \ldots, x_n]\), to be the convex hull of the \( n+1 \) points, \( \{x_0, \ldots, x_n\} \) where \( \{x_i - x_0\}_{i=1}^{n} \) are independent. Thus

\[
[x_0, \ldots, x_n] = \left\{ \sum_{i=0}^{n} t_i x_i : \sum_{i=0}^{n} t_i = 1, \ t_i \geq 0 \right\}.
\]

Since \( \{x_i - x_0\}_{i=1}^{n} \) is independent, the \( t_i \) are uniquely determined. If two of them are

\[
\sum_{i=0}^{n} t_i x_i = \sum_{i=0}^{n} s_i x_i
\]

Then

\[
\sum_{i=0}^{n} t_i (x_i - x_0) = \sum_{i=0}^{n} s_i (x_i - x_0)
\]

so \( t_i = s_i \) for \( i \geq 1 \). Since the \( s_i \) and \( t_i \) sum to 1, it follows that also \( s_0 = t_0 \).

If \( n \leq 2 \), the simplex is a triangle, line segment, or point. If \( n \leq 3 \), it is a tetrahedron, triangle, line segment or point. To say that \( \{x_i - x_0\}_{i=1}^{n} \) are independent is to say that \( \{x_i - x_r\}_{i \neq r} \) are independent for each fixed \( r \). Indeed, if \( x_i - x_r = \sum_{j \neq i, r} c_j (x_j - x_r) \), then you would have

\[
x_i - x_0 + x_0 - x_r = \sum_{j \neq i, r} c_j (x_j - x_0) + \left( \sum_{j \neq i, r} c_j \right) x_0
\]
and it follows that $\mathbf{x}_i - \mathbf{x}_0$ is a linear combination of the $\mathbf{x}_j - \mathbf{x}_0$ for $j \neq i$, contrary to assumption. A collection of simplices is a tiling of $\mathbb{R}^n$ if $\mathbb{R}^n$ is contained in their union and if $S_1, S_2$ are two simplices in the tiling, with

$$S_j = \left[ \mathbf{x}_0, \cdots, \mathbf{x}_n \right],$$

then

$$S_1 \cap S_2 = \left[ \mathbf{x}_{k_0}, \cdots, \mathbf{x}_{k_r} \right]$$

where

$$\{ \mathbf{x}_{k_0}, \cdots, \mathbf{x}_{k_r} \} \subseteq \{ \mathbf{x}_0, \cdots, \mathbf{x}_n \} \cap \{ \mathbf{x}_0^2, \cdots, \mathbf{x}_n^2 \}$$

or else the two simplices do not intersect. The collection of simplices is said to be locally finite if, for every point, there exists a ball containing that point which also intersects only finitely many of the simplices in the collection. It is left to the reader to verify that for each $\varepsilon > 0$, there exists a locally finite tiling of $\mathbb{R}^n$ which is composed of simplices which have diameters less than $\varepsilon$. The local finiteness ensures that for each $\varepsilon$ the vertices have no limit point. To see how to do this, consider the case of $\mathbb{R}^2$. Tile the plane with identical small squares and then form the triangles indicated in the following picture. It is clear something similar can be done in any dimension. Making the squares identical ensures that the little triangles are locally finite.

In general, you could consider $[0,1]^n$. The point at the center is $(1/2, \cdots, 1/2)$. Then there are $2n$ faces. Form the $2n$ pyramids having this point along with the $2^{n-1}$ vertices of the face. Then use induction on each of these faces to form smaller dimensional simplices tiling that face. Corresponding to each of these $2n$ pyramids, it is the union of the simplices whose vertices consist of the center point along with those of these new simplices tiling the chosen face. In general, you can write any $n$ dimensional cube as the translate of a scaled $[0,1]^n$. Thus one can express each of identical cubes as a tiling of $m\,(n)$ simplices of the appropriate size and thereby obtain a tiling of $\mathbb{R}^n$ with simplices. A ball will intersect only finitely many of the cubes and hence finitely many of the simplices. To get their diameters small as desired, just use $[0,r]^n$ instead of $[0,1]^n$.

Thus one can give a function any value desired on these vertices and extend appropriately to the rest of the simplex and obtain a continuous function.

The Kakutani fixed point theorem is a generalization of the Brouwer fixed point theorem from continuous single valued maps to upper semicontinuous maps which have closed convex values.
Theorem 23.4.4 Let $K$ be a compact convex subset of $\mathbb{R}^n$ and let $A : K \to \mathcal{P}(K)$ such that $Ax$ is a closed convex subset of $K$ and $A$ is upper semicontinuous. Then there exists $x$ such that $x \in Ax$. This is the "fixed point".

Proof: Let there be a locally finite tiling of $\mathbb{R}^n$ consisting of simplices having diameter no more than $\varepsilon$. Let $Px$ be the point in $K$ which is closest to $x$. For each vertex $x_k$, pick $A_\varepsilon x_k \in APx_k$ and define $A_\varepsilon$ on all of $\mathbb{R}^n$ by the following rule. If $x = \sum_{i=0}^n t_i x_i$, $t_i \in [0, 1]$, $\sum_i t_i = 1$, then

$$A_\varepsilon x \equiv \sum_{k=0}^n t_k A_\varepsilon x_k.$$  

Now by construction $A_\varepsilon x_k \in APx_k \in K$ and so $A_\varepsilon$ is a continuous map defined on $\mathbb{R}^n$ with values in $K$ thanks to the local finiteness of the collection of simplices. By the Brouwer fixed point theorem $A_\varepsilon$ has a fixed point $x_\varepsilon$ in $K$, $A_\varepsilon x_\varepsilon = x_\varepsilon$.

$x_\varepsilon = \sum_{k=0}^n t_\varepsilon^k A_\varepsilon x_\varepsilon^k$, $A_\varepsilon x_\varepsilon^k \in APx_\varepsilon^k \in K$

where a simplex containing $x_\varepsilon$ is

$$[x_0^\varepsilon, \ldots, x_n^\varepsilon]$$

Also, $x_\varepsilon \in K$ and is closer than $\varepsilon$ to each $x_k^\varepsilon$, so each $x_k^\varepsilon$ is within $\varepsilon$ of $K$. It follows that for each $k$, $|Px_k^\varepsilon - x_k^\varepsilon| < \varepsilon$ and so

$$\lim_{\varepsilon \to 0} |Px_k^\varepsilon - x_k^\varepsilon| = 0$$

By compactness of $K$, there exists a subsequence, still denoted with the subscript of $\varepsilon$ such that for each $k$, the following convergences hold as $\varepsilon \to 0$

$$t_\varepsilon^k \to t_k, A_\varepsilon x_\varepsilon^k \to y_k, Px_k^\varepsilon \to z_k, x_\varepsilon^k \to z_k$$

Any pair of the $x_k^\varepsilon$ are within $\varepsilon$ of each other. Hence, any pair of the $Px_k^\varepsilon$ are within $\varepsilon$ of each other because $P$ reduces distances. Therefore, in fact, $z_k$ does not depend on $k$.

$$\lim_{\varepsilon \to 0} Px_k^\varepsilon = \lim_{\varepsilon \to 0} x_k^\varepsilon = z, \lim_{\varepsilon \to 0} x_\varepsilon = \lim_{\varepsilon \to 0} \sum_{k=0}^n t_\varepsilon^k x_\varepsilon^k = \sum_{k=0}^n t_k z = z$$

By upper semicontinuity of $A$, for all $\varepsilon$ small enough,

$$APx_\varepsilon^k \subseteq Az + B(0, r)$$
In particular, since $A_\varepsilon x_k^* \in APx_k^*$,

$$A_\varepsilon x_k^* \in Az + B(0, r) \text{ for } \varepsilon \text{ small enough}$$

Since $r$ is arbitrary and $Az$ is closed, it follows

$$y_k \in Az.$$ 

It follows that since $K$ is closed,

$$x_\varepsilon \rightarrow z = \sum_{k=0}^{n} t_k y_k, \quad t_k \geq 0, \quad \sum_{k=0}^{n} t_k = 1$$

Now by convexity of $Az$ and the fact just shown that $y_k \in Az$,

$$z = \sum_{k=0}^{n} t_k y_k \in Az$$

and so $z \in Az$. This is the fixed point. $\blacksquare$

One can replace $\mathbb{R}^n$ with $\mathbb{C}^n$ in the above theorem because it is essentially $\mathbb{R}^{2n}$. Also the theorem holds with no change for any finite dimensional normed linear space since these are homeomorphic to $\mathbb{R}^n$ or $\mathbb{C}^n$.

**Lemma 23.4.5** Suppose $A: \mathbb{C}^n \rightarrow P(\mathbb{C}^n)$ satisfies $Ax$ is compact and convex, and $A$ is upper semicontinuous. 23.4.12 and $K$ is a nonempty compact convex set in $\mathbb{C}^n$. Then if $y \in \mathbb{C}^n$ there exists $[x, w] \in G(A)$ such that $x \in K$ and

$$\text{Re} (y - w, z - x) \leq 0$$

for all $z \in K$.

**Proof:** Tile $\mathbb{C}^n$ with $2n$ simplices such that the collection is locally finite and each simplex has diameter less than $\varepsilon < 1$. This collection of simplices is determined by a countable collection of vertices. For each vertex $x$, pick $A_\varepsilon x \in Ax$ and define $A_\varepsilon$ on all of $\mathbb{C}^n$ by the following rule. If

$$x \in [x_0, \ldots, x_{2n}],$$

so $x = \sum_{i=0}^{2n} t_i x_i$, then

$$A_\varepsilon x \equiv \sum_{k=0}^{2n} t_k A_\varepsilon x_k.$$ 

Thus $A_\varepsilon$ is a continuous map defined on $\mathbb{C}^n$ thanks to the local finiteness of the collection of simplices. Let $P_K$ denote the projection on the convex set $K$. By the Brouwer fixed point theorem, there exists a fixed point, $x_\varepsilon \in K$ such that

$$P_K (y - A_\varepsilon x_\varepsilon + x_\varepsilon) = x_\varepsilon.$$
By Corollary 17.1.9 this requires
\[
\text{Re} (y - Ax, z - x) \leq 0
\]
for all \( z \in K \).

Suppose \( x_\varepsilon \in [x_0^\varepsilon, \ldots, x_{2n}^\varepsilon] \) so \( x_\varepsilon = \sum_{k=0}^{2n} t_k^\varepsilon x_k^\varepsilon \). Then since \( x_\varepsilon \) is contained in \( K \), a compact set, and the diameter of each simplex is less than 1, it follows that \( A_\varepsilon x_k^\varepsilon \) is contained in \( A(K + B(0,1)) \), which is contained in a compact set thanks to Lemma 23.4.2. The reason is that \( A \) is assumed to take bounded sets to bounded sets and \( K + B(0,1) \) is a bounded set.

From the Heine Borel theorem, there exists a sequence \( \varepsilon \to 0 \) such that
\[
t_k^\varepsilon \to t_k, \ x_\varepsilon \to x \in K, A_\varepsilon x_k^\varepsilon \to y_k
\]
for \( k = 0, \ldots, 2n \). Since the diameter of the simplex containing \( x_\varepsilon \) converges to 0, it follows
\[
x_k^\varepsilon \to x, \ A_\varepsilon x_k^\varepsilon \to y_k.
\]
By upper semicontinuity, it follows that for all \( r > 0 \), \( A_\varepsilon x_k^\varepsilon \subseteq A_\varepsilon x_k^\varepsilon + B(0,r) \) for all \( \varepsilon \) small enough. Since \( A_\varepsilon x_k^\varepsilon \subseteq A x_k^\varepsilon \), and \( A_\varepsilon x \) is closed, this implies \( y_k \in Ax \). Since \( Ax \) is convex,
\[
\sum_{k=1}^{2n} t_k y_k \in Ax.
\]
Hence for all \( z \in K \),
\[
\text{Re} \left( y - \sum_{k=1}^{2n} t_k y_k, z - x \right) = \lim_{\varepsilon \to 0} \text{Re} \left( y - \sum_{k=1}^{2n} t_k^\varepsilon A_\varepsilon x_k^\varepsilon, z - x_\varepsilon \right)
\]
\[
= \lim_{\varepsilon \to 0} \text{Re} (y - Ax, z - x_\varepsilon) \leq 0.
\]
Let \( w = \sum_{k=1}^{2n} t_k y_k \).

You could replace \( A \) with \( A \circ P_K \) in the above and assume only that \( A \) is only defined on \( K \). This is because \( x \in K \).

**Lemma 23.4.6** Suppose in addition to 23.4.11 and 23.4.12, (compact convex valued and upper semicontinuous) \( A \) is coercive,
\[
\lim_{|x| \to \infty} \inf \left\{ \frac{\text{Re}(y,x)}{|x|} : y \in Ax \right\} = \infty.
\]
Then \( A \) is onto.

**Proof:** Let \( y \in \mathbb{C}^n \) and let \( K_r = \overline{B(0,r)} \). By Lemma 23.4.10 there exists \( x_r \in K_r \) and \( w_r \in Ax_r \) such that
\[
\text{Re} (y - w_r, z - x_r) \leq 0 \quad (23.4.13)
\]
for all \( z \in K_r \). Letting \( z = 0 \),

\[
\text{Re} (w_r, x_r) \leq \text{Re} (y, x_r).
\]

Therefore,

\[
\inf \left\{ \frac{\text{Re} (w, x_r)}{|x_r|} : w \in Ax_r \right\} \leq |y|.
\]

It follows from the assumption of coercivity that \( |x_r| \) is bounded independent of \( r \). Therefore, picking \( r \) strictly larger than this bound, (23.4.13) implies

\[
\text{Re} (y - w_r, v) \leq 0
\]

for all \( v \) in some open ball containing \( 0 \). Therefore, for all \( v \) in this ball

\[
\text{Re} (y - w_r, v) = 0
\]

and hence this holds for all \( v \in \mathbb{C}^n \) and so \( y = w_r \in Ax_r \). This proves the lemma.

**Lemma 23.4.7** Let \( F \) be a finite dimensional Banach space of dimension \( n \), and let \( T \) be a mapping from \( F \) to \( \mathcal{P}(F') \) such that (23.4.11) and (23.4.12) both hold for \( F' \) in place of \( \mathbb{C}^n \). Then if \( T \) is also coercive,

\[
\lim_{||u|| \to \infty} \inf \left\{ \frac{\text{Re} y^* (u)}{||u||} : y^* \in Tu \right\} = \infty,
\]

(23.4.14)

it follows \( T \) is onto.

**Proof:** Let \( || \cdot || \) be an equivalent norm for \( F \) such that there is an isometry of \( \mathbb{C}^n \) and \( F, \theta \). Now define \( A : \mathbb{C}^n \to \mathcal{P}(\mathbb{C}^n) \) by \( Ax = \theta^* T \theta x \).

\[
\begin{align*}
\mathcal{P}(F') & \xrightarrow{\theta^*} \mathbb{C}^n \\
T & \uparrow \circ \uparrow A \\
F & \xleftarrow{\theta} \mathbb{C}^n
\end{align*}
\]

Thus \( y \in \mathbb{C}^n \) means that there exists \( z^* \in T \theta x \) such that

\[
(w, y)_{\mathbb{C}^n} = z^* (\theta w)
\]

for all \( w \in \mathbb{C}^n \). Then \( A \) satisfies the conditions of Lemma 23.4.6 and so \( A \) is onto. Consequently \( T \) is also onto. 

With these lemmas, it is possible to prove a very useful result about a class of mappings which map a reflexive Banach space to the power set of its dual space. For more theorems about these mappings and their applications, see [90]. In the discussion below, we will use the symbol, \( \rightharpoonup \), to denote weak convergence.

**Definition 23.4.8** Let \( V \) be a Reflexive Banach space. We say \( T : V \to \mathcal{P}(V') \) is pseudomonotone if the following conditions hold.

\[
Tu \text{ is closed, nonempty, convex.}
\]

(23.4.15)
If $F$ is a finite dimensional subspace of $V$, then if $u \in F$ and $W \supseteq Tu$ for $W$ a weakly open set in $V'$, then there exists $\delta > 0$ such that

$$v \in B(u, \delta) \cap F \implies Tv \subseteq W.$$  \hspace{1cm} (23.4.16)

If $u_k \rightharpoonup u$ and if $u_k^* \in Tu_k$ is such that

$$\limsup_{k \to \infty} \Re u_k^* (u_k - u) \leq 0,$$

then for all $v \in V$, there exists $u^* (v) \in Tu$ such that

$$\liminf_{k \to \infty} \Re u_k^* (u_k - v) \geq \Re u^* (v) (u - v).$$  \hspace{1cm} (23.4.17)

We say $T$ is coercive if

$$\lim_{||v|| \to \infty} \inf \left\{ \frac{\Re z^* (v)}{||v||} : z^* \in Tv \right\} = \infty.$$  \hspace{1cm} (23.4.18)

In the case that $T$ takes bounded sets to bounded sets so it is a bounded set valued operator, it turns out you don’t have to consider the second of the above conditions about the upper semicontinuity. It follows from the other conditions. It is convenient to use the notation

$$\langle u^*, v \rangle \equiv u^* (v), u^* \in V', v \in V.$$

and this will be used interchangeably with the earlier notation from now on.

The next lemma has to do with upper semicontinuity being obtained from simpler conditions.

**Lemma 23.4.9** Let $T : X \to \mathcal{P} (X')$ satisfy conditions 23.4.15 and 23.4.17 above and suppose $T$ is bounded ($Tx$ for $x$ in a bounded set is bounded). Then if $x_n \to x$ in $X$, and if $U$ is a weakly open set containing $Tx$, then $Tx_n \subseteq U$ for all $n$ large enough. If fact the limit condition 23.4.17 can be weakened to the following more general condition: If $u_k \rightharpoonup u$, and

$$\limsup_{k \to \infty} \Re u_k^* (u_k - u) \leq 0,$$

then there exists a subsequence still denoted as $\{u_k\}$, such that if $u_k^* \in Tu_k$, then for all $v \in V$, there exists $u^* (v) \in Tu$ such that

$$\liminf_{k \to \infty} \Re u_k^* (u_k - v) \geq \Re u^* (v) (u - v).$$  \hspace{1cm} (23.4.19)

(This weaker condition says that if the lim sup condition holds for the original sequence, then there is a subsequence such that the lim inf condition holds for all $v$. In particular, for this subsequence, the lim sup condition continues to hold.)
Proof: If this is not true, there exists \( x_n \to x \), also a weakly open set \( U \), containing \( Tx \) and \( z_n \in Tx_n \), but \( z_n \notin U \). Then, taking a further subsequence, we can assume \( z_n \to z \) weakly and \( z \notin U \). Then the strong convergence implies

\[
\limsup_{n \to \infty} \text{Re} \langle z_n, x_n - x \rangle \leq 0
\]

By assumption, there is a subsequence still denoted with \( n \) such that for any \( y \),

\[
\liminf_{k \to \infty} \text{Re} \langle z_n, x_n - y \rangle \geq \text{Re} \langle z(y), x - y \rangle, \text{ some } z(y) \in T(x)
\]

Then in particular, for this subsequence,

\[
0 \geq \limsup_{n \to \infty} \text{Re} \langle z_n, x_n - x \rangle \geq \liminf_{n \to \infty} \text{Re} \langle z_n, x_n - x \rangle \geq \text{Re} \langle z(x), x - x \rangle = 0
\]

so for this subsequence,

\[
\lim_{n \to \infty} \text{Re} \langle z_n, x_n - x \rangle = 0
\]

Therefore, if \( y \in X \) there exists \( z(y) \in Tx \) such that

\[
\text{Re} \langle z, x - y \rangle = \liminf_{n \to \infty} \text{Re} \langle z_n, x_n - y \rangle \geq \text{Re} \langle z(y), x - y \rangle.
\]

Letting \( w = x - y \), this shows, since \( y \in X \) is arbitrary, that the following inequality holds for every \( w \in X \). (If you have \( w \in X \), then you just choose \( y = x - w \).)

\[
\text{Re} \langle z, w \rangle \geq \text{Re} \langle z(x - w), w \rangle, \text{ } z(x - w) \in Tx.
\]

In particular, we may replace \( w \) with \(-w\) and obtain

\[
\text{Re} \langle z, -w \rangle \geq \text{Re} \langle z(x + w), -w \rangle,
\]

which implies

\[
\text{Re} \langle z(x - w), w \rangle \leq \text{Re} \langle z, w \rangle \leq \text{Re} \langle z(x + w), w \rangle.
\]

Therefore, there exists, \( \lambda \in [0, 1] \),

\[
z_\lambda(y) \equiv \lambda z(x - w) + (1 - \lambda) z(x + w) \in Ax
\]

such that

\[
\text{Re} \langle z, w \rangle = \text{Re} \langle z_\lambda(y), w \rangle.
\]

But this is a contradiction to \( z \notin Tx \) because if \( z \notin Tx \), it follows from separation theorems there exists \( w \in X \) such that for all \( z_1 \in Tx \),

\[
\text{Re} \langle z, w \rangle > \text{Re} \langle z_1, w \rangle.
\]

You pick that \( w \) in the above. Therefore, \( z \in Tx \) which contradicts the assumption that \( z_n \) and consequently \( z \) are not contained in \( U \).
What if $T : V \to \mathcal{P}(V')$ for $V$ a finite dimensional vector space such that $T$ is upper semicontinuous and bounded? If $u_n \to u$ weakly, then this also happens strongly because the weak convergence and strong convergence are the same in finite dimensions. Therefore, by upper semicontinuity, there is a subsequence still denoted with $n$ and $z_n \in Tu_n$ such that $z_n \to z$ for some $z$. Then since $Tu$ is closed, we must have $z \in Tu$ thanks to upper semicontinuity. Then for this subsequence and arbitrary $v$,

$$
\lim \inf_{n \to \infty} \Re z_n (u_n - v) = \Re z (u - v)
$$

Also, this limit condition holds whenever $u_n \to u$ and $z_n \to z$ even without an assumption that $T$ is bounded. This mostly proves the following.

**Proposition 23.4.10** Let $V$ be finite dimensional and let $T : V \to \mathcal{P}(V)$ be upper semicontinuous with closed values. Then if $u_n \to u$ and $z_n \in Tu_n$ with $z_n \to z$, then

$$
\lim \inf_{n \to \infty} \Re z_n (u_n - v) = \Re z (u - v), \; z \in Tu
$$

If $T$ is bounded, and $u_n \to u$, then if $v$ is given,

$$
\lim \inf_{n \to \infty} \Re z_n (u_n - v) = \Re z (u - v), \; \text{some } z \in Tu
$$

**Proof:** Consider the last claim and suppose the limit condition does not hold for some $v$. Then take a subsequence such that

$$
\lim \inf_{n \to \infty} \Re z_n (u_n - v) = \lim_{n \to \infty} \Re z_n (u_n - v)
$$

By boundedness, there is a further subsequence such that $z_n \to z$. Then from upper semicontinuity and $Tu$ being closed, $z \in Tu$ and so

$$
\lim_{n \to \infty} \Re z_n (u_n - v) = \lim \inf_{n \to \infty} \Re z_n (u_n - v) = \Re z (u - v) \quad \blacksquare
$$

This more general limit condition is sometimes useful if not essential to use. The following is a definition of this more general condition used in the above lemma.

**Definition 23.4.11** Say $T : V \to \mathcal{P}(V')$ is modified bounded pseudomonotone if the following conditions hold.

$Tu$ is closed, nonempty, convex. \hfill (23.4.20)

$T$ is bounded meaning it takes bounded sets to bounded sets.

If $u_k \to u$ and if

$$
\lim \sup_{k \to \infty} \Re u_k^* (u_k - u) \leq 0,
$$

then there exists a subsequence, still denoted as $\{u_k\}$ such that if $u_k^* \in Tu_k$ then for all $v \in V$, there exists $u^* (v) \in Tu$ such that

$$
\lim \inf_{k \to \infty} \Re u_k^* (u_k - v) \geq \Re u^* (v) (u - v). \hfill (23.4.21)
$$
In this limit condition, there is a subsequence which works for all $v$. However, the preservation of lower semicontinuity happens under even less.

**Definition 23.4.12** Say $T : V \to \mathcal{P}(V')$ is generalized bounded pseudomonotone if the following conditions hold.

\[ Tu \text{ is closed, nonempty, convex.} \quad (23.4.22) \]

$T$ is bounded meaning it takes bounded sets to bounded sets.

If $u_k \to u$ and if

\[ \limsup_{k \to \infty} \Re u_k^* (u_k - u) \leq 0, \]

then if $v$ is given there exists a subsequence, still denoted as $\{u_k\}$ possibly depending on $v$, such that if $u_k^* \in Tu_k$ then, there exists $u^*(v) \in Tu$ such that

\[ \liminf_{k \to \infty} \Re u_k^* (u_k - v) \geq \Re u^* (v) (u - v). \]  

(23.4.23)

This is more general because in this situation, the subsequence depends on the choice of $v$.

In case $T$ is single valued, this condition is equivalent to type M.

**Proposition 23.4.13** A single valued bounded operator $T : V \to V'$, $V$ reflexive is generalized bounded pseudomonotone then it is bounded and type M.

**Proof:** Suppose that $u_n \to u$ weakly and $Tu_n \to \xi$ weakly and

\[ \limsup_{n \to \infty} \langle Tu_n, u_n \rangle \leq \langle \xi, u \rangle. \]

Then

\[ \limsup_{n \to \infty} \langle Tu_n, u_n - u \rangle \leq 0 \]

and so there is a subsequence depending on $v$ and a further one depending on $u$ such that

\[ \liminf_{n \to \infty} \langle Tu_n, u_n - u \rangle \geq \langle Tu, u - u \rangle = 0 \]

\[ \liminf_{n \to \infty} \langle Tu_n, u_n - v \rangle \geq \langle Tu, u - v \rangle \]

The first in the above shows with the lim sup condition that $\lim_{n \to \infty} \langle Tu_n, u_n - u \rangle = 0$. Therefore, the second condition implies

\[ \langle \xi, u - v \rangle \geq \langle Tu, u - v \rangle \]

since $v$ is arbitrary, it follows that $\xi = Tu$. Thus $T$ is type M. \( \blacksquare \)

If $T$ is generalized bounded pseudomonotone, then it is upper semicontinuous from the strong to the weak topology.
Lemma 23.4.14 Let $T : X \to \mathcal{P}(X')$ satisfy conditions 23.4.10 and 23.4.11 above and suppose $T$ is bounded. Then if $x_n \to x$ in $X$, and if $U$ is a weakly open set containing $Tx$, then $Tx_n \subseteq U$ for all $n$ large enough. If fact the limit condition 23.4.14 can be weakened to the following more general condition: If $u_k \to u$, and
\[ \limsup_{k \to \infty} \Re u_k^*(u_k - u) \leq 0, \quad (**) \]
then for each $v$, there exists a subsequence still denoted as $\{u_k\}$, possibly depending on $v$ such that if $u_k^* \in Tu_k$, then there exists $u^*(v) \in Tu$ such that
\[ \liminf_{k \to \infty} \Re u_k^*(u_k - v) \geq \Re u^*(v)(u - v). \quad (23.4.24) \]
(This weaker condition says that if the lim sup condition holds for the original sequence, then for given $v$ there is a subsequence such that the lim inf condition holds for that $v$. In particular, for this subsequence, the lim sup condition continues to hold.)

Proof: If this is not true, there exists $x_n \to x$, and a weakly open set $U$, containing $Tx$ and $z_n \in Tx_n$, but $z_n \notin U$. Then, taking a further subsequence, we can assume $z_n \to z$ weakly and $z \notin U$. Then the strong convergence implies
\[ \lim_{n \to \infty} \sup \Re \langle z_n, x_n - x \rangle \leq 0 \]
By separation theorems, there exists $w$ such that for all $\hat{w} \in T(x)$,
\[ \Re \langle z, w \rangle < \Re \langle \hat{w}, w \rangle \quad (*) \]
Thus, choose $y$ such that $w = x - y$. By assumption, there is a subsequence still denoted with $n$ such that
\[ \liminf_{k \to \infty} \Re \langle z_n, x_n - y \rangle \geq \Re \langle z(y), x - y \rangle, \text{ some } z(y) \in T(x) \]
\[ \liminf_{k \to \infty} \Re \langle z_n, x_n - x \rangle \geq \Re \langle z(x), x - x \rangle = 0, \text{ some } z(x) \in T(x) \]
To get this subsequence, get one which goes with $y$ and then note that the lim sup only gets smaller when you go to a subsequence. Hence you can apply the condition to get a further subsequence which goes with $x$. By doing so, the lim inf condition for $y$ is strengthened. Then in particular, for this subsequence,
\[ 0 \geq \limsup_{n \to \infty} \Re \langle z_n, x_n - x \rangle \geq \liminf_{n \to \infty} \Re \langle z_n, x_n - x \rangle \geq \Re \langle z(x), x - x \rangle = 0 \]
so for this subsequence,
\[ \lim_{n \to \infty} \Re \langle z_n, x_n - x \rangle = 0 \]
Therefore, from the assumed condition, there is a further subsequence such that
\[ \Re \langle z, x - y \rangle = \liminf_{n \to \infty} \Re \langle z_n, x_n - y \rangle \geq \Re \langle z(y), x - y \rangle, \text{ some } z(y) \in T(x) \]
Since $w = x - y$,
\[ \Re \langle z, w \rangle \geq \Re \langle z(y), w \rangle \]
where $z(y) \in Tx$. which contradicts *. Thus $z \in U$ as claimed. \(\blacksquare\)
23.5 Sum Of Pseudomonotone Operators

One of the nice properties of pseudomonotone maps is that when you add two of them, you get another one. I will give a proof in the case that the two pseudomonotone maps are both bounded. It is probably true in general, but as just noted, it is less trouble to verify if you don’t have to worry about as many conditions. I will also assume the spaces are all real so it will not be necessary to constantly write the real part. Actually, we do a slightly more general version which says that a bounded pseudomonotone added to a modified bounded pseudomonotone is a modified bounded pseudomonotone. First is the theorem about the sum of two bounded set valued pseudomonotone operators.

**Theorem 23.5.1** Say $A, B$ are set valued bounded pseudomonotone operators. Then their sum is also a set valued bounded pseudomonotone operator. Also, if $u_n \to u$ weakly, $z_n \to z$ weakly, $z_n \in A (u_n)$, and $w_n \to w$ weakly with $w_n \in A (u_n)$, then if \[ \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0, \] it follows that

\[ \liminf_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \geq \langle z (v) + w (v), u - v \rangle, \quad z (v) \in A (u), w (v) \in B (u), \]

and $z \in A (u), w \in B (u)$.

**Proof:** Say $z_n \in A (u_n), w_n \in B (u_n), u_n \to u$ weakly and

\[ \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0. \]

**Claim:** Both of $\limsup_{n \to \infty} \langle z_n, u_n - u \rangle, \limsup_{n \to \infty} \langle w_n, u_n - u \rangle$ are no larger than 0.

**Proof of the claim:** Suppose $\limsup_{n \to \infty} \langle w_n, u_n - u \rangle = \delta > 0$. Then take a subsequence such that the lim sup equals lim. The lim sup only gets smaller when you go to a subsequence. Thus, continuing to denote the subsequence with $n$ we still have $\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0$. But from the fact that we just took a subsequence for which the lim sup = lim,

\[
\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \limsup_{n \to \infty} \langle z_n, u_n - u \rangle + \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \\
= \limsup_{n \to \infty} \langle z_n, u_n - u \rangle + \delta \leq 0
\]

and so $\limsup_{n \to \infty} \langle z_n, u_n - u \rangle = -\delta < 0$. Therefore by the limit condition, $\liminf_{n \to \infty} \langle z_n, u_n - u \rangle \geq \langle z (u), u - u \rangle = 0$ and so

\[ 0 > -\delta \geq \limsup_{n \to \infty} \langle z_n, u_n - u \rangle \geq \liminf_{n \to \infty} \langle z_n, u_n - u \rangle \geq 0 \]

a contradiction. Thus the claim is established. We have

\[ \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \leq 0, \quad \limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0. \]
23.5. **SUM OF PSEUDOMONOTONE OPERATORS**

Thus we can apply the limit condition to the two operators separately. This refers to the original sequence now. If \( v \) is given, then

\[
\lim_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v), u - v \rangle, \quad \lim_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle
\]

where \( w(v) \in B(u) \) and \( z(v) \in A(u) \). Thus

\[
\lim_{n \to \infty} \langle z_n + w_n, u_n - v \rangle \geq \lim_{n \to \infty} \langle z_n, u_n - v \rangle + \lim_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v) + z(v), u - v \rangle
\]

and \( w(v) + z(v) \in A(u) + B(u) \) which shows that the sum is pseudomonotone.

In addition, from the claim, we know that

\[
\liminf_n \langle z_n, u_n - u \rangle \geq \langle z(u), u - u \rangle = 0,
\]

similar for \( w_n \). Thus \( \liminf_n \langle z_n, u_n - u \rangle \geq 0 \geq \limsup_n \langle z_n, u_n - u \rangle \) so \( \lim_{n \to \infty} \langle z_n, u_n - u \rangle = 0 \), similar for \( \langle w_n, u_n - u \rangle \). Therefore, if \( z_n \to z \) weakly and \( w_n \to w \) weakly,

\[
\langle z, u - v \rangle = \lim \langle z_n, u - v \rangle = \lim_{n \to \infty} [\langle z_n, u - u_n \rangle + \langle z_n, u_n - v \rangle] \geq \liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle, \quad z(v) \in A(u)
\]

It follows that \( \langle z, u - v \rangle \geq \langle z(v), u - v \rangle \) for all \( v \) which could be violated using separation theorems if \( z \) is not in \( A(u) \). Thus \( z \in A(u) \). Similarly \( w \in B(u) \). ■

The above is the main result but we can attempt to see what happens if one of the operators is only modified pseudomonotone.

Note that if \( B \) is bounded pseudomonotone, then it is certainly modified bounded pseudomonotone.

**Theorem 23.5.2** Suppose \( A, B : X \to \mathcal{P}(X') \) are both pseudomonotone and bounded. Then so is their sum. If \( A \) is bounded pseudomonotone and \( B \) is modified bounded pseudomonotone, then \( A + B \) is modified bounded pseudomonotone.

**Proof:** It is clear that \( Ax + Bx \) is closed and convex because this is true of both of the sets in the sum. It is also bounded because both terms in the sum are bounded. It only remains to verify the limit condition. Suppose then that

\[ u_n \to u \text{ weakly} \]

Will the limit condition hold for \( A + B \) when applied to this further subsequence? Suppose \( z_n \in Ax_n, w_n \in Bx_n \) and

\[
\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0 \quad (23.5.25)
\]

Is there a subsequence such that the \( \liminf \) condition holds? From the above,

\[
\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq \limsup_{n \to \infty} \langle z_n, u_n - u \rangle + \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \quad (23.5.26)
\]
and so, if the second term $\leq 0$, since $B$ is modified bounded pseudomonotone, there is a subsequence, still denoted with $n$ for which

$$\liminf_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v), u - v \rangle, \ w(v) \in B(u) \quad (*)$$

for all $v$. In particular,

$$\liminf_{n \to \infty} \langle w_n, u_n - u \rangle \geq \langle w(u), u - u \rangle = 0$$

Hence you would have $\liminf_{n \to \infty} \langle w_n, u_n - u \rangle \geq 0 \geq \limsup_{n \to \infty} \langle w_n, u_n - u \rangle$ and so $\lim_{n \to \infty} \langle w_n, u_n - u \rangle = 0$ for this subsequence still denoted with $n$. Hence for this subsequence,

$$\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0$$

Then using that $A$ is bounded pseudomonotone, $\lim_{n \to \infty} \langle z_n, u_n - u \rangle = 0$ also. If follows for any $v$,

$$\liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle$$

Then from this it is routine to establish the modified pseudomonotone limit condition for the sum $A + B$. For the subsequence just described, still denoted with $n$,

$$\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0$$

and *$. In fact, you would have for any $v$,

$$\liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle, \ z(v) \in A(u)$$

$$\liminf_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v), u - v \rangle, \ w(v) \in A(u)$$

Then you would get

$$\liminf_{n \to \infty} \langle z_n + w_n, u_n - v \rangle = \liminf_{n \to \infty} \left( \langle z_n, u_n - v \rangle + \langle w_n, u_n - v \rangle \right)$$

$$\geq \liminf_{n \to \infty} \left( \langle z_n, u_n - v \rangle \right) + \liminf_{n \to \infty} \left( \langle w_n, u_n - v \rangle \right)$$

$$\geq \langle z(v), u - v \rangle + \langle w(v), u - v \rangle$$

and $\langle z(v) + w(v), u \rangle \in (A + B)(u)$.

Returning to (23.5.26), the other case to consider is that

$$\limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0$$

Then in this case, the assumption that $A$ is pseudomonotone implies that for any $v$,

$$\liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle, \ z(v) \in A(u) \quad (***)$$
23.5. **SUM OF PSEUDOMONOTONE OPERATORS**

No subsequence here. However, if you use a subsequence, the inequality is only strengthened. In particular,

\[ 0 \geq \lim \inf_{n \to \infty} \langle z_n, u_n - u \rangle = \langle z(u), u - u \rangle = 0 \geq \lim \sup_{n \to \infty} \langle z_n, u_n - u \rangle \]

and so for the original sequence,

\[ \lim_{n \to \infty} \langle z_n, u_n - u \rangle = 0. \]

Then back to 23.5.26,

\[ \lim \sup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \lim \sup_{n \to \infty} \langle w_n, u_n - u \rangle \leq 0 \]

Now by assumption that \( B \) is modified bounded pseudomonotone, there is a subsequence, still denoted with \( n \) such that for any \( v \)

\[ \lim \inf_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v), u - v \rangle, \ w(v) \in B(u). \]  (***)

In particular, for this subsequence,

\[ 0 \geq \lim \sup_{n \to \infty} \langle w_n, u_n - u \rangle \geq \lim \inf_{n \to \infty} \langle w_n, u_n - u \rangle \geq \langle w(u), u - u \rangle = 0 \]

and so for this subsequence, \( \lim_{n \to \infty} \langle w_n, u_n - u \rangle = 0 \). Then for this subsequence, it follows from ***, and ****,

\[
\begin{align*}
\lim \inf_{n \to \infty} \langle z_n + w_n, u_n - v \rangle &= \lim \inf_{n \to \infty} (\langle z_n, u_n - v \rangle + \langle w_n, u_n - v \rangle) \\
&\geq \lim \inf_{n \to \infty} (\langle z_n, u_n - v \rangle) + \lim \inf_{n \to \infty} (\langle w_n, u_n - v \rangle) \\
&\geq \langle z(v), u - v \rangle + \langle w(v), u - v \rangle
\end{align*}
\]

We continue to be in the situation of 23.5.26 and we are asking for a subsequence such that the \( \lim \inf \) condition will hold for the subsequence. Suppose this \( \lim \inf \) condition is not obtained for any subsequence. The desired \( \lim \inf \) condition will hold for a subsequence if either \( \lim \sup_{n \to \infty} \langle z_n, u_n - u \rangle \) or \( \lim \sup_{n \to \infty} \langle w_n, u_n - u \rangle \) is \( \leq 0 \). This was shown above. Therefore, if there is no subsequence yielding the \( \lim \inf \) condition, you must have both of these strictly positive. Say \( \delta > 0 \) is smaller than both. Let \( n \) denote a subsequence such that

\[ \lim \sup_{n \to \infty} \langle z_n, u_n - u \rangle = \lim_{n \to \infty} \langle z_n, u_n - u \rangle > \delta > 0. \]

If, for this new subsequence, \( \lim \sup_{n \to \infty} \langle w_n, u_n - u \rangle < 0 \), then, since the \( \lim \sup \) gets smaller for a subsequence,

\[ \lim \sup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \lim_{n \to \infty} \langle z_n, u_n - u \rangle + \lim \sup_{n \to \infty} \langle w_n, u_n - u \rangle \leq 0 \]
then you could apply the above argument and obtain a further subsequence for which the lim inf condition would hold for the sum. Thus, we must have for this new subsequence,

$$\limsup_{n \to \infty} \langle w_n, u_n - u \rangle \geq 0.$$ 

Then, using this subsequence,

$$0 \geq \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \geq \delta + \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \geq \delta$$

which is a contradiction. Thus the lim inf condition must hold for some subsequence. In case both are bounded and pseudomonotone, things are easier. You don’t have to take a subsequence.

It is not entirely clear whether the sum of modified bounded pseudomonotone operators is modified bounded pseudomonotone. This is because when you go to a subsequence, the lim sup gets smaller and so it is not entirely clear whether the subsequence for $A$ will continue to yield the limit condition if a further subsequence is taken.

In fact, you can add a bounded pseudomonotone to a generalized bounded pseudomonotone and get a generalized bounded pseudomonotone. The proof is just like the above and is given next.

**Theorem 23.5.3** Suppose $A, B : X \to P(X')$. If $A$ is bounded pseudomonotone and $B$ is generalized bounded pseudomonotone, then $A + B$ is generalized bounded pseudomonotone.

**Proof:** It is clear that $Ax + Bx$ is closed and convex because this is true of both of the sets in the sum. It is also bounded because both terms in the sum are bounded. It only remains to verify the limit condition. Suppose then that

$$u_n \to u \text{ weakly}$$

Will the limit condition hold for $A + B$ when applied to this further subsequence? Suppose $z_n \in Ax_n, w_n \in Bx_n$ and

$$\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0 \quad (23.5.27)$$

If $v$ is given, is there a subsequence such that the lim inf condition holds? From the above,

$$\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq \limsup_{n \to \infty} \langle z_n, u_n - u \rangle + \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \quad (23.5.28)$$

and so, if the second term $\leq 0$, since $B$ is modified bounded pseudomonotone, there is a subsequence, still denoted with $n$ for which

$$\liminf_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v), u - v \rangle, \quad w(v) \in B(u) \quad (*)$$
23.5. SUM OF PSEUDOMONOTONE OPERATORS

\[
\lim_{n \to \infty} \langle w_n, u_n - u \rangle \geq \langle w(u), u - u \rangle = 0
\]

You just get a subsequence which works for \( v \) and note that the \( \limsup \) condition is only strengthened for the subsequence and then obtain a further subsequence which goes with \( u \) to get the second condition along with the first. Note that \( \liminf \) gets bigger when you go to a subsequence so if \( * \) holds for the first subsequence, then it holds even better for the second.

Hence you would have, for this subsequence depending on \( v \)

\[
\lim_{n \to \infty} \langle w_n, u_n - u \rangle \geq 0 \geq \limsup_{n \to \infty} \langle w_n, u_n - u \rangle
\]

and so \( \lim_{n \to \infty} \langle w_n, u_n - u \rangle = 0 \) for this subsequence still denoted with \( n \). Hence for this subsequence,

\[
\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0
\]

Then using that \( A \) is bounded pseudomonotone, \( \lim_{n \to \infty} \langle z_n, u_n - u \rangle = 0 \) also. If follows for any \( v \),

\[
\lim_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle
\]

Then from this it is routine to establish the modified pseudomonotone limit condition for the sum \( A + B \). For the subsequence just described, still denoted with \( n \),

\[
\limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0
\]

and \( * \). In fact, you would have

\[
\liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle, \quad z(v) \in A(u)
\]

Then you would get

\[
\liminf_{n \to \infty} \langle z_n + w_n, u_n - v \rangle = \liminf_{n \to \infty} \left( \langle z_n, u_n - v \rangle + \langle w_n, u_n - v \rangle \right) \\
\geq \liminf_{n \to \infty} \langle z_n, u_n - v \rangle + \liminf_{n \to \infty} \langle w_n, u_n - v \rangle \\
\geq \langle z(v), u - v \rangle + \langle w(v), u - v \rangle
\]

and \( z(v) + w(v) \in (A + B)(u) \).

Returning to 23.5.28, the other case to consider is that

\[
\limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0
\]

Then in this case, the assumption that \( A \) is pseudomonotone implies that for any \( v \),

\[
\liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle, \quad z(v) \in A(u) \quad (***)
\]
No subsequence here. In particular,

\[ 0 \geq \liminf_{n \to \infty} \langle z_n, u_n - u \rangle = \langle z(u), u - u \rangle = 0 \geq \limsup_{n \to \infty} \langle z_n, u_n - u \rangle \]

and so for the original sequence,

\[ \lim_{n \to \infty} \langle z_n, u_n - u \rangle = 0. \]

Then back to (23.5.28),

\[ \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \leq 0 \]

Now by assumption that \( B \) is generalized bounded pseudomonotone, there is a subsequence, still denoted with \( n \) such that for the given \( v \),

\[ \liminf_{n \to \infty} \langle w_n, u_n - v \rangle \geq \langle w(v), u - v \rangle, \ w(v) \in B(u). \quad (***) \]

Then taking a further subsequence to go with \( u \) in the third inequality below, the first inequality is preserved and

\[ 0 \geq \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \geq \liminf_{n \to \infty} \langle w_n, u_n - u \rangle \geq \langle z(u), u - u \rangle = 0 \]

and so for this further subsequence, \( \lim_{n \to \infty} \langle w_n, u_n - u \rangle = 0 \). Then for this subsequence, it follows from ***, and ****,

\[ \liminf_{n \to \infty} \langle z_n + w_n, u_n - v \rangle = \liminf_{n \to \infty} \left( \langle z_n, u_n - v \rangle + \langle w_n, u_n - v \rangle \right) \]
\[ \geq \liminf_{n \to \infty} \langle z_n, u_n - v \rangle + \liminf_{n \to \infty} \langle w_n, u_n - v \rangle \]
\[ \geq \langle z(v), u - v \rangle + \langle w(v), u - v \rangle \]

We continue to be in the situation of (23.5.27) and we are asking for a subsequence such that the \( \liminf \) condition will hold for some subsequence depending on \( v \). Suppose this \( \liminf \) condition is not obtained for any subsequence. The desired \( \liminf \) condition will hold for a subsequence if either \( \limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0 \) or \( \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \) is \( \leq 0 \). This was shown above. Therefore, if there is no subsequence yielding the \( \liminf \) condition, you must have both of these strictly positive. Say \( \delta > 0 \) is smaller than both. Let \( n \) denote a subsequence such that

\[ \limsup_{n \to \infty} \langle z_n, u_n - u \rangle = \lim_{n \to \infty} \langle z_n, u_n - u \rangle > \delta > 0. \]

If, for this new subsequence, \( \limsup_{n \to \infty} \langle w_n, u_n - u \rangle < 0 \), then, since the \( \limsup \) gets smaller for a subsequence,

\[ \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle = \lim_{n \to \infty} \langle z_n, u_n - u \rangle + \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \leq 0 \]
then you could apply the above argument and obtain a further subsequence for which the lim inf condition would hold for the sum. Thus, we must have for this new subsequence,

\[ \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \geq 0. \]

Then, using this subsequence,

\[ 0 \geq \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \geq \delta + \limsup_{n \to \infty} \langle w_n, u_n - u \rangle \geq \delta \]

which is a contradiction. Thus the lim inf condition must hold for some subsequence.

The following is mostly in [30].

**Theorem 23.5.4** Let \( V \) be a reflexive Banach space and let \( T : V \to \mathcal{P}(V') \) be pseudomonotone, bounded, and coercive. Then \( T \) is onto. More generally, the same holds if \( T \) is modified or generalized bounded pseudomonotone and coercive.

**Proof:** The proof is for modified bounded pseudomonotone since this is more general. Let \( F \) be the set of finite dimensional subspaces of \( V \) and let \( F \in F \). Then define \( T_F \) as

\[ T_F \equiv i_F^* T i_F \]

where here \( i_F \) is the identity map from \( F \) to \( V \). Then \( T_F \) satisfies the conditions of Lemma 23.4.7 thanks to Lemma 23.4.9 or Lemma 23.4.14 and so \( T_F \) is onto \( \mathcal{P}(F') \).

Let \( w^* \in V' \). Then since \( T_F \) is onto, there exists \( u_F \in F \) such that

\[ i_F^* w^* \in i_F^* T i_F u_F. \]

Thus for each finite dimensional subspace \( F \), there exists \( u_F \in F \) such that for all \( v \in F \),

\[ \langle w^*, v \rangle = \langle u_F^*, v \rangle, \quad u_F^* \in Tu_F. \tag{23.5.29} \]

Replacing \( v \) with \( u_F \), in \[23.5.29\],

\[ \frac{\langle u_F^*, u_F \rangle}{\|u_F\|} = \frac{\langle w^*, u_F \rangle}{\|u_F\|} \leq \|w^*\|. \]

Therefore, the assumption that \( T \) is coercive implies \( \{ u_F : F \in F \} \) is bounded in \( V \). Now define

\[ W_F \equiv \cup \{ u_{F'} : F' \supseteq F \}. \]

Then \( W_F \) is bounded and if \( W_F \equiv \text{weak closure of } W_F \), then

\[ \{ W_F : F \in F \} \]

is a collection of nonempty weakly compact (since \( V \) is reflexive and the \( u_F \) were just shown bounded) sets having the finite intersection property because \( W_F \neq \emptyset \) for each \( F \). (If \( F_i, i = 1, \ldots, n \) are finite dimensional subspaces, let \( F \) be a finite dimensional
subspace which contains all of these. Then \( W_F \neq \emptyset \) and \( W_F \subseteq \bigcap_{i=1}^n \overline{W_{F_i}} \). Thus there exists

\[ u \in \bigcap \{ \overline{W_F} : F \in \mathcal{F} \}. \]

I will show \( w^* \in Tu \). If \( w^* \notin Tu \), a closed convex set, there exists \( v \in V \) such that

\[ \Re \langle w^*, u - v \rangle < \Re \langle u^*, u - v \rangle \quad (23.5.30) \]

for all \( u^* \in Tu \). This follows from the separation theorems. (These theorems imply there exists \( z \in V \) such that

\[ \Re \langle w^*, z \rangle < \Re \langle u^*, z \rangle \]

for all \( u^* \in Tu \). Define \( u - v \equiv z \).)

Now let \( F \supseteq \{ u, v \} \). Since \( u \in \overline{W_F} \), a weakly sequentially compact set, there exists a sequence, \( \{ u_k \} \), such that

\[ u_k \rightharpoonup u, \quad u_k \in W_F. \]

Then since \( F \supseteq \{ u, v \} \), there exists \( u_k^* \in Tu_k \) such that

\[ \langle u_k^*, u_k - u \rangle = \langle w^*, u_k - u \rangle. \]

Therefore,

\[ \limsup_{k \to \infty} \Re \langle u_k^*, u_k - u \rangle = \limsup_{k \to \infty} \Re \langle w^*, u_k - u \rangle = 0. \]

It follows by the assumption that \( T \) is modified bounded pseudomonotone or generalized bounded pseudomonotone and the pseudomonotone limit condition, a further subsequence corresponding to \( v \) such that the following holds for the \( v \) defined above in 23.5.30.

\[ \liminf_{k \to \infty} \Re \langle u_k^*, u_k - v \rangle \geq \Re \langle u^*(v), u - v \rangle, \quad u^*(v) \in Tu. \]

But since \( v \in F, \Re \langle u_k^*, u_k - v \rangle = \Re \langle w^*, u_k - v \rangle \) and so

\[ \liminf_{k \to \infty} \Re \langle u_k^*, u_k - v \rangle = \liminf_{k \to \infty} \Re \langle w^*, u_k - v \rangle = \Re \langle w^*, u - v \rangle, \]

so from 23.5.30, \( \Re \langle w^*, u - v \rangle < \Re \langle u^*, u - v \rangle \) for all \( u^* \in Tu \),

\[ \Re \langle w^*, u - v \rangle = \liminf_{k \to \infty} \Re \langle u_k^*, u_k - v \rangle \geq \Re \langle u^*(v), u - v \rangle > \Re \langle w^*, u - v \rangle, \]

a contradiction. Thus, \( w^* \in Tu \). ■

This is likely a good place to put an extremely interesting convergence theorem. It is a version of one in Aubin and Cellina [3]. It is a perfectly marvelous use of the fact that the weak and strong closures of a convex set are the same.
Proposition 23.5.5 Let $X, Y$ be Banach spaces, and let $F : (0, T) \times X \to P(Y)$ be a multifunction such that

1. The values of $F$ are nonempty, closed and convex subsets of $Y$
2. For a.e. $t \in (0, T), F(t, \cdot)$ is upper semicontinuous from $X$ into $Y$ with the weak topology

Then let $x_n : (0, T) \to X, y_n : (0, T) \to Y$ be measurable functions such that the sequence $\{x_n\}$ converges a.e. on $(0, T)$ to a function $x : (0, T) \to X$ and $y_n$ converges weakly in $L^1(0, T; Y)$ to $y \in L^1(0, T, Y)$. If $y_n(t) \in F(t, x_n(t))$ for all $n \in \mathbb{N}$ and a.e.t, then $y(t) \in F(t, x(t))$ for a.e.t $\in (0, T)$.

Proof: It is given that $y_n \to y$ weakly in $L^1(0, T; Y)$. It is too bad that this does not confer pointwise convergence of some subsequence. However, what can be said is this:

weak closure of $\text{co} (\bigcup_{k=n}^\infty y_k) =$ strong closure of $\text{co} (\bigcup_{k=n}^\infty y_k)$

Here co signifies the convex hull. Thus something is in $\text{co} (\bigcup_{k=n}^\infty y_k)$ means it is of the form

$$v_n = \sum_{k=n}^\infty c_k^n y_k$$

where all but finitely many of the $c_k^n$ are zero and they sum to 1, each being a number in $[0, 1]$. Now it is given that $y$ is in the weak closure of $\text{co} (\bigcup_{k=n}^\infty y_k)$. In fact $y_n$ converges weakly to $y$. Therefore, from the above observation, $y$ is in the strong closure of $\text{co} (\bigcup_{k=n}^\infty y_k)$. Let $v_n$ be of the form in $*$ and let it converge in $L^1(0, T; Y)$ to $y$. Then there is a subsequence, still denoted as $v_n$ such that for a.e. $t$,

$$v_n(t) \to y(t) \text{ in } Y$$

Pick such a $t$.

If $y(t) \notin F(t, x(t))$, there exist numbers $k > l$ and $y^* \in Y'$ such that

$$\langle y^*, y(t) \rangle > k > l > \langle y^*, z \rangle \text{ for all } z \in F(t, x(t)).$$

This follows from separation theorems due to the assumption that $F(t, x(t))$ is a closed convex set. Thus for all $n$ large enough,

$$\langle y^*, v_n(t) \rangle > k > l > \langle y^*, z \rangle \text{ for all } z \in F(t, x(t)). \quad (***)$$

Let $k - l > 2\varepsilon > 0$. Consider

$$F(t, x(t)) + B_{y^*}(0, \varepsilon)$$

where the ball signifies all $z \in Y$ such that

$$|\langle y^*, z \rangle| < \varepsilon$$
By the weak upper semicontinuity assumption of $F(t,\cdot)$ and $x_n(t) \to x(t)$, it follows that for $k$ large enough,

$$y_k(t) \in F(t,x_k(t)) \subseteq F(t,x(t)) + B_{y^*}(0,\varepsilon)$$

Now $v_n$ is a convex combination of $y_k$ for $k \geq n$ and so it follows that for $n$ large enough,

$$v_n(t) \in F(t,x(t)) + B_{y^*}(0,\varepsilon)$$

which says that there exists $z_n \in F(t,x(t))$ such that

$$|\langle y^*, v_n(t) \rangle - \langle y^*, z_n \rangle| < \varepsilon$$

However, this is a contradiction to $\ast\ast$ because it says two things are closer than $\varepsilon$ and also farther than $k-l > 2\varepsilon$. Thus $y(t) \in F(t,x(t))$. ■

It does not use that the measure space is Lebesgue measure on $[0,T]$ that I can see. I think it appears to work for $[0,T]$ replaced with $\Omega$ and $t$ replaced with $\omega \in \Omega$ where $(\Omega, F, \mu)$ is just some measure space.

### 23.6 Generalized Gradients

This is an interesting theorem, but one might wonder if there are easy to verify examples of such possibly set valued mappings. In what follows consider only real spaces because the essential ideas are included in this case which is also the case of most use in applications. Of course, you might with some justification, make the claim that the following is not really very easy to verify any more than the original definition.

**Definition 23.6.1** Let $V$ be a real reflexive Banach space and let $f: V \to \mathbb{R}$ be a locally Lipschitz function, meaning that $f$ is Lipschitz near every point of $V$ although $f$ need not be Lipschitz on all of $V$. Under these conditions,

$$f^0(x,y) \equiv \lim_{\mu \to 0^+} \sup_{h \to 0} \frac{f(x+h+\mu y) - f(x+h)}{\mu}$$

and $\partial f(x) \subseteq X'$ is defined by

$$\partial f(x) \equiv \{ x^* \in X': x^*(y) \leq f^0(x,y) \text{ for all } y \in X \}.$$ (23.6.32)

The set just described is called the generalized gradient. In $\partial f(x)$ we mean the following by the right hand side.

$$\lim_{(r,\delta) \to (0,0)} \sup \left\{ \frac{f(x+h+\mu y) - f(x+h)}{\mu} : \mu \in (0,r), h \in B(0,\delta) \right\}$$

I will show, following [72], that these generalized gradients of locally Lipschitz functions are sometimes pseudomonotone. First here is a lemma.
23.6. GENERALIZED GRADIENTS

Lemma 23.6.2 Let \( f \) be as described in the above definition. Then \( \partial f(x) \) is a closed, bounded, convex, and non empty subset of \( V' \). Furthermore, for \( x^* \in \partial f(x) \),

\[
\|x^*\| \leq \text{Lip}_x(f).
\]

(23.6.33)

Proof: It is left as an exercise to verify the assertions that \( \partial f(x) \) is closed, and convex. It follows directly from the definition. To verify this set is bounded, let \( \text{Lip}_x(f) \) denote a Lipschitz constant valid near \( x \in V \) and let \( x^* \in \partial f(x) \). Then choosing \( y \) with \( \|y\| = 1 \) and \( x^*(y) \geq \frac{1}{2} \|x^*\| \),

\[
\frac{1}{2} \|x^*\| = x^*(y) \leq f^0(x,y).
\]

(23.6.34)

Also, for small \( \mu \) and \( h \),

\[
\frac{f(x+h+\mu y) - f(x+h)}{\mu} \leq \text{Lip}_x(f) \|y\| = \text{Lip}_x(f).
\]

Therefore, \( f^0(x,y) \leq \text{Lip}_x(f) \) and so \( \|x^*\| \leq 2\text{Lip}_x(f) \).

The interesting part of this Lemma is that \( \partial f(x) \neq \emptyset \). To verify this first note that the definition of \( f^0 \) implies that \( y \rightarrow f^0(x,y) \) is a gauge function. Now fix \( y \in V \) and define on \( \mathbb{R}y \) a linear map \( x_0^* \) by

\[
x_0^*(\alpha y) \equiv \alpha f^0(x,y).
\]

Then if \( \alpha \geq 0 \),

\[
x_0^*(\alpha y) = \alpha f^0(x,y) = f^0(x,\alpha y).
\]

If \( \alpha < 0 \),

\[
x_0^*(\alpha y) \equiv \alpha f^0(x,y) = \lim_{\mu \to 0^+} \inf_{h \to 0^-} \frac{(-\alpha) f(x+h) - (-\alpha) f(x+h+\mu y)}{\mu} = (-\alpha) \lim_{\mu \to 0^+} \inf_{h \to 0^-} \frac{f(x+h-\mu y) - f(x+h)}{\mu} \leq (-\alpha) f^0(x,-y) = f^0(x,\alpha y).
\]

Therefore, \( x_0^*(\alpha y) \leq f^0(x,\alpha y) \) for all \( \alpha \). By the Hahn Banach theorem there is an extension of \( x_0^* \) to all of \( V \), \( x^* \) which satisfies,

\[
x^*(y) \leq f^0(x,y)
\]

for all \( y \). It remains to verify \( x^* \) is continuous. This follows easily from

\[
\|x^*(y)\| = \max(x^*(-y),x^*(y)) \leq \max(f^0(x,y),f^0(x,-y)) \leq \text{Lip}_x(f) \|y\|,
\]

which verifies \( \text{Hahn-Banach} \) and proves the lemma.

This lemma has verified the first condition needed in the definition of pseudomonotone. The next lemma verifies that these generalized subgradients satisfy the second of the conditions needed in the definition. In fact somewhat more than is needed in the definition is shown.
**Lemma 23.6.3** Let $U$ be weakly open in $V'$ and suppose $\partial f(x) \subseteq U$. Then $\partial f(z) \subseteq U$ whenever $z$ is close enough to $x$.

**Proof:** Suppose to the contrary there exists $z_n \to x$ but $z^*_n \in \partial f(z_n) \setminus U$. From the first lemma, we may assert that $\|z^*_n\| \leq 2\text{Lip}(f)$ for all $n$ large enough. Therefore, there is a subsequence, still denoted by $n$ such that $z^*_n$ converges weakly to $z^* \notin U$.

**Claim:** $f^0(x,y) \geq \limsup_{n \to \infty} f^0(x_n,y)$.

**Proof of the claim:** There exists $\delta > 0$ such that if $\mu, ||h|| < \delta$, then
\[
\varepsilon + f^0(x,y) \geq \frac{f(x + h + \mu y) - f(x + h)}{\mu}.
\]
Thus, for $||h|| < \delta$,
\[
\varepsilon + f^0(x,y) \geq \frac{f(x_n + (x - x_n) + h + \mu y) - f(x_n + (x - x_n) + h)}{\mu}.
\]
Now let $||h'|| < \frac{\delta}{2}$ and let $n$ be so large that $||x - x_n|| < \frac{\delta}{2}$. Suppose $||h'|| < \frac{\delta}{2}$. Then choosing $h \equiv h' - (x - x_n)$, it follows the above inequality holds because $||h|| < \delta$.

Therefore, if $||h'|| < \frac{\delta}{2}$, and $n$ is sufficiently large,
\[
\varepsilon + f^0(x,y) \geq \frac{f(x_n + h' + \mu y) - f(x_n + h')}{\mu}.
\]
Consequently, for all $n$ large enough,
\[
\varepsilon + f^0(x,y) \geq f^0(x_n,y)
\]
which proves the claim.

Now with the claim,
\[
z^*(y) = \limsup_{n \to \infty} z^*_n(y) \leq \limsup_{n \to \infty} f^0(x_n,y) \leq f^0(x,y)
\]
so $z^* \in \partial f(x)$ contradicting the assumption that $z^* \notin U$. This proves the lemma.

It is necessary to assume more on $f^0$ in order to obtain the third axiom defining pseudomonotone. The following theorem describes the situation.

**Theorem 23.6.4** Let $f : V \to V'$ be locally Lipschitz and suppose it satisfies the condition that whenever $x_n$ converges weakly to $x$

and
\[
\limsup_{n \to \infty} f^0(x_n, x - x_n) \geq 0
\]

it follows that
\[
\limsup_{n \to \infty} f^0(x_n, z - x_n) \leq f^0(x, z - x)
\]
for all $z \in V$. Then $\partial f$ is pseudomonotone.
23.7. MAXIMAL MONOTONE OPERATORS

Proof: and both are satisfied thanks to Lemmas and . It remains to verify . To do so, I will adopt the convention that . Suppose

\[ \limsup_{n \to \infty} x_n^* (x_n - x) \leq 0. \]  

(23.6.35)

This implies \( \liminf_{n \to \infty} x_n^* (x - x_n) \geq 0 \). Thus,

\[ 0 \leq \liminf_{n \to \infty} x_n^* (x - x_n) \leq \liminf f^0 (x_n, x - x_n) \leq \limsup f^0 (x_n, x - x_n), \]

which implies, by the above assumption that for all \( z \),

\[ \limsup x_n^* (z - x_n) \leq \limsup f^0 (x_n, z - x_n) \leq f^0 (x, z - x). \]  

(23.6.36)

In particular, this holds for \( z = x \) and this implies \( \limsup x_n^* (x - x_n) \leq 0 \) which along with yields

\[ \lim_{n \to \infty} x_n^* (x_n - x) = 0 \]  

(23.6.37)

Now let \( z \) be arbitrary. There exists a subsequence, \( n_k \), depending on \( z \) such that

\[ \lim_{k \to \infty} x_n^* (x_n - z) = \liminf x_n^* (x_n - z). \]

Now from Lemma and its proof, the \( ||x_n^*|| \) are all bounded by \( \text{Lip}_x (f) \) whenever \( n \) is large enough. Therefore, there is a further subsequence, still denoted by \( n_k \) such that

\( x_n^* \) converges weakly to \( x^* (z) \).

We need to verify that \( x^* (z) \in \partial f (x) \). To do so, let \( y \) be arbitrary. Then from the definition,

\[ x_n^* (y - x_n) \leq f^0 (x_n, y - x_n). \]  

(23.6.38)

From we can take the \( \limsup \) of both sides and obtain, using

\[ x^* (y - x) \leq \limsup f^0 (x_n, y - x_n) \leq f^0 (x, y - x). \]

Since \( y \) is arbitrary, this shows \( x^* (z) \in \partial f (x) \) and proves the theorem.

23.7 Maximal Monotone Operators

Here it is assumed that the spaces are all real spaces to simplify the presentation.

Definition 23.7.1 Let \( A : D (A) \subseteq X \to \mathcal{P} (X) \) be a set valued map. It is said to be monotone if whenever \( y_t \in Ax_t \),

\[ \langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \]
Denote by $G(A)$ the graph of $A$ consisting of all pairs $(x,y)$ where $y \in Ax$. Such a monotone operator is said to be maximal monotone if

$$F + A$$

is onto where $F$ is the duality map with $p = 2$.

Actually, it is more usual to say that the graph is maximal monotone if the graph is monotone and there is no monotone graph which properly contains the given graph. However, the two conditions are equivalent and I am more used to using the version in the above definition.

There is a fundamental result about these which is given next.

**Theorem 23.7.2** Let $X, X'$ be reflexive and have strictly convex norms. Let $A$ be a monotone set valued map as just described. Then if $\lambda F + A$ is onto for some $\lambda > 0$, then whenever

$$\langle y - z, x - u \rangle \geq 0 \text{ for all } [x,y] \in G(A)$$

it follows that $z \in Au$ and $u \in D(A)$. That is, the graph is maximal.

**Proof:** Suppose that for all $[x,y] \in G(A)$,

$$\langle y - z, x - t \rangle \geq 0$$

Does it follow that $z \in A t$? By assumption, $z + \lambda F(t) = \lambda F\hat{x} + \hat{\xi}, \hat{\xi} \in A\hat{x}$. Then replacing $y$ with $\hat{\xi}$ and $x$ with $\hat{x}$,

$$\langle \hat{\xi} - \left(\lambda F\hat{x} + \hat{\xi} - \lambda F t\right), \hat{x} - t \rangle \geq 0$$

and so

$$\lambda \langle F t - F\hat{x}, t - \hat{x} \rangle \leq 0$$

which implies from Theorem 23.2.5 that $t = \hat{x}$ and so the graph of $A$ is indeed maximal monotone.

$$z + \lambda F(t) = \lambda F\hat{x} + \hat{\xi} \Rightarrow z = \hat{\xi} \in A\hat{x} = At$$

**Note** that this would have worked with no change if the duality map had been for arbitrary $p > 1$.

### 23.7.1 The min max Theorem

In fact, these two conditions are equivalent. This is shown in [12]. We give a proof of this here. First it is necessary to prove a min max theorem. The proof given follows Brezis [23] which is where I found it. Here is the min max theorem. A function $f$ is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$
It is concave if the inequality is turned around. It can be shown that in finite dimensions, convex functions are automatically continuous, similar for concave functions. Recall the following definition of upper and lower semicontinuous functions defined on a metric space and having values in $[-\infty, \infty]$.

**Definition 23.7.3** A function is upper semicontinuous if whenever $x_n \to x$, it follows that $f(x) \geq \limsup_{n \to \infty} f(x_n)$ and it is lower semicontinuous if $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

**Lemma 23.7.4** If $\mathcal{F}$ is a set of functions which are upper semicontinuous, then $g(x) \equiv \inf \{f(x) : f \in \mathcal{F}\}$ is also upper semicontinuous. Similarly, if $\mathcal{F}$ is a set of functions which are lower semicontinuous, then if $g(x) \equiv \sup \{f(x) : f \in \mathcal{F}\}$ it follows that $g$ is lower semicontinuous.

**Proof:** Let $f \in \mathcal{F}$ where these functions are upper semicontinuous. Then if $x_n \to x$, and $g(x) \equiv \inf \{f(x) : f \in \mathcal{F}\}$,

$$f(x) \geq \limsup_{n \to \infty} f(x_n) \geq \limsup_{n \to \infty} g(x_n)$$

Since this is true for each $f \in \mathcal{F}$, then it follows that you can take the infimum and obtain $g(x) \geq \limsup_{n \to \infty} g(x_n)$. Similarly, lower semicontinuity is preserved on taking sup.

Note that in a metric space, the above definitions up upper and lower semicontinuity in terms of sequences are equivalent to the definitions that

$$f(x) \geq \limsup_{r \to 0} \{f(y) : y \in B(x, r)\}$$

$$f(x) \leq \liminf_{r \to 0} \{f(y) : y \in B(x, r)\}$$

respectively.

Here is a technical lemma which will make the proof shorter. It seems fairly interesting also.

**Lemma 23.7.5** Suppose $H : A \times B \to \mathbb{R}$ is strictly convex in the first argument and concave in the second argument where $A, B$ are compact convex nonempty subsets of Banach spaces $E, F$ respectively and $x \to H(x, y)$ is lower semicontinuous while $y \to H(x, y)$ is upper semicontinuous. Let

$$H(g(y), y) \equiv \min_{x \in A} H(x, y)$$

Then $g(y)$ is uniquely defined and also for $t \in [0, 1]$,

$$\lim_{t \to 0} g(y + t(z - y)) = g(y).$$

**Proof:** First suppose both $z, w$ yield the definition of $g(y)$. Then

$$H\left(\frac{z + w}{2}, y\right) < \frac{1}{2} H(z, y) + \frac{1}{2} H(w, y)$$
which contradicts the definition of \( g(y) \). As to the existence of \( g(y) \) this is nothing more than the theorem that a lower semicontinuous function defined on a compact set achieves its minimum.

Now consider the last claim about "hemicontinuity". For all \( x \in A \), it follows from the definition of \( g \) that
\[
H(g(y + t(z - y)), y + t(z - y)) \leq H(x, y + t(z - y))
\]
By concavity of \( H \) in the second argument,
\[
(1 - t) H(g(y + t(z - y)), y) + tH(g(y + t(z - y)), z) \leq H(x, y + t(z - y))
\]
(23.7.39)
Now let \( t_n \to 0 \). Does \( g(y + t_n(z - y)) \to g(y) \)? Suppose not. By compactness, \( g(y + t_n(z - y)) \) is in a compact set and so there is a further subsequence, still denoted by \( t_n \) such that
\[
g(y + t_n(z - y)) \to \hat{x} \in A
\]
Then passing to a limit in (23.7.40), one obtains, using the upper semicontinuity in one and lower semicontinuity in the other the following inequality.
\[
H(\hat{x}, y) \leq \lim inf_{n \to \infty} (1 - t_n) H(g(y + t_n(z - y)), y) + \lim inf_{n \to \infty} t_n H(g(y + t_n(z - y)), z) \leq \lim sup_{n \to \infty} H(x, y + t_n(z - y)) \leq H(x, y)
\]
This shows that \( \hat{x} = g(y) \) because this holds for every \( x \). Since \( t_n \to 0 \) was arbitrary, this shows that in fact
\[
\lim_{t \to 0^+} g(y + t(z - y)) = g(y)
\]
Now with this preparation, here is the min-max theorem. A norm is called strictly convex if whenever \( x \neq y, \| \frac{x + y}{2} \| < \| \frac{x}{2} \| + \| \frac{y}{2} \| \).

**Theorem 23.7.6** Let \( E, F \) be Banach spaces with \( E \) having a strictly convex norm. Also suppose that \( A \subseteq E, B \subseteq F \) are compact and convex sets and that \( H : A \times B \to \mathbb{R} \) is such that
\[
x \to H(x, y) \text{ is convex}
y \to H(x, y) \text{ is concave}
\]
Thus \( H \) is continuous in each variable in the case of finite dimensional spaces. Here assume that \( x \to H(x, y) \) is lower semicontinuous and \( y \to H(x, y) \) is upper semicontinuous. Then
\[
\min_{x \in A} \max_{y \in B} H(x, y) = \max_{y \in B} \min_{x \in A} H(x, y)
\]
This condition is equivalent to the existence of \((x_0, y_0) \in A \times B\) such that
\[
H(x_0, y) \leq H(x_0, y_0) \leq H(x, y_0) \quad \text{for all } x, y
\tag{23.7.41}
\]

**Proof:** One part of the main equality is obvious.
\[
\max_{y \in B} H(x, y) \geq H(x, y) \geq \min_{x \in A} H(x, y)
\]
and so for each \(x,\)
\[
\max_{y \in B} H(x, y) \geq \max_{y \in B} \min_{x \in A} H(x, y)
\]
and so
\[
\min_{x \in A} \max_{y \in B} H(x, y) \geq \max_{y \in B} \min_{x \in A} H(x, y)
\tag{23.7.42}
\]
Next consider the other direction.
Define \(H_\varepsilon(x, y) \equiv H(x, y) + \varepsilon \|x\|^2\) where \(\varepsilon > 0\). Then \(H_\varepsilon\) is strictly convex in the first variable. This results from the observation that
\[
\left\| \frac{x + y}{2} \right\|^2 < \left( \frac{\|x\| + \|y\|}{2} \right)^2 \leq \frac{1}{2} \left( \|x\|^2 + \|y\|^2 \right),
\]
Then by Lemma 23.7.3 there exists a unique \(x \equiv g(y)\) such that
\[
H_\varepsilon(g(y), y) \equiv \min_{x \in A} H_\varepsilon(x, y)
\]
and also, whenever \(y, z \in A,\)
\[
\lim_{t \to 0^+} g(y + t(z - y)) = g(y).
\]
Thus \(H_\varepsilon(g(y), y) = \min_{x \in A} H_\varepsilon(x, y)\). But also this shows that \(y \to H_\varepsilon(g(y), y)\) is the minimum of functions which are upper semicontinuous and so this function is also upper semicontinuous. Hence there exists \(y^*\) such that
\[
\max_{y \in B} H_\varepsilon(g(y), y) = H_\varepsilon(g(y^*), y^*) = \max_{y \in B} \min_{x \in A} H_\varepsilon(x, y)
\tag{23.7.43}
\]
Thus from concavity in the second argument and what was just defined, for \(t \in (0, 1),\)
\[
H_\varepsilon(g(y^*), y^*) \geq H_\varepsilon(g((1-t)y^* + ty), (1-t)y^* + ty)
\]
\[
\geq (1-t) H_\varepsilon(g((1-t)y^* + ty), y^*) + t H_\varepsilon(g((1-t)y^* + ty), y)
\]
\[
\geq (1-t) H_\varepsilon(g(y^*), y^*) + t H_\varepsilon(g((1-t)y^* + ty), y)
\tag{23.7.44}
\]
This is because \(\min_{x} H_\varepsilon(x, y^*) = H_\varepsilon(g(y^*), y^*)\) so \(H_\varepsilon(g((1-t)y^* + ty), y^*) \geq H_\varepsilon(g(y^*), y^*)\). Then subtracting the first term on the right, one gets
\[
t H_\varepsilon(g(y^*), y^*) \geq t H_\varepsilon(g((1-t)y^* + ty), y)
and cancelling the $t$,

$$H_\epsilon (g (y^*), y^*) \geq H_\epsilon (g ((1 - t) y^* + ty), y)$$

Now apply Lemma 23.7.5 and let $t \to 0^+$. This along with lower semicontinuity yields

$$H_\epsilon (g (y^*), y^*) \geq \liminf_{t \to 0^+} H_\epsilon (g ((1 - t) y^* + ty), y) = H_\epsilon (g (y^*), y) \quad (23.7.45)$$

Hence for every $x, y$

$$H_\epsilon (x, y^*) \geq H_\epsilon (g (y^*), y^*) \geq H_\epsilon (g (y^*), y)$$

Thus

$$\min_x H_\epsilon (x, y^*) \geq H_\epsilon (g (y^*), y^*) \geq \max_y H_\epsilon (g (y^*), y)$$

and so

$$\max_y \min_{y \in B} H_\epsilon (x, y) \geq \min_x H_\epsilon (x, y^*) \geq \max_y H_\epsilon (g (y^*), y) \geq \min_{x \in A} \max_{y \in B} H_\epsilon (x, y)$$

Thus, letting $C \equiv \max \{||x|| : x \in A\}$

$$\varepsilon C^2 + \max_y \min_{y \in B} H (x, y) \geq \min_{x \in A} \max_{y \in B} H (x, y)$$

Since $\varepsilon$ is arbitrary, it follows that

$$\max_y \min_{y \in B} H (x, y) \geq \min_{x \in A} \max_{y \in B} H (x, y)$$

This proves the first part because it was shown above in (23.7.42) that

$$\min_{x \in A} \max_{y \in B} H (x, y) \geq \max_{y \in B} \min_{x \in A} H (x, y)$$

Now consider (23.7.41) about the existence of a “saddle point” given the equality of $\min$ $\max$ and $\max$ $\min$. Let

$$\alpha = \max_{y \in B} \min_{x \in A} H (x, y) = \min_{x \in A} \max_{y \in B} H (x, y)$$

Then from

$$y \to \min_{x \in A} H (x, y) \text{ and } x \to \max_{y \in B} H (x, y)$$

being upper semicontinuous and lower semicontinuous respectively, there exist $y_0$ and $x_0$ such that

$$\alpha = \min_{x \in A} H (x, y_0) = \max_{y \in B} \min_{x \in A} H (x, y) = \min_{x \in A} \max_{y \in B} H (x, y) = \max_{y \in B} \min_{x \in A} H (x_0, y)$$
Then
\[ \alpha = \max_{y \in B} H(x_0, y) \geq H(x_0, y_0) \]
\[ \alpha = \min_{x \in A} H(x, y_0) \leq H(x_0, y_0) \]
so in fact \( \alpha = H(x_0, y_0) \) and from the above equalities,
\[ H(x_0, y_0) = \alpha = \min_{x \in A} H(x, y_0) \leq H(x, y_0) \]
\[ H(x_0, y_0) = \alpha = \max_{y \in B} H(x_0, y) \geq H(x_0, y) \]
and so
\[ H(x_0, y) \leq H(x_0, y_0) \leq H(x, y_0) \]
Thus if the min max condition holds, then there exists a saddle point, namely \((x_0, y_0)\).

Finally suppose there is a saddle point \((x_0, y_0)\) where
\[ H(x_0, y) \leq H(x_0, y_0) \leq H(x, y_0) \]
Then
\[ \min_{x \in A} \max_{y \in B} H(x, y) \leq \max_{y \in B} H(x_0, y) \leq \min_{x \in A} H(x_0, y_0) \leq \max_{y \in B} \min_{x \in A} H(x, y) \]
However, as noted above, it is always the case that
\[ \max_{y \in B} \min_{x \in A} H(x, y) \leq \min_{x \in A} \max_{y \in B} H(x, y) \]

Of course all of this works with no change if you have \(E, F\) reflexive Banach spaces and the sets \(A, B\) are just closed and bounded and convex. Then you just use the fact that the functional is weakly lower semicontinuous in the first variable and weakly upper semicontinuous in the second. Recall that lower semicontinuous and convex implies weakly lower semicontinuity. Then just use weak convergence instead of strong convergence in the above argument. Recall that closed bounded and convex sets with the weak topology can be considered metric spaces. I think the above is most interesting in finite dimensions. Of course in this case, you can simply assume the norm is the standard Euclidean norm and there is then no need to assume one of the norms is strictly convex. It comes automatically. Just use an equivalent norm which is strictly convex.

23.7.2 Equivalent Conditions For Maximal Monotone

Next is the theorem about the graph being maximal being equivalent to the operator being maximal monotone. It is a very convenient result to have. The proof is a modified version of one in Barbu [12]. It is based on the following lemma also in Barbu. This is a little like the Browder lemma but is based on the min max theorem above. It is also a very interesting argument.
Lemma 23.7.7 Let $E$ be a finite dimensional Banach space and let $K$ be a convex and compact subset of $E$. Let $G(A)$ be a monotone subset of $E \times E'$ such that $D(A) \subseteq K$ and $B$ is a single valued monotone and continuous operator from $E$ to $E'$. Then there exists $x \in K$ such that
\[
\langle Bx + v, u - x \rangle_{E', E} \geq 0 \text{ for all } [u, v] \in G(A).
\]
If $B$ is coercive
\[
\lim_{\|x\| \to \infty} \frac{\langle Bx, x \rangle}{\|x\|} = \infty,
\]
and $0 \in D(A)$, then one can assume only that $K$ is convex and closed.

**Proof:** Let $T : E \to K$ be the multivalued operator defined by
\[
Ty = \left\{ x \in K : \langle By + v, u - x \rangle_{E', E} \geq 0 \text{ for all } [u, v] \in G(A) \right\}
\]
Here $y \in E$ and it is desired to show that $Ty \neq \emptyset$ for all $y \in K$. For $[u, v] \in G(A)$, let
\[
K_{u,v} = \left\{ x \in K : \langle By + v, u - x \rangle_{E', E} \geq 0 \right\}
\]
Then $K_{u,v}$ is a closed, hence compact subset of $K$. The thing to do is to show that $\cap_{[u,v] \in G(A)} K_{u,v} = Ty \neq \emptyset$ whenever $y \in K$. Then one argues that $T$ is set valued, has convex compact values and is upper semicontinuous. Then one applies the Kakutani fixed point theorem to get $x \in Tx$.

Since these sets $K_{u,v}$ are compact, it suffices to show that they satisfy the finite intersection property. Thus for $\{[u_i, v_i]\}_{i=1}^n$ a finite set of elements of $G(A)$, it is necessary to show that there exists a solution $x$ to the inequalities
\[
\langle u_i - x, By + v_i \rangle \geq 0, \quad i = 1, 2, \ldots, n
\]
and then it follows from finite intersection property that there exists
\[
x \in \cap_{[u,v] \in G(A)} K_{u,v}
\]
which is what was desired. Let $P_n$ be all $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ such that each $\lambda_k \geq 0$ and $\sum_{k=1}^n \lambda_k = 1$. Let $H : P_n \times P_n \to \mathbb{R}$ be given by
\[
H (\vec{\mu}, \vec{\lambda}) = \sum_{i=1}^n \mu_i \left( By_i + \sum_{j=1}^n \lambda_j u_j - u_i \right) \quad (23.7.46)
\]
Then this is both convex and concave in both $\vec{\lambda}, \vec{\mu}$ and so by Theorem [23.7.6], there exists $\vec{\mu}_0, \vec{\lambda}_0$ both in $P_n$ such that for all $\vec{\mu}, \vec{\lambda}$,
\[
H (\vec{\mu}, \vec{\lambda}_0) \leq H (\vec{\mu}_0, \vec{\lambda}_0) \leq H (\vec{\mu}_0, \vec{\lambda}) \quad (23.7.47)
\]
However, plugging in $\vec{\mu} = \vec{\lambda}$ in (23.7.46)

$$
H(\vec{\lambda}, \vec{\lambda}) = \sum_{i=1}^{n} \lambda_i \left( B y + v_i, \sum_{j=1}^{n} \lambda_j u_j - u_i \right)
$$

$$
= \sum_{i=1}^{n} \left( B y + v_i, \sum_{j=1}^{n} \lambda_i \lambda_j u_j - \lambda_i u_i \right)
$$

$$
= \sum_{i=1}^{n} \left( B y + v_i, \sum_{j=1}^{n} (\lambda_i \lambda_j u_j - \lambda_i \lambda_j u_i) \right)
$$

$$
= \sum_{i=1}^{n} \left( B y + v_i, \sum_{j=1}^{n} (\lambda_i \lambda_j u_j - \lambda_i \lambda_j u_i) \right) + \sum_{i=1}^{n} \left( v_i, \sum_{j=1}^{n} (\lambda_i \lambda_j u_j - \lambda_i \lambda_j u_i) \right)
$$

The first term obviously equals 0. Consider the second. This term equals

$$
\sum_{i}^{n} \sum_{j}^{n} \lambda_i \lambda_j (v_i, (u_j - u_i))
$$

The terms equal 0 when $j = i$ or they come in pairs

$$
\lambda_i \lambda_j (v_i, (u_j - u_i)) + \lambda_i \lambda_j (v_i, (u_i - u_j))
$$

$$
= \lambda_i \lambda_j ((v_i, (u_j - u_i)) - (v_i, (u_j - u_i)))
$$

$$
= \lambda_i \lambda_j ((v_i, (u_j - u_i)) - (v_i, (u_j - u_i))) \leq 0
$$

by monotonicity of $A$. Hence $H(\vec{\lambda}, \vec{\lambda}) \leq 0$. Then from (23.7.47) for all $\vec{\mu}$

$$
H(\vec{\mu}, \vec{\lambda}_0) \leq H(\vec{\mu}_0, \vec{\lambda}_0) \leq H(\vec{\mu}_0, \vec{\mu}_0) \leq 0
$$

It follows that

$$
\sum_{i=1}^{m} \mu_i \left( B y + v_i, \sum_{j=1}^{n} \lambda^0_j u_j - u_i \right) \leq 0
$$

$$
\sum_{i=1}^{m} \mu_i \left( B y + v_i, u_i - \sum_{j=1}^{n} \lambda^0_j u_j \right) \geq 0
$$

where $\vec{\lambda}_0 \equiv (\lambda^0_1, \cdots, \lambda^0_n)$. This is true for any choice of $\vec{\mu}$. In particular, you could let $\vec{\mu}$ equal 1 in the $i^{th}$ position and 0 elsewhere and conclude that for all $i = 1, \cdots, n$,

$$
\left( B y + v_i, u_i - \sum_{j=1}^{n} \lambda^0_j u_j \right) \geq 0
$$
so you let \( x = \sum_{j=1}^{n} \lambda_{j}^{0} u_{j} \) and this shows that \( Ty \neq \emptyset \) because the sets \( K_{u,v} \) have the finite intersection property.

Thus \( T : K \to \mathcal{P}(K) \) and for each \( y \in K, Ty \neq \emptyset \). In fact this is true for any \( y \) but we are only considering \( y \in K \). Now \( Ty \) is clearly a closed subset of \( K \). It is also clearly convex. Is it upper semicontinuous? Let \( y_{k} \to y \) and consider \( Ty + B(0,r) \). Is \( Ty_{k} \in Ty + B(0,r) \) for all \( k \) large enough? If not, then there is a subsequence, denoted as \( z_{k} \in Ty_{k} \) which is outside this open set \( Ty + B(0,r) \). Then taking a further subsequence, still denoted as \( z_{k} \), it follows that \( z_{k} \to z \in Ty + B(0,r) \). Now

\[
\langle By_{k} + u, u - z_{k} \rangle \geq 0 \text{ all } [u,v] \in \mathcal{G}(A)
\]

Therefore, from continuity of \( B \),

\[
\langle By + u, u - z \rangle \geq 0 \text{ all } [u,v] \in \mathcal{G}(A)
\]

which means \( z \in Ty \) contrary to the assumption that \( T \) is not upper semicontinuous.

Since \( T \) is upper semicontinuous and maps to compact convex sets, it follows from Theorem 23.4.4 that \( T \) has a fixed point \( x \in Tx \). Hence there exists a solution \( x \) to

\[
\langle Bx + u, u - x \rangle \geq 0 \text{ all } [u,v] \in \mathcal{G}(A)
\]

Next suppose that \( K \) is only closed and convex but \( B \) is coercive and \( 0 \in D(A) \). Then let \( K_{n} = B(0,n) \cap K \) and let \( A_{n} \) be the restriction of \( A \) to \( B(0,n) \). It follows that there exists \( x_{n} \in K_{n} \) such that for all \( [u,v] \in \mathcal{G}(A_{n}) \),

\[
\langle Bx_{n} + v, u - x_{n} \rangle \geq 0
\]

Then since \( 0 \in D(A) \), one can pick \( v_{0} \in A0 \) and obtain

\[
\langle Bx_{n} + v_{0}, -x_{n} \rangle \geq 0, \quad \langle v_{0}, -x_{n} \rangle \geq \langle Bx_{n}, x_{n} \rangle
\]

from which it follows from coercivity of \( B \) that the \( x_{n} \) are bounded independent of \( n \). Say \( \|x_{n}\| < C \). Then there is a subsequence still denoted as \( x_{n} \) such that \( x_{n} \to x \in K \), thanks to the assumption that \( K \) is closed and convex. Let \( [u,v] \in \mathcal{G}(A) \).

Then for all \( n \) large enough \( \|u\| < n \) and so

\[
\langle Bx_{n} + v, u - x_{n} \rangle \geq 0
\]

Then letting \( n \to \infty \) and using the continuity of \( B \),

\[
\langle Bx + v, u - x \rangle \geq 0
\]

Since \( [u,v] \) was arbitrary, this proves the lemma. \( \blacksquare \)

**Observation 23.7.8** If you have a monotone set valued function, then its graph can always be considered a subset of the graph of a maximal monotone graph. If \( A \) is monotone, then let \( F \) be \( \mathcal{G}(B) \) such that \( \mathcal{G}(B) \supseteq \mathcal{G}(A) \) and \( B \) is monotone. Partially order by set inclusion. Then let \( C \) be a maximal chain. Let \( \mathcal{G}(\hat{A}) = \cup C \).
If \([x_i, y_i] \in \mathcal{G} (\hat{A})\), then both are in some \(B \in \mathcal{C}\). Hence \((y_1 - y_2, x_1 - x_2) \geq 0\) so monotone and must be maximal monotone because if \((z - v, x - u) \geq 0\) for all \([u, v] \in \mathcal{G} (\hat{A})\) and \([x, z] \notin \hat{A}\), then you could include this ordered pair and contradict maximality of the chain \(\mathcal{C}\).

Next is an interesting theorem which comes from this lemma. It is an infinite dimensional version of the above lemma.

**Theorem 23.7.9** Let \(X\) be a reflexive Banach space and let \(K\) be a closed convex subset of \(X\). Let \(A, B\) be monotone such that

1. \(\text{Dom}(A) \subseteq K, 0 \in \text{Dom}(A)\).
2. \(B\) is single valued, hemicontinuous, bounded and coercive mapping \(X\) to \(X'\).

Then there exists \(x \in K\) such that

\[(Bx + v, u - x)_{X', X} \geq 0\] for all \([u, v] \in \mathcal{G}(A)\)

Before giving the proof, here is an easy lemma.

**Lemma 23.7.10** Let \(E\) be finite dimensional and let \(B : E \to E'\) be monotone and hemicontinuous. Then \(B\) is continuous.

**Proof:** The space can be considered a finite dimensional Hilbert space \((\mathbb{R}^n)\) and so weak and strong convergence are exactly the same. First it is desired to show that \(B\) is bounded. Suppose it is not. Then there exists \(\|x_k\|_E = 1\) but \(\|Bx_k\|_{E'} \to \infty\). Since finite dimensional, there is a subsequence still denoted as \(x_k\) such that \(x_k \to x, \|x\|_E = 1\).

\[(Bx_k - Bx, x_k - x) \geq 0\]

Hence

\[\frac{(Bx_k - Bx)}{\|Bx_k\|_{E'}} \cdot x_k - x \geq 0\]

Then taking another subsequence, written with index \(k\), it can be assumed that

\[Bx_k/\|Bx_k\| \to y^* \in E', \|y^*\|_{E'} = 1\]

Hence,

\[\langle y^*, x_k - x \rangle \geq 0\]

for all \(x \in E\), but this requires that \(y^* = 0\), a contradiction. Thus \(B\) is monotone, hemicontinuous, and bounded. It follows from Theorem 23.1.8 which says that monotone and hemicontinuous operators are pseudomonotone and Proposition 23.1.10 which says that bounded pseudomonotone operators are demicontinuous that \(B\) is demicontinuous, hence continuous because, as just noted above, weak and strong convergence are the same for finite dimensional spaces. In case \(B\) is
bounded, then this follows from Proposition 23.4.4 above. It is pseudomonotone and bounded hence demicontinuous and weak and strong convergence is the same in finite dimensions.

Proof of Theorem 23.7.10: Let \( \{X_n\} \) be an increasing sequence of finite dimensional subspaces. Let \( \hat{A} \) be maximal monotone on \( \cup_n X_n \) and extending \( A \). By this is meant that the graph of \( \hat{A} \) contains the graph of \( A \) restricted to \( \cup_n X_n \), \( \hat{A} \) is monotone and there is no other larger graph with these properties. See the above observation. Let \( j_n : X_n \to X \) be the inclusion map and \( j_n^* : X' \to X_n' \) be the dual map. Then \( j_n^* \hat{A}j_n \equiv A_n \) and \( j_n^* B_j \equiv B_n \) have monotone graphs from \( X_n \) to \( \mathcal{P}(X_n') \) with \( B_n \) being continuous and single valued. This follows from the hemicontinuity and the above lemma which states that on finite dimensional spaces, hemicontinuity and monotonicity imply continuity. Then

\[
[u, v] \in \mathcal{G}(A_n)
\]

means

\[
u \in D(A) \cap X_n \text{ and } v \in j_n^* \hat{A}j_n(u) = j_n^* \hat{A}(u) \text{ since } u \in X_n
\]

Then from Lemma 23.7.8, there exists \( x_n \in X_n \) such that

\[
\langle B_n x_n + v_n, u_n - x_n \rangle_{X', X} \geq 0 \text{ all } [u_n, v_n] \in \mathcal{G}(A_n)
\]

That is, there exists \( x_n \in K \cap X_n \) such that for all \( u \in D(\hat{A}) \cap X_n, [u, v] \in \mathcal{G}(\hat{A}) \)

\[
\langle Bx_n + v, u - x_n \rangle_{X', X} \geq 0 \quad (23.7.48)
\]

Then

\[
\langle v, u - x_n \rangle \geq \langle Bx_n, x_n - u \rangle \quad (23.7.49)
\]

From the assumption that \( 0 \in D(\hat{A}) \), one can let \( u = 0 \) and then pick \( v_0 \in \hat{A}0 \). Then the above reduces to

\[
\langle v_0, -x_n \rangle \geq \langle Bx_n, x_n \rangle
\]

By coercivity of \( B \), these \( x_n \) are all bounded and so by the Eberlien Smulian theorem, there is a subsequence \( \{x_n\} \) which satisfies

\[
x_n \to x \text{ weakly in } X
\]

\[
Bx_n \to y \text{ weakly in } X'
\]

Then from 23.7.8

\[
\langle v, u - x_n \rangle + \langle Bx_n, u \rangle \geq \langle Bx_n, x_n \rangle
\]

Then it follows that

\[
\langle v, u - x_n \rangle + \langle Bx_n, u \rangle - \langle Bx_n, x \rangle \geq \langle Bx_n, x_n - x \rangle
\]
It follows that
\[
\limsup_{n \to \infty} \langle Bx_n, x_n - x \rangle \leq \langle v, u - x \rangle + \langle y, u - x \rangle
\]
\[
= \langle v + y, u - x \rangle
\]

**Claim:** \( \limsup_{n \to \infty} \langle Bx_n, x_n - x \rangle \leq 0 \).

**Proof of claim:** This is so if \( \langle v + y, u - x \rangle \leq 0 \) for some \([u, v] \in G(\hat{A})\).

If \( \langle v + y, u - x \rangle \) is greater than 0 for all \([u, v]\), then since \( \hat{A} \) is maximal, it would follow that \(-y \in \hat{A}x\).

Now consider \(23.7.49\).

\[
\langle v, u - x \rangle \geq \limsup_{n \to \infty} \langle Bx_n, x_n \rangle - \langle y, u \rangle
\]

Since \( x \in D(\hat{A}) \), you could put in \(u = x\) in the above and obtain

\[
0 \geq \limsup_{n \to \infty} \langle Bx_n, x_n \rangle - \langle y, x \rangle = \limsup_{n \to \infty} \langle Bx_n, x_n - x \rangle
\]

which shows the claim is true.

Since \(B\) is monotone and hemicontinuous, it satisfies the pseudomonotone condition, Theorem 23.1.4. Hence for any \(z\),

\[
\langle y, x - z \rangle \geq \limsup_{n \to \infty} \langle Bx_n, x_n - x \rangle + \limsup_{n \to \infty} \langle Bx_n, x - z \rangle
\]

\[
\geq \limsup_{n \to \infty} (\langle Bx_n, x_n - x \rangle + \langle Bx_n, x - z \rangle)
\]

\[
\geq \liminf_{n \to \infty} (\langle Bx_n, x_n - z \rangle) \geq \langle Bx, x - z \rangle
\]

Since \(z\) is arbitrary, this shows that \(y = Bx\). It follows from 23.7.48 that for any \([u, v] \in G(\hat{A})\),

\[
\langle Bx_n + v, u - x_n \rangle = \langle Bx_n + v, u - x_n \rangle + \langle Bx_n + v, x - x_n \rangle \geq 0
\]

\[
\langle Bx_n + v, u - x \rangle \geq \langle Bx_n, x_n - x \rangle \geq \langle Bx, x_n - x \rangle
\]

Now take a limit of both sides and use the fact that \(y = Bx\) to obtain

\[
\langle Bx + v, u - x \rangle \geq 0
\]

for all \([u, v] \in G(\hat{A})\). Here \(\hat{A}\) extends \(A\) on \(\bigcup_n X_n\). Why does it follow from this that there exists an \(x\) such that the inequality holds for all \([u, v] \in G(A)\)?

Let \(V\) be a finite dimensional subspace.

\[
K_V \equiv \left\{ x \in K : \langle Bx + v, u - x \rangle_{X', X} \geq 0 \text{ for all } [u, v] \in G(A), u \in V \right\}
\]

Then from the above argument, \(K_V \neq \emptyset\). You just choose your subspaces \(X_n\) to all include \(V\). Also, from coercivity of \(B\) and the above argument, these \(K_V\) are
all bounded and weakly closed. Hence they are weakly compact. Then if you have finitely many of them, you can let your subspaces include each $V$ and conclude that these $K_V$ have finite intersection property and so there exists $x \in \bigcap V K_V$ which gives the desired $x$. ■

Note that there is only one place where $0 \in D(A)$ was used and it was to get the estimate. In the argument,

$$\langle v, u - x_n \rangle \geq \langle Bx_n, x_n - u \rangle$$

and it was convenient to be able to take $u = 0$. However, you could also assume other things on $B$ such as that it satisfies an estimate of the form

$$\|Bx\| \leq C \|x\| + C$$

and if you did this, you could also obtain the necessary estimate as follows.

$$\langle v, u - x_n \rangle \geq \langle Bx_n, x_n - u \rangle$$

and then pick some $[u,v]$. Thus the following corollary comes right away. This would have worked just as well if you had an estimate of the form

$$\|Bx\| \leq C \|x\|^{p-1} + C, \quad p > 1$$

**Corollary 23.7.11** Let $X$ be a reflexive Banach space and let $K$ be a closed convex subset of $X$. Let $A, B$ be monotone such that

1. $D(A) \subseteq K$
2. $B$ is single valued, hemicontinuous, bounded and coercive mapping $X$ to $X'$ which satisfies the estimate

$$\|Bx\| \leq C \|x\| + C \text{ or more generally } \|Bx\| \leq C \|x\|^{p-1} + C, \quad p > 1$$

Then there exists $x \in K$ such that

$$\langle Bx + v, u - x \rangle_{X', X} \geq 0 \text{ for all } [u,v] \in G(A)$$

Now here is the equivalence between maximal monotone graph and having $F + A$ be onto. It was already shown that if $\lambda F + A$ is onto, then the graph of $A$ is maximal monotone in the sense that there is no monotone operator whose graph properly contains the graph of $A$. This was Theorem 23.7.2 above which is stated here as a reminder of what it said.

**Theorem 23.7.12** Let $X, X'$ be reflexive and have strictly convex norms. Let $A$ be a monotone set valued map as just described. Then if

$$\lambda F + A \text{ onto},$$
for some \( \lambda > 0 \), then whenever
\[
\langle y - z, x - u \rangle \geq 0 \text{ for all } [x,y] \in \mathcal{G}(A)
\]
it follows that \( z \in Au \) and \( u \in D(A) \). That is, the graph is maximal.

**Theorem 23.7.13** Let \( X \) be a strictly convex reflexive Banach space. Suppose the graph of \( A : X \to P(X) \) is maximal monotone in the sense that it is monotone and no monotone graph can properly contain the graph of \( A \). Then for all \( \lambda > 0, \lambda F + A \) is onto. Conversely, if for some \( \lambda > 0, \lambda F + A \) is onto, then the graph of \( A \) is maximal with respect to being monotone.

**Proof:** In Theorem 23.7.11 let \( Bx \equiv \lambda F(x) - y_0 \). Then from the properties of the duality map, Theorem 23.7.12 above, it follows that \( B \) satisfies the necessary conditions to use the result of Corollary 23.7.11 with \( K = X \). This \( B \) is monotone hemicontinuous, and coercive. Thus there exists \( x \) such that for all \( [u,v] \in \mathcal{G}(A) \),
\[
\langle \lambda F(x) - y_0 + v, u - x \rangle_{X',X} \geq 0
\]
\[
\langle v - (y_0 - \lambda F(x)), u - x \rangle_{X',X} \geq 0
\]
By maximality of the graph, it follows that \( x \in D(A) \) and
\[
y_0 - \lambda F(x) \in A(x), \ y_0 = \lambda F(x) + A(x)
\]
so \( \lambda F + A \) is onto as claimed. The converse was proved in Theorem 23.7.11. ■

Note that this theorem holds if \( F \) is a duality map for \( p > 1 \). That is, \( \langle Fx, x \rangle = \|x\|_p \|Fx\| = \|x\|^{p-1} \).

Suppose \( A : X \to P(X) \) is maximal monotone. Then let \( z \in X \) and define a new mapping \( \hat{A} \) as follows.
\[
D(\hat{A}) \equiv \{x : x - x_0 \in D(A)\}, \ \hat{A}(x) \equiv A(x - x_0)
\]

**Proposition 23.7.14** Let \( A, \hat{A} \) be as just defined. Then \( \hat{A} \) is also maximal monotone.

**Proof:** From Theorem 23.7.11 it suffices to show that graph of \( \hat{A} \) is monotone and is maximal. Suppose then that \( x_i^* \in \hat{A}x_i \). Then
\[
\langle x_i^* - x_2^*, x_1 - x_2 \rangle = \langle x_i^* - x_2^*, x_1 - x_0 - (x_2 - x_0) \rangle
\]
by definition, \( x_i^* \in A(x_i - x_0) \) and so the above is \( \geq 0 \). Next suppose for all \( [x,x^*] \in \mathcal{G}(\hat{A}) \),
\[
\langle x^* - z^*, x - z \rangle \geq 0
\]
Does it follow that \( [z, z^*] \in \mathcal{G}(\hat{A}) \)? The above says that
\[
\langle x^* - z^*, x - x_0 - (z - x_0) \rangle \geq 0
\]
whenever \( x - x_0 \in D(A) \) and \( x^* \in A(x - x_0) \). Hence, since \( A \) is given to be maximal monotone, \( z - x_0 \in D(A) \) and \( z^* \in A(z - x_0) \) which says that \( z^* \in \hat{A}(z) \). Thus \( \hat{A} \) is maximal monotone by the Theorem 23.7.11. ■
23.7.3 Surjectivity Theorems

As an interesting example of this theorem, here is another result in Barbu \[12\]. It is interesting because it is not assumed \( B \) is bounded.

Theorem 23.7.15 Let \( B : X \to X' \) be monotone hemicontinuous. Then \( B \) is maximal monotone. If \( B \) is coercive, then \( B \) is also onto. Here \( X \) is a strictly convex reflexive Banach space.

**Proof:** Suppose \( B \) is not maximal monotone. Then there exists \((x_0, x_0^*) \in X \times X'\) such that for all \( x \),

\[
\langle Bx - x_0^*, x - x_0 \rangle \geq 0
\]

and yet \( x_0^* \neq Bx_0 \). This is going to be a contradiction. Let \( u \in X \) and consider \( x_t \equiv tx_0 + (1 - t)u \), \( t \in (0,1) \). Then consider

\[
\langle Bx_t - x_0^*, x_t - x_0 \rangle
\]

However, \( x_t - x_0 = tx_0 + (1 - t)u - x_0 = (1 - t)(u - x_0) \) and so, for each \( t \in (0,1) \),

\[
0 \leq \langle Bx_t - x_0^*, x_t - x_0 \rangle = (1 - t) \langle Bx_t - x_0^*, u - x_0 \rangle
\]

Divide by \( (1 - t) \) and then let \( t \uparrow 1 \). This yields the following by hemicontinuity.

\[
\langle Bx_0 - x_0^*, u - x_0 \rangle \geq 0
\]

which holds for all \( u \). Hence \( Bx_0 = x_0^* \) after all. Thus \( B \) is indeed maximal monotone.

Next suppose \( B \) is coercive. Let \( F \) be the duality map (or the duality map for arbitrary \( p > 1 \)). Then from Theorem \[23.7.13\] there exists a solution \( x_\lambda \) to

\[
\lambda Fx_\lambda + Bx_\lambda = x_0^* \in X'
\]

(23.7.50)

Then the \( x_\lambda \) are bounded because, doing both sides to \( x_\lambda \),

\[
\lambda \|x_\lambda\|^2 + \langle Bx_\lambda, x_\lambda \rangle = \langle x_0^*, x_\lambda \rangle
\]

and so

\[
\frac{\langle Bx_\lambda, x_\lambda \rangle}{\|x_\lambda\|} \leq \|x_0^*\|
\]

Thus the coercivity of \( B \) implies that the \( x_\lambda \) are bounded. There exists a subsequence such that

\( x_\lambda \to x \) weakly.

Then from the equation \[23.7.50\] \( \|Fx_\lambda\| = \lambda \|x_\lambda\| \) and so,

\( Bx_\lambda \to x_0^* \) strongly.
Since $B$ is monotone and hemicontinuous, it satisfies the pseudomonotone condition, Theorem 23.1.4. The above strong convergence implies

$$\lim_{\lambda \to 0} \langle Bx_\lambda, x_\lambda - x \rangle = 0$$

Hence for all $y$,

$$\liminf_{\lambda \to 0} \langle Bx_\lambda, x_\lambda - y \rangle = \liminf_{\lambda \to 0} \langle Bx_\lambda, x - y \rangle = \langle x_0^*, x - y \rangle \geq \langle Bx, x - y \rangle$$

Since $y$ is arbitrary, this shows that $x_0^* = Bx$ and so $B$ is onto as claimed.

Again, note that it really didn’t matter about the particular duality map used, although the usual one was featured in the argument.

There are some more things which can be said about maximal monotone operators. To include some of these, here is a very interesting lemma found in [12].

**Lemma 23.7.16** Let $X$ be a Banach space and suppose that

$$x_n \to 0, \quad \|x_n^*\| \to \infty$$

Then denoting by $D_r$ the closed disk centered at 0 with radius $r$. It follows that for every $D_r$, there exists $y_0 \in D_r$ and a subsequence with index $n_k$ such that

$$\langle x_{n_k}^*, x_{n_k} - y_0 \rangle \to -\infty$$

**Proof:** Suppose this is not true. Then there exists $D_r$ which has the property that for all $u \in D_r$,

$$\langle x_n^*, x_n - u \rangle \geq C_u$$

for all $n$. Now let

$$E_k \equiv \{ y \in D_r : \langle x_n^*, x_n - y \rangle \geq -k \text{ for all } n \}$$

Then this is a closed set, being the intersection of closed sets. Also, by assumption, the union of these $E_k$ equals $D_r$ which is a complete metric space. Hence one of these $E_k$ must have nonempty interior by the Bair category theorem, say for $k_0$. Say $B(y, \varepsilon) \subseteq D_r$. Then for all $\|u - y\| < \varepsilon$,

$$\langle x_n^*, x_n - u \rangle \geq -k_0 \text{ for all } n$$

Of course $-y \in D_r$ also, and so there is $C$ such that

$$\langle x_n^*, x_n + y \rangle \geq C \text{ for all } n$$

Then

$$\langle x_n^*, 2x_n + y - u \rangle \geq C - k_0 \text{ for all } n$$

whenever $\|y - u\| < \varepsilon$. Now recall that $x_n \to 0$. Consider only $u$ such that $\|y - u\| < \varepsilon/2$. Therefore, for all $n$ large enough, the expression $2x_n + y - u$ for such $u$ contains a small ball centered at the origin, say $D_\delta$. (The set of all $y - u$ for $u$ closer to $y$
than $\varepsilon/2$ is the ball $B(0, \varepsilon/2)$ and then the $2x_n$ does not move it by much provided $n$ is large enough.) Therefore,

$$\langle x_n^*, v \rangle \geq C - k_0$$

for all $\|v\| \leq \delta$. This contradicts the assumption that $\|x_n^*\| \to \infty$. ■

**Corollary 23.7.17** Let $X$ be a Banach space and suppose that

$$x_n \to x, \quad \|x_n^*\| \to \infty$$

Then denoting by $D_r$ the closed disk centered at $x$ with radius $r$. It follows that for every $D_r$, there exists $y_0 \in D_r$ and a subsequence with index $n_k$ such that

$$\langle x_{n_k}^*, x_{n_k} - y_0 \rangle \to -\infty$$

**Proof:** It follows that $x_n - x \to 0$. Therefore, from Lemma 23.7.16, for every $r > 0$, there exists $\hat{y}_0 \in B(0, r)$ and a subsequence $x_{n_k}$ such that

$$\langle x_{n_k}^*, (x_{n_k} - x) - \hat{y}_0 \rangle \to -\infty$$

Thus

$$\langle x_{n_k}^*, x_{n_k} - (x + \hat{y}_0) \rangle \to -\infty$$

Just let $y_0 = x + \hat{y}_0$. Then $y_0 \in D_r$ and satisfies the desired conditions. ■

**Definition 23.7.18** A set valued mapping $A : D(A) \to \mathcal{P}(X)$ is locally bounded at $x \in D(A)$ if whenever $x_n \to x$, $x_n \in D(A)$ it follows that

$$\limsup_{n \to \infty} \{\|x_n^*\| : x_n^* \in Ax_n\} < \infty.$$ 

**Lemma 23.7.19** A set valued operator $A$ is locally bounded at $x \in \overline{D(A)}$ if and only if there exists $r > 0$ such that $A$ is bounded on $B(x, r) \cap D(A)$.

**Proof:** Say the limit condition holds. Then if no such $r$ exists, it follows that $A$ is unbounded on every $B(x, r) \cap D(A)$. Hence, you can let $r_n \to 0$ and pick $x_n \in B(x, r_n) \cap D(A)$ with $x_n^* \in Ax_n$ such that $\|x_n^*\| > n$, violating the limit condition. Hence some $r$ exists such that $A$ is bounded on $B(x, r) \cap D(A)$. Conversely, suppose $A$ is bounded on $B(x, r) \cap D(A)$. Then if $x_n \to x$, it follows that for all $n$ large enough, $x_n \in B(x, r)$ and so if $x_n^* \in Ax_n$, $\|x_n^*\| \leq M$. Hence $\limsup_{n \to \infty} \{\|x_n^*\| : x_n^* \in Ax_n\} \leq M < \infty$ which verifies the limit condition. ■

With this definition, here is a very interesting result.

**Theorem 23.7.20** Let $A : D(A) \to X'$ be monotone. Then if $x$ is an interior point of $D(A)$, it follows that $A$ is locally bounded at $x$. 
Proof: You could use Corollary 23.7.17. If x is an interior point of D(A), and A is not locally bounded, then there exists \( x_n \rightarrow x \) and \( x^*_n \in Ax_n \) such that \( \|x^*_n\| \rightarrow \infty \). Then by Corollary 23.7.17, there exists \( y_0 \) close to \( x \), in \( D(A) \) and a subsequence \( x_{n_k} \) such that
\[
\langle x^*_{n_k}, x_{n_k} - y_0 \rangle \rightarrow -\infty
\]
Letting \( y_0^* \in Ay_0 \),
\[
\langle x^*_{n_k} - y_0^*, x_{n_k} - y_0 \rangle \geq 0
\]
and so
\[
\langle x^*_{n_k}, x_{n_k} - y_0 \rangle \geq \langle y_0^*, x_{n_k} - y_0 \rangle
\]
and the right side is bounded below because it converges to \( \langle y_0^*, x - y_0 \rangle \) and this is a contradiction. ■

Does the same proof work if \( x \) is a limit point of \( D(A) \)? No. Suppose \( x \) is a limit point of \( D(A) \). If \( A \) is not locally bounded, then there exists \( x_n \rightarrow x, x_n \in D(A) \) and \( x^*_n \in Ax_n \) and \( \|x^*_n\| \rightarrow \infty \). Then there is \( y_0 \) close to \( x \) such that \( \langle x^*_{n_k}, x_{n_k} - y_0 \rangle \rightarrow -\infty \) but now everything crashes in flames because it is not known that \( y_0 \in D(A) \).

It follows from the above theorem that if \( A \) is defined on all of \( X \) and is maximal monotone, then it is locally bounded everywhere. Now here is a very interesting result which is like the one which involves monotone and hemicontinuous conditions. It is in [52].

**Theorem 23.7.21** Let \( A : X \rightarrow \mathcal{P}(X') \) be monotone and satisfies the following conditions:

1. If \( \lambda_n \rightarrow \lambda, \lambda_n \in [0,1] \) and \( z_n \in A(u + \lambda_n(v - u)) \), then if \( B \) is any weakly open set containing 0, \( z_n \in A(u) + B \) for all \( n \) large enough. (Upper semi-continuous into weak topology along a line segment)

2. \( A(x) \) is closed and convex.

Then one can conclude that \( A \) is maximal monotone.

**Proof:** Let \( \hat{A} \) be a monotone extension of \( A \). Let \( [\hat{u}, \hat{w}] \) be such that \( \hat{w} \in \hat{A}(\hat{u}) \). Now also by assumption, \( A(x) \) is not just convex but also closed.

If \( [\hat{u}, \hat{w}] \) is not in the graph of \( A \), then by separation theorems, there is \( u \) such that
\[
\langle x^*, u \rangle < \langle \hat{w}, u \rangle \text{ for all } x^* \in A(\hat{u})
\]
Then for \( \lambda > 0 \), let \( x_\lambda \equiv \hat{u} + \lambda u \), \( x^*_\lambda \in A(x_\lambda) \). Then from monotonicity of \( \hat{A} \),
\[
0 \leq \langle x^*_\lambda - \hat{w}, x_\lambda - \hat{u} \rangle = \lambda \langle x^*_\lambda - \hat{w}, u \rangle
\]
Thus
\[
\langle x^*_\lambda - \hat{w}, u \rangle \geq 0
\]
By Theorem 23.7.20, the monotonicity of \( A \) on \( X \) implies \( A \) is locally bounded also. Thus in particular, \( Ax_\lambda \) for small \( \lambda \) is contained in a bounded set. Now by that hemicontinuity assumption, you can get a subsequence \( \lambda_n \to 0 \) for which \( x^*_\lambda \) converges weakly to \( x^* \in A \hat{u} \). Therefore, passing to the limit in the above, we get

\[
\langle x^* - \hat{w}, u \rangle \geq 0
\]

\[
\langle x^*, u \rangle \geq \langle \hat{w}, u \rangle > \langle x^*, u \rangle
\]
a contradiction. Thus there is no proper extension and this shows that \( A \) is maximal monotone.

Recall the definition of a pseudomonotone operator.

**Definition 23.7.22** A set valued operator \( B \) is quasi-bounded if whenever \( x \in D(B) \) and \( x^* \in Bx \) are such that

\[
|\langle x^*, x \rangle|, \|x\| \leq M,
\]

it follows that \( \|x^*\| \leq K_M \). Bounded would mean that if \( \|x\| \leq M \), then \( \|x^*\| \leq K_M \).

Here you only know this if there is another condition.

By Proposition 23.7.23, an example of a quasi-bounded operator is a maximal monotone operator \( G \) for which \( 0 \in \text{int} (D(G)) \).

Then there is a useful result which gives examples of quasi-bounded operators \( 23 \).

**Proposition 23.7.23** Let \( A : D(A) \subseteq X \to \mathcal{P}(X') \) be maximal monotone and suppose \( 0 \in \text{int} (D(A)) \). Then \( A \) is quasi-bounded.

**Proof:** From local boundedness, Theorem 23.7.20, there exists \( \delta, C > 0 \) such that

\[
\sup \{ \|x^*\| : x^* \in A(x) \text{ for } \|x\| \leq \delta \} < C
\]

Now suppose that \( \|x\|, |\langle x^*, x \rangle| \leq M \). Then letting \( \|y\| \leq \delta, y^* \in Ay \),

\[
0 \leq \langle x^* - y^*, x - y \rangle = \langle x^*, x \rangle - \langle x^*, y \rangle - \langle y^*, x \rangle + \langle y^*, y \rangle
\]

and so for \( \|y\| \leq \delta \),

\[
\langle x^*, y \rangle \leq \langle x^*, x \rangle - \langle y^*, x \rangle + \langle y^*, y \rangle \leq M + MC + C\delta
\]

Hence, \( \|x^*\| \leq M + MC + C\delta \equiv K_M \).

This is actually quite a restrictive requirement and leaves out a lot which would be interesting.

**Definition 23.7.24** Let \( V \) be a Reflexive Banach space. We say \( T : V \to \mathcal{P}(V') \) is pseudomonotone if the following conditions hold.

\[
Tu \text{ is closed, nonempty, convex.}
\]
23.7. MAXIMAL MONOTONE OPERATORS

If $F$ is a finite dimensional subspace of $V$, then if $u \in F$ and $W \supseteq Tu$ for $W$ a weakly open set in $V'$, then there exists $\delta > 0$ such that

$$v \in B(u, \delta) \cap F \text{ implies } Tv \subseteq W.$$  \hfill (23.7.52)

If $u_k \rightharpoonup u$ and if $u_k^* \in Tu_k$ is such that

$$\limsup_{k \to \infty} u_k^*(u_k - u) \leq 0,$$

then for all $v \in V$, there exists $u^*(v) \in Tu$ such that

$$\liminf_{k \to \infty} u_k^*(u_k - v) \geq u^*(v)(u - v).$$  \hfill (23.7.53)

Then here is an interesting result [10].

**Theorem 23.7.25** Suppose $A : X \to \mathcal{P}(X')$ is maximal monotone. That is, $D(A) = X$. Then $A$ is pseudomonotone.

**Proof:** Consider the first condition. Say $x_1^* \in Ax$. Let $u^* \in Au$. For $\lambda \in [0, 1]$,

$$\langle \lambda x_1^* + (1 - \lambda) x_2^* - u^*, x - u \rangle = \lambda \langle x_1^* - u^*, x - u \rangle + (1 - \lambda) \langle x_2^* - u^*, x - u \rangle \geq 0$$

and so, since $[u, u^*]$ is arbitrary, it follows that $\lambda x_1^* + (1 - \lambda) x_2^* \in Ax$. Thus $Ax$ is convex. Is it closed? Say $x_n^* \in Ax$ and $x_n^* \to x^*$. Is it the case that $x^* \in D(A)$? Let $[u, u^*] \in \mathcal{G}(A)$ be arbitrary. Then

$$\langle x^* - u^*, x - u \rangle = \lim_{n \to \infty} \langle x_n^* - u^*, x_n - u \rangle \geq 0$$

and so $Ax$ is also closed.

Consider the second condition. It is to show that if $x_n \to x$ in $V$ a finite dimensional subspace and if $U$ is a weakly open set containing 0, then eventually $Ax_n \subseteq Ax + U$. Suppose then that this is not the case. Then there exists $x_n^*$ outside of $Ax + U$ but in $Ax_n$. Since $A$ is locally bounded at $x$, it follows that the $\|x_n^*\|$ are bounded. Thus there is a subsequence, still denoted as $x_n$ and $x_n^*$ such that $x_n \to x^*$ weakly and $x^* \notin Ax + U$. Now let $[u, u^*] \in \mathcal{G}(A)$.

$$\langle x^* - u^*, x - u \rangle = \lim_{n \to \infty} \langle x_n^* - u^*, x_n - u \rangle \geq 0$$

and since $[u, u^*]$ is arbitrary, it follows that $x^* \in Ax$ and so is inside $Ax + U$. Thus the second condition holds also.

Consider the third. Say $x_k \to x$ weakly and letting $x_k^* \in Ax_k$, suppose

$$\limsup_{k \to \infty} \langle x_k^*, x_k - x \rangle \leq 0,$$

Is it the case that there exists $x^*(y) \in Ax$ such that

$$\lim_{k \to \infty} \inf \langle x_k^*, x_k - y \rangle \geq \langle x^*(y), x - y \rangle?
The proof goes just like it did earlier in the case of single valued pseudomonotone operators. It is just a little more complicated. First, let \( x^* \in Ax \).
\[
\langle x^*_k - x^*, x_k - x \rangle \geq 0
\]
and so
\[
\liminf_{k \to \infty} \langle x^*_k, x_k - x \rangle \geq \liminf_{k \to \infty} \langle x^*, x_k - x \rangle = 0 \geq \limsup_{k \to \infty} \langle x^*_k, x_k - x \rangle
\]
Thus
\[
\lim_{k \to \infty} \langle x^*_k, x_k - x \rangle = 0.
\]
Now let \( x^*_t \in A(x + t(y - x)), t \in (0, 1) \), where here \( y \) is arbitrary. Then
\[
\langle x^*_n - x^*_t, x_n - x + t(x - y) \rangle \geq 0
\]
Hence
\[
\liminf_{n \to \infty} \langle x^*_n, x_n - x + t(x - y) \rangle \geq \liminf_{n \to \infty} \langle x^*_t, x_n - x + t(x - y) \rangle
\]
and so from the above limit,
\[
t \liminf_{n \to \infty} \langle x^*_n, x - y \rangle \geq t \langle x^*_t, x - y \rangle
\]
Cancel the \( t \).
\[
\liminf_{n \to \infty} \langle x^*_n, x - y \rangle = \liminf_{n \to \infty} \langle x^*_n, x_n - y \rangle \geq \langle x^*_t, x - y \rangle
\]
Now you have a fixed \( y \) and \( x^*_t \in A(x + t(y - x)) \). The subspace determined by \( x, y \) is finite dimensional. Also it was shown above that \( A \) is locally bounded at \( x \) and so there is a subsequence, still denoted as \( x^*_t \) such that \( x^*_t \to x^*(y) \) weakly.
Now from the upper semicontinuity on finite dimensional spaces shown above, for every \( S \) a finite subset of \( X \) and \( \varepsilon > 0 \), it follows that for all \( t \) small enough,
\[
x^*_t \in Ax + B_S(0, \varepsilon)
\]
Thus \( x^*(y) \in Ax \). Hence, there exists \( x^*(y) \in Ax \) such that
\[
\liminf_{n \to \infty} \langle x^*_n, x_n - y \rangle \geq \langle x^*(y), x - y \rangle \]
I found this in a paper by Peng. It is a very nice result.

**Proposition 23.7.26** Let \( X \) and \( Y \) be reflexive Banach spaces with \( Y \subseteq X' \). Let \( 1 < p < \infty \) and let \( q = \frac{p}{p-1} = p' \) so \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( F : [0, T] \times X \to \mathcal{P}(Y) \) be multivalued and satisfies.

1. \( F(\cdot, x) \) has a measurable selection for each \( x \in X \)
23.7. MAXIMAL MONOTONE OPERATORS

2. $F(t, \cdot)$ is maximal monotone for a.e. $t \in [0, T]$.

3. $\|y\|_Y \leq \rho_1(t) + \rho_2 \|x\|_{X}^{p-1}$ where $y \in F(t, x)$ for a.e. $t$, and where $\rho_1 \in L^q(0, T)$ and $\rho_2 > 0$.

Let $0 \leq a < b \leq T$ with $b - a = \tau$. Define

$$F_{r}x \equiv \left\{ \frac{1}{\tau} \int_{a}^{b} y(t) \, dt : t \rightarrow y(t) \text{ is measurable} \right\}$$

and $\lambda y(t) + (1 - \lambda) \hat{y}(t) \in F(t, x)$ because $F(t, \cdot)$ is maximal monotone which implies that the set values are convex.

Next is a claim that $F_r x$ is closed and also has the property that if $z_n \in F_r x_n$ and if $x_n \rightarrow x$ strongly in $X$ and $z_n \rightarrow z$ weakly in $Y$, then $z \in F_r x$. Let $y_n(t) \in F(t, x_n)$ a.e. such that $z_n = \frac{1}{\tau} \int_{a}^{b} y_n(t) \, dt$. These $x_n$ are bounded and so by the assumed estimate, it follows that the $y_n$ are bounded in $L^q(0, T; Y)$. Therefore, there is a subsequence, still denoted with $n$ such that $y_n \rightarrow \hat{y}$ weakly in $L^q(0, T; Y)$. Now this means that $\hat{y}$ is in the weak closure of the convex hull of $\{y_k : k \geq n\}$. However, this is the same as the strong closure because convex and closed is the same as convex and weakly closed. Therefore, there are functions

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} c_k^n y_k = \hat{y} \text{ strongly in } L^q(0, T; Y) \text{ where } \sum_{k=1}^{\infty} c_k^n = 1, c_k^n \geq 0,$$

and only finitely many are nonzero. Thus a subsequence still denoted with subscript $n$ also converges to $\hat{y}(t)$ for each $t$ off a set of measure zero. The function $F(t, \cdot)$ is maximal monotone and defined on $X$ and so it is pseudomonotone by Theorem 4.7.23. The estimate also shows that it is bounded. Therefore, as shown in the section on set valued pseudomonotone operators, $x \rightarrow F(t, x)$ is upper semicontinuous from strong to weak topology. Thus, for large $n$ depending on $t$, all of the $F(t, x_n)$ are contained in $F(t, x) + B_S(0, r/2)$ where $S$ is a finite subset of points of $X$.

$$B \equiv B_S(0, r/2) \equiv \left\{ w^* : |w^*(x)| < \frac{r}{2} \text{ for all } x \in S \right\}$$

Thus, for a fixed $t$ not in the exceptional set, off which the above pointwise convergence takes place, $\hat{y}(t) \in F(t, x) + D$ where

$$D \equiv \{ w^* : |w^*(x)| \leq r \text{ for all } x \in S \}$$
Since $S, r$ are arbitrary, separation theorems imply that $\hat{y} (t) \in F (t, x)$ for $t$ off a set of measure zero: If not, there would exist $u \in X$ such that $\hat{y} (t) (u) > l > l - \delta > p (u)$ for all $p \in F (t, x)$. But then you could take $r = \delta / 2$ and $B_u (0, \delta / 2)$ and find that $\hat{y} (t) = p + w^*$ where $p \in F (t, x)$ and $|w^* (u)| \leq \delta$. Hence $p (u) = \hat{y} (t) (u) - w^* (u) > l - \delta > p (u)$ an obvious contradiction. Is $z \in F_0 x$? Certainly so if $z = \frac{1}{r} \int_a^b \hat{y} (t) \, dt$.

Letting $\phi \in X$,

$$\langle z, \phi \rangle = \lim_{n \to \infty} \langle z_n, \phi \rangle = \lim_{n \to \infty} \left\langle \frac{1}{\tau} \int_a^b y_n (t) \, dt, \phi \right\rangle = \lim_{n \to \infty} \left\langle \frac{1}{\tau} \int_a^b \sum_{k=0}^{\infty} c_k y_k (t) \, dt, \phi \right\rangle$$

$$= \left\langle \frac{1}{\tau} \int_a^b \hat{y} (t) \, dt, \phi \right\rangle$$

Since $\phi$ is arbitrary, it follows that $z = \frac{1}{\tau} \int_a^b \hat{y} (t) \, dt$ and so $z \in F_0 x$.

Is $F_0$ monotone? Say $z, \hat{z}$ are in $F_0 (x), F_0 (\hat{x})$ respectively. Consider

$$\langle z - \hat{z}, x - \hat{x} \rangle = \left\langle \frac{1}{\tau} \int_a^b y (t) - \hat{y} (t) \, dt, x - \hat{x} \right\rangle = \frac{1}{\tau} \int_a^b \langle y (t) - \hat{y} (t), x - \hat{x} \rangle \, dt$$

but $y (t) \in F (t, x)$ similar for $\hat{y}$ and so the above is $\geq 0$. Thus $F_0$ is indeed monotone. This has also shown that $F_0$ satisfies the necessary modified hemicontinuity condition of Theorem 63.7.21 to conclude that $F_0$ is indeed maximal monotone because it has convex closed values, the hemicontinuity condition, and is monotone.

Suppose $T$ is a bounded pseudomonotone operator and $S$ is a maximal monotone operator, both defined on a strictly convex reflexive Banach space. What of their sum? Is $(T + S) (x)$ convex and closed? Say $t_i \in T x$ and $s_i \in S x$ is it the case that $\theta (s_1 + t_1) + (1 - \theta) (s_2 + t_2) \in (T + S) (x)$ whenever $\theta \in [0, 1]$? Of course this is so. Thus $T + S$ has convex values. Does it have closed values? Suppose $\{s_n + t_n\}$ converges to $z \in X$, $s_n \in S x, t_n \in T x$. Is $z \in (T + S) (x)$? Taking a subsequence, and using the assumption that $T$ is bounded, it can be assumed that $t_n \to t \in T x$ weakly. Therefore, $s_n$ must also converge weakly and so it converges to some $s = z - t \in S x$. Convex and closed implies weakly closed. Thus $T + S$ has closed convex values. Is it upper semicontinuous on finite dimensional subspaces? Suppose $x_n \to x$ in a finite dimensional subspace $F$. Does it follow that

$$(S + T) x_n \subseteq (S + T) x + B (0, r)$$

for all $n$ sufficiently large? It is known that $S x_n \subseteq S x + B (0, r/2)$ and $T x_n \subseteq T x + B (0, r/2)$ whenever $n$ is sufficiently large and so it follows that

$$(S + T) x_n \subseteq (S + T) x + B (0, r/2) + B (0, r/2) \subseteq (S + T) x + B (0, r)$$

whenever $n$ is large enough.

What of the pseudomonotone condition? Suppose

$$\lim_{n \to \infty} \sup \langle u_n^* + v_n^*, x_n - x \rangle \leq 0$$
where \( u^*_n \in Sx_n \) and \( v^*_n \in Tx_n \) where \( x_n \to x \) weakly. Is it the case that for every \( y \), there exists \( u^* \in Sx \) and \( v^* \in Tx \) such that
\[
\lim_{n \to \infty} \langle u^*_n + v^*_n, x_n - y \rangle \geq \langle u^* + v^*, x - y \rangle ?
\]

By monotonicity,
\[
0 \geq \limsup_{n \to \infty} \langle u^*_n + v^*_n, x_n - x \rangle \geq \limsup_{n \to \infty} \langle u^*_n + v^*_n, x_n - x \rangle
\]

Hence
\[
\limsup_{n \to \infty} \langle v^*_n, x_n - x \rangle \leq 0
\]

which implies
\[
\liminf_{n \to \infty} \langle v^*_n, x_n - x \rangle \geq \langle v^*, x - x \rangle = 0 \geq \limsup_{n \to \infty} \langle v^*_n, x_n - x \rangle
\]

showing that
\[
\lim_{n \to \infty} \langle v^*_n, x_n - x \rangle = 0 \quad (23.7.54)
\]

It follows that if \( y \) is given, there exists \( v^* \in T(x) \) such that
\[
\liminf_{n \to \infty} \langle v^*_n, x_n - y \rangle \geq \langle v^*, x - y \rangle
\]

Now let \( u^*_t \in S(x + t(y - x)) \) for \( t > 0 \). Thus
\[
\langle u^*_n - u^*_t, x_n - x + t(x - y) \rangle \geq 0
\]

Then using the above and the convergence in \(23.7.54\),
\[
\liminf_{n \to \infty} \langle u^*_n + v^*_n, x_n - y \rangle \geq \liminf_{n \to \infty} \langle u^*_n + v^*_n, x_n - y \rangle = \langle u^*, x - y \rangle + \langle v^*, x - y \rangle
\]

Now as before where it was shown that maximal monotone and defined on \( X \) implied pseudomonotone, and the theorem which says that maximal monotone operators are locally bounded on the interior of their domains, it follows that there exists a sequence, still denoted as \( u^*_t \) which converges to something called \( u^* \). Then as before, the subspace spanned by \( x, y \) is finite dimensional and so from upper semicontinuity, for all \( t \) small enough,
\[
u^*_t \in S(x) + B(0, r)
\]

Note that weak convergence is the same as strong on finite dimensional spaces. Since this is true for all \( r \) and \( S(x) \) is closed, it follows that \( u^* \in S(x) \). Thus, passing to a limit as \( t \to 0 \) one gets \( u^* \in S(x) \), \( v^* \in T(x) \), and
\[
\liminf_{n \to \infty} \langle u^*_n + v^*_n, x_n - y \rangle \geq \langle u^* + v^*, x - y \rangle
\]

This proves the following generalization of Theorem 23.7.25.
Theorem 23.7.27 Let \( T, S : X \to \mathcal{P}(X') \) where \( X \) is a strictly convex reflexive Banach space and suppose \( T \) is bounded and pseudomonotone while \( S \) is maximal monotone. Then \( T + S \) is pseudomonotone.

Also, there is an interesting result which is based on the obvious observation that if \( A \) is maximal monotone, then so is \( \hat{A}(x) \equiv A(x_0 + x) \).

Lemma 23.7.28 Let \( A \) be maximal monotone. Then for each \( \lambda > 0 \),

\[
x \to \lambda F(x - x_0) + Ax
\]

is onto.

Proof: Let \( \hat{A}(x) \equiv A(x_0 + x) \) so as earlier, \( \hat{A} \) is maximal monotone. Then let \( y^* \in X' \). Then there exists \( y \) such that \( \hat{A}(y) + \lambda F(y) \ni y^* \). Now define \( x \equiv y + x_0 \). Then

\[
\hat{A}(y) + \lambda F(y) \ni y^*, \quad \hat{A}(x - x_0) + \lambda F(x - x_0) \ni y^*, \quad A(x) + \lambda F(x - x_0) \ni y^* \quad \blacksquare
\]

Definition 23.7.29 Let \( A : D(A) \to \mathcal{P}(X') \) be maximal monotone. Let \( A^{-1} : A(D(A)) \to \mathcal{P}(X') \) be defined as follows.

\[
x \in A^{-1}x^* \text{ if and only if } x^* \in Ax
\]

Observation 23.7.30 \( A^{-1} \) is also maximal monotone. This is easily seen as follows. \( [x, y] \in \mathcal{G}(A) \) if and only if \( [y, x] \in \mathcal{G}(A^{-1}) \).

Earlier, it was shown that if \( B \) is monotone and hemi-continuous and coercive, then it was onto. It was not necessary to assume that \( B \) is bounded. The same thing holds for \( A \) maximal monotone. This will follow from the next result. Recall that a maximal monotone operator is locally bounded at every interior point of its domain which was shown above. Also it appears to not be possible to show that a maximal monotone operator is locally bounded at a limit point of \( D(A) \). The following result is in [12] although he claims a better result than what I am proving here in which it is only necessary to verify \( A^{-1} \) is locally bounded at every point of \( A(D(A)) \). However, I was unable to follow the argument and so I am proving another theorem with the same argument he uses. It looks like a typo to me but I often have trouble following hard theorems so I am not sure. Anyway, the following is the best I can do. I think it is still a very interesting result.

Theorem 23.7.31 Suppose \( A^{-1} \) is locally bounded at every point of \( \overline{A(D(A))} \). Then in fact \( A(D(A)) = X' \) and in fact \( \overline{A(D(A))} = A(D(A)) \).

Proof: This is done by showing that \( A(D(A)) \) is both open and closed. Since it is nonempty, it must be all of \( X' \) because \( X' \) is connected. First it is shown that \( A(D(A)) \) is closed. Suppose \( y_n \in Ax_n \) and \( y_n \to y \). Does it follow that \( y \in A(D(A)) \)? Since \( y \) is a limit point of \( A(D(A)) \), it follows that \( A^{-1} \) is locally bounded at \( y \). Thus there is a subsequence still denoted by \( y_n \) such that \( y_n \to y \) and
for \(x_n \in A^{-1}y_n\) or in other words, \(y_n \in Ax_n\), it follows that \(x_n\) is bounded. Hence there exists a subsequence, still denoted with the subscript \(n\) such that \(x_n \to x\) weakly and \(y_n \to y\) strongly. Hence if \([u, v] \in G(A)\),

\[
\langle y - v, x - u \rangle = \lim_{n \to \infty} \langle y_n - v, x_n - x \rangle \geq 0
\]

Since \([u, v]\) is arbitrary and \(A\) is maximal monotone, it follows that \(y \in Ax\) or in other words, \(x \in A^{-1}y\) and \(y \in A(D(A))\). Thus \(A(D(A))\) is closed.

Next consider why \(A(D(A))\) is open. Let \(y_0 \in A(D(A))\). Then there exists \(D_r \equiv B(y_0, r)\) centered at \(y_0\) such that \(A^{-1}\) is bounded on \(D_r\). Since \(A\) is maximal monotone, for each \(y \in X\) there is a solution \(x^*\) to the inclusion

\[
y \in \varepsilon F(x^* - x_0) + Ax, \ y_\varepsilon \equiv y - \varepsilon F(x^* - x_0) \in Ax_
\]

Consider only \(y \in B(y_0, \frac{r}{2})\).

\[
\langle(y - \varepsilon F(x^* - x_0)) - y_0, x - x_0\rangle \geq 0
\]

Then using \(\langle Fz, z \rangle = \|z\|^2\),

\[
\|y - y_0\| \|x - x_0\| \geq \langle y - y_0, x - x_0\rangle \geq \varepsilon \|x^* - x_0\|^2
\]

and so \(\varepsilon \|x^* - x_0\| = \varepsilon \|F(x^* - x_0)\| \leq \|y - y_0\| < r/2\). Thus \(y_\varepsilon\) stays in \(B(y_0, r)\).

This is because \(y\) is closer to \(y_0\) than \(r/2\) while \(y_\varepsilon\) is within \(r/2\) of \(y\). It follows that the \(x^*_n\) are bounded and so \(x^*_n - x\) is bounded and so \(\varepsilon F(x^*_n - x_0) \to 0\). Thus \(y_\varepsilon \to y\) strongly. Since the \(x^*_n\) are bounded, there exists a further subsequence, still denoted as \(x^*_\varepsilon\) such that \(x^*_\varepsilon \to x\), some point of \(X\). Then if \([u, v] \in G(A)\),

\[
\langle y_\varepsilon - v, x^*_\varepsilon - u \rangle \geq 0
\]

and letting \(\varepsilon \to 0\) using the strong convergence of \(y_\varepsilon\) one obtains

\[
\langle y - v, x - u \rangle \geq 0
\]

which shows that \(y \in Ax\). Thus \(B(y_0, \frac{r}{2}) \subseteq A(D(A)) \equiv D(A^{-1})\) and so \(A(D(A))\) is open.

The proof featured the usual duality map.

Note that as part of the proof \(A(D(A))\) was shown to be closed so although it was assumed at the outset that \(A^{-1}\) was locally bounded on \(A(D(A))\), this is the same as saying that \(A^{-1}\) is locally bounded on \(A(D(A))\).

**Corollary 23.7.32** Suppose \(A : D(A) \to P(X')\) is maximal monotone and coercive. Then \(A\) is onto.

**Proof:** From Theorem [23.7.31] it suffices to show that \(A^{-1}\) is locally bounded at \(y^* \in A(D(A))\). The case of an interior point follows from Theorem [23.7.20]. Assume
then that \( y^\ast \) is a limit point of \( A(D(A)) \). Of course this includes the case of interior points. Then there exists \( y^\ast_n \rightarrow y^\ast \) where \( y^\ast_n \in Ax_n \). Then
\[
\frac{\langle y^\ast_n, x_n \rangle}{\|x_n\|} \leq \|y^\ast_n\|
\]
and the right side is bounded. Hence by coercivity, so is \( \|x_n\| \). Therefore, there is a further subsequence, still denoted as \( x_n \) such that \( x_n \rightarrow x \) weakly while \( y^\ast_n \rightarrow y^\ast \) strongly. Then letting \( [u, v^\ast] \in G(A) \),
\[
\langle y^\ast - v^\ast, x - u \rangle = \lim_{n \rightarrow \infty} \langle y^\ast_n - v^\ast, x_n - u \rangle \geq 0
\]
Hence \( y^\ast \in Ax \) and \( y^\ast \in A(D(A)) \). Thus \( A^{-1} \) is locally bounded on \( A(D(A)) \) and so \( A \) is onto from the above theorem.

### 23.7.4 Approximation Theorems

This section continues following Barbu [12]. Always it is assumed that the situation is of a real reflexive Banach space \( X \) having strictly convex norm and its dual \( X' \). As observed earlier, there exists a solution \( x_\lambda \) to the inclusion
\[
0 \in F(x_\lambda - x) + \lambda Ax_\lambda
\]
To see this, you consider \( \hat{A}(y) \equiv A(x + y) \). Then \( \hat{A} \) is also maximal monotone and so there exists a solution to
\[
0 \in F(\hat{x}) + \lambda \hat{A}(\hat{x}) = F(\hat{x}) + \lambda A(x + \hat{x})
\]
Now let \( x_\lambda = x + \hat{x} \) so \( \hat{x} = x_\lambda - x \). Hence
\[
0 \in F(x_\lambda - x) + \lambda Ax_\lambda
\]
Here you could have \( F \) the duality map for any given \( p > 1 \).

The symbol \( \limsup_{n,n \rightarrow \infty} a_{mn} \leq 0 \) means \( \lim_{N \rightarrow \infty} (\sup_{m \geq N,n \geq N} a_{mn}) \). Then here is a simple observation.

**Lemma 23.7.33** Suppose \( \limsup_{n,n \rightarrow \infty} a_{mn} \leq 0 \). Then \( \limsup_{m \rightarrow \infty} (\limsup_{n \rightarrow \infty} a_{mn}) \leq 0 \).

**Proof:** There exists \( N \) such that if both \( m, n \geq N \), \( a_{mn} \leq \varepsilon \). Then
\[
\limsup_{n \rightarrow \infty} a_{mn} = \limsup_{n \rightarrow \infty, n > N} a_{mn} \leq \varepsilon
\]
Thus also
\[
\limsup_{m \rightarrow \infty} \left( \limsup_{n \rightarrow \infty} a_{mn} \right) = \limsup_{m \rightarrow \infty, m \geq N} \left( \limsup_{n \rightarrow \infty} a_{mn} \right) \leq \varepsilon. \]

The argument will be based on the following lemma.
Lemma 23.7.34 Let $A : D(A) \to \mathcal{P}(X')$ be maximal monotone and let $v_n \in Au_n$ and $u_n \to u$, $v_n \to v$ weakly.

Also suppose that

$$\lim \sup_{m,n \to \infty} \langle v_n - v_m, u_n - u_m \rangle \leq 0$$

or

$$\lim \sup_{n \to \infty} \langle v_n - v, u_n - u \rangle \leq 0$$

Then $[u,v] \in \mathcal{G}(A)$ and $\langle v_n, u_n \rangle \to \langle v, u \rangle$.

Proof: By monotonicity,

$$\lim_{m,n \to \infty} \langle v_n - v_m, u_n - u_m \rangle = 0$$

Suppose then that $\langle v_n, u_n \rangle$ fails to converge to $\langle v, u \rangle$. Then there is a subsequence, still denoted with subscript $n$ such that $\langle v_n, u_n \rangle \to \mu \neq \langle v, u \rangle$. Let $\varepsilon > 0$. Then there exists $M$ such that if $n,m > M$, then

$$|\langle v_n, u_n \rangle - \mu| < \varepsilon, |\langle v_n - v_m, u_n - u_m \rangle| < \varepsilon$$

Then if $m,n > M$,

$$|\langle v_n - v_m, u_n - u_m \rangle| = |\langle v_n, u_n \rangle + \langle v_m, u_m \rangle - \langle v_n, u_m \rangle - \langle v_m, u_n \rangle| < \varepsilon$$

Hence it is also true that

$$|\langle v_n, u_n \rangle + \langle v_m, u_m \rangle - \langle v_n, u_m \rangle - \langle v_m, u_n \rangle| \leq |2\mu - (\langle v_n, u_m \rangle + \langle v_m, u_n \rangle)| < 3\varepsilon$$

Now take a limit first with respect to $n$ and then with respect to $m$ to obtain

$$|2\mu - (\langle v, u \rangle + \langle v, u \rangle)| < 3\varepsilon$$

Since $\varepsilon$ is arbitrary, $\mu = \langle v, u \rangle$ after all. Hence the claim that $\langle v_n, u_m \rangle \to \langle v, u \rangle$ is verified. Next suppose $[x, y] \in \mathcal{G}(A)$ and consider

$$\langle v - y, u - x \rangle = \langle v, u \rangle - \langle v, x \rangle - \langle y, u \rangle + \langle y, x \rangle$$

$$= \lim_{n \to \infty} \langle v_n, u_n \rangle - \langle v_n, x \rangle - \langle y, u_n \rangle + \langle y, x \rangle$$

$$= \lim_{n \to \infty} \langle v_n - y, u_n - x \rangle \geq 0$$

and since $[x, y]$ is arbitrary, it follows that $v \in Au$.

Next suppose $\lim \sup_{n \to \infty} \langle v_n - v, u_n - u \rangle \leq 0$. It is not known that $[u,v] \in \mathcal{G}(A)$.

$$\lim \sup_{n \to \infty} \left[ \langle v_n, u_n \rangle - \langle v, u_n \rangle - \langle v_n, u \rangle + \langle v, u \rangle \right] \leq 0$$

$$\lim \sup_{n \to \infty} \langle v_n, u_n \rangle - \langle v, u \rangle \leq 0$$
Thus $\limsup_{n \to \infty} \langle v_n, u_n \rangle \leq \langle v, u \rangle$. Now let $[x, y] \in \mathcal{G}(A)$
\[
\langle v - y, u - x \rangle = \langle v, u \rangle - \langle v, x \rangle - \langle y, u \rangle + \langle y, x \rangle
\]
\[
\geq \limsup_{n \to \infty} (\langle v_n, u_n \rangle - \langle v_n, x \rangle - \langle y, u_n \rangle + \langle y, x \rangle)
\]
\[
\geq \liminf_{n \to \infty} |\langle v_n - y, u_n - x \rangle| \geq 0
\]
Hence $[u, v] \in \mathcal{G}(A)$. Now
\[
\limsup_{n \to \infty} \langle v_n - v, u_n - u \rangle \leq 0 \leq \liminf_{n \to \infty} \langle v_n - v, u_n - u \rangle
\]
the second coming from monotonicity and the fact that $v \in Au$. Therefore,
\[
\lim_{n \to \infty} \langle v_n - v, u_n - u \rangle = 0
\]
which shows that $\lim_{n \to \infty} \langle v_n, u_n \rangle = \langle v, u \rangle$. ■

**Definition 23.7.35** Let $x_\lambda$ just defined
\[
0 \in F(x_\lambda - x) + \lambda Ax_\lambda
\]
be denoted by $J_\lambda x$ and define also
\[
A_\lambda(x) = -\lambda^{-(p-1)}F(x_\lambda - x) = -\lambda^{-(p-1)}F(J_\lambda x - x)
\]
This is for $F$ a duality map with $p > 1$. Thus for the usual duality map, you would have
\[
A_\lambda(x) = -\lambda^{-1}F(J_\lambda x - x)
\]
Recall how this $x_\lambda$ is defined. In general,
\[
0 \in F(J_\lambda x - x) + \lambda^{p-1} Ax_\lambda
\]
Thus, from the definition,
\[
A_\lambda(x) \in A(J_\lambda x)
\]
Formally, and to help remember what is going on, you are looking at a generalization of
\[
A_\lambda x = \frac{A}{1 + \lambda A} x = \frac{1}{\lambda} \left( x - (I + \lambda A)^{-1} x \right)
\]
This is in the case where $F = I$ to keep things simpler. You have $0 = x_\lambda - x + \lambda Ax_\lambda$ and so formally $x_\lambda = (I + \lambda A)^{-1} x$. Thus you are looking at $\frac{1}{\lambda} \left( x - x_\lambda \right) = \frac{1}{\lambda} \left( x - (I + \lambda A)^{-1} x \right) = A_\lambda x$. In fact, this is exactly what you do when you are in a single Hilbert space. This is just a generalization to mappings between Banach spaces and their duals.

Then there are some things which can be said about these operators. It is presented for the general duality map for $p > 1$. 

Theorem 23.7.36 The following hold. Here $X$ is a reflexive Banach space with strictly convex norm. $A : D(A) \to P(X')$ is maximal monotone. Then

1. $J_\lambda$ and $A_\lambda$ are bounded single valued operators defined on $X$. Bounded means they take bounded sets to bounded sets. Also $A_\lambda$ is a monotone operator.

2. $A_\lambda, J_\lambda$ are demicontinuous. That is, strongly convergent sequences are mapped to weakly convergent sequences.

3. For every $x \in D(A), \|A_\lambda(x)\| \leq |Ax| \equiv \inf \{\|y^*\| : y^* \in Ax\}$. For every $x \in \text{conv}(D(A))$, it follows that $\lim_{\lambda \to 0} J_\lambda(x) = x$. The new symbol means the closure of the convex hull. It is the closure of the set of all convex combinations of points of $D(A)$.

Proof: 1.) It is clear that these are single valued operators. What about the assertion that they are bounded? Let $y^* \in Ax_\lambda$ such that the inclusion defining $x_\lambda$ becomes an equality. Thus

$$F(x_\lambda - x) + \lambda^{p-1}y^* = 0$$

Then let $x_0 \in D(A)$ be given.

$$\langle F(x_\lambda - x), x_\lambda - x \rangle + \lambda^{p-1} \langle y^*, x_\lambda - x_0 \rangle + \lambda^{p-1} \langle y^*, x_0 - x \rangle = 0$$

Then by monotonicity of $A$,

$$\|x_\lambda - x\|^p + \lambda^{p-1} \langle y^*_0, x_\lambda - x_0 \rangle + \lambda^{p-1} \langle y^*, x_0 - x \rangle \leq 0$$

It follows that

$$\|x_\lambda - x\|^p \leq \lambda^{p-1} \|y^*_0\| \|x_\lambda - x_0\| + \lambda^{p-1} \|y^*\| \|x_0 - x\|$$

Hence if $x$ is in a bounded set, it follows the resulting $x_\lambda = J_\lambda x$ remain in a bounded set. Now from the definition of $A_\lambda$, it follows that this is also a bounded operator. Why is $A_\lambda$ monotone?

$$0 \leq \langle A_\lambda x - A_\lambda y, x - y \rangle = \langle A_\lambda x - A_\lambda y, x - J_\lambda x - (y - J_\lambda y) \rangle$$

$$+ \langle A_\lambda x - A_\lambda y, J_\lambda x - J_\lambda y \rangle$$

$$= \left\langle \lambda^{-(p-1)} F(J_\lambda x - x) - \lambda^{-(p-1)} F(J_\lambda y - y), J_\lambda x - x - (J_\lambda y - y) \right\rangle$$

$$+ \langle A_\lambda x - A_\lambda y, J_\lambda x - J_\lambda y \rangle$$

and both terms are nonnegative, the first because $F$ is monotone so indeed $A_\lambda$ is monotone.

2.) What of the demicontinuity of $A_\lambda$? This one is really tricky. Suppose $x_n \to x$. Does it follow that $A_\lambda x_n \to A_\lambda x$ weakly? The proof will be based on a pair of equations. These are

$$\lim_{m,n \to \infty} \langle F(J_\lambda x_n - x_n) - F(J_\lambda x_m - x_m), J_\lambda x_n - x_n - (J_\lambda x_m - x_m) \rangle = 0$$
and

\[ \lim_{m,n \to \infty} \langle A_{\lambda} (x_n) - A_{\lambda} (x_m), J_{\lambda} x_n - J_{\lambda} x_m \rangle = 0 \]

When these have been established, Lemma 23.7.34 is used to get the desired result for a subsequence. It will be shown that every sequence has a subsequence which gives the right sort of weak convergence and from this the desired weak convergence of \( A_{\lambda} x_n \) to \( A_{\lambda} x \) follows.

\[
0 \in F(J_{\lambda} x_n - x_n) + \lambda^{p-1} A(J_{\lambda} x_n)
0 \in F(J_{\lambda} x - x) + \lambda^{p-1} A(J_{\lambda} x)
\]

\[
-\lambda^{-(p-1)} F(J_{\lambda} x - x) \equiv A_{\lambda} (x) \in A(J_{\lambda} x)
-\lambda^{-(p-1)} F(J_{\lambda} x_n - x_n) \equiv A_{\lambda} (x_n) \in A(J_{\lambda} x_n)
\]

Note also that for a given \( x \) there is only one solution \( J_{\lambda} x \) to \( 0 \in F(J_{\lambda} x - x) + \lambda^{p-1} A(J_{\lambda} x) \). By monotonicity of \( F \),

\[
0 \leq \langle F(J_{\lambda} x_n - x_n) - F(J_{\lambda} x_m - x_m), x_m - x_n + J_{\lambda} x_n - J_{\lambda} x_m \rangle
\]

Then from the above,

\[
\langle F(J_{\lambda} x_n - x_n) - F(J_{\lambda} x_m - x_m), x_n - x_m \rangle \leq \langle F(J_{\lambda} x_n - x_n) - F(J_{\lambda} x_m - x_m), J_{\lambda} x_n - J_{\lambda} x_m \rangle
\]

Now from the boundedness of these operators, the left side of the above inequality converges to 0 as \( n, m \to \infty \). Thus

\[
\lim \inf_{m,n \to \infty} \langle F(J_{\lambda} x_n - x_n) - F(J_{\lambda} x_m - x_m), J_{\lambda} x_n - J_{\lambda} x_m \rangle \geq 0 \quad (23.7.55)
\]

\[
\lim \inf_{m,n \to \infty} \langle -\lambda^{p-1} A_{\lambda} (x_n) - (-\lambda^{p-1} A_{\lambda} (x_m)), J_{\lambda} x_n - J_{\lambda} x_m \rangle \geq 0
\]

\[
\lim \inf_{m,n \to \infty} \left( \lambda^{p-1} A_{\lambda} (x_m) - \lambda^{p-1} A_{\lambda} (x_n), J_{\lambda} x_n - J_{\lambda} x_m \right) \geq 0
\]

The expression on the left in the above is non positive. Multiplying by -1,

\[
0 \geq \lim \sup_{m,n \to \infty} \langle A_{\lambda} (x_n) - A_{\lambda} (x_m), J_{\lambda} x_n - J_{\lambda} x_m \rangle
\]

\[
\geq \lim \inf_{m,n \to \infty} \langle A_{\lambda} (x_n) - A_{\lambda} (x_m), J_{\lambda} x_n - J_{\lambda} x_m \rangle \geq 0 \quad (23.7.56)
\]

Thus, in fact, the expression in 23.7.55 converges to 0. By boundedness considerations and the strong convergence given,

\[
\lim_{m,n \to \infty} \langle F(J_{\lambda} x_n - x_n) - F(J_{\lambda} x_m - x_m), J_{\lambda} x_n - x_n -(J_{\lambda} x_m - x_m) \rangle = 0 \quad (23.7.57)
\]
23.7. MAXIMAL MONOTONE OPERATORS

From boundedness again, there is a subsequence still denoted with the subscript $n$ such that

$$J_{\lambda}x_n - x_n \to a - x, \quad F(J_{\lambda}x_n - x_n) \to b$$

both weakly.

Since $F$ is maximal monotone, (Theorem 23.7.44) it follows from Lemma 23.7.45 that $[a - x, b] \in \mathcal{G}(F)$ and so in fact $F(a - x) = b$. Thus this has just shown that $F(J_{\lambda}x_n - x_n) \to F(a - x)$.

Next consider $\lambda_1$ weakly and $A_{\lambda}(x_n) = -\lambda^{-(p-1)}F(J_{\lambda}x_n - x_n) \to -\lambda^{-(p-1)}b$ weakly. Then from Lemma 23.7.44 again, $[a, -\lambda^{-(p-1)}b] \in \mathcal{G}(A)$ so $-\lambda^{-(p-1)}b \in A(a)$ so $b \in -\lambda^{p-1}A(a)$.

But it was just shown that $b = F(a - x)$ and so

$$F(a - x) \in -\lambda^{p-1}A(a)$$

As noted at the beginning, there is only one solution to this inclusion for a given $x$ and it is $a = J_{\lambda}x$. This has shown that in terms of weak convergence,

$$A_{\lambda}(x_n) \to -\lambda^{-(p-1)}b = -\lambda^{-(p-1)}F(a - x) = -\lambda^{-(p-1)}F(J_{\lambda}x - x) \equiv A_{\lambda}(x)$$

This has shown that $A_{\lambda}$ is demicontinuous. Also it has shown that $J_{\lambda}$ is also demicontinuous. (This result is a lot nicer in Hilbert space.)

3.) Why is $\|A_{\lambda}(x)\| \leq |Ax|$ whenever $x \in D(A)$?

$$A_{\lambda}(x) = -\lambda^{-(p-1)}F(J_{\lambda}x - x)$$

where $0 \in F(J_{\lambda}x - x) + \lambda^{p-1}A(J_{\lambda}x)$. Therefore, $A_{\lambda}(x) \in A(J_{\lambda}x)$. Then letting $[u, v] \in \mathcal{G}(A)$,

$$0 \leq \langle v - A_{\lambda}(x), u - J_{\lambda}x \rangle$$

In particular, if $y \in Ax$

$$0 \leq \langle y - A_{\lambda}(x), x - J_{\lambda}x \rangle = \left\langle y + \lambda^{-(p-1)}F(J_{\lambda}x - x), x - J_{\lambda}x \right\rangle$$

Hence

$$\lambda^{-(p-1)} \left\| J_{\lambda}x - x \right\|^p \leq \|y\| \left\| J_{\lambda}x - x \right\|$$

and so

$$\lambda^{-(p-1)} \left\| J_{\lambda}x - x \right\|^{p-1} = \lambda^{-(p-1)} \|F(J_{\lambda}x - x)\| = \|A_{\lambda}(x)\| \leq \|y\|$$

and since $y \in Ax$ is arbitrary, $\|A_{\lambda}(x)\| \leq |Ax| = \inf \{\|y\| : y \in Ax\}$.

Next consider the claim that for all $x \in \text{conv}(D(A))$, it follows that

$$\lim_{\lambda \to 0} J_{\lambda}(x) = x.$$
Let $A$ holds for any $D$. It must be the case that $p > 1$ and so, the above results on approximation. It will include the general case of $F$ described above, is maximal monotone with domain $A$. Thus the two sets are the same and so in fact, since $x_n$ is arbitrary, it follows that for every $\varepsilon > 0$,

$$\limsup_{\lambda \to 0} \|J_\lambda x - x\| \leq \varepsilon$$

and so in fact, $\limsup_{\lambda \to \infty} \|J_\lambda x - x\| = 0$. ■

Now here is an interesting corollary.

**Corollary 23.7.37** Let $A$ be maximal monotone. $A : X \to X'$ where $X$ is a strictly convex reflexive Banach space. Then $D(A)$ is convex.

**Proof:** It is known that $J_\lambda : X \to D(A)$ for any $\lambda$. Also, if $x \in \overline{\text{conv}}(D(A))$, then it was shown that $J_\lambda x \to x$. Clearly

$$\overline{\text{conv}}(D(A)) \supseteq D(A)$$

Now if $x$ is in the set on the left, $J_\lambda x \to x$ and so in fact, $J_\lambda x \in D(A)$, it must be the case that $x \in D(A)$. Thus the two sets are the same and so in fact, $\overline{D(A)}$ is closed and convex. ■

Note that this implies that $A(D(A))$ is also convex. This is because $A^{-1}$ described above, is maximal monotone with domain $A(D(A))$.

Next is a useful generalization of some of the earlier material used to establish the above results on approximation. It will include the general case of $F$ a duality map for $p > 1$. 

$$= \left\langle v + \lambda^{-(p-1)} F(J_\lambda x - x), u - x \right\rangle + \left\langle v + \lambda^{-(p-1)} F(J_\lambda x - x), x - J_\lambda x \right\rangle$$

Thus

$$\|J_\lambda x - x\|^p \leq \lambda^{p-1} \langle v, u - x \rangle + \langle F(J_\lambda x - x), u - x \rangle + \lambda^{p-1} \langle v, x - J_\lambda x \rangle \quad (23.7.58)$$

for $x$ arbitrary and $u$ anything in $D(A)$. It follows that (23.7.58) holds for any $u \in \text{conv}(D(A))$. Say $u = x_n \in \text{conv}(D(A))$ where $x_n \to x$. Then

$$\|J_\lambda x - x\|^p \leq \lambda^{p-1} \langle v, x_n - x \rangle + \langle F(J_\lambda x - x), x_n - x \rangle + \lambda^{p-1} \langle v, x - J_\lambda x \rangle$$

$$\leq \lambda^{p-1} \|v\| \|x_n - x\| + \|J_\lambda x - x\|^{p-1} \|x_n - x\| + \lambda^{p-1} \|v\| \|J_\lambda x - x\|$$

You have something like this: $y_\lambda = \|J_\lambda x - x\|$, $a_n = \|x_n - x\|$, $y_\lambda^p \leq \lambda^{p-1} \|v\| a_n + y_\lambda^{p-1} a_n + \lambda^{p-1} \|v\| y_\lambda$, $y_\lambda \geq 0$

where $p > 1$ and $a_n \to 0$. Then

$$\limsup_{\lambda \to 0} y_\lambda^p \leq \limsup_{\lambda \to 0} y_\lambda^{p-1} a_n$$

and so,

$$\limsup_{\lambda \to 0} y_\lambda \leq a_n$$

Hence

$$\limsup_{\lambda \to 0} \|J_\lambda x - x\| \leq \|x_n - x\|$$

Since $x_n$ is arbitrary, it follows that for every $\varepsilon > 0$,

$$\limsup_{\lambda \to 0} \|J_\lambda x - x\| \leq \varepsilon$$

and so in fact, $\limsup_{\lambda \to \infty} \|J_\lambda x - x\| = 0$. ■
Proposition 23.7.38 Suppose $A : X \to \mathcal{P}(X')$ where $X$ is a reflexive Banach space with strictly convex norm. Suppose also that $A$ is maximal monotone. Then if $\lambda_n \to 0$ and if $x_n \to x$ weakly, $A_{\lambda_n}x_n \to x^*$ weakly, and
\[
\lim \sup_{n,m \to \infty} \langle A_{\lambda_n}x_n - A_{\lambda_m}x_m, x_n - x_m \rangle \leq 0
\]
Then
\[
\lim_{n,m \to \infty} \langle A_{\lambda_n}x_n - A_{\lambda_m}x_m, x_n - x_m \rangle = 0,
\]
$[x, x^*] \in \mathcal{G}(A)$, and $\langle A_{\lambda_n}x_n, x_n \rangle \to \langle x^*, x \rangle$.

Proof: Let $\alpha = \limsup_{n \to \infty} \langle A_{\lambda_n}x_n, x_n \rangle$. It is finite because the expression is bounded independent of $n$. Then
\[
\lim sup_{m \to \infty} \left( \lim sup_{n \to \infty} \left( \frac{\langle A_{\lambda_n}x_n, x_n \rangle + \langle A_{\lambda_m}x_m, x_m \rangle}{\langle A_{\lambda_n}x_n, x_m \rangle + \langle A_{\lambda_m}x_m, x_n \rangle} \right) \right) \leq 0
\]
Thus
\[
\lim sup_{m \to \infty} \left( \alpha + \langle A_{\lambda_m}x_m, x_m \rangle - \left( \langle x^*, x_m \rangle + \langle A_{\lambda_m}x_m, x \rangle \right) \right) \leq 0
\]
and so
\[
2\alpha - 2 \langle x^*, x \rangle \leq 0
\]
The next simple observation is that
\[
\|A_{\lambda_n}x_n\| = \left\| \lambda_n^{-(p-1)} F(J_{\lambda_n}x_n - x_n) \right\| \leq C
\]
due to the weak convergence. Hence $\lambda_n^{-(p-1)} \|J_{\lambda_n}x_n - x_n\|^{p-1} \leq C$ and so
\[
\|J_{\lambda_n}x_n - x_n\| \leq \lambda_n C^{1/(p-1)}.
\]
Thus if $[u, u^*] \in \mathcal{G}(A)$,
\[
\lim \inf_{n \to \infty} \langle A_{\lambda_n}x_n - u^*, x_n - u \rangle = \lim \inf_{n \to \infty} \langle A_{\lambda_n}x_n - u^*, J_{\lambda_n}x_n - u \rangle \geq 0
\]
because $A_{\lambda}x \in AJ_{\lambda}x$. However, the left side satisfies
\[
0 \leq \lim \inf_{n \to \infty} \langle A_{\lambda_n}x_n - u^*, x_n - u \rangle \leq \lim sup_{n \to \infty} \langle A_{\lambda_n}x_n - u^*, x_n - u \rangle
\]
\[
= \lim sup_{n \to \infty} \left[ \langle A_{\lambda_n}x_n, x_n \rangle - \langle A_{\lambda_n}x_n, u \rangle - \langle u^*, x_n \rangle + \langle u^*, u \rangle \right]
\]
\[
= \alpha - \langle x^*, u \rangle - \langle u^*, x \rangle \leq \langle x^*, x \rangle - \langle x^*, u \rangle - \langle u^*, x \rangle + \langle u^*, u \rangle
\]
\[
= \langle x^* - u^*, x - u \rangle
\]
and this shows that $[x, x^*] \in \mathcal{G}(A)$ since $[u, u^*]$ was arbitrary.
Next let \([u, u^*] \in \mathcal{G}(A)\). Then thanks to (23.7.59),

\[
0 \leq \liminf_{n \to \infty} \langle A_{\lambda_n} x_n - u^*, x_n - u \rangle = \liminf_{n \to \infty} \langle A_{\lambda_n} x_n - u^*, x_n - u \rangle \\
\leq \limsup_{n \to \infty} \langle A_{\lambda_n} x_n - u^*, x_n - u \rangle \\
= \limsup_{n \to \infty} \left( \langle A_{\lambda_n} x_n, x_n \rangle - \langle A_{\lambda_n} x_n, u \rangle - \langle u^*, x_n \rangle + \langle u^*, u \rangle \right) \\
= \limsup_{n \to \infty} \langle A_{\lambda_n} x_n, x_n \rangle - \langle x^*, u \rangle - \langle u^*, x \rangle + \langle u^*, u \rangle \\
\leq \langle x^*, x \rangle - \langle x^*, u \rangle - \langle u^*, x \rangle + \langle u^*, u \rangle = \langle x^* - u^*, x - u \rangle
\]

In particular, you could let \([u, u^*] = [x, x^*]\) and conclude that

\[
\lim_{n \to \infty} \langle A_{\lambda_n} x_n - x^*, x_n - x \rangle = \lim_{n \to \infty} \langle A_{\lambda_n} x_n, x_n \rangle - \langle A_{\lambda_n} x_n, x \rangle - \langle x^*, x \rangle + \langle x^*, x \rangle = 0
\]

which shows that \(\lim_{n \to \infty} \langle A_{\lambda_n} x_n, x_n \rangle = \langle x^*, x \rangle\). Then it follows from this that

\[
\lim_{n,m \to \infty} \langle A_{\lambda_n} x_n - A_{\lambda_m} x_m, x_n - x_m \rangle = 0
\]

For the rest of this, the usual duality map for \(p = 2\) will be used. It may be that one could change this, but I don’t have a need to do it right now so from now on, \(F\) will be the usual thing.

**23.7.5 Sum Of Maximal Monotone Operators**

To begin with, here is a nice lemma.

**Lemma 23.7.39** Let \(0 \in D(A)\) and let \(A\) be maximal monotone and let \(B : X \to X'\) be monotone hemicontinuous, bounded, and coercive. Then \(B + A\) is also maximal monotone. Also \(B + A\) is onto.

**Proof:** By Theorem 23.7.9, there exists \(x \in \overline{D(A)}\) such that for all \([u, u^*] \in \mathcal{G}(A)\),

\[
\langle Bx + Fx - y^* + u^*, u - x \rangle \geq 0
\]

Hence for all \([u, u^*]\),

\[
\langle u^* - (y^* - (Bx + Fx)), u - x \rangle \geq 0
\]

It follows that

\[
y^* - (Bx + Fx) \in Ax
\]

and so \(y^* \in Bx + Ax + Fx\) showing that \(B + A\) is maximal monotone because it added to \(F\) is onto. As to the last claim, just don’t add in \(F\) in the argument. Thus for all \([u, u^*]\),

\[
\langle Bx - y^* + u^*, u - x \rangle \geq 0
\]

Then the rest is as before. You find that \(y^* - Bx \in Ax\). □
Corollary 23.7.40 Suppose instead of $0 \in D(A)$, it is known that $x_0 \in D(A)$ and

$$\lim_{\|x\| \to \infty} \frac{\langle B(x_0 + x), x \rangle}{\|x\|} = \infty$$

Then if $B$ is monotone and hemicontinuous and $A$ is maximal monotone, then $B + A$ is onto.

**Proof:** Let $\hat{A}(x) \equiv A(x_0 + x)$ so in fact $0 \in D(\hat{A})$. Then letting $\hat{B}$ be defined similarly, it follows from the above lemma that if $y^* \in X'$, there exists $x$ such that $y^* \in \hat{A}x + \hat{B}x \equiv A(x_0 + x) + B(x_0 + x)$

Lemma 23.7.41 Let $0$ be on the interior of $D(A)$ and also in $D(B)$. Also let $0 \in B(0)$ and $0 \in A(0)$. Then if $A, B$ are maximal monotone, so is $A + B$.

**Proof:** Note that, since $0 \in A(0)$, if $x^* \in A x$, then $\langle x^*, x \rangle \geq 0$. Also note that $\|B_\lambda(0)\| \leq |B(0)| = 0$ and so also $\langle B_\lambda x, x \rangle \geq 0$. It is necessary to show that $F + A + B$ is onto. However, $B_\lambda$ is monotone hemicontinuous, bounded and coercive. Hence, by Lemma 23.7.20, $B_\lambda + A$ is maximal monotone. If $x^* \in X'$ is given, there exists a solution to

$$x^* \in F_{x_\lambda} + B_{x_\lambda}x_\lambda + Ax_\lambda$$

Do both sides to $x_\lambda$ and let $x_\lambda^* \in A x_\lambda$ be such that equality holds in the above.

$$x^* = Fx_\lambda + B_{x_\lambda} + x_\lambda^* \quad (\text{23.7.60})$$

Then

$$\langle x^*, x_\lambda \rangle = \|x_\lambda\|^2 + \langle x_\lambda^*, x_\lambda \rangle \geq 0$$

It follows that

$$\|x_\lambda\| \leq \|x^*\|, \quad \langle x_\lambda^*, x_\lambda \rangle \leq \langle x^*, x_\lambda \rangle \leq \|x^*\| \|x_\lambda\| \leq \|x^*\|^2 \quad (\text{23.7.61})$$

Next, $0$ is on the interior of $D(A)$ and so from Theorem 23.7.20, there exists $\rho > 0$ such that if $y^* \in Ax$ for $\|x\| \leq \rho$, then $\|y^*\| < M$ and in fact, all such $x$ are in $D(A)$. Now let

$$y_\lambda = \frac{1}{2\|x_\lambda\|} F^{-1}(x_\lambda^*) \quad \text{so} \quad \|y_\lambda\| < \rho$$

Thus $y_\lambda \in D(A)$ and if $y_\lambda^* \in Ay_\lambda$, then $\|y_\lambda^*\| < M$. Then for such bounded $y_\lambda^*$,

$$0 \leq \langle y_\lambda^* - x_\lambda^*, y_\lambda - x_\lambda \rangle = \langle y_\lambda^*, y_\lambda \rangle - \langle x_\lambda^*, y_\lambda \rangle - \langle y_\lambda^*, x_\lambda \rangle + \langle x_\lambda^*, x_\lambda \rangle$$

Then

$$\frac{1}{2} \|x_\lambda^*\| = \left(\langle x_\lambda^*, y_\lambda \rangle, \frac{1}{2\|x_\lambda\|} F^{-1}(x_\lambda^*) \right) = \langle x_\lambda^*, y_\lambda \rangle \leq \langle y_\lambda^*, y_\lambda \rangle - \langle y_\lambda^*, x_\lambda \rangle + \langle x_\lambda^*, x_\lambda \rangle \leq M \rho + M \|x_\lambda\| + \langle x_\lambda^*, x_\lambda \rangle$$
886  

CHAPTER 23. NONLINEAR OPERATORS

From \[23.7.61\]
\[
\|x^*_\lambda\| \leq 2 \left( M \rho + M \|x^*\| + \|x^*\|^2 \right)
\]
Thus from \[23.7.61\] \(x^*_\lambda, Fx^*_\lambda\) are all bounded. Hence it follows from \[23.7.60\] that \(B x^*_\lambda\) is also bounded. Therefore, there is a sequence, \(\lambda_n \to 0\) such that

- \(x^*_\lambda \to z\) weakly
- \(x^* \to w^*\) weakly
- \(Fx^*_\lambda \to u^*\) weakly
- \(B x^*_\lambda \to b^*\) weakly

Using \[23.7.60\], it follows that

\[
\langle Fx^*_\lambda + x^*_\lambda + B x^*_\lambda - (Fx^*_m + x^*_m + B x^*_m), x^*_\lambda - x^*_m \rangle = 0
\]

Thus

\[
\langle Fx^*_\lambda + x^*_\lambda - (Fx^*_m + x^*_m), x^*_\lambda - x^*_m \rangle + \langle B x^*_\lambda - B x^*_m, x^*_\lambda - x^*_m \rangle = 0
\]

(23.7.62)

Now \(F + A\) is surely monotone and so

\[
\lim \sup_{m,n \to \infty} \langle B x^*_\lambda - B x^*_m, x^*_\lambda - x^*_m \rangle \leq 0
\]

By Proposition \[23.7.38\], \(b^* \in B z\) and

\[
\lim_{m,n \to \infty} \langle B x^*_\lambda - B x^*_m, x^*_\lambda - x^*_m \rangle = 0
\]

Then returning to \[23.7.60\],

\[
\lim \sup_{m,n \to \infty} \langle Fx^*_\lambda + x^*_\lambda - (Fx^*_m + x^*_m), x^*_\lambda - x^*_m \rangle \leq 0
\]

Now from Lemma \[23.7.39\], \(F + A\) is maximal monotone. Hence Proposition \[23.7.38\] applies again and it follows that \(u^* + w^* \in Fz + Az\). Then passing to the limit as \(n \to \infty\) in

\[
x^* = F x^*_\lambda + B x^*_\lambda + x^*_\lambda
\]

it follows that

\[
x^* = u^* + b^* + w^* = Fz + Az + Bz
\]

and this shows that \(A + B\) is maximal monotone because \(x^*\) was arbitrary.

You don’t need to assume all that stuff about \(0 \in A(0), 0 \in B(0), 0\) on interior of \(D(A)\) and so forth.

**Theorem 23.7.42** Suppose \(A, B\) are maximal monotone and the interior of \(D(A)\) has nonempty intersection with \(D(B)\). Then \(A + B\) is maximal monotone.
23.7. MAXIMAL MONOTONE OPERATORS

Proof: Let $x_0$ be on the interior of $D(A)$ and also in $D(B)$. Let $\hat{A}(x) = A(x_0 + x) - x_0^*$ where $x_0^* \in A(x_0)$. Thus $0 \in D(\hat{A})$ and $0 \in \hat{A}(0)$. Do the same thing for $B$ to get $\hat{B}$ defined similarly. Are these still maximal monotone? Suppose for all $[u, u^*] \in G(\hat{A})$

$$\langle y^* - u^*, y - u \rangle \geq 0$$

Does it follow that $y^* \in \hat{A} y$? It is given that $u^* \in A(x_0 + u)$. The above implies for all $[u, u^*] \in G(\hat{A})$

$$\langle y^* + x_0^* - (u^* + x_0^*), (y + x_0) - (u + x_0) \rangle \geq 0$$

and since $u + x_0$ is a generic element of $D(A)$ for $u \in D(\hat{A})$, the above implies $y^* + x_0^* \in A(y + x_0)$ and so $y \in A(y + x_0) - x_0^* \equiv \hat{A}(y)$. Hence the graph is maximal. Similar for $\hat{B}$. Thus the lemma can be applied to $\hat{A}, \hat{B}$ to conclude that the sum of these is maximal monotone. Now a repeat of the above reasoning which shows that $\hat{A}$ is maximal monotone shows that the fact that $\hat{A} + \hat{B}$ is maximal monotone implies that $A + B$ is also. You just shift with $-x_0$ instead of $x_0$. It amounts to nothing more than the observation that maximal graphs don’t lose their maximality by shifting their ranges and domains.

Suppose $B, A$ are maximal monotone. Does there always exist a solution $x$ to

$$x^* \in Fx + B_\lambda x + Ax$$

Consider the monotone hemicontinuous and bounded operator $F + B_\lambda$. Is $\hat{F} + \hat{B}_\lambda$

defined by

$$\left(\hat{F} + \hat{B}_\lambda\right)(x) = \left(\hat{F} + \hat{B}_\lambda\right)(x + x_0)$$

also coercive for some $x_0 \in D(A)$? If so, the existence of the desired solution to the above inclusion follows from Corollary 23.7.40. Then for all $\|x\|$ large enough that $\|x + x_0\| > \|x_0\|$

$$\frac{\langle F(x + x_0) + B_\lambda(x + x_0), x \rangle}{\|x\|}$$

$$= \frac{\langle F(x + x_0), x \rangle}{\|x\|} + \frac{\langle B_\lambda(x + x_0) - B_\lambda(x_0), x \rangle}{\|x\|} + \frac{\langle B_\lambda(x_0), x \rangle}{\|x\|}$$

$$\geq \frac{1}{2} \frac{\langle F(x + x_0), x \rangle}{\|x + x_0\|} - \frac{1}{2} \frac{\langle F(x + x_0), x_0 \rangle}{\|x + x_0\|}$$

$$\geq \frac{1}{2} \frac{\langle F(x + x_0), x + x_0 \rangle}{\|x + x_0\|} - \frac{1}{2} \frac{\langle F(x + x_0), x_0 \rangle}{\|x_0\|} - \|B_\lambda(x_0)\|$$

$$\geq \frac{1}{2} \frac{\langle F(x + x_0), x + x_0 \rangle}{\|x + x_0\|} - \frac{1}{2} \frac{\langle F(x + x_0), x_0 \rangle}{\|x_0\|} - \|B_\lambda(x_0)\|$$
\[ \geq \frac{1}{2} \langle F(x + x_0), x + x_0 \rangle \frac{1}{\|x + x_0\|} - \frac{1}{2} \|x + x_0\| - \|B_0(x_0)\| \]
\[ = \frac{1}{2} \|x + x_0\|^2 - \frac{1}{2} \|x + x_0\| - \|B_0(x_0)\| \]
which shows that
\[ \lim_{\|x\| \to \infty} \frac{\langle F(x + x_0) + B_0(x + x_0), x \rangle}{\|x\|} = \infty \]
and so by Corollary 23.7.40, there exists a solution to 23.7.63. This shows half of the following interesting theorem which is another version of the above major result.

**Theorem 23.7.43** Suppose \( A, B \) are maximal monotone operators. Then for each \( x^* \in X' \), there exists a solution \( x_\lambda \) to

\[ x^* \in Fx_\lambda + B_\lambda x_\lambda + Ax_\lambda, \quad \lambda > 0 \quad (23.7.64) \]

If for \( \lambda \in (0, \delta) \), \( \{B_\lambda x_\lambda\} \) is bounded, then there exists a solution \( x \) to

\[ x^* \in Fx + Bx + Ax \]

**Proof:** The existence of a solution to the inclusion 23.7.64 comes from the above discussion. The last claim follows from almost a repeat of the last part of the proof of the above theorem. Since \( \{B_\lambda x_\lambda\} \) is given to be bounded for \( \lambda \in (0, \delta) \), there is a sequence, \( \lambda_n \to 0 \) such that

\[ x_{\lambda_n} \to z \text{ weakly} \]
\[ x_\lambda^* \to w^* \text{ weakly} \]
\[ Fx_{\lambda_n} \to u^* \text{ weakly} \]
\[ B_{\lambda_n} x_{\lambda_n} \to b^* \text{ weakly} \]

Using 23.7.13, it follows that

\[ \langle Fx_{\lambda_n} + x_{\lambda_n}^* + B_{\lambda_n} x_{\lambda_n} - (Fx_{\lambda_m} + x_{\lambda_m}^* + B_{\lambda_m} x_{\lambda_m}), x_{\lambda_n} - x_{\lambda_m} \rangle = 0 \]

Thus

\[ \langle Fx_{\lambda_n} + x_{\lambda_n}^* - (Fx_{\lambda_m} + x_{\lambda_m}^*), x_{\lambda_n} - x_{\lambda_m} \rangle \]
\[ + \langle B_{\lambda_n} x_{\lambda_n} - B_{\lambda_m} x_{\lambda_m}, x_{\lambda_n} - x_{\lambda_m} \rangle = 0 \quad (23.7.65) \]

Now \( F + A \) is surely monotone and so

\[ \limsup_{m, n \to \infty} \langle B_{\lambda_n} x_{\lambda_n} - B_{\lambda_m} x_{\lambda_m}, x_{\lambda_n} - x_{\lambda_m} \rangle \leq 0 \]

By Proposition 23.7.38, \( b^* \in Bz \) and

\[ \lim_{m, n \to \infty} \langle B_{\lambda_n} x_{\lambda_n} - B_{\lambda_m} x_{\lambda_m}, x_{\lambda_n} - x_{\lambda_m} \rangle = 0 \]
Then returning to Corollary 23.7.65, \( \lim \sup_{m,n \to \infty} \langle Fx_{\lambda_n} + x_{\lambda_n}^* - (Fx_{\lambda_m} + x_{\lambda_m}^*) \rangle \leq 0 \)

Now from Corollary 23.7.40, \( F + A \) is maximal monotone (In fact, \( F + A \) is onto). Hence Proposition 23.7.38 applies again and it follows that \( u^* + w^* \in Fz + Az \).

Then passing to the limit as \( n \to \infty \) in
\[
x^* = Fx_{\lambda_n} + B\lambda_n x_{\lambda_n} + x_{\lambda_n}^*
\]

it follows that
\[
x^* = u^* + b^* + w^* = Fz + Az + Bz \]

### 23.7.6 Convex Functions, An Example

As before, \( X \) will be a Banach space in what follows. Sometimes it will be a reflexive Banach space and in this case, it will be assumed that the norm is strictly convex.

**Definition 23.7.44** Let \( \phi : X \to (-\infty, \infty) \). Then \( \phi \) is convex if whenever \( t \in [0, 1] \), \( x, y \in X \),
\[
\phi(tx + (1 - t)y) \leq t\phi(x) + (1 - t)\phi(y)
\]

The epigraph of \( \phi \) is defined by
\[
\text{epi}(\phi) \equiv \{(x, y) : y \geq \phi(x)\}
\]

When \( \text{epi}(\phi) \) is closed in \( X \times (-\infty, \infty) \), we say that \( \phi \) is lower semicontinuous, l.s.c. The function is called proper if \( \phi(x) < \infty \) for some \( x \). The collection of all such \( x \) is called \( D(\phi) \), the domain of \( \phi \).

This definition of lower semicontinuity is equivalent to the usual definition.

**Lemma 23.7.45** The above definition of lower semicontinuity is equivalent to the assertion that whenever \( x_n \to x \), it follows that \( \phi(x) \leq \liminf_{n \to \infty} \phi(x_n) \). In case that \( \phi \) is convex, lower semicontinuity is equivalent to weak lower semicontinuity. That is \( \text{epi}(\phi) \) is closed if and only if \( \text{epi}(\phi) \) is weakly closed. In this case, the limit condition: If \( x_n \to x \) weakly, then \( \phi(x) \leq \liminf_{n \to \infty} \phi(x_n) \) is valid.

**Proof:** Suppose the limit condition holds. Why is \( \text{epi}(\phi) \) closed? Why is \( X \times (-\infty, \infty) \setminus \text{epi}(\phi) \equiv \text{epi}(\phi)^C \) open? Let \( (x, \alpha) \in \text{epi}(\phi)^C \). Then \( \alpha < \phi(x) \), \( \alpha + \delta < \phi(x) \). Consider \( B(x, r) \times \left(\alpha - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right) \). If every such open set contains a point of \( \text{epi}(\phi) \), then there exists \( x_n \to x, y_n < \alpha + \frac{\delta}{2}, y_n \geq \phi(x_n) \). Hence, from the limit condition,
\[
\phi(x) \leq \liminf_{n \to \infty} \phi(x_n) \leq \liminf_{n \to \infty} y_n \leq \alpha + \frac{\delta}{2} < \alpha + \delta < \phi(x)
\]
a contradiction. It follows that there exists \( r > 0 \) such that \( B(x, r) \times \left(\alpha - \frac{\delta}{2}, \alpha + \frac{\delta}{2}\right) \cap \text{epi}(\phi) = \emptyset \). Since \( \text{epi}(\phi)^C \) is open, it follows that \( \text{epi}(\phi) \) is closed.
Next suppose $\text{epi}(\phi)$ is closed. Why does the limit condition hold? Suppose $x_n \to x$. Then $(x_n, \phi(x_n)) \in \text{epi}(\phi)$. There is a subsequence such that
\[
\alpha \equiv \liminf_{n \to \infty} \phi(x_n) = \lim_{k \to \infty} \phi(x_{n_k})
\]
and so $(x_{n_k}, \phi(x_{n_k})) \to (x, \alpha)$. Since $\text{epi}(\phi)$ is closed, this means $(x, \alpha) \in \text{epi}(\phi)$. Hence
\[
\alpha \equiv \liminf_{n \to \infty} \phi(x_n) \geq \phi(x).
\]

Consider the last claim. In this case, $\text{epi}(\phi)$ is convex. If it is closed, then it is weakly closed thanks to separation theorems: If $(x, \alpha) \in \text{epi}(\phi)^C$, then $\alpha < \infty$ and so there exists $(x^*, \beta) \in (X \times \mathbb{R})^*$ and $l$ such that for all $(t, \gamma) \in \text{epi}(\phi)$,
\[
x^*(t) + \beta \gamma > l > x^*(x) + \alpha \beta
\]
Then $B_{(x^*, \beta)}((x, \alpha), \delta)$ is a weakly open set containing $(x, \alpha)$. For $\delta$ small enough, it does not intersect $\text{epi}(\phi)$ since if not so, there would exist $(t_n, \gamma_n) \in \text{epi}(\phi) \cap B_{(x^*, \beta)}((x, \alpha), \frac{1}{n})$ and so
\[
x^*(t_n) + \beta \gamma_n \to x^*(x) + \alpha \beta
\]
contrary to the above inequality. Thus $\text{epi}(\phi)$ is weakly closed. Also, if $\text{epi}(\phi)$ is weakly closed, then it is obviously strongly closed.

What of the limit condition using weak convergence instead of strong convergence? Say $x_n \to x$ weakly. Does it follow that if $\text{epi}(\phi)$ is weakly closed that $\phi(x) \leq \liminf_{n \to \infty} \phi(x_n)$? It is just as above. There is a subsequence such that
\[
\alpha \equiv \liminf_{n \to \infty} \phi(x_n) = \lim_{k \to \infty} \phi(x_{n_k})
\]
and so $(x_{n_k}, \phi(x_{n_k})) \to (x, \alpha)$ weakly. Since $\text{epi}(\phi)$ is weakly closed, this means $(x, \alpha) \in \text{epi}(\phi)$. Hence
\[
\alpha \equiv \liminf_{n \to \infty} \phi(x_n) \geq \phi(x).
\]

There is also another convenient characterization of what it means for a function to be lower semicontinuous.

**Lemma 23.7.46** Let $\phi : X \to (-\infty, \infty]$. Then $\phi$ is lower semicontinuous if and only if $\phi^{-1}((a, \infty])$ is open for any $a \in \mathbb{R}$.

**Proof:** Suppose first that $\text{epi}(\phi)$ is closed. Consider $x \in \phi^{-1}((a, \infty])$. Thus $\phi(x) > a$. Thus $(x, a) \in \text{epi}(\phi)^C$ because $a < \phi(x)$. Since $\text{epi}(\phi)$ is closed, there exists $r, \varepsilon > 0$ such that

\[
B(x, r) \times (a - \varepsilon, a + \varepsilon) \subseteq \text{epi}(\phi)^C
\]

Hence if $y \in B(x, r)$, it follows that $\phi(y) \geq a + \varepsilon$ since otherwise there would be a point of $\text{epi}(\phi)^C$ in this open set $B(x, r) \times (a - \varepsilon, a + \varepsilon)$. Hence $B(x, r) \subseteq \phi^{-1}((a, \infty])$. 


Conversely, suppose $\phi^{-1}((a,\infty])$ is open for any $a$ and let $(x,b) \in \text{epi}(\phi)^C$. Then $\phi(x) > b$. Thus there exists $B(x,r)$ such that for $y \in B(x,r)$, it follows that $\phi(y) > b$. That is, $y \in \phi^{-1}((b,\infty])$. So consider $B(x,r) \times (-\infty,b)$. If $(y,\alpha) \in B(x,r) \times (-\infty,b)$, then since $\phi(y) > b, \alpha < \phi(y)$ and so there is no point of intersection between $\text{epi}(\phi)$ and this open set $B(x,r) \times (-\infty,b)$.}

Of course one can define upper semicontinuous the same way that $\phi^{-1}(-\infty,a)$ is open. Thus a function is continuous if and only if it is both upper and lower semicontinuous.

In case $X$ is reflexive, the limit condition implies that $\text{epi}(\phi)$ is weakly closed. Suppose $(x,\alpha)$ is a weak limit point of $\text{epi}(\phi)$. Then by the Eberlein Smulian theorem, there is a subsequence of points of $X, (x_n,\alpha_n)$ which converges weakly to $(x,\alpha)$. Thus if the limit condition holds,

$$\phi(x) \leq \lim \inf_{n \to \infty} \phi(x_n) \leq \lim \inf_{n \to \infty} \alpha_n = \alpha$$

and so $(x,\alpha) \in \text{epi}(\phi)$. If $X$ is not reflexive, this isn’t all that clear because it is not clear that a limit point is the limit of a sequence. However, one could consider a limit condition involving nets and get a similar result.

**Definition 23.7.47** Let $\phi : X \to (-\infty,\infty]$ be convex lower semicontinuous, and proper. Then

$$\partial \phi(x) \equiv \{x^* : \phi(y) - \phi(x) \geq \langle x^*, y-x \rangle \text{ for all } y\}$$

The domain of $\partial \phi$, denoted as $D(\partial \phi)$ is just the set of all $x$ for which $\partial \phi(x) \neq \emptyset$. Note that $D(\partial \phi) \subseteq D(\phi)$ since if $x \notin D(\phi)$, the defining inequality could not hold for all $y$ because the left side would be $-\infty$ for some $y$.

**Theorem 23.7.48** For $X$ a real Banach space, let $\phi(x) \equiv \frac{1}{2}||x||^2$. Then $F(x) = \partial \phi(x)$. Here $F$ was the set valued map satisfying $x^* \in Fx$ means

$$||x^*|| = ||Fx||, \langle Fx, x \rangle = ||x||^2.$$

**Proof:** Let $x^* \in F(x)$. Then

$$\langle x^*, y-x \rangle = \langle x^*, y \rangle - \langle x^*, x \rangle \leq ||x^*|| ||y|| - ||x^*||^2 \leq \frac{1}{2}||y||^2 - \frac{1}{2}||x^*||^2.$$

This shows $F(x) \subseteq \partial \phi(x)$.

Now let $x^* \in \partial \phi(x)$. Then for all $t \in \mathbb{R}$,

$$\langle x^*, ty \rangle = \langle x^*, ty + x - x \rangle \leq \frac{1}{2} \left(||x + ty||^2 - ||x||^2\right). \quad (23.7.66)$$

Now if $t > 0$, divide both sides by $t$. This yields

$$\langle x^*, y \rangle \leq \frac{1}{2t} \left(||x|| + t ||y|| \right)^2 - ||x||^2)$$

$$= \frac{1}{2t} \left(2t ||x|| ||y|| + t^2 ||y||^2\right)$$
Letting $t \to 0$,
\[ \langle x^*, y \rangle \leq ||x|| \cdot ||y||. \]  
(23.7.67)

Next suppose $t = -s$, where $s > 0$ in (23.7.66). Then, since when you divide by a negative, you reverse the inequality, for $s > 0$

\[ \langle x^*, y \rangle \geq \frac{1}{2s} \left[ ||x||^2 - 2 ||x - sy|| ||y|| + ||sy||^2 - ||x - sy||^2 \right]. \]  
(23.7.68)

Taking a limit as $s \to 0$ yields
\[ \langle x^*, y \rangle \geq -||x|| \cdot ||y||. \]  
(23.7.70)

It follows from (23.7.67) and (23.7.68) that
\[ ||x^*|| \leq ||x|| \cdot ||y|| \]
and that, therefore, $||x^*|| \leq ||x||$ and $||\langle x^*, x \rangle|| \leq ||x||^2$. Now return to (23.7.69) and let $y = x$. Then

\[ \langle x^*, x \rangle \geq \frac{1}{2s} \left[ -2 ||x - sx|| ||sx|| + ||sx||^2 \right] \]
[23.7.69]

Letting $s \to 1$,
\[ \langle x^*, x \rangle \geq ||x||^2. \]

Since it was already shown that $||\langle x^*, x \rangle|| \leq ||x||^2$, this shows $\langle x^*, x \rangle = ||x||^2$ and also $||x^*|| \leq ||x||$. Thus
\[ ||x^*|| \geq \frac{\langle x^*, x \rangle}{||x||} = ||x|| \]
so in fact $x^* \in F(x)$. $\blacksquare$

The next result gives conditions under which the subgradient is onto. This means that if $y^* \in X'$, then there exists $x \in X$ such that $y^* \in \partial \phi(x)$.

**Theorem 23.7.49** Suppose $X$ is a reflexive Banach space and suppose $\phi : X \to (-\infty, \infty]$ is convex, proper, l.s.c., and for all $y^* \in X'$, $x \to \phi(x) - \langle y^*, x \rangle$ is coercive,

\[ \lim_{||x|| \to \infty} \phi(x) - \langle y^*, x \rangle = \infty \]

Then $\partial \phi$ is onto.
23.7. MAXIMAL MONOTONE OPERATORS

**Proof:** The function \( x \to \phi (x) - y^* (x) \equiv \psi (x) \) is convex, proper, l.s.c., and coercive. Let

\[
\lambda \equiv \inf \{ \phi (x) - \langle y^*, x \rangle : x \in X \}
\]

and let \( \{ x_n \} \) be a minimizing sequence satisfying

\[
\lambda = \lim_{n \to \infty} \phi (x_n) - \langle y^*, x_n \rangle
\]

By coercivity,

\[
\lim_{||x|| \to \infty} \phi (x) - \langle y^*, x \rangle = \infty
\]

and so this minimizing sequence is bounded. By the Eberlein Smulian theorem, Theorem 15.5.12, there is a weakly convergent subsequence \( x_{n_k} \to x \). By Lemma 23.7.45,

\[
\lambda = \phi (x) - \langle y^*, x \rangle \leq \liminf_{k \to \infty} \phi (x_{n_k}) - \langle y^*, x_{n_k} \rangle = \lambda
\]

so there exists \( x \) which minimizes \( x \to \phi (x) - \langle y^*, x \rangle \equiv \psi (x) \). Therefore, \( 0 \in \partial \psi (x) \) because

\[
\psi (y) - \psi (x) \geq 0 = \langle 0, y - x \rangle
\]

Thus, \( 0 \in \partial \psi (x) = \partial \phi (x) - y^* \). \( \blacksquare \)

Now let \( \phi \) be a convex proper lower semicontinuous function defined on \( X \) where \( X \) is a reflexive Banach space with strictly convex norm. Consider \( \partial \phi \). Is it maximal monotone? Is it the case that \( F + \partial \phi \) is onto? First of all, is \( \partial \phi \) monotone? Let \( x^* \in \partial \phi (x) \), \( y^* \in \partial \phi (y) \). Then

\[
\phi (y) - \phi (x) \geq \langle x^*, y - x \rangle
\]

\[
\phi (x) - \phi (y) \geq \langle y^*, x - y \rangle
\]

Hence adding these yields

\[
\langle y^* - x^*, x - y \rangle \leq 0, \quad \langle y^* - x^*, y - x \rangle \geq 0.
\]

Yes, \( \partial \phi \) is certainly monotone. Is it maximal monotone?

**Theorem 23.7.50** Let \( \phi \) be convex, proper, and lower semicontinuous on \( X \) where \( X \) is a reflexive Banach space having strictly convex norm. Then \( \partial \phi \) is maximal monotone.

**Proof:** It is necessary to show that \( F + \partial \phi \) is onto. To do this, let

\[
\psi (x) = \frac{1}{2} ||x||^2 + \phi (x) - \langle y^*, x \rangle
\]

where \( y^* \) is a given element of \( X' \) and the idea is to show that \( y^* \in F (x) + \partial \phi (x) \) for some \( x \). Then by separation theorems, \( \phi (x) \geq b + \langle z^*, x \rangle \) for some \( b, z^* \). Hence it is clear that \( \psi \) is convex, lower semicontinuous and coercive in the sense that

\[
\lim_{||x|| \to \infty} \psi (x) = \infty
\]
It follows that any minimizing sequence for $\psi$ is bounded. Hence by the weak lower semicontinuity, this function has a minimum at $x_0$ say. Thus

$$\frac{1}{2} \|x_0\|^2 + \phi(x_0) - \langle y^* , x_0 \rangle \leq \frac{1}{2} \|x\|^2 + \phi(x) - \langle y^* , x \rangle$$

for all $x$. Then

$$\frac{1}{2} \|x_0\|^2 - \frac{1}{2} \|x\|^2 + \langle y^* , x - x_0 \rangle \leq \phi(x) - \phi(x_0)$$

Now from Theorem 23.7.51 and so, the above reduces to

$$\langle F(x) , x_0 - x \rangle \leq \frac{1}{2} \|x_0\|^2 - \frac{1}{2} \|x\|^2$$

and so, the above reduces to

$$\langle F(x) , x_0 - x \rangle + \langle y^* , x - x_0 \rangle \leq \phi(x) - \phi(x_0)$$

Next let $x = x_0 + t(z-x_0) , t \in (0,1)$, where $z$ is arbitrary. Then

$$-t \langle F(x_0 + t(z-x_0)) , z - x_0 \rangle + t \langle y^* , z - x_0 \rangle \leq \phi(x_0 + t(z-x_0)) - \phi(x_0)$$

and so, by convexity,

$$-t \langle F(x_0 + t(z-x_0)) , z - x_0 \rangle + t \langle y^* , z - x_0 \rangle \leq (1-t) \phi(x_0) + t \phi(z) - \phi(x_0)$$

Now cancel the $t$ on both sides to obtain

$$\langle y^* , z - x_0 \rangle \leq (\phi(z) - \phi(x_0)) + \langle F(x_0 + t(z-x_0)) , z - x_0 \rangle$$

By the fact that $F$ is hemicontinuous, actually demicontinuous, one can let $t \downarrow 0$ and obtain

$$\langle y^* , z - x_0 \rangle \leq (\phi(z) - \phi(x_0)) + \langle F(x_0) , z - x_0 \rangle$$

This says that $y^* - F(x_0) \in \partial \phi(x_0)$ from the definition of what $\partial \phi(x_0)$ means. 

There is a much harder approach to this theorem which is based on a theorem about when the subgradient of a sum equals the sum of the subgradients. This major theorem is given next. Much of the above is in [12] but I don’t remember where I found the following proof.

**Theorem 23.7.51.** Let $\phi_1$ and $\phi_2$ be convex, l.s.c. and proper having values in $(-\infty, \infty]$. Then

$$\partial (\lambda \phi_1)(x) = \lambda \partial \phi_1(x) , \partial (\phi_1 + \phi_2)(x) \supseteq \partial \phi_1(x) + \partial \phi_2(x)$$

(23.7.1)

if $\lambda > 0$. If there exists $\bar{x} \in \text{dom} (\phi_1) \cap \text{dom} (\phi_2)$ and $\phi_1$ is continuous at $\bar{x}$ then for all $x \in X$,

$$\partial (\phi_1 + \phi_2)(x) = \partial \phi_1(x) + \partial \phi_2(x).$$

(23.7.2)
Proof: As is obvious so we only need to show \( \partial \phi \) is as described. It is clear \( \partial \phi \) holds whenever \( x \notin \text{dom} (\phi_1) \cap \text{dom} (\phi_2) \) since then \( \partial (\phi_1 + \phi_2) = \emptyset \). Therefore, assume

\[ x \in \text{dom} (\phi_1) \cap \text{dom} (\phi_2) \]

in what follows. Let \( x^* \in \partial (\phi_1 + \phi_2) (x) \). Is \( x^* \) is the sum of an element of \( \partial \phi_1 (x) \) and \( \partial \phi_2 (x) \)? Does there exist \( x_1^* \) and \( x_2^* \) such that for every \( y \),

\[ x^* (y - x) = x_1^* (y - x) + x_2^* (y - x) \leq \phi_1 (y) - \phi_1 (x) + \phi_2 (y) - \phi_2 (x)? \]

If so, then

\[ \phi_1 (y) - \phi_1 (x) - x^* (y - x) \geq \phi_2 (x) - \phi_2 (y). \]

Define

\[ C_1 = \{(y, a) \in X \times \mathbb{R} : \phi_1 (y) - \phi_1 (x) - x^* (y - x) \leq a\}, \]
\[ C_2 = \{(y, a) \in X \times \mathbb{R} : a \leq \phi_2 (x) - \phi_2 (y)\}. \]

I will show \( \text{int} (C_1) \cap C_2 = \emptyset \) and then by Theorem [2.13] there exists an element of \( X' \) which does something interesting.

Both \( C_1 \) and \( C_2 \) are convex and nonempty. Say \( y_1, y_2 \in C_1 \) and \( t \in [0, 1] \). Then

\[ \phi_1 ((ty_1) + (1 - t) y_2) - \phi_1 (x) - x^* (((ty_1) + (1 - t) y_2) - x) \leq t \phi (y_1) + (1 - t) \phi (y_2) - (t \phi_1 (x) + (1 - t) \phi (x)) \]
\[ - (t x^* (y_1 - x) + (1 - t) x^* (y_2 - x)) \leq t a + (1 - t) a = a \]

so \( C_1 \) is indeed convex. The case of \( C_2 \) is similar.

\( C_1 \) is nonempty because it contains \( (x, \phi_1 (x) - x^* (x - x)) \) since

\[ \phi_1 (x) - \phi_1 (x) - x^* (x - x) \leq \phi_1 (x) - \phi_1 (x) - x^* (x - x) \]

\( C_2 \) is also nonempty because it contains \( (x, \phi_2 (x) - \phi_2 (x)) \) since

\[ \phi_2 (x) - \phi_2 (x) \leq \phi_2 (x) - \phi_2 (x) \]

In addition to this,

\[ (x, \phi_1 (x) - x^* (x - x) - \phi_1 (x) + 1) \in \text{int} (C_1) \]

due to the assumed continuity of \( \phi_1 \) at \( x \) and so \( \text{int} (C_1) \neq \emptyset \). If \( (y, a) \in \text{int} (C_1) \) then

\[ \phi_1 (y) - x^* (y - x) - \phi_1 (x) \leq a - \varepsilon \]
whenever $\varepsilon$ is small enough. Therefore, if $(y, a)$ is also in $C_2$, the assumption that $x^* \in \partial (\phi_1 + \phi_2) (x)$ implies
\[
a - \varepsilon \geq \phi_1 (y) - x^* (y - x) - \phi_1 (x) \geq \phi_2 (x) - \phi_2 (y) \geq a,\]
a contradiction. Therefore $\text{int} (C_1) \cap C_2 = \emptyset$ and so by Theorem 16.2.14, there exists $(w^*, \beta) \in X' \times \mathbb{R}$ with
\[
(w^*, \beta) \neq (0, 0), \tag{23.7.73}
\]
and
\[
w^* (y) + \beta a \geq w^* (y_1) + \beta a_1, \tag{23.7.74}
\]
whenever $(y, a) \in C_1$ and $(y_1, a_1) \in C_2$.

**Claim:** $\beta > 0$.

**Proof of claim:** If $\beta < 0$ let
\[
a = \phi_1 (x) - x^* (x - x) - \phi_1 (x) + 1, \quad a_1 = \phi_2 (x) - \phi_2 (x), \quad \text{and } y = y_1 = x.
\]
Then from
\[
\beta (\phi_1 (x) - x^* (x - x) - \phi_1 (x) + 1) \geq \beta (\phi_2 (x) - \phi_2 (x)).
\]
Dividing by $\beta$ yields
\[
\phi_1 (x) - x^* (x - x) - \phi_1 (x) + 1 \leq \phi_2 (x) - \phi_2 (x)
\]
and so
\[
\phi_1 (x) + \phi_2 (x) - (\phi_1 (x) + \phi_2 (x)) + 1 \leq x^* (x - x)
\]
\[
\leq \phi_1 (x) + \phi_2 (x) - (\phi_1 (x) + \phi_2 (x)),
\]
a contradiction. Therefore, $\beta \geq 0$.

Now suppose $\beta = 0$. Letting
\[
a = \phi_1 (x) - x^* (x - x) - \phi_1 (x) + 1,
\]
then from
\[
(\pi, a) \in \text{int} (C_1),
\]
and so there exists an open set $U$ containing 0 and $\eta > 0$ such that
\[
\pi + U \times (a - \eta, a + \eta) \subseteq C_1.
\]
Therefore, applied to $(\pi + z, a) \in C_1$ and $(\pi, \phi_2 (x) - \phi_2 (x)) \in C_2$ for $z \in U$ yields
\[
w^* (\pi + z) \geq w^* (\pi)
\]
for all $z \in U$. Hence $w^* (z) = 0$ on $U$ which implies $w^* = 0$, contradicting. This proves the claim.
Now with the claim, it follows $\beta > 0$ and so, letting $z^* = w^*/\beta$, \ref{lem:23.7.16} and Lemma \ref{lem:23.7.15} implies
\begin{equation}
z^*(y) + a \geq z^*(y_1) + a_1
\end{equation}
whenever $(y, a) \in C_1$ and $(y_1, a_1) \in C_2$. In particular,
\begin{equation}
(y, \phi_1(y) - \phi_1(x) - x^*(y - x)) \in C_1
\end{equation}
because
\begin{align*}
\phi_1(y) - \phi_1(x) - x^*(y - x) &\leq \phi_1(y) - x^*(y - x) - \phi_1(x) \\
(y_1, \phi_2(x) - \phi_2(y_1)) &\in C_2.
\end{align*}
by similar reasoning so letting $y = x$,
\begin{align*}
z^*(x) + \left(\frac{0}{\phi_1(x) - x^*(x - x) - \phi_1(x)}\right) &\geq z^*(y_1) + \phi_2(x) - \phi_2(y_1).
\end{align*}
Therefore,
\begin{align*}
z^*(y_1 - x) &\leq \phi_2(y_1) - \phi_2(x)
\end{align*}
for all $y_1$ and so $z^* \in \partial\phi_2(x)$. Now let $y_1 = x$ in \ref{cor:23.7.52} and using \ref{lem:23.7.49} and \ref{lem:23.7.48}, it follows
\begin{align*}
z^*(y) + \phi_1(y) - x^*(y - x) - \phi_1(x) &\geq z^*(x) \\
\phi_1(y) - \phi_1(x) &\geq x^*(y - x) - z^*(y - x)
\end{align*}
and so $x^* - z^* \in \partial\phi_1(x)$ so $x^* = z^* + (x^* - z^*) \in \partial\phi_2(x) + \partial\phi_1(x)$.

**Corollary 23.7.52** Let $\phi : X \to (-\infty, \infty]$ be convex, proper, and lower semicontinuous. Here $X$ is a Banach space. Then $\partial\phi$ is maximal monotone.

**Proof:** Let $\psi(x) = \frac{1}{2} \|x\|^2$. There exists $x^*$ and some number $b$ such that $\phi(x) \geq b + \langle x^*, x \rangle$. Therefore, $\psi + \phi$ is convex, lower semicontinuous, and bounded. It follows $\partial(\psi + \phi)$ is onto by Theorem \ref{thm:23.7.48}. However, $\psi$ is continuous everywhere, in particular at every point of the domain of $\phi$. Therefore, $\partial\psi + \partial\phi = \partial(\phi + \psi)$ and by Theorem \ref{thm:23.7.48}, this shows that $F + \partial\phi$ is onto. 

It seems to me that the above are the most important results about convex proper lower semicontinuous functions. However, there are many other very interesting properties known.

**Proposition 23.7.53** Let $\phi : X \to (-\infty, \infty]$ be convex proper and lower semicontinuous. Then $\partial\phi$ is dense in $\partial\phi$ and so $\overline{\partial\phi} = \overline{\partial\phi}$. 

Proof: Let $x_\lambda$ be the solution to $0 \in F(x_\lambda - x) + \lambda \partial \phi (x_\lambda)$. Here $x \in D(\phi)$. Say $u_\lambda^* \in \partial \phi (x_\lambda)$ such that the inclusion becomes an equality. Then

$$0 = \langle F(x_\lambda - x) + \lambda u_\lambda^*, x_\lambda - x \rangle = \| x_\lambda - x \|^2 - \lambda \langle u_\lambda^*, x_\lambda - x \rangle$$

Hence, letting $z^*, b$ be such that $\phi (y) \geq b + \langle z^*, y - x \rangle$,

$$\lambda (\phi (x) - [b + \langle z^*, x_\lambda - x \rangle]) \geq \lambda (\phi (x) - \phi (x_\lambda)) \geq \| x_\lambda - x \|^2$$

$$\lambda \phi (x) - \lambda b \geq \| x_\lambda - x \|^2 - \lambda \| z^* \| \| x_\lambda - x \|$$

$$\geq \| x_\lambda - x \|^2 - \lambda \left( \frac{\| z^* \|^2}{2} + \frac{\| x_\lambda - x \|^2}{2} \right)$$

Thus

$$\lambda \phi (x) - \lambda b + \lambda \frac{\| z^* \|^2}{2} \geq \left( 1 - \frac{\lambda}{2} \right) \| x_\lambda - x \|^2$$

It follows that $x_\lambda \to x$. This shows that $D(\phi) \subseteq D(\partial \phi)$ and so $\overline{D(\phi)} \subseteq D(\partial \phi) \subseteq D(\phi)$. 

There is a really amazing theorem, Moreau’s theorem. It is in [23], [12] and [105]. It involves approximating a convex function with one which is differentiable, at least in the case where you have a Hilbert space. In the general case considered in this chapter, the function is continuous.

**Theorem 23.7.54** Let $\phi$ be a convex lower semicontinuous proper function defined on $X$. Define $A \equiv \partial \phi, A_\lambda = (\partial \phi)_\lambda$

$$\phi_\lambda (x) \equiv \min_{y \in X} \left( \frac{1}{2\lambda} \| x - y \|^2 + \phi (y) \right)$$

Then the function is well defined, convex, Gateaux differentiable,

$$D_z \phi_\lambda (x) \equiv \lim_{t \to 0} \frac{\phi_\lambda (x + tz) - \phi_\lambda (x)}{t} = \langle A_\lambda x, z \rangle$$

so the Gateaux derivative is just $A_\lambda x$ and for all $x \in X$,

$$\lim_{\lambda \to 0} \phi_\lambda (x) = \phi (x),$$

In addition,

$$\phi_\lambda (x) = \frac{1}{2\lambda} \| x - J_\lambda x \|^2 + \phi (J_\lambda (x)) \quad (23.7.78)$$

where $J_\lambda x$ is as before, the solution to

$$0 \in F (J_\lambda x - x) + \lambda \partial \phi (J_\lambda x)$$
**Proof:** First of all, why does the minimum take place? By the convexity, closed epigraph, and assumption that $\phi$ is proper, separation theorems apply and one can say that there exists $z^*$ such that for all $y \in H$,

$$\frac{1}{2\lambda} \|x - y\|^2 + \phi(y) \geq \frac{1}{2\lambda} \|x - y\|^2 + (z^*, y) + c \quad (23.7.79)$$

It follows easily that a minimizing sequence is bounded and so from lower semicontinuity which implies weak lower semicontinuity due to convexity, there exists $y_x$ such that

$$\min_{y \in H} \left( \frac{1}{2\lambda} \|x - y\|^2 + \phi(y) \right) = \left( \frac{1}{2\lambda} \|x - y_x\|^2 + \phi(y_x) \right)$$

Why is $\phi_\lambda$ convex? For $\theta \in [0, 1],

$$\phi_\lambda(\theta x + (1 - \theta) z) = \frac{1}{2\lambda} \|\theta x + (1 - \theta) z - y_{(\theta x + (1 - \theta) z)}\|^2 + \phi(y_{\theta x + (1 - \theta) z})$$

$$\leq \frac{1}{2\lambda} \|\theta x + (1 - \theta) z - (\theta y_x + (1 - \theta) y_z)\|^2 + \phi(\theta y_x + (1 - \theta) y_z)$$

$$\leq \frac{\theta}{2\lambda} \|x - y_x\|^2 + \frac{1 - \theta}{2\lambda} \|z - y_z\|^2 + \theta \phi(y_x) + (1 - \theta) \phi(y_z)$$

$$= \theta \phi_\lambda(x) + (1 - \theta) \phi_\lambda(z)$$

So is there a formula for $y_x$? Since it involves minimization of the functional, it follows that

$$0 \in -\frac{1}{\lambda} F(x - y_x) + \partial \phi(y_x) = \frac{1}{\lambda} F(y_x - x) + \partial \phi(y_x)$$

Recall that if $\psi(x) = \frac{1}{2} \|x\|^2$, then $\partial \psi(x) = F(x)$. Thus

$$y_x = J_\lambda x$$

because this was how $J_\lambda x$ was defined. Therefore,

$$\phi_\lambda(x) = \frac{1}{2\lambda} \|x - J_\lambda x\|^2 + \phi(J_\lambda(x)) = \frac{\lambda}{2} \|A_\lambda x\|^2 + \phi(J_\lambda x), \quad A = \partial \phi$$

It follows from this equation that

$$\phi(J_\lambda x) \leq \phi_\lambda(x) \leq \phi(x), \quad (23.7.80)$$

the second inequality following from taking $y = x$ in the definition of $\phi_\lambda$.

Next consider the claim about $\phi_\lambda(x) \uparrow \phi(x)$. First suppose that $x \in D(\phi)$. Then from Proposition 4.8.7(1), $x \in \overline{D(\partial \phi)}$ and so from the material on approximations, Theorem 4.6.3(1), it follows that $J_\lambda x \to x$. Hence from 23.7.80 and lower semicontinuity of $\phi$,

$$\phi(x) \leq \liminf_{\lambda \to 0} \phi(J_\lambda x) \leq \liminf_{\lambda \to 0} \phi_\lambda(x) \leq \limsup_{\lambda \to 0} \phi_\lambda(x) \leq \phi(x)$$
CHAPTER 23. NONLINEAR OPERATORS

showing that in this case, \( \lim_{\lambda \to 0} \phi_{\lambda}(x) = \phi(x) \). Next suppose \( x \notin D(\phi) \) so that \( \phi(x) = \infty \). Why does \( \phi_{\lambda}(x) \to \infty? \) Suppose not. Then from the description of \( \phi_{\lambda} \) given above and using the fact that the epigraph is closed and convex, there would exist a subsequence, still denoted as \( \lambda \) such that

\[
C \geq \phi_{\lambda}(x) = \frac{1}{2\lambda} ||x - J_{\lambda}x||^2 + \phi(J_{\lambda}(x)) \geq \frac{1}{2\lambda} ||x - J_{\lambda}x||^2 + \langle z^*, x - J_{\lambda}x \rangle + b
\]

Then multiplying by \( \lambda \), it follows that for a suitable constant \( M \),

\[
||x - J_{\lambda}x||^2 \leq M\lambda + \lambda M ||x - J_{\lambda}x||
\]

and so a use of the quadratic formula implies

\[
||x - J_{\lambda}x|| \leq \frac{M}{2} \left( 1 + \sqrt{5} \right) \lambda
\]

Hence \( J_{\lambda}x \to x \) and so in \( 23.7.81 \) it follows from lower semicontinuity again that

\[
\infty = \phi(x) \leq \liminf_{\lambda \to 0} \phi(J_{\lambda}x) \leq \liminf_{\lambda \to 0} \phi_{\lambda}(x) \leq \limsup_{\lambda \to 0} \phi_{\lambda}(x) \leq \phi(x)
\]

and so again, \( \lim_{\lambda \to 0} \phi_{\lambda}(x) = \infty \). Also note that if \( \lambda > \mu \), then

\[
\min_{y \in A} \left( \frac{1}{2\lambda} ||x - y||^2 + \phi(y) \right) \leq \min_{y \in A} \left( \frac{1}{2\mu} ||x - y||^2 + \phi(y) \right)
\]

because for a given \( y \), \( \frac{1}{2\lambda} ||x - y||^2 + \phi(y) \leq \frac{1}{2\mu} ||x - y||^2 + \phi(y) \). Thus \( \phi_{\lambda}(x) \uparrow \phi(x) \).

Next consider the claim about the Gateaux differentiability. Using the description \( 23.7.75 \)

\[
\phi_{\lambda}(y) - \phi_{\lambda}(x) = \frac{1}{2\lambda} ||y - J_{\lambda}y||^2 + \phi(J_{\lambda}(y)) - \left( \frac{1}{2\lambda} ||x - J_{\lambda}x||^2 + \phi(J_{\lambda}(x)) \right)
\]

(23.7.81)

Using the fact that if \( \psi(x) = ||x||^2 \), then \( \partial \psi(x) =Fx \), and that \( A_{\lambda}x \in \partial \phi(J_{\lambda}x) \),

\[
\geq \lambda^{-1} \langle F(x - J_{\lambda}x), (y - J_{\lambda}y) - (x - J_{\lambda}x) \rangle + \langle A_{\lambda}x, J_{\lambda}(y) - J_{\lambda}(x) \rangle
\]

\[
= \langle A_{\lambda}(x), (y - J_{\lambda}y) - (x - J_{\lambda}x) \rangle + \langle A_{\lambda}x, J_{\lambda}(y) - J_{\lambda}(x) \rangle = \langle A_{\lambda}x, y - x \rangle
\]

Hence

\[
(\phi_{\lambda}(y) - \phi_{\lambda}(x)) - \langle A_{\lambda}x, y - x \rangle \geq 0
\]

Also from \( 23.7.81 \)

\[
\frac{1}{2\lambda} ||y - J_{\lambda}y||^2 - \frac{1}{2\lambda} ||x - J_{\lambda}x||^2 = - \left( \frac{1}{2\lambda} ||x - J_{\lambda}x||^2 - \frac{1}{2\lambda} ||y - J_{\lambda}y||^2 \right)
\]

\[
\leq - \frac{1}{\lambda} \langle F(y - J_{\lambda}y), (x - J_{\lambda}x) - (y - J_{\lambda}y) \rangle = \langle A_{\lambda}y, (y - J_{\lambda}y) - (x - J_{\lambda}x) \rangle
\]
Similarly, from (23.7.81),
\[
\phi(J_\lambda(y)) - \phi(J_\lambda(x)) = -(\phi(J_\lambda(x)) - \phi(J_\lambda(y)))
\leq -(A_\lambda(y), J_\lambda(x) - J_\lambda(y)) = (A_\lambda(y), J_\lambda(y) - J_\lambda(x))
\]
It follows that
\[
\langle A_\lambda(y), J_\lambda(y) - J_\lambda(x) \rangle + \langle A_\lambda(y), (y - J_\lambda y) - (x - J_\lambda x) \rangle 
\geq (\phi_\lambda(y) - \phi_\lambda(x)) \geq \langle A_\lambda x, y - x \rangle
\]
and so
\[
\langle A_\lambda(y), y - x \rangle \geq (\phi_\lambda(y) - \phi_\lambda(x)) \geq \langle A_\lambda x, y - x \rangle
\]
Therefore,
\[
\langle A_\lambda(y) - A_\lambda(x), y - x \rangle \geq (\phi_\lambda(y) - \phi_\lambda(x)) - \langle A_\lambda x, y - x \rangle \geq 0
\]
Next let \( y = x + tz \) for \( t > 0 \). Then
\[
t \langle A_\lambda(x + tz) - A_\lambda(x), z \rangle \geq (\phi_\lambda(x + tz) - \phi_\lambda(x)) - t \langle A_\lambda x, z \rangle \geq 0
\]
Using the demicontinuity of \( A_\lambda \), you can divide by \( t \) and pass to a limit to obtain
\[
\lim_{t \downarrow 0} \frac{\phi_\lambda(x + tz) - \phi_\lambda(x)}{t} = \langle A_\lambda x, z \rangle.
\]
A much better theorem is available in case \( X = X' = H \) a Hilbert space. In this case \( \phi_\lambda \) is also Frechet differentiable. See Theorem \( \text{[23.7.24]} \) which is presented later. Everything is much nicer in the Hilbert space setting because \( F \) is just replaced with the identity and the approximations are defined more easily.

Then one can show that \( J_\lambda \) is Lipschitz continuous and many other nice things happen.

Next is an interesting result about when the sum of a maximal monotone operator and a subgradient is also maximal monotone. A version of this is well known in the case of a single Hilbert space. In the case of a single Hilbert space, this result can be used to produce very regular solutions to evolution equations for functions which have values in the Hilbert space. You would get this by letting \( X = X' = H \) a Hilbert space. A maximal monotone operator \( A \) would be defined on \( L^2(0, T; H) = X \) a space of Hilbert space valued functions which are square integrable. Then you could take \( Lu = u' \) with domain equal to those functions in \( X \) which are equal to 0 at the left end of the interval for example. This is done more generally later. In this case the duality map is just the identity. The next theorem includes the case of two different spaces. I am not sure whether this is a useful result at this time, in terms of evolution equations. However, it is good to have conditions which show that the sum of two maximal monotone operators is maximal monotone.
Theorem 23.7.55 Let $X$ be a reflexive Banach space with strictly convex norm and let $\Phi$ be non negative, convex, proper, and lower semicontinuous. Suppose also that $A : D(A) \to \mathcal{P}(X')$ is a maximal monotone operator and there exists $x \in D(A) \cap D(\Phi).$ \hfill (23.7.82)

Suppose also that $\Phi(J_\lambda x) \leq \Phi(x) + C\lambda \hfill (23.7.83)$

Then $A + \partial \Phi$ is maximal monotone.

Proof: Recall that $A_\lambda x = -\lambda^{-1}F(J_\lambda x - x),$ where $0 \in F(J_\lambda x - x) + \lambda \partial A(J_\lambda x)$

Let $y^* \in X'.$ From Theorem 23.7.43 there exists $x_\lambda \in H$ such that $y^* \in Fx_\lambda + A_\lambda x_\lambda + \partial \Phi(x_\lambda).$

It is desired to show that $A_\lambda x_\lambda$ is bounded. From the above, $y^* - Fx_\lambda + A_\lambda x_\lambda \in \partial \Phi(x_\lambda), \hfill (23.7.84)$

and so $y^* - Fx_\lambda + A_\lambda x_\lambda, x_\lambda \leq \Phi(J_\lambda x_\lambda) - \Phi(x_\lambda) \leq C\lambda \hfill (23.7.85)$

which implies

$\langle y^* - Fx_\lambda + A_\lambda x_\lambda, -\lambda F^{-1}(A_\lambda x) \rangle \leq \Phi(J_\lambda x_\lambda) - \Phi(x_\lambda) \leq C\lambda \hfill (23.7.84)$

and so $\langle y^* - Fx_\lambda + A_\lambda x_\lambda, -F^{-1}(A_\lambda x) \rangle \leq C$ $\hfill (23.7.86)$

I claim $\{\|x_\lambda\|\}$ are bounded independent of $\lambda.$ By 23.7.83 and monotonicity of $A_\lambda,$

$\Phi(x) - \Phi(x_\lambda) \geq \langle y^* - Fx_\lambda + A_\lambda x_\lambda, x - x_\lambda \rangle \hfill (23.7.83)$

$\geq \langle y^* - Fx_\lambda, x - x_\lambda \rangle - (A_\lambda x_\lambda, x - x_\lambda) \hfill (23.7.85)$

$\geq \langle y^* - Fx_\lambda, x_\lambda - x_\lambda \rangle - (A_\lambda x_\lambda, x - x_\lambda) \hfill (23.7.85)$

$\geq \langle y^*, x_\lambda \rangle - \langle y^*, x_\lambda \rangle - (F_{x_\lambda}, x_\lambda) + \|x_\lambda\|^2 - \|x - x_\lambda\| \|A_\lambda x_\lambda\| \hfill (23.7.85)$

$\geq -\|y^*\| \|\xi\| - \|y^*\| \|x_\lambda\| - \|x_\lambda\| \|\xi\| - \|\xi\| |A_\lambda x_\lambda| - \|x_\lambda\| |A_\lambda x_\lambda| + \|x_\lambda\|^2 \hfill (23.7.85)$

Therefore, there exist constants, $C_1$ and $C_2,$ depending on $\xi$ and $y^*$ but not on $\lambda$ such that $\Phi(x) \geq \Phi(x_\lambda) + \|x_\lambda\|^2 - C_1 \|x_\lambda\| - C_2. \hfill (23.7.85)$

Since $\Phi \geq 0,$ the above shows that $\|x_\lambda\|$ is indeed bounded. Now from 23.7.80 it follows that $\{A_\lambda x_\lambda\}$ is bounded for small positive $\lambda.$ By Theorem 23.7.83 there exists a solution $x$ to $y^* \in Fx + Ax + \partial \Phi(x) \hfill (23.7.85)$

and since $y^*$ is arbitrary, this shows that $A + \partial \Phi$ is maximal monotone. ■
23.8 Perturbation Theorems

In this section, we consider surjectivity of the sum of a pseudomonotone set valued map with a linear maximal monotone map and also with another maximal monotone operator added in. This generalizes the surjectivity results given earlier because one could have 0 for the maximal monotone linear operator. The theorems developed here lead to nice results on evolution equations because the linear maximal monotone operator can be something like a time derivative and $X$ can be some sort of an $L^p$ space for functions having values in a suitable Banach space. This is presented later in the material on Bochner integrals.

The notation $\langle z^*, u \rangle_{V', V}$ will mean $z^*(u)$ in this section. We will not worry about the order either. Thus

$$\langle u, z^* \rangle \equiv z^*(u) \equiv \langle z^*, u \rangle$$

This is just convenient in writing things down. Also, it is assumed that all Banach spaces are real to simplify the presentation. It is also usually assumed that the Banach spaces are reflexive. Thus we can regard

$$(V 	imes V')' = V' \times V$$

and $\langle (y^*, x), (u, v^*) \rangle \equiv \langle y^*, u \rangle + \langle x, v^* \rangle$. It is known that for a reflexive Banach space, there is always an equivalent strictly convex norm. It is therefore, assumed that the norm for the reflexive Banach space is strictly convex.

**Definition 23.8.1** Let $L : D(L) \subseteq V \rightarrow V'$ be a linear map where we always assume $D(L)$ is dense in $V$. Then

$$D(L^*) \equiv \{ u \in V : |\langle Lv, u \rangle| \leq C \|v\| \text{ for all } v \in D(L) \}$$

For such $u$, it follows that on a dense subset of $V$, namely $D(L), v \rightarrow \langle Lz, u \rangle$ is a continuous linear map. Hence there exists a unique element of $V'$, denoted as $L^*u$ such that for all $v \in D(L)$,

$$\langle Lv, u \rangle_{V', V} = \langle L^*u, v \rangle_{V', V}$$

Thus

$$L : D(L) \subseteq V \rightarrow V'$$

$$L^* : D(L^*) \subseteq V \rightarrow V'$$

There is an interesting description of $L^*$ in terms of $L$ which will be quite useful.

**Proposition 23.8.2** Let $\tau : V \times V' \rightarrow V' \times V$ be given by $\tau (a, b) \equiv (-b, a)$. Also for $S \subseteq X$ a reflexive Banach space,

$$S^\perp \equiv \{ z^* \in X' : \langle z^*, s \rangle = 0 \text{ for all } s \in S \}$$

Also denote by $\mathcal{G}(L) \equiv \{(x, Lx) : x \in D(L)\}$. Then

$$\mathcal{G}(L^*) = (\tau \mathcal{G}(L))^\perp$$
**Proof:** Let \((x, L^* x) \in \mathcal{G}(L^*)\). This means that
\[ |\langle L y, x \rangle| \leq C \|y\| \text{ for all } y \in \mathcal{D}(L) \]
and \(\langle L y, x \rangle = \langle L^* x, y \rangle\) for all \(y \in \mathcal{D}(L)\). Let \((y, L y) \in \mathcal{G}(L)\). Then \(\tau(y, L y) = (\langle -L y, y \rangle)\). Then
\[ \langle (x, L^* x), (L^* y, -L y) \rangle = \langle x, -L y \rangle = -\langle x, L y \rangle = \langle L^* x, y \rangle = 0 \]
Thus \(\mathcal{G}(L^*) \subseteq (\tau \mathcal{G}(L))^\perp\). Next suppose \((x, y^*) \in (\tau \mathcal{G}(L))^\perp\). This means that if \((u, L u) \in \mathcal{G}(L)\), then
\[ \langle (x, y^*), (-L u, u) \rangle = \langle x, -L u \rangle + \langle y^*, u \rangle = 0 \]
and so for all \(u \in \mathcal{D}(L)\),
\[ \langle y^*, u \rangle = \langle x, L u \rangle \]
and so \(x \in \mathcal{D}(L^*)\). Hence for all \(u \in \mathcal{D}(L)\),
\[ \langle y^*, u \rangle = \langle x, L u \rangle = \langle L^* x, u \rangle \]
Then, since \(\mathcal{D}(L)\) is dense, it follows that \(y^* = L^* x\) and so \((x, y) \in \mathcal{G}(L^*)\). Thus these are the same. ■

Theorem 23.5.4 is a very nice surjectivity result for set valued pseudomonotone operators. We recall what it said here. Recall the meaning of coercive.

\[ \lim_{\|v\| \to \infty} \inf \left\{ \frac{\langle z^*, v \rangle}{\|v\|} : z^* \in TV \right\} = \infty \]

In this section, we use the convenient notation \(\langle z^*, x \rangle_{V', V} \equiv z^*(x)\).

**Theorem 23.8.3** Let \(V\) be a reflexive Banach space and let \(T : V \to \mathcal{P}(V')\) be pseudomonotone, bounded and coercive. Then \(T\) is onto. More generally, this continues to hold if \(T\) is modified bounded pseudomonotone.

Recall the definition of pseudomonotone.

**Definition 23.8.4** For \(X\) a reflexive Banach space, we say \(A : X \to \mathcal{P}(X')\) is pseudomonotone if the following hold.

1. The set \(Au\) is nonempty, closed and convex for all \(u \in X\).
2. If \(F\) is a finite dimensional subspace of \(X\), \(u \in F\), and if \(U\) is a weakly open set in \(V'\) such that \(Au \subseteq U\), then there exists a \(\delta > 0\) such that if \(v \in B_2(u) \cap F\) then \(Av \subseteq U\). (Weakly upper semicontinuous on finite dimensional subspaces.)
3. If \(u_i \to u\) weakly in \(X\) and \(u_i^* \in Au_i\) is such that
\[ \lim_{i \to \infty} \sup u_i^*, u_i - u \leq 0, \quad (23.8.77) \]
then, for each \(v \in X\), there exists \(u^* (v) \in Au\) such that
\[ \lim_{i \to \infty} \inf u_i^*, u_i - v \geq \langle u^*(v), u - v \rangle. \quad (23.8.78) \]
Also recall the definition of modified bounded pseudomonotone. It is just the above except that the limit condition is replaced with the following condition: If \( u_i \to u \) weakly in \( X \) and
\[
\limsup_{i \to \infty} \langle u^*_i, u_i - u \rangle \leq 0, \tag{23.8.89}
\]
then there exists a subsequence, still denoted as \( \{u_i\} \) such that for each \( v \in X \), there exists \( u^*(v) \in Au \) such that
\[
\liminf_{i \to \infty} \langle u^*_i, u_i - v \rangle \geq \langle u^*(v), u - v \rangle. \tag{23.8.90}
\]

Also recall that this more general limit condition along with the assumption 1 and the assumption that \( A \) is bounded is sufficient to obtain condition 2. This was Lemma 23.4.9 proved earlier and stated here for convenience.

**Lemma 23.8.5** Let \( A : X \to \mathcal{P}(X') \) satisfy conditions \( 1 \) and \( 3 \) above and suppose \( A \) is bounded. Also suppose the condition that if \( x_n \to x \) weakly and \( \limsup_{n \to \infty} \langle z_n, x_n - x \rangle \leq 0 \) implies there exists a subsequence \( \{x_{n_k}\} \) such that for any \( y \),
\[
\liminf_{n \to \infty} \langle z_{n_k}, x_{n_k} - y \rangle \geq \langle z(y), x - y \rangle
\]
for \( z(y) \) some element of \( Ax \). Then if this weaker condition holds, you have that if \( U \) is a weakly open set containing \( Ax \), then \( Ax_n \subseteq U \) for all \( n \) large enough.

**Definition 23.8.6** Now let \( L : D(L) \subseteq V \to V' \) such that \( L \) is linear, monotone, \( D(L) \) is dense in \( V \), \( L \) is closed, and \( L^* \) is monotone. Let \( A : V \to \mathcal{P}(V') \) be a bounded operator. Then \( A \) is called \( L \) pseudomonotone if \( Au \) is closed and convex in \( V' \) and for any sequence \( \{u_n\} \subseteq D(L) \) such that \( u_n \to u \) weakly in \( V \) and \( Lu_n \to Lu \) weakly in \( V' \), and for \( z^*_n \in Au_n \),
\[
\limsup_{n \to \infty} \langle z^*_n, u_n - u \rangle \leq 0
\]
then for every \( v \in V \), there exists \( z^*(v) \in Au \) such that
\[
\liminf_{n \to \infty} \langle z^*_n, u_n - v \rangle \geq \langle z^*(v), u - v \rangle
\]
It is called \( L \) modified bounded pseudomonotone if the above \( \liminf \) condition holds for some subsequence whenever \( u_n \to u \) weakly and \( Lu_n \to Lu \) weakly and \( \limsup_{n \to \infty} \langle z^*_n, u_n - u \rangle \leq 0 \).

**Lemma 23.8.7** Suppose \( X \) is the Banach space
\[
X = D(L), \|u\|_X \equiv \|u\|_V + \|Lu\|_V,
\]
where \( L \) is as described in the above definition. Also assume that \( A \) is bounded. Then if \( A \) is \( L \) pseudomonotone, it follows that \( A \) is pseudomonotone as a map from \( X \) to \( \mathcal{P}(X') \). If \( A \) is \( L \) modified bounded pseudomonotone, then \( A \) is modified bounded pseudomonotone as a map from \( X \) to \( \mathcal{P}(X') \).
**Proof:** Is $A$ bounded? Of course, because the norm of $X$ is stronger than the norm on $V$. Is $Au$ convex and closed? This also follows because $X \subseteq V$. It is clear that $Au$ is convex. If $\{ z_n \} \subseteq Au$ and $z_n \to z$ in $X'$, then does it follow that $z \in Au$? Since $A$ is bounded, there is a further subsequence which converges weakly to $w$ in $V'$. However, $Au$ is convex and closed so it is weakly closed. Hence $w \in Au$ and also $w = z$. It only remains to verify the pseudomonotone limit condition. Suppose then that $u_n \to u$ weakly in $X$ and for $z_n^* \in Au_n$,

$$\limsup_{n \to \infty} \langle z_n^*, u_n - u \rangle \leq 0$$

Then it follows that $Lu_n \to Lu$ weakly in $V'$ and $u_n \to u$ weakly in $V$ so $u \in X$. Hence the assumption that $A$ is $L$ pseudomonotone implies that for every $v \in V$, and for every $v \in X$, there exists $z^* (v) \in Au \subseteq V' \subseteq X'$ such that

$$\liminf_{n \to \infty} \langle z_n^*, u_n - v \rangle \geq \langle z^* (v), u - v \rangle$$

The last claim goes the same way. You just have to take a subsequence. ■

Then we have the following major surjectivity result. In this theorem, we will assume for simplicity that all spaces are real spaces. Versions of this appear to be due to Brezis [2] and Lions [2]. Of course the theorem holds for complex spaces as well. You just need to use $\Re \langle \rangle$ instead of $\langle \rangle$.

**Theorem 23.8.8** Let $L : D (L) \subseteq V \to V'$ where $D (L)$ is dense, $L$ is monotone, $L$ is closed, and $L^*$ is monotone, $L$ a linear map. Let $A : V \to P (V')$ be $L$ pseudomonotone, bounded, coercive. Then $L + A$ is onto. Here $V$ is a reflexive Banach space such that the norms for $V$ and $V'$ are strictly convex. In case that $A$ is strictly monotone ($\langle Au - Av, u - v \rangle > 0$ implies $u \neq v$) the solution $u$ to $f \in Lu + Au$ is unique. If, in addition to this, $\langle Au - Av, u - v \rangle \geq r (\| u - v \|_{V'})$ where $U$ is some Banach space containing $V$, and $r$ is a positive strictly increasing function for which $\lim_{t \to \infty} r (t) = 0$, then the map $f \to u$ where $f \in Lu + Au$ is continuous as a map from $V'$ to $U$. The conclusion holds if $A$ is only $L$ modified bounded pseudomonotone.

**Proof:** Let $F$ be the duality map for $p = 2$. Consider the Banach space $X$ given by

$$X = D (L), \| u \|_X \equiv \| u \|_V + \| Lu \|_{V'}$$

This is isometric with the graph of $L$ with the graph norm and so $X$ is reflexive. Now define a set valued map $G_\varepsilon$ on $X$ as follows. $z^* \in G_\varepsilon (u)$ means there exists $w^* \in Au$ such that

$$\langle z^*, v \rangle_{X',X} = \varepsilon \langle Lv, F^{-1} (Lu) \rangle_{V',V} + \langle Lu, v \rangle_{V',V} + \langle w^*, v \rangle_{V',V}$$

It follows from Lemma [23.8.5] that $G_\varepsilon$ is the sum of a set valued $L$ modified bounded pseudomonotone operator with an operator which is demicontinuous, bounded, and
monotone, hence pseudomonotone. Thus by Lemma 23.5.2 it is $L$ modified bounded pseudomonotone. Is it coercive?

\[
\lim_{\|u\|_X \to \infty} \inf \left\{ \frac{\langle z^*, u \rangle + \varepsilon \langle F^{-1}(Lu), F^{-1}(Lu) \rangle_{V', V} + \langle Lu, u \rangle_{V', V}}{\|u\|_X} : z^* \in Au \right\} = \infty?
\]

It equals

\[
\lim_{\|u\|_X \to \infty} \inf \left\{ \frac{\langle z^*, u \rangle + \varepsilon \|F^{-1}(Lu)\|_{V'}^2}{\|u\|_X} : z^* \in Au \right\}
\]

and this is

\[
\geq \lim_{\|u\|_X \to \infty} \inf \left\{ \frac{\langle z^*, u \rangle + \varepsilon \|Lu\|_{V'}^2}{\|u\|_V + \|Lu\|_{V'}} : z^* \in Au \right\}
\]

because $L$ is monotone. Now let $M$ be an arbitrary positive number. By assumption, there exists $R$ such that if $\|u\|_V > R$, then

\[
\inf \left\{ \frac{\langle z^*, u \rangle}{\|u\|_V} : z^* \in Au \right\} > M
\]

and so for every $z^* \in Au$,

\[
\frac{\langle z^*, u \rangle}{\|u\|_V} > M, \quad (z^*, u) > M \|u\|_V
\]

Thus if $\|u\|_V > R$,

\[
\inf \left\{ \frac{\langle z^*, u \rangle + \varepsilon \|Lu\|_{V'}^2}{\|u\|_V + \|Lu\|_{V'}} : z^* \in Au \right\} \geq \frac{M \|u\|_V + \varepsilon \|Lu\|_{V'}^2}{\|u\|_V + \|Lu\|_{V'}}
\]

I claim that if $\|u\|_X$ is large enough, the above is larger than $M/2$. If not, then there exists $\{u_n\}$ such that $\|u_n\|_X \to \infty$ but the right side is less than $M/2$. First say $\|Lu_n\|$ is bounded. Then there is an obvious contradiction since the right hand side then converges to $M$. Thus it can be assumed that $\|Lu_n\|_{V'} \to \infty$. Hence, for all $n$ large enough, $\varepsilon \|Lu_n\|_{V'}^2 > M \|Lu_n\|_{V'}$. However, this implies the right side is larger than

\[
\frac{M \|u_n\|_V + M \|Lu_n\|_{V'}}{\|u_n\|_V + \|Lu_n\|_{V'}} = M > M/2
\]

This is a contradiction. Hence the right side is larger than $M/2$ for all $n$ large enough. It follows since $M$ is arbitrary, that

\[
\lim_{\|u\|_X \to \infty} \inf \left\{ \frac{\langle z^*, u \rangle + \varepsilon \|Lu\|_{V'}^2}{\|u\|_V + \|Lu\|_{V'}} : z^* \in Au \right\} = \infty
\]
It follows from Theorem 23.5.4 that if \( f \in V' \), there exists \( u_\varepsilon \) such that for all \( v \in D(L) = X \),

\[
\varepsilon \langle Lv, F^{-1}(Lu_\varepsilon) \rangle_{V',V} + \langle Lu_\varepsilon, v \rangle_{V',V} + \langle w^*_\varepsilon, v \rangle_{V',V} = \langle f, v \rangle, \quad w^*_\varepsilon \in Au_\varepsilon \quad (23.8.91)
\]

First we get an estimate.

\[
\varepsilon \langle Lu_\varepsilon, F^{-1}(Lu_\varepsilon) \rangle_{V',V} + \langle Lu_\varepsilon, u_\varepsilon \rangle_{V',V} + \langle w^*_\varepsilon, u_\varepsilon \rangle_{V',V} = \langle f, u_\varepsilon \rangle
\]

Hence it follows from the coercivity of \( A \) that \( \|u_\varepsilon\|_V \) is bounded independent of \( \varepsilon \). Thus the \( w^*_\varepsilon \) are also bounded in \( V' \) because it is assumed that \( A \) is bounded. Now from the equation solved \( 23.8.91 \), it follows that \( F^{-1}(Lu_\varepsilon) \in D(L^*) \). Thus the first term is just \( \varepsilon \langle L^*(F^{-1}(Lu_\varepsilon)), v \rangle_{V',V} \). It follows, since \( D(L) = X \) is dense in \( V \) that

\[
\varepsilon L^* (F^{-1}(Lu_\varepsilon)) + Lu_\varepsilon + w^*_\varepsilon = f \quad (23.8.92)
\]

Then act on \( F^{-1}(Lu_\varepsilon) \) on both sides. From monotonicity of \( L^* \), this yields \( \|Lu_\varepsilon\|_V \) is bounded independent of \( \varepsilon > 0 \). Thus there is a subsequence still denoted with a subscript of \( \varepsilon \) such that

\[
u_\varepsilon \rightharpoonup u \text{ in } V
\]

\[
Lu_\varepsilon \rightharpoonup Lu \text{ in } V'
\]

This because of the fact that the graph of \( L \) is closed, hence weakly closed. Thus \( u \in X \). Also

\[
w^*_\varepsilon \rightharpoonup w^* \text{ in } V'.
\]

It follows that we can pass to a limit in \( 23.8.92 \) and obtain

\[
Lu + w^* = f \quad (23.8.93)
\]

Now by assumption on \( A \), it is \( L \) modified bounded pseudomonotone and so there is a subsequence, still denoted as \( u_\varepsilon \) such that the lim inf pseudomonotone limit condition holds. This will be what is referred to in what follows. Then

\[
\langle \varepsilon L^* (F^{-1}(Lu_\varepsilon)), u_\varepsilon - u \rangle + \langle Lu_\varepsilon, u_\varepsilon - u \rangle + \langle w^*_\varepsilon, u_\varepsilon - u \rangle = \langle f, u_\varepsilon - u \rangle
\]

and so,

\[
\varepsilon \langle F^{-1}(Lu_\varepsilon), Lu_\varepsilon - Lu \rangle + \langle Lu_\varepsilon, u_\varepsilon - u \rangle + \langle w^*_\varepsilon, u_\varepsilon - u \rangle = \langle f, u_\varepsilon - u \rangle
\]

using the monotonicity of \( L \),

\[
\varepsilon \langle Lu_\varepsilon - Lu, F^{-1}(Lu_\varepsilon) - F^{-1}(Lu) \rangle + \varepsilon \langle Lu_\varepsilon - Lu, F^{-1}(Lu) \rangle
\]

\[
+ \langle Lu, u_\varepsilon - u \rangle + \langle w^*_\varepsilon, u_\varepsilon - u \rangle \leq \langle f, u_\varepsilon - u \rangle
\]
23.8. PERTURBATION THEOREMS

Now using monotonicity of $F^{-1}$,

$$
\varepsilon \langle Lu_\varepsilon - Lu, F^{-1}(Lu) \rangle + \langle Lu, u_\varepsilon - u \rangle + \langle w^*_\varepsilon, u_\varepsilon - u \rangle \leq \langle f, u_\varepsilon - u \rangle
$$

and so, passing to a limit as $\varepsilon \to 0$,

$$
\limsup_{\varepsilon \to 0} \langle w^*_\varepsilon, u_\varepsilon - u \rangle \leq 0
$$

It follows that for all $v \in X = D(L)$ there exists $w^*(v) \in Au$

$$
\liminf_{\varepsilon \to 0} \langle w^*_\varepsilon, u_\varepsilon - v \rangle \geq \langle w^*(v), u - v \rangle
$$

But the left side equals

$$
\liminf_{\varepsilon \to 0} \left( \langle w^*_\varepsilon, u_\varepsilon - u \rangle + \langle w^*_\varepsilon, u - v \rangle \right)
\leq \limsup_{\varepsilon \to 0} \langle w^*_\varepsilon, u_\varepsilon - u \rangle + \langle w^*, u - v \rangle \leq \langle w^*, u - v \rangle
$$

and so

$$
\langle w^*, u - v \rangle \geq \langle w^*(v), u - v \rangle
$$

for all $v$.

Is $w^* \in Au$? Suppose not. Then $Au$ is a closed convex set and $w^*$ is not in it. Hence, since $V$ is reflexive, there exists $z \in V$ such that whenever $y^* \in Au$, $\langle w^*, z \rangle < \langle y^*, z \rangle$. Now simply choose $v$ such that $u - v = z$ and it follows that

$$
\langle w^*(v), u - v \rangle > \langle w^*, u - v \rangle \geq \langle w^*(v), u - v \rangle
$$

which is clearly a contradiction. Hence $w^* \in Au$. Thus from [Lions-Niremberg] this has shown that $L + A$ is onto.

Consider the claim about uniqueness and continuous dependence. Say you have $f_i \in Lu_i + Au_i$, $i = 1, 2$. Let $z_i^* \in Au_i$ be such that equality holds in the two inclusions. Then

$$
f_1 - f_2 = z_1^* - z_2^* + Lu_1 - Lu_2
$$

It follows that

$$
\langle f_1 - f_2, u_1 - u_2 \rangle = \langle z_1^* - z_2^* + Lu_1 - Lu_2, u_1 - u_2 \rangle \geq r(\|u_1 - u_2\|)
$$

Thus if $f_1 = f_2$, then $u_1 = u_2$. If $f_n \to f$ in $V'$, then $r(\|u - u_n\|) \to 0$ where $u_n$ goes with $f_n$ and $u$ with $f$ as just described, and so $u_n \to u$ because the coercivity estimate given above shows that the $u_n$ and $u$ are all bounded. Thus the map just described is continuous.

The following lemma is interesting in terms of the hypotheses of the above theorem. [2]

**Lemma 23.8.9** Let $L : D(L) \to X'$ where $D(L)$ is dense and $L$ is a closed operator. Then $L$ is maximal monotone if and only if both $L, L^*$ are monotone.
Proof: Suppose both $L, L^*$ are monotone. One must show that $\lambda F + L$ is onto. However, $F$ is monotone and hemicontinuous (actually demicontinuous) and coercive. Hence the fact that $\lambda F + L$ is onto follows from Theorem 23.5.8. Next suppose $L$ is maximal monotone. If $L$ is maximal monotone, then for every $\varepsilon > 0$ there exists a solution $u_\varepsilon$ such that $\varepsilon Lu_\varepsilon + F(u_\varepsilon - u) = 0$. Here $u \in D(L^*)$. This is from Lemma 23.7.28. It is originally due to Browder [24]. Then

$$\varepsilon \langle Lu_\varepsilon, u_\varepsilon \rangle + \langle F(u_\varepsilon - u), u_\varepsilon \rangle = 0$$

and so $\langle F(u_\varepsilon - u), u_\varepsilon \rangle \leq 0$. Then

$$\langle F(u_\varepsilon - u), u_\varepsilon - u \rangle \leq \langle F(u_\varepsilon - u), u \rangle$$

so $\|u_\varepsilon - u\|^2 \leq \|u_\varepsilon - u\| \|u\|$ and so

$$\|u_\varepsilon - u\| \leq \|u\|$$

Thus the $u_\varepsilon$ are bounded.

Next let $v \in D(L)$.

$$\|u_\varepsilon - u\|^2 = \langle F(u_\varepsilon - u), u_\varepsilon - u \rangle = \langle F(u_\varepsilon - u), u_\varepsilon - v \rangle + \langle F(u_\varepsilon - u), v - u \rangle$$

$$\leq \varepsilon \langle Lv, v - u \rangle + \langle F(u_\varepsilon - u), v - u \rangle \leq \varepsilon \langle Lv, v - u \rangle + \langle F(u_\varepsilon - u), v - u \rangle$$

Hence

$$\limsup \|u_\varepsilon - u\|^2 \leq \limsup \varepsilon \langle Lv, v - u \rangle + \langle F(u_\varepsilon - u), v - u \rangle \leq \limsup \|u_\varepsilon - u\| \|v - u\|$$

and so $u_\varepsilon \to u$ strongly. Also

$$\langle F(u_\varepsilon - u), u_\varepsilon \rangle = -\varepsilon \langle Lu_\varepsilon, u_\varepsilon \rangle \leq 0$$

Then

$$\langle L^*u, u \rangle = \lim_{\varepsilon \to 0} \langle L^*u, u_\varepsilon \rangle = \lim_{\varepsilon \to 0} \langle Lu_\varepsilon, u \rangle = \frac{1}{\varepsilon} \langle -F(u_\varepsilon - u), u \rangle$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle F(u_\varepsilon - u), u_\varepsilon - u \rangle - \frac{1}{\varepsilon} \langle F(u_\varepsilon - u), u_\varepsilon \rangle$$

Both of these last terms are nonnegative, the first obviously and the second from the above where it was shown that $\langle F(u_\varepsilon - u), u_\varepsilon \rangle \leq 0$. $\blacksquare$

In the hypotheses of Theorem 23.5.8, one could have simply said that $L$ is closed, linear, densely defined and maximal monotone. One can also show that if $L$ is maximal monotone, then it must be densely defined. This is done in [24].

One can go further in obtaining a perturbation theorem like the above. Let linear $L$ be densely defined with $L$ closed and $L, L^*$ monotone. In short, $L$ is densely defined and maximal monotone, $L : X \to X'$. Let $A$ be a set valued $L$ pseudomonotone operator which is coercive and bounded. Also let $B : D(B) \to P(X)$ be maximal monotone. It is of interest to consider whether $L + A + B$ is onto $X'$. In considering this, I will add further assumptions as needed. First note that $\langle Lx, x \rangle = \langle Lx - L0, x - 0 \rangle \geq 0$. 

23.8. PERTURBATION THEOREMS

Definition 23.8.10 Define
\[ \limsup_{m,n \to \infty} a_{m,n} \equiv \lim_{k \to \infty} \sup \{ a_{m,n} : \min(m, n) \geq k \} \]

Then \( \limsup_{m,n \to \infty} a_{m,n} \geq \limsup_{m \to \infty} (\limsup_{n \to \infty} a_{m,n}) \). To see this, suppose \( a > \limsup_{m,n \to \infty} a_{m,n} \). Then there exist \( k \) such that whenever \( m, n > k \),
\[ a_{m,n} < a \]
It follows that for \( m \geq k \),
\[ \limsup_{n \to \infty} a_{m,n} \leq a \]
Hence
\[ \limsup_{m \to \infty} \left( \limsup_{n \to \infty} a_{m,n} \right) \leq a \]
Since \( a > \limsup_{m,n \to \infty} a_{m,n} \) is arbitrary, it follows that
\[ \limsup_{m \to \infty} \left( \limsup_{n \to \infty} a_{m,n} \right) \leq \limsup_{m,n \to \infty} a_{m,n}. \]

Then the following lemma is useful. I found this result in a paper by Gasinski, Migorski and Ochal [51]. They begin with the following interesting lemma or something like it which is similar to some of the ideas used in the section on approximation of maximal monotone operators.

Lemma 23.8.11 Suppose \( A \) is a set valued operator, \( A : X \to P(X) \) and \( u^*_n \in A u_n \). Suppose also that \( u_n \to u \) weakly and \( u^*_n \to u^* \) weakly. Suppose also that
\[ \lim \sup_{m,n \to \infty} \langle u^*_n - u^*_m, u_n - u_m \rangle \leq 0 \]
Then one can conclude that
\[ \limsup_{n \to \infty} \langle u^*_n, u_n - u \rangle \leq 0 \]

Proof: Let \( \alpha \equiv \limsup_{n \to \infty} \langle u^*_n, u_n \rangle \). It is a finite number because these sequences are bounded. Then using the weak convergence,
\[ 0 \geq \limsup_{m \to \infty} \left( \limsup_{n \to \infty} \langle u^*_n - u^*_m, u_n - u_m \rangle \right) \]
\[ = \limsup_{m \to \infty} \left( \limsup_{n \to \infty} \left( \langle u^*_n, u_n \rangle + \langle u^*_m, u_m \rangle - \langle u^*_n, u_m \rangle - \langle u^*_m, u_n \rangle \right) \right) \]
\[ = \limsup_{m \to \infty} \left( \alpha + \langle u^*_m, u_m \rangle - \langle u^*, u \rangle - \langle u^*_m, u \rangle \right) \]
\[ = \alpha + \langle u^*, u \rangle - \langle u^*, u \rangle = 2\alpha - 2\langle u^*, u \rangle \]
Now
\[ \limsup_{n \to \infty} \langle u^*_n, u_n - u \rangle = \alpha - \langle u^*, u \rangle \leq 0. \]
To begin with, consider the approximate problem which is to determine whether \( L + A + B \lambda \) is onto. Here \( B \lambda x = \lambda^{-1} F(x_{\lambda} - x) \) where \( 0 \in F(x_{\lambda} - x) + \lambda B x \).

In the notation given above, \( B \lambda x = \lambda^{-1} F(J_{\lambda} x - x) \). Then by Theorem 23.7.36, \( B \lambda \) is monotone, demicontinuous, and bounded. In addition, we assume \( 0 \in D(B) \).

Then

\[
\langle B \lambda x, x \rangle \geq \langle B \lambda 0, x \rangle \geq -|B(0)| \|x\| \quad (23.8.94)
\]

**Lemma 23.8.12** Let \( A \) be pseudomonotone, bounded and coercive and let \( 0 \in D(B) \). Then if \( y^* \in X' \), there exists a solution \( x_{\lambda} \) to

\[
y^* \in L x_{\lambda} + A x_{\lambda} + B \lambda x_{\lambda}
\]

**Proof:** From the inequality \( 23.8.94 \), \( A + B \lambda \) is coercive. It is also bounded and pseudomonotone. It is pseudomonotone from Theorem 23.7.27. Therefore, there exists a solution \( x_{\lambda} \) by Theorem 23.8.8.

Acting on \( x_{\lambda} \) and using the inequality \( 23.8.94 \), it follows that these solutions \( x_{\lambda} \) lie in a bounded set. The details follow. Letting \( z^*_{\lambda} \in A x_{\lambda} \) be such that equality holds in the above inclusion,

\[
y^* = L x_{\lambda} + z^*_{\lambda} + B \lambda x_{\lambda} \quad (23.8.95)
\]

Thus, from coercivity, \( \|x_{\lambda}\| \) are bounded. Then since \( A \) is bounded, the \( z^*_{\lambda} \) are all bounded also independent of \( \lambda \). The top line shows also that

\[
\|y^*\| \geq \frac{\langle y^*, x_{\lambda} \rangle}{\|x_{\lambda}\|} \geq \frac{\langle L x_{\lambda}, x_{\lambda} \rangle + \langle z^*_{\lambda}, x_{\lambda} \rangle + \langle B \lambda x_{\lambda}, x_{\lambda} \rangle}{\|x_{\lambda}\|} \geq \frac{\langle L x_{\lambda}, x_{\lambda} \rangle + \langle z^*_{\lambda}, x_{\lambda} \rangle - |B(0)| \|x_{\lambda}\|}{\|x_{\lambda}\|} \geq \|z^*_{\lambda} - B(0)\| \|x_{\lambda}\| - |B(0)|
\]

Thus, from coercivity, \( \|x_{\lambda}\| \) are bounded. Then since \( A \) is bounded, the \( z^*_{\lambda} \) are all bounded also independent of \( \lambda \). The top line shows also that

\[
\langle y^*, x_{\lambda} \rangle = \langle L x_{\lambda}, x_{\lambda} \rangle + \langle z^*_{\lambda}, x_{\lambda} \rangle + \langle B \lambda x_{\lambda}, x_{\lambda} \rangle \geq \langle z^*_{\lambda}, x_{\lambda} \rangle + \langle B \lambda x_{\lambda}, x_{\lambda} \rangle \geq \langle B \lambda x_{\lambda}, x_{\lambda} \rangle - M \geq -|B(0)| \|x_{\lambda}\| - M \quad (23.8.96)
\]

where \( |\langle z^*_{\lambda}, x_{\lambda} \rangle| \leq M \) for all \( \lambda \). Hence there is a constant \( M \) such that

\[
|\langle B \lambda x_{\lambda}, x_{\lambda} \rangle| \leq M
\]

**Definition 23.8.13** A set valued operator \( B \) is quasi-bounded if whenever \( x \in D(B) \) and \( x^* \in B x \) are such that

\[
|\langle x^*, x \rangle|, \|x\| \leq M,
\]

it follows that \( \|x^*\| \leq K_M \). Bounded would mean that if \( \|x\| \leq M, \) then \( \|x^*\| \leq K_M \). Here you only know this if there is another condition.
Lemma 23.8.14 In the above situation, suppose the maximal monotone operator $B$ is quasi-bounded and $|\langle B_\lambda x_\lambda, x_\lambda \rangle| \leq M$. Then the $B_\lambda x_\lambda$ are bounded. Also

$$\|J_\lambda x_\lambda - x_\lambda\|^2 \leq M\lambda$$

Proof: Now $B_\lambda x_\lambda \in BJ_\lambda x_\lambda$

$$-|B(0)||x_\lambda| \leq \langle B_\lambda x_\lambda, J_\lambda x_\lambda \rangle + \langle B_\lambda x_\lambda, x_\lambda - J_\lambda x_\lambda \rangle$$

$$= \langle B_\lambda x_\lambda, J_\lambda x_\lambda \rangle + \langle \lambda^{-1}F(J_\lambda x_\lambda - x_\lambda), J_\lambda x_\lambda - x_\lambda \rangle$$

$$= \langle B_\lambda x_\lambda, J_\lambda x_\lambda \rangle + \lambda^{-1}\|J_\lambda x_\lambda - x_\lambda\|^2 \leq M$$

This inequality shows that $J_\lambda x_\lambda - x_\lambda \to 0$ and so $J_\lambda x_\lambda$ is bounded as is $x_\lambda$ which was shown above. Also $B_\lambda x_\lambda \in BJ_\lambda x_\lambda$ and since $B$ is quasi-bounded, it follows that $B_\lambda x_\lambda$ is bounded.

Assume from now on that $B$ is quasi-bounded. Then the estimate and this lemma shows that $B_\lambda x_\lambda$ is also bounded independent of $\lambda$. Thus, adjusting the constants, there exists an estimate of the form

$$\|x_\lambda\| + \|J_\lambda x_\lambda\| + \|B_\lambda x_\lambda\| + \|z_\lambda^*\| + \|Lx_\lambda\| \leq C, \quad \|x_\lambda - J_\lambda x_\lambda\| \leq \sqrt{\lambda}M \quad (23.8.97)$$

Let $\lambda = 1/n$. Also denote by $J_n$ the the operator $J_{1/n}$ to save notation. There exists a subsequence

$$x_n \to x \text{ weakly},$$

$$J_n x_n \to x \text{ weakly},$$

$$B_n x_n \to g^* \text{ weakly},$$

$$z_n^* \to z^* \text{ weakly},$$

$$L x_n \to Lx \text{ weakly}$$

Now from the inclusion satisfied,

$$0 = \langle z_n^* - z_m^*, x_n - x_m \rangle + \langle B_n x_n - B_m x_m, x_n - x_m \rangle \quad (23.8.98)$$

Consider that last term. $B_n x_n \in BJ_n x_n$ similar for $B_m x_m$. Hence this term is of the form

$$\langle B_n x_n - B_m x_m, x_n - x_m \rangle \geq \langle B_n x_n - B_m x_m, J_n x_n - J_m x_m \rangle$$

From the estimate and

$$\langle B_n x_n - B_m x_m, x_n - x_m \rangle \geq \langle B_n x_n - B_m x_m, (x_n - J_n x_n) - (x_m - J_m x_m) \rangle$$
and
\[ |\langle B_nx_n - B_mx_m, (x_n - J_nx_n) - (x_m - J_mx_m) \rangle| \leq 2C \left( \sqrt{\frac{1}{n}} + \sqrt{\frac{1}{m}} \right) \]

Then from (23.8.98),
\[ 0 \geq \langle z_n^* - z_m^*, x_n - x_m \rangle + e_{n,m} \]
where \( e_{n,m} \to 0 \) as \( n, m \to \infty \). Hence
\[ \limsup_{n,m \to \infty} \langle z_n^* - z_m^*, x_n - x_m \rangle \leq 0 \]

From Lemma (23.8.11),
\[ \limsup_{n \to \infty} \langle z_n^*, x_n - x_0 \rangle \leq 0 \]
Hence, since \( A \) is pseudomonotone, for every \( y \), there exists \( z^*(y) \in Ax \) such that
\[ \liminf_{n \to \infty} \langle z_n^*, x_n - y \rangle \geq \langle z^*(y), x - y \rangle \]
In particular, if \( x = y \), this shows that
\[ \liminf_{n \to \infty} \langle z_n^*, x_n - x \rangle \geq 0 \geq \limsup_{n \to \infty} \langle z_n^*, x_n - x \rangle \]

showing that
\[ \lim_{n \to \infty} \langle z_n^*, x_n \rangle = \langle z^*, x \rangle \]

Next, returning to the inclusion solved,
\[ 0 = Lx_n + z_n^* + B_nx_n \]
Act on \( (x_n - x) \). Then from monotonicity of \( L \),
\[ 0 \geq \langle Lx, x_n - x \rangle + \langle z_n^*, x_n - x \rangle + \langle B_nx_n, x_n - x \rangle \]
Thus, taking \( \limsup \) of both sides,
\[ \limsup_{n \to \infty} \langle B_nx_n, x_n - x \rangle = \limsup_{n \to \infty} \langle B_nx_n, J_nx_n - x \rangle \leq 0 \]
Hence
\[ \limsup_{n \to \infty} \langle B_nx_n, J_nx_n \rangle \leq \langle g^*, x \rangle \]
Letting \([a, b^*] \in \mathcal{G}(B)\),
\[ \langle B_nx_n - b^*, J_nx_n - a \rangle = \langle B_nx_n, J_nx_n \rangle - \langle B_nx_n, a \rangle - \langle b^*, J_nx_n \rangle + \langle b^*, a \rangle \]
Then taking \( \limsup \),
\[ 0 \leq \limsup_{n \to \infty} \langle B_nx_n - b^*, J_nx_n - a \rangle \]
\[ \leq \langle g^*, x \rangle - \langle g^*, a \rangle - \langle b^*, x \rangle + \langle b^*, a \rangle = \langle g^* - b^*, x - a \rangle \]
It follows that $g^* \in B(x)$ and $x \in D(B)$.

Thus, passing to the limit in the equation where, as explained $\lambda = 1/n$, one obtains

$$y^* = Lu + z^* + g^*$$

where $z^* \in Ax$ and $g^* \in Bx$. This proves the following nice generalization of the above perturbation theorem.

**Theorem 23.8.15** Let $B$ be maximal monotone from $X$ to $\mathcal{P}(X')$, $0 \in D(B)$, and $B$ is quasi-bounded as explained above. Let $A : X \to \mathcal{P}(X')$ be pseudomonotone, bounded, and coercive. Also let $L$ be a densely defined linear operator such that both $L$ and $L^*$ are monotone. (That is, $L$ is linear and maximal monotone.) Then $L + A + B$ is onto $X'$. 
Chapter 24

Integrals And Derivatives

24.1 The Fundamental Theorem Of Calculus

The version of the fundamental theorem of calculus found in Calculus has already been referred to frequently. It says that if $f$ is a Riemann integrable function, the function

$$x \rightarrow \int_a^x f(t) \, dt,$$

has a derivative at every point where $f$ is continuous. It is natural to ask what occurs for $f$ in $L^1$. It is an amazing fact that the same result is obtained aside from a set of measure zero even though $f$, being only in $L^1$, may fail to be continuous anywhere. Proofs of this result are based on some form of the Vitali covering theorem presented above. In what follows, the measure space is $(\mathbb{R}^n, \mathcal{S}, m)$ where $m$ is $n$-dimensional Lebesgue measure although the same theorems can be proved for arbitrary Radon measures $\mu$. To save notation, $m$ is written in place of $m_n$.

By Lemma 10.1.9 on Page 259 and the completeness of $m$, the Lebesgue measurable sets are exactly those measurable in the sense of Caratheodory. Also, to save on notation $m$ is also the name of the outer measure defined on all of $\mathcal{P}(\mathbb{R}^n)$ which is determined by $m_n$. Recall

$$B(p, r) = \{ x : |x - p| < r \}. \quad (24.1.1)$$

Also define the following.

If $B = B(p, r)$, then $\hat{B} = B(p, 5r). \quad (24.1.2)$

The first version of the Vitali covering theorem presented above will now be used to establish the fundamental theorem of calculus. The space of locally integrable functions is the most general one for which the maximal function defined below makes sense.
Definition 24.1.1 \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) means \( f \chi_{B(0,R)} \in L^1(\mathbb{R}^n) \) for all \( R > 0 \). For \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), the Hardy Littlewood Maximal Function, \( Mf \), is defined by

\[
Mf(x) \equiv \sup_{r > 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|dy.
\]

Theorem 24.1.2 If \( f \in L^1(\mathbb{R}^n) \), then for \( \alpha > 0 \),

\[
m([Mf > \alpha]) \leq \frac{5^n}{\alpha} \|f\|_1.
\]

(Here and elsewhere, \([Mf > \alpha] \equiv \{x \in \mathbb{R}^n : Mf(x) > \alpha\}\) with other occurrences of \([\ ]\) being defined similarly.)

Proof: Let \( S \equiv [Mf > \alpha] \). For \( x \in S \), choose \( r_x > 0 \) with

\[
\frac{1}{m(B(x,r_x))} \int_{B(x,r_x)} |f| \, dm > \alpha.
\]

The \( r_x \) are all bounded because

\[
m(B(x,r_x)) < \frac{1}{\alpha} \int_{B(x,r_x)} |f| \, dm < \frac{1}{\alpha} \|f\|_1.
\]

By the Vitali covering theorem, there are disjoint balls \( B(x_i,r_i) \) such that

\[
S \subseteq \bigcup_{i=1}^\infty B(x_i,5r_i)
\]

and

\[
\frac{1}{m(B(x_i,r_i))} \int_{B(x_i,r_i)} |f| \, dm > \alpha.
\]

Therefore

\[
\overline{m}(S) \leq \sum_{i=1}^\infty m(B(x_i,5r_i)) = 5^n \sum_{i=1}^\infty m(B(x_i,r_i)) \leq \frac{5^n}{\alpha} \sum_{i=1}^\infty \int_{B(x_i,r_i)} |f| \, dm \leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f| \, dm,
\]

the last inequality being valid because the balls \( B(x_i,r_i) \) are disjoint. This proves the theorem.

Note that at this point it is unknown whether \( S \) is measurable. This is why \( \overline{m}(S) \) and not \( m(S) \) is written.

The following is the fundamental theorem of calculus from elementary calculus.
24.1. THE FUNDAMENTAL THEOREM OF CALCULUS

Lemma 24.1.3 Suppose $g$ is a continuous function. Then for all $x$,
\[
\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} g(y)dy = g(x).
\]

Proof: Note that
\[
g(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} g(y)dy
\]
and so
\[
\left| g(x) - \frac{1}{m(B(x,r))} \int_{B(x,r)} g(y)dy \right|
\]
\[
= \left| \frac{1}{m(B(x,r))} \int_{B(x,r)} (g(y) - g(x))dy \right|
\]
\[
\leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(y) - g(x)|dy.
\]
Now by continuity of $g$ at $x$, there exists $r > 0$ such that if $|x - y| < r$, $|g(y) - g(x)| < \varepsilon$. For such $r$, the last expression is less than
\[
\frac{1}{m(B(x,r))} \int_{B(x,r)} \varepsilon dy < \varepsilon.
\]
This proves the lemma.

Definition 24.1.4 Let $f \in L^1(\mathbb{R}^k, m)$. A point, $x \in \mathbb{R}^k$ is said to be a Lebesgue point if
\[
\lim \sup_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dm = 0.
\]
Note that if $x$ is a Lebesgue point, then
\[
\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) dm = f(x).
\]
and so the symmetric derivative exists at all Lebesgue points.

Theorem 24.1.5 (Fundamental Theorem of Calculus) Let $f \in L^1(\mathbb{R}^k)$. Then there exists a set of measure 0, $N$, such that if $x \notin N$, then
\[
\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| dy = 0.
\]
Proof: Let $\lambda > 0$ and let $\varepsilon > 0$. By density of $C_c(\mathbb{R}^k)$ in $L^1(\mathbb{R}^k, m)$ there exists $g \in C_c(\mathbb{R}^k)$ such that $||g - f||_{L^1(\mathbb{R}^k)} < \varepsilon$. Now since $g$ is continuous,

\[
\limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm
\]

\[
= \limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm - \lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |g(y) - g(x)| \, dm
\]

\[
= \limsup_{r \to 0} \left( \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| - |g(y) - g(x)| \, dm \right)
\]

\[
\leq \limsup_{r \to 0} \left( \frac{1}{m(B(x, r))} \int_{B(x, r)} ||f(y) - f(x)| - |g(y) - g(x)|| \, dm \right)
\]

\[
\leq \limsup_{r \to 0} \left( \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - g(y) - (f(x) - g(x))| \, dm \right)
\]

\[
\leq \limsup_{r \to 0} \left( \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - g(y)| \, dm \right) + |f(x) - g(x)|
\]

\[
\leq M((|f - g|)(x) + |f(x) - g(x)|).
\]

Therefore,

\[
\left[ x : \limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm > \lambda \right]
\]

\[
\subseteq \left[ M(|f - g|) > \frac{\lambda}{2} \right] \cup \left[ |f - g| > \frac{\lambda}{2} \right]
\]

Now

\[
\varepsilon > \int |f - g| \, dm \geq \int_{[|f - g| > \frac{\lambda}{2}]} |f - g| \, dm
\]

\[
\geq \frac{\lambda}{2} m\left( \left[ |f - g| > \frac{\lambda}{2} \right] \right)
\]

This along with the weak estimate of Theorem 24.1.2 implies

\[
m\left( \left[ x : \limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm > \lambda \right] \right)
\]

\[
< \left( \frac{2}{\lambda} 5^k + \frac{2}{\lambda} \right) ||f - g||_{L^1(\mathbb{R}^k)}
\]

\[
< \left( \frac{2}{\lambda} 5^k + \frac{2}{\lambda} \right) \varepsilon.
\]
24.1. THE FUNDAMENTAL THEOREM OF CALCULUS

Since $\varepsilon > 0$ is arbitrary, it follows

$$m_n \left( \left[ x : \limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm > \lambda \right] \right) = 0.$$ 

Now let

$$N = \left[ x : \limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm > 0 \right]$$

and

$$N_n = \left[ x : \limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm > \frac{1}{n} \right]$$

It was just shown that $m(N_n) = 0$. Also, $N = \bigcup_{n=1}^{\infty} N_n$. Therefore, $m(N) = 0$ also. It follows that for $x \notin N$,

$$\limsup_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dm = 0$$

and this proves a.e. point is a Lebesgue point.

Of course it is sufficient to assume $f$ is only in $L^1_{loc}(\mathbb{R}^k)$.

**Corollary 24.1.6 (Fundamental Theorem of Calculus)** Let $f \in L^1_{loc}(\mathbb{R}^k)$. Then there exists a set of measure 0, $N$, such that if $x \notin N$, then

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy = 0.$$ 

**Proof:** Consider $B(0, n)$ where $n$ is a positive integer. Then $f_n \equiv f \chi_{B(0, n)} \in L^1(\mathbb{R}^k)$ and so there exists a set of measure 0, $N_n$ such that if $x \in B(0, n) \setminus N_n$, then

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f_n(y) - f_n(x)| \, dy = 0.$$ 

Let $N = \bigcup_{n=1}^{\infty} N_n$. Then if $x \notin N$, the above equation holds.

**Corollary 24.1.7** If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \to 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dy = f(x) \text{ a.e. } x. \quad (24.1.3)$$
CHAPTER 24. INTEGRALS AND DERIVATIVES

Proof:

\[
\left| \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) \, dy - f(x) \right| \\
\leq \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| \, dy
\]

and the last integral converges to 0 a.e. \(x\).

**Definition 24.1.8** For \(N\) the set of Theorem 24.1.5 or Corollary 24.1.6, \(N^C\) is called the Lebesgue set or the set of Lebesgue points.

The next corollary is a one dimensional version of what was just presented.

**Corollary 24.1.9** Let \(f \in L^1(\mathbb{R})\) and let

\[F(x) = \int_{-\infty}^{x} f(t) \, dt.\]

Then for a.e. \(x\), \(F'(x) = f(x)\).

**Proof:** For \(h > 0\)

\[
\frac{1}{h} \int_{x}^{x+h} |f(y) - f(x)| \, dy \leq 2 \left( \frac{1}{2h} \right) \int_{x-h}^{x+h} |f(y) - f(x)| \, dy
\]

By Theorem 24.1.5, this converges to 0 a.e. Similarly

\[
\frac{1}{h} \int_{x-h}^{x} |f(y) - f(x)| \, dy
\]

converges to 0 a.e. \(x\).

\[
\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{h} \int_{x}^{x+h} |f(y) - f(x)| \, dy \tag{24.1.4}
\]

and

\[
\left| \frac{F(x) - F(x-h)}{h} - f(x) \right| \leq \frac{1}{h} \int_{x-h}^{x} |f(y) - f(x)| \, dy. \tag{24.1.5}
\]

Now the expression on the right in (24.1.4) and (24.1.5) converges to zero for a.e. \(x\). Therefore, by (24.1.4), for a.e. \(x\) the derivative from the right exists and equals \(f(x)\) while from (24.1.5) the derivative from the left exists and equals \(f(x)\) a.e. It follows

\[
\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \text{ a.e. } x
\]

This proves the corollary.
24.2 Absolutely Continuous Functions

Definition 24.2.1 Let \([a, b]\) be a closed and bounded interval and let \(F : [a, b] \to \mathbb{R}\). Then \(F\) is said to be absolutely continuous if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that if \(\sum_{i=1}^{m} |y_i - x_i| < \delta\) where the intervals \((x_i, y_i)\) are non-overlapping, then \(\sum_{i=1}^{m} |F(y_i) - F(x_i)| < \varepsilon\).

Definition 24.2.2 A finite subset, \(P\) of \([a, b]\) is called a partition of \([x, y] \subseteq [a, b]\) if \(P = \{x_0, x_1, \cdots, x_n\}\) where
\[x = x_0 < x_1 < \cdots, < x_n = y.\]

For \(f : [a, b] \to \mathbb{R}\) and \(P = \{x_0, x_1, \cdots, x_n\}\) define
\[V_P [x, y] = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.
\]

Denoting by \(P [x, y]\) the set of all partitions of \([x, y]\) define
\[V [x, y] = \sup_{P \in P [x, y]} V_P [x, y].\]

For simplicity, \(V [a, x]\) will be denoted by \(V(x)\). It is called the total variation of the function, \(f\).

There are some simple facts about the total variation of an absolutely continuous function, \(f\) which are contained in the next lemma.

Lemma 24.2.3 Let \(f\) be an absolutely continuous function defined on \([a, b]\) and let \(V\) be its total variation function as described above. Then \(V\) is an increasing bounded function. Also if \(P\) and \(Q\) are two partitions of \([x, y]\) with \(P \subseteq Q\), then \(V_P [x, y] \leq V_Q [x, y]\) and if \([x, y] \subseteq [z, w]\),
\[V [x, y] \leq V [z, w]\] (24.2.6)

If \(P = \{x_0, x_1, \cdots, x_n\}\) is a partition of \([x, y]\), then
\[V [x, y] = \sum_{i=1}^{n} V [x_i, x_{i-1}]\] (24.2.7)

Also if \(y > x\),
\[V (y) - V (x) \geq |f (y) - f (x)|\] (24.2.8)
and the function, \(x \to V (x) - f (x)\) is increasing. The total variation function, \(V\) is absolutely continuous.
Proof: The claim that \(V\) is increasing is obvious as is the next claim about \(P \subseteq Q\) leading to \(V_P [x, y] \leq V_Q [x, y]\). To verify this, simply add in one point at a time and verify that from the triangle inequality, the sum involved gets no smaller. The claim that \(V\) is increasing consistent with set inclusion of intervals is also clearly true and follows directly from the definition.

Now let \(t < V [x, y]\) where \(P_0 = \{x_0, x_1, \ldots, x_n\}\) is a partition of \([x, y]\). There exists a partition, \(P\) of \([x, y]\) such that \(t < V_P [x, y]\). Without loss of generality it can be assumed that \(\{x_0, x_1, \ldots, x_n\} \subseteq P\) since if not, you can simply add in the points of \(P_0\) and the resulting sum for the total variation will get no smaller. Let \(P_i\) be those points of \(P\) which are contained in \([x_{i-1}, x_i]\). Then

\[
t < V_P [x, y] = \sum_{i=1}^{n} V_{P_i} [x_{i-1}, x_i] \leq \sum_{i=1}^{n} V [x_{i-1}, x_i].
\]

Since \(t < V [x, y]\) is arbitrary,

\[
V [x, y] \leq \sum_{i=1}^{n} V [x_i, x_{i-1}] \tag{24.2.9}
\]

Note that 24.2.9 does not depend on \(f\) being absolutely continuous. Suppose now that \(f\) is absolutely continuous. Let \(\delta\) correspond to \(\varepsilon = 1\). Then if \([x, y]\) is an interval of length no larger than \(\delta\), the definition of absolute continuity implies

\[
V [x, y] < 1.
\]

Then from 24.2.9

\[
V [a, n\delta] \leq \sum_{i=1}^{n} V [a + (i - 1) \delta, a + i\delta] < \sum_{i=1}^{n} 1 = n.
\]

Thus \(V\) is bounded on \([a, b]\). Now let \(P_i\) be a partition of \([x_{i-1}, x_i]\) such that

\[
V_{P_i} [x_{i-1}, x_i] > V [x_{i-1}, x_i] - \frac{\varepsilon}{n}
\]

Then letting \(P = \bigcup P_i\),

\[
-\varepsilon + \sum_{i=1}^{n} V [x_{i-1}, x_i] < \sum_{i=1}^{n} V_{P_i} [x_{i-1}, x_i] = V_P [x, y] \leq V [x, y].
\]

Since \(\varepsilon\) is arbitrary, 24.2.7 follows from this and 24.2.9.

Now let \(x < y\)

\[
V (y) - f (y) - (V (x) - f (x)) = V (y) - V (x) - (f (y) - f (x)) \geq V (y) - V (x) - |f (y) - f (x)| \geq 0.
\]

It only remains to verify that \(V\) is absolutely continuous.
24.2. ABSOLUTELY CONTINUOUS FUNCTIONS

Let \(\varepsilon > 0\) be given and let \(\delta\) correspond to \(\varepsilon/2\) in the definition of absolute continuity applied to \(f\). Suppose \(\sum_{i=1}^{n} |y_i - x_i| < \delta\) and consider \(\sum_{i=1}^{n} |V(y_i) - V(x_i)|\).

By Lemma 24.2.9 this last is no larger than \(\sum_{i=1}^{n} V[x_i, y_i]\).

Now let \(P_i\) be a partition of \([x_i, y_i]\) such that \(V_{P_i}[x_i, y_i] + \varepsilon/2n > V[x_i, y_i]\). Then by the definition of absolute continuity,

\[
\sum_{i=1}^{n} |V(y_i) - V(x_i)| = \sum_{i=1}^{n} V[x_i, y_i] \\
\leq \sum_{i=1}^{n} V_{P_i}[x_i, y_i] + \eta < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

and shows \(V\) is absolutely continuous as claimed.

**Lemma 24.2.4** Suppose \(f : [a, b] \to \mathbb{R}\) is absolutely continuous and increasing. Then \(f'\) exists a.e., is in \(L^1([a, b])\), and

\[
f(x) = f(a) + \int_{a}^{x} f'(t) \, dt.
\]

**Proof:** Define \(L\), a positive linear functional on \(C([a, b])\) by

\[
Lg \equiv \int_{a}^{b} gdf
\]

where this integral is the Riemann Stieltjes integral with respect to the integrating function, \(f\). By the Riesz representation theorem for positive linear functionals, there exists a unique Radon measure, \(\mu\) such that \(Lg = \int g \, d\mu\). Now consider the following picture for \(g_n \in C([a, b])\) in which \(g_n\) equals 1 for \(x\) between \(x + 1/n\) and \(y\).

Then \(g_n(t) \to \mathcal{X}_{(x, y)}(t)\) pointwise. Therefore, by the dominated convergence theorem,

\[
\mu((x, y]) = \lim_{n \to \infty} \int g_n \, d\mu.
\]
However, 

\[
\left( f(y) - f \left( x + \frac{1}{n} \right) \right) 
\leq \int g_n d\mu = \int_a^b g_n df \leq \left( f \left( y + \frac{1}{n} \right) - f(y) \right) 
+ \left( f(y) - f \left( x + \frac{1}{n} \right) \right) + \left( f \left( x + \frac{1}{n} \right) - f(x) \right)
\]

and so as \( n \to \infty \) the continuity of \( f \) implies

\[
\mu((x, y]) = f(y) - f(x).
\]

Similarly, \( \mu(x, y) = f(y) - f(y) \) and \( \mu([x, y]) = f(y) - f(x) \), the argument used to establish this being very similar to the above. It follows in particular that

\[
f(x) - f(a) = \int_{[a, x]} d\mu.
\]

Note that up till now, no reference has been made to the absolute continuity of \( f \). Any increasing continuous function would be fine.

Now if \( E \) is a Borel set such that \( m(E) = 0 \), then the outer regularity of \( m \) implies there exists an open set, \( V \) containing \( E \) such that \( m(V) < \delta \) where \( \delta \) corresponds to \( \varepsilon \) in the definition of absolute continuity of \( f \). Then letting \( \{I_k\} \) be the connected components of \( V \) it follows \( E \subseteq \bigcup_{k=1}^\infty I_k \) with \( \sum_k m(I_k) = m(V) < \delta \).

Therefore, from absolute continuity of \( f \), it follows that for \( I_k = (a_k, b_k) \) and each \( n \)

\[
\mu(\bigcup_{k=1}^n I_k) = \sum_{k=1}^n \mu(I_k) = \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon
\]

and so letting \( n \to \infty \),

\[
\mu(E) \leq \mu(V) = \sum_{k=1}^\infty |f(b_k) - f(a_k)| \leq \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, it follows \( \mu(E) = 0 \). Therefore, \( \mu \ll m \) and so by the Radon Nikodym theorem there exists a unique \( h \in L^1([a, b]) \) such that

\[
\mu(E) = \int_E hdm.
\]

In particular,

\[
\mu([a, x]) = f(x) - f(a) = \int_{[a, x]} hdm.
\]

From the fundamental theorem of calculus \( f''(x) = h(x) \) at every Lebesgue point of \( h \). Therefore, writing in usual notation,

\[
f(x) = f(a) + \int_a^x f'(t) dt
\]
24.2. ABSOLUTELY CONTINUOUS FUNCTIONS

as claimed. This proves the lemma.

With the above lemmas, the following is the main theorem about absolutely continuous functions.

**Theorem 24.2.5** Let \( f : [a, b] \to \mathbb{R} \) be absolutely continuous if and only if \( f'(x) \) exists a.e., \( f' \in L^1([a, b]) \) and

\[
 f(x) = f(a) + \int_a^x f'(t) \, dt.
\]

**Proof:** Suppose first that \( f \) is absolutely continuous. By Lemma 24.2.3 the total variation function, \( V \) is absolutely continuous and \( f(x) = V(x) - (V(a) - f(a)) \) where both \( V \) and \( V - f \) are increasing and absolutely continuous. By Lemma 24.2.4

\[
 f(x) - f(a) = \int_a^x V'(t) \, dt - \int_a^x (V - f)'(t) \, dt.
\]

Now \( f' \) exists and is in \( L^1 \) because \( f = V - (V - f) \) and \( V \) and \( V - f \) have derivatives in \( L^1 \). Therefore, \( (V - f)' = V' - f' \) and so the above reduces to

\[
 f(x) - f(a) = \int_a^x f'(t) \, dt.
\]

This proves one half of the theorem.

Now suppose \( f' \in L^1 \) and \( f(x) = f(a) + \int_a^x f'(t) \, dt \). It is necessary to verify that \( f \) is absolutely continuous. But this follows easily from Lemma 9.5.2 on Page 243 which implies that a single function, \( f' \) is uniformly integrable. This lemma implies that if \( \sum_i |y_i - x_i| \) is sufficiently small then

\[
 \sum_i \left| \int_{x_i}^{y_i} f'(t) \, dt \right| = \sum_i |f(y_i) - f(x_i)| < \varepsilon.
\]

The following simple corollary is a case of Rademacher’s theorem.

**Corollary 24.2.6** Suppose \( f : [a, b] \to \mathbb{R} \) is Lipschitz continuous,

\[
 |f(x) - f(y)| \leq K |x - y|.
\]

Then \( f'(x) \) exists a.e. and

\[
 f(x) = f(a) + \int_a^x f'(t) \, dt.
\]

**Proof:** It is easy to see that \( f \) is absolutely continuous. Therefore, Theorem 24.2.5 applies.
24.3 Weak Derivatives

A related concept is that of weak derivatives. Let \( \Omega \subseteq \mathbb{R}^n \). A distribution on \( \Omega \) is defined to be a linear functional on \( C^\infty_c(\Omega) \), called the space of test functions. The space of all such linear functionals will be denoted by \( D^*(\Omega) \). Actually, more is sometimes done here. One imposes a topology on \( C^\infty_c(\Omega) \) making it into a topological vector space, and when this has been done, \( D'(\Omega) \) is defined as the dual space of this topological vector space. To see this, consult the book by Yosida [115] or the book by Rudin [103].

**Example:** The space \( L^1_{loc}(\Omega) \) may be considered as a subset of \( D^*(\Omega) \) as follows.

\[
\int_{\Omega} f(x) \phi(x) \, dx
\]

for all \( \phi \in C^\infty_c(\Omega) \). Recall that \( f \in L^1_{loc}(\Omega) \) if \( f \chi_K \in L^1(\Omega) \) whenever \( K \) is compact.

The following lemma is the main result which makes this identification possible.

**Lemma 24.3.1** Suppose \( f \in L^1_{loc}(\mathbb{R}^n) \) and suppose

\[
\int f \phi \, dx = 0
\]

for all \( \phi \in C^\infty_c(\mathbb{R}^n) \). Then \( f(x) = 0 \) a.e. \( x \).

**Proof:** Without loss of generality \( f \) is real-valued. Let

\[
E \equiv \{ x : f(x) > \epsilon \}
\]

and let

\[
E_m = E \cap B(0, m).
\]

We show that \( m(E_m) = 0 \). If not, there exists an open set, \( V \), and a compact set \( K \) satisfying

\[
K \subseteq E_m \subseteq V \subseteq B(0, m), \ m(V \setminus K) < 4^{-1} m(E_m),
\]

\[
\int_{V \setminus K} |f| \, dx < \epsilon 4^{-1} m(E_m).
\]

Let \( H \) and \( W \) be open sets satisfying

\[
K \subseteq H \subseteq \overline{H} \subseteq W \subseteq \overline{W} \subseteq V
\]

and let

\[
\overline{H} \prec g \prec W
\]

where the symbol, \( \prec \), has the same meaning as it does in Chapter [103]. Then let \( \phi_\delta \) be a mollifier and let \( h = g * \phi_\delta \) for \( \delta \) small enough that

\[
K \prec h \prec V.
\]
Thus
\[ 0 = \int fhdx = \int_K f dx + \int_{V \setminus K} fhdx \]
\[ \geq \epsilon m(K) - \epsilon 4^{-1} m(E_m) \]
\[ \geq \epsilon (m(E_m) - 4^{-1} m(E_m)) - \epsilon 4^{-1} m(E_m) \]
\[ \geq 2^{-1} \epsilon m(E_m). \]

Therefore, \( m(E_m) = 0 \), a contradiction. Thus
\[ m(E) \leq \sum_{m=1}^{\infty} m(E_m) = 0 \]

and so, since \( \epsilon > 0 \) is arbitrary,
\[ m(\{ x : f(x) > 0 \}) = 0. \]

Similarly \( m(\{ x : f(x) < 0 \}) = 0 \). This proves the lemma.

**Example:** \( \delta_x \in \mathcal{D}^* (\Omega) \) where \( \delta_x (\phi) \equiv \phi(x) \).

It will be observed from the above two examples and a little thought that \( \mathcal{D}^*(\Omega) \) is truly enormous. We shall define the derivative of a distribution in such a way that it agrees with the usual notion of a derivative on those distributions which are also continuously differentiable functions. With this in mind, let \( f \) be the restriction to \( \Omega \) of a smooth function defined on \( \mathbb{R}^n \). Then \( D_x f \) makes sense and for \( \phi \in C^\infty_c(\Omega) \)
\[ D_x f(\phi) = -\int \left( D_x f(\phi) \phi(x) \right) dx \]
\[ = -\int f D_x \phi \, dx = -f(D_x \phi). \]

Motivated by this, here is the definition of a weak derivative.

**Definition 24.3.2** For \( T \in \mathcal{D}^* (\Omega) \)
\[ D_x T (\phi) \equiv -T(D_x \phi). \]

Of course one can continue taking derivatives indefinitely. Thus,
\[ D_{x_1 x_2} T = D_{x_1} (D_{x_2} T) \]
and it is clear that all mixed partial derivatives are equal because this holds for the functions in \( C_c^\infty(\Omega) \). Thus one can differentiate virtually anything, even functions that may be discontinuous everywhere. However the notion of “derivative” is very weak, hence the name, “weak derivatives”.

**Example:** Let \( \Omega = \mathbb{R} \) and let
\[ H(x) \equiv \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \]

Then
\[ DH(\phi) = -\int H(x) \phi'(x) \, dx = \phi(0) = \delta_0(\phi). \]

Note that in this example, \( DH \) is not a function.

What happens when \( Df \) is a function?
Theorem 24.3.3 Let $\Omega = (a,b)$ and suppose that $f$ and $Df$ are both in $L^1(a,b)$. Then $f$ is equal to a continuous function a.e., still denoted by $f$ and
\[ f(x) = f(a) + \int_a^x Df(t) \, dt. \]

The proof of Theorem 24.3.3 depends on the following lemma.

Lemma 24.3.4 Let $T \in D^*(a,b)$ and suppose $DT = 0$. Then there exists a constant $C$ such that
\[ T(\phi) = \int_a^b C\phi \, dx. \]

**Proof:** $T(D\phi) = 0$ for all $\phi \in C_c^\infty(a,b)$ from the definition of $DT = 0$. Let
\[ \phi_0 \in C_c^\infty(a,b), \quad \int_a^b \phi_0(x) \, dx = 1, \]
and let
\[ \psi_\phi(x) = \int_a^x [\phi(t) - \left(\int_a^b \phi(y) \, dy\right) \phi_0(t)] \, dt \]
for $\phi \in C_c^\infty(a,b)$. Thus $\psi_\phi \in C_c^\infty(a,b)$ and
\[ D\psi_\phi = \phi - \left(\int_a^b \phi(y) \, dy\right) \phi_0. \]

Therefore,
\[ \phi = D\psi_\phi + \left(\int_a^b \phi(y) \, dy\right) \phi_0 \]
and so
\[ T(\phi) = T(D\psi_\phi) + \left(\int_a^b \phi(y) \, dy\right) T(\phi_0) = \int_a^b T(\phi_0) \phi(y) \, dy. \]

Let $C = T\phi_0$. This proves the lemma.

**Proof of Theorem 24.3.3** Since $f$ and $Df$ are both in $L^1(a,b)$,
\[ Df(x) - \int_a^b Df(x) \phi(x) \, dx = 0. \]
Consider
\[ f(\cdot) - \int_a^{(\cdot)} Df(t) \, dt \]
and let $\phi \in C_c^\infty(a,b)$.
\[ D\left(f(\cdot) - \int_a^{(\cdot)} Df(t) \, dt\right)(\phi) \]
\[ \equiv - \int_a^b f(x) \phi'(x) \, dx + \int_a^b \left( \int_a^x Df(t) \, dt \right) \phi'(x) \, dx \]

\[ = Df(\phi) + \int_a^b \int_a^b Df(t) \phi'(x) \, dx \, dt \]

\[ = Df(\phi) - \int_a^b Df(t) \phi(t) \, dt = 0. \]

By Lemma 24.3.2.3, there exists a constant, \( C \), such that

\[ \left( f(\cdot) - \int_a^{(\cdot)} Df(t) \, dt \right)(\phi) = \int_a^b C\phi(x) \, dx \]

for all \( \phi \in C_c^\infty(a,b) \). Thus

\[ \int_a^b \{ f(x) - \int_a^x Df(t) \, dt \} - C \phi(x) \, dx = 0 \]

for all \( \phi \in C_c^\infty(a,b) \). It follows from Lemma 24.3.1 in the next section that

\[ f(x) - \int_a^x Df(t) \, dt - C = 0 \text{ a.e. } x. \]

Thus we let \( f(a) = C \) and write

\[ f(x) = f(a) + \int_a^x Df(t) \, dt. \]

This proves Theorem 33.2.2.

Theorem 33.2.2 says that

\[ f(x) = f(a) + \int_a^x Df(t) \, dt \]

whenever it makes sense to write \( \int_a^x Df(t) \, dt \), if \( Df \) is interpreted as a weak derivative. Somehow, this is the way it ought to be. It follows from the fundamental theorem of calculus that \( f'(x) \) exists for a.e. \( x \) in the classical sense where the derivative is taken in the sense of a limit of difference quotients and \( f'(x) = Df(x) \). This raises an interesting question. Suppose \( f \) is continuous on \([a,b]\) and \( f'(x) \) exists in the classical sense for a.e. \( x \). Does it follow that

\[ f(x) = f(a) + \int_a^x f'(t) \, dt? \]

The answer is no. You can build such an example from the Cantor function which is increasing and has a derivative a.e. which equals 0 a.e. and yet climbs from 0 to 1. Thus this function is not recovered from integrating its classical derivative. Thus, in a sense weak derivatives are more agreeable than the classical ones.
24.4 Lipschitz Functions

Definition 24.4.1  A function \( f : [a, b] \to \mathbb{R} \) is Lipschitz if there is a constant \( K \) such that for all \( x, y \),

\[
|f(x) - f(y)| \leq K |x - y|.
\]

More generally, \( f \) is Lipschitz on a subset of \( \mathbb{R}^n \) if for all \( x, y \) in this set,

\[
|f(x) - f(y)| \leq K |x - y|.
\]

Lemma 24.4.2  Suppose \( f : [a, b] \to \mathbb{R} \) is Lipschitz continuous and increasing. Then \( f' \) exists a.e., is in \( L^1 ([a, b]) \), and

\[
f(x) = f(a) + \int_a^x f'(t) \, dt.
\]

If \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz, then it is in \( L^1_{loc} (\mathbb{R}) \).

Proof: The Dini derivates are defined as follows.

\[
D^+ f(x) \equiv \lim_{h \to 0+} \sup \frac{f(x+h) - f(x)}{h}, D^- f(x) \equiv \lim_{h \to 0+} \inf \frac{f(x+h) - f(x)}{h}
\]

For convenience, just let \( f \) equal \( f(a) \) for \( x < a \) and equal \( f(b) \) for \( x > b \). Let \( (a, b) \) be an open interval and let

\[
N_{ab} = \{ x \in (a, b) : D^+ f(x) > q > p > D^- f(x) \}
\]

Let \( V \subseteq (a, b) \) be an open set containing \( N_{pq} \) such that \( m(V) < m(N_{pq}) + \varepsilon \). By assumption, if \( x \in N_{pq} \), there exist arbitrarily small \( h \) such that

\[
\frac{f(x+h) - f(x)}{h} < p,
\]

These intervals \([x, x+h]\) are then a Vitali covering of \( N_{pq} \). It follows from Corollary 11.4.6 that there is a disjoint union of countably many, \( \{ [x_i, x_i + h_i] \}_{i=1}^{\infty} \) which cover all of \( N_{pq} \) except for a set of measure zero. Thus also the open intervals \( \{ (x_i, x_i + h_i) \}_{i=1}^{\infty} \) also cover all of \( N_{pq} \) except for a set of measure zero. Now for points \( x' \) of \( N_{pq} \) so covered, there are arbitrarily small \( h \) such that

\[
\frac{f(x' + h') - f(x')}{h'} > q
\]

and \([x', x' + h']\) is contained in one of these original open intervals \((x_i, x_i + h_i)\). By the Vitali covering theorem again, Corollary 11.4.6 it follows that there exists a countable disjoint sequence \( \{ [x_j', x_j' + h'_j] \} \) which covers all of \( N_{pq} \) except for a
set of measure zero, each of these \([x_j', x_j' + h_j']\) being contained in some \((x_i, x_i + h_i)\). Then it follows that
\[
qm(N_{pq}) \leq q \sum_j h_j' \leq \sum_j f(x_j' + h_j') - f(x_j') \leq \sum_i f(x_i + h_i) - f(x_i)
\]
\[
\leq p \sum_i h_i \leq pm(V) \leq p(m(N_{pq}) + \varepsilon)
\]
Since \(\varepsilon > 0\) is arbitrary, this shows that \(qm(N_{pq}) \leq pm(N_{pq})\) and so \(m(N_{pq}) = 0\). Now taking the union of all \(N_{pq}\) for \(p, q \in \mathbb{Q}\), it follows that for a.e. \(x\), \(D^+f(x) = D^-f(x) = D^-f(x)\) and so the derivative from the right exists. Similar reasoning shows that off a set of measure zero the derivative from the left also exists. You just do the same argument using \(D^-f(x)\) and \(D^-f(x)\) to obtain the existence of a derivative from the left. Next you can use the same argument to verify that \(D^-f(x) = D^-f(x)\) off a set of measure zero. This is outlined next. Define a new \(N_{pq}\),
\[
N_{pq} = \{x \in (a, b) : D_+f(x) > q > p > D^-f(x)\}
\]
Let \(V\) be an open set containing \(N_{pq}\) such that \(m(V) < m(N_{pq}) + \varepsilon\). For each \(x \in N_{pq}\) there are arbitrarily small \(h\) such that
\[
\frac{f(x) - f(x - h)}{h} < p
\]
Then as before, there is a countable disjoint sequence of closed intervals contained in \(V, \{[x_i - h_i, x_i]\}_{i=1}^\infty\) such that their union includes all of \(N_{pq}\) except a set of measure zero. Thus this is also true of the open intervals \(\{(x_i - h_i, x_i)\}_{i=1}^\infty\). Then for the points of \(N_{pq}\) covered by these open intervals \(x'\), there are arbitrarily small \(h'\) such that
\[
\frac{f(x' + h') - f(x')}{h'} > q.
\]
and each \([x', x' + h']\) is contained in an interval \((x_i - h_i, x_i)\). Then by the Vitali covering theorem again, Corollary 24.4.2 there are countably many disjoint closed intervals \(\{[x_j', x_j' + h_j']\}_{j=1}^\infty\) whose union includes all of \(N_{pq}\) except for a set of measure zero such that each of these is contained in some \((x_i - h_i, x_i)\) described earlier. Then as before,
\[
qm(N_{pq}) \leq q \sum_j h_j' \leq \sum_j f(x_j' + h_j') - f(x_j') \leq \sum_i f(x_i) - f(x_i - h_i)
\]
\[
\leq p \sum_i h_i \leq pm(V) \leq p(m(N_{pq}) + \varepsilon)
\]
Then as before, this shows that \(qm(N_{pq}) \leq pm(N_{pq})\) and so \(m(N_{pq}) = 0\). Then taking the union of all such for \(p, q \in \mathbb{Q}\) yields \(D_+f(x) = D^-f(x)\) for a.e. \(x\). Taking the union of all these sets of measure zero and considering points not in this union, it follows that \(f'(x)\) exists for a.e. \(x\). Thus \(f'(t) \geq 0\) and is a limit of
measurable even continuous functions for a.e. \( x \) so \( f' \) is clearly measurable. The issue is whether \( f(y) - f(x) = \int X_{[x,y]}(t) f'(t) \, dm \). Up to now, the only thing used has been that \( f \) is increasing.

Let \( h > 0 \).

\[
\int_a^x \frac{f(t) - f(t-h)}{h} \, dt = \frac{1}{h} \int_a^x f(t) \, dt - \frac{1}{h} \int_a^{x-h} f(t) \, dt
\]

\[
= \frac{1}{h} \int_{a-h}^x f(t) \, dt - \frac{1}{h} \int_{a-h}^a f(t) \, dt
\]

\[
= \frac{1}{h} \int_{x-h}^x f(t) \, dt - f(a)
\]

Therefore, by continuity of \( f \) it follows from Fatou’s lemma that

\[
\int_a^x D_- f(t) \, dt = \int_a^x f'(t) \, dt \leq \liminf_{h \to 0^+} \int_a^x \frac{f(t) - f(t-h)}{h} \, dt = f(x) - f(a)
\]

and this shows that \( f' \) is in \( L^1 \). This part only used the fact that \( f \) is increasing and continuous. That \( f \) is Lipschitz has not been used.

If it were known that there is a dominating function for \( t \to \frac{f(t) - f(t-h)}{h} \), then you could simply apply the dominated convergence theorem in the above inequality instead of Fatou’s lemma and get the desired result. But from Lipschitz continuity, you have

\[
\left| \frac{f(t) - f(t-h)}{h} \right| \leq K
\]

and so one can indeed apply the dominated convergence theorem and conclude that

\[
\int_a^x f'(t) \, dt = f(x) - f(a)
\]

The last claim follows right away from consideration of intervals since the restriction of a Lipschitz function is Lipschitz.

With the above lemmas, the following is the main theorem about absolutely continuous functions.

The following simple corollary is a case of Rademacher’s theorem.

**Corollary 24.4.3** Suppose \( f : [a,b] \to \mathbb{R} \) is Lipschitz continuous,

\[
\lvert f(x) - f(y) \rvert \leq K \lvert x - y \rvert.
\]

Then \( f'(x) \) exists a.e. and

\[
f(x) = f(a) + \int_a^x f'(t) \, dt.
\]
24.5. RADEMACHER’S THEOREM FIRST VERSION

**Proof:** If \( f \) were increasing, this would follow from the above lemma. Let \( g(x) = 2Kx - f(x) \). Then \( g \) is Lipschitz with a different Lipschitz constant and also if \( x < y \),

\[
g(y) - g(x) = 2Ky - f(y) - (2Kx - f(x)) \\
\geq 2K(y - x) - K|y - x| = k|y - x| \geq 0
\]

and so Lemma 24.4.3 applies to \( g \) and this shows that \( f'(t) \) exists for a.e. \( t \) and \( g'(x) = 2K - f'(x) \). Also

\[
2K(x - a) - (f(x) - f(a)) \\
= g(x) - g(a) = 2Kx - f(x) - (2Ka - f(a)) = \int_a^x (2K - f'(t)) \\
= 2K(x - a) - \int_a^x f'(t) \, dt
\]

showing that \( f(x) - f(a) = \int_a^x f'(t) \, dt. \]

**24.5 Rademacher’s Theorem First Version**

It turns out that Lipschitz functions on \( \mathbb{R}^n \) can be differentiated a.e. This is called Rademacher’s theorem. It also can be shown to follow from the Lebesgue theory of differentiation. We denote \( D_v f(x) \) the directional derivative of \( f \) in the direction \( v \). Here \( v \) is a unit vector. In the following lemma, notation is abused slightly. The symbol \( f(x + tv) \) will mean \( t \to f(x + tv) \) and \( \frac{d}{dt} f(x + tv) \) will refer to the derivative of this function of \( t \).

**Lemma 24.5.1** Let \( f: \mathbb{R}^n \to \mathbb{R} \) be a Lipschitz function with constant \( K \) and let \( v \) be a unit vector. Then the following hold.

1. \( D_v f(x) \) exists for a.e. \( x \). Also off a set of measure zero, \( |D_v f(x)| \leq K \).
2. For every \( x \),

\[
f(x+tv) - f(x) = \int_0^t D_v f(x + sv) \, ds
\]

3. \( D_v f(x) = \nabla f(x) \cdot v \).
4. For a given \( x \) and for \( \sigma \) the measure on \( S^{n-1} \) of Section 11.9 and for \( w \in S^{n-1} \),

\[
D_v f(x + tw) = \nabla f(x + tw) \cdot w
\]

for a.e. \( t \) for \( \sigma \) a.e. \( w \). Thus

\[
\int_{S^{n-1}} \int_0^t s^{n-1} D_w f(x + tw) \, dtd\sigma(w) \\
= \int_{S^{n-1}} \int_0^t s^{n-1} \nabla f(x + tw) \cdot w dt d\sigma(w)
\]
Proof: Let \( h(t) \equiv f(x + tv) \). Then from Corollary 24.4.3,
\[
f(x+tv) - f(x) = h(t) - h(0) = \int_0^t h'(s) \, ds = \int_0^t D_v f(x + sv) \, ds
\]
This verifies 2. To verify 1., let \( Q \) be an orthogonal matrix such that \( Qe_n = v \) and \( Q(\text{span} \left( e_1, \cdots, e_{n-1} \right)) = v^\perp \) where this denotes the orthogonal complement of \( v \), meaning all vectors perpendicular to \( v \). Now let
\[
g(x) \equiv f(Qx)
\]
Then from Corollary 24.4.3 for an arbitrary \( y = (y_1, \cdots, y_{n-1}, 0) \)
\[
\frac{d}{dt} g(y + te_n) \equiv D_{e_n} g(y + te_n)
\]
events for a.e. \( t \). By the definition of product measure, this shows that \( D_{e_n} g(x) \) exists for \( x \) off some set of measure zero \( N \). Now
\[
D_v f(Qy + tv) = D_v f(Q(y + te_n)) \equiv \frac{d}{dt} f(Qy + tv)
\]
\[
= \frac{d}{dt} g(y + te_n) = D_{e_n} g(y + te_n)
\]
Since every point of \( \mathbb{R}^n \) is of the form \( y + te_n \) for suitable \( t \), it follows that for \( x \not\in Q(N) \), a set of measure zero, \( D_v f(x) \) exists. The estimate for \( D_v f(x) \) wherever the limit exists is obvious from the Lipschitz condition. This proves part 1.

Finally, consider part 3. Let \( \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \).
\[
\int_{\mathbb{R}^n} f(x + tv) \frac{\phi(x)}{t} \, dx = \int_{\mathbb{R}^n} \phi(x - tv) - \phi(x) \frac{f(x)}{t} \, dx
\]
Then passing to the limit using the dominated convergence theorem,
\[
\int_{\mathbb{R}^n} D_v f(x) \phi(x) \, dx = -\int_{\mathbb{R}^n} f(x) \nabla \phi \cdot v \, dx \equiv -\int_{\mathbb{R}^n} f(x) \sum_i \partial_i \phi(x) \, v_i \, dx
\]
\[
= \lim_{h \to 0} -\int_{\mathbb{R}^n} f(x) \sum_i \frac{\phi(x + he_i) - \phi(x)}{h} \, v_i \, dx
\]
\[
= \lim_{h \to 0} \int_{\mathbb{R}^n} \sum_i \frac{f(x) - f(x - he_i)}{h} \, v_i \phi(x) \, dx
\]
Using the dominated convergence theorem and what was just shown about existence of directional derivatives a.e.,
\[
\int_{\mathbb{R}^n} D_v f(x) \phi(x) \, dx = \int_{\mathbb{R}^n} (\nabla f(x) \cdot v) \phi(x) \, dx
\]
By Lemma 24.3.1 this implies that $D_v f(x) = \nabla f(x) \cdot v$ a.e.

For $|w| = 1$, denote the measure of Section 11.9 defined on the unit sphere $S^{n-1}$ as $\sigma$. Let $N_w$ be defined as those $t \in [0, \infty)$ for which $D_w f(x + tw) \neq \nabla f(x + tw) \cdot w$.

$$B \equiv \{ w \in S^{n-1} : N_w \text{ has positive measure} \}$$

The set of points of $\mathbb{R}^n$ for which $D_w f(x) \neq \nabla f(x) \cdot w$ consists of all points $x + tw$ where $t \in N_w$ and $w \in S^{n-1}$. Thus from Section 11.9 the measure of this set is

$$\int_B \int_{N_w} \rho^{n-1} dp d\sigma(w)$$

This must equal zero from what was just shown and so $\sigma(B) = 0$. The claimed formula follows from this.

The following lemma gives an interesting inequality due to Morrey.

**Lemma 24.5.2** Let $u$ be Lipschitz continuous on $\mathbb{R}^n$. Then there exists a constant $C$, depending only on $n$ such that for any $x, y \in \mathbb{R}^n$,

$$|u(x) - u(y)| \leq C \left( \int_{B(x,2|x-y|)} |\nabla u(z)|^p dz \right)^{1/p} \left( |x - y|^{1-n/p} \right). \quad (24.5.10)$$

Here $p > n$.

**Proof:** In the argument $C$ will be a generic constant which depends on $n$. Consider the following picture.

![Diagram of two balls intersecting](image)

This is a picture of two balls of radius $r$ in $\mathbb{R}^n$, $U$ and $V$ having centers at $x$ and $y$ respectively, which intersect in the set $W$. The center of $U$ is on the boundary of $V$ and the center of $V$ is on the boundary of $U$ as shown in the picture. There exists a constant, $C$, independent of $r$ depending only on $n$ such that

$$\frac{m(W)}{m(U)} = \frac{m(W)}{m(V)} = \frac{1}{C}.$$  

You could compute this constant if you desired but it is not important here.
\[ |u(x) - u(y)| = \frac{1}{m(W)} \int_W |u(x) - u(y)| \, dz \]
\[ \leq \frac{1}{m(W)} \int_W |u(x) - u(z)| \, dz + \frac{1}{m(W)} \int_W |u(z) - u(y)| \, dz \]
\[ = \frac{C}{m(U)} \left[ \int_W |u(x) - u(z)| \, dz + \int_W |u(z) - u(y)| \, dz \right] \]
\[ \leq \frac{C}{m(U)} \left[ \int_W |u(x) - u(z)| \, dz + \int_W |u(y) - u(z)| \, dz \right] \]

Now consider these two terms. Using spherical coordinates and letting \( U_0 \) denote the ball of the same radius as \( U \) but with center at \( 0 \),
\[ \frac{1}{m(U)} \int_U |u(x) - u(z)| \, dz \]
\[ = \frac{1}{m(U_0)} \int_{U_0} |u(x) - u(z + x)| \, dz \]

Now using spherical coordinates, Section 11.9, and letting \( C \) denote a generic constant which depends on \( n \), and also using Lemma 24.5.1
\[ \leq \frac{C}{r} \int_0^r \int_{S^{n-1}} |\nabla u (x + z) \cdot w| \, d\sigma (w) \, d\rho \]
\[ = \frac{C}{r} \int_{S^{n-1}} \int_0^r |\nabla u (x + z) \cdot w| \, d\sigma (w) \, d\rho \]
\[ \leq \frac{C}{r} \int_0^r \int_{S^{n-1}} |\nabla u (x + z) \cdot w| \, d\sigma (w) \, d\rho \]
\[ = C \int_{S^{n-1}} |\nabla u (x + z)| |z|^{-\frac{n}{p'}} \, dz \]
\[ \leq C \left( \int_U |\nabla u (z)|^p \, dz \right)^{1/p} \left( \int_U |z|^{p'-np'} \right)^{1/p'} \]
\[ = C \left( \int_U |\nabla u (z)|^p \, dz \right)^{1/p} \left( \int_{S^{n-1}} \int_0^r \frac{1}{\rho^{p-1}} \, d\rho \, d\sigma \right)^{(p-1)/p} \]
\[ = C \left( \int_U |\nabla u (z)|^p \, dz \right)^{1/p} \left( \int_{S^{n-1}} \int_0^r \frac{1}{\rho^{p-1}} \, d\rho \, d\sigma \right)^{(p-1)/p} \]
Suppose \( \nabla g \) \( \text{ceptional points being in a set of measure zero. Then} \)
\[ g = C \left( \frac{p - 1}{p - n} \right)^{(p-1)/p} \left( \int_U |\nabla u(z)|^p \, dz \right)^{1/p} r^{1 - \frac{n}{p}} \]
\[ = C \left( \frac{p - 1}{p - n} \right)^{(p-1)/p} \left( \int_U |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1 - \frac{n}{p}} \]

Similarly,
\[ \frac{1}{m(V)} \int u(y) - u(z) \, dz \leq C \left( \frac{p - 1}{p - n} \right)^{(p-1)/p} \left( \int_V |\nabla u(z)|^p \, dz \right)^{1/p} \]

Therefore,
\[ |u(x) - u(y)| \leq C \left( \frac{p - 1}{p - n} \right)^{(p-1)/p} \left( \int_{B(x,2|x-y|)} |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1 - \frac{n}{p}} \]

because \( B(x, 2|x-y|) \supseteq V \cup U \). 

Here is Rademacher’s theorem.

**Theorem 24.5.3** Suppose \( u \) is Lipschitz with constant \( K \) then if \( x \) is a point where \( \nabla u(x) \) exists,
\[ |u(y) - u(x) - \nabla u(x) \cdot (y - x)| \]
\[ \leq C \left( \frac{1}{m(B(x, 2|x-y|))} \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x - y|. \] (24.5.11)

Also \( u \) is differentiable at a.e. \( x \) and also
\[ u(x+tv) - u(x) = \int_0^t Dv u(x + sv) \, ds \] (24.5.12)

**Proof:** This follows easily from letting \( g(y) \equiv u(y) - u(x) - \nabla u(x) \cdot (y - x) \). As explained above, \( |\nabla u(x)| \leq \sqrt{n}K \) at every point where \( \nabla u \) exists, the exceptional points being in a set of measure zero. Then \( g(x) = 0 \), and \( \nabla g(y) = \nabla u(y) - \nabla u(x) \) at the points \( y \) where the gradient of \( g \) exists. From Lemma 24.4.
Now this is no larger than

\[ \leq C \left( \frac{1}{m(B(x, 2|x-y|))} \int_{B(x, 2|x-y|)} |\nabla u(z) - \nabla u(x)| \left(2\sqrt{nK}\right)^{p-1} \, dz \right)^{1/p} |x-y| \]

It follows that at Lebesgue points of \( \nabla u \), the above expression is \( o(|x-y|) \) and so at all such points \( u \) is differentiable. As to \( 24.5.12 \), this follows from Lemma \( 24.5.2 \).

In the above major theorem, the function \( u \) is defined on all of \( \mathbb{R}^n \). However, it is always the case that Lipschitz functions can be extended off a given set. Thus if a Lipschitz function is defined on some set \( \Omega \), then it can always be considered the restriction to \( \Omega \) of a Lipschitz map defined on all of \( \mathbb{R}^n \).

**Theorem 24.5.4** If \( h: \Omega \to \mathbb{R}^m \) is Lipschitz, then there exists \( \overline{h}: \mathbb{R}^n \to \mathbb{R}^m \) which extends \( h \) and is also Lipschitz.

**Proof:** It suffices to assume \( m = 1 \) because if this is shown, it may be applied to the components of \( h \) to get the desired result. Suppose

\[ |h(x) - h(y)| \leq K|x-y|. \quad (24.5.13) \]

Define

\[ \overline{h}(x) \equiv \inf \{ h(w) + K|x-w| : w \in \Omega \}. \quad (24.5.14) \]

If \( x \in \Omega \), then for all \( w \in \Omega \),

\[ h(w) + K|x-w| \geq h(x) \]

by \( 24.5.14 \). This shows \( h(x) \leq \overline{h}(x) \). But also you could take \( w = x \) in \( 24.5.14 \) which yields \( \overline{h}(x) \leq h(x) \). Therefore \( \overline{h}(x) = h(x) \) if \( x \in \Omega \).

Now suppose \( x, y \in \mathbb{R}^n \) and consider \( |\overline{h}(x) - \overline{h}(y)| \). Without loss of generality assume \( \overline{h}(x) \geq \overline{h}(y) \). (If not, repeat the following argument with \( x \) and \( y \) interchanged.) Pick \( w \in \Omega \) such that

\[ h(w) + K|y-w| - \varepsilon < \overline{h}(y). \]

Then

\[ |\overline{h}(x) - \overline{h}(y)| = \overline{h}(x) - \overline{h}(y) \leq h(w) + K|x-w| - \varepsilon < \overline{h}(y) + K|y-w| + \varepsilon. \]

Since \( \varepsilon \) is arbitrary,

\[ |\overline{h}(x) - \overline{h}(y)| \leq K|x-y| \]

**24.6 Rademacher’s Theorem**

It turns out that Lipschitz functions on \( \mathbb{R}^n \) can be differentiated a.e. This is called Rademacher’s theorem. It also can be shown to follow from the Lebesgue theory of differentiation.
24.6. RADEMACHER’S THEOREM

24.6.1 Morrey’s Inequality

The following inequality will be called Morrey’s inequality. It relates an expression which is given pointwise to an integral of the \( p \)th power of the derivative.

Lemma 24.6.1 Let \( u \in C^1(\mathbb{R}^n) \) and \( p > n \). Then there exists a constant, \( C \), depending only on \( n \) such that for any \( x, y \in \mathbb{R}^n \),

\[
|u(x) - u(y)| \leq C \left( \int_{B(x, 2|x-y|)} |\nabla u(z)|^p dz \right)^{1/p} \left( |x - y|^{(1-n/p)} \right).
\]

(24.6.15)

Proof: In the argument \( C \) will be a generic constant which depends on \( n \).

Consider the following picture.

![Picture of two balls](image)

This is a picture of two balls of radius \( r \) in \( \mathbb{R}^n \), \( U \) and \( V \) having centers at \( x \) and \( y \) respectively, which intersect in the set, \( W \). The center of \( U \) is on the boundary of \( V \) and the center of \( V \) is on the boundary of \( U \) as shown in the picture. There exists a constant, \( C \), independent of \( r \) depending only on \( n \) such that

\[
\frac{m(W)}{m(U)} = \frac{m(W)}{m(V)} = C.
\]

You could compute this constant if you desired but it is not important here.

Define the average of a function over a set, \( E \subseteq \mathbb{R}^n \) as follows.

\[
\int_E f dx \equiv \frac{1}{m(E)} \int_E f dx.
\]

Then

\[
|u(x) - u(y)| = \int_W |u(x) - u(y)| dz
\]

\[
\leq \int_W |u(x) - u(z)| dz + \int_W |u(z) - u(y)| dz
\]

\[
= \frac{C}{m(U)} \left[ \int_W |u(x) - u(z)| dz + \int_W |u(z) - u(y)| dz \right]
\]

\[
\leq C \left[ \int_U |u(x) - u(z)| dz + \int_V |u(y) - u(z)| dz \right]
\]
Now consider these two terms. Using spherical coordinates and letting $U_0$ denote the ball of the same radius as $U$ but with center at $0$,

\[
\begin{align*}
&\int_U |u(x) - u(z)| \, dz \\
&= \frac{1}{m(U_0)} \int_{U_0} |u(x) - u(z)| \, dz \\
&= \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} |u(x) - u(\rho w + x)| \, d\sigma \, d\rho \\
&\leq \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} \int_0^{\rho} |\nabla u(x + tw)| \, dt \, d\sigma \, d\rho \\
&\leq C \frac{1}{r} \int_0^r \int_{S^{n-1}} \int_0^{\rho} |\nabla u(x + tw)| \, dt \, d\sigma \, d\rho \\
&= C \frac{1}{r} \int_0^r \int_{S^{n-1}} \int_0^{\rho} \frac{|\nabla u(x + tw)|}{t^{n-1}} \, t^{n-1} \, dt \, d\sigma \, d\rho \\
&= C \int_{S^{n-1}} \int_0^r \frac{|\nabla u(x + tw)|}{t^{n-1}} \, t^{n-1} \, dt \, d\sigma \\
&= C \int_{U_0} \frac{|\nabla u(x + z)|}{|z|^{n-1}} \, dz \\
&\leq C \left( \int_{U_0} |\nabla u(x + z)|^p \, dz \right)^{1/p} \left( \int_U |z|^{p' - np'} \right)^{1/p'} \\
&= C \left( \int_U |\nabla u(z)|^p \, dz \right)^{1/p} \left( \int_{S^{n-1}} \int_0^r \frac{1}{\rho^{n-1}} \, d\rho \, d\sigma \right)^{(p-1)/p} \\
&= C \left( \int_U |\nabla u(z)|^p \, dz \right)^{1/p} \left( \int_{S^{n-1}} \int_0^{\rho} \frac{1}{\rho^{n-1}} \, d\rho \, d\sigma \right)^{(p-1)/p} \\
&= C \left( \int_{U_0} |\nabla u(z)|^p \, dz \right)^{1/p} \left( \int_U |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1 - \frac{n}{p}} \\
&= C \left( \int_{U_0} |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1 - \frac{n}{p}}
\end{align*}
\]

Similarly,

\[
\begin{align*}
\int_V |u(y) - u(z)| \, dz &\leq C \left( \int_{U_0} |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1 - \frac{n}{p}}
\end{align*}
\]
Therefore,

\[ |u(x) - u(y)| \leq C \left( \frac{p-1}{p-n} \right) \left( \int_{B(x,2|x-y|)} |\nabla u(z)|^p \, dz \right)^{1/p} |x-y|^{1-\frac{n}{p}} \]

because \( B(x,2|x-y|) \supseteq V \cup U \). This proves the lemma.

The following corollary is also interesting

**Corollary 24.6.2** Suppose \( u \in C^1(\mathbb{R}^n) \). Then

\[ |u(y) - u(x) - \nabla u(x) \cdot (y-x)| \]

\[ \leq C \left( \frac{1}{m(B(x,2|x-y|))} \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x-y| \]

(24.6.16)

**Proof:** This follows easily from letting \( g(y) \equiv u(y) - u(x) - \nabla u(x) \cdot (y-x) \). Then \( g \in C^1(\mathbb{R}^n) \), \( g(x) = 0 \), and \( \nabla g(z) = \nabla u(z) - \nabla u(x) \). From Lemma 24.6.1,

\[ |u(y) - u(x) - \nabla u(x) \cdot (y-x)| \]

\[ = |g(y)| = |g(y) - g(x)| \]

\[ \leq C \left( \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x-y|^{1-\frac{n}{p}} \]

\[ = C \left( \frac{1}{m(B(x,2|x-y|))} \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x-y| \]

This proves the corollary.

It may be interesting at this point to recall the definition of differentiability on Page 122. If you knew the above inequality held for \( \nabla u \) having components in \( L^1_{loc}(\mathbb{R}^n) \), then at Lebesgue points of \( \nabla u \), the above would imply \( Du(x) \) exists.

### 24.6.2 Rademacher’s Theorem

**Lemma 24.6.3** Let \( u \) be a Lipschitz continuous function which vanishes outside some compact set. Then there exists a unique \( u_i \in L^\infty(\mathbb{R}^n) \) such that

\[ \lim_{h \to 0} \frac{u(\cdot + h) - u(\cdot)}{h} = u_i \quad \text{weak* in } L^\infty(\mathbb{R}^n). \]

**Proof:** By the Lipschitz condition, the above difference quotient is bounded in \( L^\infty \) by \( K \) the Lipschitz constant of \( u \). It follows from the Banach Aloglu theorem and Corollary 24.5.6 on Page 148 that there exists a subsequence \( h_k \to 0 \) and \( g \in L^\infty(\mathbb{R}^n) \) such that

\[ \frac{u(\cdot + h_k) - u(\cdot)}{h_k} \to g \quad \text{weak* in } L^\infty(\mathbb{R}^n) \]
Letting $\phi \in C^\infty_c (\mathbb{R}^n)$, it follows
\[
\int g \phi \, dx = \lim_{k \to \infty} \int \frac{u(\cdot + h_k) - u(\cdot)}{h_k} \phi \, dx = - \int u \phi_{,i} \, dx
\]
This also shows that $g$ must vanish outside some compact set because the integral on the right shows that if $\text{spt} \phi$ does not intersect $\text{spt} u$, then $\int g \phi \, dx = 0$. Thus $g \in L^2 (\mathbb{R}^n)$. If $g_1$ is a weak $*$ limit of another subsequence $h_j \to 0$, the same result follows. Thus for any $\phi \in C^\infty_c (\mathbb{R}^n)$
\[
\int (g - g_1) \phi \, dx = 0
\]
and since $C^\infty_c (\mathbb{R}^n)$ is dense in $L^2 (\mathbb{R}^n)$, this requires $g = g_1$ in $L^2$ and so they are equal a.e. Since every sequence of $h \to 0$ has a subsequence which when applied to the difference quotient, always converges to the same thing, it follows the claimed limit exists. This is called $u_{,i}$. This proves the lemma.

**Lemma 24.6.4** Let $u$ be a Lipschitz continuous function which vanishes outside a compact set and let $u_{,i}$ be described above. For $\phi_\varepsilon$ a mollifier and $u_\varepsilon \equiv u * \phi_\varepsilon$,
\[
u_{\varepsilon,i} = u_{,i} * \phi_\varepsilon
\]
where the symbol $u_{,i}$ means the usual partial derivative with respect to the $i^{th}$ variable. Also for any $p > n$,
\[
u_{\varepsilon,i} \to u_{,i} \text{ in } L^p (\mathbb{R}^n).
\]
**Proof:** This follows from a computation and Lemma 24.6.3
\[
u_{\varepsilon,i} (x) \equiv \lim_{h \to 0} \int \frac{u(x - y + he_i) - u(x - y)}{h} \phi_\varepsilon (y) \, dy
\]
\[
= \lim_{h \to 0} \int \frac{u(z + he_i) - u(z)}{h} \phi_\varepsilon (x - z) \, dz
\]
\[
= \int u_{,i} (z) \phi_\varepsilon (x - z) \, dz = u_{,i} * \phi_\varepsilon (x)
\]
It remains to verify the last assertion. Note that $u_{,i} \in L^p (\mathbb{R}^n)$ for any $p > 1$ because it is bounded and vanishes outside some compact set. By the first part,
\[
\left( \int |u_{\varepsilon,i} - u_{,i}|^p \, dx \right)^{1/p} \leq \left( \int \left( \int (u_{,i} (x - y) - u_{,i} (x)) \phi_\varepsilon (y) \, dy \right)^p \, dx \right)^{1/p}
\]
and by Minkowski’s inequality,
\[
\leq \int \phi_\varepsilon (y) \left( \int |(u_{,i} (x - y) - u_{,i} (x))^p \, dx \right)^{1/p} \, dy
\]
\[
= \int_{B(0,\varepsilon)} \phi_\varepsilon (y) \left\| (u_{,i})_y - u_{,i} \right\|_{L^p(\mathbb{R}^n)} \, dy
\]
which converges to 0 from continuity of translation. This proves the lemma.

Now from Corollary 24.6.2 applied to $u_\varepsilon$ just described and letting $y - x = v$

$$|u_\varepsilon (x + v) - u_\varepsilon (x) - \nabla u_\varepsilon (x) \cdot v|$$

$$\leq C \left( \frac{1}{m(\Omega)} \int_{\Omega} \left| \nabla u_\varepsilon (z) - \nabla u_\varepsilon (x) \right|^p dz \right)^{1/p} |v|.$$  

From Lemma 24.6.3, there is a subsequence, still denoted as $\varepsilon$ such that for each $i$, $u_{\varepsilon,i} \to u_i$ pointwise a.e. and in $L^p(\mathbb{R}^n)$ where $p > n$ is given. Denote as $\nabla u$ the vector $(u_1, u_2, \cdots, u_n)^T$. Then passing to the limit as $\varepsilon \to 0$, for a.e. $x$,

$$|u (x + v) - u (x) - \nabla u (x) \cdot v|$$

$$\leq C \left( \frac{1}{m(\Omega)} \int_{\Omega} \left| \nabla u (z) - \nabla u (x) \right|^p dz \right)^{1/p} |v|.$$  

At every Lebesgue point $x$ of $\nabla u$, the above shows $u (x + v) - u (x) - \nabla u (x) \cdot v = o (v)$. Thus this has proved the following.

**Lemma 24.6.5** Let $u$ be Lipschitz continuous and vanish outside some bounded set. Then $Du(x)$ exists for a.e. $x$.

This is a good result but it is easy to give an even easier to use result. First here is a theorem which says you can extend a Lipschitz map.

**Theorem 24.6.6** If $h : \Omega \to \mathbb{R}^m$ is Lipschitz, then there exists $\overline{h} : \mathbb{R}^n \to \mathbb{R}^m$ which extends $h$ and is also Lipschitz.

**Proof:** It suffices to assume $m = 1$ because if this is shown, it may be applied to the components of $h$ to get the desired result. Suppose

$$|h (x) - h (y)| \leq K |x - y|.$$  

(24.6.17)

Define

$$\overline{h} (x) \equiv \inf \{ h (w) + K |x - w| : w \in \Omega \}.$$  

(24.6.18)

If $x \in \Omega$, then for all $w \in \Omega$,

$$h (w) + K |x - w| \geq h (x)$$

by (24.6.17). This shows $h (x) \leq \overline{h} (x)$. But also you could take $w = x$ in (24.6.18) which yields $\overline{h} (x) \leq h (x)$. Therefore $\overline{h} (x) = h (x)$ if $x \in \Omega$.

Now suppose $x, y \in \mathbb{R}^n$ and consider $|\overline{h} (x) - \overline{h} (y)|$. Without loss of generality assume $\overline{h} (x) \geq \overline{h} (y)$. (If not, repeat the following argument with $x$ and $y$ interchanged.) Pick $w \in \Omega$ such that

$$h (w) + K |y - w| - \varepsilon < \overline{h} (y).$$
Then
\[
|\bar{f}(x) - \bar{f}(y)| = |f(x) - f(y)| \leq h(w) + K|x - w| -
|h(w) + K|y - w| - \varepsilon| \leq K|x - y| + \varepsilon.
\]
Since \(\varepsilon\) is arbitrary,
\[
|\bar{f}(x) - \bar{f}(y)| \leq K|x - y|
\]
and this proves the theorem.

With this theorem, here is the main result called Rademacher’s theorem.

**Theorem 24.6.7** Let \(h : \Omega \to \mathbb{R}^m\) be Lipschitz on \(\Omega\) where \(\Omega\) is some nonempty measurable set in \(\mathbb{R}^n\). Then \(Dh(x)\) exists for a.e. \(x \in \Omega\). If \(\Omega = \mathbb{R}^n\), then for each \(e_i\),
\[
\lim_{h \to 0} \frac{h(\cdot + he_i) - h(\cdot)}{h} = h_i \text{ weak * in } L^\infty(\mathbb{R}^n)
\]
and whenever \(\phi_\varepsilon\) is a mollifier,
\[
(h * \phi_\varepsilon)_i \to h_i \text{ in } L^p(\mathbb{R}^n; \mathbb{R}^m).
\]

**Proof:** The last two claims follow from the above argument applied to the components of \(h\). By Theorem 24.6.7 the function can be extended to a Lipschitz function defined on all of \(\mathbb{R}^n\), still denoted as \(h\). Let \(\Omega_r \equiv \Omega \cap B(0, r)\). Now let \(\psi \in C^\infty_c(B(0, 2r))\) such that \(\psi = 1\) on \(B(0, \frac{3}{2}r)\). Then \(\psi h\) is Lipschitz on \(\mathbb{R}^n\) and vanishes off a bounded set. It follows from Lemma 24.6.8 applied to the components of \(h\) that this function has a derivative off a set of measure zero \(N_r\). If \(x \in \Omega_r \setminus N_r\) it follows since \(\psi = 1\) near \(x\) that \(Dh(x)\) exists. Letting \(N = \bigcup_{r=1}^\infty N_r\), it follows that if \(x \in \Omega \setminus N\), then \(Dh(x)\) exists. This proves the theorem.

For \(u\) Lipschitz as described above, the limit of the difference quotient \(u_{ij}\) is called the weak partial derivative of \(u\). For \(p > n\) and an assertion that the difference quotients are bounded in \(L^p\) everything done above would work out the same way and one can therefore generalize parts of the above theorem. The extension is problematic but one can give the following results with essentially the same proof as the above.

**Lemma 24.6.8** Let \(u \in L^p(\mathbb{R}^n)\). There exists \(u_{ij} \in L^p(\mathbb{R}^n)\) such that
\[
\lim_{h \to 0} \frac{u(\cdot + he_i) - u(\cdot)}{h} = u_{ij} \text{ weakly in } L^p(\mathbb{R}^n)
\]
if and only if the difference quotients \(u(\cdot + he_i) - u(\cdot)/h\) are bounded in \(L^p(\mathbb{R}^n)\) for all nonzero \(h\).

**Proof:** If the weak limit exists, then the difference quotients must be bounded. This follows from the uniform boundedness theorem, Theorem 24.6.8. Here is why. Denote the difference quotient by \(D_h\) to save space. Weak convergence requires \(\int D_h f \to \int u_{ij} f\) for all \(f \in L^p\). Could there exist \(h_k\) such that \(\|D_h\|_{L^p} \to \infty\)? Not unless a subsequence satisfies \(h_k \to 0\) because if this sequence is bounded away from
0, the formula for $D_h$ will yield the difference quotients are bounded. However, if $h_k \to 0$, then for each $f \in L^p$, 
\[
\sup_k \int D_{h_k} f < \infty
\]
because in fact, $\lim_{k \to \infty} \int D_{h_k} f$ exists so it must be bounded. Now $D_{h_k}$ can be considered in $(L^p)'$ and this shows it is pointwise bounded on $L^p$. Therefore, $D_{h_k}$ is bounded in $(L^p)'$ but the norm on this is the same as the norm in $L^p$. Thus $D_{h_k}$ is bounded after all.

Conversely, if the difference quotients are bounded, the same argument used earlier, involving convergence of a subsequence, this time coming from the Eberlein Smulian theorem, Theorem 15.5.12 and showing that every subsequence converges to the same thing, shows the difference quotients converge weakly in $L^p(\mathbb{R}^n)$ to something we can call $u_i$. This proves the lemma.

**Definition 24.6.9** A function $f \in L^p(\mathbb{R}^n)$ is said to have weak partial derivatives in $L^p(\mathbb{R}^n)$ if the difference quotients $u_i(\cdot+he_i)-u_i(\cdot)/h$ for each $i = 1, 2, \cdots, n$ are bounded for $h \neq 0$. If $f \in L^p(\mathbb{R}^n;\mathbb{R}^m)$, it has weak partial derivatives in $L^p(\mathbb{R}^n;\mathbb{R}^m)$ if each component function has weak partial derivatives in $L^p(\mathbb{R}^n)$.

This following theorem may also be referred to as Rademacher’s theorem.

**Theorem 24.6.10** Let $h$ be in $L^p(\mathbb{R}^n;\mathbb{R}^m)$, $p > n$, and suppose it has weak derivatives $h_i \in L^p(\mathbb{R}^n;\mathbb{R}^m)$ for $i = 1, \cdots, n$. Then $Dh(x)$ exists a.e. and $h$ is almost everywhere equal to a continuous function. Also if $\phi_\varepsilon$ is a mollifier, 
\[
\langle h*\phi_\varepsilon \rangle_i = h_i * \phi_\varepsilon, \quad \langle h*\phi_\varepsilon \rangle_i \to h_i
\]
in $L^p(\mathbb{R}^n;\mathbb{R}^m)$.

**Proof:** As before,
\[
\langle h*\phi_\varepsilon \rangle_i(x) \equiv \lim_{h \to 0} \int h(x + he_i - y) - h(x - y) \phi_\varepsilon(y) dy
\]
\[
= \lim_{h \to 0} \int h(z + he_i) - h(z) \phi_\varepsilon(x - z) dy \equiv \int h_i(z) \phi_\varepsilon(x - y) dy
\]
\[
= h_i * \phi_\varepsilon(x)
\]
and now $\langle h*\phi_\varepsilon \rangle_i \to h_i$ follows as before from a use of Minkowski’s inequality. Letting $u$ be one of the component functions of $h$, Morrey’s inequality holds for $u_\varepsilon \equiv u * \phi_\varepsilon$. Thus
\[
|u_\varepsilon(x) - u_\varepsilon(y)| \leq C \left( \int_{B(x,2|x-y|)} |\nabla u_\varepsilon(z)|^p dz \right)^{1/p} \left( |x-y|^{1-n/p} \right)
\]
Now there exists a subsequence such that \( u_\varepsilon \to u \) pointwise a.e. and also each \( u_\varepsilon, i \to u, i \) pointwise a.e. as well as in \( L^p \). Therefore, for \( x, y \) not in a set of measure zero,

\[
|u(x) - u(y)| \leq C \left( \int_{B(x, 2|x-y|)} |\nabla u(z)|^p \, dz \right)^{1/p} \left( |x - y|^{1-n/p} \right)
\]

which shows the claim about \( u \) being equal to a continuous function off a set of measure zero. Thus \( h \) is also continuous off a set of measure zero.

As before, letting \( g(y) = u_\varepsilon(y) - u_\varepsilon(x) - \nabla u_\varepsilon(x) \cdot (y - x) \) and writing Morrey’s inequality,

\[
|u_\varepsilon(y) - u_\varepsilon(x) - \nabla u_\varepsilon(x) \cdot (y - x)| \leq C \left( \int_{B(x, 2|y-x|)} |\nabla u_\varepsilon(z) - \nabla u_\varepsilon(x)|^p \, dz \right)^{1/p} \left( |y - x|^{1-n/p} \right)
\]

Then taking a suitable subsequence and passing to the limit while also letting \( v = y - x \), it follows

\[
|u(x + v) - u(x) - \nabla u(x) \cdot v| \leq C \left( \int_{B(x, 2|v|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |v|^{1-n/p}
\]

\[
= C \left( \frac{1}{|v|} \int_{B(x, 2|v|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |v|^{1-n/p}
\]

\[
= C' \left( \frac{1}{|v|^n} \int_{B(x, 2|v|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |v|
\]

for all \( x, x + v \notin N \), a set of measure zero. Defining \( u, \nabla u \) at the points of \( N \) so that the inequality continues to hold, \( Du(x) \) exists at every Lebesgue point of \( \nabla u \). Also \( Dh \) exists a.e. because this is true of the component functions. This proves the theorem.

### 24.7 Differentiation Of Measures With Respect To Lebesgue Measure

Recall the Vitali covering theorem in Corollary (11.4.5) on Page 335.

**Corollary 24.7.1** Let \( E \subset \mathbb{R}^n \) and let \( \mathcal{F} \), be a collection of open balls of bounded radii such that \( \mathcal{F} \) covers \( E \) in the sense of Vitali. Then there exists a countable collection of disjoint balls from \( \mathcal{F} \), \( \{B_j\}_{j=1}^\infty \), such that \( m(E \setminus \bigcup_{j=1}^\infty B_j) = 0 \).
Definition 24.7.2 Let $\mu$ be a Radon measure defined on $\mathbb{R}^n$. Then

$$ \frac{d\mu}{dm}(x) \equiv \lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} $$

whenever this limit exists.

It turns out this limit exists for $m$ a.e. $x$. To verify this here is another definition.

Definition 24.7.3 Let $f(r)$ be a function having values in $[-\infty, \infty]$. Then

$$ \limsup_{r \to 0^+} f(r) \equiv \lim_{r \to 0^+} \sup \{ f(t) : t \in [0,r] \} $$

$$ \liminf_{r \to 0^+} f(r) \equiv \lim_{r \to 0^+} \inf \{ f(t) : t \in [0,r] \} $$

This is well defined because the function $r \to \inf \{ f(t) : t \in [0,r] \}$ is increasing and $r \to \sup \{ f(t) : t \in [0,r] \}$ is decreasing. Also note that $\lim_{r \to 0^+} f(r)$ exists if and only if

$$ \limsup_{r \to 0^+} f(r) = \liminf_{r \to 0^+} f(r) $$

and if this happens

$$ \lim_{r \to 0^+} f(r) = \liminf_{r \to 0^+} f(r) = \limsup_{r \to 0^+} f(r). $$

The claims made in the above definition follow immediately from the definition of what is meant by a limit in $[-\infty, \infty]$ and are left for the reader.

Theorem 24.7.4 Let $\mu$ be a Borel measure on $\mathbb{R}^n$ then $\frac{d\mu}{dm}(x)$ exists in $[-\infty, \infty]$ $m$ a.e.

Proof: Let $p < q$ and let $p,q$ be rational numbers. Define

$$ N_{pq}(M) \equiv \left\{ x \in \mathbb{R}^n \text{ such that } \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} > q \right\} $$

$$ N_{pq}(\infty) \equiv \left\{ x \in \mathbb{R}^n \text{ such that } \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} > q \right\} $$

$$ N \equiv \left\{ x \in \mathbb{R}^n \text{ such that } \liminf_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} > \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} \right\}. $$
I will show \( m(\mathbb{N}_{pq}(M)) = 0 \). Use outer regularity to obtain an open set, \( V \) containing \( \mathbb{N}_{pq}(M) \) such that
\[
m(\mathbb{N}_{pq}(M)) + \varepsilon > m(V).
\]

From the definition of \( \mathbb{N}_{pq}(M) \), it follows that for each \( x \in \mathbb{N}_{pq}(M) \) there exist arbitrarily small \( r > 0 \) such that
\[
m(B(x, r)) < p m(B(x, r)).
\]
Only consider those \( r \) which are small enough to be contained in \( B(0, M) \) so that the collection of such balls has bounded radii. This is a Vitali cover of \( \mathbb{N}_{pq}(M) \) and so by Corollary 24.7.1 there exists a sequence of disjoint balls of this sort, \( \{B_i\}_{i=1}^{\infty} \) such that
\[
m(B_i) < p m(B_i), m(\mathbb{N}_{pq}(M) \cup \bigcup_{i=1}^{\infty} B_i) = 0. \quad (24.7.19)
\]
Now for \( x \in \mathbb{N}_{pq}(M) \cap (\bigcup_{i=1}^{\infty} B_i) \) (most of \( \mathbb{N}_{pq}(M) \)), there exist arbitrarily small balls, \( B(x, r) \), such that \( B(x, r) \) is contained in some set of \( \{B_i\}_{i=1}^{\infty} \) and
\[
m(B(x, r)) > q m(B(x, r)).
\]
This is a Vitali cover of \( \mathbb{N}_{pq}(M) \cap (\bigcup_{i=1}^{\infty} B_i) \) and so there exists a sequence of disjoint balls of this sort, \( \{B'_j\}_{j=1}^{\infty} \) such that
\[
m((\mathbb{N}_{pq}(M) \cap (\bigcup_{i=1}^{\infty} B_i)) \setminus \bigcup_{i=1}^{\infty} B'_i) = 0, m(B'_j) > q m(B'_j). \quad (24.7.20)
\]
It follows from 24.7.19 and 24.7.20 that
\[
m(\mathbb{N}_{pq}(M)) \leq m((\mathbb{N}_{pq}(M) \cap (\bigcup_{i=1}^{\infty} B_i)) \leq m(\bigcup_{i=1}^{\infty} B'_i) \quad (24.7.21)
\]
Therefore,
\[
\sum_j m(B'_j) \geq \sum_j m(B'_j) \geq m(\mathbb{N}_{pq}(M)) \quad (24.7.22)
\]
Since \( \varepsilon \) is arbitrary, \( m(\mathbb{N}_{pq}(M)) = 0 \). Now \( \mathbb{N}_{pq} \subseteq \bigcup_{M=1}^{\infty} \mathbb{N}_{pq}(M) \) and so \( m(\mathbb{N}_{pq}) = 0 \). Now
\[
N = \bigcup_{p, q \in \mathbb{Q}} \mathbb{N}_{pq}
\]
and since this is a countable union of sets of measure zero, \( m(N) = 0 \) also. This proves the theorem.

From Theorem 18.2.5 on Page 588 it follows that if \( \mu \) is a complex measure then \( |\mu| \) is a finite measure. This makes possible the following definition.
Definition 24.7.5 Let $\mu$ be a real measure. Define the following measures. For $E$ a measurable set,
\[
\mu^+(E) \equiv \frac{1}{2} (|\mu| + \mu)(E),
\]
\[
\mu^-(E) \equiv \frac{1}{2} (|\mu| - \mu)(E).
\]
These are measures thanks to Theorem 18.2.3 on Page 586 and $\mu^+ - \mu^- = \mu$. These measures have values in $[0, \infty)$. They are called the positive and negative parts of $\mu$ respectively. For $\mu$ a complex measure, define $\text{Re} \mu$ and $\text{Im} \mu$ by
\[
\text{Re} \mu(E) \equiv \frac{1}{2} (\mu(E) + \overline{\mu(E)})
\]
\[
\text{Im} \mu(E) \equiv \frac{1}{2i} (\mu(E) - \overline{\mu(E)})
\]
Then $\text{Re} \mu$ and $\text{Im} \mu$ are both real measures. Thus for $\mu$ a complex measure,
\[
\mu = \text{Re} \mu^+ - \text{Re} \mu^- + i (\text{Im} \mu^+ - \text{Im} \mu^-)
\]
where each $\nu_i$ is a real measure having values in $[0, \infty)$.

Then there is an obvious corollary to Theorem 24.7.4.

Corollary 24.7.6 Let $\mu$ be a complex Borel measure on $\mathbb{R}^n$. Then $\frac{d\mu}{dm}(x)$ exists a.e.

Proof: Letting $\nu_i$ be defined in Definition 24.7.5. By Theorem 24.7.4, for $m$ a.e. $x$, $\frac{d\nu_i}{dm}(x)$ exists. This proves the corollary because $\mu$ is just a finite sum of these $\nu_i$.

Theorem 18.1.2 on Page 579, the Radon Nikodym theorem, implies that if you have two finite measures, $\mu$ and $\lambda$, you can write $\lambda$ as the sum of a measure absolutely continuous with respect to $\mu$ and one which is singular to $\mu$ in a unique way. The next topic is related to this. It has to do with the differentiation of a measure which is singular with respect to Lebesgue measure.

Theorem 24.7.7 Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and suppose there exists a $\mu$ measurable set, $N$ such that for all Borel sets, $E$, $\mu(E) = \mu(E \cap N)$ where $m(N) = 0$. Then
\[
\frac{d\mu}{dm}(x) = 0 \text{ m. a.e.}
\]

Proof: For $k \in \mathbb{N}$, let
\[
B_k(M) \equiv \left\{ x \in N^\mathbb{C} : \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} > \frac{1}{k} \right\} \cap B(0,M),
\]
\[
B_k \equiv \left\{ x \in N^\mathbb{C} : \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} > \frac{1}{k} \right\},
\]
\[
B \equiv \left\{ x \in N^\mathbb{C} : \limsup_{r \to 0^+} \frac{\mu(B(x,r))}{m(B(x,r))} > 0 \right\}.
\]
Let $\varepsilon > 0$. Since $\mu$ is regular, there exists $H$, a compact set such that $H \subseteq N \cap B(0, M)$ and 

$$
\mu(N \cap B(0, M) \setminus H) < \varepsilon.
$$

For each $x \in B_k(M)$, there exist arbitrarily small $r > 0$ such that $B(x, r) \subseteq B(0, M) \setminus H$ and 

$$
\frac{\mu(B(x, r))}{m(B(x, r))} > \frac{1}{k}.
$$

Two such balls are illustrated in the above picture. This is a Vitali cover of $B_k(M)$ and so there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^\infty$ such that 

$$
\overline{m}(B_k(M) \cup \bigcup_i B_i) = 0.
$$

Therefore, 

$$
\overline{m}(B_k(M)) \leq \overline{m}(B_k(M) \cap (\bigcup_i B_i)) \leq \sum_i \overline{m}(B_i) \leq k \sum_i \mu(B_i)
$$

$$
= k \sum_i \mu(B_i \cap N) = k \sum_i \mu(B_i \cap N \cap B(0, M))
$$

$$
\leq k \mu(N \cap B(0, M) \setminus H) < \varepsilon k
$$

Since $\varepsilon$ was arbitrary, this shows $\overline{m}(B_k(M)) = 0$.

Therefore, 

$$
\overline{m}(B_k) \leq \sum_{M=1}^\infty \overline{m}(B_k(M)) = 0
$$

and $\overline{m}(B) \leq \sum_k \overline{m}(B_k) = 0$. Since $\overline{m}(N) = 0$, this proves the theorem.

It is easy to obtain a different version of the above theorem. This is done with the aid of the following lemma.

**Lemma 24.7.8** Suppose $\mu$ is a Borel measure on $\mathbb{R}^n$ having values in $[0, \infty)$. Then there exists a Radon measure, $\mu_1$ such that $\mu_1 = \mu$ on all Borel sets.
24.7. DIFFERENTIATION OF MEASURES WITH RESPECT TO LEBESGUE MEASURE

Proof: By assumption, $\mu(\mathbb{R}^n) < \infty$ and so it is possible to define a positive linear functional, $L$ on $C_c(\mathbb{R}^n)$ by

$$Lf \equiv \int fd\mu.$$ 

By the Riesz representation theorem for positive linear functionals of this sort, there exists a unique Radon measure, $\mu_1$ such that for all $f \in C_c(\mathbb{R}^n)$,

$$\int fd\mu_1 = Lf = \int fd\mu.$$ 

Now let $V$ be an open set and let $K_k \equiv \{x \in V : \text{dist } (x,V^C) \leq 1/k\} \cap \overline{B(0,k)}$. Then $\{K_k\}$ is an increasing sequence of compact sets whose union is $V$. Let $f_k \prec f_k \prec V$. Then $f_k(x) \to \chi_V(x)$ for every $x$. Therefore,

$$\mu_1(V) = \lim_{k \to \infty} \int f_k d\mu_1 = \lim_{k \to \infty} \int f_k d\mu = \mu(V)$$

and so $\mu = \mu_1$ on open sets. Now if $K$ is a compact set, let

$$V_k \equiv \{x \in \mathbb{R}^n : \text{dist } (x,K) < 1/k\}.$$ 

Then $V_k$ is an open set and $\cap_k V_k = K$. Letting $K \prec f_k \prec V_k$, it follows that $f_k(x) \to \chi_K(x)$ for all $x \in \mathbb{R}^n$. Therefore, by the dominated convergence theorem with a dominating function, $\chi_{\mathbb{R}^n}$

$$\mu_1(K) = \lim_{k \to \infty} \int f_k d\mu_1 = \lim_{k \to \infty} \int f_k d\mu = \mu(K)$$

and so $\mu$ and $\mu_1$ are equal on all compact sets. It follows $\mu = \mu_1$ on all countable unions of compact sets and countable intersections of open sets.

Now let $E$ be a Borel set. By regularity of $\mu_1$, there exist sets, $H$ and $G$ such that $H$ is the countable union of an increasing sequence of compact sets, $G$ is the countable intersection of a decreasing sequence of open sets, $H \subseteq E \subseteq G$, and $\mu_1(H) = \mu_1(G) = \mu_1(E)$. Therefore,

$$\mu_1(H) = \mu(H) \leq \mu(E) \leq \mu(G) = \mu_1(G) = \mu_1(E) = \mu_1(H).$$

Therefore, $\mu(E) = \mu_1(E)$ and this proves the lemma.

Corollary 24.7.9 Suppose $\mu$ is a complex Borel measure defined on $\mathbb{R}^n$ for which there exists a $\mu$ measurable set, $N$ such that for all Borel sets, $E$, $\mu(E) = \mu(E \cap N)$ where $m(N) = 0$. Then

$$\frac{d\mu}{dm}(x) = 0 \text{ m a.e.}$$

Proof: Each of $\text{Re } \mu^+$, $\text{Re } \mu^-$, $\text{Im } \mu^+$, and $\text{Im } \mu^-$ are real measures having values in $[0, \infty)$ and so by Lemma 24.7.8 each is a Radon measure having the same property that $\mu$ has in terms of being supported on a set of $m$ measure zero. Therefore, for $\nu$ equal to any of these, $\frac{d\nu}{dm}(x) = 0 \text{ m a.e.}$ This proves the corollary.
24.8 Exercises

1. Suppose $A$ and $B$ are sets of positive Lebesgue measure in $\mathbb{R}^n$. Show that $A - B$ must contain $B(c, \varepsilon)$ for some $c \in \mathbb{R}^n$ and $\varepsilon > 0$.

$$A - B \equiv \{a - b : a \in A \text{ and } b \in B\}.$$

**Hint:** First assume both sets are bounded. This creates no loss of generality. Next there exist $a_0 \in A$, $b_0 \in B$ and $\delta > 0$ such that

$$\int_{B(a_0, \delta)} X_A(t) \, dt > \frac{3}{4} m(B(a_0, \delta)), \quad \int_{B(b_0, \delta)} X_B(t) \, dt > \frac{3}{4} m(B(b_0, \delta)).$$

Now explain why this implies

$$m(A - a_0 \cap B(0, \delta)) > \frac{3}{4} m(B(0, \delta))$$

and

$$m(B - b_0 \cap B(0, \delta)) > \frac{3}{4} m(B(0, \delta)).$$

Explain why

$$m((A - a_0) \cap (B - b_0)) > \frac{1}{2} m(B(0, \delta)) > 0.$$

Let

$$f(x) \equiv \int X_{A-a_0}(x + t) X_{B-b_0}(t) \, dt.$$

Explain why $f(0) > 0$. Next explain why $f$ is continuous and why $f(x) > 0$ for all $x \in B(0, \varepsilon)$ for some $\varepsilon > 0$. Thus if $|x| < \varepsilon$, there exists $t$ such that $x + t \in A - a_0$ and $t \in B - b_0$. Subtract these.

2. Show $Mf$ is Borel measurable by verifying that $[Mf > \lambda] \equiv E_\lambda$ is actually an open set. **Hint:** If $x \in E_\lambda$ then for some $r$, $\int_{B(x, r)} |f| \, dm > \lambda m(B(x, r))$.

Then for $\delta$ a small enough positive number, $\int_{B(x, r)} |f| \, dm > \lambda m(B(x, r + 2\delta))$.

Now pick $y \in B(x, \delta)$ and argue that $B(y, \delta + r) \supseteq B(x, r)$. Therefore show that,

$$\int_{B(y, \delta + r)} |f| \, dm > \int_{B(x, r)} |f| \, dm > \lambda B(x, r + 2\delta) \geq \lambda m(B(y, r + \delta)).$$

Thus $B(x, \delta) \subseteq E_\lambda$.

3. Consider the following nested sequence of compact sets, $\{P_n\}$. Let $P_1 = [0, 1]$, $P_2 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, etc. To go from $P_n$ to $P_{n+1}$, delete the open interval which is the middle third of each closed interval in $P_n$. Let $P = \bigcap_{n=1}^\infty P_n$.

By the finite intersection property of compact sets, $P \neq \emptyset$. Show $m(P) = 0$. If you feel ambitious also show there is a one to one onto mapping of $[0, 1]$
24.8. EXERCISES

955
to $P$. The set $P$ is called the Cantor set. Thus, although $P$ has measure zero, it has the same number of points in it as $[0, 1]$ in the sense that there is a one to one and onto mapping from one to the other. **Hint:** There are various ways of doing this last part but the most enlightenment is obtained by exploiting the topological properties of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0, 1]$. If you do this right and remember the theorems about characterizations of compact metric spaces, Proposition [Page 146] on Page 146, you may get a pretty good idea why every compact metric space is the continuous image of the Cantor set.

4. Consider the sequence of functions defined in the following way. Let $f_1(x) = x$ on $[0, 1]$. To get from $f_n$ to $f_{n+1}$, let $f_{n+1} = f_n$ on all intervals where $f_n$ is constant. If $f_n$ is nonconstant on $[a, b]$, let $f_{n+1}(a) = f_n(a)$, $f_{n+1}(b) = f_n(b)$, $f_{n+1}$ is piecewise linear and equal to $\frac{1}{2}(f_n(a) + f_n(b))$ on the middle third of $[a, b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.

Show $\{f_n\}$ converges uniformly on $[0, 1]$. If $f(x) = \lim_{n \to \infty} f_n(x)$, show that $f(0) = 0$, $f(1) = 1$, $f$ is continuous, and $f'(x) = 0$ for all $x \notin P$ where $P$ is the Cantor set of Problem 3. This function is called the Cantor function. It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1. Explain why this interesting function is not absolutely continuous although it is continuous. **Hint:** This isn’t too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.

5. A function, $f : [a, b] \to \mathbb{R}$ is Lipschitz if $|f(x) - f(y)| \leq K|x - y|$. Show that every Lipschitz function is absolutely continuous. Thus every Lipschitz function is differentiable a.e., $f' \in L^1$, and $f(y) - f(x) = \int_x^y f'(t) \, dt$.

6. Suppose $f, g$ are both absolutely continuous on $[a, b]$. Show the product of these functions is also absolutely continuous. Explain why $(fg)' = f'g + g'f$ and show the usual integration by parts formula

$$f(b)g(b) - f(a)g(a) = \int_a^b fg' \, dt = \int_a^b f'g \, dt.$$  

7. In Problem 4 $f'$ failed to give the expected result for $\int_a^b f' \, dx$ but at least

---

\[\text{In this example, you only know that } f' \text{ exists a.e.}\]
A sequence of sets, \( \{E_i\} \) containing the point \( x \) is said to shrink to \( x \) nicely if there exists a sequence of positive numbers, \( \{r_i\} \) and a positive constant, \( \alpha \) such that \( r_i \to 0 \) and

\[
m(E_i) \geq \alpha m(B(x, r_i)), \quad E_i \subseteq B(x, r_i).
\]

Show the above theorems about differentiation of measures with respect to Lebesgue measure all have a version valid for \( E_i \) replacing \( B(x, r) \).

9. Suppose \( F(x) = \int_a^x f(t) \, dt \). Using the concept of nicely shrinking sets in Problem 8 show \( F'(x) = f(x) \) a.e.

10. A random variable, \( X \) is a measurable real valued function defined on a measure space, \((\Omega, \mathcal{S}, P)\) where \( P \) is just a measure with \( P(\Omega) = 1 \) called a probability measure. The distribution function for \( X \) is the function, \( F(x) = P([X \leq x]) \) in words, \( F(x) \) is the probability that \( X \) has values no larger than \( x \). Show that \( F \) is a right continuous increasing function with the property that \( \lim_{x \to -\infty} F(x) = 0 \) and \( \lim_{x \to \infty} F(x) = 1 \).

11. Suppose \( F \) is an increasing right continuous function.

(a) Show that \( Lf = \int_a^b f \, dF \) is a well defined positive linear functional on \( C_c(\mathbb{R}) \) where here \([a, b]\) is a closed interval containing the support of \( f \in C_c(\mathbb{R}) \).

(b) Using the Riesz representation theorem for positive linear functionals on \( C_c(\mathbb{R}) \), let \( \mu \) denote the Radon measure determined by \( L \). Show that \( \mu([a, b]) = F(b) - F(a) \) and \( \mu(\{b\}) = F(b) - F(b-) \) where \( F(b-) = \lim_{x \to b-} F(x) \).

(c) Review Corollary 18.4.4 on Page 93 at this point. Show that the conditions of this corollary hold for \( \mu \) and \( m \). Consider \( \mu_+ + \mu_- \), the Lebesgue decomposition of \( \mu \) where \( \mu_- \ll m \) and there exists a set of \( m \) measure zero, \( N \) such that \( \mu_-(E) = \mu_-(E \cap N) \). Show \( \mu(\{0, x]\} = \mu_+((0, x]) + \int_0^x h \, dt \) for some \( h \in L^1(m) \). Using Theorem 18.4.4 show \( h(x) = F'(x) \) a.e. Explain why \( F(x) = F(0) + S(x) + \int_0^x F'(t) \, dt \) for some function, \( S(x) \) which is increasing but has \( S'(x) = 0 \) a.e. Note this shows in particular that a right continuous increasing function has a derivative a.e.

12. Suppose now that \( G \) is just an increasing function defined on \( \mathbb{R} \). Show that \( G'(x) \) exists a.e. **Hint:** You can mimic the proof of Theorem 18.4.3. The Dini derivates are defined as

\[
D_+ G(x) \equiv \lim_{h \to 0^+} \inf \frac{G(x+h) - G(x)}{h},
\]

\[
D_+ G(x) \equiv \lim_{h \to 0^+} \sup \frac{G(x+h) - G(x)}{h}.
\]
\[ D_- G(x) \equiv \lim_{h \to 0^+} \inf \frac{G(x) - G(x-h)}{h}, \]
\[ D^+ G(x) \equiv \lim_{h \to 0^+} \sup \frac{G(x) - G(x-h)}{h}. \]

When \( D^+ G(x) = D^+ G(x) \) the derivative from the right exists and when \( D^- G(x) = D^- G(x) \), then the derivative from the left exists. Let \((a,b)\) be an open interval and let
\[ N_{pq} \equiv \{ x \in (a,b) : D^+ G(x) > q > p > D^- G(x) \} . \]

Let \( V \subseteq (a,b) \) be an open set containing \( N_{pq} \) such that \( m(V) < m(N_{pq}) + \varepsilon \). Show using a Vitali covering theorem there is a disjoint sequence of intervals contained in \( V \), \( \{(x_i, x_i + h_i)\}_{i=1}^\infty \) such that
\[ \frac{G(x_i + h_i) - G(x_i)}{h_i} < p. \]

Next show there is a disjoint sequence of intervals \( \{(x'_j, x'_j + h'_j)\}_{j=1}^\infty \) such that each of these is contained in one of the former intervals and
\[ \frac{G(x'_j + h'_j) - G(x'_j)}{h'_j} > q, \sum_{j} h'_j \geq m(N_{pq}) . \]

Then
\[ qm(N_{pq}) \leq q \sum_{j} h'_j \leq \sum_{j} G(x'_j + h'_j) - G(x'_j) \leq \sum_{i} G(x_i + h_i) - G(x_i) \leq p \sum_{i} h_i \leq pm(V) \leq p(m(N_{pq}) + \varepsilon). \]

Since \( \varepsilon \) was arbitrary, this shows \( m(N_{pq}) = 0 \). Taking a union of all \( N_{pq} \) for \( p, q \) rational, shows the derivative from the right exists a.e. Do a similar argument to show the derivative from the left exists a.e. and then show the derivative from the left equals the derivative from the right a.e. using a similar argument. Thus \( G'(x) \) exists on \((a, b)\) a.e. and so it exists a.e. on \( \mathbb{R} \) because \((a, b)\) was arbitrary.
Chapter 25

Orlitz Spaces

25.1 Basic Theory

All the theorems about the $L^p$ spaces have generalizations to something called an Orlitz space. Instead of the convex function, $A(t) = t^p / p$, one considers a more general convex increasing function called an $N$ function.

Definition 25.1.1 $A : [0, \infty) \to [0, \infty)$ is an $N$ function if the following two conditions hold.

1. $A$ is convex and strictly increasing
   \[ \lim_{t \to 0^+} \frac{A(t)}{t} = 0, \quad \lim_{t \to \infty} \frac{A(t)}{t} = \infty. \]  

For $A$ an $N$ function,

\[ \tilde{A}(s) \equiv \max\{st - A(t) : t \geq 0\}. \]  

As an example see the following picture of a typical $N$ function.

Note that from the assumption, the maximum in the definition of $\tilde{A}$ must exist. This is because for $t \neq 0$

\[ (s - A(t) / t) t \]

is negative for all $t$ large enough. On the other hand, it equals 0 when $t = 0$ and so it suffices to consider only $t$ in a compact set.
Lemma 25.1.2 Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then $\phi$ is Lipschitz continuous on $[a, b]$.

**Proof:** Since it is convex, the difference quotients,

$$\frac{\phi(t) - \phi(a)}{t - a}$$

are increasing because by convexity, if $a < t < x$

$$\frac{t - a}{x - a} \phi(x) + \left(1 - \frac{t - a}{x - a}\right) \phi(a) \geq \phi(t)$$

and this reduces to

$$\frac{\phi(t) - \phi(a)}{t - a} \leq \frac{\phi(x) - \phi(a)}{x - a}.$$  

Also these difference quotients are bounded below by

$$\frac{\phi(a) - \phi(a - 1)}{1} = \phi(a) - \phi(a - 1).$$

Let

$$A \equiv \inf \left\{ \frac{\phi(t) - \phi(a)}{t - a} : t \in (a, b) \right\}.$$  

Then $A$ is some finite real number. Similarly there exists a real number $B$ such that for all $t \in (a, b)$,

$$B \geq \frac{\phi(b) - \phi(t)}{b - t}.$$  

Now let $a \leq s < t \leq b$. Then

$$\frac{\phi(t) - \phi(s)}{t - s} \geq \frac{\phi(t) - \theta \phi(a) - (1 - \theta) \phi(t)}{t - s}$$

where $\theta$ is such that $\theta a + (1 - \theta) t = s$. Thus

$$\theta = \frac{t - s}{t - t_1}$$

and so the above implies

$$\frac{\phi(t) - \phi(s)}{t - s} \geq \frac{t - s}{t - t_1} \frac{\phi(t) - \phi(a)}{t - t_1} = \frac{\phi(t) - \phi(s)}{t - s} \geq A.$$  

Similarly,

$$\frac{\phi(t) - \phi(s)}{t - s} \leq \frac{\theta \phi(b) + (1 - \theta) \phi(s) - \phi(s)}{t - s}$$

$$= \frac{t - s \phi(b) - \phi(s)}{b - s} \frac{t - s}{t - s} \leq B.$$
It follows
\[ |\phi(t) - \phi(s)| \leq (|A| + |B|)|t - s| \]
and this proves the lemma.

The following is like the inequality, \( st \leq t^p/p + s^q/q \), important in the study of \( L^p \) spaces.

**Proposition 25.1.3** If \( A \) is an \( N \) function, then so is \( \widetilde{A} \) and
\[ A(t) = \max \left\{ ts - \widetilde{A}(s) : s \geq 0 \right\}, \tag{25.1.4} \]
so \( \widetilde{A} = A \). Also
\[ st \leq A(t) + \widetilde{A}(s) \text{ for all } s, t \geq 0 \tag{25.1.5} \]
and for all \( s > 0 \),
\[ A\left( \frac{\widetilde{A}(s)}{s} \right) \leq \widetilde{A}(s). \tag{25.1.6} \]

**Proof:** First consider the claim \( \widetilde{A} \) is convex. Let \( \lambda \in [0,1] \).
\[
\widetilde{A}(\lambda s_1 + (1-\lambda)s_2) \equiv \max \left\{ [s_1 \lambda + (1-\lambda)s_2]t - A(t) : t \geq 0 \right\}
\leq \lambda \max \{ s_1 t - A(t) : t \geq 0 \} + (1-\lambda) \max \{ s_2 t - A(t) : t \geq 0 \}
= \lambda \tilde{A}(s_1) + (1-\lambda) \tilde{A}(s_2).
\]

It is obvious \( \tilde{A} \) is strictly increasing because \( st \) is strictly increasing in \( s \). Next consider \( 25.1.7 \).

For \( s > 0 \) let \( t_s \) denote the number where the maximum is achieved. That is,
\[ \tilde{A}(s) = st_s - A(t_s). \]
Thus
\[ \frac{\tilde{A}(s)}{s} = t_s - \frac{A(t_s)}{s} \geq 0. \tag{25.1.7} \]
It follows from this that
\[ \lim_{s \to 0^+} t_s = 0 \]
since otherwise, a contradiction results to \( 25.1.7 \), the expression becoming negative for small enough \( s \). Thus
\[ t_s \geq \frac{\tilde{A}(s)}{s} \geq 0 \]
and this shows
\[ \lim_{s \to 0^+} \frac{\tilde{A}(s)}{s} = 0. \]
which shows \( 25.1.2 \).
To verify the second part of \textsection{25.1.2}, let $t_s$ be as just described. Then for any $t > 0$

$$
\frac{\bar{A}(s)}{s} = t_s - \frac{A(t_s)}{s} \geq t - \frac{A(t)}{s}
$$

It follows

$$
\liminf_{s \to \infty} \frac{\bar{A}(s)}{s} \geq t.
$$

Since $t$ is arbitrary, this proves the second part of \textsection{25.1.2}.

The inequality \textsection{25.1.5} follows from the definition of $\bar{A}(s)$.

Next consider \textsection{25.1.4}. It must be shown that

$$
A(t_0) = \max \left\{ t_0 s - \bar{A}(s) : s \geq 0 \right\}.
$$

To do so, first note

$$
\bar{A}(s) = \max \left\{ st - A(t) : t \geq 0 \right\} \geq st_0 - A(t_0).
$$

Hence

$$
\max \left\{ t_0 s - \bar{A}(s) : s \geq 0 \right\} \leq \max \left\{ t_0 s - [st_0 - A(t_0)] \right\} = A(t_0).
$$

Now let

$$
s_0 \equiv \inf \left\{ \frac{A(t) - A(t_0)}{t - t_0} : t > t_0 \right\}.
$$

By convexity, the above difference quotients are nondecreasing in $t$ and so

$$
s_0 (t - t_0) \leq A(t) - A(t_0)
$$

for all $t \neq t_0$. Hence for all $t$,

$$
s_0 t - A(t) \leq s_0 t_0 - A(t_0)
$$

and so

$$
\bar{A}(s_0) = s_0 t_0 - A(t_0)
$$

implying

$$
A(t_0) = s_0 t_0 - \bar{A}(s_0) \leq \max \left\{ st_0 - \bar{A}(s) : s \geq 0 \right\} \leq A(t_0).
$$

Therefore, \textsection{25.1.4} holds.

Consider \textsection{25.1.6} next. To do so, let $a = A'$ so that

$$
A(t) = \int_0^t a(r) \, dr, \; a \text{ increasing}.
$$
This is possible by Rademacher’s theorem, Corollary 24.4.3 and the fact that since
$A$ is convex, it is locally Lipshitz found in Lemma 25.1.2 above. That $a$ is increasing
follows from convexity of $A$. Here is why. For a.e. $s,t \geq 0$, and letting $\lambda \in [0,1]$,
\[
\frac{A(s + \lambda(t - s)) - A(s)}{\lambda} \leq \frac{(1 - \lambda) A(s) + \lambda A(t) - A(s)}{\lambda} = A(t) - A(s)
\]
Then passing to a limit as $\lambda \to 0+$,
\[a(s)(t - s) \leq A(t) - A(s).
\]
Similarly
\[a(t)(s - t) \leq A(s) - A(t)
\]
and so
\[(a(t) - a(s))(t - s) \geq 0.
\]
(If you like, you can simply assume from the beginning that $A(t)$ is given this way
as an integral of a positive increasing function, $a$, and verify directly that such an $A$
is convex and satisfies the properties of an $N$ function. There is no loss of generality
in doing so.) Thus geometrically, $A(t)$ equals the area under the curve defined by
$a$ and above the $x$ axis from $x = 0$ to $x = t$. In the definition of $\bar{A}(s)$ let $t_s$ be
the point where the maximum is achieved. Then
\[\bar{A}(s) = st_s - A(t_s)
\]
and so at this point, $\bar{A}(s) + A(t_s) = st_s$. This means that $\bar{A}(s)$ is the area to the
left of the graph of $a$ which is to the right of the $y$ axis for $y$ between 0 and $a(t_s)$
and that in fact $a(t_s) = s$. The following picture illustrates the reasoning which
follows.

\[
\begin{array}{c}
\text{graph of } a \\
\hline
0 \quad t_0
\end{array}
\]

\[
\begin{array}{c}
\text{graph of } a \\
\hline
0 \quad t_0
\end{array}
\]

Therefore,
\[
\frac{\bar{A}(s)}{s} = t_s - \frac{A(t_s)}{s} = t_s - \frac{1}{s} \int_0^{t_s} a(r) \, dr
\]
\[
= t_s - \frac{1}{a(t_s)} \int_0^{t_s} a(r) \, dr = \frac{1}{a(t_s)} \left( t_s s - \int_0^{t_s} a(r) \, dr \right)
\]
and so
\[
A \left( \frac{\tilde{A}(s)}{s} \right) = \int_0^{\tilde{A}(s)/s} a(r) \, dr = \int_0^{\tilde{A}(s)/s} \int_0^{s-a(r)} a(\tau) \, d\tau = \int_0^{\tilde{A}(s)/s} s - a(r) \, dr = st - A(t_s) = \tilde{A}(s).
\]

The inequality results from replacing \(a(\tau)\) with \(a(t_s)\) in the last integral on the top line.

An example of an \(N\) function is \(A(t) = t^p\) for \(t \geq 0\) and \(p > 1\). For this example, \(\tilde{A}(s) = \frac{x^p}{p}\) where \(\frac{1}{p} + \frac{1}{p'} = 1\).

**Definition 25.1.4** Let \(A\) be an \(N\) function and let \((\Omega, S, \mu)\) be a measure space. Define
\[
K_A(\Omega) \equiv \{ u \text{ measurable such that } \int_{\Omega} A(|u|) \, d\mu < \infty \}. \tag{25.1.8}
\]
This is called the Orlitz class. Also define
\[
L_A(\Omega) \equiv \{ \lambda u : u \in K_A(\Omega) \text{ and } \lambda \in \mathbb{F} \} \tag{25.1.9}
\]
where \(\mathbb{F}\) is the field of scalars, assumed to be either \(\mathbb{R}\) or \(\mathbb{C}\).

The pair \((A, \Omega)\) is called \(\Delta\) regular if either of the following conditions hold.
\[
A(rx) \leq K_r A(x) \text{ for all } x \in [0, \infty) \tag{25.1.10}
\]
or \(\mu(\Omega) < \infty\) and for all \(r > 0\), there exists \(M_r\) and \(K_r > 0\) such that
\[
A(rx) \leq K_r A(x) \text{ for all } x \geq M_r. \tag{25.1.11}
\]

Note there are \(N\) functions which are not \(\Delta\) regular. For example, consider
\[
A(x) \equiv e^{x^2} - 1.
\]
It can't be \(\Delta\) regular because
\[
\lim_{r \to \infty} \frac{e^{rx^2} - 1}{e^{x^2} - 1} = \infty.
\]
However, functions like \(x^p/p\) for \(p > 1\) are \(\Delta\) regular.

Then the following proposition is important.

**Proposition 25.1.5** If \((A, \Omega)\) is \(\Delta\) regular, then \(K_A(\Omega) = L_A(\Omega)\). In any case, \(L_A(\Omega)\) is a vector space and \(K_A(\Omega) \subseteq L_A(\Omega)\).
Proof: Suppose \((A, \Omega)\) is \(\Delta\) regular. Then I claim \(K_A(\Omega)\) is a vector space. This will verify \(K_A(\Omega) = L_A(\Omega)\). Let \(f, g \in K_A(\Omega)\) and suppose \(\|f\| + \|g\| \leq M\). Then
\[
A(|f + g|) = A \left( 2 \left( \frac{|f + g|}{2} \right) \right) \leq K_2 A \left( \frac{|f + g|}{2} \right) \leq K_2 \frac{1}{2} [A(\|f\|) + A(\|g\|)]
\]
so \(f + g \in K_A(\Omega)\) in this case. Now suppose \(\int_\Omega A(|f + g|) \, d\mu = \int_\Omega [A(\|f\|) + A(\|g\|)] \, d\mu < \infty\).

Thus \(f + g \in K_A(\Omega)\) in this case also.

Next consider scalar multiplication. First consider the case of \(\|f\| + \|g\| \leq M\). If \(f \in K_A(\Omega)\) and \(\alpha \in \mathbb{F}\),
\[
\int_\Omega A(|\alpha| |f|) \, d\mu \leq K_{|\alpha|} A(\|f\|) \, d\mu \leq A(M_{|\alpha|}) \mu(\Omega) + \int_\Omega K_2 \frac{1}{2} [A(\|f\|) + A(\|g\|)] \, d\mu < \infty.
\]
This establishes the first part of the proposition.

Next consider the claim that \(L_A(\Omega)\) is always a vector space. First note \(K_A(\Omega)\) is always convex due to convexity of \(A\). Let \(\lambda u, \alpha v \in L_A(\Omega)\) where \(u, v \in K_A(\Omega)\) and let \(a, b\) be scalars in \(\mathbb{F}\). Then
\[
a\lambda u + b\alpha v = |a\lambda| \omega u + |b\alpha| \theta v
\]
where \(|\omega| = |\theta| = 1\). Then
\[
= (|a\lambda| + |b\alpha|) \left( \frac{|a\lambda| \omega u + |b\alpha| \theta v}{|a\lambda| + |b\alpha|} \right)
\]
which exhibits \(a\lambda u + b\alpha v\) as a multiple of a convex combination of two elements of \(K_A(\Omega)\), \(\omega u\) and \(\theta v\). Thus \(L_A(\Omega)\) is closed with respect to linear combinations. This shows it is a vector space. This proves the proposition.

The following norm for \(L_A(\Omega)\) is due to Luxemburg \cite{Luxemburg}. You might compare this to the definition of a Minkowski functional. The definition of \(L_A(\Omega)\) above was cooked up so that the following norm does make sense.

**Definition 25.1.6** Define
\[
\|u\|_A = \|u\|_{A, \Omega} \equiv \inf \left\{ t > 0 : \int_\Omega A \left( \frac{|u(x)|}{t} \right) \, d\mu \leq 1 \right\}.
\]
If two functions of $L_A(\Omega)$ are equal a.e. they are considered to be the same in the usual way.

**Proposition 25.1.7** The number defined in Definition 25.1.6 is a norm on $L_A(\Omega)$.

Also, if $\Omega_1 \subseteq \Omega$, then

$$||u||_{A, \Omega_1} \leq ||u||_{A, \Omega}.$$ 

**Proof:** Clearly $||u||_A \geq 0$. Is $||u||_A$ finite for $u \in L_A(\Omega)$? Let $u \in L_A(\Omega)$ so $u = \lambda v$ where $v \in K_A(\Omega)$. Then for $s > 0$

$$\int_\Omega A\left(\frac{|u|}{s|\lambda|}\right) d\mu = \int_\Omega A\left(\frac{|v|}{s}\right) d\mu < \infty$$

whenever $s > 1$. Therefore, from the dominated convergence theorem, if $s$ is large enough,

$$\int_\Omega A\left(\frac{|u|}{s|\lambda|}\right) d\mu \leq 1$$

and this shows there are values of $t > 0$ such that

$$\int_\Omega A\left(\frac{|u(x)|}{t}\right) d\mu \leq 1.$$ 

Thus $||u||_A$ is finite as hoped.

Now suppose $||u||_A = 0$ and let

$$E_n = \left\{ x : |u(x)| \geq \frac{1}{n} \right\}.$$ 

Then for arbitrarily small values of $t$,

$$\int_{E_n} A\left(\frac{(1/n)}{t}\right) d\mu = \int_\Omega A\left(\frac{|u(x)|}{t}\right) d\mu \leq 1$$

and so for arbitrarily small values of $t$,

$$A\left(\frac{(1/n)}{t}\right) \mu(E_n) \leq 1.$$ 

Letting $t \to 0^+$ yields a contradiction unless $\mu(E_n) = 0$. Now

$$\mu(|u(x)| > 0) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0.$$ 

Thus $u = 0$ as claimed.

Consider the other axioms of a norm. Let $u, v \in L_A(\Omega)$ and let $\alpha, \beta$ be scalars. Then

$$||\alpha u + \beta v||_A = \inf \left\{ t > 0 : \int_\Omega A\left(\frac{|u(x) + v(x)|}{t}\right) d\mu \leq 1 \right\}$$
Without loss of generality \( ||u||_A, ||v||_A < \infty \) since otherwise there is nothing to prove.

\[
||u + v||_A \equiv \inf \left\{ t > 0 : \int_{\Omega} A \left( \frac{|\alpha u(x) + \beta v(x)|}{t} \right) d\mu \leq 1 \right\}.
\]

\[
\leq \inf \left\{ t > 0 : \int_{\Omega} A \left( \frac{|\alpha| |u| + |\beta| |v|}{|\alpha| + |\beta|} \right) d\mu \leq 1 \right\}
\]

\[
= \inf \left\{ t > 0 : \int_{\Omega} A \left( \frac{|\alpha| |u| + |\beta| |v|}{|\alpha| + |\beta|} \right) d\mu \leq 1 \right\}
\]

\[
\leq \inf \left\{ t > 0 : \frac{|\alpha|}{|\alpha| + |\beta|} \int_{\Omega} A \left( \frac{|u|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\}
\]

\[
+ \inf \left\{ t > 0 : \frac{|\beta|}{|\alpha| + |\beta|} \int_{\Omega} A \left( \frac{|v|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\}
\]

\[
= |\alpha| \inf \left\{ t/(|\alpha| + |\beta|) > 0 : \int_{\Omega} A \left( \frac{|u|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\}
\]

\[
+ |\beta| \inf \left\{ t/(|\alpha| + |\beta|) > 0 : \int_{\Omega} A \left( \frac{|v|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\}
\]

\[
= |\alpha| ||u||_A + |\beta|| ||v||_A.
\]

Now let \( \Omega_1 \subseteq \Omega \).

\[
||u||_{A,\Omega_1} \equiv \inf \left\{ t > 0 : \int_{\Omega_1} A \left( \frac{|u(x)|}{t} \right) d\mu \leq 1 \right\}
\]

\[
\leq \inf \left\{ t > 0 : \int_{\Omega} A \left( \frac{|u(x)|}{t} \right) d\mu \leq 1 \right\} \equiv ||u||_{A,\Omega}.
\]

This occurs because if \( t \) is in the second set, then it is in the first so the infimum of the second is no smaller than that of the first. This proves the proposition.

Next it is shown that \( L_A(\Omega) \) is a Banach space.

**Theorem 25.1.8** \( L_A(\Omega) \) is a Banach space and every Cauchy sequence has a subsequence which also converges pointwise a.e.

**Proof:** Let \( \{f_n\} \) be a Cauchy sequence in \( L_A(\Omega) \) and select a subsequence \( \{f_{n_k}\} \) such that

\[
||f_{n_{k+1}} - f_{n_k}||_A \leq 2^{-k}.
\]

Thus

\[
f_{n_m}(x) = f_{n_1}(x) + \sum_{k=1}^{m-1} f_{n_{k+1}}(x) - f_{n_k}(x).
\]
Let
\[ g_m(x) \equiv |f_{n_1}(x)| + \sum_{k=1}^{m-1} |f_{n_{k+1}}(x) - f_{n_k}(x)|. \]

Then
\[ ||g_m||_A \leq ||f_{n_1}||_A + \sum_{k=1}^{\infty} 2^{-k} \equiv K < \infty. \]

Let
\[ g(x) \equiv \lim_{m \to \infty} g_m(x) \equiv |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|. \]

Now \( K > ||g_m||_A \) so
\[ 1 \geq \int_{\Omega} A \left( \frac{|g_m(x)|}{K} \right) d\mu. \]

By the monotone convergence theorem,
\[ 1 \geq \int_{\Omega} A \left( \frac{|g(x)|}{K} \right) d\mu \]
showing \( g(x) < \infty \) a.e., say for all \( x \notin E \) where \( E \) is a measurable set having measure zero. Let
\[ f(x) \equiv \mathcal{X}_{KC}(x) \left( f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)) \right) = \lim_{m \to \infty} \mathcal{X}_{KC}(x) f_{n_m}(x). \]

Thus \( f \) is measurable and \( f_{n_m}(x) \to f(x) \) a.e. as \( m \to \infty \).

For \( l > k \),
\[ ||f_{n_k} - f_{n_l}||_A < \frac{1}{2^{k-2}} \]
and so
\[ 1 \geq \int_{\Omega} A \left( \frac{|f_{n_l}(x) - f_{n_k}(x)|}{(2^{k-2})} \right) d\mu. \]

By Fatou’s lemma, let \( l \to \infty \) and obtain
\[ 1 \geq \int_{\Omega} A \left( \frac{|f(x) - f_{n_k}(x)|}{(2^{k-2})} \right) d\mu \]
and so \( (f - f_{n_k}) 2^{k-2} \in K_A(\Omega) \) and so \( f - f_{n_k} \in L_A(\Omega) \), \( f_{n_k} \in L_A(\Omega) \). Since \( L_A(\Omega) \) is a vector space, this shows \( f \in L_A(\Omega) \). Also
\[ ||f - f_{n_k}||_A \leq \frac{1}{2^{k-2}}, \]
showing that \( f_{n_k} \to f \) in \( L_A(\Omega) \). Since a subsequence converges in \( L_A(\Omega) \), it follows the original Cauchy sequence also converges to \( f \) in \( L_A(\Omega) \). This proves the theorem.

Next consider the space, \( E_A(\Omega) \) which will be a subspace of the Orlicz class, \( K_A(\Omega) \) just as \( L_A(\Omega) \) is a vector space containing the Orlicz class.

**Definition 25.1.9** Let \( S \) denote the set of simple functions, \( s \), such that 
\[
\mu \left( \{ x : s(x) \neq 0 \} \right) < \infty.
\]
Then define
\[
E_A(\Omega) \equiv \text{the closure in } L_A(\Omega) \text{ of } S.
\]

**Proposition 25.1.10** \( E_A(\Omega) \subseteq K_A(\Omega) \subseteq L_A(\Omega) \) and they are all equal if \((A,\Omega)\) is \( \Delta \) regular.

**Proof:** First note that \( S \subseteq K_A(\Omega) \cap E_A(\Omega) \). Let \( f \in E_A(\Omega) \). Then by the definition of \( E_A(\Omega) \), there exists \( s_n \in S \) such that 
\[
||s_n - f||_A \to 0.
\]

Therefore, for \( n \) large enough, 
\[
||s_n - f||_A < \frac{1}{2}
\]
and so 
\[
\int_{\Omega} A \left( \frac{|f - s_n|}{\frac{1}{2}} \right) d\mu = \int_{\Omega} A (|2f - 2s_n|) d\mu < \infty.
\]

Since \( S \subseteq K_A(\Omega) \), 
\[
\int_{\Omega} A (2|s_n|) d\mu < \infty.
\]

Therefore, \( 2f - 2s_n \in K_A(\Omega) \) and \( 2s_n \in K_A(\Omega) \) and so, since \( K_A(\Omega) \) is convex, 
\[
\frac{2f - 2s_n}{2} + \frac{2s_n}{2} = f \in K_A(\Omega).
\]

This shows \( E_A(\Omega) \subseteq K_A(\Omega) \).

Next consider the claim these spaces are all equal in the case that \((A,\Omega)\) is \( \Delta \) regular. It was already shown in Proposition 25.1.5 that in this case, 
\[
K_A(\Omega) = L_A(\Omega)
\]
so it remains to show \( E_A(\Omega) = K_A(\Omega) \). Is every \( f \in K_A(\Omega) \) the limit in \( L_A(\Omega) \) of functions from \( S \)? First suppose \( \mu(\Omega) = \infty \). Then \( A(r|f|) \leq K_rA(|f|) \) and so \( A(r|f|) \in L^1(\Omega) \) for any \( r \). Let \( \varepsilon > 0 \) be given and let 
\[
\Omega_\delta = \{ x : |f(x)| \geq \delta \}
\]
Then by the dominated convergence theorem,
\[
\lim_{\delta \to 0^+} \int_{\Omega \setminus \Omega_\delta} A \left( \frac{|f|}{\varepsilon} \right) d\mu = 0.
\]
Choose \(\delta\) such that
\[
\int_{\Omega \setminus \Omega_\delta} A \left( \frac{|f|}{\varepsilon} \right) d\mu < \frac{1}{2}
\]
and let \(s_n \to f\chi_{\Omega_\delta}\) pointwise with \(|s_n| \leq |f\chi_{\Omega_\delta}|\). Then \(s_n = 0\) on \(\Omega \setminus \Omega_\delta\) and so
\[
\int_{\Omega} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu = \int_{\Omega_\delta} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu
\]
\[
+ \int_{\Omega \setminus \Omega_\delta} A \left( \frac{|f|}{\varepsilon} \right) d\mu \leq \int_{\Omega_\delta} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu + \frac{1}{2}.
\]
By the dominated convergence theorem,
\[
\int_{\Omega_\delta} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu < \frac{1}{2}
\]
for all \(n\) large enough. Therefore, for such \(n\),
\[
\int_{\Omega} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu < 1
\]
and so \(\|f - s_n\|_A \leq \varepsilon\) showing that in this case \(E_A(\Omega) \supseteq K_A(\Omega)\) since \(\varepsilon > 0\) is arbitrary.

Now suppose \(\mu(\Omega) < \infty\). In this case, only assume \(A(rt) \leq K_A(t)\) for \(t\) large enough, say for \(t \geq M_r\). However, this is enough to conclude \(A(r|f|) \in L^1(\Omega)\) for any \(r > 0\) because \(\mu(\Omega) < \infty\) and \(f \in K_A(\Omega)\). Let \(s_n \to f\) pointwise with \(|s_n| \leq |f|\), and \(s\) simple. Then
\[
A \left( \frac{|f - s_n|}{\varepsilon} \right) \leq A \left( \frac{2}{\varepsilon} |f| \right) \in L^1(\Omega)
\]
and so the dominated convergence theorem implies
\[
\lim_{n \to \infty} \int_{\Omega} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu = 0.
\]
Hence
\[
\int_{\Omega} A \left( \frac{|f - s_n|}{\varepsilon} \right) d\mu < 1
\]
for all \(n\) large enough and so for such \(n\),
\[
\|f - s_n\|_A \leq \varepsilon
\]
which proves the proposition.

It turns out \(E_A(\Omega)\) is the largest linear subspace of \(K_A(\Omega)\).
Proposition 25.1.11 \( E_A(\Omega) \) is the maximal linear subspace of \( K_A(\Omega) \).

**Proof:** Let \( M \) be a subspace of \( K_A(\Omega) \). Is \( M \subseteq E_A(\Omega) \)? For \( f \in M, f/\varepsilon \in K_A(\Omega) \) for all \( \varepsilon > 0 \) because of the fact that \( M \) is a subspace and \( f \in M \). Thus \( A(|f|/\varepsilon) \) is in \( L^1(\Omega) \). Let \( \varepsilon > 0 \) be given, choose \( \delta > 0 \) and let

\[
F_\delta \equiv \{ x : |f(x)| \leq \delta \}
\]

By the dominated convergence theorem there exists \( \delta \) small enough that

\[
\int_{F_\delta} A\left( \frac{2|f|}{\varepsilon} \right) d\mu < \frac{1}{2}.
\]

Let \( |s_n| \leq |f|X_{F_\delta}^c \) and \( s_n \to fX_{F_\delta}^c \) pointwise for \( s_n \) a simple function. Thus \( s_n = 0 \) on \( F_\delta \) and so \( s_n \in \mathcal{S} \) because \( \mu(F_\delta^c) < \infty \). Now

\[
\int_{\Omega} A\left( \frac{|f - s_n|}{\varepsilon} \right) d\mu = \int_{F_\delta} A\left( \frac{|f|}{\varepsilon} \right) d\mu + \int_{F_\delta^c} A\left( \frac{|f - s_n|}{\varepsilon} \right) d\mu
\]

\[
< \frac{1}{2} + \int_{F_\delta^c} A\left( \frac{|f - s_n|}{\varepsilon} \right) d\mu.
\]

The integrand in the last integral is no larger than \( \frac{2|f|}{\varepsilon} \) and so by the dominated convergence theorem, this integral converges to 0 as \( n \to \infty \). In particular, it is eventually less than \( \frac{1}{2} \). Therefore, for such \( n \),

\[
||f - s_n||_A \leq r.
\]

Since \( r \) is arbitrary, this shows that \( f \in E_A(\Omega) \) which proves the proposition.

Next is a comparison of these function spaces for different choices of the \( N \) function. The notation \( X \hookrightarrow Y \) for two normed linear spaces means \( X \) is a subset of \( Y \) and the identity map is continuous.

**Proposition 25.1.12** \( L_B(\Omega) \hookrightarrow L_A(\Omega) \) if either

\[
B(t) \geq A(t) \text{ for all } t \geq 0 \quad (25.1.12)
\]

or if

\[
B(t) \geq A(t) \text{ for all } t > M \quad (25.1.13)
\]

and \( \mu(\Omega) < \infty \).

**Proof:** Let \( f \in L_B(\Omega) \) and let

\[
\int_{\Omega} B\left( \frac{|f|}{t} \right) d\mu \leq 1.
\]

Then if \( 25.1.12 \) holds, it follows

\[
\int_{\Omega} A\left( \frac{|f|}{t} \right) d\mu \leq 1.
\]
Thus if \( t \geq \|f\|_B \) then \( t \geq \|f\|_A \) which implies \( \|f\|_B \geq \|f\|_A \).

Now suppose \ref{25.1.13} holds and \( \mu(\Omega) < \infty \). Then \( \max(A, B) \) is an \( N \) function dominating both \( A \) and \( B \) for all \( t \). By what was just shown \( L_{\max(A,B)}(\Omega) \hookrightarrow L_B(\Omega) \). Then let \( f \in L_B(\Omega) \) and let

\[
\int_{\Omega} B\left(\frac{|f|}{t}\right) \, d\mu < 1.
\]

Then

\[
\int_{\Omega} \max(A, B)\left(\frac{|f|}{t}\right) \, d\mu = \int_{\left[\frac{|f|}{t} \geq M\right]} B\left(\frac{|f|}{t}\right) \, d\mu + \int_{\left[\frac{|f|}{t} \leq M\right]} \max(A, B)\left(\frac{|f|}{t}\right) \, d\mu
\]

\[
\leq \int_{\Omega} B\left(\frac{|f|}{t}\right) \, d\mu + \mu(\Omega) \max(A, B)(M) < \infty.
\]

It follows \( \frac{|f|}{t} \in K_{\max(A,B)}(\Omega) \) and so \( f \in L_{\max(A,B)}(\Omega) \). Hence \( L_B(\Omega) = L_{\max(A,B)}(\Omega) \) and the identity map from \( L_{\max(A,B)}(\Omega) \) to \( L_B(\Omega) \) is continuous. Therefore, by the open mapping theorem, the norms \( \|\cdot\|_B \) and \( \|\cdot\|_{\max(A,B)} \) are equivalent. Hence for \( f \in L_B(\Omega) \),

\[
\|f\|_A \leq \|f\|_{\max(A,B)} \leq C\|f\|_B.
\]

This proves the proposition.

**Corollary 25.1.13** Suppose there exists \( C > 0 \), a constant such that either

\[
CB(t) \geq A(t)
\]

for all \( t \geq 0 \) or

\[
CB(t) \geq A(t)
\]

for all \( t > M \) and \( \mu(\Omega) < \infty \). Then

\( L_B(\Omega) \hookrightarrow L_A(\Omega) \).

**Proof:** If \( f \in L_B(\Omega) \) then \( f = \lambda u \) where \( u \in K_B(\Omega) = K_{CB}(\Omega) \). Hence \( L_{CB}(\Omega) = L_B(\Omega) \) and the two norms on \( L_B(\Omega) \),

\[
\|\cdot\|_{CB}, \ \text{and} \ \|\cdot\|_B,
\]

are equivalent norms by the open mapping theorem. Hence by the Proposition \ref{25.1.12}, if \( f \in L_B(\Omega) \),

\[
\|f\|_A \leq C_1 \|f\|_{CB} \leq C_2 \|f\|_B
\]

which proves the corollary.
25.1. BASIC THEORY

Definition 25.1.14 A increases essentially more slowly than B if for all a > 0,

\[ \lim_{t \to \infty} \frac{A(at)}{B(t)} = 0 \]

The next theorem gives added information on how these spaces are related in case that one function increases essentially more slowly than the other.

Theorem 25.1.15 Suppose \( \mu(\Omega) < \infty \) and A increases essentially more slowly than B. Then

\[ L_B(\Omega) \hookrightarrow E_A(\Omega) \]

Proof: Let \( f \in L_B(\Omega) \). Then there exists \( \lambda > 0 \) such that

\[ \int_\Omega B\left(\frac{|f|}{\lambda}\right) d\mu \leq 1. \]

Let \( r \) be such that for \( t \geq r \),

\[ A(|\lambda| t) \leq B(t) . \]

Then

\[ \int_\Omega A(|f|) d\mu = \int_{[|f| \geq r]} A(|f|) d\mu + \int_{[|f| < r]} A(|f|) d\mu \]
\[ \leq \int_\Omega B\left(\frac{|f|}{|\lambda|}\right) d\mu + A(r) \mu(\Omega) \]
\[ < 1 + A(r) \mu(\Omega) . \]

Therefore, \( L_B(\Omega) \) is a linear space contained in \( K_A(\Omega) \). It follows from Proposition 25.1.11 that \( L_B(\Omega) \subseteq E_A(\Omega) \). This proves the theorem.

The norm of \( E_A(\Omega) \) is the same as the norm on \( L_A(\Omega) \) so this shows \( L_B(\Omega) \hookrightarrow L_A(\Omega) \).

Note that for \( 1 < p < q \) and \( A(t) = t^p/p, B(t) = t^q/q \),

\[ A(rt) = r^p t^p \]

and

\[ \lim_{t \to \infty} \frac{A(rt)}{B(t)} = 0 \]

showing this case is covered by the above theorem.

If A is increasing essentially more slowly than B and \( \mu(\Omega) < \infty \), this has shown the following inclusions

\[ E_B(\Omega) \subseteq K_B(\Omega) \subseteq L_B(\Omega) \hookrightarrow E_A(\Omega) \subseteq K_A(\Omega) \subseteq L_A(\Omega) . \]

In the case of \( A(t) = t^p/p, B(t) = t^q/q \) both \( A, \Omega) \) and \( B, \Omega) \) are \( \Delta \) regular and so in this case or any other case where the \( N \) functions are \( \Delta \) regular, the above sequence of inclusions reduces to

\[ E_B(\Omega) = K_B(\Omega) = L_B(\Omega) \hookrightarrow E_A(\Omega) = K_A(\Omega) = L_A(\Omega) . \]
25.2 Dual Spaces In Orlitz Space

Recall that for \( s, t \geq 0 \),
\[
st \leq A(t) + \tilde{A}(s).
\]

Let \( v \in L_{\tilde{A}}(\Omega) \) and \( u \in L_A(\Omega) \). Then there is a version of Holder’s inequality as follows. For \( \varepsilon > 0 \),
\[
\frac{|v|}{||v||_{\tilde{A}} + \varepsilon} \in K_{\tilde{A}}(\Omega), \quad \frac{|u|}{||u||_A + \varepsilon} \in K_A(\Omega).
\]

Therefore,
\[
\int_{\Omega} \left( \frac{|u|}{||u||_A + \varepsilon} \right) \left( \frac{|v|}{||v||_{\tilde{A}} + \varepsilon} \right) d\mu \leq \int_{\Omega} A \left( \frac{|u|}{||u||_A + \varepsilon} \right) d\mu
\]
\[
+ \int_{\Omega} \tilde{A} \left( \frac{|v|}{||v||_{\tilde{A}} + \varepsilon} \right) d\mu \leq 2
\]
and so \( uv \in L^1(\Omega) \) and
\[
\int_{\Omega} uv d\mu \leq \int_{\Omega} |u| |v| d\mu \leq 2 (||v||_{\tilde{A}} + \varepsilon) (||u||_A + \varepsilon).
\]

Since \( \varepsilon \) is arbitrary this shows
\[
\int_{\Omega} uv d\mu \leq \int_{\Omega} |u| |v| d\mu \leq 2 ||v||_{\tilde{A}} ||u||_A.
\] (25.2.14)

Defining \( L_v \) for \( v \in \tilde{A} \) by
\[
L_v(u) \equiv \int_{\Omega} uv d\mu,
\]
it follows \( L_v \in L_A(\Omega)' \). From now on assume the measure space is \( \sigma \) finite. That is, there exist measurable sets, \( \Omega_k \) satisfying the following:
\[
\Omega = \bigcup_{k=1}^{\infty} \Omega_k, \quad \mu(\Omega_k) < \infty, \quad \Omega_k \subseteq \Omega_{k+1}.
\]

Then

**Proposition 25.2.1** For \( v \in L_{\tilde{A}}(\Omega) \), the following inequality holds.
\[
||v||_{\tilde{A}} \leq ||L_v|| \leq 2 ||v||_{\tilde{A}}.
\]

Here \( L_v \) is considered as either an element of \( E_A(\Omega)' \) or \( L_A(\Omega)' \) and \( ||L_v|| \) refers to the operator norm in either dual space.
Proof: The inequality \[25.2.14\] implies \[|L_v| \leq 2 v_{\bar{A}}\]. It remains to show the other half of the inequality. If \(L_v = 0\) there is nothing to show because this would imply that \(v = 0\) so assume \(|L_v| > 0\). Define a measurable function, \(u\), as follows.

Letting \(r \in (0, 1)\),

\[
u(x) = \begin{cases} \tilde{A} \left( \frac{r |v(x)|}{|L_v|} \right) / \frac{|v(x)|}{|L_v|} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0. \end{cases} \tag{25.2.15}\]

Now let

\[F_n \equiv \{ x : |u(x)| \leq n \} \cap \Omega_n \cap \{ x : v(x) \neq 0 \} \tag{25.2.16}\]

and define

\[u_n(x) \equiv u(x) \mathcal{A}_{F_n}(x). \tag{25.2.17}\]

Thus \(u_n\) is bounded and equals zero off a set which has finite measure. It follows that

\[A \left( \frac{|u_n|}{\alpha} \right) \in L^1(\Omega)\]

for all \(\alpha > 0\). I claim that \(|u_n|_{\alpha} \leq 1\). If not, there exists \(\varepsilon > 0\) such that \(|u_n|_{\alpha - \varepsilon} > 1\). Then since \(A\) is convex,

\[1 < \int_{\Omega} A \left( \frac{|u_n|}{|u_n|_{\alpha} - \varepsilon} \right) d\mu \leq \frac{1}{|u_n|_{\alpha} - \varepsilon} \int_{\Omega} A(|u_n|) d\mu.\]

Taking \(\varepsilon \to 0^+\), using \[25.2.15\] and convexity of \(A\) along with \[25.2.14\] and \[25.2.17\],

\[|u_n|_{\alpha} \leq \int_{\Omega} A(|u_n|) d\mu = \int_{F_n} A \left( \frac{r \tilde{A} \left( \frac{r |v(x)|}{|L_v|} \right)}{\frac{|v(x)|}{|L_v|}} \right) d\mu \leq r \int_{F_n} \tilde{A} \left( \frac{r |v(x)|}{|L_v|} \right) d\mu \leq r \int_{\Omega} u_n(x) v(x) d\mu \leq r \frac{1}{|L_v|} \int_{\Omega} u_n(x) v(x) d\mu = r |u_n|_{\alpha},\]

a contradiction since \(r < 1\). Therefore, from \[25.2.15\],

\[|L_v| \geq |L_v(u_n)| \equiv \int_{F_n} v(x) u(x) d\mu = |L_v| \int_{F_n} \tilde{A} \left( \frac{r |v(x)|}{|L_v|} \right) d\mu\]

and so

\[1 \geq \int_{F_n} \tilde{A} \left( \frac{r |v(x)|}{|L_v|} \right) d\mu\]

By the monotone convergence theorem, letting \(n \to \infty\),

\[1 \geq \int_{\Omega} \tilde{A} \left( \frac{r |v(x)|}{|L_v|} \right) d\mu\]

showing that

\[|v|_{\tilde{A}} \leq \frac{|L_v|}{r}.\]
Since this holds for all \( r \in (0, 1) \), it follows \( ||L_o|| \geq ||v||_A \) as claimed. This proves the proposition.

Now what follows is the Riesz representation theorem for the dual space of \( E_A(\Omega) \).

**Theorem 25.2.2** Suppose \( \mu(\Omega) < \infty \) and suppose \( L \in E_A(\Omega)' \). Then the map \( v \to L_v \) from \( L_A(\Omega) \) to \( E_A(\Omega)' \) is one to one continuous, linear, and onto. If \((\Omega, A)\) is \( \Delta \)-regular then \( v \to L_v \) is one to one, linear, onto and continuous as a map from \( L_A(\Omega) \) to \( L_A(\Omega)' \).

**Proof:** It is obvious this map is linear. From Proposition 25.2.1 it is continuous and one to one. It remains only to verify that it is onto. Let \( L \in E_A(\Omega)' \) and define a complex valued function, \( \lambda \), mapping the measurable sets to \( \mathbb{C} \) as follows.

\[
\lambda(F) = L(X_F).
\]

In case \( \mu(F) \neq 0 \),

\[
\int_{\Omega} A(k) \left( \frac{1}{\mu(F)} \right) d\mu = \int_F A^{-1} \left( \frac{1}{\mu(F)} \right) d\mu = \int_F \frac{1}{\mu(F)} d\mu = 1
\]

and so

\[
\|X_F\|_A \leq \frac{1}{A^{-1} \left( \frac{1}{\mu(F)} \right)}. \tag{25.2.18}
\]

In fact, \( \lambda \) is actually a complex measure. To see this, suppose \( F_i \uparrow F \). Then from the formula just derived,

\[
\|X_{F_i} - X_F\|_A = \lim_{m \to \infty} \|X_{F_i \setminus F_i} - X_{F \setminus F_i}\|_A \leq \frac{1}{A^{-1} \left( \frac{1}{\mu(F \setminus F_i)} \right)}
\]

which converges to zero as \( i \to \infty \). Therefore, if the \( F_i \) are disjoint and \( F = \bigcup_{i=1}^{\infty} F_i \), let \( S_m = \bigcup_{i=1}^{m} F_i \) so that \( S_m \uparrow F \). Then since \( X_{S_m} \to X_F \) in \( E_A(\Omega) \) and \( L \) is continuous,

\[
\lambda(F) = \lim_{m \to \infty} L(X_{S_m}) = \lim_{m \to \infty} \sum_{i=1}^{m} L(X_{F_i}) = \sum_{i=1}^{\infty} \lambda(F_i) .
\]

Next observe that \( \lambda \) is absolutely continuous with respect to \( \mu \). To see this, suppose \( \mu(F) = 0 \). Then if \( t > 0 \),

\[
\int_{\Omega} A \left( \frac{X_F(x)}{t} \right) d\mu = 0 < 1
\]
for all \( t > 0 \) and so \( ||X_F||_A = 0 \). Therefore, \( \lambda(F) \equiv L(X_F) = 0 \).

It follows by the Radon Nikodym theorem there exists \( v \in L^1(\Omega) \) such that

\[
L(X_F) = \lambda(F) = \int_F v d\mu.
\]

Therefore, for all \( s \in \mathbb{S} \),

\[
L(s) = \int_F s v d\mu. \tag{25.2.19}
\]

I need to show that \( v \) is actually in \( L^e_A(\Omega) \). If \( v = 0 \) a.e., there is nothing to prove so assume this is not so. Let \( u \) be defined by

\[
u (x) \equiv \begin{cases} \tilde{A} \left( \frac{r|v(x)|}{||L||} \right) / \frac{v(x)}{||L||} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0. \end{cases} \tag{25.2.20}\]

for \( r \in (0, 1) \). Now let

\[F_n \equiv \{x : |u(x)| \leq n\} \cap \{x : v(x) \neq 0\} \tag{25.2.21}\]

and define

\[u_n (x) \equiv u(x) X_{F_n}(x). \tag{25.2.22}\]

I claim \( ||u_n||_A \leq 1 \). It is clear that since \( \mu(\Omega) < \infty \), \( u_n \in E_A(\Omega) \). If \( ||u_n||_A > 1 \), then for \( \varepsilon \) small enough,

\[||u_n||_A - \varepsilon > 1\]

and so, by convexity of \( A \) and the fact that \( A(0) = 0 \),

\[
1 < \int_{\Omega} A \left( \frac{|u_n(x)|}{||u_n||_A - \varepsilon} \right) d\mu \leq \frac{1}{||u_n||_A - \varepsilon} \int_{\Omega} A(|u_n(x)|) d\mu
\]

and so, letting \( \varepsilon \to 0^+ \) and using \( 25.1.6 \) and convexity of \( A \) as in the proof of the preceding proposition,

\[
||u_n||_A \leq \int_{\Omega} A(|u_n(x)|) d\mu \leq \int_{F_n} A \left( r\tilde{A} \left( \frac{r|v(x)|}{||L||} \right) / \frac{r|v(x)|}{||L||} \right) d\mu \leq \int_{F_n} A \left( r\tilde{A} \left( \frac{r|v(x)|}{||L||} \right) / \frac{r|v(x)|}{||L||} \right) d\mu \leq r \frac{1}{||L||} \int_{F_n} u(x) v(x) d\mu = \frac{r}{||L||} \int_{\Omega} u_n v d\mu. \tag{25.2.23}\]

Now by Theorem \( 16.1.3 \) applied to the positive and negative parts of real and imaginary parts, there exists a uniformly bounded sequence of simple functions, \( \{s_k\} \) converging uniformly to \( u_n \), implying convergence in \( E_A(\Omega) \), and so

\[
Lu_n = \lim_{k \to \infty} Ls_k = \lim_{k \to \infty} \int_{\Omega} s_k v d\mu = \int_{\Omega} u_n v d\mu. \tag{25.2.24}\]
therefore, from

\[ \|u_n\|_A \leq \frac{r}{\|L\|} \int_{\Omega} u_n v d\mu = \frac{r}{\|L\|} L(u_n) \leq \frac{r}{\|L\|} \|L\| \|u_n\|_A, \]

which is a contradiction since \( r < 1 \). Therefore, \( \|u_n\|_A \leq 1 \) and from

\[ \|L\| \geq \|L\| \|u_n\|_A \geq |Lu_n| = \int_{\Omega} u_n v d\mu \]

\[ \geq \|L\| \int_{F_n} A \left( \frac{r |v(x)|}{\|L\|} \right) d\mu. \]

Letting \( n \to \infty \) the monotone convergence theorem and the above imply

\[ \int_{\Omega} A \left( \frac{r |v(x)|}{\|L\|} \right) d\mu \leq 1 \]

which shows that \( v \in L_A(\Omega) \) and \( \|v\|_A \leq \frac{\|L\|}{r} \) for all \( r \in (0,1) \). Therefore, \( \|v\|_A \leq \frac{\|\|L\|}{r} \).

Since \( v \in L_A(\Omega) \) it follows \( L_v = L \) on \( S \) and so \( L_v = L \) because \( S \) is dense in \( E_A(\Omega) \). The last assertion follows from Proposition (25.1.10) This completes the proof.
Chapter 26

Hausdorff Measure

26.1 Definition Of Hausdorff Measures

This chapter is on Hausdorff measures. First I will discuss some outer measures. In all that is done here, \( \alpha(n) \) will be the volume of the ball in \( \mathbb{R}^n \) which has radius 1.

**Definition 26.1.1** For a set \( E \), denote by \( r(E) \) the number which is half the diameter of \( E \). Thus

\[
r(E) \equiv \frac{1}{2} \sup \{|x - y| : x, y \in E\} \equiv \frac{1}{2} \text{diam}(E)
\]

Let \( E \subseteq \mathbb{R}^n \).

\[
\mathcal{H}_\delta^s(E) \equiv \inf \left\{ \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s : E \subseteq \bigcup_{j=1}^{\infty} C_j, \text{diam}(C_j) \leq \delta \right\}
\]

\[
\mathcal{H}^s(E) \equiv \lim_{\delta \to 0^+} \mathcal{H}_\delta^s(E).
\]

Note that \( \mathcal{H}_\delta^s(E) \) is increasing as \( \delta \to 0^+ \) so the limit clearly exists.

In the above definition, \( \beta(s) \) is an appropriate positive constant depending on \( s \). It will turn out that for \( n \) an integer, \( \beta(n) = \alpha(n) \) where \( \alpha(n) \) is the Lebesgue measure of the unit ball, \( B(0,1) \) where the usual norm is used to determine this ball.

**Lemma 26.1.2** \( \mathcal{H}^s \) and \( \mathcal{H}_\delta^s \) are outer measures.

**Proof:** It is clear that \( \mathcal{H}^s(\emptyset) = 0 \) and if \( A \subseteq B \), then \( \mathcal{H}^s(A) \leq \mathcal{H}^s(B) \) with similar assertions valid for \( \mathcal{H}_\delta^s \). Suppose \( E = \bigcup_{i=1}^{\infty} E_i \) and \( \mathcal{H}_\delta^s(E_i) < \infty \) for each \( i \). Let \( \{C_j^i\}_{j=1}^{\infty} \) be a covering of \( E_i \) with

\[
\sum_{j=1}^{\infty} \beta(s)(r(C_j^i))^s - \varepsilon/2^i < \mathcal{H}_\delta^s(E_i)
\]
and \( \text{diam}(C_j^i) \leq \delta \). Then

\[
\mathcal{H}_s^\delta(E) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta(s) (r(C_j^i))^s \\
\leq \sum_{i=1}^{\infty} \mathcal{H}_s^\delta(E_i) + \varepsilon/2^i \\
\leq \varepsilon + \sum_{i=1}^{\infty} \mathcal{H}_s^\delta(E_i).
\]

It follows that since \( \varepsilon > 0 \) is arbitrary,

\[
\mathcal{H}_s^\delta(E) \leq \sum_{i=1}^{\infty} \mathcal{H}_s^\delta(E_i)
\]

which shows \( \mathcal{H}_s^\delta \) is an outer measure. Now notice that \( \mathcal{H}_s^\delta(E) \) is increasing as \( \delta \to 0 \). Picking a sequence \( \delta_k \) decreasing to 0, the monotone convergence theorem implies

\[
\mathcal{H}_s(E) \leq \sum_{i=1}^{\infty} \mathcal{H}_s(E_i).
\]

The outer measure \( \mathcal{H}_s \) is called \( s \) dimensional Hausdorff measure when restricted to the \( \sigma \) algebra of \( \mathcal{H}_s \) measurable sets.

Next I will show the \( \sigma \) algebra of \( \mathcal{H}_s \) measurable sets includes the Borel sets. This is done by the following very interesting condition known as Carathéodory’s criterion.

### 26.1.1 Properties Of Hausdorff Measure

**Definition 26.1.3** For two sets, \( A, B \) in a metric space, we define

\[
\text{dist} (A, B) \equiv \inf \{d(x, y) : x \in A, y \in B\}.
\]

**Theorem 26.1.4** Let \( \mu \) be an outer measure on the subsets of \((X, d)\), a metric space. If

\[
\mu(A \cup B) = \mu(A) + \mu(B)
\]

whenever \( \text{dist}(A, B) > 0 \), then the \( \sigma \) algebra of measurable sets contains the Borel sets.

**Proof:** It suffices to show that closed sets are in \( \mathcal{S} \), the \( \sigma \)-algebra of measurable sets, because then the open sets are also in \( \mathcal{S} \) and consequently \( \mathcal{S} \) contains the Borel sets. Let \( K \) be closed and let \( S \) be a subset of \( \Omega \). Is \( \mu(S) \geq \mu(S \cap K) + \mu(S \setminus K) \)? It suffices to assume \( \mu(S) < \infty \). Let

\[
K_n \equiv \{x : \text{dist}(x, K) \leq \frac{1}{n}\}
\]
By Lemma 6.1.7 on Page 145, \( x \to \text{dist} (x, K) \) is continuous and so \( K_n \) is closed. By the assumption of the theorem,

\[
\mu(S) \geq \mu((S \cap K) \cup (S \setminus K_n)) = \mu(S \cap K) + \mu(S \setminus K_n)
\]

(26.1.1) since \( S \cap K \) and \( S \setminus K_n \) are a positive distance apart. Now

\[
\mu(S \setminus K_n) \leq \mu(S \setminus K) \leq \mu(S \setminus K_n) + \mu((K_n \setminus K) \cap S). \tag{26.1.2}
\]

If \( \lim_{n \to \infty} \mu((K_n \setminus K) \cap S) = 0 \) then the theorem will be proved because this limit along with (26.1.2) implies \( \lim_{n \to \infty} \mu(S \setminus K_n) = \mu(S \setminus K) \) and then taking a limit in (26.1.1), \( \mu(S) \geq \mu(S \cap K) + \mu(S \setminus K) \) as desired. Therefore, it suffices to establish this limit.

Since \( K \) is closed, a point, \( x \notin K \) must be at a positive distance from \( K \) and so

\[
K_n \setminus K = \bigcup_{k=n}^{\infty} K_k \setminus K_{k+1}.
\]

Therefore

\[
\mu(S \cap (K_n \setminus K)) \leq \sum_{k=n}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})). \tag{26.1.3}
\]

If

\[
\sum_{k=1}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})) < \infty, \tag{26.1.4}
\]

then \( \mu(S \cap (K_n \setminus K)) \to 0 \) because it is dominated by the tail of a convergent series so it suffices to show (26.1.4).

\[
\sum_{k=1}^{M} \mu(S \cap (K_k \setminus K_{k+1})) = \\
\sum_{k \text{ even, } k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd, } k \leq M} \mu(S \cap (K_k \setminus K_{k+1})). \tag{26.1.5}
\]

By the construction, the distance between any pair of sets, \( S \cap (K_k \setminus K_{k+1}) \) for different even values of \( k \) is positive and the distance between any pair of sets, \( S \cap (K_k \setminus K_{k+1}) \) for different odd values of \( k \) is positive. Therefore,

\[
\sum_{k \text{ even, } k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd, } k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) \leq \\
\mu\left( \bigcup_{k \text{ even}} S \cap (K_k \setminus K_{k+1}) \right) + \mu\left( \bigcup_{k \text{ odd}} S \cap (K_k \setminus K_{k+1}) \right) \leq 2\mu(S) < \infty
\]

and so for all \( M, \sum_{k=1}^{M} \mu(S \cap (K_k \setminus K_{k+1})) \leq 2\mu(S) \) showing (26.1.4).

With the above theorem, the following theorem is easy to obtain.

**Theorem 26.1.5** The \( \sigma \) algebra of \( \mathcal{H}^s \) measurable sets contains the Borel sets and \( \mathcal{H}^s \) has the property that for all \( E \subseteq \mathbb{R}^n \), there exists a Borel set \( F \supseteq E \) such that \( \mathcal{H}^s(F) = \mathcal{H}^s(E) \).
Proof: Let $\text{dist}(A, B) = 2\delta_0 > 0$. Is it the case that

$$\mathcal{H}^s(A) + \mathcal{H}^s(B) = \mathcal{H}^s(A \cup B)?$$

This is what is needed to use Caratheodory’s criterion.

Let $\{C_j\}_{j=1}^{\infty}$ be a covering of $A \cup B$ such that $\text{diam}(C_j) \leq \delta < \delta_0$ for each $j$ and

$$\mathcal{H}^s_\delta(A \cup B) + \varepsilon > \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s.$$

Thus

$$\mathcal{H}^s_\delta(A \cup B) + \varepsilon > \sum_{j \in J_1} \beta(s)(r(C_j))^s + \sum_{j \in J_2} \beta(s)(r(C_j))^s$$

where

$$J_1 = \{j : C_j \cap A \neq \emptyset\}, \quad J_2 = \{j : C_j \cap B \neq \emptyset\}.$$

Recall $\text{dist}(A, B) = 2\delta_0$, $J_1 \cap J_2 = \emptyset$. It follows

$$\mathcal{H}^s_\delta(A \cup B) + \varepsilon > \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).$$

Letting $\delta \to 0$, and noting $\varepsilon > 0$ was arbitrary, yields

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).$$

Equality holds because $\mathcal{H}^s$ is an outer measure. By Caratheodory’s criterion, $\mathcal{H}^s$ is a Borel measure.

To verify the second assertion, note first there is no loss of generality in letting $\mathcal{H}^s(E) < \infty$. Let

$$E \subseteq \bigcup_{j=1}^{\infty} C_j, \quad r(C_j) < \delta,$$

and

$$\mathcal{H}^s_\delta(E) + \delta > \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s.$$

Let

$$F_\delta = \bigcup_{j=1}^{\infty} \overline{C_j}.$$

Thus $F_\delta \supseteq E$ and

$$\mathcal{H}^s_\delta(E) \leq \mathcal{H}^s_\delta(F_\delta) \leq \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s$$

$$= \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s < \delta + \mathcal{H}^s_\delta(E).$$

Let $\delta_k \to 0$ and let $F = \cap_{k=1}^{\infty} F_{\delta_k}$. Then $F \supseteq E$ and

$$\mathcal{H}^s_{\delta_k}(E) \leq \mathcal{H}^s_{\delta_k}(F) \leq \mathcal{H}^s_{\delta_k}(F_\delta) \leq \delta_k + \mathcal{H}^s_{\delta_k}(E).$$

Letting $k \to \infty$,

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(F) \leq \mathcal{H}^s(E) \blacksquare$$

A measure satisfying the conclusion of Theorem 26.1.5 is called a Borel regular measure.
26.2 \( \mathcal{H}^n \) And \( m_n \)

Next I will compare \( \mathcal{H}^n \) and \( m_n \). To do this, recall the following covering theorem which is a summary of Corollaries 11.4.4 and 11.4.5 in [11.4.5]

**Theorem 26.2.1** Let \( E \subseteq \mathbb{R}^n \) and let \( \mathcal{F} \) be a collection of balls of bounded radii such that \( \mathcal{F} \) covers \( E \) in the sense of Vitali. Then there exists a countable collection of disjoint balls from \( \mathcal{F} \), \( \{B_j\}_{j=1}^{\infty} \), such that \( m_n(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0 \).

In the next lemma, the balls are the usual balls taken with respect to the usual distance in \( \mathbb{R}^n \).

**Lemma 26.2.2** If \( m_n(S) = 0 \) then \( \mathcal{H}^n(S) = \mathcal{H}_0^n(S) = 0 \). Also, there exists a constant, \( k \) such that \( \mathcal{H}^n(E) \leq km_n(E) \) for all \( E \) Borel. Also, if \( Q_0 \equiv [0,1)^n \), the unit cube, then \( \mathcal{H}^n([0,1)^n) > 0 \).

**Proof:** Suppose first \( m_n(S) = 0 \). Without loss of generality, \( S \) is bounded. Then by outer regularity, there exists a bounded open \( V \) containing \( S \) and \( m_n(V) < \varepsilon \). For each \( x \in S \), there exists a ball \( B_x \) such that \( B_x \subseteq V \) and \( \delta > r(B_x) \). By the Vitali covering theorem there is a sequence of disjoint balls \( \{B_k\} \) such that \( \{B_k\} \) covers \( S \). Here \( B_k \) has the same center as \( B_k \) but 5 times the radius. Then letting \( \alpha(n) \) be the Lebesgue measure of the unit ball in \( \mathbb{R}^n \)

\[
\mathcal{H}_0^n(S) \leq \sum_k \beta(n) r(B_k)^n = \frac{\beta(n)}{\alpha(n)} 5^n \sum_k \alpha(n) r(B_k)^n
\]

\[
\leq \frac{\beta(n)}{\alpha(n)} 5^n m_n(V) < \frac{\beta(n)}{\alpha(n)} 5^n \varepsilon
\]

Since \( \varepsilon \) is arbitrary, this shows \( \mathcal{H}_0^n(S) = 0 \) and now it follows \( \mathcal{H}^n(S) = 0 \).

Letting \( U \) be an open set and \( \delta > 0 \), consider all balls \( B \) contained in \( U \) which have diameters less than \( \delta \). This is a Vitali covering of \( U \) and therefore by Theorem 26.2.1 there exists \( \{B_i\} \), a sequence of disjoint balls of radii less than \( \delta \) contained in \( U \) such that \( \bigcup_{i=1}^{\infty} B_i \) differs from \( U \) by a set of Lebesgue measure zero. Let \( \alpha(n) \) be the Lebesgue measure of the unit ball in \( \mathbb{R}^n \). Then from what was just shown,

\[
\mathcal{H}_0^n(U) = \mathcal{H}_0^n(\bigcup_i B_i) \leq \sum_{i=1}^{\infty} \beta(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} \alpha(n) r(B_i)^n
\]

\[
= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} m_n(B_i) = \frac{\beta(n)}{\alpha(n)} m_n(U) \equiv km_n(U), \quad k = \frac{\beta(n)}{\alpha(n)}
\]

Now letting \( E \) be Borel, it follows from the outer regularity of \( m_n \) there exists a decreasing sequence of open sets, \( \{V_i\} \) containing \( E \) such that \( m_n(V_i) \to m_n(E) \). Then from the above,

\[
\mathcal{H}_0^n(E) \leq \lim_{i \to \infty} \mathcal{H}_0^n(V_i) \leq \lim_{i \to \infty} km_n(V_i) = km_n(E).
\]
CHAPTER 26. HAUSDORFF MEASURE

Since \( \delta > 0 \) is arbitrary, it follows that also

\[
H^n(E) \leq k m_n(E).
\]

This proves the first part of the lemma.

To verify the second part, note that it is obvious \( H^n_\delta \) and \( H^n \) are translation invariant because diameters of sets do not change when translated. Therefore, if \( H^n ([0, 1]^n) = 0 \), it follows \( H^n (\mathbb{R}^n) = 0 \) because \( \mathbb{R}^n \) is the countable union of translates of \( Q_0 = [0, 1]^n \). Since each \( H^n_\delta \) is no larger than \( H^n \), the same must hold for \( H^n \). Therefore, there exists a sequence of sets, \( \{ C_i \} \) each having diameter less than \( \delta \) such that the union of these sets equals \( \mathbb{R}^n \) but

\[
1 > \sum_{i=1}^{\infty} \beta(n) r(C_i)^n.
\]

Now let \( B_i \) be a ball having radius equal to \( \text{diam}(C_i) = 2r(C_i) \) which contains \( C_i \). It follows

\[
m_n(B_i) = \alpha(n) 2^n r(C_i)^n = \alpha(n) \beta(n) r(C_i)^n
\]

which implies

\[
1 > \sum_{i=1}^{\infty} \beta(n) r(C_i)^n = \sum_{i=1}^{\infty} \frac{\beta(n)}{\alpha(n)} 2^n m_n(B_i) = \infty,
\]

a contradiction. \( \blacksquare \)

**Theorem 26.2.3** By choosing \( \beta(n) \) properly, one can obtain \( H^n = m_n \) on all Lebesgue measurable sets.

**Proof:** I will show \( H^n \) is a positive multiple of \( m_n \) for any choice of \( \beta(n) \). Define

\[
k = \frac{m_n(Q_0)}{H^n(Q_0)}
\]

where \( Q_0 = [0, 1]^n \) is the half open unit cube in \( \mathbb{R}^n \). I will show \( k H^n(E) = m_n(E) \) for any Lebesgue measurable set. When this is done, it will follow that by adjusting \( \beta(n) \) the multiple can be taken to be 1.

Let \( Q = \prod_{i=1}^{n} [a_i, a_i + 2^{-k}] \) be a half open box where \( a_i = l2^{-k} \). Thus \( Q_0 \) is the union of \( (2^k)^n \) of these identical half open boxes. By translation invariance, of \( H^n \) and \( m_n \)

\[
(2^k)^n H^n(Q) = H^n(Q_0) = \frac{1}{k} m_n(Q_0) = \frac{1}{k} (2^k)^n m_n(Q).
\]

Therefore, \( k H^n(Q) = m_n(Q) \) for any such half open box and by translation invariance, for the translation of any such half open box. It follows from Lemma \( \boxed{\text{L1.2.2}} \) that \( k H^n(U) = m_n(U) \) for all open sets. It follows immediately, since every compact set is the countable intersection of open sets that \( k H^n = m_n \) on compact sets.

984
sets. Therefore, they are also equal on all closed sets because every closed set is the countable union of compact sets. Now let $F$ be an arbitrary Lebesgue measurable set. I will show that $F$ is $\mathcal{H}^n$ measurable and that $k\mathcal{H}^n(F) = m_n(F)$. Let $F_i = B(0, l) \cap F$. Then there exists $H$ a countable union of compact sets and $G$ a countable intersection of open sets such that

$$H \subseteq F_i \subseteq G$$

(26.2.6)

and $m_n(G \setminus H) = 0$ which implies by Lemma 26.2.2

$$m_n(G \setminus H) = k\mathcal{H}^n(G \setminus H) = 0.$$  

(26.2.7)

To do this, let $\{G_i\}$ be a decreasing sequence of bounded open sets containing $F_i$ and let $\{H_i\}$ be an increasing sequence of compact sets contained in $F_i$ such that

$$k\mathcal{H}^n(G_i \setminus H_i) = m_n(G_i \setminus H_i) < 2^{-i}$$

Then letting $G = \bigcap_i G_i$ and $H = \bigcup_i H_i$ this establishes Lemma 26.2.2 and Lemma 26.2.3. Then by completeness of $\mathcal{H}^n$ it follows $F_i$ is $\mathcal{H}^n$ measurable and

$$k\mathcal{H}^n(F_i) = k\mathcal{H}^n(H) = m_n(H) = m_n(F_i).$$

Now taking $l \rightarrow \infty$, it follows $F$ is $\mathcal{H}^n$ measurable and $k\mathcal{H}^n(F) = m_n(F)$. Therefore, adjusting $\beta(n)$ it can be assumed the constant, $k$ is 1.

The exact determination of $\beta(n)$ is more technical.

26.3 Technical Considerations

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$. Thus the volume of $B(0, r)$ in $\mathbb{R}^n$ is $\alpha(n)r^n$ from the change of variables formula. There is a very important and interesting inequality known as the isodiametric inequality which says that if $A$ is any set in $\mathbb{R}^n$, then

$$\overline{m}(A) \leq \alpha(n)(2^{-1}\text{diam}(A))^n = \alpha(n)r(A)^n.$$  

This inequality may seem obvious at first but it is not really. The reason it is not is that there are sets which are not subsets of any sphere having the same diameter as the set. For example, consider an equilateral triangle.

Lemma 26.3.1 Let $f : \mathbb{R}^{n-1} \rightarrow [0, \infty)$ be Borel measurable and let

$$S = \{(x,y) : |y| < f(x)\}.$$  

Then $S$ is a Borel set in $\mathbb{R}^n$.

Proof: Set $s_k$ be an increasing sequence of Borel measurable functions converging pointwise to $f$.

$$s_k(x) = \sum_{m=1}^{N_k} c_{mk}^k \mathcal{X}_{E_k^m}(x).$$
Let
\[ S_k = \bigcup_{m=1}^{N_k} E_m^k \times (-c_m^k, c_m^k). \]
Then \((x, y) \in S_k\) if and only if \(f(x) > 0\) and \(|y| < s_k(x) \leq f(x)\). It follows that
\[ S_k \subseteq S_{k+1} \quad \text{and} \quad S = \bigcup_{k=1}^{\infty} S_k. \]
But each \(S_k\) is a Borel set and so \(S\) is also a Borel set. This proves the lemma.

Let \(P_i\) be the projection onto
\[ \text{span} \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\} \]
where the \(e_k\) are the standard basis vectors in \(\mathbb{R}^n\), \(e_k\) being the vector having a 1 in the \(k\)th slot and a 0 elsewhere. Thus \(P_i x = \sum_{j \neq i} x_j e_j\). Also let
\[ A_{P_i x} = \{x_i : (x_1, \ldots, x_i, \ldots, x_n) \in A\} \]

**Lemma 26.3.2** Let \(A \subseteq \mathbb{R}^n\) be a Borel set. Then \(P_i x \mapsto m(A_{P_i x})\) is a Borel measurable function defined on \(P_i(\mathbb{R}^n)\).

**Proof:** Let \(\mathcal{K}\) be the \(\pi\) system consisting of sets of the form \(\prod_{j=1}^{n} A_j\) where \(A_i\) is Borel. Also let \(\mathcal{G}\) denote those Borel sets of \(\mathbb{R}^n\) such that if \(A \in \mathcal{G}\) then
\[ P_i x \mapsto m((A \cap R_k)_{P_i x}) \]
is Borel measurable because it is of the form
\[ m((R_k)_{P_i x}) - m((A \cap R_k)_{P_i x}) \]
and these are Borel measurable functions of \(P_i x\). Also, if \(\{A_i\}\) is a disjoint sequence of sets in \(\mathcal{G}\) then
\[ m\left((\bigcup_i A_i \cap R_k)_{P_i x}\right) = \sum_i m((A_i \cap R_k)_{P_i x}) \]
and each function of \(P_i x\) is Borel measurable. Thus by the lemma on \(\pi\) systems, Lemma 10.12.3, \(\mathcal{G} = \mathcal{B}(\mathbb{R}^n)\) and this proves the lemma.

Now let \(A \subseteq \mathbb{R}^n\) be Borel. Let \(P_i\) be the projection onto
\[ \text{span} \{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\} \]
and as just described,
\[ A_{P_i x} = \{y \in \mathbb{R} : P_i x + ye_i \in A\} \]
26.3. TECHNICAL CONSIDERATIONS

Thus for \( x = (x_1, \ldots, x_n) \),
\[
A_{P_i x} = \{ y \in \mathbb{R} : (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in A \}.
\]
Since \( A \) is Borel, it follows from Lemma 26.3.1 that
\[
P_i x \to m(A_{P_i x})
\]
is a Borel measurable function on \( P_i \mathbb{R}^n = \mathbb{R}^{n-1} \).

26.3.1 Steiner Symmetrization

Define
\[
S(A, e_i) \equiv \{ x = P_i x + y e_i : |y| < 2^{-1} m(A_{P_i x}) \}
\]

Lemma 26.3.3 Let \( A \) be a Borel subset of \( \mathbb{R}^n \). Then \( S(A, e_i) \) satisfies
\( P_i x + y e_i \in S(A, e_i) \) if and only if \( P_i x - y e_i \in S(A, e_i) \),

\[
S(A, e_i) \text{ is a Borel set in } \mathbb{R}^n,
\]
\[
m_n(S(A, e_i)) = m_n(A), \quad (26.3.8)
\]
\[
diam(S(A, e_i)) \leq diam(A). \quad (26.3.9)
\]

Proof: The first assertion is obvious from the definition. The Borel measurability of \( S(A, e_i) \) follows from the definition and Lemmas 26.3.2 and 26.3.1. To show Formula 26.3.8.

\[
m_n(S(A, e_i)) = \int_{P_i \mathbb{R}^n} 2^{-1} m(A_{P_i x}) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n
\]
\[
= \int_{P_i \mathbb{R}^n} m(A_{P_i x}) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n
\]
\[
= m(A).
\]

Now suppose \( x_1 \) and \( x_2 \in S(A, e_i) \)
\[
x_1 = P_i x_1 + y_1 e_i, \quad x_2 = P_i x_2 + y_2 e_i.
\]

For \( x \in A \) define
\[
l(x) = \sup \{ y : P_i x + y e_i \in A \},
\]
\[
g(x) = \inf \{ y : P_i x + y e_i \in A \}.
\]

Then it is clear that
\[
l(x_1) - g(x_1) \geq m(A_{P_i x_1}) \geq 2|y_1|, \quad (26.3.10)
\]
\[
l(x_2) - g(x_2) \geq m(A_{P_i x_2}) \geq 2|y_2|. \quad (26.3.11)
\]
Claim: \( |y_1 - y_2| \leq |l(x_1) - g(x_2)| \) or \( |y_1 - y_2| \leq |l(x_2) - g(x_1)| \).

Proof of Claim: If not,

\[
2|y_1 - y_2| > |l(x_1) - g(x_2)| + |l(x_2) - g(x_1)|
\]

\[
\geq |l(x_1) - g(x_1) + l(x_2) - g(x_2)|
\]

\[
= l(x_1) - g(x_1) + l(x_2) - g(x_2).
\]

\[
\geq 2|y_1| + 2|y_2|
\]

by 26.3.10 and 26.3.11 contradicting the triangle inequality.

Now suppose \( |y_1 - y_2| \leq |l(x_1) - g(x_2)| \). From the claim,

\[
|x_1 - x_2| = (|P_i x_1 - P_i x_2|^2 + |y_1 - y_2|^2)^{1/2}
\]

\[
\leq (|P_i x_1 - P_i x_2|^2 + |l(x_1) - g(x_2)|^2)^{1/2}
\]

\[
\leq (|P_i x_1 - P_i x_2|^2 + (|z_1 - z_2| + 2\varepsilon)^2)^{1/2}
\]

\[
\leq \text{diam}(A) + O(\sqrt{\varepsilon})
\]

where \( z_1 \) and \( z_2 \) are such that \( P_i x_1 + z_i e_i \in A, P_i x_2 + z_2 e_i \in A \), and \( |z_1 - l(x_1)| < \varepsilon \) and \( |z_2 - g(x_2)| < \varepsilon \).

If \( |y_1 - y_2| \leq |l(x_2) - g(x_1)| \), then we use the same argument but let \( |z_1 - g(x_1)| < \varepsilon \) and \( |z_2 - l(x_2)| < \varepsilon \).

Since \( x_1, x_2 \) are arbitrary elements of \( S(A, e_i) \) and \( \varepsilon \) is arbitrary, this proves Lemma 26.3.4.

The next lemma says that if \( A \) is already symmetric with respect to the \( j \)th direction, then this symmetry is not destroyed by taking \( S(A, e_i) \).

Lemma 26.3.4 Suppose \( A \) is a Borel set in \( \mathbb{R}^n \) such that \( P_j x + e_j x_j \in A \) if and only if \( P_j x + (-x_j) e_j \in A \). Then if \( i \neq j \), \( P_j x + e_j x_j \in S(A, e_i) \) if and only if \( P_j x + (-x_j) e_j \in S(A, e_i) \).

Proof: By definition,

\[
P_j x + e_j x_j \in S(A, e_i)
\]

if and only if

\[
|x_i| < 2^{-1} m(A_{P_i(P_j x + e_j x_j)}).
\]

Now

\[
x_i \in A_{P_i(P_j x + e_j x_j)}
\]

if and only if

\[
x_i \in A_{P_i(P_j x + (-x_j) e_j)}
\]

by the assumption on \( A \) which says that \( A \) is symmetric in the \( e_j \) direction. Hence

\[
P_j x + e_j x_j \in S(A, e_i)
\]

if and only if

\[
|x_i| < 2^{-1} m(A_{P_i(P_j x + (-x_j) e_j)}).
\]

This proves the lemma.
26.3.2 The Isodiametric Inequality

The next theorem is called the isodiametric inequality. It is the key result used to compare Lebesgue and Hausdorff measures.

**Theorem 26.3.5** Let $A$ be any Lebesgue measurable set in $\mathbb{R}^n$. Then

$$m_n(A) \leq \alpha(n)(r(A))^n.$$ 

**Proof:** Suppose first that $A$ is Borel. Let $A_1 = S(A, e_1)$ and let $A_k = S(A_{k-1}, e_k)$. Then by the preceding lemmas, $A_n$ is a Borel set, $\text{diam}(A_n) \leq \text{diam}(A)$, $m_n(A_n) = m_n(A)$, and $A_n$ is symmetric. Thus $x \in A_n$ if and only if $-x \in A_n$. It follows that $A_n \subseteq B(0, r(A_n))$.

(If $x \in A_n \setminus B(0, r(A_n))$, then $-x \in A_n \setminus B(0, r(A_n))$ and so $\text{diam}(A_n) \geq 2|x| > \text{diam}(A_n)$.) Therefore,

$$m_n(A_n) \leq \alpha(n)(r(A_n))^n \leq \alpha(n)(r(A))^n.$$

It remains to establish this inequality for arbitrary measurable sets. Letting $A$ be such a set, let $\{K_n\}$ be an increasing sequence of compact subsets of $A$ such that

$$m(A) = \lim_{k \to \infty} m(K_k).$$

Then

$$m(A) = \lim_{k \to \infty} m(K_k) \leq \limsup_{k \to \infty} \alpha(n)(r(K_k))^n \\
\leq \alpha(n)(r(A))^n.$$ 

This proves the theorem.

26.4 The Proper Value Of $\beta(n)$

I will show that the proper determination of $\beta(n)$ is $\alpha(n)$, the volume of the unit ball. Since $\beta(n)$ has been adjusted such that $k = 1$, $m_n(B(0, 1)) = H^n(B(0, 1))$. There exists a covering of $B(0, 1)$ of sets of radii less than $\delta, \{C_i\}_{i=1}^\infty$ such that

$$\mathcal{H}^n_\delta(B(0, 1)) + \varepsilon > \sum_i \beta(n) r(C_i)^n$$

Then by Theorem 26.3.3, the isodiametric inequality,

$$\mathcal{H}^n_\delta(B(0, 1)) + \varepsilon > \sum_i \beta(n) r(C_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_i \alpha(n) r(C_i)^n \\
\geq \frac{\beta(n)}{\alpha(n)} \sum_i m_n(C_i) \geq \frac{\beta(n)}{\alpha(n)} m_n(B(0, 1)) = \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(0, 1)).$$
Now taking the limit as $\delta \to 0$,
\[
\mathcal{H}^n (B(0,1)) + \varepsilon \geq \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n (B(0,1))
\]
and since $\varepsilon > 0$ is arbitrary, this shows $\alpha(n) \geq \beta(n)$.

By the Vitali covering theorem, there exists a sequence of disjoint balls, $\{B_i\}$ such that
\[
B(0,1) = (\bigcup_{i=1}^{\infty} B_i) \cup N
\]
where $m_n(N) = 0$. Then $\mathcal{H}^n_\delta (N) = 0$ can be concluded because $\mathcal{H}^n_\delta \leq \mathcal{H}^n$ and Lemma 26.2.2. Using $m_n(B(0,1)) = \mathcal{H}^n(B(0,1))$ again,
\[
\mathcal{H}^n_\delta (B(0,1)) = \mathcal{H}^n_\delta (\bigcup_i B_i) \leq \sum_{i=1}^{\infty} \beta(n) r(B_i)^n
\]
which implies $\alpha(n) \leq \beta(n)$ and so the two are equal. This proves that if $\alpha(n) = \beta(n)$, then the $\mathcal{H}^n = m_n$ on the measurable sets of $\mathbb{R}^n$.

This gives another way to think of Lebesgue measure which is a particularly nice way because it is coordinate free, depending only on the notion of distance.

For $s < n$, note that $\mathcal{H}^s$ is not a Radon measure because it will not generally be finite on compact sets. For example, let $n = 2$ and consider $\mathcal{H}^1(L)$ where $L$ is a line segment joining $(0,0)$ to $(1,0)$. Then $\mathcal{H}^1(L)$ is no smaller than $\mathcal{H}^1(L)$ when $L$ is considered a subset of $\mathbb{R}^1, n = 1$. Thus by what was just shown, $\mathcal{H}^1(L) \geq 1$. Hence $\mathcal{H}^1([0,1] \times [0,1]) = \infty$. The situation is this: $L$ is a one-dimensional object inside $\mathbb{R}^2$ and $\mathcal{H}^1$ is giving a one-dimensional measure of this object. In fact, Hausdorff measures can make such heuristic remarks as these precise. Define the Hausdorff dimension of a set, $A$, as

\[
\dim(A) = \inf\{s : \mathcal{H}^s(A) = 0\}
\]

### 26.4.1 A Formula For $\alpha(n)$

What is $\alpha(n)$? Recall the gamma function which makes sense for all $p > 0$.
\[
\Gamma(p) \equiv \int_0^\infty e^{-t} t^{p-1} dt.
\]

**Lemma 26.4.1** The following identities hold.
\[
p\Gamma(p) = \Gamma(p + 1),
\]
26.4. THE PROPER VALUE OF $\beta(N)$

$$\Gamma(p)\Gamma(q) = \left(\int_0^1 x^{p-1}(1-x)^{q-1}dx\right)\Gamma(p+q),$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

**Proof**: Using integration by parts,

$$\Gamma(p+1) = \int_0^\infty e^{-tx}dt = -e^{-tp}\big|_0^\infty + p\int_0^\infty e^{-tp}dt = p\Gamma(p)$$

Next

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-tx}dt \int_0^\infty e^{-sy}ds$$

$$= \int_0^\infty \int_0^\infty e^{-(t+s)xy}dsdt$$

$$= \int_0^\infty \int_0^\infty e^{-y(x-s)^{p-1}s^{q-1}}duds$$

$$= \int_0^\infty \int_0^u e^{-y(x-s)^{p-1}s^{q-1}}dsdu$$

$$= \int_0^\infty \int_0^1 e^{-y(u-ux)^{p-1}(ux)^{q-1}}udxdu$$

$$= \int_0^\infty \int_0^1 e^{-yux^{p+q-1}(1-x)^{p-1}x^{q-1}dx}$$

$$= \Gamma(p+q)\left(\int_0^1 x^{p-1}(1-x)^{q-1}dx\right).$$

It remains to find $\Gamma\left(\frac{1}{2}\right)$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-tx^{1/2}}dt = \int_0^\infty e^{-u^2}du = 2\int_0^\infty e^{-u^2}du$$

Now

$$\left(\int_0^\infty e^{-x^2}dx\right)^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$$

$$= \int_0^{\pi/2} \int_0^\infty e^{-r^2}rdrdr = \frac{1}{4}\pi$$

and so

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}du = \sqrt{\pi}$$

This proves the lemma.

Next let $n$ be a positive integer.
Theorem 26.4.2 \( \alpha(n) = \pi^{n/2}(\Gamma(n/2 + 1))^{-1} \) where \( \Gamma(s) \) is the gamma function

\[
\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt.
\]

Proof: First let \( n = 1 \).

\[
\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.
\]

Thus \( \pi^{1/2}(\Gamma(1/2 + 1))^{-1} = \frac{2}{\sqrt{\pi}} \sqrt{\pi} = 2 = \alpha(1) \).

and this shows the theorem is true if \( n = 1 \).

Assume the theorem is true for \( n \) and let \( B_{n+1} \) be the unit ball in \( \mathbb{R}^{n+1} \). Then by the result in \( \mathbb{R}^n \),

\[
m_{n+1}(B_{n+1}) = \int_{-1}^{1} \alpha(n)(1 - x_{n+1}^2)^{n/2}dx_{n+1}
\]

\[
= 2\alpha(n) \int_{0}^{1} (1 - t^2)^{n/2}dt.
\]

Doing an integration by parts and using Lemma 26.4.1

\[
= 2\alpha(n) n \int_{0}^{1} t^2(1 - t^2)^{(n-2)/2}dt
\]

\[
= 2\alpha(n) n \int_{0}^{1} u^{1/2}(1 - u)^{n/2-1}du
\]

\[
= n\alpha(n) \int_{0}^{1} u^{3/2-1}(1 - u)^{n/2-1}du
\]

\[
= n\alpha(n) \Gamma(3/2)\Gamma(n/2)(\Gamma((n+3)/2))^{-1}
\]

\[
= n\pi^{n/2}(\Gamma(n/2 + 1))^{-1}(\Gamma((n + 3)/2))^{-1}\Gamma(3/2)\Gamma(n/2)
\]

\[
= n\pi^{n/2}(\Gamma(n/2)(n/2))^{-1}(\Gamma((n + 1)/2 + 1))^{-1}\Gamma(3/2)\Gamma(n/2)
\]

\[
= 2\pi^{n/2}\Gamma(3/2)(\Gamma((n + 1)/2 + 1))^{-1}
\]

\[
= \pi(n+1/2)(\Gamma((n + 1)/2 + 1))^{-1}.
\]

This proves the theorem.

From now on, in the definition of Hausdorff measure, it will always be the case that \( \beta(s) = \alpha(s) \). As shown above, this is the right thing to have \( \beta(s) \) to equal if \( s \) is a positive integer because this yields the important result that Hausdorff measure is the same as Lebesgue measure. Note the formula, \( \pi^{s/2}(\Gamma(s/2 + 1))^{-1} \) makes sense for any \( s \geq 0 \).
26.4. THE PROPER VALUE OF $\beta(N)$

26.4.2 Hausdorff Measure And Linear Transformations

Hausdorff measure makes possible a unified development of $n$ dimensional area. As in the case of Lebesgue measure, the first step in this is to understand basic considerations related to linear transformations. Recall that for $L \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^l)$, $L^*$ is defined by

$$(Lu, v) = (u, L^*v).$$

Also recall Theorem 4.13.6 on Page 99 which is stated here for convenience. This theorem says you can write a linear transformation as the composition of two linear transformations, one which preserves length and the other which distorts, the right polar decomposition. The one which distorts is the one which will have a nontrivial interaction with Hausdorff measure while the one which preserves lengths does not change Hausdorff measure. These ideas are behind the following theorems and lemmas.

**Theorem 26.4.3** Let $F$ be an $n \times m$ matrix where $m \geq n$. Then there exists an $m \times n$ matrix $R$ and an $n \times n$ matrix $U$ such that

$$F = RU, \quad U = U^*,$$

all eigenvalues of $U$ are non negative,

$$U^2 = F^*F, \quad R^*R = I,$$

and $|Rx| = |x|$.

**Lemma 26.4.4** Let $R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, and $R^*R = I$. Then if $A \subseteq \mathbb{R}^n$,

$$\mathcal{H}^n(RA) = \mathcal{H}^n(A).$$

In fact, if $P : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $|Px - Py| = |x - y|$, then

$$\mathcal{H}^n(PA) = \mathcal{H}^n(A).$$

**Proof:** Note that

$$|R(x - y)|^2 = (R(x - y), R(x - y)) = (R^*R(x - y), x - y) = |x - y|^2$$

Thus $R$ preserves lengths.

Now let $P$ be an arbitrary mapping which preserves lengths and let $A$ be bounded, $P(A) \subseteq \bigcup_{j=1}^{\infty} C_j$, $r(C_j) < \delta$, and

$$\mathcal{H}_\delta^n(PA) + \varepsilon > \sum_{j=1}^{\infty} a(n)(r(C_j))^n.$$
Since $P$ preserves lengths, it follows $P$ is one to one on $P (\mathbb{R}^n)$ and $P^{-1}$ also preserves lengths on $P (\mathbb{R}^n)$. Replacing each $C_j$ with $C_j \cap (PA)$,

$$
\mathcal{H}^n_\delta (PA) + \varepsilon > \sum_{j=1}^{\infty} \alpha(n) r(C_j \cap (PA))^n
$$

$$
= \sum_{j=1}^{\infty} \alpha(n) r(P^{-1}(C_j \cap (PA)))^n
$$

$$
\geq \mathcal{H}^n_\delta (A).
$$

Thus $\mathcal{H}^n_\delta (PA) \geq \mathcal{H}^n_\delta (A)$.

Now let $A \subseteq \bigcup_{j=1}^{\infty} C_j$, $\text{diam}(C_j) \leq \delta$, and

$$
\mathcal{H}^n_\delta (A) + \varepsilon \geq \sum_{j=1}^{\infty} \alpha(n) (r(C_j))^n
$$

Then

$$
\mathcal{H}^n_\delta (A) + \varepsilon \geq \sum_{j=1}^{\infty} \alpha(n) (r(C_j))^n
$$

$$
= \sum_{j=1}^{\infty} \alpha(n) (r(PC_j))^n
$$

$$
\geq \mathcal{H}^n_\delta (PA).
$$

Hence $\mathcal{H}^n_\delta (PA) = \mathcal{H}^n_\delta (A)$. Letting $\delta \to 0$ yields the desired conclusion in the case where $A$ is bounded. For the general case, let $A_r = A \cap B(0, r)$. Then $\mathcal{H}^n(PA_r) = \mathcal{H}^n(A_r)$. Now let $r \to \infty$. ■

**Lemma 26.4.5** Let $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, and let $F = RU$ where $R$ and $U$ are described in Theorem 4.13.6 on Page 99. Then if $A \subseteq \mathbb{R}^n$ is Lebesgue measurable,

$$
\mathcal{H}^n(FA) = \det(U)m_n(A).
$$

**Proof:** Using Theorem 4.13.7 on Page 339 and Theorem 26.2.3

$$
\mathcal{H}^n(FA) = \mathcal{H}^n(RU A)
$$

$$
= \mathcal{H}^n(UA) = m_n(UA) = \det(U)m_n(A). ■
$$

**Definition 26.4.6** Define $J$ to equal $\det(U)$. Thus

$$
J = \det((F^*F)^{1/2}) = (\det(F^*F))^{1/2}.
$$
26.5 The Area Formula

I am very grateful to those who have found errors in earlier versions of this material, some which were quite egregious. I was unlikely to find these mistakes because I never teach this material and do not use it in my research. If there are still errors, it is my fault.

26.5.1 Preliminary Results

It was shown in Lemma 26.4.5 that

\[ \mathcal{H}^n(FA) = \det(U)m_n(A) \]

where \( F = RU \) with \( R \) preserving distances and \( U \) a symmetric matrix having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where \( F \) is replaced by a nonlinear mapping, \( h \). It contains as a special case the earlier change of variables formula. There are two parts to this development. The first part is to generalize Lemma 26.4.5 to the case of nonlinear maps. When this is done, the area formula can be presented.

In the first part of this, \( h \) will be a Lipschitz function,

\[ |h(x) - h(y)| \leq K|x - y| \]

deﬁned on \( \mathbb{R}^n \) which is one to one on \( G \), a measurable subset of \( \mathbb{R}^n \). This is no loss of generality because of Theorem 24.5.4.

The following lemma states that Lipschitz maps take sets of measure zero to sets of measure zero.

**Lemma 26.5.1** If \( h \) is Lipschitz and \( m_n(T) = 0 \), then

\[ \mathcal{H}^n(h(T)) = 0 \]

**Proof:** Let \( \varepsilon > 0 \) be given. By outer regularity, there exists an open set \( V \) containing \( T \) such that \( m_n(V) < \frac{\varepsilon}{K^n} \). For \( x \in T \) it follows there exists \( r_x < 1 \) such that the ball centered at \( x \) with radius \( 5r_x \) is contained in \( V \) and \( 5Kr_x < \delta \) for \( \delta > 0 \). Here \( K \) is the Lipschitz constant of \( h \). Then by the Vitali covering theorem, there are disjoint balls \( \{ B(x_i, r_i) \} \) such that the enlarged balls \( \hat{B}_i \) having ﬁve times the radius cover \( T \), each being contained in \( V \). Then

\[ \mathcal{H}^\delta(h(T)) \leq \mathcal{H}^\delta(h(\bigcup_{i=1}^\infty \hat{B}_i)) \leq \mathcal{H}^\delta(\bigcup_{i=1}^\infty h(\hat{B}_i)) \]

\[ \leq \sum_{i=1}^\infty \mathcal{H}^\delta(h(\hat{B}_i)) \leq \sum_{i=1}^\infty \mathcal{H}^\delta(B(h(x_i), 5Kr_{x_i})) \]
11.4.6

336

26.5.1

on Page

26.5.1

says that

Let

If

h

and so it is also true that

H

comes from

Lemma

h

so

m

F

there exists a set

F

come from some norm, open or not, but having bounded radii such that

Hence,

δ

larger than

there are disjoint balls

\{F_i\}

such open balls having radius less than

δ

Then

h

Lemma 26.5.3

having disjoint closures, denoted by

\{F_i\}

in the sense of Vitali. Then there exists a countable collection of balls from

Corollary 26.5.2

Since

ε > 996

CHAPTER 26. HAUSDORFF MEASURE

point of

S

S

containing

S

which is the countable union of compact sets and so it is Borel. By Lemma

n

measurable. Also, if

h

is Lipschitz then

h(S)

is

H^n

measurable. Also, if

h

is Lipschitz with constant

K,

\[ H^n(h(S)) \leq K^n m_n(S) \]

Proof: Let

S_k = S \cap B(0, k), k \in \mathbb{N}.

By inner regularity of Lebesgue measure, there exists a set

F

, which is the countable union of compact sets and a set

T

with

m_n(T) = 0

such that

F \cup T = S_k.

Then

h(F) \subseteq h(S_k) \subseteq h(F) \cup h(T).

By continuity of

h

, \( h(F) \) is a countable union of compact sets and so it is Borel. By Lemma 26.5.1,

\[ H^n(h(T)) = 0 \]

and so \( h(S_k) \) is \( H^n \) measurable because of completeness of Hausdorff measure, which comes from \( H^n \) being obtained from an outer measure. Now \( h(S) = \bigcup_{k=1}^{\infty} h(S_k) \) and so it is also true that \( h(S) \) is \( H^n \) measurable.

Consider the estimate. Assume first that \( S \subseteq B(0, R) \). Let \( V \) be an open set containing \( S \) which is contained in \( B(0, R) \) such that \( m_n(V \setminus S) < \varepsilon \). Then each point of \( S \) is contained in an open ball which is contained in \( V \). Considering all such open balls having radius less than \( \frac{\delta}{K} \), this forms a Vitali cover of \( S \). Therefore, there are disjoint balls \( \{B_i\} \) centered at points of \( S \) such that \( m_n(S \setminus \bigcup_i B_i) = 0 \). Lemma 26.5.1 says that \( H^n(h(S \setminus \bigcup_i B_i)) = 0 \). Thus \( h(B_i) \) is a set of radius no larger than \( \delta \). Also,

\[ h(S) = h(S \setminus \bigcup_i B_i) + h(\bigcup_i B_i) \]

Hence,

\[ H_\delta^n(h(S)) \leq H_\delta^n(h(S \setminus \bigcup_i B_i)) + \sum_i \beta(n) r(h(B_i))^n \]

\[ \leq H_\delta^n(h(S \setminus \bigcup_i B_i)) + \sum_i \beta(n) K^n r(B_i)^n \]

\[ \leq H_\delta^n(h(S \setminus \bigcup_i B_i)) + K^n m_n(V) \]
Then take the limit as $\delta \to 0$ to obtain
\[
\mathcal{H}^n (\mathcal{h} (S)) \leq K^n \mathcal{m}_n (V) \leq K^n \mathcal{m}_n (S) + \varepsilon
\]

Since $\varepsilon$ is arbitrary, this verifies the inequality. For the general case,
\[
\mathcal{H}^n (\mathcal{h} (S)) = \lim_{m \to \infty} \mathcal{H}^n (\mathcal{h} (S \cap B (0, m))) \\
\leq \liminf_{m \to \infty} K^n \mathcal{m}_n (S \cap B (0, m)) = K^n \mathcal{m}_n (S)
\]

By Theorem 4.13.6 on Page 99, when $D\mathcal{h} (x)$ exists,
\[
D\mathcal{h} (x) = R (x) U (x)
\]

where $(U (x) u, v) = (U (x) v, u), (U (x) u, u) \geq 0$ and $R^* R = I$ so $R$ preserves lengths. This convention will be used in what follows.

Lemma 26.5.4 In this situation, $|R^* u| \leq |u|$.

Proof: First note that
\[
(u - RR^* u, RR^* u) = (u, RR^* u) - |RR^* u|^2 \\
= |R^* u|^2 - |R^* u|^2 = 0,
\]

and so
\[
|u|^2 = |u - RR^* u + RR^* u|^2 \\
= |u - RR^* u|^2 + |RR^* u|^2 \\
= |u - RR^* u|^2 + |R^* u|^2. \quad \blacksquare
\]

As discussed earlier, there is a convenient estimate involving Lipschitz maps.

Lemma 26.5.5 If $|Px - Py| \leq L|x - y|$, then for $E$ a set in $\mathbb{R}^n$,
\[
\mathcal{H}^n (PE) \leq L^n \mathcal{H}^n (E).
\]

Proof: Without loss of generality, assume $\mathcal{H}^n (E) < \infty$. Let $\delta > 0$ and let $(C_i)_{i=1}^\infty$ be a covering of $E$ such that $r (C_i) \leq \delta$ for each $i$ and
\[
\sum_{i=1}^\infty \alpha (n) r (C_i)^n \leq \mathcal{H}^n_\delta (E) + \varepsilon.
\]

Then $(PC_i)_{i=1}^\infty$ is a covering of $PE$ such that $r (PC_i) \leq L\delta$. Therefore,
\[
\mathcal{H}^{n\delta} (PE) \leq \sum_{i=1}^\infty \alpha (n) r (PC_i)^n \\
\leq L^n \sum_{i=1}^\infty \alpha (n) r (C_i)^n \leq L^n \mathcal{H}^n_\delta (E) + L^n \varepsilon \\
\leq L^n \mathcal{H}^n (E) + L^n \varepsilon.
Letting $\delta \to 0$,

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E) + L^n \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the Lemma. ■

Then the following corollary follows from Lemma 26.5.4.

**Corollary 26.5.6** Let $T \subseteq \mathbb{R}^m$. Then

$$\mathcal{H}^n(T) \geq \mathcal{H}^n(R^nT).$$

**a decomposition**

First is a simple lemma which is fairly interesting.

**Lemma 26.5.7** Let $S,T$ be $n \times n$ matrices which are invertible. Then

$$o(Tv) = o(Sv) = o(v)$$

and if $L$ is a continuous linear transformation such that for $a < b$,

$$\sup_{v \neq 0} \frac{|Lv|}{|Sv|} < b, \quad \inf_{v \neq 0} \frac{|Lv|}{|Sv|} > a$$

If $\|S - T\|$ is small enough, it follows that the same inequalities hold with $S$ replaced with $T$. Here $\|\cdot\|$ denotes the operator norm.

**Proof:** Consider the first claim. For

$$|v| = |T^{-1}Tv| \leq \|T^{-1}\||Tv|,$$

$$|Tv| \leq \|T\||v|$$

and so

$$\frac{1}{\|T\|} |Tv| \leq |v| \leq \|T^{-1}\||Tv|$$

so $|v| \to 0$ is the same as saying that $|Tv| \to 0$. Similar considerations apply to $S$. Thus the first claim is clearly true.

Consider the second claim. To say that $\|S - T\|$ is small is the same as saying that $\|S - T\|_F$ is small where this refers to the Frobenius norm in which the $n \times n$ matrix is regarded as an element of $\mathbb{R}^{n^2}$, with the Euclidean norm, since all norms are equivalent. Then

$$|Sv| = |ST^{-1}Tv| \leq \|ST^{-1}\||Tv|$$

$$|Tv| = |TS^{-1}Sv| \leq \|TS^{-1}\||Sv|$$

Hence

$$\frac{1}{\|TS^{-1}\|} |Tv| \leq |Sv| \leq \|ST^{-1}\||Tv|$$
Thus the second result follows if it is the case that for $\|T - S\|$ sufficiently small, both $\|TS^{-1}\|$, $\|ST^{-1}\|$ close to 1. This is because there is $\hat{a} > a$ such that

$$\inf_{v \neq 0} \frac{|Lv|}{|Sv|} > \hat{a} > a$$

provided $\|TS^{-1}\|$ is close enough to 1. The other inequality is similar.

From the formula for the inverse in terms of the determinant, the entries of $T^{-1}$ are close to the corresponding entries of $S^{-1}$ provided $\|T - S\|$ is small enough. Thus by continuity,

$$\|ST^{-1} - I\|_F, \|TS^{-1} - I\|_F$$

and consequently $\|ST^{-1}\|$, $\|TS^{-1}\|$ are both close to 1, the operator norm of $I$. This is because being close to $I$ in the Frobenius norm is equivalent to being close to $I$ in the operator norm.

The following is a simplified version of an argument in [13]. In what follows, it is assumed also that $h$ is one to one. In particular, we assume the following:

$h$ is one to one on $G$ a measurable subset of $\mathbb{R}^n$ \hspace{1cm} (26.5.12)

$$Dh(x) \text{ exists at } a.e. x \in G \text{ say at all } x \in A \subseteq G \hspace{1cm} (26.5.13)$$

By Rademacher’s theorem, these conditions are satisfied if $h : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz and this situation is the one of main interest here. The conditions \hspace{1cm} (26.5.12) - \hspace{1cm} (26.5.13) could likely be obtained in other situations also.

For $x \in A$, let $Dh(x) \equiv R(x)U(x)$ where $R(x)$ preserves lengths and $U(x) \equiv (Dh(x)^T Dh(x))^{1/2}$. Let $A^+$ denote those points of $A$ for which $U(x)^{-1}$ exists. Thus this is a measurable subset of $A$.

Let $B$ be a Lebesgue measurable subset of $A^+$ and let $b \in B$. Let $S$ be a countable dense subset of the space of symmetric invertible matrices and let $C$ be a countable dense subset of $B$. Let $t > 1$ and let $\varepsilon$ be so small that

$$\frac{1}{t} + \varepsilon < 1 < t - \varepsilon$$

Now since $U(b)$ is invertible, Lemma \hspace{1cm} (26.5.7) implies $o(a - b) = o(U(b)(a - b))$ and so

$$|h(a) - h(b) - Dh(b)(a - b)| < \varepsilon |U(b)(a - b)| \hspace{1cm} (26.5.14)$$

provided that $a \in B(b, \frac{2}{t})$ assuming $i$ is sufficiently large. In addition to this,

$$\sup_{v \neq 0} \frac{|Dh(b)v|}{|U(b)v|} = \sup_{v \neq 0} \frac{|R(b)U(b)v|}{|U(b)v|} = \sup_{v \neq 0} \frac{|U(b)v|}{|U(b)v|} < (t - \varepsilon)$$

$$\inf_{v \neq 0} \frac{|Dh(b)v|}{|U(b)v|} = \inf_{v \neq 0} \frac{|R(b)U(b)v|}{|U(b)v|} = \inf_{v \neq 0} \frac{|U(b)v|}{|U(b)v|} > \frac{1}{t} + \varepsilon \hspace{1cm} (26.5.15)$$
By Lemma 26.5.7, the inequalities 26.5.14 continue to hold if $U(b)$ is replaced by another linear one to one and onto symmetric mapping $T$ provided $T$ is sufficiently close to $U(b)$. Let $T \in S$ be such a linear transformation. Also let $i$ be large enough that for all $a \in B(b, \frac{2}{i})$,

$$|h(a) - h(b) - Dh(b)(a - b)| < \varepsilon |T(a - b)|$$  \hspace{1cm} (26.5.16)

Now let $c \in C$ be close enough to $b$ that $b \in B(c, \frac{1}{i})$. Thus $b \in E(T, c, i)$ where for $i \in \mathbb{N}, c \in C, T \in S, E(T, c, i)$ consists of those $b \in B(c, \frac{1}{i})$ such that for all $a \in B(b, \frac{2}{i})$ 26.5.16 holds and also

$$\inf_{v \neq 0} \frac{|Dh(b)v|}{|Tv|} > \frac{1}{t} + \varepsilon, \quad \sup_{v \neq 0} \frac{|Dh(b)v|}{|Tv|} < (t - \varepsilon)$$  \hspace{1cm} (26.5.17)

Thus there are countably many of these sets $E(T, c, i)$, some may be empty but as just shown, their union includes all of $B$. They are Borel measurable sets because $b \to Dh(b)$ is Borel measurable.

For $a, b \in E(T, c, i)$ it follows from 26.5.11

$$|h(a) - h(b)| < |Dh(b)(a - b)| + \varepsilon |T(a - b)| < |T(a - b)|$$

and also

$$\left(\frac{1}{t} + \varepsilon\right) |T(a - b)| < |Dh(b)(a - b)| < |h(a) - h(b)| + \varepsilon |T(a - b)|$$

and so

$$t |T(a - b)| > |h(a) - h(b)| > \frac{1}{t} |T(a - b)|$$

It follows from this that

$$|h(T^{-1}(x)) - h(T^{-1}(y))| \leq t |x - y|$$  \hspace{1cm} (26.5.18)

$$\frac{1}{t} |T(h^{-1}(x)) - T(h^{-1}(y))| \leq |x - y|$$  \hspace{1cm} (26.5.19)

Here the functions are defined on the appropriate sets, $T(E(T, c, i))$ in the first inequality and $h(E(T, c, i))$ in the second.

Now let $(E_k, T_k)$ result from a disjoint union of measurable subsets of the countably many $E(T, c, i)$ such that $B = \cup_k E_k$. Thus the above Lipschitz conditions hold for $T_k$ in place of $T$. This proves most of the following lemma.
**Lemma 26.5.8** There are disjoint measurable sets $E_k$ whose union equals $B$ and symmetric linear transformations $T_k$ such that

\[
|h(T_k^{-1}(x)) - h(T_k^{-1}(y))| \leq t|x - y| \tag{26.5.20}
\]

\[
|T_k(h^{-1}(x)) - T_k(h^{-1}(y))| \leq t|x - y| \tag{26.5.21}
\]
on $T_k(E_k)$ and $h(E_k)$ respectively. Also, for $b \in E_k$

\[
\left(\frac{1}{t + \varepsilon}\right) |T_k v| < |Dh(b)v| = |U(b)v| < (t - \varepsilon)|T_k v| \tag{26.5.22}
\]

One can also conclude that for $b \in E_k$,

\[
t^{-n} |\det(T_k)| \leq \det(U(b)) \leq t^n |\det(T_k)| \tag{26.5.23}
\]

**Proof:** It only remains to verify the last claim. However, this follows right away from 26.5.22. This formula implies that

\[
\frac{1}{t^n} |v| < |U(b) T_k^{-1} v| < t^n |v|
\]

and so

\[
B\left(0, \frac{1}{t}\right) \subseteq U(b) T_k^{-1} B(0,1) \subseteq B(0, t)
\]

This implies

\[
\alpha(n) \frac{1}{t^n} \leq \det(U(b) T_k^{-1}) \leq \alpha(n) t^n
\]

and so

\[
\frac{1}{t^n} |\det(T_k)| \leq \det(U(b)) \leq t^n |\det(T_k)|
\]

This lemma, along with Lemma 26.5.1 about the relationship between Hausdorff measure and Lipschitz mappings, implies the following.

\[
\mathcal{H}^n(h(E_k)) = \mathcal{H}^n(h \circ T_k^{-1}(T_k(E_k))) \leq t^n \mathcal{H}^n(T_k(E_k)) = t^n m_n(T_k(E_k))
\]

\[
m_n(T_k(E_k)) = m_n((T_k \circ h^{-1}(h(E_k)))) \leq t^n \mathcal{H}^n(h(E_k))
\]

Summarizing,

\[
t^n \mathcal{H}^n(h(E_k)) \geq m_n(T_k(E_k)) \geq t^{-n} \mathcal{H}^n(h(E_k))
\]

Then the above inequality and 26.5.16 implies the following.

\[
t^{-2n} \mathcal{H}^n(h(E_k)) \leq t^{-n} m_n(T_k(E_k)) \leq t^{-n} |\det(T_k)| m_n(E_k)
\]

\[
\leq \int_{E_k} \det(U(x)) dm_n \leq t^n |\det(T_k)| m_n(E_k) = t^n m_n(T_k E_k) \leq t^{2n} \mathcal{H}^n(h(E_k))
\]
Summing over all $E_k$ yields the following thanks to the assumption that $h$ is one to one.

$$t^{-2n} \mathcal{H}^n(h(B)) \leq \int_B \det(U(x)) \, dx \leq t^{2n} \mathcal{H}^n(h(B))$$

Now $B$ was completely arbitrary. Let it equal $B(x, r) \cap A^+$ where $x \in A^+$. Then

$$t^{-2n} \mathcal{H}^n(h(B(x, r) \cap A^+)) \leq \int_{B(x, r)} X_{A^+}(y) \det(U(y)) \, dm_n(y) \leq t^{2n} \mathcal{H}^n(h(B(x, r) \cap A^+))$$

Divide by $m_n(B(x, r))$ and use the fundamental theorem of calculus. This yields that for $x$ off a set of measure zero,

$$t^{-2n} \lim_{r \to 0^+} \sup \mathcal{H}^n(h(B(x, r) \cap A^+)) \leq X_{A^+}(x) \det(U(x)) \leq t^{2n} \lim_{r \to 0^+} \inf \mathcal{H}^n(h(B(x, r) \cap A^+))$$

However, $t > 1$ was completely arbitrary and this shows that off a set of measure zero,

$$\lim_{r \to 0^+} \frac{\mathcal{H}^n(h(B(x, r) \cap A^+))}{m_n(B(x, r))} = X_{A^+}(x) \det(U(x)).$$

This has proved the following lemma.

**Lemma 26.5.9** There is a set of measure zero $N$ such that for $x \in A^+ \setminus N$,

$$\lim_{r \to 0^+} \frac{\mathcal{H}^n(h(B(x, r) \cap A^+))}{m_n(B(x, r))} = \det(U(x)).$$

The next theorem removes the assumption that $U(x)^{-1}$ exists. From now on

$$J_*(x) \equiv \det(U(x)).$$

**Theorem 26.5.10** Let $h : U \to \mathbb{R}^m$ for $n \leq m$, $U$ an open set in $\mathbb{R}^n$, and suppose $h$ Lipschitz. Then for a.e. $x \in A$, the set in $G$ where $Dh(x)$ exists,

$$J_*(x) = \lim_{r \to 0} \frac{\mathcal{H}^n(h(B(x, r) \cap A))}{m_n(B(x, r))},$$

(26.5.24)

where $J_*(x) \equiv \det(U(x)) = \det(Dh(x)^* Dh(x))^{1/2}$.

**Proof:** The above argument shows that the conclusion of the theorem holds when $J_*(x) \neq 0$ at least with $A$ replaced with $A^+$. I will apply this to a modified function in which the corresponding $U(x)$ always has an inverse. Let

$$k : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$
26.5. THE AREA FORMULA

be defined as
\[ k(x) \equiv \begin{pmatrix} h(x) \\ \varepsilon x \end{pmatrix}. \]

Then
\[ Dk(x)^* Dk(x) = Dh(x)^* Dh(x) + \varepsilon^2 I_n \]
and so
\[
J, k(x)^2 = \det (Dh(x)^* Dh(x) + \varepsilon^2 I_n) = \det (Q^* DQ + \varepsilon^2 I_n) > 0
\]
where \( D \) is a diagonal matrix having the nonnegative eigenvalues of \( Dh(x)^* Dh(x) \) down the main diagonal, \( Q \) an orthogonal matrix.

Therefore, what was just shown applies to \( k \).

Let
\[
T \equiv \{ (h(w), 0)^T : w \in B(x, r) \cap A \},
\]
\[
T_\varepsilon \equiv \{ (h(w), \varepsilon w)^T : w \in B(x, r) \cap A \}
\equiv k(B(x, r) \cap A),
\]
then
\[ T = PT_\varepsilon \]
where \( P \) is the projection map defined by
\[
P \begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} x \\ 0 \end{pmatrix}.
\]

Since \( P \) decreases distances, it follows from Lemma 26.5.5
\[
\mathcal{H}^n(h(B(x, r) \cap A)) = \mathcal{H}^n(T) = \mathcal{H}^n(PT_\varepsilon)
\leq \mathcal{H}^n(T_\varepsilon) = \mathcal{H}^n(k(B(x, r) \cap A)).
\]

Now from what was shown earlier,
\[
\det (Dk(x)^* Dk(x))^{1/2} = \lim_{r \to 0} \mathcal{H}^n(k(B(x, r) \cap A)) / m_n(B(x, r))
\]
for a.e. \( x \in A \). This is because
\[
\{ x : Dk(x) \text{ exists} \} = A
\]
and there is no exceptional set where \( Dk(x)^* Dk(x) \) fails to have an inverse. Thus for a.e. \( x \),
\[
\det (Dk(x)^* Dk(x))^{1/2} = \lim_{r \to 0} \mathcal{H}^n(k(B(x, r) \cap A)) / m_n(B(x, r))
\geq \limsup_{r \to 0} \mathcal{H}^n(h(B(x, r) \cap A)) / m_n(B(x, r))
Now let $\varepsilon_n \to 0^+$ and pass to a limit. This yields that for a.e. $x$, those not in the union of the exceptional sets corresponding to the exceptional sets which correspond to each $\varepsilon_n$ in the sequence $\varepsilon_n$,

$$\det (Dh(x)^* Dh(x))^{1/2} \geq \sup_{r \to 0} \frac{\mathcal{H}^n (h(B(x,r) \cap A))}{m_n(B(x,r))}$$

If $x \in A^+$, then it was shown above, that

$$\det (Dh(x)^* Dh(x))^{1/2} = \lim_{r \to 0} \frac{\mathcal{H}^n (h(B(x,r) \cap A))}{m_n(B(x,r))}$$

and if $x \notin A^+$, the above shows that

$$\det (Dh(x)^* Dh(x))^{1/2} = 0 \geq \lim_{r \to 0} \frac{\mathcal{H}^n (h(B(x,r) \cap A))}{m_n(B(x,r))} \geq 0$$

and so this has shown that for a.e. $x$,

$$\lim_{r \to 0} \frac{\mathcal{H}^n (h(B(x,r) \cap A))}{m_n(B(x,r))} = J_*(x)$$

Define the following set for future reference.

$$S = \{ x \in A, Dh(x) \text{ exists and } U(x)^{-1} \text{ does not exist} \} \quad (26.5.25)$$

### 26.5.2 The Area Formula

Since $h$ is one to one on $G$, and Lipschitz on $G$, Lemma 26.5.3 implies one can define a measure $\nu$, on the $\sigma$– algebra of Lebesgue measurable subsets of $G$ as follows.

$$\nu(E) \equiv \mathcal{H}^n (h(E \cap A)).$$

Recall that $A$ is the set in $G$ on which $Dh(x)$ exists. This is all except a set of measure zero and so one could actually replace the right side with $\mathcal{H}^n (h(E))$ because the new material has $\mathcal{H}^n$ measure zero. By Lemma 26.5.3 this is a measure and $\nu \ll m$. If $m_n(E) = 0$, then by Lemma 26.5.3 $\mathcal{H}^n (h(E \cap A)) = 0$. In fact, by this lemma, $\mathcal{H}^n (h(E \cap A)) \leq K^n m_n(E)$ so it is also clear that $\nu$ is $\sigma$ finite.

Therefore by the corollary to the Radon Nikodym theorem, Corollary 18.1.3 on Page 583, there exists $f \geq 0$, $f(x) = 0$ if $x \notin A$, and

$$\nu(E) = \int_E f \, dm = \int_{A \cap E} f \, dm.$$

In fact,

$$f \in L^1_{\text{loc}}(\mathbb{R}^n)$$

Indeed, for a ball $B$,

$$\infty > K^n m_n(B) \geq \mathcal{H}^n (h(B \cap A)) \equiv \nu(B) = \int_B f \, dm$$
What is \( f \)? I will show that \( f(x) = J_*(x) = \det(U(x)) \) a.e. Here \( U(x) \equiv (Dh(x)^* \, Dh(x))^{1/2} \). Theorem 26.5.10 and the fundamental theorem of calculus implies that for a.e. \( x \in G \),

\[
\lim_{r \to 0} \frac{1}{m_n(B(x,r))} \int_{B(x,r)} f(y) \, dm
\]

\[
= \lim_{r \to 0} \frac{H^n(h(B(x,r) \cap A))}{m_n(B(x,r))} = J_*(x).
\]

Therefore, \( f(x) = J_*(x) \) a.e.

Note that one could define a measure \( \nu_R \) as \( H^n(h(E \cap A \cap B(0,R))) \) and it would still be the case that \( \nu_R \ll m_n \) and the Radon Nikodym derivative would still be \( J_*(x) \) for \( x \in A \cap B(0,R) \). You would just apply the above Theorem 26.5.10 for \( G = A \cap B(0,R) \) and the new \( A \) would be \( A \cap B(0,R) \).

Now let \( F \) be a Borel set in \( \mathbb{R}^m \). Recall this implies \( F \) is \( H^n \) measurable. Then

\[
\int_{h^{-1}(A)} X_F(y) \, dH^n = \int_{h^{-1}(A)} X_F \cap h(A) (y) \, dH^n
\]

\[
= H^n(h^{-1}(F) \cap A)
\]

\[
= \nu(h^{-1}(F)) = \int_{A \cap h^{-1}(F)} (x) \, J_*(x) \, dm
\]

\[
= \int_A X_F(h(x)) \, J_*(x) \, dm. \tag{26.5.26}
\]

Similarly, if \( A_R \equiv A \cap B(0,R) \),

\[
\int_{h^{-1}(A_R)} X_F(y) \, dH^n = \int_{h^{-1}(A_R)} X_F \cap h(A_R) (y) \, dH^n = \int_{A_R} X_F(h(x)) \, J_*(x) \, dm.
\]

Note there are no measurability questions in the above formula because \( h^{-1}(F) \) is a Borel set due to the continuity of \( h \). The Borel measurability of \( J_*(x) \) also follows from the observation that \( h \) is continuous and therefore, the partial derivatives are Borel measurable, being the limit of continuous functions. Then \( J_*(x) \) is just a continuous function of these partial derivatives. However, things are not so clear if \( F \) is only assumed \( H^n \) measurable. Is there a similar formula for \( F \) only \( H^n \) measurable?

First consider the case where \( E \) is only \( H^n \) measurable but

\[
H^n(E \cap h(A)) = 0.
\]

By Theorem 26.1.3 on Page 1091, there exists a Borel set \( F \supseteq E \cap h(A) \) such that

\[
H^n(F) = H^n(E \cap h(A)) = 0.
\]

Then from 26.5.26

\[
X_{A \cap h^{-1}(F)} (x) \, J_*(x) = 0 \text{ a.e.}
\]
But
\[ 0 \leq \mathcal{X}_{A \cap h^{-1}(E)}(x) J_*(x) \leq \mathcal{X}_{A \cap h^{-1}(F)}(x) J_*(x) \]  
(26.5.27)

which shows the two functions in (26.5.24) are equal a.e. Therefore by completeness of Lebesgue measure, \( \mathcal{X}_{A \cap h^{-1}(E)}(x) J_*(x) \) is Lebesgue measurable and so from (26.5.26)
\[ 0 = \int \mathcal{X}_{E \cap h(A)}(y) \, d\mathcal{H}^n = \int \mathcal{X}_{F \cap h(A)}(y) \, d\mathcal{H}^n \]
\[ = \int \mathcal{X}_{A \cap h^{-1}(F)}(x) J_*(x) \, dm_n = \int \mathcal{X}_{A \cap h^{-1}(E)}(x) J_*(x) \, dm_n, \]  
(26.5.28)

which shows (26.5.20) holds in this case where \( E \) is \( \mathcal{H}^n \) measurable and
\[ \mathcal{H}^n \left( E \cap h(A) \right) = 0. \]

Now let \( A_R \equiv A \cap B(0,R) \) for large \( R \) and let \( E \) be \( \mathcal{H}^n \) measurable. By Theorem 46.5.39, there exists \( F \supseteq E \cap h(A_R) \) such that \( F \) is Borel and
\[ \mathcal{H}^n \left( F \setminus \left( E \cap h(A_R) \right) \right) = 0. \]  
(26.5.29)

Then
\[ (E \cap h(A_R)) \cup (F \setminus (E \cap h(A_R)) \cap h(A_R)) = F \cap h(A_R) \]
and so
\[ \mathcal{X}_{A_R \cap h^{-1}(F)} J_*(x) = \mathcal{X}_{A_R \cap h^{-1}(E)} J_*(x) + \mathcal{X}_{A_R \cap h^{-1}(F \setminus (E \cap h(A_R)))} J_*(x) \]

where from (26.5.31) and (26.5.32), the second function on the right of the equal sign is Lebesgue measurable and equals zero a.e. Therefore, the first function on the right of the equal sign is also Lebesgue measurable and equals the function on the left a.e. Thus,
\[ \int \mathcal{X}_{E \cap h(A_R)}(y) \, d\mathcal{H}^n = \int \mathcal{X}_{F \cap h(A_R)}(y) \, d\mathcal{H}^n = \int_{A_R} \mathcal{X}_F(h(x)) \, J_*(x) \, dm \]
\[ = \int \mathcal{X}_{A_R \cap h^{-1}(F)}(x) J_*(x) \, dm_n = \int \mathcal{X}_{A_R \cap h^{-1}(E)}(x) J_*(x) \, dm_n. \]  
(26.5.30)

Since this holds for any \( R \), it holds for (26.5.31) with \( A \) replacing \( A_R \) and the function
\[ x \to \mathcal{X}_{A \cap h^{-1}(E)}(x) J_*(x) \]
is Lebesgue measurable. Writing this in a more familiar form yields
\[ \int_{h(A)} \mathcal{X}_E(y) \, d\mathcal{H}^n = \int_A \mathcal{X}_E(h(x)) \, J_*(x) \, dm_n. \]  
(26.5.31)

From this, it follows that if \( s \) is a nonnegative, \( \mathcal{H}^n \) measurable simple function, (26.5.34) continues to be valid with \( s \) in place of \( \mathcal{X}_E \). Then approximating an arbitrary nonnegative \( \mathcal{H}^n \) measurable function \( g \) by an increasing sequence of simple functions, it follows that (26.5.31) holds with \( g \) in place of \( \mathcal{X}_E \) and there are no measurability problems because \( x \to g(h(x)) J_*(x) \) is Lebesgue measurable. This proves the following theorem which is the area formula.
Theorem 26.5.11 Let \( h : \mathbb{R}^n \to \mathbb{R}^m \) be Lipschitz continuous. Also let \( h \) be one to one on a measurable set \( G \subseteq \mathbb{R}^n \) and let \( m \geq n \). Let \( A \subseteq G \) be the set of \( x \in G \) on which \( D h (x) \) exists which is all but a set of measure zero, and let \( g : h (A) \to [0, \infty] \) be \( \mathcal{H}^n \) measurable. Then
\[
 x \to (g \circ h) (x) J_* (x)
\]
is Lebesgue measurable and
\[
\int_{h(A)} g(y) d\mathcal{H}^n = \int_A g(h(x)) J_* (x) dm_n
\]
where \( J_* (x) = \det (U (x)) = \det (D h (x)^* D h (x))^{1/2} \).

Since \( \mathcal{H}^n = m_n \) on \( \mathbb{R}^n \), this is just a generalization of the usual change of variables formula except that here, one does not even need to know that \( h \) is \( C^1 \) so this is much better and in addition it is not limited to \( h \) having values in \( \mathbb{R}^n \). Also note that you could replace \( A \) with \( G \) since they differ by a set of measure zero.

Note that if you assume that \( h \) is Lipschitz on \( G \) then it has a Lipschitz extension to \( \mathbb{R}^n \). The conclusion has to do with integrals over \( G \). It is not really necessary to have \( h \) be Lipschitz continuous on \( \mathbb{R}^n \), but you might as well assume this because of the existence of the Lipschitz extension. However, it can all be generalized to a situation in which \( h \) is not known to be Lipschitz on \( G \). An assumption of locally Lipschitz is sufficient. The definition follows.

Definition 26.5.12 Let \( h : \mathbb{R}^n \to \mathbb{R}^m \). This function is said to be locally Lipschitz if for every \( x \in \mathbb{R}^n \), there exists a ball \( B_x \) containing \( x \) and a constant \( K_x \) such that for all \( y, z \in B_x \),
\[
|h(z) - h(y)| \leq K_x |z - y|
\]
The proof uses a little generalization of Lemma 26.5.1.

Lemma 26.5.13 If \( h \) is locally Lipschitz and \( m_n (T) = 0 \), then
\[
\mathcal{H}^n (h (T)) = 0
\]
Proof: Let
\[
T_k \equiv \{ x \in T : h \text{ has Lipschitz constant } k \text{ near } x \}.
\]
Thus \( T = \bigcup_k T_k \). I will show \( h (T_k) \) has \( \mathcal{H}^n \) measure zero and then it will follow that
\[
h (T) = \bigcup_{k=1}^\infty h (T_k), \text{ the } h (T_k) \text{ increasing in } k,
\]
must also have measure zero.

Let \( \varepsilon > 0 \) be given. By outer regularity, there exists an open set \( V \) containing \( T_k \) such that \( m_n (V) < \frac{\varepsilon}{k^2 \varepsilon^2} \). For \( x \in T_k \) it follows there exists \( r_x < 1 \) such that the ball centered at \( x \) with radius \( 5r_x \) is contained in \( V \) and in this ball, \( h \) has Lipschitz constant \( k \) and \( 5kr_x < \delta \) for \( \delta > 0 \). Then by the Vitali covering theorem, there
are disjoint balls \( \{ B(x_i, r_i) \} \) such that the enlarged balls \( \hat{B}_i \) having five times the radius cover \( T_k \), each being contained in \( V \). Then

\[
\mathcal{H}^n_\delta (h(T_k)) \leq \mathcal{H}^n_\delta (h(\bigcup_{i=1}^\infty \hat{B}_i)) \leq \sum_{i=1}^\infty \mathcal{H}^n_\delta (h(\hat{B}_i)) \\
\leq \sum_{i=1}^\infty \mathcal{H}^n_\delta (B(h(x_i), 5kr_{x_i})) \\
\leq \sum_{i=1}^\infty \alpha(n)(5kr_{x_i})^n = (5k)^n \sum_{i=1}^\infty \alpha(n) r_{x_i}^n \\
= (5k)^n \sum_{i=1}^\infty m_n (B(x_i, r_{x_i})) \\
\leq (5k)^n m_n (V) \leq (5k)^n \varepsilon \frac{\varepsilon}{k^n 6^n} < \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this shows \( \mathcal{H}^n_\delta (h(T_k)) = 0 \). Since \( \delta \) is arbitrary, this implies \( \mathcal{H}^n (h(T_k)) = 0 \). Now

\[
\mathcal{H}^n (h(T)) = \lim_{k \to \infty} \mathcal{H}^n (h(T_k)) = 0. \blacksquare
\]

Then an easy generalization of the above formula is the following.

**Theorem 26.5.14** Let \( h : \mathbb{R}^n \to \mathbb{R}^m \) be locally Lipschitz. Also suppose that \( h \) is one to one on \( G \), a measurable subset of \( \mathbb{R}^n \). Then let \( g : h(G) \to [0, \infty] \) be \( \mathcal{H}^n \) measurable. It follows that

\[
x \to (g \circ h)(x) J_*(x)
\]

is Lebesgue measurable and

\[
\int_{h(G)} g(y) \, d\mathcal{H}^n = \int_G g(h(x)) J_*(x) \, dm_n
\]

where \( J_*(x) = \det (U(x)) = \det (Dh(x)^* Dh(x))^{1/2} \).

**Proof:** Let \( \mathcal{C} \) consist of balls of radius less than 1 covering \( G \) such that for \( B \in \mathcal{C} \), \( h \) is Lipschitz continuous on \( B \). By the Vitali covering theorem, there exists a sequence of these balls \( \{ B_i \} \) such that they are disjoint and \( G \setminus \bigcup_i B_i \) has measure zero. Then, using Theorem 26.5.11 and the above Lemma 26.5.13

\[
\int_{h(G)} g(y) \, d\mathcal{H}^n = \int_{h(\bigcup_i B_i)} g(y) \, d\mathcal{H}^n = \sum_i \int_{h(B_i)} g(y) \, d\mathcal{H}^n \\
= \sum_i \int_{B_i} g(h(x)) J_*(x) \, dm_n \\
= \int_{\bigcup_i B_i} g(h(x)) J_*(x) \, dm_n = \int_G g(h(x)) J_*(x) \, dm_n \blacksquare
\]
26.5.3 Mappings That Are Not One To One

Let \( h : \mathbb{R}^n \to \mathbb{R}^m \) be Lipschitz. We drop the requirement that \( h \) be one to one. This follows \([70]\). See also \([43]\) which is where I read it originally. This reference considers this case originally instead of splitting it into a case where \( h \) is one to one and a case where it is not. Again, let \( A \) be the set on which \( Dh(x) \) exists.

Thus \( m_n(N) = 0 \) where \( N \) is the set where \( Dh(x) \) does not exist and \( H^n(h(S)) = 0 \) and so by Lemma 26.5.3

\[
H^n(h(S \cup N)) \leq H^n(h(S)) + H^n(h(N)) = 0. \tag{26.5.32}
\]

Let \( B = \mathbb{R}^n \setminus (S \cup N) \).

A similar lemma to the following was proved in the section on the change of variables formula for a \( C^1 \) map. There the proof was based on the inverse function theorem. However, this is no longer possible so a slightly more technical argument is required.

Let \( S \) be given by

\[
S \equiv \{ x \in A, \text{ such that } U(x)^{-1} \text{ does not exist} \}
\]

Recall that \( Dh(x) = R(x)U(x) \) where \( R \) preserves distances.

**Lemma 26.5.15** For \( S \) defined above, \( H^n(h(S)) = 0 \).

**Proof:** From Theorem 26.5.10 for a.e. \( x \in S \) and \( r \) is small enough,

\[
\frac{H^n(h(B(x,r) \cap A))}{m_n(B(x,r))} = \frac{H^n(h(B(x,r)))}{m_n(B(x,r))} < \varepsilon.
\]

Therefore, whenever \( x \in S \) and \( r \) small enough,

\[
H^n(h(B(x,r))) \leq \varepsilon \alpha(n)r^n. \tag{26.5.33}
\]

Let \( S_k = S \cap B(0,k) \) and for each \( x \in S_k \), let \( r_x \) be such that 26.5.33 holds with \( r \) replaced by \( 5r_x \) and

\[
B(x,r_x) \subseteq B(0,k).
\]

By the Vitali covering theorem, there is a disjoint subsequence of these balls, \( \{B(x_i, r_i)\} \), with the property that \( \{B(x_i, 5r_i)\} \equiv \{\widetilde{B}_i\} \) covers \( S_k \). Then by the way these balls were defined, with 26.5.33 holding for \( r = 5r_i \),

\[
H^n(h(S_k)) \leq \sum_{i=1}^{\infty} H^n(h(\widetilde{B}_i)) \leq 5^n \varepsilon \sum_{i=1}^{\infty} \alpha(n)r_i^n
\]

\[= 5^n \varepsilon \sum_{i=1}^{\infty} m_n(B(x_i, r_i)) \leq 5^n \varepsilon m_n(B(0,k)).\]

Since \( \varepsilon \) is arbitrary, this shows \( H^n(h(S_k)) = 0 \). Now letting \( k \to \infty \), this shows \( H^n(h(S)) = 0 \).
Lemma 26.5.16 There exists a sequence of disjoint measurable sets, \( \{ F_i \} \), such that
\[
\bigcup_{i=1}^\infty F_i = B
\]
and \( h \) is one to one on \( F_i \).

Proof: \( L(\mathbb{R}^n, \mathbb{R}^n) \) is a finite dimensional normed linear space. Let \( I \) be the elements of \( L(\mathbb{R}^n, \mathbb{R}^n) \) which are invertible and let \( F \) be a countable dense subset of \( I \). Also let \( C \) be a countable dense subset of \( B \equiv \mathbb{R}^n \setminus (S \cup N) \). For \( c \in C \) and \( T \in F \),
\[
E(c, T, i) \equiv \{ b \in B(c, 2^{-i}) \cap B \text{ such that (a.) and (b.) hold} \}
\]
where the conditions (a.) and (b.) are as follows.

- \[
\frac{1}{1 + \varepsilon} |Tv| \leq |U(b)v| \text{ for all } v \quad \text{(a.)}
\]
- \[
|h(a) - h(b) - Dh(b)(a - b)| \leq \varepsilon |T(a - b)| \quad \text{(b.)}
\]
for all \( a \in B(b, 2^{-i}) \). Here \( 0 < \varepsilon < 1/2 \).

![Diagram](image)

Obviously, there are countably many \( E(c, T, i) \). Now suppose \( a, b \in E(c, T, i) \) and \( h(a) = h(b) \). Then
\[
|a - b| \leq |a - c| + |c - b| < \frac{2}{i}.
\]
Therefore, from (a.) and (b.),
\[
\frac{1}{1 + \varepsilon} |T(a - b)| \leq |U(b)(a - b)| = |Dh(b)(a - b)|
= |h(a) - h(b) - Dh(b)(a - b)| \leq \varepsilon |T(a - b)|.
\]
Since \( T \) is one to one, this shows that \( a = b \). Thus \( h \) is one to one on \( E(c, T, i) \).

Now let \( b \in B \). Choose \( T \in F \) such that
\[
\|U(b) - T\| < \varepsilon \left\| U(b)^{-1} \right\|^{-1}.
\]
Then for all \( v \in \mathbb{R}^n \),
\[
|Tv - U(b)v| \leq \varepsilon \left\| U(b)^{-1} \right\|^{-1} |v| \leq \varepsilon |U(b)v|.
\]
and so

\[ |Tv| \leq (1 + \varepsilon) |U(b)v| \]

which yields (a). Now choose \( i \) large enough that for \( |a - b| < 2i^{-1} \),

\[ |h(a) - h(b) - Dh(b)(a - b)| < \frac{\varepsilon}{||T^{-1}||}|a - b| \]

\[ \leq \varepsilon |T(a - b)| \]

and pick \( c \in C \cap B(b,i^{-1}) \). Then \( b \in E(c,T,i) \) and this shows that \( B \) equals the union of these sets.

Let \( \{E_i\} \) be an enumeration of these sets and define \( F_1 \equiv E_1 \), and if \( F_1, \cdots, F_n \) have been chosen, \( F_{n+1} \equiv E_{n+1} \setminus \bigcup_{i=1}^{n} F_i \). Then \( \{F_i\} \) satisfies the conditions of the lemma and this proves the lemma.

The following corollary will not be needed right away but it is of interest.

**Corollary 26.5.17** For each \( E_i \) in Lemma 26.5.16, \( h^{-1} \) is Lipschitz on \( h(E_i) \).

**Proof:** Pick \( a, b \in E_i \). Then by condition a. and b.,

\[ |h(a) - h(b)| \geq |Dh(b)(a - b)| - \varepsilon |T(a - b)| \]

\[ \geq \left( \frac{1}{1 + \varepsilon} - \varepsilon \right) |T(a - b)| \geq r |a - b| \]

for some \( r > 0 \) by the equivalence of all norms on a finite dimensional space. Therefore,

\[ |h^{-1}(h(a)) - h^{-1}(h(b))| \leq \frac{1}{r} |h(a) - h(b)| \]

and this proves the corollary. \( \blacksquare \)

Now let \( g : h(\mathbb{R}^n) \to [0, \infty] \) be \( \mathcal{H}^n \) measurable. By Theorem 26.3.11,

\[ \int_{h(A)} \lambda_{h(F)}(y) g(y) d\mathcal{H}^n = \int_{F} g(h(x)) J_*(x) dm. \quad (26.5.34) \]

Now define

\[ n(y) = \sum_{i=1}^{\infty} \lambda_{h(F_i)}(y). \]

By Lemma 26.5.3, \( h(F_i) \) is \( \mathcal{H}^n \) measurable and so \( n \) is a \( \mathcal{H}^n \) measurable function. For each \( \mathbf{y} \in B \), \( n(y) \) gives the number of elements in \( h^{-1}(y) \cap B \). From 26.5.34,

\[ \int_{h(\mathbb{R}^n)} n(y) g(y) d\mathcal{H}^n = \int_{B} g(h(x)) J_*(x) dm. \quad (26.5.35) \]

Now define \( \#(y) \equiv \text{number of elements in } h^{-1}(y) \).
Theorem 26.5.18 Let \( h : \mathbb{R}^n \to \mathbb{R}^m \) be Lipschitz. Then the function \( y \to \#(y) \) is \( \mathcal{H}^n \)-measurable and if

\[
g : h(\mathbb{R}^n) \to [0, \infty]
\]

is \( \mathcal{H}^n \)-measurable, then

\[
\int_{h(\mathbb{R}^n)} g(y) \#(y) \, d\mathcal{H}^n = \int_{\mathbb{R}^n} g(h(x)) J_*(x) \, dm.
\]

Proof: If \( y \notin h(S \cup N) \), then \( n(y) = \#(y) \). By 26.5.32

\[
\mathcal{H}^n(h(S \cup N)) = 0
\]

and so \( n(y) = \#(y) \) a.e. Since \( \mathcal{H}^n \) is a complete measure, \( \#(\cdot) \) is \( \mathcal{H}^n \)-measurable. Letting

\[
G \equiv h(\mathbb{R}^n) \setminus h(S \cup N),
\]

26.5.35 implies

\[
\int_{h(\mathbb{R}^n)} g(y) \#(y) \, d\mathcal{H}^n = \int_{G} g(y) n(y) \, d\mathcal{H}^n = \int_B g(h(x)) J_*(x) \, dm = \int_{\mathbb{R}^n} g(h(x)) J_*(x) \, dm.
\]

As in Theorem 26.5.14, there is an easy generalization based on the Vitali covering theorem to the case where \( h \) is only locally Lipschitz on \( \mathbb{R}^n \).

Theorem 26.5.19 Let \( h : \mathbb{R}^n \to \mathbb{R}^m \) be locally Lipschitz. Then the function \( y \to \#(y) \) is \( \mathcal{H}^n \)-measurable and if

\[
g : h(\mathbb{R}^n) \to [0, \infty]
\]

is \( \mathcal{H}^n \)-measurable, then

\[
\int_{h(\mathbb{R}^n)} g(y) \#(y) \, d\mathcal{H}^n = \int_{\mathbb{R}^n} g(h(x)) J_*(x) \, dm.
\]

26.6 The Divergence Theorem

As an important application of the area formula I will give a general version of the divergence theorem. It will always be assumed \( n \geq 2 \). Actually it is not necessary to make this assumption but what results in the case where \( n = 1 \) is nothing more than the fundamental theorem of calculus and the considerations necessary to draw this conclusion seem unnecessarily tedious. You have to consider \( \mathcal{H}^0 \), zero dimensional Hausdorff measure. It is left as an exercise but I will not present it.

It will be convenient to have some lemmas and theorems in hand before beginning the proof. First recall the Tietze extension theorem on Page 161. It is stated next for convenience.
Theorem 26.6.1 Let $M$ be a closed nonempty subset of a metric space $(X, d)$ and let $f : M \to [a, b]$ be continuous at every point of $M$. Then there exists a function, $g$ continuous on all of $X$ which coincides with $f$ on $M$ such that $g(X) \subseteq [a, b]$.

The next topic needed is the concept of an infinitely differentiable partition of unity.

Definition 26.6.2 Let $\mathcal{C}$ be a set whose elements are subsets of $\mathbb{R}^n$. Then $\mathcal{C}$ is said to be locally finite if for every $x \in \mathbb{R}^n$, there exists an open set, $U_x$ containing $x$ such that $U_x$ has nonempty intersection with only finitely many sets of $\mathcal{C}$.

Lemma 26.6.3 Let $\mathcal{C}$ be a set whose elements are open subsets of $\mathbb{R}^n$ and suppose $\cup \mathcal{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\{U_i\}_{i=1}^\infty$ such that each $U_i$ is bounded, each $U_i$ is a subset of some set of $\mathcal{C}$, and $\cup_{i=1}^\infty U_i \supseteq H$.

Proof: Let $W_k = B(0, k)$, $W_0 = W_1 = \emptyset$. For each $x \in H \cap \overline{W_k}$ there exists an open set, $U_x$ such that $U_x$ is a subset of some set of $\mathcal{C}$ and $U_x \subseteq W_{k+1} \setminus \overline{W_k}$. Then since $H \cap \overline{W_k}$ is compact, there exist finitely many of these sets, $\{U_i^k\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_k}$. If $H \cap \overline{W_k} = \emptyset$, let $m(k) = 0$ and there are no such sets obtained. The desired countable list of open sets is $\cup_{i=1}^\infty \{U_i^k\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $x \in \mathbb{R}^n$, then $x \in W_k$ where $k$ is the first positive integer with $x \in W_k$. Then $W_k \setminus \overline{W_{k-1}}$ is an open set containing $x$ and this open set can have nonempty intersection only with a set of $\{U_i^k\}_{i=1}^{m(k)} \cup \{U_i^{k-1}\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\cup_{i=1}^\infty \{U_i^k\}_{i=1}^{m(k)}$ is locally finite.

The set, $\{U_i\}_{i=1}^\infty$ is said to be a locally finite cover of $H$. The following lemma gives some important reasons why a locally finite list of sets is so significant. First of all consider the rational numbers, $\{r_i\}_{i=1}^\infty$ each rational number is a closed set.

The set of rational numbers is definitely not locally finite.

Lemma 26.6.4 Let $\mathcal{C}$ be locally finite. Then

$$
\overline{\mathcal{C}} = \cup \{H : H \in \mathcal{C}\}
$$

Next suppose the elements of $\mathcal{C}$ are open sets and that for each $U \in \mathcal{C}$, there exists a differentiable function, $\psi_U$ having $\text{spt} (\psi_U) \subseteq U$. Then you can define the following finite sum for each $x \in \mathbb{R}^n$

$$
f(x) = \sum \{\psi_U(x) : x \in U \in \mathcal{C}\}.
$$

Furthermore, $f$ is also a differentiable function and

$$
Df (x) = \sum \{D\psi_U(x) : x \in U \in \mathcal{C}\}.
$$

¹The definition applies with no change to a general topological space in place of $\mathbb{R}^n$.
²If each $\psi_U$ were only continuous, one could conclude $f$ is continuous. Here the main interest is differentiable.
CHAPTER 26. HAUSDORFF MEASURE

Proof: Let \( p \) be a limit point of \( \cup \mathcal{C} \) and let \( W \) be an open set which intersects only finitely many sets of \( \mathcal{C} \). Then \( p \) must be a limit point of one of these sets. It follows \( p \in \bigcup \{ \overline{H} : H \in \mathcal{C} \} \) and so \( \overline{\mathcal{C}} \subseteq \bigcup \{ \overline{H} : H \in \mathcal{C} \} \). The inclusion in the other direction is obvious.

Now consider the second assertion. Letting \( x \in \mathbb{R}^n \), there exists an open set, \( W \) intersecting only finitely many open sets of \( \mathcal{C}, U_1, U_2, \ldots, U_m \). Then for all \( y \in W \),

\[
 f(y) = \sum_{i=1}^{m} \psi_{U_i}(y)
\]

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

**Definition 26.6.5** Let \( K \) be a closed subset of an open set, \( U \). \( K \prec f \prec U \) if \( f \) is continuous, has values in \([0, 1]\), equals 1 on \( K \), and has compact support contained in \( U \).

**Lemma 26.6.6** Let \( U \) be a bounded open set and let \( K \) be a closed subset of \( U \). Then there exist an open set, \( W \), such that \( W \subseteq \overline{W} \subseteq U \) and a function, \( f \in C_c^\infty(U) \) such that \( K \prec f \prec U \).

**Proof:** The set, \( K \) is compact so is at a positive distance from \( U^C \). Let

\[
 W = \{ x : \text{dist}(x, K) < 3^{-1} \text{dist}(K, U^C) \}.
\]

Also let

\[
 W_1 = \{ x : \text{dist}(x, K) < 2^{-1} \text{dist}(K, U^C) \}
\]

Then it is clear

\[
 K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W}_1 \subseteq U
\]

Now consider the function,

\[
 h(x) = \frac{\text{dist}(x, W_1^C)}{\text{dist}(x, W_1^C) + \text{dist}(x, \overline{W})}
\]

Since \( \overline{W} \) is compact it is at a positive distance from \( W_1^C \) and so \( h \) is a well defined continuous function which has compact support contained in \( W_1 \), equals 1 on \( W \), and has values in \([0, 1]\). Now let \( \phi_k \) be a mollifier. Letting

\[
 k^{-1} < \min \left( \text{dist}(K, W^C), 2^{-1} \text{dist}(\overline{W}_1, U^C) \right),
\]

it follows that for such \( k \), the function, \( h * \phi_k \in C_c^\infty(U) \), has values in \([0, 1]\), and equals 1 on \( K \). Let \( f = h * \phi_k \).

The above lemma is used repeatedly in the following.
Lemma 26.6.7 Let \( K \) be a closed set and let \( \{ V_i \}_{i=1}^\infty \) be a locally finite list of bounded open sets whose union contains \( K \). Then there exist functions, \( \psi_i \in C_c^\infty (V_i) \) such that for all \( x \in K \),
\[
1 = \sum_{i=1}^\infty \psi_i (x)
\]
and the function \( f (x) \) given by
\[
f (x) = \sum_{i=1}^\infty \psi_i (x)
\]
is in \( C^\infty (\mathbb{R}^n) \).

Proof: Let \( K_1 = K \setminus \bigcup_{i=2}^\infty V_i \). Thus \( K_1 \) is compact because \( K_1 \subseteq V_1 \). Let \( W_1 \) be an open set having compact closure which satisfies
\[
K_1 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1
\]
Thus \( W_1, V_2, \ldots, V_n \) covers \( K \) and \( \overline{W}_1 \subseteq V_1 \). Suppose \( W_1, \ldots, W_r \) have been defined such that \( \overline{W}_i \subseteq V_i \) for each \( i \), and \( W_1, \ldots, W_r, V_{r+1}, \ldots, V_n \) covers \( K \). Then let
\[
K_{r+1} = K \setminus (\bigcup_{i=r+2}^\infty V_i) \cup (\bigcup_{j=1}^r W_j).
\]
It follows \( K_{r+1} \) is compact because \( K_{r+1} \subseteq V_{r+1} \). Let \( W_{r+1} \) satisfy
\[
K_{r+1} \subseteq W_{r+1} \subseteq \overline{W}_{r+1} \subseteq V_{r+1}, \overline{W}_{r+1} \text{ is compact}
\]
Continuing this way defines a sequence of open sets, \( \{ W_i \}_{i=1}^\infty \) having compact closures with the property
\[
\overline{W}_i \subseteq V_i, K \subseteq \bigcup_{i=1}^\infty W_i.
\]
Note \( \{ W_i \}_{i=1}^\infty \) is locally finite because the original list, \( \{ V_i \}_{i=1}^\infty \) was locally finite. Now let \( U_i \) be open sets which satisfy
\[
\overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i, \overline{U}_i \text{ is compact.}
\]
Similarly, \( \{ U_i \}_{i=1}^\infty \) is locally finite.

Since the sets, \( \{ W_i \}_{i=1}^\infty \) are locally finite, it follows \( \bigcup_{i=1}^\infty \overline{W}_i = \overline{\bigcup_{i=1}^\infty W_i} \) and so it is possible to define \( \phi_i \) and \( \gamma_i \), infinitely differentiable functions having compact support such that
\[
U_i \prec \phi_i \prec V_i, \overline{W}_i \prec \gamma_i \prec U_i.
\]
Now define
\[\psi_i(x) = \begin{cases} 
\frac{\gamma_i(x)\phi_i(x)}{\sum_{j=1}^{\infty} \phi_j(x)} & \text{if } \sum_{j=1}^{\infty} \phi_j(x) \neq 0, \\
0 & \text{if } \sum_{j=1}^{\infty} \phi_j(x) = 0.
\end{cases}\]

Note how the sum makes sense because of local finiteness. If \(x\) is such that \(\sum_{j=1}^{\infty} \phi_j(x) = 0\), then \(x \not\in \bigcup_{i=1}^{\infty} U_i\) because \(\phi_i\) equals one on \(U_i\). Consequently, for each \(i\), \(\gamma_i(y) = 0\) for all \(y\) near \(x\) thanks to the fact that \(U_i\) is closed and so \(\psi_i(y) = 0\) for all \(y\) near \(x\). Hence \(\psi_i\) is infinitely differentiable at such \(x\). If \(\sum_{j=1}^{\infty} \phi_j(x) \neq 0\), this situation persists near \(x\) because each \(\phi_j\) is continuous and by local finiteness, so is the sum. So \(\psi_i\) is infinitely differentiable at such points also thanks to Lemma 26.6.4. Therefore \(\psi_i\) is infinitely differentiable. If \(x \in K\), then \(\gamma_i(x) = 1\) for each \(i\) and so \(\sum_{j=1}^{\infty} \psi_j(x) = 1\). Clearly \(0 \leq \psi_i(x) \leq 1\) and \(\text{spt}(\psi_j) \subseteq \text{V}_j\). This proves the theorem.

The functions, \(\{\psi_i\}\) are called a \(C^\infty\) partition of unity.

The method of proof of this lemma easily implies the following useful corollary.

**Corollary 26.6.8** If \(H\) is a compact subset of \(\text{V}_i\) for some \(\text{V}_i\) there exists a partition of unity such that \(\psi_i(x) = 1\) for all \(x \in H\) in addition to the conclusion of Lemma 26.6.7.

**Proof:** Keep \(V_i\) the same but replace \(V_j\) with \(\widetilde{V}_j \equiv V_j \setminus H\). Now in the proof above, applied to this modified collection of open sets, if \(j \neq i, \phi_j(x) = 0\) whenever \(x \in H\). Therefore, \(\psi_i(x) = 1\) on \(H\).

**Lemma 26.6.9** Let \(\Omega\) be a metric space with the closed balls compact and suppose \(\mu\) is a measure defined on the Borel sets of \(\Omega\) which is finite on compact sets. Then there exists a unique Radon measure, \(\overline{\mu}\) which equals \(\mu\) on the Borel sets. In particular \(\mu\) must be both inner and outer regular on all Borel sets.

**Proof:** Define a positive linear functional, \(\Lambda(f) = \int f d\mu\). Let \(\overline{\mu}\) be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all \(f\) continuous,
\[\int f d\mu = \int f d\overline{\mu}.
\]

If \(V\) is an open set, let \(\{f_n\}\) be a sequence of continuous functions which is increasing and converges to \(\chi_V\) pointwise. Then applying the monotone convergence theorem,
\[\int \chi_V d\mu = \mu(V) = \int \chi_V d\overline{\mu} = \overline{\mu}(V)\]
and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let \(B(a,n)\) be a ball of radius \(n\) and let \(E\) be a Borel set contained in this ball. Then by regularity of \(\overline{\mu}\) there exist sets \(F,G\) such that \(G\) is a countable intersection
of open sets and \( F \) is a countable union of compact sets such that \( F \subseteq E \subseteq G \) and \( \overline{\mu(G \setminus F)} = 0 \). Now \( \mu(G) = \overline{\mu(G)} \) and \( \mu(F) = \overline{\mu(F)} \). Thus
\[
\overline{\mu(G \setminus F)} = \overline{\mu(G)} = \mu(G) = \mu(G \setminus F) + \mu(F)
\]
and so \( \mu(G \setminus F) = \overline{\mu(G \setminus F)} \). Thus
\[
\mu(E) = \mu(F) = \overline{\mu(F)} = \overline{\mu(G)} = \mu(E).
\]
If \( E \) is an arbitrary Borel set, then
\[
\mu(E \cap B(a,n)) = \overline{\mu(E \cap B(a,n))}
\]
and letting \( n \to \infty \), this yields \( \mu(E) = \overline{\mu(E)} \).

One more lemma will be useful.

**Lemma 26.6.10** Let \( V \) be a bounded open set and let \( X \) be the closed subspace of \( C(V) \), the space of continuous functions defined on \( V \), which is given by the following.
\[
X = \{ u \in C(V) : u(x) = 0 \text{ on } \partial V \}.
\]
Then \( C_c^\infty(V) \) is dense in \( X \) with respect to the norm given by
\[
||u|| = \max \{ |u(x)| : x \in V \}
\]

**Proof:** Let \( O \subseteq \overline{O} \subseteq W \subseteq \overline{W} \subseteq V \) be such that \( \text{dist} (\overline{O}, V^C) < \eta \) and let \( \psi_\delta(\cdot) \) be a mollifier. Let \( u \in X \) and consider \( \lambda_W u \ast \psi_\delta \). Let \( \epsilon > 0 \) be given and let \( \eta \) be small enough that \( |u(x)| < \epsilon/2 \) whenever \( x \in V \setminus \overline{O} \). Then if \( \delta \) is small enough \( |\lambda_W u \ast \psi_\delta(x) - u(x)| < \epsilon \) for all \( x \in \overline{O} \) and \( \lambda_W u \ast \psi_\delta \) is in \( C_c^\infty(V) \). For \( x \in V \setminus \overline{O} \), \( |\lambda_W u \ast \psi_\delta(x) - u(x)| \leq \epsilon/2 \) and so for such \( x \),
\[
|\lambda_W u \ast \psi_\delta(x) - u(x)| \leq \epsilon.
\]
This proves the lemma since \( \epsilon \) was arbitrary.

**Definition 26.6.11** A bounded open set, \( U \subseteq \mathbb{R}^n \) is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, \( Q_1, \cdots, Q_N \),
\[
Q_i = \prod_{j=1}^n (a_{ij}, b_{ij})
\]
such that \( \partial U \equiv \overline{U} \setminus U \) is contained in their union. Also, for each \( Q_i \), there exists \( k \) and a Lipschitz function, \( g_i \) such that \( U \cap Q_i \) is of the form
\[
\left\{ x : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} (a^i_j, b^i_j) \times \prod_{j=k+1}^n (a^i_j, b^i_j) \text{ and } a^i_k < x_k < g_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \right\}
\]
or else of the form

\[
\begin{cases}
  x : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \\
  \prod_{j=k+1}^n (a_j^i, b_j^i) \text{ and } g_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) < x_k < b_j^i
\end{cases}
\]  

(26.6.37)

The function, \(g_i\), has a derivative on \(A_i \subseteq \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i)\) where

\[
m_{n-1} \left( \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i) \setminus A_i \right) = 0.
\]

Also, there exists an open set, \(Q_0\) such that \(Q_0 \subseteq U \subseteq Q_0 \cup Q_1 \cup \cdots \cup Q_N\).

Note that since there are only finitely many \(Q_i\) and each \(g_i\) is Lipschitz, it follows from an application of Lemma 26.5.5 that \(H^{n-1}(\partial U) < \infty\). Also from Lemma 26.6.9 \(H^{n-1}\) is inner and outer regular on \(\partial U\).

**Lemma 26.6.12** Suppose \(U\) is a bounded open set as described above. Then there exists a unique function in \(L^\infty(\partial U, H^{n-1})\) such that \(|n(y)| = 1, n\) is \(H^{n-1}\) measurable, (meaning each component of \(n\) is \(H^{n-1}\) measurable) and for every \(w \in \mathbb{R}^n\) satisfying \(|w| = 1\), and for every \(f \in C^1_c(\mathbb{R}^n)\),

\[
\lim_{t \to 0} \int_U f(x + tw) - f(x) \frac{dx}{t} = \int_{\partial U} f(n \cdot w) dH^{n-1}
\]

**Proof:** Let \(U \subseteq V \subseteq \overline{V} \subseteq \bigcup_{i=0}^N Q_i\), and let \(\{\psi_i\}_{i=0}^N\) be a \(C^\infty\) partition of unity on \(\overline{V}\) such that \(\text{spt}(\psi_i) \subseteq Q_i\). Then for all \(t\) small enough and \(x \in U\),

\[
\frac{f(x + tw) - f(x)}{t} = \frac{1}{t} \sum_{i=0}^N \psi_i f(x + tw) - \psi_i f(x).
\]

Thus using the dominated convergence theorem,

\[
\lim_{t \to 0} \int_U \frac{f(x + tw) - f(x)}{t} dx = \int_U \left( \frac{1}{t} \sum_{i=0}^N \psi_i f(x + tw) - \psi_i f(x) \right) dx
\]

\[
= \int_U \sum_{i=0}^N \sum_{j=1}^n D_j(\psi_i f)(x) w_j dx
\]
26.6. THE DIVERGENCE THEOREM

\[ \int_{\Omega} \sum_{j=1}^{n} D_j (\psi_i f) (x) w_j dx + \sum_{i=1}^{N} \int_{\Omega} \sum_{j=1}^{n} D_j (\psi_i f) (x) w_j dx \]

(26.6.38)

Since \( \text{spt} (\psi_0) \subseteq Q_0 \), it follows the first term in the above equals zero. In the second term, fix \( i \). Without loss of generality, suppose the \( k \) in the above definition equals \( n \) and \( 26.6.39 \) holds. This just makes things a little easier to write. Thus \( g_i \) is a function of

\[ (x_1, \cdots, x_{n-1}) \in \prod_{j=1}^{n-1} (a_j^{(i)}, b_j^{(i)}) \equiv B_i \]

Then

\[ \int_{\Omega} \sum_{j=1}^{n} D_j (\psi_i f) (x) w_j dx \]

= \( \int_{B_i} \int_{a_i^{(j)}}^{b_i^{(j)}} \sum_{j=1}^{n} D_j (\psi_i f) (x) w_j dx \]

= \( \int_{B_i} \int_{-\infty}^{0} \sum_{j=1}^{n} D_j (\psi_i f) (x_1, \cdots, x_{n-1}, y + g_i (x_1, \cdots, x_{n-1})) \cdot w_j dx \cdot dx_{n-1} \)

= \( \int_{B_i} \int_{-\infty}^{0} \sum_{j=1}^{n} D_j (\psi_i f) (x_1, \cdots, x_{n-1}, y + g_i (x_1, \cdots, x_{n-1})) \cdot w_j dy \cdot dx_{n-1} \)

Letting \( x_n = y + g_i (x_1, \cdots, x_{n-1}) \) and changing the variable, this equals

\[ \int_{B_i} \int_{-\infty}^{0} \sum_{j=1}^{n} D_j (\psi_i f) (x_1, \cdots, x_{n-1}, y + g_i (x_1, \cdots, x_{n-1})) \cdot w_j dy \cdot dx_{n-1} \]

Recall \( D_j \) refers to the partial derivative taken with respect to the entry in the \( j \)-th slot. In the \( n \)-th slot is found not just \( x_n \) but \( y + g_i (x_1, \cdots, x_{n-1}) \) so a differentiation with respect to \( x_j \) will not be the same as \( D_j \). In fact, it will introduce another term involving \( g_{i,j} \). Thus from the chain rule,

\[ \int_{A_i} \int_{-\infty}^{0} \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f (x_1, \cdots, x_{n-1}, y + g_i (x_1, \cdots, x_{n-1}))) w_j - D_n (\psi_i f) (x_1, \cdots, x_{n-1}, y + g_i (x_1, \cdots, x_{n-1})) \cdot \]

\[ g_{i,j} (x_1, \cdots, x_{n-1}) w_j dy dx_1 \cdots dx_{n-1} \]

\[ + \int_{A_i} \int_{-\infty}^{0} D_n (\psi_i f) (x_1, \cdots, x_{n-1}, y + g_i (x_1, \cdots, x_{n-1})) \cdot \]

\[ w_n dy dx_1 \cdots dx_{n-1} \]

(26.6.39)
Consider the term
\[
\int_{A_i} \int_{-\infty}^{0} \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f (x_1, \cdots, x_n, y + g_i (x_1, \cdots, x_n))) \cdot w_j \, dy \, dx_1 \cdots dx_{n-1}.
\]
This equals
\[
\int_{B_i} \int_{-\infty}^{0} \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f (x_1, \cdots, x_n, y + g_i (x_1, \cdots, x_n))) \cdot w_j \, dy \, dx_1 \cdots dx_{n-1},
\]
and now interchanging the order of integration and using the fact that \(\text{spt} (\psi_i) \subseteq Q_i\), it follows this term equals zero. (The reason this is valid is that \(x_j \to \psi_i f (x_1, \cdots, x_n, y + g_i (x_1, \cdots, x_n)))\)
is the composition of Lipschitz functions and is therefore Lipschitz. Therefore, this function is absolutely continuous and can be recovered by integrating its derivative.)

Then, changing the variable back to \(x_n\) it follows (26.6.39) reduces to
\[
= - \int_{A_i} \int_{-\infty}^{0} \sum_{j=1}^{n-1} D_n (\psi_i f) (x_1, \cdots, x_n) \cdot g_{i,j} (x_1, \cdots, x_n) \, dx_n \, dx_1 \cdots dx_{n-1}
\]
\[+ \int_{A_i} \int_{-\infty}^{0} D_n (\psi_i f) (x_1, \cdots, x_n) \cdot w_n \, dx_n \, dx_1 \cdots dx_{n-1}.
\]
Doing the integrals using the observation that \(g_{i,j} (x_1, \cdots, x_n)\) does not depend on \(x_n\), this reduces further to
\[
\int_{A_i} (\psi_i f) (x_1, \cdots, x_n) \cdot N_i (x_1, \cdots, x_n, g_i (x_1, \cdots, x_n)) \cdot w \, dm_{n-1}
\]
where \(N_i (x_1, \cdots, x_n, g_i (x_1, \cdots, x_n))\) is given by
\[
(-g_{i,1} (x_1, \cdots, x_n), -g_{i,2} (x_1, \cdots, x_n), \cdots, -g_{i,n-1} (x_1, \cdots, x_n), 1).
\]
(26.6.41)
At this point I need a technical lemma which will allow the use of the area formula. The part of the boundary of \(U\) which is contained in \(Q_i\) is the image of the map, \(h_i (x_1, \cdots, x_n)\) given by \((x_1, \cdots, x_n, g_i (x_1, \cdots, x_n))\) for \((x_1, \cdots, x_n) \in A_i\). I need a formula for
\[
\det (Dh_i (x_1, \cdots, x_n))^* \, Dh_i (x_1, \cdots, x_n))^{1/2}.
\]
To avoid interrupting the argument, I will state the lemma here and prove it later.
Lemma 26.6.13

\[
\det \left( D\mathbf{h} \left( x_1, \ldots, x_{n-1} \right) \right)^{1/2} = \sqrt{1 + \sum_{j=1}^{n-1} g_{i,j} \left( x_1, \ldots, x_{n-1} \right)^2} \equiv J_i \left( x_1, \ldots, x_{n-1} \right).
\]

For \( y = (x_1, \ldots, x_{n-1}, g_i (x_1, \ldots, x_{n-1})) \in \partial U \cap Q_i \) and \( \mathbf{n} \) defined by

\[
\mathbf{n}_i(y) = \frac{1}{J_i(x_1, \ldots, x_{n-1})} \mathbf{N}_i(y)
\]

it follows from the description of \( J_i(x_1, \ldots, x_{n-1}) \) given in the above lemma, that \( \mathbf{n}_i \) is a unit vector. All components of \( \mathbf{n}_i \) are continuous functions of limits of continuous functions. Therefore, \( \mathbf{n}_i \) is Borel measurable and so it is \( \mathcal{H}^{n-1} \) measurable.

Now (26.6.40) reduces to

\[
\int_{A_i} \left( \psi f \right) (x_1, \ldots, x_{n-1}, g_i (x_1, \ldots, x_{n-1})) \times
\mathbf{n}_i (x_1, \ldots, x_{n-1}, g_i (x_1, \ldots, x_{n-1})) \cdot w J_i (x_1, \ldots, x_{n-1}) \, dm_{n-1}.
\]

By the area formula this equals

\[
\int_{\mathbf{n}(A_i)} \psi_i f (y) \mathbf{n}_i(y) \cdot w d\mathcal{H}^{n-1}.
\]

Now by Lemma 26.5.5 and the equality of \( m_{n-1} \) and \( \mathcal{H}^{n-1} \) on \( \mathbb{R}^{n-1} \), the above integral equals

\[
\int_{\partial U \cap Q_i} \psi_i f (y) \mathbf{n}_i(y) \cdot w d\mathcal{H}^{n-1} = \int_{\partial U} \psi_i f (y) \mathbf{n}_i(y) \cdot w d\mathcal{H}^{n-1}.
\]

Returning to (26.6.40) similar arguments apply to the other terms and therefore,

\[
\lim_{t \to 0} \int_U \frac{f(x + tw) - f(x)}{t} \, dm_n
= \sum_{i=1}^{N} \int_{\partial U} \psi_i f (y) \mathbf{n}_i(y) \cdot w d\mathcal{H}^{n-1}
= \int_{\partial U} f(y) \sum_{i=1}^{N} \psi_i (y) \mathbf{n}_i(y) \cdot w d\mathcal{H}^{n-1}
= \int_{\partial U} f(y) \mathbf{n}(y) \cdot w d\mathcal{H}^{n-1}
\]

(26.6.42)

Then let \( \mathbf{n}(y) = \sum_{i=1}^{N} \psi_i (y) \mathbf{n}_i(y) \).
I need to show first there is no other \( n \) which satisfies \[ 40.6.3 \] and then I need to show that \( |n(y)| = 1 \). Note that it is clear \( |n(y)| \leq 1 \) because each \( n_i \) is a unit vector and this is just a convex combination of these. Suppose then that \( n_1 \in L^\infty(\partial U, H^{n-1}) \) also works in \[ 40.5.4.2 \]. Then for all \( f \in C^1_c(\mathbb{R}^n) \),

\[
\int_{\partial U} f(y) n(y) \cdot w dH^{n-1} = \int_{\partial U} f(y) n_1(y) \cdot w dH^{n-1}.
\]

Suppose \( h \in C(\partial U) \). Then by the Tietze extension theorem, there exists \( f \in C_c(\mathbb{R}^n) \) such that the restriction of \( f \) to \( \partial U \) equals \( h \). Now by Lemma \[ 40.6.14 \] applied to a bounded open set containing the support of \( f \), there exists a sequence \( \{f_m\} \) of functions in \( C^1_c(\mathbb{R}^n) \) converging uniformly to \( f \). Therefore,

\[
\int_{\partial U} h(y) n(y) \cdot w dH^{n-1} = \lim_{m \to \infty} \int_{\partial U} f_m(y) n(y) \cdot w dH^{n-1} = \lim_{m \to \infty} \int_{\partial U} f_m(y) n_1(y) \cdot w dH^{n-1} = \int_{\partial U} h(y) n_1(y) \cdot w dH^{n-1}.
\]

Now \( H^{n-1} \) is a Radon measure on \( \partial U \) and so the continuous functions on \( \partial U \) are dense in \( L^1(\partial U, H^{n-1}) \). It follows \( n \cdot w = n_1 \cdot w \) a.e. Now let \( \{w_m\}_{m=1}^\infty \) be a countable dense subset of the unit sphere. From what was just shown, \( n \cdot w_m = n_1 \cdot w_m \) except for a set of measure zero, \( N_m \). Letting \( N = \cup_m N_m \), it follows that for \( y \notin N, n(y) \cdot w_m = n_1(y) \cdot w_m \) for all \( m \). Since the set is dense, it follows \( n(y) \cdot w = n_1(y) \cdot w \) for all \( y \notin N \) and for all \( w \) a unit vector. Therefore, \( n(y) = n_1(y) \) for all \( y \notin N \) and this shows \( n \) is unique. In particular, although it appears to depend on the partition of unity \( \{\psi_i\} \) from its definition, this is not the case.

It only remains to verify \( |n(y)| = 1 \) a.e. I will do this by showing how to compute \( n \). In particular, I will show that \( n = n_i \) a.e. on \( \partial U \cap Q_i \). Let \( W \subseteq \mathbb{W} \subseteq Q_i \cap \partial U \) where \( W \) is open in \( \partial U \). Let \( O \) be an open set such that \( O \cap \partial U = W \) and \( \overline{O} \subseteq Q_i \). Using Corollary \[ 40.6.5 \] there exists a \( C^\infty \) partition of unity \( \{\psi_m\} \) such that \( \psi_i = 1 \) on \( \overline{O} \). Therefore, if \( m \neq i, \psi_m = 0 \) on \( \overline{O} \). Then if \( f \in C^1_c(O) \),

\[
\int_W f w \cdot n dH^{n-1} = \int_{\partial U} f w \cdot n dH^{n-1} = \int_U \nabla f \cdot w dm_n = \int_U \nabla (\psi_i f) \cdot w dm_n
\]

which by the first part of the argument given above equals

\[
\int_W \psi_i f n_i \cdot w dH^{n-1} = \int_W f w \cdot n_i dH^{n-1}.
\]
Thus for all $f \in C^1_c (O)$,
\[
\int_W f \cdot w \, dH^{n-1} = \int_W f \cdot n_i \, dH^{n-1}
\] (26.6.43)

Since $C^1_c (O)$ is dense in $C_c (O)$, the above equation is also true for all $f \in C_c (O)$. Now letting $h \in C_c (W)$, the Tietze extension theorem implies there exists $f_1 \in C (\overline{O})$ whose restriction to $W$ equals $h$. Let $f$ be defined by
\[
f_1 (x) = \frac{\text{dist} (x, O^C)}{\text{dist} (x, \text{spt} (h)) + \text{dist} (x, O^C)} = f (x).
\]

Then $f = h$ on $W$ and so this has shown that for all $h \in C_c (W)$, (26.6.43) holds for $h$ in place of $f$. But as observed earlier, $H^{n-1}$ is outer and inner regular on $\partial U$ and so $C_c (W)$ is dense in $L^1 (W, H^{n-1})$ which implies $w \cdot n (y) = w \cdot n_i (y)$ for a.e. $y$. Considering a countable dense subset of the unit sphere as above, this implies $n (y) = n_i (y)$ a.e. $y$. This proves $|n (y)| = 1$ a.e. and in fact $n (y)$ can be computed by using the formula for $n_i (y)$. This proves the lemma.

It remains to prove Lemma 26.6.13.

**Proof of Lemma 26.6.13**: Let $h (x) = (x_1, \ldots, x_{n-1}, g (x_1, \ldots, x_{n-1}))^T$

\[
Dh (x) = \begin{pmatrix}
1 & 0 \\
\vdots & \ddots & \vdots \\
0 & 1 & g_{x_1} & \cdots & g_{x_{n-1}}
\end{pmatrix}
\]

Therefore,
\[
J (x) = (\det (Dh (x)^T Dh (x)))^{1/2}.
\]

Therefore, $J (x)$ is the square root of the determinant of the following $(n-1) \times (n-1)$ matrix.

\[
\begin{pmatrix}
1 + (g_{x_1})^2 & g_{x_1} g_{x_2} & \cdots & g_{x_1} g_{x_{n-1}} \\
g_{x_2} g_{x_1} & 1 + (g_{x_2})^2 & \cdots & g_{x_2} g_{x_{n-1}} \\
\vdots & \ddots & \ddots & \vdots \\
g_{x_{n-1}} g_{x_1} & g_{x_{n-1}} g_{x_2} & \cdots & 1 + (g_{x_{n-1}})^2
\end{pmatrix}
\] (26.6.44)

I need to show the determinant of the above matrix equals
\[
1 + \sum_{i=1}^{n-1} (g_{x_i} (x))^2.
\]

This is implied by the following claim. To simplify the notation I will replace $n-1$ with $n$. 

Claim: Let $a_1, \cdots, a_n$ be real numbers and let $A(a_1, \cdots, a_n)$ be the matrix which has $1 + a_i^2$ in the $ii^{th}$ slot and $a_i a_j$ in the $ij^{th}$ slot when $i \neq j$. Then
\[
\det A = 1 + \sum_{i=1}^{n} a_i^2.
\]

Proof of the claim: The matrix, $A(a_1, \cdots, a_n)$ is of the form
\[
A(a_1, \cdots, a_n) = \begin{pmatrix}
1 + a_1^2 & a_1 a_2 & \cdots & a_1 a_n \\
a_1 a_2 & 1 + a_2^2 & \cdots & a_2 a_n \\
\vdots & \vdots & \ddots & \vdots \\
a_1 a_n & a_2 a_n & \cdots & 1 + a_n^2
\end{pmatrix}
\]
Now consider the product of a matrix and its transpose, $B^T B$ below.
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & a_1 \\
0 & 1 & \cdots & 0 & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -a_2 & \cdots & -a_n & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_n
\end{pmatrix}
\]
This product equals a matrix of the form
\[
\begin{pmatrix}
A(a_1, \cdots, a_n) & 0 \\
0 & 1 + \sum_{i=1}^{n} a_i^2
\end{pmatrix}
\]
Therefore, $(1 + \sum_{i=1}^{n} a_i^2) \det (A(a_1, \cdots, a_n)) = \det (B)^2 = \det (B^T)^2$. However, using row operations,
\[
\det B^T = \det \begin{pmatrix}
1 & 0 & \cdots & 0 & a_1 \\
0 & 1 & \cdots & 0 & a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & a_n
\end{pmatrix}
\]
\[
= 1 + \sum_{i=1}^{n} a_i^2
\]
and therefore,
\[
\left(1 + \sum_{i=1}^{n} a_i^2\right) \det (A(a_1, \cdots, a_n)) = \left(1 + \sum_{i=1}^{n} a_i^2\right)^2
\]
which shows $\det (A(a_1, \cdots, a_n)) = \left(1 + \sum_{i=1}^{n} a_i^2\right)$. This proves the claim.
Now the above lemma implies the divergence theorem.
**Theorem 26.6.14** Let $U$ be a bounded open set with a Lipschitz boundary which lies on one side of its boundary. Then if $f \in C^1_c(\mathbb{R}^n)$,

$$
\int_U f_k(x) \, dm_n = \int_{\partial U} f_n d\mathcal{H}^{n-1}
$$

(26.6.46)

where $n = (n_1, \cdots, n_n)$ is the $\mathcal{H}^{n-1}$ measurable unit vector of Lemma 26.6.12. Also, if $F$ is a vector field such that each component is in $C^1_c(\mathbb{R}^n)$, then

$$
\int_U \nabla \cdot F(x) \, dm_n = \int_{\partial U} F \cdot n d\mathcal{H}^{n-1}.
$$

(26.6.47)

**Proof:** To obtain (26.6.46) apply Lemma 26.6.12 to $w = e_k$. Then to obtain from this,

$$
\int_U \nabla \cdot F(x) \, dm_n \\
= \sum_{j=1}^n \int_U F_{j,j} \, dm_n = \sum_{j=1}^n \int_{\partial U} F_{jn} d\mathcal{H}^{n-1} \\
= \int_{\partial U} \sum_{j=1}^n F_{jn} d\mathcal{H}^{n-1} = \int_{\partial U} F \cdot n d\mathcal{H}^{n-1}.
$$

This proves the theorem.

What is the geometric significance of the vector, $n$? Recall that in the part of the boundary contained in $Q_i$, this vector points in the same direction as the vector

$$
N_i(x_1, \cdots, x_{n-1}, g_i(x_1, \cdots, x_{n-1}))
$$

given by

$$
(-g_{i,1}(x_1, \cdots, x_{n-1}), -g_{i,2}(x_1, \cdots, x_{n-1}), \cdots, -g_{i,n-1}(x_1, \cdots, x_{n-1}), 1)
$$

(26.6.48)

in the case where $k = n$. This vector is the gradient of the function,

$$
x_n - g_i(x_1, \cdots, x_{n-1})
$$

and so is perpendicular to the level surface given by

$$
x_n - g_i(x_1, \cdots, x_{n-1}) = 0
$$

in the case where $g_i$ is $C^1$. It also points away from $U$ so the vector $n$ is the unit outer normal. The other cases work similarly.

The divergence theorem is valid in situations more general than for Lipschitz boundaries. What you need is essentially the ability to say that the functions, $g_i$ above can be differentiated a.e. and more importantly that these functions can be recovered by integrating their partial derivatives. In other words, you need absolute continuity in each variable. Later in the chapter on weak derivatives, examples of such functions which are more general than Lipschitz functions will be discussed. However, the Lipschitz functions are pretty general and will suffice for now.
26.7 Integration And The Degree

There is a very interesting application of the degree to integration \([\mathbb{R}]\). Recall Lemma 21.3.5. I want to generalize this to the case where \(h : \mathbb{R}^n \to \mathbb{R}^n\) is only Lipschitz continuous, vanishing outside a bounded set. In the following proposition, let \(\phi_\varepsilon\) be a symmetric nonnegative mollifier,

\[
\phi_\varepsilon (x) \equiv \frac{1}{\varepsilon^n} \phi \left( \frac{x}{\varepsilon} \right), \quad \text{spt} \phi \subseteq B(0,1).
\]

\(\Omega\) will be a bounded open set. By Theorem 24.6.7, \(h\) satisfies

\[
Dh(x) \text{ exists } a.e.,
\]

For any \(p > n\),

\[
\lim_{m \to \infty} D(h \ast \psi_m) = Dh \text{ in } L^p(\mathbb{R}^n; \mathbb{R}^{n \times n})
\]

where \(\psi_m\) is a mollifier.

**Proposition 26.7.1** Let \(S \subseteq h(\partial \Omega)^C\) such that

\[
\text{dist } (S, h(\partial \Omega)) > 0
\]

where \(\Omega\) is a bounded open set and also let \(h\) be Lipschitz continuous, vanishing outside some bounded set. Then whenever \(\varepsilon > 0\) is small enough,

\[
d(h, \Omega, y) = \int_{\Omega} \phi_\varepsilon (h(x) - y) \det Dh(x) \, dx
\]

for all \(y \in S\).

**Proof:** Let \(\varepsilon_0 > 0\) be small enough that for all \(y \in S\),

\[
B(y, 5\varepsilon_0) \cap h(\partial \Omega) = \emptyset.
\]

Now let \(\psi_m\) be a mollifier as \(m \to \infty\) with support in \(B(0, m^{-1})\) and let

\[
h_m \equiv h \ast \psi_m.
\]

Thus \(h_m \in C^\infty(\overline{\Omega}; \mathbb{R}^n)\) and for any \(p > n\),

\[
\|h_m - h\|_{L^\infty(\Omega)}, \|Dh_m - Dh\|_{L^p(\Omega)} \to 0
\]

as \(m \to \infty\). The first claim above is obvious and the second follows by 26.7.50. Choose \(M\) such that for \(m \geq M\),

\[
\|h_m - h\|_{\infty} < \varepsilon_0.
\]

Thus \(h_m \in \mathcal{U}_y \cap C^2(\overline{\Omega}; \mathbb{R}^n)\) for all \(y \in S\).
For \( y \in S \), let \( z \in B(y, \varepsilon) \) where \( \varepsilon < \varepsilon_0 \) and suppose \( x \in \partial \Omega \), and \( k, m \geq M \). Then for \( t \in [0,1] \),

\[
|\( (1-t)h_m(x) + h_k(x) t - z \) | \geq |h_m(x) - z| - t |h_k(x) - h_m(x)| > 2\varepsilon_0 - t2\varepsilon_0 \geq 0
\]

showing that for each \( y \in S \), \( B(y, \varepsilon) \cap ((1-t)h_m + th_k)(\partial \Omega) = \emptyset \). By Lemma 21.3.5, for all \( y \in S \),

\[
\int_{\Omega} \phi_\varepsilon(h_m(x) - y) \det(Dh_m(x)) \, dx = \int_{\Omega} \phi_\varepsilon(h_k(x) - y) \det(Dh_k(x)) \, dx
\]

(26.7.53)

for all \( k, m \geq M \). By this lemma again, which says that for small enough \( \varepsilon \) the integral is constant and the definition of the degree in Definition 21.2.6,

\[
d(y, \partial \Omega, h_m) = \int_{\Omega} \phi_\varepsilon(h_m(x) - y) \det(Dh_m(x)) \, dx
\]

(26.7.54)

for all \( \varepsilon \) small enough. For \( x \in \partial \Omega \), \( y \in S \), and \( t \in [0,1] \),

\[
|\( (1-t)h(x) + h_m(x) t - y \) | \geq |h(x) - y| - t |h(x) - h_m(x)| > 3\varepsilon_0 - t2\varepsilon_0 > 0
\]

and so by Theorem 21.3.7 the part about homotopy, for each \( y \in S \),

\[
d(y, \Omega, h) = d(y, \Omega, h_m) = \int_{\Omega} \phi_\varepsilon(h_m(x) - y) \det(Dh_m(x)) \, dx
\]

whenever \( \varepsilon \) is small enough. Fix such an \( \varepsilon < \varepsilon_0 \) and use 26.7.53 to conclude the right side of the above equation is independent of \( m > M \).

By 26.7.53, there exists a subsequence still denoted by \( m \) such that \( Dh_m(x) \to Dh(x) \) a.e. Since \( p > n \), \( \det(Dh_m) \) is bounded in \( L^r(\Omega) \) for some \( r > 1 \) and so the integrands in the following are uniformly integrable. By the Vitali convergence theorem, one can pass to the limit as follows.

\[
d(y, \Omega, h) = \lim_{m \to \infty} \int_{\Omega} \phi_\varepsilon(h_m(x) - y) \det(Dh_m(x)) \, dx
\]

\[= \int_{\Omega} \phi_\varepsilon(h(x) - y) \det(Dh(x)) \, dx.
\]

This proves the proposition.

Next is an interesting change of variables theorem. Let \( \Omega \) be a bounded open set with the property that \( \partial \Omega \) has measure zero and let \( h \) be Lipschitz continuous on \( \mathbb{R}^n \). Then from Lemma 26.5.3, \( h(\partial \Omega) \) also has measure zero.
Now suppose \( f \in C_c\left(h(\partial\Omega)^C\right) \). There are finitely many components of \( h(\partial\Omega)^C \) which have nonempty intersection with \( \text{spt}(f) \). From the Proposition above,

\[
\int f(y) d(y,\Omega,h) dy = \int f(y) \lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon(h(x) - y) \det Dh(x) dx dy
\]

Actually, there exists an \( \varepsilon \) small enough that for all \( y \in \text{spt}(f) \),

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon(h(x) - y) \det Dh(x) dx = \int_{\Omega} \phi_\varepsilon(h(x) - y) \det Dh(x) dx = d(y,\Omega,h)
\]

This is because \( \text{spt}(f) \) is at a positive distance from the compact set \( h(\partial\Omega)^C \). Therefore, for all \( \varepsilon \) small enough,

\[
\int f(y) d(y,\Omega,h) dy = \int \int_{\Omega} f(y) \phi_\varepsilon(h(x) - y) \det Dh(x) dx dy = \int_{\Omega} \det Dh(x) \int f(y) \phi_\varepsilon(h(x) - y) dy dx
\]

Using the uniform continuity of \( f \), you can now pass to a limit and obtain using the fact that \( \det Dh(x) \) is in \( L^r(\mathbb{R}^n) \) for some \( r > 1 \),

\[
\int f(y) d(y,\Omega,h) dy = \int_{\Omega} \phi(h(x)) \det Dh(x) dx
\]

This has proved the following interesting lemma.

**Lemma 26.7.2** Let \( f \in C_c\left(h(\partial\Omega)^C\right) \) for \( \Omega \) a bounded open set and let \( h \) be Lipschitz on \( \mathbb{R}^n \). Say \( \partial\Omega \) has measure zero so that \( h(\partial\Omega) \) has measure zero. Then everything is measurable which needs to be and

\[
\int f(y) d(y,\Omega,h) dy = \int_{\Omega} \det (Dh(x)) f(h(x)) dx.
\]

Note that \( h \) is not necessarily one to one. Next is a simple corollary which replaces \( C_c(\mathbb{R}^n) \) with \( L^1_{\text{loc}}(\mathbb{R}^n) \) in the case that \( h \) is one to one. Also another assumption is made on there being finitely many components.

**Corollary 26.7.3** Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and let \( h \) be one to one and satisfy [26.7.49 - 26.7.50] \( \partial\Omega \) has measure zero for \( \Omega \) a bounded open set and \( h(\partial\Omega)^C \) has finitely many components. Then everything is measurable which needs to be and

\[
\int f(y) d(y,\Omega,h) dy = \int_{\Omega} \det Dh(x) f(h(x)) dx.
\]
Proof: Since \( d(y, \Omega, h) = 0 \) for all \(|y| \) large enough due to \( y \notin h(\Omega) \) for large \( y \), there is no loss of generality in assuming \( f \) is in \( L^1(\mathbb{R}^n) \). For all \( y \notin h(\partial \Omega) \), a set of measure zero, \( d(y, \Omega, h) \) is bounded by some constant, depending on the maximum of the degree on the various components of \( h(\partial \Omega)^C \). Then from Proposition 26.7.1.1

\[
\int f(y) d(y, \Omega, h) \, dy = \int f(y) \lim_{\epsilon \to 0} \int_{\Omega} \phi_\epsilon (h(x) - y) \det Dh(x) \, dx \, dy \quad (26.7.55)
\]

This time, use the area formula to write

\[
\left| \int_{\Omega} \phi_\epsilon (h(x) - y) \det Dh(x) \, dx \right| \leq \int_{\mathbb{R}^n} \phi_\epsilon (h(x) - y) |\det Dh(x)| \, dx
\]

and so using the dominated convergence theorem in 26.5.1, it equals

\[
\lim_{\epsilon \to 0} \int_{\Omega} \det Dh(x) \int f(y) \phi_\epsilon (h(x) - y) \, dy \, dx
\]

\[= \lim_{\epsilon \to 0} \int_{\Omega} \det Dh(x) \int f(h(x) - y) \phi_\epsilon (y) \, dy \, dx \]

\[= \lim_{\epsilon \to 0} \int_{\Omega} \det Dh(x) \int_{B(0,1)} f(h(x) - \epsilon u) \phi(u) \, du \, dx \]

Now

\[
\left| \int_{\Omega} \det Dh(x) \int_{B(0,1)} f(h(x) - \epsilon u) \phi(u) \, du \, dx \right| - \int_{\Omega} \det Dh(x) f(h(x)) \, dx \right| \leq \left| \int_{B(0,1)} \int_{\Omega} \det Dh(x) |f(h(x) - \epsilon u) - f(h(x))| \, dx \phi(u) \, du \right|
\]

which needs to converge to 0 as \( \epsilon \to 0 \). However, from the area formula, Theorem 26.5.1 applied to the inside integral, the above equals

\[
\int_{B(0,1)} \int_{h(\Omega)} |f(y - \epsilon u) - f(y)| \, dy \phi(u) \, du \leq \int_{B(0,1)} \|f_{\epsilon u} - f\|_{L^1(\mathbb{R}^n)} \phi(u) \, du
\]

which converges to 0 by continuity of translation in \( L^1(\mathbb{R}^n) \). Thus as in the lemma,

\[
\int f(y) d(y, \Omega, h) \, dy = \lim_{\epsilon \to 0} \int f(y) \lim_{\epsilon \to 0} \int_{\Omega} \phi_\epsilon (h(x) - y) \det Dh(x) \, dx \, dy
\]

\[= \lim_{\epsilon \to 0} \int_{\Omega} \det Dh(x) \int_{B(0,1)} f(h(x) - \epsilon u) \phi(u) \, du \, dx \]

\[= \int_{\Omega} \det Dh(x) f(h(x)) \, dx
\]

and this proves the corollary.

Note that in this corollary \( h \) is one to one.
26.8 The Case Of $W^{1,p}$

There is a very interesting application of the degree to integration [49]. Recall Lemma 21.3.5. I want to generalize this to the case where $h : \mathbb{R}^n \to \mathbb{R}^n$ has the property that its weak partial derivatives and $h$ are in $L^p(\mathbb{R}^n, \mathbb{R}^n), p > n$. This is denoted by saying

$$h \in W^{1,p}(\mathbb{R}^n, \mathbb{R}^n).$$

In the following proposition, let $\phi_x$ be a symmetric nonnegative mollifier,

$$\phi_x(x) \equiv \frac{1}{\varepsilon^n} \phi \left( \frac{x}{\varepsilon} \right), \text{ spt } \phi \subseteq B(0, 1).$$

$\Omega$ will be a bounded open set. By Theorem 24.6.10, $h$ may be considered continuous and it satisfies

$$Dh(x) \text{ exists a.e.,} \quad (26.8.56)$$

For any $p > n$,

$$\lim_{m \to \infty} D(h \ast \psi_m) = Dh \text{ in } L^p(\mathbb{R}^n, \mathbb{R}^{n \times n}) \quad (26.8.57)$$

where $\psi_m$ is a mollifier. Here $\mathbb{R}^{n \times n}$ denotes the $n \times n$ matrices with any norm you like.

**Proposition 26.8.1** Let $S \subseteq h(\partial \Omega)^C$ such that

$$\text{dist} (S, h(\partial \Omega)) > 0$$

where $\Omega$ is a bounded open set and also let $h$ be in $W^{1,p}(\mathbb{R}^n, \mathbb{R}^n)$. Then whenever $\varepsilon > 0$ is small enough,

$$d(h, \Omega, y) = \int_{\Omega} \phi_x(h(x) - y) \det Dh(x) \, dx$$

for all $y \in S$.

**Proof:** Let $\varepsilon_0 > 0$ be small enough that for all $y \in S$,

$$B(y, 3\varepsilon_0) \cap h(\partial \Omega) = \emptyset.$$

Now let $\psi_m$ be a mollifier as $m \to \infty$ with support in $B(0, m^{-1})$ and let

$$h_m \equiv h \ast \psi_m.$$ 

Thus $h_m \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$ and,

$$||h_m - h||_{L^\infty(\Omega)}, ||Dh_m - Dh||_{L^p(\Omega)} \to 0 \quad (26.8.58)$$

as $m \to \infty$. The first claim above follows from the definition of convolution and the uniform continuity of $h$ on the compact set $\overline{\Omega}$ and the second follows by 26.8.57. Choose $M$ such that for $m \geq M$,

$$||h_m - h||_{L^\infty(\Omega)} < \varepsilon_0. \quad (26.8.59)$$
26.8. THE CASE OF $W^{1,p}$

Thus $h_m \in \mathcal{U}_y \cap C^2(\overline{\Omega}; \mathbb{R}^n)$ for all $y \in S$.

For $y \in S$, let $z \in B(y, \varepsilon)$ where $\varepsilon < \varepsilon_0$ and suppose $x \in \partial\Omega$, and $k, m \geq M$. Then for $t \in [0, 1]$,

$$|(1 - t) h_m(x) + h_k(x)t - z| \geq |h_m(x) - z| - t|h_k(x) - h_m(x)|$$

$$\geq 2\varepsilon_0 - t2\varepsilon_0 \geq 0$$

showing that for each $y \in S$, $B(y, \varepsilon) \cap ((1 - t) h_m + t h_k)(\partial\Omega) = \emptyset$. By Lemma 21.3.5, for all $y \in S$,

$$\int_{\Omega} \phi_\varepsilon (h_m(x) - y) \det(Dh_m(x)) \, dx =$$

$$\int_{\Omega} \phi_\varepsilon (h_k(x) - y) \det(Dh_k(x)) \, dx$$

(26.8.60)

for all $k, m \geq M$. By this lemma again, which says that for small enough $\varepsilon$ the integral is constant and the definition of the degree in Definition 21.2.6,

$$d(y, \Omega, h_m) = \int_{\Omega} \phi_\varepsilon (h_m(x) - y) \det(Dh_m(x)) \, dx$$

(26.8.61)

for all $\varepsilon$ small enough. For $x \in \partial\Omega$, $y \in S$, and $t \in [0, 1]$,

$$|(1 - t) h(x) + h_m(x)t - y| \geq |h(x) - y| - t|h(x) - h_m(x)|$$

$$> 3\varepsilon_0 - t2\varepsilon_0 > 0$$

and so by Theorem 21.3.7, the part about homotopy, for each $y \in S$,

$$d(y, \Omega, h) = d(y, \Omega, h_m) =$$

$$\int_{\Omega} \phi_\varepsilon (h_m(x) - y) \det(Dh_m(x)) \, dx$$

whenever $\varepsilon$ is small enough. Fix such an $\varepsilon < \varepsilon_0$ and use 26.8.60 to conclude the right side of the above equation is independent of $m > M$.

By 26.8.65, there exists a subsequence still denoted by $m$ such that $Dh_m(x) \rightarrow Dh(x)$ a.e. Since $p > n$, $\det(Dh_m)$ is bounded in $L^r(\Omega)$ for some $r > 1$ and so the integrands in the following are uniformly integrable. By the Vitali convergence theorem, one can pass to the limit as follows.

$$d(y, \Omega, h) = \lim_{m \rightarrow \infty} \int_{\Omega} \phi_\varepsilon (h_m(x) - y) \det(Dh_m(x)) \, dx =$$

$$\int_{\Omega} \phi_\varepsilon (h(x) - y) \det(Dh(x)) \, dx.$$  

This proves the proposition.

Next is an interesting change of variables theorem. Let $\Omega$ be a bounded open set and let $h \in W^{1,p}(\mathbb{R}^n)$. Also assume

$$m(h(\partial\Omega)) = 0.$$
From Proposition 26.8.1, for \( y \not\in h(\partial \Omega) \),

\[
d(y, \Omega, h) = \lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon(h(x) - y) \det D h(x) \, dx,
\]

showing that \( y \to d(y, \Omega, h) \) is a measurable function since it is the limit of continuous functions off the set of measure zero \( h(\partial \Omega) \).

Now suppose \( f \in C_c \left(h(\partial \Omega)^C\right) \). There are finitely many components of \( h(\partial \Omega)^C \) which have nonempty intersection with \( \text{spt}(f) \). From the Proposition above,

\[
\int f(y) \, d(y, \Omega, h) \, dy = \int f(y) \lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon(h(x) - y) \det D h(x) \, dx \, dy
\]

Actually, from Proposition 26.8.1 there exists an \( \varepsilon \) small enough that for all \( y \in \text{spt}(f) \),

\[
\lim_{\varepsilon \to 0} \int_{\Omega} \phi_\varepsilon(h(x) - y) \det D h(x) \, dx = \int_{\Omega} \phi_\varepsilon(h(x) - y) \det D h(x) \, dx
\]

This is because \( \text{spt}(f) \) is at a positive distance from \( h(\partial \Omega)^C \). Therefore, for all \( \varepsilon \) small enough,

\[
\int f(y) \, d(y, \Omega, h) \, dy = \int \int \phi_\varepsilon(h(x) - y) \det D h(x) \, dx \, dy
\]

Using the uniform continuity of \( f \), you can now pass to a limit as \( \varepsilon \to 0 \) and obtain, using the fact that \( \det D h(x) \) is in \( L^r(\mathbb{R}^n) \) for some \( r > 1 \),

\[
\int f(y) \, d(y, \Omega, h) \, dy = \int f(h(x)) \det D h(x) \, dx
\]

This has proved the following interesting lemma.

**Lemma 26.8.2** Let \( f \in C_c \left(h(\partial \Omega)^C\right) \) and let \( h \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n), p > n \), \( h(\partial \Omega) \) has measure zero for \( \Omega \) a bounded open set. Then everything is measurable which needs to be and

\[
\int f(y) \, d(y, \Omega, h) \, dy = \int \det(D h(x)) \, f(h(x)) \, dx.
\]

Note that \( h \) is not necessarily one to one. The difficult issue is handling \( d(y, \Omega, h) \) which has integer values constant on each component of \( h(\partial \Omega)^C \) and
Let \( n \) two components, three if \( \Omega \) have measure zero for terms in the above sum although two of them give 0. This proves the proposition.

The following is a version of the area formula.

**Proposition 26.8.3** Let \( \Omega \) be an open connected bounded set in \( \mathbb{R}^n \), \( n \geq 1 \) such that \( \mathbb{R}^n \setminus \partial \Omega \) consists of two, three if \( n = 1 \), connected components. Let \( f \in C(\overline{\Omega}; \mathbb{R}^n) \) be continuous and one to one. Then \( f(\Omega) \) is the bounded component of \( \mathbb{R}^n \setminus f(\partial \Omega) \) and for \( y \in f(\Omega) \), \( d(f, \Omega, y) \) either equals 1 or \(-1\).

**Proof:** First suppose \( n \geq 2 \). By the Jordan separation theorem, \( \mathbb{R}^n \setminus f(\partial \Omega) \) consists of two components, a bounded component \( B \) and an unbounded component \( U \). Using the Tietze extension theorem, there exists \( g \) defined on \( \mathbb{R}^n \) such that \( g = f^{-1} \) on \( f(\Omega) \). Thus on \( \partial \Omega \), \( g \circ f = \text{id} \). It follows from this and the product formula that

\[
1 = d(\text{id}, \Omega, g(y)) = d(g \circ f, \Omega, g(y))
\]

\[
= d(g, B, g(y)) d(f, \Omega, B) + d(f, \Omega, U) d(g, U, g(y))
\]

\[
= d(g, B, g(y)) d(f, \Omega, B)
\]

Therefore, \( d(f, \Omega, B) \neq 0 \) and so for every \( z \in B \), it follows \( z \in f(\Omega) \). Thus \( B \subseteq f(\Omega) \). On the other hand, \( f(\Omega) \) cannot have points in both \( U \) and \( B \) because it is a connected set. Therefore \( f(\Omega) \subseteq B \) and this shows \( B = f(\Omega) \). Thus \( d(f, \Omega, B) = d(f, \Omega, y) \) for each \( y \in B \) and the above formula shows this equals either 1 or \(-1\) because the degree is an integer. In the case where \( n = 1 \), the argument is similar but here you have 3 components in \( \mathbb{R}^1 \setminus f(\partial \Omega) \) so there are more terms in the above sum although two of them give 0. This proves the proposition.

The following is a version of the area formula.

**Lemma 26.8.4** Let \( h \in W^{1,p}(\mathbb{R}^n; \mathbb{R}^n), p > n \) where \( h \) is one to one, \( h(\partial \Omega), \partial \Omega \) have measure zero for \( \Omega \) a bounded open connected set in \( \mathbb{R}^n \). Then \( h(\partial \Omega)^C \) has two components, three if \( n = 1 \), and for \( y \in h(\Omega) \), and \( f \in C_c(\mathbb{R}^n) \).

\[
\int_{h(\Omega)} f(y) \, dy = \int_{\Omega} |\det(Dh(x))| f(h(x)) \, dx
\]

If \( O \) is an open set, it is also true that

\[
\int_{h(\Omega)} \chi_O(y) \, dy = \int_{\Omega} |\det(Dh(x))| \chi_O(h(x)) \, dx
\]
Also if \( f \) is any nonnegative Borel measurable function

\[
\int_{h(\Omega)} f(y) \, dy = \int_{\Omega} |\det (Dh(x))| f(h(x)) \, dx
\]

**Proof:** Consider the first claim. Let \( \delta \) be such that \( B(x_1, \delta) \subseteq \Omega \) and let \( \{f_j(y)\}_{j=1}^{\infty} \) be nonnegative, increasing in \( j \) and converging pointwise to \( \chi_{h(B(x_1, \delta))}(y) \). This can be done because \( h(B(x_1, \delta)) \) is an open bounded set thanks to invariance of domain, Theorem \( \text{26.8.3} \). By Proposition \( \text{26.8.2} \), \( d(y, \Omega, h) \) either equals 1 or \(-1\). Suppose it equals \(-1\). Then from Lemma \( \text{26.7.2} \)

\[
\int_{h(\Omega)} f_j(y) \, dy = -\int_{\Omega} \det(Dh(x)) f_j(h(x)) \, dx
\]

The integrand on the right is uniformly integrable thanks to the fact the \( f_j \) are bounded and \( \det(Dh(x)) \) is in \( L^r(\Omega) \) for some \( r > 1 \). Therefore, by the Vitali convergence theorem and the monotone convergence theorem,

\[
\int_{h(\Omega)} \chi_{h(B(x_1, \delta))}(y) \, dy = -\int_{\Omega} \det(Dh(x)) \chi_{B(x_1, \delta)}(x) \, dx
\]

so

\[
m(h(B(x_1, \delta))) \frac{1}{m(B(x_1, \delta))} = -\frac{1}{m(B(x_1, \delta))} \int_{B(x_1, \delta)} \det(Dh(x)) \, dx
\]

If \( x_1 \) is a Lebesgue point of \( \det(Dh(x)) \), then you can pass to the limit as \( \delta \to 0 \) and conclude

\[-\det(Dh(x_1)) \geq 0\]

Since a.e. point is a Lebesgue point, it follows that in the case where \( d(y, \Omega, h) = -1 \),

\[-\det(Dh(x)) = |\det(Dh(x))| \text{ a.e. } x \in \Omega\]

The case where the degree equals 1 is similar. Thus \( \det(Dh(x)) \) has the same sign on \( h(\Omega) \).

Now let \( O \) be an open set. Then by invariance of domain, \( h(O) \) is also an open set. Let \( V_k \) denote a decreasing sequence of open sets, \( V_k \supseteq V_{k+1} \) whose intersection is the compact set \( h(\partial \Omega) \) such that \( m(V_k) < 1/k \). Then if \( f \prec h(O) \setminus V_k \), it follows since \( h(O) \setminus V_k \) is an open set which is at a positive distance from \( h(\partial \Omega) \), Lemma \( \text{26.7.2} \) implies

\[
\int_{h(\Omega)} f(y) \, dy = \int_{\Omega} |\det(Dh(x))| f(h(x)) \, dx
\]

Taking a sequence of such \( f \) increasing to \( \chi_{h(O) \setminus V_k} \), it follows from monotone convergence theorem in the above that

\[
\int_{h(\Omega)} \chi_{h(O) \setminus V_k}(y) \, dy = \int_{\Omega} |\det(Dh(x))| \chi_{h(O) \setminus V_k}(h(x)) \, dx
\]

\[
= \int_{\Omega} |\det(Dh(x))| \chi_{h^{-1}(V_k)}(x) \, dx
\]
26.8. THE CASE OF $W^{1,p}$

Now letting $k \to \infty$, it follows from the monotone convergence theorem that

$$\int_{h(\Omega)} X_{\Omega \setminus \partial \Omega} (y) \, dy = \int_\Omega |\det (Dh(x))| X_{\partial \Omega} (x) \, dx$$

Since both $\partial \Omega$ and $h(\partial \Omega)$ have measure zero, this implies

$$\int_{h(\Omega)} X_{h(\Omega) \setminus h(\partial \Omega)} (y) \, dy = \int_\Omega |\det (Dh(x))| X_{\partial \Omega} (x) \, dx$$

Now let $G$ denote the Borel sets $E$ with the property that

$$\int_{h(\Omega)} X_{h(\Omega)} (y) \, dy = \int_\Omega |\det (Dh(x))| X_{\partial \Omega} (x) \, dx$$

It follows easily that if $E \in G$ then so does $E^C$. This is because $h(\Omega)$ has finite measure and $|\det (Dh(x))|$ is in $L^1(\Omega)$. If $E_i$ is a sequence of disjoint sets of $G$ then the monotone convergence theorem implies $\bigcup E_i$ is also in $G$. It was shown above that the $\pi$ system of open sets is contained in $G$. Therefore, it follows from the lemma on $\pi$ systems, Lemma 10.12.3 on Page 1032, $G$ equals the Borel sets. Now the desired result follows from approximating $f \geq 0$ and Borel measurable with a sequence of Borel simple functions which converge pointwise to $f$. This proves the theorem.

The following corollary follows right away by splitting $f$ into positive and negative parts of real and imaginary parts.

**Corollary 26.8.5** Let $h$ be one to one and in $W^{1,p}(\mathbb{R}^n; \mathbb{R}^n), p > n$. Let $\Omega$ be a bounded, open, connected set in $\mathbb{R}^n$ and suppose $\partial \Omega, h(\partial \Omega)$ have measure zero. Let $f \in L^1(h(\Omega))$ where $f$ is also Borel measurable. Then

$$\int_{h(\Omega)} f(y) \, dy = \int_\Omega |\det (Dh(x))| f(h(x)) \, dx$$

It can also be written in the form

$$\int f(y) \, d(y, \Omega, h) \, dy = \int_\Omega |\det (Dh(x))| f(h(x)) \, dx$$

Note this is a general area formula under somewhat more restrictive hypotheses than the usual area formula because it involves an assumption that $\Omega$ is connected and a troublesome condition on the measure of $h(\partial \Omega), \partial \Omega$ being zero, but it does not require $h$ to be Lipschitz. It looks like a strange result because $|\det Dh(x)|$ is not in $L^\infty$ and so it is not clear why the integral on the right should even be finite just because $f$ is in $L^1$. If the result is correct, it is surprising.

The condition on the measure of $\partial \Omega$ and $h(\partial \Omega)$ is not necessary. Neither is it necessary to assume $\Omega$ is connected. This is shown next.
CHAPTER 26. HAUSDORFF MEASURE

Theorem 26.8.6 Let $h$ be one to one on $\Omega$ and in $W^{1,p}(\mathbb{R}^n;\mathbb{R}^n)$, $p > n$. Let $\Omega$ be a bounded, open set in $\mathbb{R}^n$. Let $f \in L^1(h(\Omega))$ where $f$ is also Borel measurable. Then

$$\int_{h(\Omega)} f(y) \, dy = \int_{\Omega} |\det(Dh(x))| f(h(x)) \, dx$$

It can also be written in the form

$$\int f(y) \, d(y, \Omega, h) \, dy = \int_{\Omega} \det(Dh(x)) f(h(x)) \, dx$$

Proof: Let $\Omega \subseteq [-R, R]^n \equiv Q$. Let $p_i(x) \equiv x_i$ where $x = (x_1, \ldots, x_n)^T$. Then here is a claim.

Claim: Let $b$ be given. There exists $a$, $|a - b| \leq 4^{-k}$ such that

$$m(h([p_i, x = a] \cap Q)) = 0.$$ 

Here $[p_i, x = a]$ is short for $\{x : p_i, x = a\}$.

Proof of claim: If this is not so, then for every $a$ in an interval centered at $b$,

$$m(h([p_i, x = a] \cap Q)) > 0$$

However,

$$m(\bigcup_{a \in [b-4^{-k}, b+4^{-k}]} h([p_i, x = a] \cap Q)) = m(h(\prod_j A_j))$$

where $A_j = [-R, R]$ if $j \neq i$ and $A_i = [b - 4^{-k}, b + 4^{-k}]$. This is finite because $h(\prod_j A_j)$ is a compact set, being the continuous image of such a set. Since $h$ is one to one, this compact set would then be the union of uncountably many disjoint sets, each having positive measure. Thus for some $1/l > 0$ there must be infinitely many of these disjoint sets having measure larger than $1/l$, $l \in \mathbb{N}$ a contradiction to the set having finite measure. This proves the claim.

Now from the claim, consider $b_k, k \in \mathbb{Z}$ given by $b_k = k2^{-(m+1)}$ and for each $i = 1, \ldots, n$, let $a_{ik}$ denote a value of the claim, $|a_{ik} - b_k| \leq 4^{-m}$. Thus

$$m(h([p_i, x = a_{ik}] \cap Q)) = 0, i = 1, \ldots, n$$

Thus also

$$|a_{ik} - a_{i(k+1)}| \leq |a_{ik} - b_k| + |b_{k+1} - a_{i(k+1)}| + |b_{k+1} - b_k|$$

$$\leq 4^{-m} + 4^{-m} + 2^{-(m+1)} < 2^{-m}$$

Consider boxes of the form $\prod_{i=1}^n [a_{ik}, a_{i(k+1)}]$. Denote these boxes as $B_m$. Thus they are non overlapping boxes the sides of which are of length less than $2^{-m}$. Let $\Omega_1$ denote the union of the finitely many boxes of $B_1$ which are contained in $\Omega$. 

Next let \( \Omega_2 \) denote the union of the boxes of \( \mathcal{B}_2 \cup \mathcal{B}_1 \) which are contained in \( \Omega \) and so forth. Then \( \bigcup_{k=1}^{\infty} \Omega_k \subseteq \Omega \). Suppose now that \( p \in \Omega \). Then it is at positive distance from \( \partial \Omega \). Let \( k \) be the first such that \( p \) is contained in a box of \( \mathcal{B}_k \) which is contained in \( \Omega \). Therefore, this has shown that \( \Omega \) is a countable union of non overlapping closed boxes \( B \) which have the property that \( \partial B, h(\partial B) \) have measure zero. Denote these boxes as \( \{ \mathcal{B}_k \} \).

First assume \( f \) is nonnegative and Borel measurable. Then from Corollary 26.8.5,

\[
\int_{h(B_k)} f(y) \, dy = \int_{B_k} |\det(Dh(x))| f(h(x)) \, dx
\]

Since \( h(\partial B_k) \) has measure zero,

\[
\int_{h(\bigcup_{k=1}^{m} B_k)} f(y) \, dy = \sum_{k=1}^{m} \int_{h(B_k)} f(y) \, dy = \sum_{k=1}^{m} \int_{B_k} |\det(Dh(x))| f(h(x)) \, dx = \int_{\bigcup_{k=1}^{m} B_k} |\det(Dh(x))| f(h(x)) \, dx
\]

and now letting \( m \to \infty \) and using the monotone convergence theorem,

\[
\int_{h(\Omega)} f(y) \, dy = \int_{\Omega} |\det(Dh(x))| f(h(x)) \, dx \quad (26.8.62)
\]

Next assume in addition that \( f \) is also in \( L^1(h(\Omega)) \). Recall that from properties of the degree, \( d(y, U, h) \) is constant on \( h(U) \) for \( U \) a component of \( \Omega \). Since \( h \) is one to one, Proposition 24.6.4 implies this constant is either \(-1\) or \(1\). Let the components of \( \Omega \) be \( \{U_i\}_{i=1}^{\infty} \). Also from Theorem 24.3.7 and the assumption that \( h \) is one to one, if \( y \in h(U_i) \), then \( y \notin h(\bigcup_{j \neq i} U_j) \) and

\[
d(y, U_i, h) + d(y, \bigcup_{j \neq i} U_j, h) = d(y, \Omega, h)
\]

Since \( y \notin h(\bigcup_{j \neq i} U_j) \) the second term on the left is 0 and so \( d(y, U_i, h) = d(y, \Omega, h) \). Therefore, by Corollary 26.8.3,

\[
\int_{h(\Omega)} f(y) \, d(y, \Omega, h) \, dy = \sum_{i=1}^{\infty} \int_{h(U_i)} f(y) \, d(y, \Omega, h) \, dy = \sum_{i=1}^{\infty} \int_{U_i} f(x) \, d(h(x)) \, \det(Dh(x)) \, dx \quad (26.8.63)
\]
From (26.8.62)
\[ \sum_{i=1}^{\infty} \int_{\mathcal{U}_i} f(h(x)) |\det(Dh(x))| \, dx \]

\[ = \int \sum_{i=1}^{\infty} \mathcal{U}_i \left( x \right) f(h(x)) |\det(Dh(x))| \, dx \]

\[ = \int \mathcal{U}_\Omega f(h(x)) |\det(Dh(x))| \, dx < \infty \]

and so by Fubini's theorem, the sum and the integral may be interchanged in (26.8.63) to obtain from the dominated convergence theorem,

\[ \int \sum_{i=1}^{\infty} \mathcal{U}_i \left( x \right) f(h(x)) |\det(Dh(x))| \, dx \]

\[ = \int f(h(x)) |\det(Dh(x))| \, dx \]

which shows

\[ \int_{h(\Omega)} f(y) \, d(y, \Omega, h) \, dy = \int_{\Omega} f(h(x)) |\det(Dh(x))| \, dx \]

(26.8.64)

Now if \( f \) is Borel measurable and in \( L^1(\Omega) \), the above may be applied to the positive parts of the real and imaginary parts of \( f \) to obtain (26.8.65) for such \( f \). This proves the theorem.

Not surprisingly, it is not necessary to assume \( f \) is Borel measurable.

**Corollary 26.8.7** Let \( h \) be one to one on \( \Omega \) and in \( W^{1,p}(\mathbb{R}^n; \mathbb{R}^n) \), \( p > n \). Let \( \Omega \) be a bounded, open set in \( \mathbb{R}^n \). Let \( f \in L^1(h(\Omega)) \) where \( f \) is Lebesgue measurable. Then \( x \to |\det(Dh(x))| f(h(x)) \) is Lebesgue measurable and

\[ \int_{h(\Omega)} f(y) \, dy = \int_{\Omega} |\det(Dh(x))| f(h(x)) \, dx \]

(26.8.65)

It can also be written in the form

\[ \int f(y) \, d(y, \Omega, h) \, dy = \int_{\Omega} \det(Dh(x)) f(h(x)) \, dx \]

(26.8.66)

**Proof:** Let \( E \) be a Lebesgue measurable subset of \( h(\Omega) \). By regularity of the measure, there exist Borel sets \( F \subseteq E \subseteq G \) such that \( F \) and \( G \) are both Borel measurable sets contained in \( h(\Omega) \) with \( m(G \setminus F) = 0 \). Then by Theorem 26.8.6

\[ \int_{\Omega} |\det(Dh(x))| \chi_F(h(x)) \, dx = \int_{h(\Omega)} \chi_F(y) \, dy \]

\[ = \int_{h(\Omega)} \chi_E(y) \, dy = \int_{h(\Omega)} \chi_G(y) \, dy \]

\[ = \int_{\Omega} |\det(Dh(x))| \chi_G(h(x)) \, dx \]

(26.8.67)
which shows that
\[
|\det(Dh(x))| \chi_E(h(x)) = |\det(Dh(x))| \chi_G(h(x)) = |\det(Dh(x))| \chi_E(h(x))
\]
a.e. and so, by completeness, it follows \(x \rightarrow |\det(Dh(x))| \chi_E(h(x))\) must be Lebesgue measurable. This is because the function \(x \rightarrow |\det(Dh(x))| \chi_G(h(x))\) is Borel measurable due to the continuity of \(h\) which forces \(x \rightarrow |\det(Dh(x))| \chi_G(h(x))\) to be Borel measurable, and the other function in the product is of the form \(\chi_{h^{-1}(G)}(x)\) and since \(G\) is Borel, so is \(h^{-1}(G)\). Now the desired result follows because
\[
\int_\Omega |\det(Dh(x))| \chi_E(h(x)) \, dx
\]
is between the ends of 26.8.67. The rest of the argument involves the usual technique of approximating a nonnegative function with an increasing sequence of simple functions followed by consideration of the positive and negative parts of the real and imaginary parts of an arbitrary function in \(L^1(\Omega)\). The other version of the formula follows as in the proof of Theorem 26.8.6. This proves the corollary.
Chapter 27

Integration Of Differential Forms

27.1 Manifolds

Manifolds are sets which resemble $\mathbb{R}^n$ locally. To make the concept of a manifold more precise, here is a definition.

**Definition 27.1.1** Let $\Omega \subseteq \mathbb{R}^m$. A set, $U$, is open in $\Omega$ if it is the intersection of an open set from $\mathbb{R}^m$ with $\Omega$. Equivalently, a set, $U$ is open in $\Omega$ if for every point, $x \in U$, there exists $\delta > 0$ such that if $|x - y| < \delta$ and $y \in \Omega$, then $y \in U$. A set, $H$, is closed in $\Omega$ if it is the intersection of a closed set from $\mathbb{R}^m$ with $\Omega$. Equivalently, a set, $H$, is closed in $\Omega$ if whenever, $y$ is a limit point of $H$ and $y \in \Omega$, it follows $y \in H$.

Recall the following definition.

**Definition 27.1.2** Let $V \subseteq \mathbb{R}^n$. $C^k(V; \mathbb{R}^m)$ is the set of functions which are restrictions to $V$ of some function defined on $\mathbb{R}^n$ which has $k$ continuous derivatives and compact support. When $k = 0$, it means the restriction to $V$ of continuous functions with compact support.

**Definition 27.1.3** A closed and bounded subset of $\mathbb{R}^m$, $\Omega$, will be called an $n$-dimensional manifold with boundary, $n \geq 1$, if there are finitely many sets, $U_i$, open in $\Omega$ and continuous one to one functions, $R_i \in C^0(U_i; \mathbb{R}^n)$ such that $R_i U_i$ is relatively open in $\mathbb{R}^n_{\leq} \equiv \{ u \in \mathbb{R}^n : u_1 \leq 0 \}$, $R_i^{-1}$ is continuous. These mappings, $R_i$, together with their domains, $U_i$, are called charts and the totality of all the charts, $(U_i, R_i)$ just described is called an atlas for the manifold. Define

$$\text{int}(\Omega) \equiv \{ x \in \Omega : \text{ for some } i, R_i x \in \mathbb{R}^n_{\leq} \}$$
where \( R_{<}^{n} \equiv \{ v \in \mathbb{R}^{n} : u_{1} < 0 \} \). Also define

\[
\partial \Omega \equiv \{ x \in \Omega : \text{for some } i, R_{i}x \in R_{<}^{n} \}
\]

where

\[
R_{0}^{n} \equiv \{ u \in \mathbb{R}^{n} : u_{1} = 0 \}
\]

and \( \partial \Omega \) is called the boundary of \( \Omega \). Note that if \( n = 1 \), \( R_{0}^{n} \) is just the single point \( 0 \). By convention, we will consider the boundary of such a 0 dimensional manifold to be empty.

This definition is a little too restrictive. In general the collection of sets, \( U_{i} \) is not finite. However, in the case where \( \Omega \) is closed and bounded, compactness of \( \Omega \) can be used to get a finite covering and since this is the case of most interest here, the assumption that the collection of sets, \( U_{i} \), is finite is made. However, most of what is presented here can be generalized to the case of a locally finite atlas.

**Theorem 27.1.4** Let \( \partial \Omega \) and \( \text{int}(\Omega) \) be as defined above. Then \( \text{int}(\Omega) \) is open in \( \Omega \) and \( \partial \Omega \) is closed in \( \Omega \). Furthermore, \( \partial \Omega \cap \text{int}(\Omega) = \emptyset \), \( \Omega = \partial \Omega \cup \text{int}(\Omega) \), and for \( n \geq 2 \), \( \partial \Omega \) is an \( n - 1 \) dimensional manifold for which \( \partial(\partial \Omega) = \emptyset \). The property of being in \( \text{int}(\Omega) \) or \( \partial \Omega \) does not depend on the choice of atlas.

**Proof:** It is clear that \( \Omega = \partial \Omega \cup \text{int}(\Omega) \). First consider the claim that \( \partial \Omega \cap \text{int}(\Omega) = \emptyset \). Suppose this does not happen. Then there would exist \( x \in \partial \Omega \cap \text{int}(\Omega) \). Therefore, there would exist two mappings \( R_{i} \) and \( R_{j} \) such that \( R_{j}x \in R_{<}^{n} \) and \( R_{i}x \in \mathbb{R}_{>}^{n} \) with \( x \in U_{i} \cap U_{j} \). Now consider the map, \( R_{j} \circ R_{i}^{-1} \), a continuous one to one map from \( R_{<}^{n} \) to \( R_{>}^{n} \) having a continuous inverse. By continuity, there exists \( r > 0 \) small enough that,

\[
R_{i}^{-1}B(R_{i}x, r) \subseteq U_{i} \cap U_{j}.
\]

Therefore, \( R_{j} \circ R_{i}^{-1} \) \( B(R_{i}x, r) \) \( \subseteq R_{<}^{n} \) and contains a point on \( R_{0}^{n} \), \( R_{i}x \). However, this cannot occur because it contradicts the theorem on invariance of domain, Theorem 27.1.3, which requires that \( R_{j} \circ R_{i}^{-1} \) \( B(R_{i}x, r) \) must be an open subset of \( \mathbb{R}^{n} \) and this one isn’t because of the point on \( R_{0}^{n} \). Therefore, \( \partial \Omega \cap \text{int}(\Omega) = \emptyset \) as claimed. This argument shows that the property of being in \( \text{int}(\Omega) \) or \( \partial \Omega \) does not depend on the choice of the atlas.

To verify that \( \partial(\partial \Omega) = \emptyset \), let \( S_{i} \) be the restriction of \( R_{i} \) to \( \partial \Omega \cap U_{i} \). Thus

\[
S_{i}(x) = (0, (R_{i}x)_{2}, \cdots, (R_{i}x)_{n})
\]

and the collection of such points for \( x \in \partial \Omega \cap U_{i} \) is an open bounded subset of \( \{ u \in \mathbb{R}^{n} : u_{1} = 0 \} \), identified with \( \mathbb{R}^{n-1} \). \( S_{i}(\partial \Omega \cap U_{i}) \) is bounded because \( S_{i} \) is the restriction of a continuous function defined on \( \mathbb{R}^{m} \) and \( \partial \Omega \cap U_{i} \equiv V_{i} \) is contained in the compact set \( \Omega \). Thus if \( S_{i} \) is modified slightly, to be of the form

\[
S_{i}^{'}(x) = ((R_{i}x)_{2} - k_{i}, \cdots, (R_{i}x)_{n})
\]
where $k_i$ is chosen sufficiently large enough that $(R_i(V_i))_2 - k_i < 0$, it follows that

$$\{(V_i, S_i^j)\}$$

is an atlas for $\partial \Omega$ as an $n-1$ dimensional manifold such that every point of $\partial \Omega$ is sent to $\mathbb{R}^{n-1}_\infty$ and none gets sent to $\mathbb{R}^{n-1}_0$. It follows $\partial \Omega$ is an $n-1$ dimensional manifold with empty boundary. In case $n = 1$, the result follows by definition of the boundary of a 0 dimensional manifold.

Next consider the claim that $\text{int} (\Omega)$ is open in $\Omega$. If $x \in \text{int} (\Omega)$, are all points of $\Omega$ which are sufficiently close to $x$ also in $\text{int} (\Omega)$? If this were not true, there would exist $\{x_n\}$ such that $x_n \in \partial \Omega$ and $x_n \to x$. Since there are only finitely many charts of interest, this would imply the existence of a subsequence, still denoted by $x_n$ and a single map, $R_i$ such that $R_i (x_n) \in \mathbb{R}^n_0$. But then $R_i (x_n) \to R_i (x)$ and so $R_i (x) \in \mathbb{R}^n_0$ showing $x \in \partial \Omega$, a contradiction to $\text{int} (\Omega) \cap \partial \Omega = \emptyset$. Now it follows that $\partial \Omega$ is closed in $\Omega$ because $\partial \Omega = \Omega \setminus \text{int} (\Omega)$. This proves the Theorem.

**Definition 27.1.5** An $n$ dimensional manifold with boundary, $\Omega$ is a $C^k$ manifold with boundary for some $k \geq 1$ if

$$R_j \circ R_i^{-1} \in C^k \left( \mathbb{R}^n \right)$$

and $R_i^{-1} \in C^k \left( \mathbb{R}^n \right)$. It is called a continuous or Lipschitz manifold with boundary if the mappings, $R_j \circ R_i^{-1}, R_i^{-1}, R_i$ are respectively continuous or Lipschitz continuous. In the case where $\Omega$ is a $C^k, k \geq 1$ manifold, it is called orientable if in addition to this there exists an atlas, $(U_i, R_i)$, such that whenever $U_i \cap U_j \neq \emptyset$,

$$\det \left( D \left( R_j \circ R_i^{-1} \right) \right) (u) > 0 \text{ for all } u \in R_i (U_i \cap U_j)$$

(27.1.1)

The mappings, $R_i \circ R_i^{-1}$ are called the overlap maps. In the case where $k = 0$, the $R_i$ are only assumed continuous so there is no differentiability available and in this case, the manifold is oriented if whenever $A$ is an open connected subset of $\text{int} (R_i (U_i \cap U_j))$ whose boundary has measure zero and separates $\mathbb{R}^n$ into two components,

$$d (y, A, R_j \circ R_i^{-1}) \in \{1, 0\}$$

(27.1.2)

depending on whether $y \in R_j \circ R_i^{-1} (A)$. An atlas satisfying (27.1.1) or more generally (27.1.2) is called an oriented atlas.

It follows from Proposition 27.1.4 the degree in (27.1.2) is either undefined if $y \in R_j \circ R_i^{-1} \partial A$ or it is 1, -1, or 0.

The study of manifolds is really a generalization of something with which everyone who has taken a normal calculus course is familiar. We think of a point in three dimensional space in two ways. There is a geometric point and there are coordinates associated with this point. There are many different coordinate systems which describe a point. There are spherical coordinates, cylindrical coordinates and rectangular coordinates to name the three most popular coordinate systems. These coordinates are like the vector $u$. The point, $x$ is like the geometric point although it is always assumed $x$ has rectangular coordinates in $\mathbb{R}^m$ for some $m$. Under fairly general conditions, it can be shown there is no loss of generality in making such an assumption. Next is some algebra.
27.2 The Binet Cauchy Formula

The Binet Cauchy formula is a generalization of the theorem which says the determinant of a product is the product of the determinants. The situation is illustrated in the following picture.

\[
\begin{array}{c}
B \\
A
\end{array}
\]

**Theorem 27.2.1** Let \( A \) be an \( n \times m \) matrix with \( n \geq m \) and let \( B \) be a \( m \times n \) matrix. Also let \( A_i \), \( i = 1, \ldots, C(n, m) \) be the \( m \times m \) submatrices of \( A \) which are obtained by deleting \( n - m \) rows and let \( B_i \) be the \( m \times m \) submatrices of \( B \) which are obtained by deleting corresponding \( n - m \) columns. Then

\[
\det (BA) = \sum_{k=1}^{C(n,m)} \det (B_k) \det (A_k)
\]

**Proof:** This follows from a computation. By Corollary 4.7.5 on Page 76, \( \det (BA) = \)

\[
\frac{1}{m!} \sum_{i_1 \cdots i_m} \sum_{j_1 \cdots j_m} \text{sgn} (i_1 \cdots i_m) \text{sgn} (j_1 \cdots j_m) (BA)_{i_1j_1} (BA)_{i_2j_2} \cdots (BA)_{i_mj_m}
\]

\[
\frac{1}{m!} \sum_{i_1 \cdots i_m} \sum_{j_1 \cdots j_m} \text{sgn} (i_1 \cdots i_m) \text{sgn} (j_1 \cdots j_m) \cdot 
\]

\[
\sum_{r_1=1}^{n} B_{i_1r_1} A_{r_1j_1} \sum_{r_2=1}^{n} B_{i_2r_2} A_{r_2j_2} \cdots \sum_{r_m=1}^{n} B_{i_mr_m} A_{r_mj_m}
\]

Now denote by \( I_k \) one subsets of \( \{1, \ldots, n\} \) having \( m \) elements. Thus there are \( C(n, m) \) of these. Then the above equals

\[
= \sum_{k=1}^{C(n,m)} \sum_{\{r_1, \ldots, r_m\}=I_k} \frac{1}{m!} \sum_{i_1 \cdots i_m} \sum_{j_1 \cdots j_m} \text{sgn} (i_1 \cdots i_m) \text{sgn} (j_1 \cdots j_m) \cdot 
\]

\[
B_{i_1r_1} A_{r_1j_1} B_{i_2r_2} A_{r_2j_2} \cdots B_{i_mr_m} A_{r_mj_m}
\]

\[
= \sum_{k=1}^{C(n,m)} \sum_{\{r_1, \ldots, r_m\}=I_k} \frac{1}{m!} \sum_{i_1 \cdots i_m} \text{sgn} (i_1 \cdots i_m) B_{i_1r_1} B_{i_2r_2} \cdots B_{i_mr_m} \cdot 
\]

\[
\sum_{\{j_1, \ldots, j_m\}} \text{sgn} (j_1 \cdots j_m) A_{r_1j_1} A_{r_2j_2} \cdots A_{r_mj_m}
\]
27.3. **INTEGRATION OF DIFFERENTIAL FORMS ON MANIFOLDS**

\[ C^{(n,m)} \sum_{k=1} \sum_{\{r_1, \cdots, r_m\} = I_k} \frac{1}{m!} \operatorname{sgn}(r_1 \cdots r_m)^2 \det(B_k) \det(A_k) \]

\[ = C^{(n,m)} \sum_{k=1} \det(B_k) \det(A_k) \]

since there are \(m!\) ways of arranging the indices \(\{r_1, \cdots, r_m\}\).

---

**27.3 Integration Of Differential Forms On Manifolds**

This section presents the integration of differential forms on manifolds. This topic is a higher dimensional version of what is done in calculus in finding the work done by a force field on an object which moves over some path. There you evaluated line integrals. Differential forms are just a higher dimensional version of this idea and it turns out they are what it makes sense to integrate on manifolds. The following lemma, on Page 1043 used in establishing the definition of the degree and in giving a proof of the Brouwer fixed point theorem is also a fundamental result in discussing the integration of differential forms.

**Lemma 27.3.1** Let \(g : U \to V\) be \(C^2\) where \(U\) and \(V\) are open subsets of \(\mathbb{R}^n\). Then

\[ \sum_{j=1}^n (\operatorname{cof}(Dg))_{ij,j} = 0, \]

where here \((Dg)_{ij} = \frac{\partial g_i}{\partial x_j}\).

Also recall the interesting relation of the degree to integration in Corollary 26.7.3.

**Corollary 27.3.2** Let \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\) for \(p \geq 1\) and let \(h\) be Lipschitz where \(\partial U\) has measure zero for \(U\) a bounded open set and \(h(\partial U)^c\) has finitely many components. Then everything is measurable which needs to be and

\[ \int f(y) d(y, U, h) dy = \int_U \det(Dh(x)) f(h(x)) dx. \]

(Recall that if \(y \notin h(U)\), then \(d(y, U, h) = 0\).)

Recall Proposition 21.6.9.

**Proposition 27.3.3** Let \(\Omega\) be an open connected bounded set in \(\mathbb{R}^n\) such that \(\mathbb{R}^n \setminus \partial \Omega\) consists of two, three if \(n = 1\), connected components. Let \(f \in C(\overline{\Omega}, \mathbb{R}^n)\) be continuous and one to one. Then \(f(\Omega)\) is the bounded component of \(\mathbb{R}^n \setminus f(\partial \Omega)\) and for \(y \in f(\Omega)\), \(d(f, \Omega, y)\) either equals 1 or \(-1\).
Also recall the following fundamental lemma on partitions of unity in Corollary 13.5.9.

Lemma 27.3.4 Let $K$ be a compact set in $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^{\infty}$ be an open cover of $K$. Then there exist functions, $\psi_k \in C^\infty_c(U_i)$ such that $\psi_i \prec U_i$ and for all $x \in K$,

$$\sum_{i=1}^{\infty} \psi_i(x) = 1.$$ 

If $K$ is a compact subset of $U_1$ ($U_i$) there exist such functions such that also $\psi_1(x) = 1$ ($\psi_i(x) = 1$) for all $x \in K$.

With the above, what follows is the definition of what a differential form is and how to integrate one.

**Definition 27.3.5** Let $I$ denote an ordered list of $n$ indices taken from the set, $\{1, \ldots, m\}$. Thus $I = (i_1, \ldots, i_n)$. It is an ordered list because the order matters.

A differential form of order $n$ in $\mathbb{R}^m$ is a formal expression,

$$\omega = \sum_I a_I(x) \, dx^I$$

where $a_I$ is at least Borel measurable $dx^I$ is short for the expression

$$dx^{i_1} \wedge \cdots \wedge dx^{i_n},$$

and the sum is taken over all ordered lists of indices taken from the set, $\{1, \ldots, m\}$. For $\Omega$ an orientable $n$ dimensional manifold with boundary, define

$$\int_{\Omega} \omega$$

according to the following procedure in which it is assumed the integrals which occur make sense. Let $\{(U_i, R_i)\}$ be an oriented atlas for $\Omega$. Each $U_i$ is the intersection of an open set in $\mathbb{R}^m$, $O_i$, with $\Omega$ and so there exists a $C^\infty$ partition of unity subordinate to the open cover, $\{O_i\}$ which sums to 1 on $\Omega$. Thus $\psi_i \in C^\infty_c(O_i)$, has values in $[0, 1]$ and satisfies $\sum_i \psi_i(x) = 1$ for all $x \in \Omega$. Then define 27.3.3 by

$$\int_{\Omega} \omega \equiv \sum_{i=1}^{p} \sum_I \int_{R_i U_i} \psi_i(R_i^{-1}(u)) a_I(R_i^{-1}(u)) \frac{\partial(x_{i_1} \cdots x_{i_n})}{\partial(u_1 \cdots u_n)} \, du$$

(27.3.4)

where that symbol at the end denotes

$$\det \begin{pmatrix} x_{i_1, u_1} & x_{1_1, u_2} & \cdots & x_{i_1, u_n} \\ x_{i_2, u_1} & x_{i_2, u_2} & \cdots & x_{i_2, u_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i_n, u_1} & x_{i_n, u_2} & \cdots & x_{i_n, u_n} \end{pmatrix}(u)$$

for $(x_1, x_2, \ldots, x_n) = R_i^{-1}(u)$. 
27.3.3

Suppose which is makes no mention of partitions of unity or a particular atlas. What if you had a different atlas and a different partition of unity? Would \( \int_{\Omega} \omega \) change? In general, the answer is yes. However, there is a sense in which \( \int_{\Omega} \omega \) is well defined. This involves the concept of orientation. This looks a lot like the concept of an oriented manifold.

**Definition 27.3.6** Suppose \( \Omega \) is an \( n \) dimensional orientable manifold with boundary and let \((U_i, R_i)\) and \((V_i, S_i)\) be two oriented atlases of \( \Omega \). They have the same orientation if for all open connected sets \( A \subseteq S_j (V_j \cap U_i) \) with \( \partial A \) having measure zero and separating \( \mathbb{R}^n \) into two components,

\[
d \left( u, R_i \circ S_j^{-1}, A \right) \in \{0, 1\}
\]

depending on whether \( u \in R_i \circ S_j^{-1} (A) \).

The above definition of \( \int_{\Omega} \omega \) is well defined in the sense that any two atlases which have the same orientation deliver the same value for this symbol.

**Theorem 27.3.7** Suppose \( \Omega \) is an \( n \) dimensional Lipschitz orientable manifold with boundary and let \((U_i, R_i)\) and \((V_i, S_i)\) be two oriented atlases of \( \Omega \). Suppose the two atlases have the same orientation. Then if \( \int_{\Omega} \omega \) is computed with respect to the two atlases the same number is obtained.

**Proof:** Let \( \{ \psi_i \} \) be a partition of unity as described in Lemma 27.3.3 which is associated with the atlas \((U_i, R_i)\) and let \( \{ \eta_i \} \) be a partition of unity associated in the same manner with the atlas \((V_i, S_i)\). First note the following.

\[
\sum_I \int_{R_i U_i} \psi_i (R_i^{-1} (u)) a_f (R_i^{-1} (u)) \frac{\partial (x_i \cdots x_n)}{\partial (u_1 \cdots u_n)} du \quad (27.3.5)
\]

\[
= \sum_{j=1}^q \sum_I \int_{R_i (U_i \cap V_j)} \eta_j (R_i^{-1} (u)) \psi_i (R_i^{-1} (u)) a_f (R_i^{-1} (u)) \frac{\partial (x_i \cdots x_n)}{\partial (u_1 \cdots u_n)} du
\]

\[
= \sum_{j=1}^q \sum_I \int_{\text{int } R_i (U_i \cap V_j)} \eta_j (R_i^{-1} (u)) \psi_i (R_i^{-1} (u)) a_f (R_i^{-1} (u)) \frac{\partial (x_i \cdots x_n)}{\partial (u_1 \cdots u_n)} du
\]

The reason this can be done is that points not on the interior of \( R_i \) \((U_i \cap V_j)\) are on the plane \( u_1 = 0 \) which is a set of measure zero.

Now let \( A \) be an open connected set contained in \( S_j (U_j \cap V_j) \) whose boundary \( \partial A \) separates \( \mathbb{R}^n \) into two components. Then by assumption,

\[
\int_{R_i \circ S_j^{-1} (A)} \eta_j (R_i^{-1} (u)) \psi_i (R_i^{-1} (u)) a_f (R_i^{-1} (u)) \frac{\partial (x_i \cdots x_n)}{\partial (u_1 \cdots u_n)} du \quad (27.3.6)
\]
CHAPTER 27. INTEGRATION OF DIFFERENTIAL FORMS

\[ \int_{R_{i} \circ S_{j}^{-1}(A)} \eta_{j} \left( R_{i}^{-1}(u) \right) \psi_{i} \left( R_{i}^{-1}(u) \right) a_{f} \left( R_{i}^{-1}(u) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (u_{1} \cdots u_{n})} \, du \]

because that degree is given to be 1. (Unless \( u \in R_{i} \circ S_{j}^{-1}(A) \), the above degree equals 0.) By Corollary \[7.3.2\], this equals

\[ \int_{A} \eta_{j} \left( S_{j}^{-1}(v) \right) \psi_{i} \left( S_{j}^{-1}(v) \right) a_{f} \left( S_{j}^{-1}(v) \right) \cdot \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (u_{1} \cdots u_{n})} \left( R_{i} \circ S_{j}^{-1}(v) \right) \, \det \left( D \left( R_{i} \circ S_{j}^{-1} \right)(v) \right) \, dv \]

and by the chain rule and Rademacher’s theorem, Theorem \[7.3.1\], this equals

\[ \int_{A} \eta_{j} \left( S_{j}^{-1}(v) \right) \psi_{i} \left( S_{j}^{-1}(v) \right) a_{f} \left( S_{j}^{-1}(v) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (v_{1} \cdots v_{n})} \, dv \] (27.3.7)

Thus for every open \( A \) of the sort described \[7.3.3\] = \[7.3.4\]. By the Vitali covering theorem, there exists a sequence of disjoint open balls \( \{ B_{k} \} \) whose union fills up \( \text{int} \left( S_{j}(U_{i} \cap V_{j}) \right) \) except for a set of measure zero \( N \). Since \( R_{i} \circ S_{j}^{-1} \) is Lipschitz, it follows \( R_{i} \circ S_{j}^{-1}(N) \) also has measure zero. Therefore,

\[ \int_{R_{i}(U_{i} \cap V_{j})} \eta_{j} \left( R_{i}^{-1}(u) \right) \psi_{i} \left( R_{i}^{-1}(u) \right) a_{f} \left( R_{i}^{-1}(u) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (u_{1} \cdots u_{n})} \, du \] (27.3.8)

\[ = \int_{\text{int} \left( R_{i}(U_{i} \cap V_{j}) \right)} \eta_{j} \left( R_{i}^{-1}(u) \right) \psi_{i} \left( R_{i}^{-1}(u) \right) a_{f} \left( R_{i}^{-1}(u) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (u_{1} \cdots u_{n})} \, du \]

\[ = \sum_{k=1}^{\infty} \int_{R_{i} \circ S_{j}^{-1}(B_{k})} \eta_{j} \left( R_{i}^{-1}(u) \right) \psi_{i} \left( R_{i}^{-1}(u) \right) a_{f} \left( R_{i}^{-1}(u) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (u_{1} \cdots u_{n})} \, du \]

\[ = \sum_{k=1}^{\infty} \int_{B_{k}} \eta_{j} \left( S_{j}^{-1}(v) \right) \psi_{i} \left( S_{j}^{-1}(v) \right) a_{f} \left( S_{j}^{-1}(v) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (v_{1} \cdots v_{n})} \, dv \]

\[ = \int_{\text{int} \left( S_{j}(U_{i} \cap V_{j}) \right)} \eta_{j} \left( S_{j}^{-1}(v) \right) \psi_{i} \left( S_{j}^{-1}(v) \right) a_{f} \left( S_{j}^{-1}(v) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (v_{1} \cdots v_{n})} \, dv \] (27.3.9)

The equality of \[7.3.8\] and \[7.3.9\] was the goal. With this, the definition of \( \int \omega \) using the atlas \( (U_{i}, R_{i}) \) and partition of unity \( \{ \psi_{i} \}_{i=1}^{p} \) given in \[7.3.1\] is

\[ \sum_{i=1}^{p} \sum_{f} \int_{R_{i}U_{i}} \psi_{i} \left( R_{i}^{-1}(u) \right) a_{f} \left( R_{i}^{-1}(u) \right) \frac{\partial (x_{i_{1}} \cdots x_{i_{n}})}{\partial (u_{1} \cdots u_{n})} \, du \]
27.3.8 Let \( \omega \) be a differential form such that \( R \) and \( \psi \) has weak partial derivatives. Then define \( d\omega \) by replacing \( d\omega \equiv \sum_{k=1}^{m} \frac{\partial a_I(x)}{\partial x_k} dx_k \) (27.3.10) and putting a wedge after the \( dx_k \). Therefore,

\[
d\omega = \sum_{I} \sum_{k=1}^{m} \frac{\partial a_I(x)}{\partial x_k} dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}.
\] (27.3.11)

27.4 Stoke’s Theorem And The Orientation Of \( \partial \Omega \)

Here \( \Omega \) will be an \( n \) dimensional orientable Lipschitz manifold with boundary in \( \mathbb{R}^m \). Let an oriented manifold for it be \( \{U_i, R_i\}_{i=1}^{p} \) and let a \( C^\infty \) partition of unity be \( \{\psi_i\}_{i=1}^{p} \). Also let

\[
\omega = \sum_{I} a_I(x) dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}
\]

be a differential form such that \( a_I \) is \( C^1 \) (\( \Omega \)). Since \( \sum_{i} \psi_i(x) = 1 \) on \( \Omega \),

\[
d\omega = \sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \frac{\partial (\psi_j a_I)}{\partial x_k} (x) dx_k \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}
\]

It follows

\[
\int d\omega = \sum_{I} \sum_{k=1}^{m} \sum_{j=1}^{p} \int_{R_j(u)} \frac{\partial (\psi_j a_I)}{\partial x_k} (R_j^{-1}(u)) \frac{\partial (x_k, x_{i_1} \cdots x_{i_{n-1}})}{\partial (u_1, \cdots, u_n)} du
\]
27.4.12

where those last two expressions sum to \( e(\varepsilon) \) which converges to 0 as \( \varepsilon \to 0 \) for a suitable subsequence. Here is why.

\[
\frac{\partial (\psi_{a_I})}{\partial x_k}(R_{j\varepsilon}^{-1}(u)) \to \frac{\partial (\psi_{a_I})}{\partial x_k}(R_j^{-1}(u))
\]

because of the uniform convergence of \( R_{j\varepsilon}^{-1} \) to \( R_j^{-1} \). In addition to this,

\[
\frac{\partial (x_{k\varepsilon}, x_{i_{1\varepsilon}} \cdot \cdot \cdot x_{i_{n-1\varepsilon}})}{\partial (u_1, \ldots, u_n)} \to \frac{\partial (x_k, x_{i_1} \cdot \cdot \cdot x_{i_{n-1}})}{\partial (u_1, \ldots, u_n)}
\]

in \( L_r(R_j(U_j)) \) for any \( r > 0 \) and so a suitable subsequence converges pointwise.

The integrands are also uniformly integrable. Thus the Vitali convergence theorem can be applied to each of the integrals in the above sum and obtain that for a suitable subsequence, \( e(\varepsilon) \to 0 \).

Then (27.4.13) equals

\[
= \sum_{l=1}^n \sum_{j=1}^p \int \frac{\partial (\psi_{a_I})}{\partial x_k}(R_{j\varepsilon}^{-1}(u)) \sum_{l=1}^m \frac{\partial x_{k\varepsilon}}{\partial u_l} A_{1l} du + e(\varepsilon)
\]

where \( A_{1l} \) is the \( l^{th} \) cofactor for the determinant

\[
\frac{\partial (x_{k\varepsilon}, x_{i_{1\varepsilon}} \cdot \cdot \cdot x_{i_{n-1\varepsilon}})}{\partial (u_1, \ldots, u_n)}
\]

which is determined by a particular \( I \). I am suppressing the \( \varepsilon \) for the sake of notation. Then the above reduces to

\[
= \sum_{l=1}^n \sum_{j=1}^p \int \frac{\partial (\psi_{a_I})}{\partial x_k}(R_{j\varepsilon}^{-1}(u)) \sum_{l=1}^m \frac{\partial x_{k\varepsilon}}{\partial u_l} A_{1l} du + e(\varepsilon)
\]

(27.4.13)

(Note \( l \) goes up to \( n \) not \( m \).) Recall \( R_j(U_j) \) is relatively open in \( \mathbb{R}_n^\leq \). Consider the integral where \( l > 1 \). Integrate first with respect to \( u_l \). In this case the boundary term vanishes because of \( \psi_j \) and you get

\[
- \int_{R_j(U_j)} A_{1l,l} (\psi_{a_I} \circ R_{j\varepsilon}^{-1})(u) du
\]

(27.4.14)
Next consider the case where \( l = 1 \). Integrating first with respect to \( u_1 \), the term reduces to

\[
\int_{R_j V_j} \psi_j a_I \circ R_{j e}^{-1} (0, u_2, \ldots, u_n) A_{11} \, du_1 - \int_{R_j (U_j)} A_{11,1} \left( \psi_j a_I \circ R_{j e}^{-1} \right) (u) \, du
\]  

(27.4.15)

where \( R_j V_j \) is an open set in \( \mathbb{R}^{n-1} \) consisting of

\[
\left\{ (u_2, \ldots, u_n) \in \mathbb{R}^{n-1} : (0, u_2, \ldots, u_n) \in R_j (U_j) \right\}
\]

and \( du_1 \) represents \( du_2 du_3 \ldots du_n \) on \( R_j V_j \) for short. Thus \( V_j \) is just the part of \( \partial \Omega \) which is in \( U_j \) and the mappings \( S_j^{-1} \) given on \( R_j V_j = R_j (U_j \cap \partial \Omega) \) by

\[
S_j^{-1} (u_2, \ldots, u_n) = R_j^{-1} (0, u_2, \ldots, u_n)
\]

are such that \( \{(S_j, V_j)\} \) is an atlas for \( \partial \Omega \). Then if \( \{U_j, \psi_j\} \) and \( \{V_j, \psi_j\} \) are placed in \( \mathcal{U} \), then it follows from Lemma \( 27.4.14 \) that this reduces to

\[
\sum \sum_{l=1}^{p} \int_{R_j V_j} \psi_j a_I \circ R_{j e}^{-1} (0, u_2, \ldots, u_n) A_{11} \, du_1 + \varepsilon (\varepsilon)
\]

Now as before, there exists a subsequence, still denoted as \( \varepsilon \) such that each \( \partial x_s / \partial u_r \) converges pointwise to \( \partial x_s / \partial u_r \) and then using that these are bounded in every \( L^p \), one can use the Vitali convergence theorem to pass to a limit obtaining finally

\[
\sum \sum_{l=1}^{p} \int_{R_j V_j} \psi_j a_I \circ R_{j e}^{-1} (0, u_2, \ldots, u_n) A_{11} \, du_1 = \sum \sum_{l=1}^{p} \int_{S_j V_j} \psi_j a_I \circ S_j^{-1} (u_2, \ldots, u_n) A_{11} \, du_1
\]

\[
= \sum \sum_{l=1}^{p} \int_{S_j V_j} \psi_j a_I \circ S_j^{-1} (u_2, \ldots, u_n) \frac{\partial (x_1 \cdots x_{n-1})}{\partial (u_2, \ldots, u_n)} (0, u_2, \ldots, u_n) \, du_1
\]

(27.4.16)

This of course is the definition of \( \int_{\partial \Omega} \omega \) provided \( \partial \Omega \) is orientable. This is shown next.

What if \( \text{spt } a_I \subseteq K \subseteq U \cap U_j \) for each \( I \)? Then using Lemma \( 27.4.14 \) it can be shown that \( \int du \)

\[
\sum \int_{S_j (V_j \cap V_j)} a_I \circ S_j^{-1} (u_2, \ldots, u_n) \frac{\partial (x_1 \cdots x_{n-1})}{\partial (u_2, \ldots, u_n)} (0, u_2, \ldots, u_n) \, du_1
\]

This is done by using a partition of unity which has the property that \( \psi_j \) equals 1 on \( K \) which forces all the other \( \psi_k \) to equal zero there. Using the same trick involving a judicious choice of the partition of unity, \( \int du \) is also equal to

\[
\sum \int_{S_j (V_j \cap V_j)} a_I \circ S_i^{-1} (v_2, \ldots, v_n) \frac{\partial (x_1 \cdots x_{i-1})}{\partial (v_2, \ldots, v_n)} (0, v_2, \ldots, v_n) \, dv_1
\]
Similarly if $A$ is an open connected subset of $S_i(V_j \cap V_j)$ whose measure zero boundary separates $\mathbb{R}^n$ into two components, and $K$ is a compact subset of $S_i^{-1}(A)$, containing spt $a_I$ for all $I$, $\int d\omega$ equals each of the above.

$$\sum_I \int_A a_I \circ S_i^{-1}(v_2, \cdots, v_n) \frac{\partial (x_{i_1} \cdots x_{i_{n-1}})}{\partial (v_2, \cdots, v_n)} (0, v_2, \cdots, v_n) dv_1$$  \hspace{1cm} (27.4.17)

$$\sum_I \int_{S_j \circ S_i^{-1}(A)} a_I \circ S_j^{-1}(u_2, \cdots, u_n) \frac{\partial (x_{i_1} \cdots x_{i_{n-1}})}{\partial (u_2, \cdots, u_n)} du_1$$  \hspace{1cm} (27.4.18)

By Corollary 27.4.1 applied to $S_j \circ S_i^{-1}(v_1) = u_1$, the expression in (27.4.18) equals

$$\sum_I \int_{S_j \circ S_i^{-1}(A)} a_I \circ S_j^{-1}(u_2, \cdots, u_n) \frac{\partial (x_{i_1} \cdots x_{i_{n-1}})}{\partial (u_2, \cdots, u_n)} d (u_1, A, S_j \circ S_i^{-1}) du_1$$

and so, subtracting (27.4.17) and the above,

$$\sum_I \int_{S_j \circ S_i^{-1}(A)} a_I \circ S_j^{-1}(u_2, \cdots, u_n) \frac{\partial (x_{i_1} \cdots x_{i_{n-1}})}{\partial (u_2, \cdots, u_n)} \left(1 - d (u_1, A, S_j \circ S_i^{-1})\right) du_1 = 0$$  \hspace{1cm} (27.4.19)

Now by invariance of domain, it follows $S_j \circ S_i^{-1}(A)$ is an open connected set contained in a single component of $(S_j \circ S_i^{-1}(A))^C$ and so the above degree is constant on $S_j \circ S_i^{-1}(A)$. If this degree is not 1 then it follows that for any choice of the $a_I$ having compact support in $S_i^{-1}(A)$,

$$\sum_I \int_{S_j \circ S_i^{-1}(A)} a_I \circ S_j^{-1}(u_2, \cdots, u_n) \frac{\partial (x_{i_1} \cdots x_{i_{n-1}})}{\partial (u_2, \cdots, u_n)} du_1 = 0$$  \hspace{1cm} (27.4.19)

Next let $I$ always denote an increasing list of indices. Note that $S_j \circ S_i^{-1}$ maps the open set $A$ to an open set which therefore has positive Lebesgue measure. It follows from the area formula that

$$\det \left(D \left(S_j \circ S_i^{-1}\right)\right) = \det \left(\frac{1}{D (S_j \circ S_i^{-1}(u))} D S_i^{-1}(u)\right)$$  \hspace{1cm} (27.4.20)

must be nonzero on a set of positive measure. It follows that at least some

$$\frac{\partial (x_{i_1} \cdots x_{i_{n-1}})}{\partial (u_2, \cdots, u_n)}$$

must be nonzero since by the Binet Cauchy theorem, the above determinant in (27.4.20) is the sum of products of these multiplied by other determinants which
27.4. STOKES' THEOREM AND THE ORIENTATION OF ∂Ω

come from deleting corresponding columns in the matrix for $D(S_j(S^{-1}_j(u)))$. It follows that

$$
\sum_I \left( \frac{\partial(x_{i_1} \cdots x_{i_{n-1}})}{\partial(u_2, \cdots, u_n)} \right)^2
$$

is positive on a set of positive measure. Let $\lim_{p \to \infty} a_{lp} \circ S^{-1}_j = \frac{\partial(x_{i_1} \cdots x_{i_{n-1}})}{\partial(u_2, \cdots, u_n)}$ in $L^2(S_j \circ S^{-1}_j(A))$ for each $I = (i_1, \cdots, i_{n-1})$. Replacing $a_I \circ S^{-1}_j$ with $a_{lp} \circ S^{-1}_j$ in Theorem 27.4.1 and passing to the limit, it follows

$$
0 = \lim_{p \to \infty} \int_{S_j \circ S^{-1}_j(A)} \sum_I a_{lp} \circ S^{-1}_j(u) \frac{\partial(x_{i_1} \cdots x_{i_{n-1}})}{\partial(u_2, \cdots, u_n)} du_1
$$

$$
= \int_{S_j \circ S^{-1}_j(A)} \left( \frac{\partial(x_{i_1} \cdots x_{i_{n-1}})}{\partial(u_2, \cdots, u_n)} \right)^2 du_1 > 0
$$

a contradiction. Therefore, $d(u_1, A, S_j \circ S^{-1}_j) = 1$ and this shows the atlas is an oriented atlas for $\partial \Omega$. This has proved a general Stokes theorem.

**Theorem 27.4.1** Let $\Omega$ be an oriented Lipschitz manifold and let

$$\omega = \sum_I a_I(x) dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}},$$

where each $a_I$ is $C^1(\overline{\Omega})$. For $\{U_j, R_j\}_{j=0}^p$ an oriented atlas for $\Omega$ where $R_j(U_j)$ is a relatively open set in $\{u \in \mathbb{R}^n : u_1 \leq 0\}$, define an atlas for $\partial \Omega, \{V_j, S_j\}$ where $V_j \equiv \partial \Omega \cap U_j$ and $S_j$ is just the restriction of $R_j$ to $V_j$. Then this is an oriented atlas for $\partial \Omega$ and

$$\int_{\partial \Omega} \omega = \int_\Omega d\omega$$

where the two integrals are taken with respect to the given oriented atlas.

What if $a_I$ is only the restriction to $\Omega$ of a function in $W^{1,p}(\mathbb{R}^m), p > 1$? Would the same formula still hold? Let $\phi_\varepsilon$ be a mollifier and let $a_{I\varepsilon} \equiv a_I * \phi_\varepsilon$. Then Stoke’s theorem applies to the mollified situation and it follows

$$
\int_\Omega d\omega_\varepsilon
$$

$$
= \sum_I \sum_{k=1}^m \sum_{j=1}^p \int_{R_j(U_j)} \frac{\partial(\psi_j a_{I\varepsilon})}{\partial x_k} (R_j^{-1}(u)) \frac{\partial(x_k, x_{i_1} \cdots x_{i_{n-1}})}{\partial(u_1, \cdots, u_n)} du
$$

$$
= \sum_I \sum_{j=1}^p \int_{S_j V_j} \psi_j a_{I\varepsilon} \circ S_j^{-1}(u_2, \cdots, u_n) \frac{\partial(x_{i_1} \cdots x_{i_{n-1}})}{\partial(u_2, \cdots, u_n)} (0, u_2, \cdots, u_n) du_1
$$

$$
= \int_{\partial \Omega} \omega_\varepsilon
$$
CHAPTER 27. INTEGRATION OF DIFFERENTIAL FORMS

Now if you let $\varepsilon \to 0$, it follows from the definition of convolution that

$$\frac{\partial (\psi_j a_{I\varepsilon})}{\partial x_k} \to \frac{\partial (\psi_j a_I)}{\partial x_k} \text{ in } L^p(\mathbb{R}^m)$$

and so there is a subsequence such that for each $k$,

$$\frac{\partial (\psi_j a_{I\varepsilon})}{\partial x_k}(x) \to \frac{\partial (\psi_j a_I)}{\partial x_k}(x)$$

pointwise a.e. Since $R_j^{-1}, R_j$ are Lipschitz, they take sets of measure zero to sets of measure zero. Hence

$$\frac{\partial (\psi_j a_{I\varepsilon})}{\partial x_k} \circ R_j^{-1} \to \frac{\partial (\psi_j a_I)}{\partial x_k} \circ R_j^{-1}$$

pointwise a.e. on $R_j(U_j)$. Similar considerations apply to $a_{I\varepsilon}$. Using the Vitali convergence theorem in $\int_\Omega d\omega_\varepsilon, \int_\Omega \omega_\varepsilon$, it is possible to pass to the limit. This is because the integrands are bounded in $L^p$ and so they are uniformly integrable. This proves the following corollary.

**Corollary 27.4.2** Let $\Omega$ be an oriented Lipschitz manifold and let

$$\omega = \sum_I a_I(x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}}.$$ 

where each $a_I$ is in $W^{1,p}(\mathbb{R}^m)$ where $p > 1$. For $\{U_j, R_j\}_{j=0}^p$ an oriented atlas for $\Omega$ where $R_j(U_j)$ is a relatively open set in

$$\{u \in \mathbb{R}^n : u_1 \leq 0\},$$

define an atlas for $\partial \Omega, \{V_j, S_j\}$ where $V_j \equiv \partial \Omega \cap U_j$ and $S_j$ is just the restriction of $R_j$ to $V_j$. Then this is an oriented atlas for $\partial \Omega$ and

$$\int_{\partial \Omega} \omega = \int_\Omega d\omega$$

where the two integrals are taken with respect to the given oriented atlas.

### 27.5 Green’s Theorem

Green’s theorem is a well known result in calculus and it pertains to a region in the plane. I am going to generalize to an open set in $\mathbb{R}^n$ with sufficiently smooth boundary using the methods of differential forms described above.
27.5. GREEN’S THEOREM

27.5.1 An Oriented Manifold

A bounded open subset, Ω, of \( \mathbb{R}^n, n \geq 2 \) has Lipschitz boundary and lies locally on one side of its boundary if it satisfies the following conditions.

For each \( p \in \partial \Omega \equiv \Omega \setminus \Omega \), there exists an open set, \( Q \), containing \( p \), an open interval \((a, b)\), a bounded open set \( B \subseteq \mathbb{R}^{n-1} \), and an orthogonal transformation \( R \) such that \( \det R = 1 \),

\[
B \times (a, b) = RQ,
\]

and letting \( W = Q \cap \Omega \),

\[
RW = \{ u \in \mathbb{R}^n : a < u_1 < g(u_2, \ldots, u_n), (u_2, \ldots, u_n) \in B \}
\]

where \( g \) is Lipschitz continuous on \( \mathbb{R}^{n-1} \), \( g(u_2, \ldots, u_n) < b \) for \( (u_2, \ldots, u_n) \in B \), and \( g \) vanishing outside some compact set in \( \mathbb{R}^{n-1} \).

\[
R(\partial \Omega \cap Q) = \{ u \in \mathbb{R}^n : u_1 = g(u_2, \ldots, u_n), (u_2, \ldots, u_n) \in B \}.
\]

Note that finitely many of these sets \( Q \) cover \( \partial \Omega \) because \( \partial \Omega \) is compact. The following picture describes the situation.

Define \( P_1 : \mathbb{R}^n \to \mathbb{R}^{n-1} \) by

\[
P_1u \equiv (u_2, \ldots, u_n)
\]

and \( \Sigma : \mathbb{R}^n \to \mathbb{R}^n \) given by

\[
\Sigma u \equiv u - g(P_1u)e_1
\]

\[
\equiv u - g(u_2, \ldots, u_n)e_1
\]

\[
\equiv (u_1 - g(u_2, \ldots, u_n), u_2, \ldots, u_n)
\]

Thus \( \Sigma \) is invertible and

\[
\Sigma^{-1}u = u + g(P_1u)e_1
\]

\[
\equiv (u_1 + g(u_2, \ldots, u_n), u_2, \ldots, u_n)
\]

For \( x \in \partial \Omega \cap Q \), it follows the first component of \( Rx \) is \( g(P_1(Rx)) \). Now define \( R : W \to \mathbb{R}_x^n \) as

\[
u \equiv Rx \equiv Rx - g(P_1(Rx))e_1 \equiv \Sigma Rx
\]
and so it follows

\[ \mathbb{R}^{-1} = R^* \Sigma^{-1}. \]

These mappings \( R \) involve first a rotation followed by a variable sheer in the direction of the \( u_1 \) axis.

Since \( \partial \Omega \) is compact, there are finitely many of these open sets, \( Q_1, \ldots, Q_p \) which cover \( \partial \Omega \). Let the orthogonal transformations and other quantities described above also be indexed by \( k \) for \( k = 1, \ldots, p \). Also let \( Q_0 \) be an open set with \( Q_0 \subseteq \Omega \) and \( \Omega \) is covered by \( Q_0, Q_1, \ldots, Q_p \). Let \( u \equiv R_0 x = x - ke_1 \) where \( k \) is large enough that \( R_0 Q_0 \subseteq \mathbb{R}^n \). Thus in this case, the orthogonal transformation \( R_0 \) equals \( I \) and \( \Sigma_0 x = x - ke_1 \). I claim \( \Omega \) is an oriented manifold with boundary and the charts are \( (W_i, R_i) \).

Letting \( A \) be an open set contained in \( R_i (W_i \cap W_j) \) such that \( \partial A \) has measure 0 and \( \partial A \) separates \( \mathbb{R}^n \) into two components, consider

\[ d (u, A, R_j \circ R_i^{-1}) \, u \notin R_j \circ R_i^{-1} (\partial A) \]

By convolving \( g \) with a mollifier, there exists a sequence of infinitely differentiable functions \( g_\varepsilon \) which converge uniformly to \( g \) on all of \( \mathbb{R}^{n-1} \) as \( \varepsilon \to 0 \). Therefore, letting \( \Sigma_\varepsilon \) be the corresponding functions defined above with \( g \) replaced with \( g_\varepsilon \), it follows the \( \Sigma_\varepsilon \) will converge uniformly to \( \Sigma \) and \( \Sigma_\varepsilon^{-1} \) will converge uniformly to \( \Sigma^{-1} \). Thus from the above descriptions of \( R_j^{-1} \), it follows \( R_j^{-1} \) converges uniformly to \( R_j^{-1} \) for each \( j \). Therefore, if \( \varepsilon \) is small enough, \( u \notin (\partial R_\varepsilon \circ R_\varepsilon^{-1} + (1 - t) R_j \circ R_i^{-1}) (\partial A) \) and so from properties of the degree, the mappings \( R_j \) and \( R_i^{-1} \) can be replaced with smooth ones in computing the degree. To save on notation, I will drop the \( \varepsilon \).

The mapping involved is

\[ \Sigma_j R_j R_i^* \Sigma_i^{-1} \]

and it is a one to one mapping. What is the determinant of its derivative? By the chain rule,

\[
D (\Sigma_j R_j R_i^* \Sigma_i^{-1}) = D \Sigma_j (R_j R_i^* \Sigma_i^{-1}) D R_j (R_i^* \Sigma_i^{-1}) D R_i^* (\Sigma_i^{-1}) D \Sigma_i^{-1}
\]

However,

\[
\det (D \Sigma_j) = 1 = \det (D \Sigma_i^{-1})
\]

and \( \det (R_i) = \det (R_i^*) = 1 \) by assumption. Therefore, if \( u \in (R_j \circ R_i^{-1}) (A) \), the above degree is 1 and if \( u \) is not in this set, the above degree is 0 or undefined if \( u \) is on \( (R_j \circ R_i^{-1}) (\partial A) \). By Definition \( \mathbb{R}^n \) \( \Omega \) is indeed an oriented manifold.

### 27.5.2 Green’s Theorem

The general Green’s theorem is the following. It follows from Corollary \( \mathbb{R}^n \).

**Theorem 27.5.1** Let \( \Omega \) be a bounded open set having Lipschitz boundary as described above. Also let

\[ \omega = \sum_I a_I (x) dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} \]
27.6.  THE DIVERGENCE THEOREM

be a differential form where \( a_I \) is assumed to be the restriction to \( \Omega \) of a function in \( W^{1,p}(\mathbb{R}^n) \), \( p > 1 \). Then

\[
\int_{\partial \Omega} \omega = \int_{\Omega} d\omega
\]

It can be shown that, since the boundary is Lipschitz, it would have sufficed to assume \( u \in W^{1,p}(\Omega) \) and then it is automatically the restriction of one in \( W^{1,p}(\mathbb{R}^n) \). However, these terms have not all been defined and the necessary results are not proved till the topic of Sobolev spaces is discussed.

Another thing to notice is that, while the above result is pretty general, including the usual calculus result in the plane as a special case, it does not have the generality of the best results in the plane which involve only a rectifiable simple closed curve. The issue whether \( \partial \Omega \) is an oriented manifold was dealt with in the general Stokes theorem described above.

Next is a general version of the divergence theorem which comes from choosing the differential form in an auspicious manner.

27.6  The Divergence Theorem

From Green’s theorem, one can quickly obtain a general Divergence theorem for \( \Omega \) as described above in Section 27.5.1. First note that from the above description of the \( R_j \),

\[
\frac{\partial (x_k, x_{i_1}, \ldots, x_{i_{n-1}})}{\partial (u_1, \ldots, u_n)} = \text{sgn} (k, i_1, \ldots, i_{n-1}).
\]

So let \( F(x) \) be a Lipschitz vector field. Say \( F = (F_1, \ldots, F_n) \). Consider the differential form

\[
\omega(x) \equiv \sum_{k=1}^{n} F_k(x) (-1)^{k-1} \, dx_1 \wedge \cdots \wedge \hat{dx_k} \wedge \cdots \wedge dx_n
\]

where the hat means \( dx_k \) is being left out. Here it is assumed \( F_k \) is the restriction to \( \Omega \) of a function in \( W^{1,p}(\mathbb{R}^n) \) where \( p > 1 \). Then

\[
d\omega(x) = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{\partial F_k}{\partial x_j} (-1)^{k-1} \, dx_j \wedge dx_1 \wedge \cdots \wedge \hat{dx_k} \wedge \cdots \wedge dx_n
\]

\[
= \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k} \, dx_1 \wedge \cdots \wedge dx_k \wedge \cdots \wedge dx_n
\]

\[
= \text{div}(F) \, dx_1 \wedge \cdots \wedge dx_k \wedge \cdots \wedge dx_n
\]

The assertion between the first and second lines follows right away from properties of determinants and the definition of the integral of the above wedge products in terms of determinants. From Green’s theorem and the change of variables formula applied to the individual terms in the description of \( \int_{\Omega} d\omega \)

\[
\int_{\Omega} \text{div}(F) \, dx =
\]
by dividing by its magnitude. Then it

\[ \sum_{j=1}^{p} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} (\psi_j F_k) \circ R_j^{-1} (0, u_2, \ldots, u_n) \, du_1, \]

du_1 \text{ short for } du_2 du_3 \cdots du_n. \text{ Also, this shows the result on the right of the equal sign does not depend on the choice of partition of unity or on the atlas.}

I want to write this in a more attractive manner which will give more insight in terms of the Hausdorff measure on \( \partial \Omega \). The above involves a particular partition of unity, the functions being the \( \psi_j \). Replace \( F \) in the above with \( \psi \circ F \). Next let \( \{ \eta_j \} \) be a partition of unity \( \eta_j \prec O_j \) such that \( \eta_s = 1 \) on \( \text{spt } \psi_s \). This partition of unity exists by Lemma 27.6.21. Then

\[ \int \Omega \text{div}(\psi \circ F) \, dx = \]

\[ \sum_{j=1}^{p} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} (\eta_j \psi_s F_k) \circ R_j^{-1} (0, u_2, \ldots, u_n) \, du_1 \]

\[ = \int \sum_{k=1}^{n} \sum_{j=1}^{p} (-1)^{k-1} \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} (\psi_s F_k) \circ R_s^{-1} (0, u_2, \ldots, u_n) \, du_1 \quad (27.6.21) \]

because since \( \eta_s = 1 \) on \( \text{spt } \psi_s \), it follows all the other \( \eta_j \) equal zero there. Consider the vector defined for \( u_1 \in \mathbb{R}_s (W_s) \cap \mathbb{R}^n \), whose \( k \)-th component is

\[ (-1)^{k-1} \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} = (-1)^{k+1} \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} \quad (27.6.22) \]

Suppose you dot this vector with a “tangent” vector \( \partial R_s^{-1}/\partial u_i \). For each \( j \) this yields

\[ \sum_{j} (-1)^{k+1} \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} \frac{\partial x_k}{\partial u_i} = 0 \]

because it is the expansion of

\[
\begin{vmatrix}
  x_{1,i} & x_{1,2} & \cdots & x_{1,n} \\
x_{2,i} & x_{2,2} & \cdots & x_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{n,i} & x_{n,2} & \cdots & x_{n,n}
\end{vmatrix},
\]

a determinant with two equal columns. Thus this vector is at least in some sense normal to \( \Omega \). Since it works in the divergence theorem, it is called the exterior normal.

One could normalize the vector of 27.6.22 by dividing by its magnitude. Then it would be the unit exterior normal \( \mathbf{n} \). Letting \( J(u_1) \) be its usual Euclidean norm, this equals

\[ J(u_1)^2 = \sum_{k=1}^{n} \left( \frac{\partial (x_1, \ldots \hat{x}_k \ldots \cdot, x_n)}{\partial (u_2, \ldots, u_n)} \right)^2 \]
27.6. THE DIVERGENCE THEOREM

and by the Binet Cauchy theorem this equals

\[ \det \left( DR_s^{-1}(u) \right) \cdot DR_s^{-1}(u) \]^{1/2} \]

Thus the expression in \[27.6.21\] reduces to

\[ \int_{B_s} (\psi_s F \circ R_s^{-1}(u_1)) \cdot n( R_s^{-1}(u_1)) J(u_1) du_1. \]

By the area formula, Theorem \[26.5.11\] this reduces to

\[ \int_{\partial \Omega \cap W_s} \psi_s F \cdot n d\mathcal{H}^{n-1} = \int_{\partial \Omega} \psi_s F \cdot n d\mathcal{H}^{n-1}. \]

It follows upon summing over \( s \) and using that the \( \psi_s \) add to 1,

\[ \int_{\partial \Omega} F \cdot n d\mathcal{H}^{n-1} = \int_{\Omega} \sum_{s=1}^p \text{div} (\psi_s F) dx \]

\[ = \int_{\Omega} \sum_{s=1}^p \psi_s F_k + \psi_s \text{div} (F) dx = \int_{\Omega} F_k \left( \sum_{s=1}^p \psi_s \right)_k + \psi_s \text{div} (F) dx \]

\[ = \int_{\Omega} \text{div} (F) dx \]

This proves the following general divergence theorem.

**Theorem 27.6.1** Let \( \Omega \) be a bounded open set having Lipschitz boundary as described above. Also let \( F \) be a vector field with the property that for each component function of \( F \), \( F_k \) is the restriction to \( \Omega \) of a function in \( W^{1,p}(\mathbb{R}^n) \), \( p > 1 \). Then there exists a normal vector \( n \) which is defined a.e. on \( \partial \Omega \) such that

\[ \int_{\partial \Omega} F \cdot n d\mathcal{H}^{n-1} = \int_{\Omega} \text{div} (F) dx \]

It is clear \( n \) is unique \( \mathcal{H}^{n-1} \) a.e. since if there were two, then a simple manipulation shows for all such \( F \),

\[ \int_{\partial \Omega} F \cdot (n - n_1) d\mathcal{H}^{n-1} = 0 \]

Thus \( n - n_1 = 0 \) a.e.
Chapter 28

Differentiation Of Radon Measures

This is a brief chapter on certain important topics on the differentiation theory for general Radon measures. For different proofs and some results which are not discussed here, a good source is [44] which is where I first read some of these things. The Besicovitch covering theorem is one of the most amazing and insightful ideas that I have ever encountered. It is simultaneously elegant, elementary and profound. The next section is an attempt to present this wonderful result.

28.1 Besicovitch Covering Theorem

When dealing with probability distribution functions or some other Radon measure, it is necessary to have a better covering theorem than the Vitali covering theorem which works well for Lebesgue measure. However, for a Radon measure, if you enlarge the ball by making the radius larger, you don’t know what happens to the measure of the enlarged ball except that its measure does not get smaller. Thus the thing required is a covering theorem which does not depend on enlarging balls.

The first fundamental observation is found in the following lemma which holds for the context illustrated by the following picture. This picture is drawn such that the balls come from the usual Euclidean norm, but the norm could be any norm on $\mathbb{R}^n$. 

![Diagram of balls and points](image-url)
The idea is to consider balls $B_i$ which intersect a given ball $B$ such that $B$ contains no center of any $B_i$ and no $B_i$ contains the center of another $B_j$. There are two cases to consider, the case where the balls have large radii and the case where the balls have small radii.

**Intersections with big balls**

**Lemma 28.1.1** Let the balls $B_a, B_x, B_y$ be as shown having radii $r, r_x, r_y$ respectively. Suppose the centers of $B_x$ and $B_y$ are not both in any of the balls shown, and suppose $r_y \geq r_x \geq 0$ or where $\alpha$ is a number larger than 1. Also let $P_x \equiv a + r \frac{x-a}{|x-a|}$ with $P_y$ being defined similarly. Then it follows that $|P_x - P_y| \geq \frac{\alpha - 1}{\alpha + 1} r$. There exists a constant $L(n, \alpha)$ depending on $\alpha$ and the dimension, such that if $B_1, \ldots, B_m$ are all balls such that any pair are in the same situation relative to $B_a$ as $B_x$ and $B_y$, then $m \leq L(n, \alpha)$.

**Proof:** From the definition,

$$|P_x - P_y| = r \left| \frac{x-a}{|x-a|} - \frac{y-a}{|y-a|} \right|$$

$$= r \left| \frac{(x-a) |y-a| - (y-a) |x-a|}{|x-a| |y-a|} \right|$$

$$= r \left| \frac{|y-a| (x-y) + (y-a) (|y-a| - |x-a|)}{|x-a| |y-a|} \right|$$

$$\geq r \left| \frac{|x-y| - r |y-a| |x-a| - |y-a|}{|x-a| |y-a|} \right|$$

$$= r \left( \frac{|x-y|}{|x-a|} - \frac{r}{|x-a|} |y-a| - |x-a| \right). \quad (28.1.1)$$

There are two cases. First suppose that $|y-a| - |x-a| \geq 0$. Then the above

$$= r \frac{|x-y|}{|x-a|} - \frac{r}{|x-a|} |y-a| + r.$$

From the assumptions, $|x-y| \geq r_y$ and also $|y-a| \leq r + r_y$. Hence the above

$$\geq r \left( \frac{r_y}{|x-a|} - \frac{r + r_y}{|x-a|} + 1 \right) \geq r \left( 1 - \frac{r}{|x-a|} \right)$$

$$\geq r \left( 1 - \frac{r}{r_x} \right) \geq r \left( 1 - \frac{1}{\alpha} \right) = r \left( \frac{\alpha - 1}{\alpha} \right) \geq \frac{\alpha - 1}{\alpha + 1}.$$

The other case is that $|y-a| - |x-a| < 0$ in 28.1.1. Then in this case 28.1.1 reduces to

$$r \left( \frac{|x-y|}{|x-a|} - \frac{1}{|x-a|} (|x-a| - |y-a|) \right)$$

$$= r \left( \frac{|x-y|}{|x-a|} - 1 + \frac{|y-a|}{|x-a|} \right).$$
Let not be \( \alpha \) of the points \((B_1, \ldots, B_m)\) be a ball having radius \(r\) and \(\gamma\) the application of this lemma, we will have \(\alpha r\) in contrast to the above lemma in which the radii of the balls are larger than \(\alpha r\) given ball. Note that in the statement of this lemma, the radii are smaller than \(\alpha r\). Next is a lemma which has to do with relatively small balls intersecting \(L\) be any larger than \(1\) compact and so there exists a \(\alpha\). How many points on the unit sphere can be pairwise this far apart? This set is \(P\). Then since \(\parallel x - a \parallel \leq r + r_x\), \(\parallel x - y \parallel \geq r_y\), \(\parallel x - a \parallel \leq r + r_x\), \(\parallel y - a \parallel \geq r_y\),

\[
\geq \frac{r}{r_x + r} (r_y - (r + r_x) + r_y) \geq \frac{r}{r_x + r} (r_y - (r + r_y) + r_y)
\]

\[
\geq \frac{r}{r_x + r} (r_y - r) \geq \frac{r}{r_x + r} (r_x - r) \geq \frac{r}{r_x + r} (r_x - \frac{\alpha}{\gamma} r_x)
\]

\[
= \frac{r}{1 + (1/\alpha)} (1 - 1/\alpha) = \frac{\alpha - 1}{\alpha + 1}.
\]

This proves the estimate between \(P_x\) and \(P_y\). Finally, in the case of the balls \(B_i\) having centers at \(x_i\), then as above, let \(P_{x_i} = a + r \frac{x_i - a}{\parallel x_i - a \parallel} \). Then \((P_{x_i} - a)^{-1}\) is on the unit sphere having center \(0\). Furthermore,

\[
\parallel (P_{x_i} - a)^{-1} - (P_{y_i} - a)^{-1} \parallel = r^{-1} \parallel P_{x_i} - P_{y_i} \parallel \geq r^{-1} \frac{\alpha - 1}{\alpha + 1} = \frac{\alpha - 1}{\alpha + 1}.
\]

How many points on the unit sphere can be pairwise this far apart? This set is compact and so there exists a \(\frac{1}{4} \left( \frac{\alpha - 1}{\alpha + 1} \right)\) net having \(L(n, \alpha)\) points. Thus \(m\) cannot be any larger than \(L(n, \alpha)\) because if it were, then by the pigeon hole principal, two of the points \((P_{x_i} - a)^{-1}\) would lie in a single ball \(B \left( p, \frac{1}{4} \left( \frac{\alpha - 1}{\alpha + 1} \right) \right)\) so they could not be \(\frac{\alpha - 1}{\alpha + 1}\) apart. \(\blacksquare\)

The above lemma has do do with balls which are relatively large intersecting a given ball. Next is a lemma which has to do with relatively small balls intersecting a given ball. Note that in the statement of this lemma, the radii are smaller than \(\alpha r\) in contrast to the above lemma in which the radii of the balls are larger an \(\alpha r\). In the application of this lemma, we will have \(\gamma = 4/3\) and \(\beta = 1/3\). These constants will come from a construction, while \(\alpha\) is just something larger than \(1\) which we will take here to equal \(10\).

**Intersections with small but comparable balls**

**Lemma 28.1.2** Let \(B\) be a ball having radius \(r\) and suppose \(B\) has nonempty intersection with the balls \(B_1, \ldots, B_m\) having radii \(r_1, \ldots, r_m\) respectively, and as before, no \(B_i\) has the center of any other and the centers of the \(B_i\) are not contained in \(B\). Suppose \(\alpha, \gamma > 1\) and the \(r_i\) are comparable with \(r\) in the sense that

\[
\frac{1}{\gamma} r \leq r_i \leq \alpha r.
\]
Let $B'_i$ have the same center as $B_i$ with radius equal to $r'_i = \beta r_i$ for some $\beta < 1$. If the $B'_i$ are disjoint, then there exists a constant $M(n, \alpha, \beta, \gamma)$ such that $m \leq M(n, \alpha, \beta, \gamma)$. Letting $\alpha = 10$, $\beta = 1/3$, $\gamma = 4/3$, it follows that $m \leq 60^n$.

**Proof:** Let the volume of a ball of radius $r$ be given by $\alpha(n) r^n$ where $\alpha(n)$ depends on the norm used and on the dimension $n$ as indicated. The idea is to enlarge $B$, till it swallows all the $B'_i$. Then, since they are disjoint and their radii are not too small, there can’t be too many of them.

This can be done for a single $B'_i$ by enlarging the radius of $B$ to $r + r_i + r'_i$.

Then to get all the $B_i$, you would just enlarge the radius of $B$ to $r + \alpha r + \beta \alpha r = (1 + \alpha + \alpha \beta) r$. Then, using the inequality which makes $r_i$ comparable to $r$, it follows that

$$
\sum_{i=1}^{m} \alpha(n) \left( \frac{\beta r_i}{\gamma} \right)^n \leq \sum_{i=1}^{m} \alpha(n) (\beta r_i)^n \leq \alpha(n) (1 + \alpha + \alpha \beta)^n r^n
$$

Therefore,

$$
m \left( \frac{\beta}{\gamma} \right)^n \leq (1 + \alpha + \alpha \beta)^n
$$

and so $m \leq (1 + \alpha + \alpha \beta)^n \left( \frac{2}{3} \right)^n \equiv M(n, \alpha, \beta, \gamma)$.

From now on, let $\alpha = 10$ and let $\beta = 1/3$ and $\gamma = 4/3$. Then

$$
M(n, \alpha, \beta, \gamma) \leq \left( \frac{172}{3} \right)^n \leq 60^n
$$

Thus $m \leq 60^n$. ■

The next lemma gives a construction which yields balls which are comparable as described in the above lemma. $r(B)$ will denote the radius of the ball $B$.

**A construction of a sequence of balls**

**Lemma 28.1.3** Let $\mathcal{F}$ be a nonempty set of nonempty balls in $\mathbb{R}^n$ with

$$
\sup \{ \text{diam}(B) : B \in \mathcal{F} \} \leq D < \infty
$$

and let $A$ denote the set of centers of these balls. Suppose $A$ is bounded. Define a sequence of balls from $\mathcal{F}$, $\{B_j\}_{j=1}^J$ where $J \leq \infty$ such that

$$
r(B_1) \geq \frac{3}{4} \sup \{ r(B) : B \in \mathcal{F} \}
$$

(28.1.2)
and if
\[ A_m \equiv A \setminus (\cup_{i=1}^{m} B_i) \neq \emptyset, \]
then \( B_{m+1} \in \mathcal{F} \) is chosen with center in \( A_m \) such that
\[ r_{m+1} \equiv r(B_{m+1}) \geq \frac{3}{4} \sup \{ r : B(a, r) \in \mathcal{F}, a \in A_m \}. \]
Then letting \( B_j = B(a_j, r_j) \), this sequence satisfies
\[ r(B_k) \leq \frac{4}{3} r(B_j) \text{ for } j < k, \]
\[ \{B(a_j, r_j/3)\}_{j=1}^{J} \text{ are disjoint,} \]
\[ A \subseteq \cup_{i=1}^{J} B_i. \]

**Proof:** Consider (28.1.3). First note the sets \( A_m \) form a decreasing sequence. Thus from the definition of \( B_j \), for \( j < k \),
\[ r(B_k) \leq \sup \{ r : B(a, r) \in \mathcal{F}, a \in A_{k-1} \} \]
\[ \leq \sup \{ r : B(a, r) \in \mathcal{F}, a \in A_{j-1} \} \leq \frac{4}{3} r(B_j) \]
because the construction gave
\[ r(B_j) \geq \frac{3}{4} \sup \{ r : B(a, r) \in \mathcal{F}, a \in A_{j-1} \} \]

Next consider (28.1.6). If \( x \in B(a_j, r_j/3) \cap B(a_i, r_i/3) \) where these balls are two which are chosen by the above scheme such that \( j > i \), then from what was just shown
\[ ||a_j - a_i|| \leq ||a_j - x|| + ||x - a_i|| \leq \frac{r_j}{3} + \frac{r_i}{3} \leq \left( \frac{4}{9} + \frac{1}{3} \right) r_i = \frac{7}{9} r_i < r_i \]
and this contradicts the construction because \( a_j \) is not covered by \( B(a_i, r_i) \).

Finally consider the claim that \( A \subseteq \cup_{i=1}^{J} B_i \). Pick \( B_1 \) satisfying (28.1.2). If \( B_1, \ldots, B_m \) have been chosen, and \( A_m \) is given in (28.1.3), then if it equals \( \emptyset \), it follows \( A \subseteq \cup_{i=1}^{m} B_i \). Set \( J = m \). Now let \( a \) be the center of \( B_a \in \mathcal{F} \). If \( a \in A_m \) for all \( m \), (That is \( a \) does not get covered by the \( B_i \)) then \( r_{m+1} \geq \frac{3}{4} r(B_a) \) for all \( m \), a contradiction since the balls \( B(a_j, r_j/3) \) are disjoint and \( A \) is bounded, implying that \( r_j \to 0 \). Thus \( a \) must fail to be in some \( A_m \) which means it got covered by some ball in the sequence.

As explained above, in this sequence of balls from the above lemma, if \( j < k \)
\[ \frac{3}{4} r(B_k) \leq r(B_j) \]
Then there are two cases to consider,
\[ r(B_j) \geq 10r(B_k), \ r(B_j) \leq 10r(B_k) \]
In the first case, we use Lemma \(28.1.1\) to estimate the number of intersections of \(B_k\) with \(B_j\) for \(j < k\). In the second case, we use Lemma \(28.1.2\) to estimate the number of intersections of \(B_k\) with \(B_j\) for \(j < k\).

Now here is the Besicovitch covering theorem.

**Theorem 28.1.4** There exists a constant \(N_n\), depending only on \(n\) with the following property. If \(F\) is any collection of nonempty balls in \(\mathbb{R}^n\) with

\[
\sup \{\text{diam}(B) : B \in F\} < D < \infty
\]

and if \(A\) is the set of centers of the balls in \(F\), then there exist subsets of \(F\), \(\mathcal{H}_1, \ldots, \mathcal{H}_{N_n}\), such that each \(\mathcal{H}_i\) is a countable collection of disjoint balls from \(F\) (possibly empty) and

\[
A \subseteq \bigcup_{i=1}^{N_n} \bigcup \{B : B \in \mathcal{H}_i\}.
\]

**Proof:** To begin with, suppose \(A\) is bounded. Let \(L(n, 10)\) be the constant of Lemma \(28.1.1\) and let \(M_n = L(n, 10) + 60^n + 1\). Define the following sequence of subsets of \(F\), \(\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{M_n}\). Referring to the sequence \(\{B_k\}\) just considered, let \(B_1 \in \mathcal{G}_1\) and if \(B_1, \ldots, B_m\) have been assigned, each to a \(\mathcal{G}_i\), place \(B_{m+1}\) in the first \(\mathcal{G}_j\) such that \(B_{m+1}\) intersects no set already in \(\mathcal{G}_j\). The existence of such a \(j\) follows from Lemmas \(28.1.1\) and \(28.1.2\). Here is why. \(B_{m+1}\) can intersect at most \(L(n, 10)\) sets of \(\{B_1, \ldots, B_m\}\) which have radii at least as large as \(10B_{m+1}\) thanks to Lemma \(28.1.1\). It can intersect at most \(60^n\) sets of \(\{B_1, \ldots, B_m\}\) which have radius smaller than \(10B_{m+1}\) thanks to Lemma \(28.1.4\). Thus each \(\mathcal{G}_j\) consists of disjoint sets of \(F\) and the set of centers is covered by the union of these \(\mathcal{G}_j\). This proves the theorem in case the set of centers is bounded.

Now let \(R_1 = B(0, 5D)\) and if \(R_m\) has been chosen, let

\[
R_{m+1} = B(0, (m + 1)5D) \setminus R_m
\]

Thus, if \(|k - m| \geq 2\), no ball from \(F\) having nonempty intersection with \(R_m\) can intersect any ball from \(F\) which has nonempty intersection with \(R_k\). This is because all these balls have radius less than \(D\). Now let \(A_m \equiv A \cap R_m\) and apply the above result for a bounded set of centers to those balls of \(F\) which intersect \(R_m\) to obtain sets of disjoint balls \(\mathcal{G}_1(R_m), \mathcal{G}_2(R_m), \ldots, \mathcal{G}_{M_n}(R_m)\) covering \(A_m\). Then simply define \(\mathcal{G}_j \equiv \bigcup_{i=1}^{\infty} \mathcal{G}_j(R_{2k}), \mathcal{G}_j \equiv \bigcup_{i=1}^{\infty} \mathcal{G}_j(R_{2k-1})\). Let \(N_n = 2M_n\) and

\[
\{\mathcal{H}_1, \ldots, \mathcal{H}_{N_n}\} \equiv \{\mathcal{G}_1, \ldots, \mathcal{G}_{M_n}, \mathcal{G}_1, \ldots, \mathcal{G}_{M_n}\}
\]

Note that the balls in \(\mathcal{G}_j\) are disjoint. This is because those in \(\mathcal{G}_j(R_{2k})\) are disjoint and if you consider any ball in \(\mathcal{G}_j(R_{2k})\), it cannot intersect a ball of \(\mathcal{G}_j(R_{2m})\) for \(m \neq k\) because \(|2k - 2m| \geq 2\). Similar considerations apply to the balls of \(\mathcal{G}_j\).

### 28.2 Fundamental Theorem Of Calculus For Radon Measures

In this section the Besicovitch covering theorem will be used to give a generalization of the Lebesgue differentiation theorem to general Radon measures. In what follows,
28.2. FUNDAMENTAL THEOREM OF CALCULUS FOR RADON MEASURES

\( \mu \) will be a Radon measure, 
\[ Z \equiv \{ x \in \mathbb{R}^n : \mu(B(x,r)) = 0 \text{ for some } r > 0 \}. \]

**Lemma 28.2.1** Z is measurable and \( \mu(Z) = 0 \).

**Proof:** For each \( x \in Z \), there exists a ball \( B(x,r) \) with \( \mu(B(x,r)) = 0 \). Let \( \mathcal{C} \) be the collection of these balls. Since \( \mathbb{R}^n \) has a countable basis, a countable subset, \( \tilde{\mathcal{C}} \), of \( \mathcal{C} \) also covers \( Z \). Let 
\[ \tilde{\mathcal{C}} = \{ B_i \}_{i=1}^{\infty}. \]

Then letting \( \overline{\mu} \) denote the outer measure determined by \( \mu \), 
\[ \overline{\mu}(Z) \leq \sum_{i=1}^{\infty} \overline{\mu}(B_i) = \sum_{i=1}^{\infty} \mu(B_i) = 0. \]

Therefore, \( Z \) is measurable and has measure zero as claimed. \( \blacksquare \)

Let \( Mf : \mathbb{R}^n \to [0, \infty] \) by 
\[ Mf(x) \equiv \begin{cases} \sup_{r \leq 1} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu & \text{if } x \notin Z, \\ 0 & \text{if } x \in Z. \end{cases} \]

**Theorem 28.2.2** Let \( \mu \) be a Radon measure and let \( f \in L^1(\mathbb{R}^n, \mu) \). Then for a.e. \( x \), 
\[ \lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu(y) = 0. \]

**Proof:** First consider the following claim which is a weak type estimate of the same sort used when differentiating with respect to Lebesgue measure.

**Claim 1:** The following inequality holds for \( N_n \) the constant of the Besicovitch covering theorem. 
\[ \overline{\mu}([Mf > \varepsilon]) \leq N_n \varepsilon^{-1} \|f\|_1. \]

**Proof:** First note \( [Mf > \varepsilon] \cap Z = \emptyset \) and without loss of generality, you can assume \( \overline{\mu}([Mf > \varepsilon]) > 0 \). Next, for each \( x \in [Mf > \varepsilon] \) there exists a ball \( B_x = B(x,r_x) \) with \( r_x \leq 1 \) and 
\[ \mu(B_x)^{-1} \int_{B(x,r_x)} |f| \, d\mu > \varepsilon. \]

Let \( \mathcal{F} \) be this collection of balls so that \([Mf > \varepsilon]\) is the set of centers of balls of \( \mathcal{F} \). By the Besicovitch covering theorem,
\[ [Mf > \varepsilon] \subseteq \bigcup_{i=1}^{N_n} \{ B : B \in \mathcal{G}_i \} \]
where \( \mathcal{G}_i \) is a collection of disjoint balls of \( \mathcal{F} \). Now for some \( i \),
\[ \overline{\mu}([Mf > \varepsilon]) / N_n \leq \mu(\bigcup \{ B : B \in \mathcal{G}_i \}) \]
because if this is not so, then
\[
\mathbb{p}(\{M > \varepsilon\}) \leq \sum_{i=1}^{N_n} \mu(\cup\{B : B \in \mathcal{G}_i\}) < \sum_{i=1}^{N_n} \mathbb{p}(\{M > \varepsilon\}) = \frac{\mathbb{p}(\{M > \varepsilon\})}{N_n},
\]
a contradiction. Therefore for this \(i\),
\[
\mathbb{p}(\{M > \varepsilon\}) \leq \mu(\cup\{B : B \in \mathcal{G}_i\}) = \sum_{B \in \mathcal{G}_i} \mu(B) \leq \sum_{B \in \mathcal{G}_i} \varepsilon^{-1} \int |f| \, d\mu \leq \varepsilon^{-1} \int_{\mathbb{R}^n} |f| \, d\mu = \varepsilon^{-1} ||f||_1.
\]
This shows Claim 1.

**Claim 2:** If \(g\) is any continuous function defined on \(\mathbb{R}^n\), then for \(x \in Z\),
\[
\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| \, d\mu(y) = 0
\]
and
\[
\lim_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} g(y) \, d\mu(y) = g(x). \tag{28.2.8}
\]

**Proof:** Since \(g\) is continuous at \(x\), whenever \(r\) is small enough,
\[
\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| \, d\mu(y) \leq \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \varepsilon \, d\mu(y) = \varepsilon.
\]
follows from the above and the triangle inequality. This proves the claim.

Now let \(g \in C_c(\mathbb{R}^n)\) and \(x \in Z\). Then from the above observations about continuous functions,
\[
\mathbb{p}\left(\{x \notin Z : \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu(y) > \varepsilon\} \right) \tag{28.2.9}
\]
\[
\leq \mathbb{p}\left(\{x \notin Z : \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - g(y)| \, d\mu(y) > \frac{\varepsilon}{2}\} \right) + \mathbb{p}\left(\{|g(x) - f(x)| > \frac{\varepsilon}{2}\}\right),
\]
\[
\leq \mathbb{p}\left(\{|f-g| > \frac{\varepsilon}{2}\}\right) + \mathbb{p}\left(\{|f-g| > \frac{\varepsilon}{2}\}\right) \tag{28.2.10}
\]
Now
\[
\int_{\{|f-g| > \frac{\varepsilon}{2}\}} |f-g| \, d\mu \geq \frac{\varepsilon}{2} \mathbb{p}\left(\{|f-g| > \frac{\varepsilon}{2}\}\right)
\]
and so from Claim 1 and hence is dominated by
\[
\left( \frac{2}{\varepsilon} + \frac{N_n}{\varepsilon} \right) ||f - g||_{L^1(\mathbb{R}^n, \mu)}.
\]

But by regularity of Radon measures, \( C_c(\mathbb{R}^n) \) is dense in \( L^1(\mathbb{R}^n, \mu) \), and so since \( g \) in the above is arbitrary, this shows equals 0. Now
\[
\mu \left( \left\{ x \notin Z : \limsup_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu(y) > 0 \right\} \right) = 0
\]

By completeness of \( \mu \) this implies
\[
\left\{ x \notin Z : \limsup_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu(y) > 0 \right\}
\]
is a set of \( \mu \) measure zero.  

The following corollary is the main result referred to as the Lebesgue Besicovitch Differentiation theorem.

**Corollary 28.2.3** If \( f \in L^1_{loc}(\mathbb{R}^n, \mu) \), then for a.e. \( x \notin Z \),
\[
\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu(y) = 0. \tag{28.2.11}
\]

**Proof:** If \( f \) is replaced by \( f \chi_{B(0,k)} \) then the conclusion holds for all \( x \notin F_k \) where \( F_k \) is a set of \( \mu \) measure 0. Letting \( k = 1, 2, \cdots, \) and \( F \equiv \bigcup_{k=1}^{\infty} F_k \), it follows that \( F \) is a set of measure zero and for any \( x \notin F \), and \( k \in \{1, 2, \cdots\} \), holds if \( f \) is replaced by \( f \chi_{B(0,k)} \). Picking any such \( x \), and letting \( k > |x| + 1 \), this shows
\[
\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f(x)| d\mu(y)
\]
\[
= \lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f \chi_{B(0,k)}(y) - f \chi_{B(0,k)}(x)| d\mu(y) = 0. \]

28.3 Slicing Measures

Let \( \mu \) be a finite Radon measure. I will show here that a formula of the following form holds.
\[
\mu(F) = \int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_F(x,y) d\nu_x(y) d\alpha(x)
\]
where \( \alpha(E) = \mu(E \times \mathbb{R}^m) \). When this is done, the measures, \( \nu_x \), are called slicing measures and this shows that an integral with respect to \( \mu \) can be written as an
iterated integral in terms of the measure \( \alpha \) and the slicing measures, \( \nu_x \). This is like going backwards in the construction of product measure. One starts with a measure \( \mu \), defined on the Cartesian product and produces \( \alpha \) and an infinite family of slicing measures from it whereas in the construction of product measure, one starts with two measures and obtains a new measure on a \( \sigma \) algebra of subsets of the Cartesian product of two spaces. These slicing measures are dependent on \( x \). Later, this will be tied to the concept of independence or not of random variables. First here are two technical lemmas.

**Lemma 28.3.1** The space \( C^c_c(\mathbb{R}^m) \) with the norm

\[
||f|| \equiv \sup \{|f(y)| : y \in \mathbb{R}^m\}
\]

is separable.

**Proof:** Let \( D_l \) consist of all functions which are of the form

\[
\sum_{|\alpha| \leq N} a_{\alpha} y^\alpha \left( \text{dist} \left( y, B(0,l+1)^C \right) \right)^{n_{\alpha}}
\]

where \( a_{\alpha} \in \mathbb{Q}, \alpha \) is a multi-index, and \( n_{\alpha} \) is a positive integer. Consider \( D \equiv \bigcup_l D_l \). Then \( D \) is countable. If \( f \in C^c_c(\mathbb{R}^n) \), then choose \( l \) large enough that \( \text{spt}(f) \subseteq B(0,l+1) \), a locally compact space, \( f \in C^0_0(B(0,l+1)) \). Then since \( D_l \) separates the points of \( B(0,l+1) \) is closed with respect to conjugates, and annihilates no point, it is dense in \( C^0_0(B(0,l+1)) \) by the Stone Weierstrass theorem. Alternatively, \( D \) is dense in \( C^0_0(\mathbb{R}^n) \) by Stone Weierstrass and \( C^c_c(\mathbb{R}^n) \) is a subspace so it is also separable. So is \( C^c_c(\mathbb{R}^n)^+ \), the nonnegative functions in \( C^c_c(\mathbb{R}^n) \).

From the regularity of Radon measures, the following lemma follows.

**Lemma 28.3.2** If \( \mu \) and \( \nu \) are two Radon measures defined on \( \sigma \) algebras, \( S_\mu \) and \( S_\nu \), of subsets of \( \mathbb{R}^n \) and if \( \mu(V) = \nu(V) \) for all \( V \) open, then \( \mu = \nu \) and \( S_\mu = S_\nu \).

**Proof:** Every compact set is a countable intersection of open sets so the two measures agree on every compact set. Hence it is routine that the two measures agree on every \( G_\delta \) and \( F_\sigma \) set. (Recall \( G_\delta \) sets are countable intersections of open sets and \( F_\sigma \) sets are countable unions of closed sets.) Now suppose \( E \in S_\nu \) is a bounded set. Then by regularity of \( \nu \) there exists \( G \) a \( G_\delta \) set and \( F \) an \( F_\sigma \) set such that \( F \subseteq E \subseteq G \) and \( \nu(G \setminus F) = 0 \). Then it is also true that \( \mu(G \setminus F) = 0 \). Hence \( E = F \cup (E \setminus F) \) and \( E \setminus F \) is a subset of \( G \setminus F \), a set of \( \mu \) measure zero. By completeness of \( \mu \), it follows \( E \in S_\mu \) and

\[
\mu(E) = \mu(F) = \nu(F) = \nu(E).
\]

If \( E \in S_\nu \) not necessarily bounded, let \( E_m = E \cap B(0,m) \) and then \( E_m \in S_\mu \) and \( \mu(E_m) = \nu(E_m) \). Letting \( m \to \infty \), \( E \in S_\mu \) and \( \mu(E) = \nu(E) \). Similarly, \( S_\mu \subseteq S_\nu \) and the two measures are equal on \( S_\mu \).

The main result in the section is the following theorem.
Theorem 28.3.3  Let \( \mu \) be a finite Radon measure on \( \mathbb{R}^{n+m} \) defined on a \( \sigma \)-algebra, \( \mathcal{F} \). Then there exists a unique finite Radon measure \( \alpha \), defined on a \( \sigma \)-algebra \( \mathcal{S} \), of sets of \( \mathbb{R}^n \) which satisfies

\[
\alpha(E) = \mu(E \times \mathbb{R}^m) \quad (28.3.12)
\]

for all \( E \) Borel. There also exists a Borel set of \( \alpha \) measure zero \( N \), such that for each \( x \notin N \), there exists a Radon probability measure \( \nu_x \) such that if \( f \) is a nonnegative \( \mu \) measurable function or a \( \mu \) measurable function in \( L^1(\mu) \),

\[
y \to f(x, y) \text{ is } \nu_x \text{ measurable } \alpha \text{ a.e.}
\]

\[
x \to \int_{\mathbb{R}^m} f(x, y) \, d\nu_x(y) \text{ is } \alpha \text{ measurable} \quad (28.3.13)
\]

and

\[
\int_{\mathbb{R}^{n+m}} f(x, y) \, d\mu = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} f(x, y) \, d\nu_x(y) \right) \, d\alpha(x). \quad (28.3.14)
\]

If \( \tilde{\nu}_x \) is any other collection of Radon measures satisfying (28.3.12) and (28.3.14), then \( \tilde{\nu}_x = \nu_x \) for \( \alpha \) a.e. \( x \).

Proof:

Existence and uniqueness of \( \alpha \)

First consider the uniqueness of \( \alpha \). Suppose \( \alpha_1 \) is another Radon measure satisfying (28.3.12). Then in particular, \( \alpha_1 \) and \( \alpha \) agree on open sets and so the two measures are the same by Lemma 28.3.2.

To establish the existence of \( \alpha \), define \( \alpha_0 \) on Borel sets by

\[
\alpha_0(E) = \mu(E \times \mathbb{R}^m).
\]

Thus \( \alpha_0 \) is a finite Borel measure and so it is finite on compact sets. Lemma 12.2.3 on Page 367 implies the existence of the Radon measure \( \alpha \) extending \( \alpha_0 \).

Uniqueness of \( \nu_x \)

Next consider the uniqueness of \( \nu_x \). Suppose \( \nu_x \) and \( \tilde{\nu}_x \) satisfy all conclusions of the theorem with exceptional sets denoted by \( N \) and \( \tilde{N} \) respectively. Then, enlarging \( N \) and \( \tilde{N} \), one may also assume, using Lemma 28.2.1, that for \( x \notin N \cup \tilde{N} \), \( \alpha(B(x, r)) > 0 \) whenever \( r > 0 \). Now let

\[
A = \prod_{i=1}^{m} [a_i, b_i]
\]

where \( a_i \) and \( b_i \) are rational. Thus there are countably many such sets. Then from the conclusion of the theorem, if \( x_0 \notin N \cup \tilde{N} \),

\[
\frac{1}{\alpha(B(x_0, r))} \int_{B(x_0, r)} \int_{\mathbb{R}^m} X_A(y) \, d\nu_x(y) \, d\alpha
\]
and by the Lebesgue Besicovitch Differentiation theorem, there exists a set of measure zero, \( E_A \), such that if \( x_0 \notin E_A \cup N \cup \tilde{N} \), then the limit in the above exists as \( r \to 0 \) and yields

\[
\nu_{x_0}(A) = \tilde{\nu}_{x_0}(A).
\]

Letting \( E \) denote the union of all the sets \( E_A \) for \( A \) as described above, it follows that \( E \) is a set of measure zero and if \( x_0 \notin E \cup N \cup \tilde{N} \), then \( \nu_{x_0}(A) = \tilde{\nu}_{x_0}(A) \) for all such sets \( A \). But every open set can be written as a disjoint union of sets of this form and so for all such \( x_0 \), \( \nu_{x_0}(V) = \tilde{\nu}_{x_0}(V) \) for all \( V \) open. By Lemma 28.3.2, this shows the two measures are equal and proves the uniqueness assertion for \( \nu_\alpha \).

It remains to show the existence of the measures \( \nu_\alpha \).

**Existence of \( \nu_\alpha \)**

For \( f \geq 0 \), \( f, g \in C_c(\mathbb{R}^m) \) and \( C_c(\mathbb{R}^n) \) respectively, define

\[
g \rightarrow \int_{\mathbb{R}^{n+m}} g(x) f(y) \, d\mu
\]

Since \( f \geq 0 \), this is a positive linear functional on \( C_c(\mathbb{R}^n) \). Therefore, there exists a unique Radon measure \( \nu_f \) such that for all \( g \in C_c(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^{n+m}} g(x) f(y) \, d\mu = \int_{\mathbb{R}^n} g(x) \, d\nu_f.
\]

I claim that \( \nu_f \ll \alpha \), the two being considered as measures on \( B(\mathbb{R}^n) \). Suppose then that \( K \) is a compact set and \( \alpha(K) = 0 \). Then let \( K \prec g \prec V \) where \( V \) is open.

\[
\nu_f(K) = \int_{\mathbb{R}^n} \mathcal{X}_K(x) \, d\nu_f(x) \leq \int_{\mathbb{R}^n} g(x) \, d\nu_f(x) = \int_{\mathbb{R}^{n+m}} g(x) f(y) \, d\mu
\]

\[
\leq \int_{\mathbb{R}^{n+m}} \mathcal{X}_{V \times \mathbb{R}^m}(x,y) f(y) \, d\mu \leq \|f\|_\infty \mu(V \times \mathbb{R}^m) = \|f\|_\infty \alpha(V)
\]

Then for any \( \varepsilon > 0 \), one can choose \( V \) such that the right side is less than \( \varepsilon \). Therefore, \( \nu_f(K) = 0 \) also. By regularity considerations, \( \nu_f \ll \alpha \) as claimed.

It follows from the Radon Nikodym theorem the existence of a function \( h_f \in L^1(\alpha) \) such that for all \( g \in C_c(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^{n+m}} g(x) f(y) \, d\mu = \int_{\mathbb{R}^n} g(x) \, d\nu_f = \int_{\mathbb{R}^n} g(x) h_f(x) \, d\alpha. \tag{28.3.15}
\]

It is obvious from the formula that the map from \( f \in C_c(\mathbb{R}^m) \) to \( L^1(\alpha) \) given by \( f \rightarrow h_f \) is linear. However, this is not sufficiently specific because functions in \( L^1(\alpha) \) are only determined a.e. However, for \( h_f \in L^1(\alpha) \), you can specify a particular representative \( \alpha \) a.e. By the fundamental theorem of calculus,

\[
\widehat{h_f}(x) = \lim_{r \to 0} \frac{1}{\alpha(B(x,r))} \int_{B(x,r)} h_f(z) \, d\alpha(z) \tag{28.3.16}
\]
exists off some set of measure zero $Z_f$. Note that since this involves the integral over a ball, it does not matter which representative of $h_f$ is placed in the formula. Therefore, $\hat{h}_f(x)$ is well defined pointwise for all $x$ not in some set of measure zero $Z_f$. Since $\hat{h}_f = h_f$ a.e. it follows that $\hat{h}_f$ is well defined and will work in the formula

$$Z = \cup \{Z_f : f \in D\}$$

where $D$ is a countable dense subset of $C_c(\mathbb{R}^m)^+$. Of course it is desired to have the limit hold for all $f$, not just $f \in D$. We will show that this limit holds for all $x \notin Z$. Thus, we will have $x \to \hat{h}_f(x)$ defined by the above limit off $Z$ and so, since $\hat{h}_f(x) = h_f(x)$ a.e., it follows that

$$\int_{\mathbb{R}^{n+m}} g(x) f(y) d\mu = \int_{\mathbb{R}^n} g(x) d\nu_f = \int_{\mathbb{R}^n} g(x) \hat{h}_f(x) d\alpha$$

One could then take $\hat{h}_f(x)$ to be defined as 0 for $x \notin Z$.

For $f$ an arbitrary function in $C_c(\mathbb{R}^m)^+$ and $f' \in D$, a dense countable subset of $C_c(\mathbb{R}^n)^+$, it follows from

$$\left| \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha \right| \leq \|f - f'\|_\infty \int_{\mathbb{R}^{n+m}} |g(x)| d\mu$$

Let $g_k(x) \uparrow \chi_{B(z,r)}(x)$ where $z \notin Z$. Then by the dominated convergence theorem, the above implies

$$\left| \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha \right| \leq \|f - f'\|_\infty \int_{B(z,r) \times \mathbb{R}^m} d\mu = ||f - f'||_\infty \alpha(B(z,r))$$

Dividing by $\alpha(B(z,r))$, it follows that if $\alpha(B(z,r)) > 0$ for all $r > 0$, then for all $r > 0$,

$$\left| \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha \right| \leq ||f - f'||_\infty$$

It follows that for $f \in C_c(\mathbb{R}^m)^+$ arbitrary and $z \notin Z$,

$$\limsup_{r \to 0} \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} h_f(x) d\alpha - \liminf_{r \to 0} \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} h_f(x) d\alpha$$

$$= \limsup_{r \to 0} \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha(x)$$

$$- \liminf_{r \to 0} \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha(x)$$

$$\leq \left| \limsup_{r \to 0} \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha(x) \right|$$

$$+ \left| \liminf_{r \to 0} \frac{1}{\alpha(B(z,r))} \int_{B(z,r)} (h_f(x) - h_{f'}(x)) d\alpha(x) \right|$$

$$\leq 2 ||f - f'||_\infty$$
and since \( f' \) is arbitrary, it follows that the limit of \( (28.3.16) \) holds for all \( f \in C_c (\mathbb{R}^m)^+ \) whenever \( z \notin Z \), the above set of measure zero.

Now for \( f \) an arbitrary real valued function of \( C_c (\mathbb{R}^m) \), simply apply the above result to positive and negative parts to obtain \( h_f = h_{f+} - h_{f-} \) and \( \hat{h}_f = \hat{h}_{f+} - \hat{h}_{f-} \). Then it follows that for all \( f \in C_c (\mathbb{R}^m) \) and \( g \in C_c (\mathbb{R}^m) \)

\[
\int_{\mathbb{R}^{n+m}} g (\mathbf{x}) \, f (\mathbf{y}) \, d\mu = \int_{\mathbb{R}^n} g (\mathbf{x}) \, \hat{h}_f (\mathbf{x}) \, d\alpha.
\]

It is obvious from the description given above that for each \( \mathbf{x} \notin Z \), the set of measure zero given above, that \( f \to \hat{h}_f (\mathbf{x}) \) is a positive linear functional. It is clear that it acts like a linear map for nonnegative \( f \) and so the usual trick just described above is well defined and delivers a positive linear functional. Hence by the Riesz representation theorem, there exists a unique \( \nu_\mathbf{x} \) such that for all \( \mathbf{x} \)

\[
\hat{h}_f (\mathbf{x}) = \int_{\mathbb{R}^m} f (\mathbf{y}) \, d\nu_\mathbf{x} (\mathbf{y}).
\]

It follows that

\[
\int_{\mathbb{R}^{n+m}} g (\mathbf{x}) \, f (\mathbf{y}) \, d\mu = \int_{\mathbb{R}^n} g (\mathbf{x}) \, \int_{\mathbb{R}^m} f (\mathbf{y}) \, d\nu_\mathbf{x} (\mathbf{y}) \, d\alpha (\mathbf{x})
\]

(28.3.17)

and \( \mathbf{x} \to \int_{\mathbb{R}^m} f (\mathbf{y}) \, d\nu_\mathbf{x} \) is \( \alpha \) measurable and \( \nu_\mathbf{x} \) is a Radon measure.

Now let \( f_k \uparrow \mathcal{X}_{\mathbb{R}^m} \) and \( g \geq 0 \). Then by monotone convergence theorem,

\[
\int_{\mathbb{R}^{n+m}} g (\mathbf{x}) \, d\mu = \int_{\mathbb{R}^n} g (\mathbf{x}) \int_{\mathbb{R}^m} d\nu_\mathbf{x} \, d\alpha
\]

If \( g_k \uparrow \mathcal{X}_{\mathbb{R}^m} \), the monotone convergence theorem shows that \( \mathbf{x} \to \int_{\mathbb{R}^m} d\nu_\mathbf{x} \) is \( L^1 (\alpha) \).

Next let \( g_k \uparrow \mathcal{X}_{\mathcal{B}(\mathbf{x},r)} \) and use monotone convergence theorem to write

\[
\alpha (B(\mathbf{x},r)) = \int_{B(\mathbf{x},r) \times \mathbb{R}^m} d\mu = \int_{B(\mathbf{x},r)} \int_{\mathbb{R}^m} d\nu_\mathbf{x} \, d\alpha
\]

Then dividing by \( \alpha (B(\mathbf{x},r)) \) and taking a limit as \( r \to 0 \), it follows that for \( \alpha \) a.e. \( \mathbf{x} \), \( 1 = \nu_\mathbf{x} (\mathbb{R}^m) \), so these \( \nu_\mathbf{x} \) are probability measures off a set of \( \alpha \) measure zero. Letting \( g_k (\mathbf{x}) \uparrow \mathcal{X}_A (\mathbf{x}) \), \( f_k (\mathbf{y}) \uparrow \mathcal{X}_B (\mathbf{y}) \) for \( A, B \) open, it follows that \( (28.3.16) \) is valid for \( g (\mathbf{x}) \) replaced with \( \mathcal{X}_A (\mathbf{x}) \) and \( f (\mathbf{y}) \) replaced with \( \mathcal{X}_B (\mathbf{y}) \).

Now let \( \mathcal{G} \) denote the Borel sets \( F \) of \( \mathbb{R}^{n+m} \) such that

\[
\int_{\mathbb{R}^{n+m}} \mathcal{X}_F (\mathbf{x}, \mathbf{y}) \, d\mu (x,y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_F (\mathbf{x}, \mathbf{y}) \, d\nu_\mathbf{x} (\mathbf{y}) \, d\alpha (x)
\]

and that all the integrals make sense. As just explained, this includes all Borel sets of the form \( F = A \times B \) where \( A, B \) are open. It is clear that \( \mathcal{G} \) is closed with respect to countable disjoint unions and complements, while sets of the form \( A \times B \) for \( A, B \) open form a \( \pi \) system. Therefore, by Lemma \( \text{II.13.1} \), \( \mathcal{G} \) contains the Borel
sets which is the smallest $\sigma$ algebra which contains such products of open sets. It follows from the usual approximation with simple functions that if $f \geq 0$ and is Borel measurable, then

$$\int_{\mathbb{R}^{n+m}} f(x, y) \, d\mu(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x, y) \, d\nu_x(y) \, d\alpha(x)$$

with all the integrals making sense.

This proves the theorem in the case where $f$ is Borel measurable and non-negative. It just remains to extend this to the case where $f$ is only $\mu$ measurable. However, from regularity of $\mu$ there exist Borel measurable functions $g, h, g \leq f \leq h$ such that

$$\int_{\mathbb{R}^{n+m}} f(x, y) \, d\mu(x, y) = \int_{\mathbb{R}^{n+m}} g(x, y) \, d\mu(x, y) = \int_{\mathbb{R}^{n+m}} h(x, y) \, d\mu(x, y)$$

It follows

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} g(x, y) \, d\nu_x(y) \, d\alpha(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} h(x, y) \, d\nu_x(y) \, d\alpha(x)$$

and so, since for $\alpha$ a.e. $x, y \to g(x, y)$ and $y \to h(x, y)$ are $\nu_x$ measurable with

$$0 = \int_{\mathbb{R}^m} (h(x, y) - g(x, y)) \, d\nu_x(y)$$

and $\nu_x$ is a Radon measure, hence complete, it follows for $\alpha$ a.e. $x, y \to f(x, y)$ must be $\nu_x$ measurable because it is equal to $y \to g(x, y), \nu_x$ a.e. Therefore, for $\alpha$ a.e. $x$, it makes sense to write

$$\int_{\mathbb{R}^m} f(x, y) \, d\nu_x(y).$$

Similar reasoning applies to the above function of $x$ being $\alpha$ measurable due to $\alpha$ being complete. It follows

$$\int_{\mathbb{R}^{n+m}} f(x, y) \, d\mu(x, y) = \int_{\mathbb{R}^{n+m}} g(x, y) \, d\mu(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} g(x, y) \, d\nu_x(y) \, d\alpha(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x, y) \, d\nu_x(y) \, d\alpha(x)$$

with everything making sense. ■
28.4 Vitali Coverings

There is another covering theorem which may also be referred to as the Besicovitch covering theorem. As before, the balls can be taken with respect to any norm on $\mathbb{R}^n$. At first, the balls will be closed but this assumption will be removed.

**Definition 28.4.1** A collection of balls, $\mathcal{F}$, covers a set, $E$, in the sense of Vitali if whenever $x \in E$ and $\varepsilon > 0$, there exists a ball $B \in \mathcal{F}$ whose center is $x$ having diameter less than $\varepsilon$.

I will give a proof of the following theorem.

**Theorem 28.4.2** Let $\mu$ be a Radon measure on $\mathbb{R}^n$ and let $E$ be a set with $\mu(E) < \infty$. Where $\mu$ is the outer measure determined by $\mu$. Suppose $\mathcal{F}$ is a collection of closed balls which cover $E$ in the sense of Vitali. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that

$$\mu(E \setminus \bigcup_{i=1}^{\infty} B_i) = 0.$$ 

**Proof:** Let $N_n$ be the constant of the Besicovitch covering theorem. Choose $r > 0$ such that

$$(1 - r)^{-1} \left(1 - \frac{1}{2N_n + 2}\right) \equiv \lambda < 1.$$ 

If $\mu(E) = 0$, there is nothing to prove so assume $\mu(E) > 0$. Let $U_1$ be an open set containing $E$ with $(1 - r) \mu(U_1) < \mu(E)$ and $2\mu(E) > \mu(U_1)$, and let $\mathcal{F}_1$ be those sets of $\mathcal{F}$ which are contained in $U_1$ whose centers are in $E$. Thus $\mathcal{F}_1$ is also a Vitali cover of $E$. Now by the Besicovitch covering theorem proved earlier, there exist balls, $B$, of $\mathcal{F}_1$ such that

$$E \subseteq \bigcup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}$$

where $\mathcal{G}_i$ consists of a collection of disjoint balls of $\mathcal{F}_1$. Therefore,

$$\mu(E) \leq \sum_{i=1}^{N_n} \sum_{B \in \mathcal{G}_i} \mu(B)$$

and so, for some $i \leq N_n$,

$$(N_n + 1) \sum_{B \in \mathcal{G}_i} \mu(B) > \mu(E).$$

It follows there exists a finite set of balls of $\mathcal{G}_i$, $\{B_1, \cdots, B_{m_1}\}$ such that

$$(N_n + 1) \sum_{i=1}^{m_1} \mu(B_i) > \mu(E) \quad (28.4.18).$$
and so
\[
(2N_n + 2) \sum_{i=1}^{m_1} \mu(B_i) > 2\pi(E) > \mu(U_1).
\]
Since \(2\pi(E) \geq \mu(U_1)\), \(28.3.18\) implies
\[
\frac{\mu(U_1)}{2N_2 + 2} \leq \frac{2\pi(E)}{2N_2 + 2} = \frac{\pi(E)}{N_2 + 1} < \sum_{i=1}^{m_1} \mu(B_i).
\]
Also \(U_1\) was chosen such that \((1 - r) \mu(U_1) < \bar{\mu}(E)\), and so
\[
\lambda \bar{\mu}(E) \geq \lambda (1 - r) \mu(U_1) = \left(1 - \frac{1}{2N_n + 2}\right) \mu(U_1)
\]
\[
\geq \mu(U_1) - \sum_{i=1}^{m_1} \mu(B_i) = \mu(U_1) - \mu(\bigcup_{j=1}^{m_1} B_j)
\]
\[
= \mu(U_1 \setminus \bigcup_{j=1}^{m_1} B_j) \geq \pi(E \setminus \bigcup_{j=1}^{m_1} B_j).
\]
Since the balls are closed, you can consider the sets of \(F\) which have empty intersection with \(\bigcup_{j=1}^{m_1} B_j\) and this new collection of sets will be a Vitali cover of \(E \setminus \bigcup_{j=1}^{m_1} B_j\).

Letting this collection of balls play the role of \(F\) in the above argument and letting \(E \setminus \bigcup_{j=1}^{m_1} B_j\) play the role of \(E\), repeat the above argument and obtain disjoint sets of \(F\),
\[
\{B_{m_1+1}, \cdots, B_{m_2}\},
\]
such that
\[
\lambda \bar{\mu}(E \setminus \bigcup_{j=1}^{m_1} B_j) > \pi((E \setminus \bigcup_{j=1}^{m_1} B_j) \setminus \bigcup_{j=m_1+1}^{m_2} B_j) = \pi(E \setminus \bigcup_{j=1}^{m_2} B_j),
\]
and so
\[
\lambda^2 \pi(E) > \pi(E \setminus \bigcup_{j=1}^{m_2} B_j).
\]
Continuing in this way, yields a sequence of disjoint balls \(\{B_i\}\) contained in \(F\) and
\[
\pi(E \setminus \bigcup_{j=1}^{m_k} B_j) \leq \pi(E \setminus \bigcup_{j=1}^{m_k} B_j) < \lambda^k \pi(E)
\]
for all \(k\). Therefore, \(\pi(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0\) and this proves the Theorem.

It is not necessary to assume \(\pi(E) < \infty\).

**Corollary 28.4.3** Let \(\mu\) be a Radon measure on \(\mathbb{R}^n\). Letting \(\pi\) be the outer measure determined by \(\mu\), suppose \(F\) is a collection of closed balls which cover \(E\) in the sense of Vitali. Then there exists a sequence of disjoint balls, \(\{B_i\} \subseteq F\) such that
\[
\pi(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0.
\]
Proof: Since $\mu$ is a Radon measure it is finite on compact sets. Therefore, there are at most countably many numbers, $\{b_i\}_{i=1}^{\infty}$ such that $\mu(\partial B(0,b_i)) > 0$. It follows there exists an increasing sequence of positive numbers, $\{r_i\}_{i=1}^{\infty}$ such that $\lim_{i \to \infty} r_i = \infty$ and $\mu(\partial B(0,r_i)) = 0$. Now let

$$D_1 = \{x : ||x|| < r_1\}, D_2 = \{x : r_1 < ||x|| < r_2\},$$

$$\cdots, D_m = \{x : r_{m-1} < ||x|| < r_m\}, \cdots.$$

Let $F_m$ denote those closed balls of $F$ which are contained in $D_m$. Then let $E_m$ denote $E \cap D_m$, if $F_m$ is a Vitali cover of $E_m$, $\overline{\mu}(E_m) < \infty$, and so by Theorem 28.4.2, there exists a countable sequence of balls from $F_m \{B_j^m\}_{j=1}^{\infty}$, such that $\overline{\mu}(E_m \cup \bigcup_{j=1}^{\infty} B_j^m) = 0$. Then consider the countable collection of balls, $\{B_j^m\}_{j,m=1}^{\infty}$.

$$\overline{\mu}(E \cup \cup_{m=1}^{\infty} \cup_{j=1}^{\infty} B_j^m) \leq \overline{\mu}(\cup_{j=1}^{\infty} \partial B(0,r_i)) +$$

$$+ \sum_{m=1}^{\infty} \overline{\mu}(E_m \cup \cup_{j=1}^{\infty} B_j^m) = 0$$

This proves the corollary.

You don’t need to assume the balls are closed. In fact, the balls can be open closed or anything in between and the same conclusion can be drawn.

Corollary 28.4.4 Let $\mu$ be a Radon measure on $\mathbb{R}^n$. Letting $\overline{\mu}$ be the outer measure determined by $\mu$, suppose $F$ is a collection of balls which cover $E$ in the sense of Vitali, open closed or neither. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq F$ such that

$$\overline{\mu}(E \cup \cup_{j=1}^{\infty} B_j) = 0.$$

Proof: Let $x \in E$. Thus $x$ is the center of arbitrarily small balls from $F$. Since $\mu$ is a Radon measure, at most countably many radii, $r$ of these balls can have the property that $\mu(\partial B(0,r)) = 0$. Let $F'$ denote the closures of the balls of $F$, $B(x,r)$ with the property that $\mu(\partial B(x,r)) = 0$. Since for each $x \in E$ there are only countably many exceptions, $F'$ is still a Vitali cover of $E$. Therefore, by Corollary 28.4.3 there is a disjoint sequence of these balls of $F'$, $\{\overline{B}_i\}_{i=1}^{\infty}$ for which

$$\overline{\mu}(E \cup \cup_{j=1}^{\infty} \overline{B}_j) = 0$$

However, since their boundaries have $\mu$ measure zero, it follows

$$\overline{\mu}(E \cup \cup_{j=1}^{\infty} B_j) = 0.$$

This proves the corollary.

28.5 Differentiation Of Radon Measures

This section is a generalization of earlier material in which a measure was differentiated with respect to Lebesgue measure. Here an arbitrary Radon measure will
be differentiated with respect to another arbitrary Radon measure. In this section, \( B(\mathbf{x}, r) \) will denote a ball with center \( \mathbf{x} \) and radius \( r \). Also, let \( \lambda \) and \( \mu \) be Radon measures and as above, \( Z \) will denote a \( \mu \) measure zero set off of which \( \mu(B(\mathbf{x}, r)) > 0 \) for all \( r > 0 \).

**Definition 28.5.1** For \( \mathbf{x} \not\in Z \), define the upper and lower symmetric derivatives as

\[
\overline{D}_\mu \lambda(\mathbf{x}) \equiv \limsup_{r \to 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}, \quad \underline{D}_\mu \lambda(\mathbf{x}) \equiv \liminf_{r \to 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}.
\]

respectively. Also define

\[
D_\mu \lambda(\mathbf{x}) \equiv \overline{D}_\mu \lambda(\mathbf{x}) = \underline{D}_\mu \lambda(\mathbf{x})
\]
in the case when both the upper and lower derivatives are equal.

**Lemma 28.5.2** Let \( \lambda \) and \( \mu \) be Radon measures. If \( A \) is a bounded subset of \( \{ \mathbf{x} \not\in Z : \overline{D}_\mu \lambda(\mathbf{x}) \geq a \} \), then

\[
\lambda(A) \geq a \overline{\mu}(A)
\]
and if \( A \) is a bounded subset of \( \{ \mathbf{x} \not\in Z : \underline{D}_\mu \lambda(\mathbf{x}) \leq a \} \), then

\[
\lambda(A) \leq a \overline{\mu}(A)
\]

**Proof:** Suppose first that \( A \) is a bounded subset of \( \{ \mathbf{x} \not\in Z : \overline{D}_\mu \lambda(\mathbf{x}) \geq a \} \), let \( \varepsilon > 0 \), and let \( V \) be a bounded open set with \( V \supseteq A \) and \( \lambda(V) - \varepsilon < \overline{\mu}(A), \mu(V) - \varepsilon < \overline{\mu}(A) \). Then if \( \mathbf{x} \in A \),

\[
\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))} > a - \varepsilon, \quad B(\mathbf{x}, r) \subseteq V,
\]
for infinitely many values of \( r \) which are arbitrarily small. Thus the collection of such balls constitutes a Vitali cover for \( A \). By Corollary 28.4.4 there is a disjoint sequence of these balls \( \{B_i\} \) such that

\[
\overline{\mu}(A \setminus \bigcup_{i=1}^{\infty} B_i) = 0.
\]

Therefore,

\[
(a - \varepsilon) \sum_{i=1}^{\infty} \mu(B_i) < \sum_{i=1}^{\infty} \lambda(B_i) \leq \lambda(V) < \varepsilon + \overline{\lambda}(A)
\]
and so

\[
a \sum_{i=1}^{\infty} \mu(B_i) \leq \varepsilon + \varepsilon \mu(V) + \overline{\lambda}(A)
\]

\[
\leq \varepsilon + \varepsilon (\overline{\mu}(A) + \varepsilon) + \overline{\lambda}(A)
\]

Now

\[
\overline{\mu}(A \setminus \bigcup_{i=1}^{\infty} B_i) + \overline{\mu}(\bigcup_{i=1}^{\infty} B_i) \geq \mu(A)
\]
and so by (28.5.14) and the fact the $B_i$ are disjoint,

\[
\alpha(A) \leq \alpha(\bigcup_{i=1}^{\infty} B_i) = a \sum_{i=1}^{\infty} \mu(B_i) \\
\leq \varepsilon + \varepsilon (\mu(A) + \varepsilon) + \tilde{\lambda}(A). \tag{28.5.21}
\]

Hence $\alpha(A) \leq \tilde{\lambda}(A)$ since $\varepsilon > 0$ was arbitrary.

Now suppose $A$ is a bounded subset of \{\(x \notin Z : D_\mu \lambda(x) \leq a\}\} and let $V$ be a bounded open set containing $A$ with $\mu(V) - \varepsilon < \alpha(A)$. Then if $x \in A$,

\[
\frac{\lambda(B(x,r))}{\mu(B(x,r))} < a + \varepsilon, \ B(x,r) \subseteq V
\]

for values of $r$ which are arbitrarily small. Therefore, by Corollary (28.4.4) again, there exists a disjoint sequence of these balls, $\{B_i\}$ satisfying this time,

\[
\lambda(A \cup \bigcup_{i=1}^{\infty} B_i) = 0.
\]

Then by arguments similar to the above,

\[
\tilde{\lambda}(A) \leq \sum_{i=1}^{\infty} \lambda(B_i) < (a + \varepsilon) \mu(V) < (a + \varepsilon) (\alpha(A) + \varepsilon).
\]

Since $\varepsilon$ was arbitrary, this proves the lemma.

**Theorem 28.5.3** There exists a set of measure zero, $N$ containing $Z$ such that for $x \notin N$, $D_\mu \lambda(x)$ exists and also $X_{NC}(\cdot) D_\mu \lambda(\cdot)$ is a $\mu$ measurable function. Furthermore, $D_\mu \lambda(x) < \infty$ $\mu$ a.e.

**Proof:** First I show $D_\mu \lambda(x)$ exists a.e. Let $0 \leq a < b < \infty$ and let $A$ be any bounded subset of

\[
N(a,b) \equiv \{x \notin Z : D_\mu \lambda(x) > b > a > D_\mu \lambda(x)\}.
\]

By Lemma (28.5.14),

\[
\alpha(A) \geq \tilde{\lambda}(A) \geq b\alpha(A)
\]

and so $\mu(A) = 0$ and $A$ is $\mu$ measurable. It follows $\mu(N(a,b)) = 0$ because

\[
\mu(N(a,b)) \leq \sum_{m=1}^{\infty} \mu(N(a,b) \cap B(0,m)) = 0.
\]

Define

\[
N_0 \equiv \{x \notin Z : D_\mu \lambda(x) > D_\mu \lambda(x)\}.
\]

Thus $\mu(N_0) = 0$ because

\[
N_0 \subseteq \bigcup \{N(a,b) : 0 \leq a < b, \text{ and } a,b \in \mathbb{Q}\}
\]
Therefore, $N_0$ is also $\mu$ measurable and has $\mu$ measure zero. Letting $N \equiv N_0 \cup Z$, it follows $D_\mu \lambda (x)$ exists on $N^C$. It remains to verify $X_{N^C} (\cdot) D_\mu \lambda (\cdot)$ is finite a.e. and is $\mu$ measurable.

Let

$$I = \{x : D_\mu \lambda (x) = \infty\}.$$ 

Then by Lemma 28.3.2

$$\lambda(I \cap B(0,m)) \geq a \mu(I \cap B(0,m))$$

for all $a$ and since $\lambda$ is finite on bounded sets, the above implies $\lambda(I \cap B(0,m)) = 0$ for each $m$ which implies that $I$ is $\mu$ measurable and has $\mu$ measure zero since

$$I = \bigcup_{m=1}^{\infty} I_m.$$

Letting $\eta$ be an arbitrary Radon measure, let $r > 0$, and suppose $\eta(\partial B(x,r)) = 0$. (Since $\eta$ is finite on every ball, there are only countably many $r$ such that $\eta(\partial B(x,r)) > 0$) and let $V$ be an open set containing $\overline{B(x,r)}$. Then whenever $y$ is close enough to $x$, it follows that $B(y,r)$ is also a subset of $V$. Since $V$ is an arbitrary open set containing $\overline{B(x,r)}$, it follows

$$\eta(B(x,r)) = \eta(\overline{B(x,r)}) \geq \lim sup_{y \to x} \eta(B(y,r))$$

and so $y \to \eta(B(y,r))$ an upper semicontinuous real valued function of $x$, one which satisfies

$$f(x) \geq \lim sup_{n \to \infty} f(x_n)$$

whenever $x_n \to x$. Now it is routine to verify that a function $f$ is upper semicontinuous if and only if $f^{-1} ((-\infty, a])$ is open for all $a \in \mathbb{R}$. Therefore, $f^{-1} ((-\infty, a])$ is a Borel set for all $a \in \mathbb{R}$ and so $f$ is Borel measurable by Lemma 9.1.6. Now the measurability of $X_{N^C} (\cdot) D_\mu \lambda (\cdot)$ follows from

$$X_{N^C} (x) D_\mu \lambda (x) = \lim_{r_i \to 0} \frac{\lambda(B(x,r_i))}{\mu(B(x,r_i))} X_{N^C} (x)$$

where $r_i$ is such that $\partial B(x,r_i)$ has $\mu$ and $\lambda$ measure zero.

### 28.6 The Radon Nikodym Theorem For Radon Measures

The above theory can be used to give an alternative treatment of the Radon Nikodym theorem for Radon measures. Recall first the definition of absolute continuity.

**Definition 28.6.1** Let $\lambda, \mu$ be two Radon measures defined on $F$. Then $\lambda \ll \mu$ means that whenever $\mu(E) = 0$, it follows that $\lambda(E) = 0$. 
Theorem 28.6.2 Let $\lambda$ and $\mu$ be Radon measures and suppose $\lambda \ll \mu$. Then for all $E$ a $\mu$ measurable set,

$$\lambda(E) = \int_E (D\mu \lambda) \, d\mu.$$ 

Proof: Let $t > 1$ and let $E$ be a $\mu$ measurable set which is bounded and a subset of $N^C$ where $N$ is the exceptional set of $\mu$ measure zero in Theorem 28.5.3 off of which $\mu(B(x,r)) > 0$ for all $r > 0$ and $D\mu \lambda(x)$ exists. Consider

$$E_m \equiv E \cap \{x \in N^C : t^m \leq D\mu \lambda(x) < t^{m+1}\}$$

for $m \geq 0$ and let

$$E_m \equiv E \cap \{x \in N^C : t^{m+1} < D\mu \lambda(x) \leq t^m\}$$

for $m < 0$. Here $m$ is an integer. First note that

$E \cap \{x \in N^C : D\mu \lambda(x) = 0\}$

has $\lambda$ measure zero because by Lemma 28.5.2,

$$\lambda(E \cap \{x \in N^C : D\mu \lambda(x) = 0\}) \leq a\mu(E)$$

for all $a > 0$ and $\mu(E)$ is finite due to the assumption that $E$ is bounded and $\mu$ is a Radon measure. Therefore, by Lemma 28.5.2,

$$\lambda(E) = \sum_{m=0}^{\infty} \lambda(E_m) + \sum_{m=-\infty}^{-1} \lambda(E_m)$$

$$\leq \sum_{m=0}^{\infty} t^{m+1} \mu(E_m) + \sum_{m=-\infty}^{-1} t^m \mu(E_m) = t \sum_{m=0}^{\infty} t^m \mu(E_m) + \sum_{m=-\infty}^{-1} t^m \mu(E_m)$$

$$= t \sum_{m=0}^{\infty} \mu(E_m) + t^{-1} \sum_{m=-\infty}^{0} \mu(E_m) + \sum_{m=-\infty}^{-1} \mu(E_m) + \sum_{m=0}^{\infty} t \int_{E_m} D\mu \lambda(x) \, d\mu + t^{-1} \int_{E_m} D\mu \lambda(x) \, d\mu$$

where $E_-$ is the union of the $E_m$ for $m < 0$. Thus $E_-$ is the set of $x$ in $E$ such that $D\mu \lambda(x) < 1$ and $E_+$ is the union of the $E_m$ for $m \geq 0$ so it is the set of $x$ in $E$ out of the exceptional set where $D\mu \lambda(x) \geq 1$. Hence $E_-$ and $E_+$ do not depend on $m$. Also by this same lemma,

$$\lambda(E) = \sum_{m=0}^{\infty} \lambda(E_m) + \sum_{m=-\infty}^{-1} \lambda(E_m) \geq \sum_{m=0}^{\infty} t^m \mu(E_m) + \sum_{m=-\infty}^{-1} t^{m+1} \mu(E_m)$$
28.6. THE RADON NIKODYM THEOREM FOR RADON MEASURES

\[ \geq t^{-1} \sum_{m=0}^{\infty} t^{m+1} \mu(E_m) + t \sum_{m=-\infty}^{-1} t^{m} \mu(E_m) \]

\[ \geq t^{-1} \sum_{m=0}^{\infty} \int_{E_m} D_\mu \lambda(x) \, d\mu + t \sum_{m=-\infty}^{-1} \int_{E_m} D_\mu \lambda(x) \, d\mu \]

\[ = t^{-1} \int_{E_+} D_\mu \lambda(x) \, d\mu + t \int_{E_-} D_\mu \lambda(x) \, d\mu \]

Thus,

\[ t \int_{E_+} D_\mu \lambda(x) \, d\mu + t^{-1} \int_{E_-} D_\mu \lambda(x) \, d\mu \]

\[ \geq \lambda(E) \geq t^{-1} \int_{E_+} D_\mu \lambda(x) \, d\mu + t \int_{E_-} D_\mu \lambda(x) \, d\mu \]

and letting \( t \to 1 \), it follows

\[ \lambda(E) = \int_E D_\mu \lambda(x) \, d\mu. \quad (28.6.22) \]

Now if \( E \) is an arbitrary measurable set, contained in \( N^C \), this formula holds with \( E \) replaced with \( E \cap B(0,k) \). Letting \( k \to \infty \) and using the monotone convergence theorem, the above formula holds for all \( E \subseteq N^C \). Since \( N \) is a set of \( \mu \) measure zero, it follows \( N \) is also a set of \( \lambda \) measure zero due to the assumption of absolute continuity. Therefore (28.6.22) continues to hold for arbitrary \( \mu \) measurable sets, \( E \) even if they are not contained in \( N^C \). This is where the \( \lambda \ll \mu \) was used.

What if \( \lambda \) and \( \mu \) are just two arbitrary Radon measures defined on \( F \)? What then? It was shown above that \( D_\mu \lambda(x) \) exists for \( \mu \) a.e. \( x \). Also, it was shown above in the proof that if \( E \subseteq N^C \), then

\[ \lambda(E) = \int_E D_\mu \lambda(x) \, d\mu \]

Define for arbitrary \( E \in F \),

\[ \lambda_\mu(E) \equiv \int_E D_\mu \lambda(x) \, d\mu \]

Then you could let

\[ \lambda_\perp(E) \equiv \lambda(E) - \lambda_\mu(E) \]

Letting \( N \) be the set on which the derivative \( D_\mu \lambda(x) \) does not exist, it was shown above that \( \mu(N) = 0 \). Then

\[ \lambda(E) = \lambda(E \cap N) + \lambda(E \cap N^C) \]
\[ \lambda_\perp (E) + \lambda_\mu (E) = \lambda (E) = \lambda (E \cap N) + \lambda (E \cap N^C) \]
\[ = \lambda (E \cap N) + \int_{E \cap N^C} D_\mu \lambda (x) \, d\mu \]
\[ = \lambda (E \cap N) + \int_{E} D_\mu \lambda (x) \, d\mu \equiv \lambda (E \cap N) + \lambda_\mu (E) \]

and this shows that \( \lambda_\perp (E) = \lambda (E \cap N) \). This shows most of the following corollary.

**Corollary 28.6.3** Let \( \mu, \lambda \) be two Radon measures. Then there exist two measures, \( \lambda_\mu, \lambda_\perp \) such that
\[ \lambda_\mu \ll \mu, \lambda = \lambda_\mu + \lambda_\perp \]
and a set of \( \mu \) measure zero \( N \) such that
\[ \lambda_\perp (E) = \lambda (E \cap N) \]

Also \( \lambda_\mu \) is given by the formula
\[ \lambda_\mu (E) = \int_{E} D_\mu \lambda (x) \, d\mu \]

**Proof:** It only remains to verify that \( \lambda_\mu \) given above satisfies \( \lambda_\mu \ll \mu \). However, this is obvious because if \( \mu (E) = 0 \), then clearly \( \int_{E} D_\mu \lambda (x) \, d\mu = 0 \).
Chapter 29

Fourier Transforms

29.1 An Algebra Of Special Functions

First recall the following definition of a polynomial.

**Definition 29.1.1** \( \alpha = (\alpha_1, \ldots, \alpha_n) \) for \( \alpha_1 \cdots \alpha_n \) positive integers is called a multi-index. For \( \alpha \) a multi-index, \( |\alpha| = \alpha_1 + \cdots + \alpha_n \) and if \( x \in \mathbb{R}^n \),

\[ x = (x_1, \ldots, x_n), \]

and \( f \) a function, define

\[ x^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \]

A polynomial in \( n \) variables of degree \( m \) is a function of the form

\[ p(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha. \]

Here \( \alpha \) is a multi-index as just described and \( a_\alpha \in \mathbb{C} \). Also define for \( \alpha = (\alpha_1, \ldots, \alpha_n) \) a multi-index

\[ D^\alpha f(x) \equiv \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}. \]

**Definition 29.1.2** Define \( \mathcal{G}_1 \) to be the functions of the form \( p(x) e^{-a|x|^2} \) where \( a > 0 \) and \( p(x) \) is a polynomial. Let \( \mathcal{G} \) be all finite sums of functions in \( \mathcal{G}_1 \). Thus \( \mathcal{G} \) is an algebra of functions which has the property that if \( f \in \mathcal{G} \) then \( f \in \mathcal{G} \).

It is always assumed, unless stated otherwise that the measure will be Lebesgue measure.

**Lemma 29.1.3** \( \mathcal{G} \) is dense in \( C_0(\mathbb{R}^n) \) with respect to the norm,

\[ \|f\|_{\infty} = \sup \{ |f(x)| : x \in \mathbb{R}^n \} \]
Proof: By the Weierstrass approximation theorem, it suffices to show \( \mathcal{G} \) separates the points and annihilates no point. It was already observed in the above definition that \( f \in \mathcal{G} \) if \( f \in \mathcal{G} \). If \( y_1 \neq y_2 \) suppose first that \( |y_1| 
eq |y_2| \). Then in this case, you can let \( f(x) \equiv e^{-|x|^2} \) and \( f \in \mathcal{G} \) and \( f(y_1) \neq f(y_2) \). If \( |y_1| = |y_2| \), then suppose \( y_{1k} \neq y_{2k} \). This must happen for some \( k \) because \( y_1 \neq y_2 \). Then let \( f(x) \equiv x_k e^{-|x|^2} \). Thus \( \mathcal{G} \) separates points. Now \( e^{-|x|^2} \) is never equal to zero and so \( \mathcal{G} \) annihilates no point of \( \mathbb{R}^n \). This proves the lemma.

These functions are clearly quite specialized. Therefore, the following theorem is somewhat surprising.

Theorem 29.1.4 For each \( p \geq 1, p < \infty \), \( \mathcal{G} \) is dense in \( L^p(\mathbb{R}^n) \).

Proof: Let \( f \in L^p(\mathbb{R}^n) \). Then there exists \( g \in C_c(\mathbb{R}^n) \) such that \( \|f - g\|_p < \varepsilon \).

Now let \( b > 0 \) be large enough that

\[
\int_{\mathbb{R}^n} \left( e^{-b|x|^2} \right)^p \, dx < \varepsilon^p.
\]

Then \( x \rightarrow g(x) e^{b|x|^2} \) is in \( C_c(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n) \). Therefore, from Lemma 29.1.3 there exists \( \psi \in \mathcal{G} \) such that

\[
\left\| g e^{b|x|^2} - \psi \right\|_\infty < 1
\]

Therefore, letting \( \phi(x) \equiv e^{-b|x|^2} \psi(x) \) it follows that \( \phi \in \mathcal{G} \) and for all \( x \in \mathbb{R}^n \),

\[
|g(x) - \phi(x)| < e^{-b|x|^2}
\]

Therefore,

\[
\left( \int_{\mathbb{R}^n} |g(x) - \phi(x)|^p \, dx \right)^{1/p} \leq \left( \int_{\mathbb{R}^n} \left( e^{-b|x|^2} \right)^p \, dx \right)^{1/p} < \varepsilon.
\]

It follows

\[
\|f - \phi\|_p \leq \|f - g\|_p + \|g - \phi\|_p < 2\varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, this proves the theorem.

The following lemma is also interesting even if it is obvious.

Lemma 29.1.5 For \( \psi \in \mathcal{G} \), \( p \) a polynomial, and \( \alpha, \beta \) multiindices, \( D^\alpha \psi \in \mathcal{G} \) and \( p\psi \in \mathcal{G} \). Also

\[
\sup \{|x^\beta D^\alpha \psi(x)| : x \in \mathbb{R}^n \} < \infty
\]

29.2 Fourier Transforms Of Functions In \( \mathcal{G} \)

Definition 29.2.1 For \( \psi \in \mathcal{G} \) Define the Fourier transform, \( F \) and the inverse Fourier transform, \( F^{-1} \) by

\[
F\psi(t) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \psi(x) \, dx,
\]
29.2. FOURIER TRANSFORMS OF FUNCTIONS IN $G$

$$F^{-1}\psi(t) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it\cdot x} \psi(x) dx.$$  

where $t \cdot x \equiv \sum_{i=1}^{n} t_i x_i$. Note there is no problem with this definition because $\psi$ is in $L^1(\mathbb{R}^n)$ and therefore, 

$$\left| e^{it\cdot x} \psi(x) \right| \leq |\psi(x)|,$$

an integrable function.

One reason for using the functions, $G$ is that it is very easy to compute the Fourier transform of these functions. The first thing to do is to verify $F$ and $F^{-1}$ map $G$ to $G$ and that $F^{-1} \circ F(\psi) = \psi$.

**Lemma 29.2.2** The following formulas are true. ($c > 0$)

$$\int_{\mathbb{R}} e^{-ct^2} e^{-ist} dt = \int_{\mathbb{R}} e^{-ct^2} e^{ist} dt = e^{-\frac{s^2}{4c}} \sqrt{\frac{\pi}{c}}. \quad (29.2.1)$$

$$\int_{\mathbb{R}^n} e^{-c|t|^2} e^{-ist} dt = \int_{\mathbb{R}^n} e^{-c|t|^2} e^{i|s|^2} dt = e^{-\frac{|s|^2}{4c}} \left( \sqrt{\frac{\pi}{c}} \right)^n. \quad (29.2.2)$$

**Proof:** Consider the first one. Let $h(s)$ be given by the left side. Then

$$H(s) \equiv \int_{\mathbb{R}} e^{-ct^2} e^{-ist} dt = \int_{\mathbb{R}} e^{-ct^2} \cos(st) dt$$

Then using the dominated convergence theorem to differentiate,

$$H'(s) = \int_{\mathbb{R}} -e^{-ct^2} t \sin(st) dt = \frac{e^{-ct^2}}{2c} \sin(st) \bigg|_{-\infty}^{\infty} - \frac{s}{2c} \int_{\mathbb{R}} e^{-ct^2} \cos(st) dt = -\frac{s}{2c} H(s).$$

Also $H(0) = \int_{\mathbb{R}} e^{-ct^2} dt$. Thus $H(0) = \int_{\mathbb{R}} e^{-ct^2} dx \equiv I$ and so

$$I^2 = \int_{\mathbb{R}^2} e^{-c(x^2+y^2)} dxdy = \int_0^\infty \int_0^{2\pi} e^{-cr^2} r drd\theta = \frac{\pi}{c}.$$

Hence

$$H'(s) + \frac{s}{2c} H(s) = 0, \quad H(0) = \sqrt{\frac{\pi}{c}}.$$

It follows that $H(s) = e^{-\frac{s^2}{4c}} \sqrt{\frac{\pi}{c}}$. The second formula follows right away from Fubini’s theorem. ■

With these formulas, it is easy to verify $F, F^{-1}$ map $G$ to $G$ and $F \circ F^{-1} = F^{-1} \circ F = id.$

**Theorem 29.2.3** Each of $F$ and $F^{-1}$ map $G$ to $G$. Also $F^{-1} \circ F(\psi) = \psi$ and $F \circ F^{-1}(\psi) = \psi$. 
Proof: The first claim will be shown if it is shown that $F\psi \in G$ for $\psi(x) = x^\alpha e^{-b|x|^2}$ because an arbitrary function of $G$ is a finite sum of scalar multiples of functions such as $\psi$. Using Lemma 29.2.2, 

\[ F\psi(t) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} x^\alpha e^{-b|x|^2} dx \]

\[ = \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D^\alpha_t \left(\int_{\mathbb{R}^n} e^{-it \cdot x} e^{-b|x|^2} dx\right) \]

\[ = \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D^\alpha_t \left(e^{-\frac{|t|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) \]

and this is clearly in $G$ because it equals a polynomial times $e^{-\frac{|t|^2}{4b}}$.

It remains to verify the other assertion. As in the first case, it suffices to consider $\psi(x) = x^\alpha e^{-b|x|^2}$.

\[ F^{-1} \circ F(\psi)(s) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{is \cdot t} F(\psi)(t) dt \]

\[ = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{is \cdot t} \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D^\alpha_t \left(e^{-\frac{|t|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) dt \]

\[ = \left(\frac{1}{2\pi}\right)^n (i)^{-|\alpha|} \int_{\mathbb{R}^n} e^{is \cdot t} D^\alpha_t \left(e^{-\frac{|t|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) dt \]

\[ = \left(\frac{1}{2\pi}\right)^n (i)^{-|\alpha|} \int_{\mathbb{R}^n} (i)^{|\alpha|} s^\alpha e^{is \cdot t} \left(e^{-\frac{|t|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) dt \]

and by Lemma 29.2.2,

\[ = \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \int_{\mathbb{R}^n} s^\alpha e^{is \cdot t} \left(e^{-\frac{|t|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) dt \]

\[ = \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{1/4b}}\right)^n s^\alpha e^{-b|s|^2} = \psi(s). \]

29.3 Fourier Transforms Of Just About Anything

29.3.1 Fourier Transforms Of $G^*$

Definition 29.3.1 Let $G^*$ denote the vector space of linear functions defined on $G$ which have values in $\mathbb{C}$. Thus $T \in G^*$ means $T: G \to \mathbb{C}$ and $T$ is linear, 

\[ T(a\psi + b\phi) = aT(\psi) + bT(\phi) \text{ for all } a, b \in \mathbb{C}, \ \psi, \phi \in G \]
Let $\psi \in \mathcal{G}$. Then we can regard $\psi$ as an element of $\mathcal{G}^*$ by defining

$$\psi(\phi) \equiv \int_{\mathbb{R}^n} \psi(x) \phi(x) \, dx.$$  

Then we have the following important lemma.

**Lemma 29.3.2** The following is obtained for all $\phi, \psi \in \mathcal{G}$.

$$F\psi(\phi) = \psi(F\phi), \quad F^{-1}\psi(\phi) = \psi(F^{-1}\phi)$$

Also if $\psi \in \mathcal{G}$ and $\psi = 0$ in $\mathcal{G}^*$ so that $\psi(\phi) = 0$ for all $\phi \in \mathcal{G}$, then $\psi = 0$ as a function.

**Proof:**

$$F\psi(\phi) = \int_{\mathbb{R}^n} F\psi(t) \phi(t) \, dt$$

$$= \int_{\mathbb{R}^n} (1/2\pi)^{n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \psi(x) \phi(t) \, dx \, dt$$

$$= \int_{\mathbb{R}^n} \psi(x) (1/2\pi)^{n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \phi(t) \, dt \, dx$$

$$= \int_{\mathbb{R}^n} \psi(x) F\phi(x) \, dx \equiv \psi(F\phi)$$

The other claim is similar.

Suppose now $\psi(\phi) = 0$ for all $\phi \in \mathcal{G}$. Then

$$\int_{\mathbb{R}^n} \psi(x) \phi(x) \, dx = 0$$

for all $\phi \in \mathcal{G}$. Therefore, this is true for $\phi = \psi$ and so $\psi = 0$.

This lemma suggests a way to define the Fourier transform of something in $\mathcal{G}^*$.

**Definition 29.3.3** For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by

$$FT(\phi) \equiv T(F\phi), \quad F^{-1}T(\phi) \equiv T(F^{-1}\phi)$$

**Lemma 29.3.4** $F$ and $F^{-1}$ are both one to one, onto, and are inverses of each other.

**Proof:** First note $F$ and $F^{-1}$ are both linear. This follows directly from the definition. Suppose now $FT = 0$. Then $FT(\phi) = T(F\phi) = 0$ for all $\phi \in \mathcal{G}$. But $F$ and $F^{-1}$ map $\mathcal{G}$ onto $\mathcal{G}$ because if $\psi \in \mathcal{G}$, then as shown above, $\psi = F\left(F^{-1}(\psi)\right)$. Therefore, $T = 0$ and so $F$ is one to one. Similarly $F^{-1}$ is one to one. Now

$$F^{-1}(FT)(\phi) \equiv (FT)(F^{-1}\phi) \equiv T\left(F\left(F^{-1}(\phi)\right)\right) = T\phi.$$  

Therefore, $F^{-1} \circ F(T) = T$. Similarly, $F \circ F^{-1}(T) = T$. Thus both $F$ and $F^{-1}$ are one to one and onto and are inverses of each other as suggested by the notation.

Probably the most interesting things in $\mathcal{G}^*$ are functions of various kinds. The following lemma will be useful in considering this situation.
Lemma 29.3.5 If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^n} f \phi dx = 0 \) for all \( \phi \in C_c(\mathbb{R}^n) \), then \( f = 0 \) a.e.

Proof: For \( r > 0 \), let 
\[
E \equiv \{ x : f(x) \geq r \}, \quad E_R \equiv E \cap B(0,R).
\]
Let \( K_m \) be an increasing sequence of compact sets, and let \( V_m \) be a decreasing sequence of open sets satisfying
\[
K_m \subseteq E_R \subseteq V_m, \quad m_n(V_m) \leq m_n(K_m) + 2^{-m}, V_1 \subseteq B(0,R).
\]
Therefore,
\[
m_n(V_m \setminus K_m) \leq 2^{-m}.
\]
Let \( \phi_m \in C_c(V_m), \quad K_m \prec \phi_m \prec V_m. \)
The statement \( K_m \prec \phi_m \prec V_m \) means that \( \phi_m \) equals 1 on \( K_m \), has compact support in \( V_m \), maps into \([0,1]\), and is continuous. Then \( \phi_m(x) \to \chi_{E_R}(x) \) a.e. because the set where \( \phi_m(x) \) fails to converge to this set is contained in the set of all \( x \) which are in infinitely many of the sets \( V_m \setminus K_m \). This set has measure zero because
\[
\sum_{m=1}^{\infty} m_n(V_m \setminus K_m) < \infty
\]
Thus \( \phi_m \) converges pointwise a.e to \( \chi_{E_R} \) and so, by the dominated convergence theorem,
\[
0 = \lim_{m \to \infty} \int_{\mathbb{R}^n} f \phi_m dx = \lim_{m \to \infty} \int_{V_1} f \phi_m dx = \int_{E_R} f dx \geq rm(E_R).
\]
Thus, \( m_n(E_R) = 0 \) and therefore \( m_n(E) = \lim_{R \to \infty} m_n(E_R) = 0. \) Since \( r > 0 \) is arbitrary, it follows
\[
m_n([f > 0]) = \bigcup_{k=1}^{\infty} m_n([f > k^{-1}]) = \bigcup_{k=1}^{\infty} m_n([f^+ > k^{-1}]) = m_n([f^+ > 0]) = 0.
\]
Hence \( f^+ = 0 \) a.e. It follows that \( \int f^- \phi dx = 0 \) for all \( \phi \in C_c(\mathbb{R}^n) \) because
\[
\int f^- \phi dx = \int f^+ \phi - \int f \phi = 0.
\]
Thus from what was just shown, with \( f^- \) taking the place of \( f \), it follows \( \frac{|f^-| + f^-}{2} = 0 \) and so \( f^- = 0 \) a.e. also. \( \blacksquare \)

Corollary 29.3.6 Let \( f \in L^1(\mathbb{R}^n) \) and suppose
\[
\int_{\mathbb{R}^n} f(x) \phi(x) dx = 0
\]
for all \( \phi \in G \). Then \( f = 0 \) a.e.
29.3. FOURIER TRANSFORMS OF JUST ABOUT ANYTHING

Proof: Let \( \psi \in C_c(\mathbb{R}^n) \). Then by the Stone Weierstrass approximation theorem, there exists a sequence of functions, \( \{\phi_k\} \subseteq \mathcal{G} \) such that \( \phi_k \to \psi \) uniformly. Then by the dominated convergence theorem,
\[
\int f \psi dx = \lim_{k \to \infty} \int f \phi_k dx = 0.
\]
By Lemma 29.3.5, \( f = 0 \).

The next theorem is the main result of this sort.

Theorem 29.3.7 Let \( f \in L^p(\mathbb{R}^n) \), \( p \geq 1 \), or suppose \( f \) is measurable and has polynomial growth,
\[
|f(x)| \leq K \left(1 + |x|^2\right)^m
\]
for some \( m \in \mathbb{N} \). Then if
\[
\int f \psi dx = 0
\]
for all \( \psi \in \mathcal{G} \), then it follows \( f = 0 \).

Proof: First note that if \( f \in L^p(\mathbb{R}^n) \) or has polynomial growth, then it makes sense to write the integral \( \int f \psi dx \) described above. This is obvious in the case of polynomial growth. In the case where \( f \in L^p(\mathbb{R}^n) \) it also makes sense because
\[
\int |f| \psi dx \leq \left(\int |f|^p dx\right)^{1/p} \left(\int |\psi|^{p'} dx\right)^{1/p'} < \infty
\]
due to the fact mentioned above that all these functions in \( \mathcal{G} \) are in \( L^p(\mathbb{R}^n) \) for every \( p \geq 1 \). Suppose now that \( f \in L^p, p \geq 1 \). The case where \( f \in L^1(\mathbb{R}^n) \) was dealt with in Corollary 29.3.6. Suppose \( f \in L^p(\mathbb{R}^n) \) for \( p > 1 \). Then
\[
|f|^{p-2} f \in L^{p'}(\mathbb{R}^n), \quad p' = q, \frac{1}{p} + \frac{1}{q} = 1
\]
and by density of \( \mathcal{G} \) in \( L^{p'}(\mathbb{R}^n) \) (Theorem 29.1.3), there exists a sequence \( \{g_k\} \subseteq \mathcal{G} \) such that
\[
\left\|g_k - |f|^{p-2} f\right\|_{p'} \to 0.
\]
Then
\[
\int_{\mathbb{R}^n} |f|^p dx = \int_{\mathbb{R}^n} f \left(|f|^{p-2} f - g_k\right) dx + \int_{\mathbb{R}^n} f g_k dx
\]
\[
= \int_{\mathbb{R}^n} f \left(|f|^{p-2} f - g_k\right) dx
\]
\[
\leq ||f||_{L^p} \left\|g_k - |f|^{p-2} f\right\|_{p'}
\]
which converges to 0. Hence \( f = 0 \).
It remains to consider the case where $f$ has polynomial growth. Thus $x \rightarrow f(x) e^{-|x|^2} \in L^1(\mathbb{R}^n)$. Therefore, for all $\psi \in \mathcal{G}$,

$$0 = \int f(x) e^{-|x|^2} \psi(x) \, dx$$

because $e^{-|x|^2} \psi(x) \in \mathcal{G}$. Therefore, by the first part, $f(x) e^{-|x|^2} = 0$ a.e. □

The following theorem shows that you can consider most functions you are likely to encounter as elements of $\mathcal{G}^*$.

**Theorem 29.3.8** Let $f$ be a measurable function with polynomial growth,

$$|f(x)| \leq C \left(1 + |x|^2\right)^N$$

for some $N$, or let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$. Then $f \in \mathcal{G}^*$ if

$$f(\phi) \equiv \int f \phi \, dx.$$

**Proof:** Let $f$ have polynomial growth first. Then the above integral is clearly well defined and so in this case, $f \in \mathcal{G}^*$.

Next suppose $f \in L^p(\mathbb{R}^n)$ with $\infty > p \geq 1$. Then it is clear again that the above integral is well defined because of the fact that $\phi$ is a sum of polynomials times exponentials of the form $e^{-c|x|^2}$ and these are in $L^{p'}(\mathbb{R}^n)$. Also $\phi \rightarrow f(\phi)$ is clearly linear in both cases. □

This has shown that for nearly any reasonable function, you can define its Fourier transform as described above. You could also define the Fourier transform of a finite Borel measure $\mu$ because for such a measure

$$\psi \rightarrow \int_{\mathbb{R}^n} \psi \, d\mu$$

is a linear functional on $\mathcal{G}$. This includes the very important case of probability distribution measures. The theoretical basis for this assertion will be given a little later.

### 29.3.2 Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$

First suppose $f \in L^1(\mathbb{R}^n)$.

**Theorem 29.3.9** Let $f \in L^1(\mathbb{R}^n)$. Then $F f(\phi) = \int_{\mathbb{R}^n} g \phi \, dt$ where

$$g(t) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i t \cdot x} f(x) \, dx$$

and $F^{-1} f(\phi) = \int_{\mathbb{R}^n} g \phi \, dt$ where $g(t) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i t \cdot x} f(x) \, dx$. In short,

$$F f(t) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i t \cdot x} f(x) \, dx,$$
29.3. **FOURIER TRANSFORMS OF JUST ABOUT ANYTHING**

\[ F^{-1} f(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot x} f(x) dx. \]

**Proof:** From the definition and Fubini’s theorem,

\[
Ff(\phi) = \int_{\mathbb{R}^n} f(t) F\phi(t) dt = \int_{\mathbb{R}^n} f(t) \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \phi(x) dx dt
\]

\[
= \int_{\mathbb{R}^n} \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(t) e^{-it \cdot x} dx \phi(x) dt.
\]

Since \( \phi \in \mathcal{G} \) is arbitrary, it follows from Theorem 29.3.7 that \( Ff(x) \) is given by the claimed formula. The case of \( F^{-1} \) is identical.

Here are interesting properties of these Fourier transforms of functions in \( L^1 \).

**Theorem 29.3.10** If \( f \in L^1(\mathbb{R}^n) \) and \( \|f_k - f\|_1 \to 0 \), then \( Ff_k \) and \( F^{-1} f_k \) converge uniformly to \( Ff \) and \( F^{-1} f \) respectively. If \( f \in L^1(\mathbb{R}^n) \), then \( F^{-1} f \) and \( Ff \) are both continuous and bounded. Also,

\[
\lim_{|x| \to \infty} F^{-1} f(x) = \lim_{|x| \to \infty} Ff(x) = 0. \quad (29.3.3)
\]

Furthermore, for \( f \in L^1(\mathbb{R}^n) \) both \( Ff \) and \( F^{-1} f \) are uniformly continuous.

**Proof:** The first claim follows from the following inequality.

\[
|Ff_k(t) - Ff(t)| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{it \cdot x} f_k(x) - e^{-it \cdot x} f(x)| dx
\]

\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f_k(x) - f(x)| dx
\]

\[
= (2\pi)^{-n/2} \|f - f_k\|_1.
\]

which a similar argument holding for \( F^{-1} \).

Now consider the second claim of the theorem.

\[
|Ff(t) - Ff(t')| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{it \cdot x} - e^{-it' \cdot x}| |f(x)| dx
\]

The integrand is bounded by \( 2|f(x)| \), a function in \( L^1(\mathbb{R}^n) \) and converges to 0 as \( t' \to t \) and so the dominated convergence theorem implies \( Ff \) is continuous. To see \( Ff(t) \) is uniformly bounded,

\[
|Ff(t)| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(x)| dx < \infty.
\]

A similar argument gives the same conclusions for \( F^{-1} \).

It remains to verify \( \text{adj.} \) and the claim that \( Ff \) and \( F^{-1} f \) are uniformly continuous.

\[
|Ff(t)| \leq \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} f(x) dx \right|
\]
Now let \( \varepsilon > 0 \) be given and let \( g \in C_c^\infty(\mathbb{R}^n) \) such that \((2\pi)^{-n/2} \|g - f\|_1 < \varepsilon/2\). Then

\[
|F f (t)| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(x) - g(x)| \, dx + \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} g(x) \, dx \right| \\
\leq \varepsilon/2 + (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{-it \cdot x} g(x)| \, dx.
\]

Now integrating by parts, it follows that for \( ||t||_\infty \equiv \max \{ |t_j| : j = 1, \cdots, n \} \geq 0 \)

\[
|F f (t)| \leq \varepsilon/2 + (2\pi)^{-n/2} \left( 1/||t||_\infty \right) \int_{\mathbb{R}^n} \sum_{j=1}^n \left| \frac{\partial g(x)}{\partial x_j} \right| \, dx 
\]

and this last expression converges to zero as \( ||t||_\infty \to \infty \). The reason for this is that if \( t_j \neq 0 \), integration by parts with respect to \( x_j \) gives

\[
(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} g(x) \, dx = (2\pi)^{-n/2} \frac{1}{-it_j} \int_{\mathbb{R}^n} e^{-it \cdot x} \frac{\partial g(x)}{\partial x_j} \, dx.
\]

Therefore, choose the \( j \) for which \( ||t||_\infty = |t_j| \) and the result of (29.3.4) holds. Therefore, from (29.3.4), if \( ||t||_\infty \) is large enough, \( |F f (t)| < \varepsilon \). Similarly, \( \lim_{||t||_\infty \to \infty} F^{-1} (t) = 0 \). Consider the claim about uniform continuity. Let \( \varepsilon > 0 \) be given. Then there exists \( R \) such that if \( ||t||_\infty > R \), then \( |F f (t)| < \varepsilon/2 \). Since \( F f \) is continuous, it is uniformly continuous on the compact set \([-R - 1, R + 1]^n\). Therefore, there exists \( \delta_1 \) such that if \( ||t - t'||_\infty < \delta_1 \) for \( t, t' \in [-R - 1, R + 1]^n \), then

\[
|F f (t) - F f (t')| < \varepsilon/2.
\]

Now let \( 0 < \delta < \min (\delta_1, 1) \) and suppose \( ||t - t'||_\infty < \delta \). If both \( t, t' \) are contained in \([-R, R]^n\), then (29.3.5) holds. If \( t \in [-R, R]^n \) and \( t' \notin [-R, R]^n \), then both are contained in \([-R - 1, R + 1]^n\) and so this verifies (29.3.5) in this case. The other case is that neither point is in \([-R, R]^n\) and in this case,

\[
|F f (t) - F f (t')| \leq |F f (t)| + |F f (t')| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

There is a very interesting relation between the Fourier transform and convolutions.

**Theorem 29.3.11** Let \( f, g \in L^1(\mathbb{R}^n) \). Then \( f * g \in L^1 \) and \( F(f * g) = (2\pi)^{n/2} Ff Fg \).

**Proof:** Consider

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y) g(y)| \, dy \, dx.
\]
The function, \((x, y) \rightarrow |f(x - y)g(y)|\) is Lebesgue measurable and so by Fubini’s theorem,
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)g(y)| \, dy \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - y)g(y)| \, dx \, dy = \|f\|_1 \|g\|_1 < \infty.
\]
It follows that for a.e. \(x\), \(\int_{\mathbb{R}^n} |f(x - y)g(y)| \, dy < \infty\) and for each of these values of \(x\), it follows that \(\int_{\mathbb{R}^n} f(x - y)g(y) \, dy\) exists and equals a function of \(x\) which is in \(L^1(\mathbb{R}^n)\), \(f \ast g(x)\). Now
\[
F(f \ast g)(t) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} f \ast g(x) \, dx
\]

\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \int_{\mathbb{R}^n} f(x - y)g(y) \, dy \, dx
\]

\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \int_{\mathbb{R}^n} e^{-i(x-y) \cdot x} f(x - y) \, dy \, dx
\]

\[
= (2\pi)^{n/2} Ff(t) Fg(t). \quad \Box
\]

There are many other considerations involving Fourier transforms of functions in \(L^1(\mathbb{R}^n)\).

### 29.3.3 Fourier Transforms Of Functions In \(L^2(\mathbb{R}^n)\)

Consider \(Ff\) and \(F^{-1}f\) for \(f \in L^2(\mathbb{R}^n)\). First note that the formula given for \(Ff\) and \(F^{-1}f\) when \(f \in L^1(\mathbb{R}^n)\) will not work for \(f \in L^2(\mathbb{R}^n)\) unless \(f\) is also in \(L^1(\mathbb{R}^n)\). Recall that \(a + ib = a - ib\).

**Theorem 29.3.12** For \(\phi \in \mathcal{G}\), \(\|F\phi\|_2 = \|F^{-1}\phi\|_2 = \|\phi\|_2\).

**Proof:** First note that for \(\psi \in \mathcal{G}\),
\[
F(\overline{\psi}) = \overline{F^{-1}(\psi)}, \quad F^{-1}(\overline{\psi}) = \overline{F(\psi)}.
\]
(29.3.6)

This follows from the definition. For example,
\[
F\overline{\psi}(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot x} \overline{\psi}(x) \, dx
\]

\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot x} \psi(x) \, dx
\]

Let \(\phi, \psi \in \mathcal{G}\). It was shown above that
\[
\int_{\mathbb{R}^n} (F\phi)\psi(t) \, dt = \int_{\mathbb{R}^n} \phi(F\psi) \, dx.
\]
Similarly,
\[
\int_{\mathbb{R}^n} \phi(F^{-1}\psi) \, dx = \int_{\mathbb{R}^n} (F^{-1}\phi)\psi(t) \, dt. \quad (29.3.7)
\]
Now, \( \int |\phi|^2 \, dx = \int \phi F^{-1}(F\phi) \, dx = \int \phi F(\phi) \, dx \)

\[
= \int F\phi(\phi) \, dx = \int |\phi|^2 \, dx.
\]

Similarly

\[
||\phi||_2 = ||F^{-1}\phi||_2.
\]

**Lemma 29.3.13** Let \( f \in L^2(\mathbb{R}^n) \) and let \( \phi_k \to f \) in \( L^2(\mathbb{R}^n) \) where \( \phi_k \in \mathcal{G} \). (Such a sequence exists because of density of \( \mathcal{G} \) in \( L^2(\mathbb{R}^n) \).) Then \( Ff \) and \( F^{-1}f \) are both in \( L^2(\mathbb{R}^n) \) and the following limits take place in \( L^2 \).

\[
\lim_{k \to \infty} F(\phi_k) = F(f), \quad \lim_{k \to \infty} F^{-1}(\phi_k) = F^{-1}(f).
\]

**Proof:** Let \( \psi \in \mathcal{G} \) be given. Then

\[
Ff(\psi) \equiv f(F\psi) \equiv \int f(x)F\psi(x) \, dx \equiv \lim_{k \to \infty} \int \phi_k(x)F\psi(x) \, dx = \lim_{k \to \infty} \int F\phi_k(x)\psi(x) \, dx.
\]

Also by Theorem 29.3.12 \( \{F\phi_k\}_{k=1}^\infty \) is Cauchy in \( L^2(\mathbb{R}^n) \) and so it converges to some \( h \in L^2(\mathbb{R}^n) \). Therefore, from the above,

\[
Ff(\psi) = \int h(x)\psi(x)
\]

which shows that \( F(f) \in L^2(\mathbb{R}^n) \) and \( h = F(f) \). The case of \( F^{-1} \) is entirely similar.

Since \( Ff \) and \( F^{-1}f \) are in \( L^2(\mathbb{R}^n) \), this also proves the following theorem.

**Theorem 29.3.14** If \( f \in L^2(\mathbb{R}^n) \), \( Ff \) and \( F^{-1}f \) are the unique elements of \( L^2(\mathbb{R}^n) \) such that for all \( \phi \in \mathcal{G} \),

\[
\int f(x)\phi(x) \, dx = \int f(x)F\phi(x) \, dx, \quad (29.3.8)
\]

\[
\int f(x)\phi(x) \, dx = \int f(x)F^{-1}\phi(x) \, dx. \quad (29.3.9)
\]

**Theorem 29.3.15** (Plancherel)

\[
||f||_2 = ||Ff||_2 = ||F^{-1}f||_2. \quad (29.3.10)
\]
29.3. FOURIER TRANSFORMS OF JUST ABOUT ANYTHING

**Proof:** Use the density of $G$ in $L^2(\mathbb{R}^n)$ to obtain a sequence, $\{\phi_k\}$ converging to $f$ in $L^2(\mathbb{R}^n)$. Then by Lemma 29.3.13

$$||Ff||_2 = \lim_{k \to \infty} ||F\phi_k||_2 = \lim_{k \to \infty} ||\phi_k||_2 = ||f||_2.$$

Similarly,

$$||f||_2 = ||F^{-1}f||_2.$$

The following corollary is a simple generalization of this. To prove this corollary, use the following simple lemma which comes as a consequence of the Cauchy Schwarz inequality.

**Lemma 29.3.16** Suppose $f_k \to f$ in $L^2(\mathbb{R}^n)$ and $g_k \to g$ in $L^2(\mathbb{R}^n)$. Then

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} f_k g_k dx = \int_{\mathbb{R}^n} f g dx$$

**Proof:**

$$\left| \int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} f g dx \right| \leq \left| \int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} f_k g dx \right| + \left| \int_{\mathbb{R}^n} f_k g dx - \int_{\mathbb{R}^n} f g dx \right|$$

$$\leq ||f_k||_2 ||g - g_k||_2 + ||g||_2 ||f_k - f||_2.$$

Now $||f_k||_2$ is a Cauchy sequence and so it is bounded independent of $k$. Therefore, the above expression is smaller than $\varepsilon$ whenever $k$ is large enough. ■

**Corollary 29.3.17** For $f, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f \overline{g} dx = \int_{\mathbb{R}^n} Ff \overline{Fg} dx = \int_{\mathbb{R}^n} F^{-1}f \overline{F^{-1}g} dx.$$

**Proof:** First note the above formula is obvious if $f, g \in G$. To see this, note

$$\int_{\mathbb{R}^n} Ff \overline{Fg} dx = \int_{\mathbb{R}^n} Ff(x) \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \overline{g(t)} dt dx$$

$$= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} Ff(x) dx \overline{g(t)} dt dx = \int_{\mathbb{R}^n} (F^{-1} \circ F) f(t) \overline{g(t)} dt$$

$$= \int_{\mathbb{R}^n} f(t) \overline{g(t)} dt.$$

The formula with $F^{-1}$ is exactly similar.

Now to verify the corollary, let $\phi_k \to f$ in $L^2(\mathbb{R}^n)$ and let $\psi_k \to g$ in $L^2(\mathbb{R}^n)$. Then by Lemma 29.3.13

$$\int_{\mathbb{R}^n} Ff \overline{Fg} dx = \lim_{k \to \infty} \int_{\mathbb{R}^n} F\phi_k \overline{F\psi_k} dx = \lim_{k \to \infty} \int_{\mathbb{R}^n} \phi_k \overline{\psi_k} dx = \int_{\mathbb{R}^n} f \overline{g} dx$$

A similar argument holds for $F^{-1}$. ■

How does one compute $Ff$ and $F^{-1}f$?
CHAPTER 29. FOURIER TRANSFORMS

Theorem 29.3.18 For \( f \in L^2(\mathbb{R}^n) \), let \( f_r = f \chi_{E_r} \) where \( E_r \) is a bounded measurable set with \( E_r \uparrow \mathbb{R}^n \). Then the following limits hold in \( L^2(\mathbb{R}^n) \).

\[
FF = \lim_{r \to \infty} FF_r, \quad F^{-1}f = \lim_{r \to \infty} F^{-1}f_r.
\]

Proof: \( ||f - f_r||_2 \to 0 \) and so \( ||Ff - Ff_r||_2 \to 0 \) and \( ||F^{-1}f - F^{-1}f_r||_2 \to 0 \) by Plancherel’s Theorem. 

What are \( FF_r \) and \( F^{-1}f_r \)? Let \( \phi \in \mathcal{G} \)

\[
\int_{\mathbb{R}^n} FF_r \phi dx = \int_{\mathbb{R}^n} F_r \phi dx
\]

\[
= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_r(x) e^{-ix \cdot y} \phi(y) dy dx
\]

\[
= \int_{\mathbb{R}^n} |(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(x) e^{-ix \cdot y} dx| \phi(y) dy.
\]

Since this holds for all \( \phi \in \mathcal{G} \), a dense subset of \( L^2(\mathbb{R}^n) \), it follows that

\[
FF_r(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(x) e^{-ix \cdot y} dx.
\]

Similarly

\[
F^{-1}f_r(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(x) e^{ix \cdot y} dx.
\]

This shows that to take the Fourier transform of a function in \( L^2(\mathbb{R}^n) \), it suffices to take the limit as \( r \to \infty \) in \( L^2(\mathbb{R}^n) \) of \( (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(x) e^{-ix \cdot y} dx \). A similar procedure works for the inverse Fourier transform.

Note this reduces to the earlier definition in case \( f \in L^1(\mathbb{R}^n) \). Now consider the convolution of a function in \( L^2 \) with one in \( L^1 \).

Theorem 29.3.19 Let \( h \in L^2(\mathbb{R}^n) \) and let \( f \in L^1(\mathbb{R}^n) \). Then \( h \ast f \in L^2(\mathbb{R}^n) \),

\[
F^{-1}(h \ast f) = (2\pi)^{n/2} F^{-1}h F^{-1}f,
\]

and

\[
||h \ast f||_2 \leq ||h||_2 ||f||_1.
\]  \hspace{1cm} (29.3.11)

Proof: An application of Minkowski’s inequality yields

\[
\left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |h(x - y)||f(y)| dy \right)^2 dx \right)^{1/2} \leq ||f||_1||h||_2.
\]  \hspace{1cm} (29.3.12)

Hence \( \int |h(x - y)||f(y)| dy < \infty \) a.e. \( x \) and

\[
x \to \int h(x - y) f(y) dy
\]
is in $L^2(\mathbb{R}^n)$. Let $E_r \uparrow \mathbb{R}^n$, $m(E_r) < \infty$. Thus,

$$h_r \equiv \chi_{E_r}, h \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

and letting $\phi \in \mathcal{G}$,

$$\int F(h_r * f)(\phi) \, dx$$

$$= \int (h_r * f)(F\phi) \, dx$$

$$= (2\pi)^{-n/2} \int \int h_r(x - y) f(y) e^{-ix \cdot t} \phi(t) \, dt \, dy \, dx$$

$$= (2\pi)^{-n/2} \int \left( \int h_r(x - y) e^{-i(x - y) \cdot t} \, dx \right) f(y) e^{-iy \cdot t} \, dy \phi(t) \, dt$$

$$= (2\pi)^{-n/2} Fh_r(t) Ff(t) \phi(t) \, dt.$$
Thus $f \in \mathcal{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and
\[
\sup\{|x^\beta D^\alpha f(x)| : x \in \mathbb{R}^n\} < \infty
\] (29.3.13)
for all multi-indices $\alpha$ and $\beta$.

Also note that if $f \in \mathcal{S}$, then $p(f) \in \mathcal{S}$ for any polynomial, $p$ with $p(0) = 0$ and that
\[
\mathcal{S} \subseteq L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)
\]
for any $p \geq 1$. To see this assertion about the $p(f)$, it suffices to consider the case of the product of two elements of the Schwartz class. If $f, g \in \mathcal{S}$, then $D^\alpha (fg)$ is a finite sum of derivatives of $f$ times derivatives of $g$. Therefore, $\rho_N (fg) < \infty$ for all $N$. You may wonder about examples of things in $\mathcal{S}$. Clearly any function in $C^\infty_c(\mathbb{R}^n)$ is in $\mathcal{S}$. However there are other functions in $\mathcal{S}$. For example $e^{-|x|^2}$ is in $\mathcal{S}$ as you can verify for yourself and so is any function from $\mathcal{G}$. Note also that the density of $C^\infty_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ shows that $\mathcal{S}$ is dense in $L^p(\mathbb{R}^n)$ for every $p$.

Recall the Fourier transform of a function in $L^1(\mathbb{R}^n)$ is given by
\[
Ff(t) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i t \cdot x} f(x) dx.
\] Therefore, this gives the Fourier transform for $f \in \mathcal{S}$. The nice property which $\mathcal{S}$ has in common with $\mathcal{G}$ is that the Fourier transform and its inverse map $\mathcal{S}$ one to one onto $\mathcal{S}$. This means I could have presented the whole of the above theory in terms of $\mathcal{S}$ rather than in terms of $\mathcal{G}$. However, it is more technical.

**Theorem 29.3.21** If $f \in \mathcal{S}$, then $Ff$ and $F^{-1}f$ are also in $\mathcal{S}$.

**Proof:** To begin with, let $\alpha = e_j = (0, 0, \cdots, 1, 0, \cdots, 0)$, the 1 in the $j$th slot.
\[
\frac{F^{-1}f(t + he_j) - F^{-1}f(t)}{h} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot x} f(x) \left( \frac{e^{ihx_j} - 1}{h} \right) dx. \tag{29.3.14}
\]
Consider the integrand in (29.3.14)
\[
\left| e^{it \cdot x} f(x) \left( \frac{e^{ihx_j} - 1}{h} \right) \right| = |f(x)| \left| \frac{e^{ihx_j} - e^{-i(h/2)x_j}}{h} \right|
= |f(x)| \left| \frac{i \sin \left( (h/2)x_j \right)}{h/2} \right| \leq |f(x)||x_j|
\]
and this is a function in $L^1(\mathbb{R}^n)$ because $f \in \mathcal{S}$. Therefore by the Dominated Convergence Theorem,
\[
\frac{\partial F^{-1}f(t)}{\partial t_j} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot x} x_j f(x) dx
= i(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot x} e_j f(x) dx.
\]
29.3. FOURIER TRANSFORMS OF JUST ABOUT ANYTHING

Now \( x^\alpha f(x) \in \mathcal{S} \) and so one can continue in this way and take derivatives indefinitely. Thus \( F^{-1}f \in C^\infty(\mathbb{R}^n) \) and from the above argument,

\[
D^\alpha F^{-1}f(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it\cdot x} (i\alpha)^\alpha f(x) dx.
\]

To complete showing \( F^{-1}f \in \mathcal{S} \),

\[
t^\beta D^\alpha F^{-1}f(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it\cdot x} t^\beta (i\alpha)^\alpha f(x) dx.
\]

Integrate this integral by parts to get

\[
t^\beta D^\alpha F^{-1}f(t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it\cdot x} D^\beta ((i\alpha)^\alpha f(x)) dx.
\] (29.3.15)

Here is how this is done.

\[
\int_{\mathbb{R}} e^{it_j x_j} t_j^\beta ((i\alpha)^\alpha f(x)) dx_j = \int_{\mathbb{R}} e^{it_j x_j} t_j^\beta ((i\alpha)^\alpha f(x)) |_{-\infty}^{\infty} + i \int_{\mathbb{R}} e^{it_j x_j} t_j^\beta D^\alpha ((i\alpha)^\alpha f(x)) dx_j
\]

where the boundary term vanishes because \( f \in \mathcal{S} \). Returning to 29.3.15, use the fact that \(|e^{ia}| = 1\) to conclude

\[
|t^\beta D^\alpha F^{-1}f(t)| \leq C \int_{\mathbb{R}^n} |D^\beta ((i\alpha)^\alpha f(x))| dx < \infty.
\]

It follows \( F^{-1}f \in \mathcal{S} \). Similarly \( Ff \in \mathcal{S} \) whenever \( f \in \mathcal{S} \). ■

Of course \( \mathcal{S} \) can be considered a subset of \( \mathcal{G}^* \) as follows. For \( \psi \in \mathcal{S} \),

\[
\psi(\phi) \equiv \int_{\mathbb{R}^n} \psi \phi dx
\]

**Theorem 29.3.22** Let \( \psi \in \mathcal{S} \). Then \( (F \circ F^{-1})(\psi) = \psi \) and \( (F^{-1} \circ F)(\psi) = \psi \) whenever \( \psi \in \mathcal{S} \). Also \( F \) and \( F^{-1} \) map \( \mathcal{S} \) one to one and onto \( \mathcal{S} \).

**Proof:** The first claim follows from the fact that \( F \) and \( F^{-1} \) are inverses of each other on \( \mathcal{G}^* \) which was established above. For the second, let \( \psi \in \mathcal{S} \). Then \( \psi = F(F^{-1}\psi) \). Thus \( F \) maps \( \mathcal{S} \) onto \( \mathcal{S} \). If \( F\psi = 0 \), then do \( F^{-1} \) to both sides to conclude \( \psi = 0 \). Thus \( F \) is one to one and onto. Similarly, \( F^{-1} \) is one to one and onto. ■

29.3.5 Convolution

To begin with it is necessary to discuss the meaning of \( \phi f \) where \( f \in \mathcal{G}^* \) and \( \phi \in \mathcal{G} \). What should it mean? First suppose \( f \in L^p (\mathbb{R}^n) \) or measurable with polynomial growth. Then \( \phi f \) also has these properties. Hence, it should be the case that \( \phi f (\psi) = \int_{\mathbb{R}^n} \phi f \psi dx = \int_{\mathbb{R}^n} f (\phi \psi) dx \). This motivates the following definition.
Definition 29.3.23 Let $T \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then $\phi T \equiv T \phi \in \mathcal{G}^*$ will be defined by

$$\phi T (\psi) \equiv T (\phi \psi).$$

The next topic is that of convolution. It was just shown that

$$F (f * \phi) = (2\pi)^{n/2} F\phi Ff,$$

whenever $f \in L^2 (\mathbb{R}^n)$ and $\phi \in \mathcal{G}$ so the same definition is retained in the general case because it makes perfect sense and agrees with the earlier definition.

Definition 29.3.24 Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then define the convolution of $f$ with an element of $\mathcal{G}$ as follows.

$$f * \phi \equiv (2\pi)^{n/2} F^{-1} (F\phi Ff) \in \mathcal{G}^*$$

There is an obvious question. With this definition, is it true that $F^{-1} (f * \phi) = (2\pi)^{n/2} F^{-1} \phi F^{-1} f$ as it was earlier?

Theorem 29.3.25 Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$.

$$F (f * \phi) = (2\pi)^{n/2} F\phi Ff,$$  \hspace{1cm} (29.3.16)

$$F^{-1} (f * \phi) = (2\pi)^{n/2} F^{-1} \phi F^{-1} f.$$  \hspace{1cm} (29.3.17)

Proof: Note that (29.3.16) follows from Definition 29.3.24 and both assertions hold for $f \in \mathcal{G}$. Consider (29.3.17). Here is a simple formula involving a pair of functions in $\mathcal{G}$.

$$(\psi * F^{-1} F^{-1} \phi) (x)$$

$$= \left( \int \int \int \psi (x - y) e^{iy \cdot y_1} e^{iy_1 \cdot z} \phi (z) \, dz \, dy_1 \, dy \right) (2\pi)^n$$

$$= \left( \int \int \int \psi (x - y) e^{-iy \cdot y} e^{-iy_1 \cdot z} \phi (z) \, dz \, dy_1 \, dy \right) (2\pi)^n$$

$$= (\psi * FF\phi) (x).$$

Now for $\psi \in \mathcal{G}$,

$$(2\pi)^{n/2} F (F^{-1} F^{-1} \phi) (\psi) \equiv (2\pi)^{n/2} (F^{-1} \phi F^{-1} f) (F\psi) \equiv$$

$$(2\pi)^{n/2} F^{-1} f (F^{-1} \phi F\psi) \equiv (2\pi)^{n/2} f (F^{-1} \phi F\psi) \equiv$$

$$f \left( (2\pi)^{n/2} F^{-1} \left( (FF^{-1} F^{-1} \phi) (F\psi) \right) \right) \equiv$$

$$f (\psi * F^{-1} F^{-1} \phi) = f (\psi * FF\phi)$$  \hspace{1cm} (29.3.18)
29.4. Exercises

Also
\[ (2\pi)^{n/2} F^{-1} (F\phi Ff) (\psi) \equiv (2\pi)^{n/2} (F\phi Ff) (F^{-1}\psi) \equiv \]
\[ (2\pi)^{n/2} F \circ (F\phi F^{-1}) \equiv (2\pi)^{n/2} f \circ (F\phi F^{-1}) \]
\[ = f \left( F \left( (2\pi)^{n/2} (F\phi F^{-1}) \right) \right) \]
\[ = f \left( F \left( (2\pi)^{n/2} FF\phi F^{-1}\psi \right) \right) = f \left( F \left( (FF\phi) \right) \right) \]
\[ = f (FF\phi \ast \psi) = f (\psi \ast FF\phi). \quad (29.3.19) \]

The last line follows from the following.
\[
\int F\phi (x - y) \psi (y) dy = \int F\phi (x - y) F\psi (y) dy
\]
\[ = \int F\psi (x - y) F\phi (y) dy
\]
\[ = \int \psi (x - y) FF\phi (y) dy. \]

From 29.3.18 and 29.3.19, since \( \psi \) was arbitrary,
\[ (2\pi)^{n/2} F (F^{-1} \phi F^{-1} f) = (2\pi)^{n/2} F^{-1} (F\phi Ff) \equiv f \ast \phi \]
which shows 29.3.17.

29.4 Exercises

1. For \( f \in L^1 (\mathbb{R}^n) \), show that if \( F^{-1} f \in L^1 \) or \( Ff \in L^1 \), then \( f \) equals a continuous bounded function a.e.

2. Suppose \( f, g \in L^1 (\mathbb{R}) \) and \( Ff = Fg \). Show \( f = g \) a.e.

3. Show that if \( f \in L^1 (\mathbb{R}^n) \), then \( \lim_{|x| \to \infty} Ff (x) = 0 \).

4. Suppose \( f \ast f = f \) or \( f \ast f = 0 \) and \( f \in L^1 (\mathbb{R}) \). Show \( f = 0 \).

5. For this problem define \( \int_a^\infty f (t) \, dt \equiv \lim_{r \to \infty} \int_a^r f (t) \, dt \). Note this coincides with the Lebesgue integral when \( f \in L^1 (a, \infty) \). Show

(a) \( \int_0^\infty \frac{\sin (u)}{u} \, du = \frac{\pi}{2} \)

(b) \( \lim_{r \to \infty} \int_0^r \frac{\sin (u)}{u} \, du = 0 \) whenever \( \delta > 0 \).

(c) If \( f \in L^1 (\mathbb{R}) \), then \( \lim_{r \to \infty} \int_\mathbb{R} \sin (ru) f (u) \, du = 0 \).

Hint: For the first two, use \( \frac{1}{u} = \int_0^\infty e^{-ut} \, dt \) and apply Fubini’s theorem to \( \int_0^\infty \sin u \int_\mathbb{R} e^{-ut} \, dt \, du \). For the last part, first establish it for \( f \in C_c^\infty (\mathbb{R}) \) and then use the density of this set in \( L^1 (\mathbb{R}) \) to obtain the result. This is sometimes called the Riemann Lebesgue lemma.
6. Suppose that \( g \in L^1(\mathbb{R}) \) and that at some \( x > 0 \), \( g \) is locally Holder continuous from the right and from the left. This means
\[
\lim_{r \to 0^+} g(x + r) \equiv g(x+)
\]
exists, and there exist constants \( K, \delta > 0 \) and \( r \in (0, 1] \) such that for \( |x - y| < \delta \),
\[
|g(x+) - g(y)| < K |x - y|^r
\]
for \( y > x \) and
\[
|g(x-) - g(y)| < K |x - y|^r
\]
for \( y < x \). Show that under these conditions,
\[
\lim_{R \to \infty} \frac{2}{\pi} \int_0^\infty \sin(ur) \frac{u}{2} \left( \frac{g(x - u) + g(x + u)}{2} \right) du = \frac{g(x+) + g(x-)}{2}.
\]

7. Let \( g \in L^1(\mathbb{R}) \) and suppose \( g \) is locally Holder continuous from the right and from the left at \( x \). Show that then
\[
\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^R e^{\gamma it} \int_{-\infty}^\infty e^{-iuy} g(y) dy dt = \frac{g(x+) + g(x-)}{2}.
\]
This is very interesting. If \( g \in L^2(\mathbb{R}) \), this shows \( F^{-1}(Fg)(x) = \frac{g(x+) + g(x-)}{2} \), the midpoint of the jump in \( g \) at the point, \( x \). In particular, if \( g \in \mathcal{G} \), \( F^{-1}(Fg) = g \). Hint: Show the left side of the above equation reduces to
\[
\frac{2}{\pi} \int_0^\infty \sin(ur) \frac{u}{2} \left( \frac{g(x - u) + g(x + u)}{2} \right) du
\]
and then use Problem 6 to obtain the result.

8. A measurable function \( g \) defined on \((0, \infty)\) has exponential growth if \( |g(t)| \leq Ce^{\eta t} \) for some \( \eta \). For \( \operatorname{Re}(s) > \eta \), define the Laplace Transform by
\[
Lg(s) \equiv \int_0^\infty e^{-su} g(u) du.
\]
Assume that \( g \) has exponential growth as above and is Holder continuous from the right and from the left at \( t \). Pick \( \gamma > \eta \). Show that
\[
\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^R e^{\gamma it} e^{iyt} Lg(\gamma + iy) dy = \frac{g(t+) + g(t-)}{2}.
\]
This formula is sometimes written in the form
\[ \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} Lg(s) \, ds \]
and is called the complex inversion integral for Laplace transforms. It can be used to find inverse Laplace transforms. **Hint:**
\[ \frac{1}{2\pi} \int_{-R}^{R} e^{\gamma t} e^{iyt} Lg(\gamma + iy) \, dy = \]
\[ \frac{1}{2\pi} \int_{-R}^{R} e^{\gamma t} e^{iyt} \int_{0}^{\infty} e^{-(\gamma + iy)u} g(u) \, du \, dy. \]
Now use Fubini’s theorem and do the integral from \(-R\) to \(R\) to get this equal to
\[ \frac{e^{\gamma t}}{\pi} \int_{-\infty}^{\infty} e^{-\gamma u} \overline{g}(u) \frac{\sin (R(t-u))}{t-u} \, du \]
where \(\overline{g}\) is the zero extension of \(g\) off \([0, \infty)\). Then this equals
\[ \frac{e^{\gamma t}}{\pi} \int_{-\infty}^{\infty} e^{-\gamma (t-u)} g(t-u) \frac{\sin (Ru)}{u} du \]
which equals
\[ \frac{2e^{\gamma t}}{\pi} \int_{0}^{\infty} \overline{g}(t-u) e^{-\gamma (t-u)} + \overline{g}(t+u) e^{-\gamma (t+u)} \frac{\sin (Ru)}{u} \, du \]
and then apply the result of Problem 6.

9. Suppose \(f \in \mathcal{S}\). Show \(F(f_{x_j})(t) = it_j Ff(t)\).

10. Let \(f \in \mathcal{S}\) and let \(k\) be a positive integer.
\[ \|f\|_{k,2} \equiv (\|f\|_{2}^2 + \sum_{|\alpha| \leq k} \|D^\alpha f\|_{2}^2)^{1/2} \]
One could also define
\[ \|\|f\|\|_{k,2} \equiv \left( \int_{\mathbb{R}^n} |Ff(x)|^2 (1 + |x|^2)^k \, dx \right)^{1/2} \]
Show both \(\|\|_{k,2}\) and \(\|\| \|_{k,2}\) are norms on \(\mathcal{S}\) and that they are equivalent. These are Sobolev space norms. For which values of \(k\) does the second norm make sense? How about the first norm?

11. Define \(H^k(\mathbb{R}^n), k \geq 0\) by \(f \in L^2(\mathbb{R}^n)\) such that
\[ \left( \int |Ff(x)|^2 (1 + |x|^2)^k \, dx \right)^{1/2} < \infty, \]
CHAPTER 29. FOURIER TRANSFORMS

\[ |||f|||_{k,2} \equiv (\int |Ff(x)|^2 (1 + |x|^2)^k dx)^{\frac{1}{2}}. \]

Show \( H^k(\mathbb{R}^n) \) is a Banach space, and that if \( k \) is a positive integer, \( H^k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : \text{there exists } \{u_j\} \subseteq \mathcal{G} \text{ with } ||u_j - f||_2 \to 0 \text{ and } \{u_j\} \text{ is a Cauchy sequence in } ||||_{k,2} \text{ of Problem 11} \} \). This is one way to define Sobolev Spaces. **Hint:** One way to do the second part of this is to define a new measure, \( \mu \) by

\[ \mu(E) \equiv \int_E \left(1 + |x|^2\right)^k dx. \]

Then show \( \mu \) is a Radon measure and show there exists \( \{g_m\} \) such that \( g_m \in \mathcal{G} \) and \( g_m \to Ff \) in \( L^2(\mu) \). Thus \( g_m = Ff_m, f_m \in \mathcal{G} \) because \( F \) maps \( \mathcal{G} \) onto \( \mathcal{G} \). Then by Problem 11, \( \{f_m\} \) is Cauchy in the norm \( ||||_{k,2} \).

12. If \( 2k > n \), show that if \( f \in H^k(\mathbb{R}^n) \), then \( f \) equals a bounded continuous function a.e. **Hint:** Show that for \( k \) this large, \( Ff \in L^1(\mathbb{R}^n) \), and then use Problem 1. To do this, write

\[ |Ff(x)| = |Ff(x)|(1 + |x|^2)^{\frac{k}{2}}(1 + |x|^2)^{-\frac{k}{2}}, \]

So

\[ \int |Ff(x)|dx = \int |Ff(x)|(1 + |x|^2)^{\frac{k}{2}}(1 + |x|^2)^{-\frac{k}{2}} dx. \]

Use the Cauchy Schwarz inequality. This is an example of a Sobolev imbedding Theorem.

13. Let \( u \in \mathcal{G} \). Then \( Fu \in \mathcal{G} \) and so, in particular, it makes sense to form the integral,

\[ \int_{\mathbb{R}} Fu(x', x_n) dx_n \]

where \( (x', x_n) = x \in \mathbb{R}^n \). For \( u \in \mathcal{G} \), define \( \gamma u(x') \equiv u(x', 0) \). Find a constant such that \( F(\gamma u)(x') \) equals this constant times the above integral. **Hint:** By the dominated convergence theorem

\[ \int_{\mathbb{R}} Fu(x', x_n) dx_n = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-\varepsilon x_n^2} Fu(x', x_n) dx_n. \]

Now use the definition of the Fourier transform and Fubini’s theorem as required in order to obtain the desired relationship.

14. Recall the Fourier series of a function in \( L^2(-\pi, \pi) \) converges to the function in \( L^2(-\pi, \pi) \). Prove a similar theorem with \( L^2(-\pi, \pi) \) replaced by \( L^2(-m\pi, m\pi) \) and the functions

\[ \{(2\pi)^{-1/2} e^{inx}\}_{n \in \mathbb{Z}} \]
used in the Fourier series replaced with

\[ \left\{ (2m\pi)^{-1/2} e^{i\frac{\pi}{2} x} \right\}_{n \in \mathbb{Z}} \]

Now suppose \( f \) is a function in \( L^2(\mathbb{R}) \) satisfying \( Ff(t) = 0 \) if \( |t| > m\pi \). Show that if this is so, then

\[ f(x) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} f\left( -\frac{n}{m} \right) \sin \left( \pi \left( mx + n \right) \right) \]

Here \( m \) is a positive integer. This is sometimes called the Shannon sampling theorem. **Hint:** First note that since \( Ff \in L^2 \) and is zero off a finite interval, it follows \( Ff \in L^1 \). Also

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-m\pi}^{m\pi} e^{itx} Ff(x) \, dx \]

and you can conclude from this that \( f \) has all derivatives and they are all bounded. Thus \( f \) is a very nice function. You can replace \( Ff \) with its Fourier series. Then consider carefully the Fourier coefficient of \( Ff \). Argue it equals \( f\left( -\frac{n}{m} \right) \) or at least an appropriate constant times this. When you get this the rest will fall quickly into place if you use \( Ff \) is zero off \([-m\pi, m\pi] \).
Chapter 30

Fourier Analysis In $\mathbb{R}^n$ An Introduction

The purpose of this chapter is to present some of the most important theorems on Fourier analysis in $\mathbb{R}^n$. These theorems are the Marcinkiewicz interpolation theorem, the Calderon Zygmund decomposition, and Mihlin’s theorem. They are all fundamental results whose proofs depend on the methods of real analysis.

30.1 The Marcinkiewicz Interpolation Theorem

Let $(\Omega, \mu, \mathcal{S})$ be a measure space.

**Definition 30.1.1** $L^p(\Omega) + L^1(\Omega)$ will denote the space of measurable functions, $f$, such that $f$ is the sum of a function in $L^p(\Omega)$ and $L^1(\Omega)$. Also, if $T : L^p(\Omega) + L^1(\Omega) \to$ space of measurable functions, $T$ is subadditive if

$$|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|.$$

$T$ is of type $(p, p)$ if there exists a constant independent of $f \in L^p(\Omega)$ such that

$$||Tf||_p \leq A \|f\|_p, \ f \in L^p(\Omega).$$

$T$ is weak type $(p, p)$ if there exists a constant $A$ independent of $f$ such that

$$\mu\{|x : |Tf(x)| > \alpha\} \leq \left(\frac{A}{\alpha} \|f\|_p\right)^p, \ f \in L^p(\Omega).$$

The following lemma involves writing a function as a sum of a functions whose values are small and one whose values are large.

**Lemma 30.1.2** If $p \in [1, r]$, then $L^p(\Omega) \subseteq L^1(\Omega) + L^r(\Omega)$. 

1109
**Proof**: Let $\lambda > 0$ and let $f \in L^p(\Omega)$

$$f_1(x) \equiv \begin{cases} f(x) & \text{if } |f(x)| \leq \lambda \\
0 & \text{if } |f(x)| > \lambda \end{cases}, \quad f_2(x) \equiv \begin{cases} f(x) & \text{if } |f(x)| > \lambda \\
0 & \text{if } |f(x)| \leq \lambda \end{cases}.$$

Thus $f(x) = f_1(x) + f_2(x)$.

$$\int |f_1(x)|^r \, d\mu = \int_{|f| \leq \lambda} |f(x)|^r \, d\mu \leq \lambda^{r-p} \int |f(x)|^p \, d\mu < \infty.$$ 

Therefore, $f_1 \in L^r(\Omega)$.

$$\int |f_2(x)| \, d\mu = \int_{|f| > \lambda} |f(x)| \, d\mu \leq \mu(|f| > \lambda)^{1/p} \left( \int |f|^p \, d\mu \right)^{1/p} < \infty.$$ 

This proves the lemma since $f = f_1 + f_2$, $f_1 \in L^r$ and $f_2 \in L^1$.

For $f$ a function having nonnegative real values, $\alpha \to \mu([f > \alpha])$ is called the distribution function.

**Lemma 30.1.3** Let $\phi(0) = 0$, $\phi$ is strictly increasing, and $C^1$. Let $f : \Omega \to [0, \infty)$ be measurable. Then

$$\int_{\Omega} (\phi \circ f) \, d\mu = \int_{0}^{\infty} \phi'(\alpha) \mu([f > \alpha]) \, d\alpha. \quad (30.1.1)$$

**Proof**: First suppose

$$f = \sum_{i=1}^{m} a_i X_{E_i}$$

where $a_i > 0$ and the $a_i$ are all distinct nonzero values of $f$, the sets, $E_i$ being disjoint. Thus,

$$\int_{\Omega} (\phi \circ f) \, d\mu = \sum_{i=1}^{m} \phi(a_i) \mu(E_i).$$

Suppose without loss of generality $a_1 < a_2 < \cdots < a_m$. Observe

$$\alpha \to \mu([f > \alpha])$$

is constant on the intervals $[0, a_1), [a_1, a_2), \cdots$. For example, on $[a_i, a_{i+1})$, this function has the value

$$\sum_{j=i+1}^{m} \mu(E_j).$$

The function equals zero on $[a_m, \infty)$. Therefore,

$$\alpha \to \phi'(\alpha) \mu([f > \alpha])$$
is Lebesgue measurable and letting $a_0 = 0$, the second integral in (30.1.1) equals

$$
\int_0^\infty \phi' (\alpha) \mu ([f > \alpha]) \, d\alpha = \sum_{i=1}^m \int_{a_{i-1}}^{a_i} \phi' (\alpha) \mu ([f > \alpha]) \, d\alpha
$$

$$
= \sum_{i=1}^m \sum_{j=1}^m \mu (E_j) \int_{a_{i-1}}^{a_i} \phi' (\alpha) \, d\alpha
$$

$$
= \sum_{j=1}^m \sum_{i=1}^m \mu (E_j) (\phi (a_i) - \phi (a_{i-1}))
$$

$$
= \sum_{j=1}^m \mu (E_j) \phi (a_j) = \int_\Omega (\phi \circ f) \, d\mu
$$

and so this establishes (30.1.1) in the case when $f$ is a nonnegative simple function. Since every measurable nonnegative function may be written as the pointwise limit of such simple functions, the desired result will follow by the Monotone convergence theorem and the next claim.

**Claim:** If $f_n \uparrow f$, then for each $\alpha > 0$,

$$
\mu ([f > \alpha]) = \lim_{n \to \infty} \mu ([f_n > \alpha]).
$$

**Proof of the claim:** $[f_n > \alpha] \uparrow [f > \alpha]$ because if $f (x) > \alpha$ then for large enough $n$, $f_n (x) > \alpha$ and so

$$
\mu ([f_n > \alpha]) \uparrow \mu ([f > \alpha]).
$$

This proves the lemma. (Note the importance of the strict inequality in $[f > \alpha]$ in proving the claim.)

The next theorem is the main result in this section. It is called the Marcinkiewicz interpolation theorem.

**Theorem 30.1.4** Let $(\Omega, \mu, S)$ be a $\sigma$ finite measure space, $1 < r < \infty$, and let

$$
T : L^1 (\Omega) + L^r (\Omega) \to \text{space of measurable functions}
$$

be subadditive, weak $(r, r)$, and weak $(1, 1)$. Then $T$ is of type $(p, p)$ for every $p \in (1, r)$ and

$$
\|Tf\|_p \leq A_p \|f\|_p
$$

where the constant $A_p$ depends only on $p$ and the constants in the definition of weak $(1, 1)$ and weak $(r, r)$.

**Proof:** Let $\alpha > 0$ and let $f_1$ and $f_2$ be defined as in Lemma (30.1.2).

$$
 f_1 (x) = \begin{cases} f (x) & \text{if } |f (x)| \leq \alpha \\ 0 & \text{if } |f (x)| > \alpha \end{cases}, \quad f_2 (x) = \begin{cases} f (x) & \text{if } |f (x)| > \alpha \\ 0 & \text{if } |f (x)| \leq \alpha \end{cases}
$$
Thus $f = f_1 + f_2$ where $f_1 \in L^r$ and $f_2 \in L^1$. Since $T$ is subadditive,

$$||Tf|| > \alpha \subseteq ||Tf_1|| > \alpha/2 \cup ||Tf_2|| > \alpha/2.$$ 

Let $p \in (1, r)$. By Lemma 30.1.3

$$\int |Tf|^p \, d\mu \leq p \int_0^\infty \alpha^{p-1} \mu(|Tf_1| > \alpha/2) \, d\alpha + p \int_0^\infty \alpha^{p-1} \mu(|Tf_2| > \alpha/2) \, d\alpha.$$ 

Therefore, since $T$ is weak $(1, 1)$ and weak $(r, r)$,

$$\int |Tf|^p \, d\mu \leq p \int_0^\infty \alpha^{p-1} \left(\frac{2A_r}{\alpha} ||f_1||_r\right)^r \, d\alpha + p \int_0^\infty \alpha^{p-1} \frac{2A_1}{\alpha} ||f_2||_1 \, d\alpha. \quad (30.1.2)$$

Therefore, the right side of (30.1.2) equals

$$p (2A_r)^r \int_0^\infty \alpha^{p-1-r} \int_{\Omega} |f_1|^r \, d\alpha \, d\mu + 2A_1p \int_0^\infty \alpha^{p-2} \int_{\Omega} |f_2|^r \, d\alpha \, d\mu =$$

$$p (2A_r)^r \int_0^\infty \alpha^{p-1-r} \int_{\Omega} |f_1|^r \, d\alpha \, d\mu + 2A_1p \int_0^\infty \alpha^{p-2} \int_{\Omega} |f_2|^r \, d\alpha \, d\mu.$$ 

Now $f_1 (x) = 0$ unless $|f_1 (x)| \leq \alpha$ and $f_2 (x) = 0$ unless $|f_2 (x)| > \alpha$ so this equals

$$p (2A_r)^r \int_{|f(x)|}^\infty |f(x)|^r \int_{|f(x)|}^\infty \alpha^{p-1-r} \, d\alpha \, d\mu + 2A_1p \int_{|f(x)|}^\infty |f(x)|^r \int_{|f(x)|}^\infty \alpha^{p-2} \, d\alpha \, d\mu$$

which equals

$$\frac{2r A_r^p}{r - p} \int_{\Omega} |f(x)|^p \, d\mu + \frac{2pA_1}{p - 1} \int_{\Omega} |f(x)|^p \, d\mu$$

$$\leq \max \left(\frac{2r A_r^p}{r - p}, \frac{2pA_1}{p - 1}\right) ||f||_{L^p(\Omega)}^p$$

and this proves the theorem.

### 30.2 The Calderon Zygmund Decomposition

For a given nonnegative integrable function, $\mathbb{R}^n$ can be decomposed into a set where the function is small and a set which is the union of disjoint cubes on which the average of the function is under some control. The measure in this section will always be Lebesgue measure on $\mathbb{R}^n$. This theorem depends on the Lebesgue theory of differentiation.
Theorem 30.2.1 Let \( f \geq 0, \int f dx < \infty \), and let \( \alpha \) be a positive constant. Then there exist sets \( F \) and \( \Omega \) such that
\[
\mathbb{R}^n = F \cup \Omega, \ F \cap \Omega = \emptyset
\]  
(30.2.3)
\[
f(x) \leq \alpha \text{ a.e. on } F
\]  
(30.2.4)
\[
\Omega = \bigcup_{k=1}^{\infty} Q_k \text{ where the interiors of the cubes are disjoint and for each cube, } Q_k,
\]  
\[
\alpha < \frac{1}{m(Q_k)} \int_{Q_k} f(x) \, dx \leq 2^n \alpha.
\]  
(30.2.5)

Proof: Let \( S_0 \) be a tiling of \( \mathbb{R}^n \) into cubes having sides of length \( M \) where \( M \) is chosen large enough that if \( Q \) is one of these cubes, then
\[
\frac{1}{m(Q)} \int_{Q} f \, dm \leq \alpha.
\]  
(30.2.6)
Suppose \( S_0, \ldots, S_m \) have been chosen. To get \( S_{m+1} \), replace each cube of \( S_m \) by the \( 2^n \) cubes obtained by bisecting the sides. Then \( S_{m+1} \) consists of exactly those cubes of \( S_m \) for which (30.2.6) holds and let \( T_{m+1} \) consist of the bisected cubes from \( S_m \) for which (30.2.6) does not hold. Now define
\[
F \equiv \{ x : x \text{ is contained in some cube from } S_m \text{ for all } m \},
\]
\[
\Omega \equiv \mathbb{R}^n \setminus F = \bigcup_{m=1}^{\infty} \bigcup \{ Q : Q \in T_m \}
\]
Note that the cubes from \( T_m \) have pair wise disjoint interiors and also the interiors of cubes from \( T_m \) have empty intersections with the interiors of cubes of \( T_k \) if \( k \neq m \).
Let \( x \) be a point of \( \Omega \) and let \( x \) be in a cube of \( T_m \) such that \( m \) is the first index for which this happens. Let \( Q \) be the cube in \( S_{m-1} \) containing \( x \) and let \( Q^* \) be the cube in the bisection of \( Q \) which contains \( x \). Therefore (30.2.6) does not hold for \( Q^* \). Thus
\[
\alpha < \frac{1}{m(Q^*)} \int_{Q^*} f \, dx \leq \frac{m(Q)}{m(Q^*)} \frac{1}{m(Q)} \int_{Q} f \, dx \leq 2^n \alpha
\]
which shows \( \Omega \) is the union of cubes having disjoint interiors for which (30.2.6) holds.
Now a.e. point of \( F \) is a Lebesgue point of \( f \). Let \( x \) be such a point of \( F \) and suppose \( x \in Q_k \) for \( Q_k \in S_k \). Let \( d_k \equiv \text{diameter of } Q_k \). Thus \( d_k \to 0 \).
\[
\frac{1}{m(Q_k)} \int_{Q_k} |f(y) - f(x)| \, dy \leq \frac{1}{m(Q_k)} \int_{B(x,d_k)} |f(y) - f(x)| \, dy
\]
\[
= \frac{m(B(x,d_k))}{m(Q_k)} \frac{1}{m(B(x,d_k))} \int_{B(x,d_k)} |f(y) - f(x)| \, dy
\]
\[
\leq K_n \frac{1}{m(B(x,d_k))} \int_{B(x,d_k)} |f(x) - f(y)| \, dy
\]
where $K_n$ is a constant which depends on $n$ and measures the ratio of the volume of a ball with diameter $2d$ and a cube with diameter $d$. The last expression converges to 0 because $x$ is a Lebesgue point. Hence

$$f(x) = \lim_{k \to \infty} \frac{1}{m(Q_k)} \int_{Q_k} f(y) \, dy \leq \alpha$$

and this shows $f(x) \leq \alpha$ a.e. on $F$. This proves the theorem.

### 30.3 Mihlin’s Theorem

In this section, the Marcinkiewicz interpolation theorem and Calderon Zygmund decomposition will be used to establish a remarkable theorem of Mihlin, a generalization of Plancherel’s theorem to the $L^p$ spaces. It is of fundamental importance in the study of elliptic partial differential equations and can also be used to give proofs for the theory of singular integrals. Mihlin’s theorem involves a conclusion which is of the form

$$F^{-1} \rho * \phi(x) \leq A \|\phi\|_p$$

for $p > 1$ and $\phi \in \mathcal{G}$. Thus $F^{-1} \rho *$ extends to a continuous linear map defined on $L^p$ because of the density of $\mathcal{G}$. It is proved by showing various weak type estimates and then applying the Marcinkiewicz Interpolation Theorem to get an estimate like the above.

Recall that by Corollary 29.3.19, if $f \in L^2(\mathbb{R}^n)$ and if $\phi \in \mathcal{G}$, then $f * \phi \in L^2(\mathbb{R}^n)$ and

$$F(f * \phi)(x) = (2\pi)^{n/2} F\phi(x) Ff(x).$$

The next lemma is essentially a weak $(1,1)$ estimate. The inequality is established under the condition and then it is shown there exist conditions which are easier to verify which imply condition. I think the approach used here is due to Hormander and is found in Berg and Lofstrom. For many more references and generalizations, you might look in Triebel. Functions, $\rho$ which yield an inequality of the sort in are called $L^p$ multipliers.

**Lemma 30.3.1** Suppose $\rho \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and suppose also there exists a constant $C_1$ such that

$$\int_{|x| \geq 2|y|} |F^{-1} \rho(x - y) - F^{-1} \rho(x)| \, dx \leq C_1. \quad (30.3.8)$$

Then there exists a constant $A$ depending only on $C_1, \|\rho\|_{\infty}$, and $n$ such that

$$m\left(\{x : |F^{-1} \rho * \phi(x)| > \alpha\}\right) \leq \frac{A}{\alpha \|\phi\|_1}$$

for all $\phi \in \mathcal{G}$. 
30.3. MIHLIN’S THEOREM

Proof: Let \( \phi \in \mathcal{G} \) and use the Calderon decomposition to write \( \mathbb{R}^n = E \cup \Omega \) where \( \Omega \) is a union of cubes, \( \{Q_i\} \) with disjoint interiors such that

\[
\alpha m(Q_i) \leq \int_{Q_i} |\phi(x)| \, dx \leq 2^n \alpha m(Q_i), \quad |\phi(x)| \leq \alpha \text{ a.e. on } E. \quad (30.3.9)
\]

The proof is accomplished by writing \( \phi \) as the sum of a good function and a bad function and establishing a similar weak inequality for these two functions separately. Then this information is used to obtain the desired conclusion.

\[
g(x) = \begin{cases} 
\phi(x) & \text{if } x \in E \\
\frac{1}{m(Q_i)} \int_{Q_i} \phi(y) \, dy & \text{if } x \in Q_i \subseteq \Omega \end{cases}, \quad g(x) + b(x) = \phi(x). \quad (30.3.10)
\]

Thus

\[
\int_{Q_i} b(x) \, dx = \int_{Q_i} (\phi(x) - g(x)) \, dx = \int_{Q_i} \phi(x) \, dx - \int_{Q_i} \phi(x) \, dx = 0 \quad (30.3.11)
\]

Claim:

\[
\|g\|_2 \leq \alpha (1 + 4^n) \|\phi\|_1, \quad \|g\|_1 \leq \|\phi\|_1. \quad (30.3.13)
\]

Proof of claim:

\[
\|g\|_2^2 = \|g\|_{L^2(E)}^2 + \|g\|_{L^2(\Omega)}^2.
\]

Thus

\[
\|g\|_{L^2(\Omega)}^2 = \sum_i \int_{Q_i} |g(x)|^2 \, dx \\
\leq \sum_i \int_{Q_i} \left( \frac{1}{m(Q_i)} \int_{Q_i} |\phi(y)| \, dy \right)^2 \, dx \\
\leq \sum_i \int_{Q_i} (2^n \alpha)^2 \, dx \leq 4^n \alpha^2 \sum_i m(Q_i) \\
\leq 4^n \alpha^2 \sum_i \int_{Q_i} |\phi(x)| \, dx \leq 4^n \alpha \|\phi\|_1.
\]

\[
\|g\|_{L^2(E)}^2 = \int_E |\phi(x)|^2 \, dx \leq \alpha \int_E |\phi(x)| \, dx = \alpha \|\phi\|_1.
\]

Now consider the second of the inequalities in (30.3.13).

\[
\|g\|_1 = \int_E |g(x)| \, dx + \int_\Omega |g(x)| \, dx \\
= \int_E |\phi(x)| \, dx + \sum_i \int_{Q_i} |g| \, dx \\
\leq \int_E |\phi(x)| \, dx + \sum_i \int_{Q_i} \frac{1}{m(Q_i)} \int_{Q_i} |\phi(x)| \, dm(x) \, dm \\
= \int_E |\phi(x)| \, dx + \sum_i \int_{Q_i} |\phi(x)| \, dm(x) = \|\phi\|_1.
\]
This proves the claim. From the claim, it follows that $b \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Because of Corollary 30.3.13, $g \in L^1(\mathbb{R}^n)$ and so $F^{-1}\rho * g \in L^2(\mathbb{R}^n)$. (Since $\rho \in L^2$, it follows $F^{-1}\rho \in L^2$ and so this convolution is indeed in $L^2$.) By Plancherel’s theorem,

$$||F^{-1}\rho * g||_2 = ||F(F^{-1}\rho * g)||_2.$$  

By Corollary 29.3.19 on Page 1098, the expression on the right equals

$$(2\pi)^{n/2} ||\rho Fg||_2$$  

and so

$$||F^{-1}\rho * g||_2 = (2\pi)^{n/2} ||\rho Fg||_2 \leq C_n ||\rho||_{\infty} ||g||_2.$$  

From this and Corollary 30.3.11,

$$m \left( \{ F^{-1}\rho * g \geq \alpha/2 \} \right) \leq C_n ||\rho||_{\infty}^2 \alpha (1 + 4^n) \||\phi||_1 = C_n \alpha^{-1} ||\phi||_1. \quad (30.3.14)$$  

This is what is wanted so far as $g$ is concerned. Next it is required to estimate $m \left( \{ F^{-1}\rho * b \geq \alpha/2 \} \right)$.

If $Q$ is one of the cubes whose union is $\Omega$, let $Q^*$ be the cube with the same center as $Q$ but whose sides are $2\sqrt{n}$ times as long.

Let

$$\Omega^* = \cup_{i=1}^{\infty} Q_i^*$$

and let

$$E^* = \mathbb{R}^n \setminus \Omega^*.$$  

Thus $E^* \subseteq E$. Let $x \in E^*$. Then because of Corollary 30.3.11,

$$\int_{Q_i} F^{-1}\rho (x-y) b(y) \, dy = \int_{Q_i} \left[ F^{-1}\rho (x-y) - F^{-1}\rho (x-y_i) \right] b(y) \, dy, \quad (30.3.15)$$  

where $y_i$ is the center of $Q_i$. Consequently if the sides of $Q_i$ have length $2t/\sqrt{n}$,

$$\int_{E^*} \left| \int_{Q_i} F^{-1}\rho (x-y) b(y) \, dy \right| \, dx \leq \quad (30.3.16)$$  

30.3. MIHLIN’S THEOREM

\[
\int_{E^*} \int_{Q_i} |F^{-1} \rho (x - y) - F^{-1} \rho (x - y) - F^{-1} \rho (x - y)| |b(y)| \, dy \, dx
\]

\[
= \int_{Q_i} \int_{E^*} |F^{-1} \rho (x - y) - F^{-1} \rho (x - y) - F^{-1} \rho (x - y) - F^{-1} \rho (x - y)| \, dx \, |b(y)| \, dy \quad (30.3.17)
\]

\[
\leq \int_{Q_i} \int_{|x - y| \geq 2t} |F^{-1} \rho (x - y) - F^{-1} \rho (x - y)| \, dx \, |b(y)| \, dy \quad (30.3.18)
\]

since if \( x \in E^* \), then \(|x - y| \geq 2t\). Now for \( y \in Q_i \),

\[
|y - y_i| \leq \left( \sum_{j=1}^{n} \left( \frac{t}{\sqrt{n}} \right)^2 \right)^{1/2} = t.
\]

From 30.3.8 and the change of variables \( u = x - y_i \) imply

\[
\int_{E^*} \int_{Q_i} F^{-1} \rho (x - y) b(y) \, dy \, dx \leq C_1 \int_{Q_i} |b(y)| \, dy. \quad (30.3.19)
\]

Now from 30.3.19 and the fact that \( b = 0 \) off \( \Omega \),

\[
\int_{E^*} |F^{-1} \rho \ast b(x)| \, dx = \int_{E^*} \left| \int_{\mathbb{R}^n} F^{-1} \rho (x - y) b(y) \, dy \right| \, dx
\]

\[
= \int_{E^*} \sum_{i=1}^{\infty} \int_{Q_i} F^{-1} \rho (x - y) b(y) \, dy \, dx
\]

\[
\leq \int_{E^*} \sum_{i=1}^{\infty} \left| \int_{Q_i} F^{-1} \rho (x - y) b(y) \, dy \right| \, dx
\]

\[
= \sum_{i=1}^{\infty} \int_{E^*} \int_{Q_i} F^{-1} \rho (x - y) b(y) \, dy \, dx
\]

\[
\leq \sum_{i=1}^{\infty} C_1 \int_{Q_i} |b(y)| \, dy = C_1 ||b||_1.
\]

Thus, by 30.3.13,

\[
\int_{E^*} |F^{-1} \rho \ast b(x)| \, dx \leq C_1 ||b||_1
\]

\[
\leq C_1 [||\phi||_1 + ||g||_1]
\]

\[
\leq C_1 [||\phi||_1 + ||\phi||_1]
\]

\[
\leq 2C_1 ||\phi||_1.
\]

Consequently,

\[
m \left( \left[ |F^{-1} \rho \ast b| \geq \frac{\alpha}{2} \right] \cap E^* \right) \leq \frac{4C_1}{\alpha} ||\phi||_1.
\]
\[ m \left[ \left| F^{-1} \rho * \phi \right| > \alpha \right] \leq m \left[ \left| F^{-1} \rho * g \right| \geq \frac{\alpha}{2} \right] + m \left[ \left| F^{-1} \rho * b \right| \geq \frac{\alpha}{2} \right] \]
\[ \leq C_n \frac{||\phi||_1}{\alpha} + \frac{m \left( \left[ \left| F^{-1} \rho * b \right| \geq \frac{\alpha}{2} \right] \cap E^* \right)}{\alpha} + m (\Omega^*) \]
\[ \leq C_n \frac{||\phi||_1}{\alpha} + \frac{4C_1}{\alpha} ||\phi||_1 + C_n m (\Omega) \leq \frac{A}{\alpha} ||\phi||_1 \]

because
\[ m (\Omega) \leq \alpha^{-1} ||\phi||_1 \]

by \[ 30.3.9 \]. This proves the lemma.

The next lemma extends this lemma by giving a weak (2, 2) estimate and a (2, 2) estimate.

**Lemma 30.3.2** Suppose \( \rho \in L^\infty (\mathbb{R}^n) \cap L^2 (\mathbb{R}^n) \) and suppose also that there exists a constant \( C_1 \) such that
\[ \int_{|x|>2|y|} \left| F^{-1} \rho (x-y) - F^{-1} \rho (x) \right| dx \leq C_1. \] (30.3.20)

Then \( F^{-1} \rho * \) maps \( L^1 (\mathbb{R}^n) + L^2 (\mathbb{R}^n) \) to measurable functions and there exists a constant \( A \) depending only on \( C_1, n, ||\rho||_{\infty} \) such that
\[ m \left( \left| F^{-1} \rho * f \right| > \alpha \right) \leq A \frac{||f||_1}{\alpha} \text{ if } f \in L^1 (\mathbb{R}^n), \] (30.3.21)
\[ m \left( \left| F^{-1} \rho * f \right| > \alpha \right) \leq \left( A \frac{||f||_2}{\alpha} \right)^2 \text{ if } f \in L^2 (\mathbb{R}^n). \] (30.3.22)

Thus, \( F^{-1} \rho * \) is weak type \((1, 1)\) and weak type \((2, 2)\). Also
\[ \left| \left| F^{-1} \rho * f \right|_2 \right| \leq A ||f||_2 \text{ if } f \in L^2 (\mathbb{R}^n). \] (30.3.23)

**Proof:** By Plancherel’s theorem \( F^{-1} \rho \) is in \( L^2 (\mathbb{R}^n) \). If \( f \in L^1 (\mathbb{R}^n) \), then by Minkowski’s inequality,
\[ F^{-1} \rho * f \in L^2 (\mathbb{R}^n). \]

Now let \( g \in L^2 (\mathbb{R}^n) \). By Holder’s inequality,
\[ \int \left| F^{-1} \rho (x-y) \right| \left| g (y) \right| dy \leq \left( \int \left| F^{-1} \rho (x-y) \right|^2 dy \right)^{1/2} \left( \int \left| g (y) \right|^2 dy \right)^{1/2} < \infty \]
and so the following is well defined a.e.
\[ F^{-1} \rho * g (x) \equiv \int F^{-1} \rho (x-y) g (y) dy \]
also,

\[ |F^{-1} \rho \ast g(x) - F^{-1} \rho \ast g(x')| \leq \int |F^{-1} \rho(x - y) - F^{-1} \rho(x' - y)| |g(y)| \, dy \]

\[ \leq \|F^{-1} \rho - F^{-1} \rho_{x' - x}\|_{L^2} \|g\|_{L^2} \]

and by continuity of translation in \( L^2(\mathbb{R}^n) \), this shows \( x \to F^{-1} \rho \ast g(x) \) is continuous. Therefore, \( F^{-1} \rho \ast \) maps \( L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) \) to the space of measurable functions. (Continuous functions are measurable.) It is clear that \( F^{-1} \rho \ast \) is subadditive.

If \( \phi \in G \), Plancherel's theorem implies as before,

\[ (2\pi)^{n/2} \|\rho F\phi\|_2 \leq (2\pi)^{n/2} \|\rho\|_\infty \|\phi\|_2. \tag{30.3.24} \]

Now let \( f \in L^2(\mathbb{R}^n) \) and let \( \phi_k \in G \), with \( \|\phi_k - f\|_2 \to 0 \).

Then by Hölder's inequality,

\[ \int F^{-1} \rho(x - y) f(y) \, dy = \lim_{k \to \infty} \int F^{-1} \rho(x - y) \phi_k(y) \, dy \]

and so by Fatou's lemma, Plancherel's theorem, and (30.3.24),

\[ \|F^{-1} \rho \ast f\|_2 = \left( \int \left( \int F^{-1} \rho(x - y) f(y) \, dy \right)^2 \, dx \right)^{1/2} \leq \]

\[ \leq \lim \inf_{k \to \infty} \left( \int \left( \int F^{-1} \rho(x - y) \phi_k(y) \, dy \right)^2 \, dx \right)^{1/2} = \lim \inf_{k \to \infty} \|F^{-1} \rho \ast \phi_k\|_2 \]

\[ \leq \|\rho\|_\infty (2\pi)^{n/2} \lim \inf_{k \to \infty} \|\phi_k\|_2 = \|\rho\|_\infty (2\pi)^{n/2} \|f\|_2. \]

Thus, (30.3.24) holds with \( A = \|\rho\|_\infty (2\pi)^{n/2} \). Consequently,

\[ A \|f\|_2 \geq \left( \int_{|F^{-1} \rho \ast f| > \alpha} |F^{-1} \rho \ast f(x)|^2 \, dx \right)^{1/2} \]

\[ \geq \alpha \|m (|F^{-1} \rho \ast f| > \alpha)\|^{1/2} \]

and so (30.3.24) follows.

It remains to prove (30.3.24) which holds for all \( f \in G \) by Lemma (30.3.1). Let \( f \in L^1(\mathbb{R}^n) \) and let \( \phi_k \to f \) in \( L^1(\mathbb{R}^n) \), \( \phi_k \in G \). Without loss of generality, assume that both \( f \) and \( F^{-1} \rho \) are Borel measurable. Therefore, by Minkowski's inequality, and Plancherel's theorem,

\[ \|F^{-1} \rho \ast \phi_k - F^{-1} \rho \ast f\|_2 \]
CHAPTER 30. FOURIER ANALYSIS IN $\mathbb{R}^N$ AN INTRODUCTION

\[ \left( \int \int F^{-1} \rho (x - y) (\phi_k (y) - f (y)) \, dy \, dx \right)^{1/2} \leq \|\phi_k - f\|_1 \|\rho\|_2 \]

which shows that $F^{-1} \rho * \phi_k$ converges to $F^{-1} \rho * f$ in $L^2 (\mathbb{R}^n)$. Therefore, there exists a subsequence such that the convergence is pointwise a.e. Then, denoting the subsequence by $k$,

\[ \chi_{\{|F^{-1} \rho * f| > \alpha\}} (x) \leq \liminf_{k \to \infty} \chi_{\{|F^{-1} \rho * \phi_k| > \alpha\}} (x) \text{ a.e. } x. \]

Thus by Lemma 30.3.2, and Fatou's lemma, there exists a constant, $A$, depending on $C_1, n,$ and $\|\rho\|_{\infty}$ such that

\[ m (\{|F^{-1} \rho * f| > \alpha\}) \leq \liminf_{k \to \infty} m (\{|F^{-1} \rho * \phi_k| > \alpha\}) \leq \liminf_{k \to \infty} A \frac{\|\phi_k\|_1}{\alpha} = A \frac{\|f\|_1}{\alpha}. \]

This shows 30.3.21 and proves the lemma.

**Theorem 30.3.3** Let $\rho \in L^2 (\mathbb{R}^n) \cap L^\infty (\mathbb{R}^n)$ and suppose

\[ \int_{|x| \geq 2|y|} |F^{-1} \rho (x - y) - F^{-1} \rho (x)| \, dx \leq C_1. \]

Then for each $p \in (1, \infty)$, there exists a constant, $A_p$, depending only on $p, n, \|\rho\|_{\infty}$, and $C_1$ such that for all $\phi \in \mathcal{G}$,

\[ \|F^{-1} \rho * \phi\|_p \leq A_p \|\phi\|_p. \]

**Proof:** From Lemma 30.3.2, $F^{-1} \rho *$ is weak $(1, 1)$, weak $(2, 2)$, and maps $L^1 (\mathbb{R}^n) + L^2 (\mathbb{R}^n)$ to measurable functions. Therefore, by the Marcinkiewicz interpolation theorem, there exists a constant $A_p$ depending only on $p, C_1, n$, and $\|\rho\|_{\infty}$ for $p \in (1, 2]$, such that for $f \in L^p (\mathbb{R}^n)$, and $p \in (1, 2]$,

\[ \|F^{-1} \rho * f\|_p \leq A_p \|f\|_p. \]

Thus the theorem is proved for these values of $p$. Now suppose $p > 2$. Then $p' < 2$ where

\[ \frac{1}{p} + \frac{1}{p'} = 1. \]
By Plancherel’s theorem and Theorem 62.25,
\[
\int F^{-1} \rho * \phi(x) \psi(x) \, dx = (2\pi)^{n/2} \int \rho(x) F\phi(x) F\psi(x) \, dx \\
= \int F(F^{-1} \rho * \psi) \phi \, dx \\
= \int (F^{-1} \rho * \psi)(\phi) \, dx.
\]

Thus by the case for \( p \in (1,2) \) and Holder’s inequality,
\[
\left| \int F^{-1} \rho * \phi(x) \psi(x) \, dx \right| = \left| \int (F^{-1} \rho * \psi)(\phi) \, dx \right| \\
\leq ||F^{-1} \rho * \psi||_p ||\phi||_p \\
\leq A_p ||\phi||_p.
\]

Letting \( L \psi \equiv \int F^{-1} \rho * \phi(x) \psi(x) \, dx \), this shows \( L \in L^{p'}(\mathbb{R}^n) \) and \( ||L||_{(L^{p'})'} \leq A_p ||\phi||_p \), which implies by the Riesz representation theorem that \( F^{-1} \rho * \phi \) represents \( L \) and
\[
||L||_{(L^{p'})'} = ||F^{-1} \rho * \phi||_{L^p} \leq A_p ||\phi||_p.
\]

Since \( p' = p/(p-1) \), this proves the theorem.

It is possible to give verifiable conditions on \( \rho \) which imply 62.25. The condition on \( \rho \) which is presented here is the existence of a constant, \( C_0 \) such that
\[
C_0 \geq \sup \{ |x|^{|\alpha|} |D^\alpha \rho(x)| : |\alpha| \leq L, x \in \mathbb{R}^n \setminus \{0\}, L > n/2. \} \quad (30.3.25)
\]

\( \rho \in C^L(\mathbb{R}^n \setminus \{0\}) \) where \( L \) is an integer.

Here \( \alpha \) is a multi-index and \( |\alpha| = \sum_{i=1}^n \alpha_i \). The condition says roughly that \( \rho \) is pretty smooth away from \( 0 \) and all the partial derivatives vanish pretty fast as \( |x| \to \infty \). Also recall the notation
\[
x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]
where \( \alpha = (\alpha_1 \cdots \alpha_n) \). For more general conditions, see 62.

**Lemma 30.3.4** Let 62.25 hold and suppose \( \psi \in C^\infty_c(\mathbb{R}^n \setminus \{0\}) \). Then for each \( \alpha, |\alpha| \leq L \), there exists a constant \( C \equiv C(\alpha,n,\psi) \) independent of \( \epsilon \) such that
\[
\sup_{x \in \mathbb{R}^n} |x|^{|\alpha|} |D^\alpha (\rho(x) \psi(2^k x))| \leq CC_0.
\]

**Proof:**
\[
|x|^{|\alpha|} |D^\alpha (\rho(x) \psi(2^k x))| \leq |x|^{|\alpha|} \sum_{\beta+\gamma=\alpha} |D^\beta \rho(x)| 2^{k|\gamma|} |D^\gamma \psi(2^k x)|
\]
\[ \sum_{\beta + \gamma = \alpha} |x|^{|\beta|} |D^\beta \rho(x)| |2^k x|^{\gamma_1} |D^{\gamma_1} \psi(2^k x)| \leq C_0 C(\alpha, n) \sum_{|\gamma| \leq |\alpha|} \sup \{|x|^{\gamma_1} |D^{\gamma_1} \psi(z)| : z \in \mathbb{R}^n \} = C_0 C(\alpha, n, \psi) \]

and this proves the lemma.

**Lemma 30.3.5** There exists \( \phi \in C^\infty_c \left( \left[ x : 4^{-1} < |x| < 4 \right] \right) \), \( \phi(x) \geq 0 \)

and

\[ \sum_{k=-\infty}^{\infty} \phi(2^k x) = 1 \]

for each \( x \neq 0 \).

**Proof:** Let \( \psi \geq 0, \psi = 1 \) on \( [2^{-1} \leq |x| \leq 2] \), \( \text{spt} (\psi) \subseteq [4^{-1} < |x| < 4] \).

Consider

\[ g(x) = \sum_{k=-\infty}^{\infty} \psi(2^k x). \]

Then for each \( x \), only finitely many terms are not equal to 0. Also, \( g(x) > 0 \) for all \( x \neq 0 \). To verify this last claim, note that for some \( k \) an integer, \( |x| \in [2^{k+1}, 2^{-1}] \).

Therefore, choose \( k \) an integer such that \( 2^k |x| \in [2^{-1}, 2] \). For example, let \( k = l-1 \). This works because \( 2^k |x| \in [2^{l-1}, 2^{l+1}-2^{l-1}] = [2^{-1}, 2] \).

Therefore, for this value of \( k \), \( \psi(2^k x) = 1 \) so \( g(x) > 0 \).

Now notice that

\[ g(2^r x) = \sum_{k=-\infty}^{\infty} \psi(2^k 2^r x) = \sum_{k=-\infty}^{\infty} \psi(2^k x) = g(x). \]

Let \( \phi(x) \equiv \psi(x) g(x)^{-1} \). Then

\[ \sum_{k=-\infty}^{\infty} \phi(2^k x) = \sum_{k=-\infty}^{\infty} \frac{\psi(2^k x)}{g(2^k x)} = g(x)^{-1} \sum_{k=-\infty}^{\infty} \psi(2^k x) = 1 \]

for each \( x \neq 0 \). This proves the lemma.

Now define

\[ \rho_m(x) \equiv \sum_{k=-m}^{m} \rho(x) \phi(2^k x), \quad \gamma_k(x) \equiv \rho(x) \phi(2^k x). \]
Let $t > 0$ and let $|y| \leq t$. Consider the problem of estimating

$$
\int_{|x| \geq 2t} \left| F^{-1} \gamma_k (x - y) - F^{-1} \gamma_k (x) \right| \, dx.
$$

(30.3.26)

In the following estimates, $C (a, b, \cdots, d)$ will denote a generic constant depending only on the indicated objects, $a, b, \cdots, d$. For the first estimate, note that since $|y| \leq t$, (30.3.26) is no larger than

$$
2 \int_{|x| \geq t} \left| F^{-1} \gamma_k (x) \right| \, dx = 2 \int_{|x| \geq t} \left| F^{-1} \gamma_k (x) \right| |x|^{-L} |x|^L \, dx
$$

$$
\leq 2 \left( \int_{|x| \geq t} |x|^{-2L} \, dx \right)^{1/2} \left( \int_{|x| \geq t} |x|^{2L} \left| F^{-1} \gamma_k (x) \right|^2 \, dx \right)^{1/2}
$$

Using spherical coordinates and Plancherel’s theorem,

$$
\leq C (n, L) t^{n/2 - L} \left( \int \left| \sum_{j=1}^n |x_j|^{-2L} \left| F^{-1} \gamma_k (x) \right|^2 \, dx \right)^{1/2}
$$

$$
\leq C (n, L) t^{n/2 - L} \left( \sum_{j=1}^n \int \left| F^{-1} D_j^L \gamma_k (x) \right|^2 \, dx \right)^{1/2}
$$

$$
= C (n, L) t^{n/2 - L} \left( \sum_{j=1}^n \int_{S_k} \left| D_j^L \gamma_k (x) \right|^2 \, dx \right)^{1/2}
$$

(30.3.27)

where

$$
S_k \equiv \{ x : 2^{-2-k} < |x| < 2^{2-k} \},
$$

(30.3.28)

a set containing the support of $\gamma_k$. Now from the definition of $\gamma_k$,

$$
|D_j^L \gamma_k (z)| = |D_j^L (\rho (z) \phi (2^k z))|.
$$

By Lemma 30.3.26, this is no larger than

$$
C (L, n, \phi) C_0 |z|^{-L}.
$$

(30.3.29)

It follows, using polar coordinates, that the last expression in (30.3.27) is no larger than

$$
C (n, L, \phi, C_0) t^{n/2 - L} \left( \int_{S_k} |z|^{-2L} \, dz \right)^{1/2} \leq C (n, L, \phi, C_0) t^{n/2 - L}.
$$

(30.3.30)

$$
\left( \int_{2^{-2-k}}^{2^{2-k}} \rho^{n-1-2L} \, d\rho \right)^{1/2} \leq C (n, L, \phi, C_0) t^{n/2 - L} 2^{k (L-n/2)}.
$$

Now estimate (30.3.27) in another way. The support of $\gamma_k$ is in $S_k$, a bounded set, and so $F^{-1} \gamma_k$ is differentiable. Therefore,

$$
\int_{|x| \geq 2t} \left| F^{-1} \gamma_k (x - y) - F^{-1} \gamma_k (x) \right| \, dx =
$$
\[ \int_{|x|\geq 2t} \left| \int_0^1 \sum_{j=1}^n D_j F^{-1} \gamma_k (x-sy) \, dy \right| \, dx \]

\[ \leq t \int_{|x|\geq 2t} \left| \int_0^1 \sum_{j=1}^n |D_j F^{-1} \gamma_k (x-sy)| \, ds \, dx \right| \]

\[ \leq t \int \sum_{j=1}^n |D_j F^{-1} \gamma_k (x)| \, dx \]

\[ \leq t \sum_{j=1}^n \left( \int \left( 1 + |2^{-k}x|^2 \right)^{-L} \, dx \right)^{1/2} \]

\[ \cdot \left( \int \left( 1 + |2^{-k}x|^2 \right)^L |D_j F^{-1} \gamma_k (x)|^2 \, dx \right)^{1/2} \]

\[ \leq C (n, L) t 2^{kn/2} \sum_{j=1}^n \left( \int \left( 1 + |2^{-k}x|^2 \right)^L |D_j F^{-1} \gamma_k (x)|^2 \, dx \right)^{1/2} \]. \quad (30.3.31)\]

Now consider the \( j \)th term in the last sum in (30.3.31):

\[ \int \left( 1 + |2^{-k}x|^2 \right)^L |D_j F^{-1} \gamma_k (x)|^2 \, dx \leq C (n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} x^{2\alpha} |D_j F^{-1} \gamma_k (x)|^2 \, dx \]

\[ = C (n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int x^{2\alpha} |F^{-1} (\pi_j \gamma_k) (x)|^2 \, dx \]

where \( \pi_j (z) \equiv z_j \). This last assertion follows from

\[ D_j \int e^{-ix \cdot y} \gamma_k (y) \, dy = \int (-i) e^{-ix \cdot y} y_j \gamma_k (y) \, dy. \]

Therefore, a similar computation and Plancherel’s theorem implies equals

\[ = C (n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int |F^{-1} D^\alpha (\pi_j \gamma_k) (x)|^2 \, dx \]

\[ = C (n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int |D^\alpha (z_j \gamma_k (z))|^2 \, dz \] \quad (30.3.33)

where \( S_k \) is given in (30.3.28). Now

\[ |D^\alpha (z_j \gamma_k (z))| = 2^{-k} |D^\alpha (\rho (z) z_j 2^k \phi (2^k z))| \]

\[ = 2^{-k} |D^\alpha (\rho (z) \psi_j (2^k z))| \]
where \( \psi_j(z) \equiv z_j \phi(z) \). By Lemma 30.3.4, this is dominated by

\[
2^{-k} C(\alpha, n, \phi, j, C_0) |z|^{-|\alpha|}.
\]

Therefore, \(30.3.33\) is dominated by

\[
C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int_{S_k} 2^{-2k} |z|^{-2|\alpha|} \, dz
\]

\[
\leq C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} 2^{-2k} (2^{-2-k})^{(-2|\alpha|)} (2^{2-k})^n.
\]

\[
\leq C(L, n, \phi, j, C_0) 2^{-kn-2k}.
\]

It follows that \(30.3.33\) is no larger than

\[
C(L, n, \phi, C_0) t^{2^{kn/2}2^{-kn-2k}} = C(L, n, \phi, C_0) t^{2^{-k}}.
\]  \(30.3.34\)

It follows from \(30.3.34\) and \(30.3.30\) that if \(|y| \leq t\),

\[
\int_{|x| \geq 2t} \left| F^{-1} \gamma_k(x) - F^{-1} \gamma_k(x) \right| \, dx \leq C(L, n, \phi, C_0) \min \left( t^{2^{-k}}, (2^{-k}t)^{n/2-L} \right).
\]

With this inequality, the next lemma which is the desired result can be obtained.

**Lemma 30.3.6** There exists a constant depending only on the indicated objects, \(C_1 = C(L, n, \phi, C_0)\) such that when \(|y| \leq t\),

\[
\int_{|x| \geq 2t} \left| F^{-1} \rho(x - y) - F^{-1} \rho(x) \right| \, dx \leq C_1
\]

\[
\int_{|x| \geq 2t} \left| F^{-1} \rho_m(x - y) - F^{-1} \rho_m(x) \right| \, dx \leq C_1.
\]  \(30.3.35\)

**Proof:** \(F^{-1} \rho = \lim_{m \to \infty} F^{-1} \rho_m\) in \(L^2(\mathbb{R}^n)\). Let \(m_k \to \infty\) be such that convergence is pointwise a.e. Then if \(|y| \leq t\), Fatou’s lemma implies

\[
\int_{|x| \geq 2t} \left| F^{-1} \rho(x - y) - F^{-1} \rho(x) \right| \, dx \leq \lim \inf_{l \to \infty} \int_{|x| \geq 2t} \left| F^{-1} \rho_m(x - y) - F^{-1} \rho_m(x) \right| \, dx
\]
\[ \leq \lim_{t \to \infty} \inf \frac{m_{t}}{k=-m_{t}} \int_{|x| \geq 2t} |F^{-1} \gamma_{k}(x-y) - F^{-1} \gamma_{k}(x)| \, dx \]

\[ \leq C(L, n, \phi, C_{0}) \sum_{k=-\infty}^{\infty} \min \left( t2^{-k}, (2^{-k}t)^{n/2-L} \right). \quad (30.3.36) \]

Now consider the sum in \(30.3.36\),

\[ \sum_{k=-\infty}^{\infty} \min \left( t2^{-k}, (2^{-k}t)^{n/2-L} \right). \quad (30.3.37) \]

\( t2^{j} = \min \left( t2^{j}, (2^{j}t)^{n/2-L} \right) \) exactly when \( t2^{j} \leq 1 \). This occurs if and only if \( j \leq -\ln(t)/\ln(2) \). Therefore \(30.3.37\) is no larger than

\[ \sum_{j \leq -\ln(t)/\ln(2)} 2^{j}t + \sum_{j \geq -\ln(t)/\ln(2)} (2^{j}t)^{n/2-L}. \]

Letting \( a = L - n/2 \), this equals

\[ t \sum_{k \geq \ln(t)/\ln(2)} 2^{-k} + t^{-a} \sum_{j \geq -\ln(t)/\ln(2)} (2^{-a}j)^{n/2-L} \]

\[ \leq 2t \left( \frac{1}{2} \right)^{\ln(t)/\ln(2)} + t^{-a} \left( \frac{1}{2^{a}} \right)^{-\ln(t)/\ln(2)} \]

\[ = 2t \left( \frac{1}{2} \right)^{\log_{2}(t)} + t^{-a} \left( \frac{1}{2^{a}} \right)^{-\log_{2}(t)} \]

\[ = 2 + 1 = 3. \]

Similarly, \(30.3.36\) holds. This proves the lemma.

Now it is possible to prove Mihlin’s theorem.

**Theorem 30.3.7** (Mihlin’s theorem) Suppose \( \rho \) satisfies

\[ C_{0} \geq \sup \{ |x|^{\alpha} |D^{\alpha} \rho(x)| : |\alpha| \leq L, \ x \in \mathbb{R}^{n} \setminus \{0\} \}, \]

where \( L \) is an integer greater than \( n/2 \) and \( \rho \in C^{L}(\mathbb{R}^{n} \setminus \{0\}) \). Then for every \( p > 1 \), there exists a constant \( A_{p} \) depending only on \( p, C_{0}, \phi, n, \) and \( L \), such that for all \( \psi \in G \),

\[ \left\| F^{-1} \rho * \psi \right\|_{p} \leq A_{p} \left\| \psi \right\|_{p}. \]

**Proof:** Since \( \rho_{m} \) satisfies \(30.3.36\) and is obviously in \( L^{2}(\mathbb{R}^{n}) \cap L^{\infty}(\mathbb{R}^{n}) \), Theorem \(30.3.36\) implies there exists a constant \( A_{p} \) depending only on \( p, n, \|\rho_{m}\|_{\infty}, \) and \( C_{1} \) such that for all \( \psi \in G \) and \( p \in (1, \infty) \),

\[ \left\| F^{-1} \rho_{m} * \psi \right\|_{p} \leq A_{p} \left\| \psi \right\|_{p}. \]
Now \(|\rho_m|_\infty \leq |\rho|_\infty\) because

\[
|\rho_m(x)| \leq |\rho(x)| \sum_{k=-m}^{m} \phi(2^k x) \leq |\rho(x)|. \tag{30.3.38}
\]

Therefore, since \(C_1 = C_1 (L, n, \phi, C_0)\) and \(C_0 \geq |\rho|_\infty\),

\[
\|F^{-1} \rho_m * \psi\|_p \leq A_p (L, n, \phi, C_0, p) \|\psi\|_p.
\]

In particular, \(A_p\) does not depend on \(m\). Now, by (30.3.38) the observation that \(\rho \in L^\infty(\mathbb{R}^n)\), \(\lim_{m \to \infty} \rho_m(y) = \rho(y)\) and the dominated convergence theorem, it follows that for \(\theta \in \mathcal{G}\).

\[
\left| \left( F^{-1} \rho \ast \psi \right) (\theta) \right| = \left| \frac{(2\pi)^{n/2}}{(2\pi)^{n/2}} \int \rho(x) F\psi(x) F^{-1} \theta(x) \, dx \right|
\]

\[
= \lim_{m \to \infty} \left( F^{-1} \rho_m \ast \psi \right) (\theta) \leq \lim_{m \to \infty} \sup_{m \to \infty} \left| F^{-1} \rho_m \ast \psi \right|_p \|\theta\|_{p'}.
\]

Hence \(F^{-1} \rho \ast \psi \in L^p(\mathbb{R}^n)\) and \(\|F^{-1} \rho \ast \psi\|_p \leq A_p \|\psi\|_p\). This proves the theorem.

### 30.4 Singular Integrals

If \(K \in L^1(\mathbb{R}^n)\) then when \(p > 1\),

\[
\|K * f\|_p \leq \|f\|_p.
\]

It turns out that some meaning can be assigned to \(K * f\) for some functions \(K\) which are not in \(L^1\). This involves assuming a certain form for \(K\) and exploiting cancellation. The resulting theory of singular integrals is very useful. To illustrate, an application will be given to the Helmholtz decomposition of vector fields in the next section. Like Mihlin’s theorem, the theory presented here rests on Theorem 30.3.3, restated here for convenience.

**Theorem 30.4.1** Let \(\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) and suppose

\[
\int_{|x| \geq 2|y|} \left| F^{-1} \rho(x - y) - F^{-1} \rho(x) \right| \, dx \leq C_1.
\]

Then for each \(p \in (1, \infty)\), there exists a constant, \(A_p\), depending only on \(p, n, |\rho|_\infty\),

and \(C_1\) such that for all \(\phi \in \mathcal{G}\),

\[
\|F^{-1} \rho \ast \phi\|_p \leq A_p \|\phi\|_p.
\]
Lemma 30.4.2 Suppose

\[ K \in L^2(\mathbb{R}^n), \|FK\|_\infty \leq B < \infty, \]  

and

\[ \int_{|x| > 2|y|} |K(x-y) - K(x)| \, dx \leq B. \]  

Then for all \( p > 1 \), there exists a constant, \( A(p,n,B) \), depending only on the indicated quantities such that

\[ \|K * f\|_p \leq A(p,n,B) \|f\|_p \]

for all \( f \in \mathcal{G} \).

Proof: Let \( FK = \rho \) so \( F^{-1}\rho = K \). Then from Lemma 30.4.3, \( \rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) and \( K = F^{-1}\rho \). By Theorem 30.3.3 listed above,

\[ \|K * f\|_p = \|F^{-1}\rho * f\|_p \leq A(p,n,B) \|f\|_p \]

for all \( f \in \mathcal{G} \). This proves the lemma.

The next lemma provides a situation in which the above conditions hold.

Lemma 30.4.3 Suppose

\[ |K(x)| \leq B |x|^{-n}, \]  

\[ \int_{a<|x|<b} K(x) \, dx = 0, \]  

\[ \int_{|x| > 2|y|} |K(x-y) - K(x)| \, dx \leq B. \]  

Define

\[ K_\varepsilon(x) = \begin{cases} K(x) & \text{if } |x| \geq \varepsilon, \\ 0 & \text{if } |x| < \varepsilon. \end{cases} \]

Then there exists a constant \( C(n) \) such that

\[ \int_{|x| > 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)| \, dx \leq C(n) B \]

and

\[ \|FK_\varepsilon\|_\infty \leq C(n) B. \]

Proof: In the argument, \( C(n) \) will denote a generic constant depending only on \( n \). Consider first. The integral is broken up according to whether \( |x|, |x-y| > \varepsilon \).
30.4. SINGULAR INTEGRALS

\[ \int_{|x| \geq 2y} |K \varepsilon (x - y) - K \varepsilon (x)| \, dx = \]
\[ \int_{|x| \geq 2y, |x-y| > \varepsilon, |x| < \varepsilon} |K \varepsilon (x - y) - K \varepsilon (x)| \, dx + \]
\[ \int_{|x| \geq 2y, |x-y| < \varepsilon, |x| \geq \varepsilon} |K \varepsilon (x - y) - K \varepsilon (x)| \, dx + \]
\[ \int_{|x| \geq 2y, |x-y| > \varepsilon, |x| > \varepsilon} |K \varepsilon (x - y) - K \varepsilon (x)| \, dx + \]
\[ \int_{|x| \geq 2y, |x-y| < \varepsilon, |x| < \varepsilon} |K \varepsilon (x - y) - K \varepsilon (x)| \, dx. \] (30.4.46)

Now consider the terms in the above expression. The last integral in (30.4.46) equals 0 from the definition of \( K \varepsilon \). The third integral on the right is no larger than \( B \) by the definition of \( K \varepsilon \) and (30.4.45). Consider the second integral on the right. This integral is no larger than

\[ \int_{|x| \geq 2y, |x-y| > \varepsilon, |x| < \varepsilon} B |x|^{-n} \, dx. \]

Now \( |x| \leq |y| + \varepsilon \leq |x|/2 + \varepsilon \) and so \( |x| < 2\varepsilon \). Thus this is no larger than

\[ \int_{\varepsilon \leq |x| \leq 2\varepsilon} B |x|^{-n} \, dx = B \int_{S_{n-1}} \int_{\varepsilon}^{2\varepsilon} \rho^{n-1} \frac{1}{\rho^{n}} \, d\rho \, d\sigma \leq BC(n) \ln 2 = C(n) B. \]

It remains to estimate the first integral on the right in (30.4.46). This integral is bounded by

\[ \int_{|x| \geq 2y, |x-y| > \varepsilon, |x| < \varepsilon} B |x-y|^{-n} \, dx \]

In the integral above, \( |x| < \varepsilon \) and so \( |x-y| < |y| < \varepsilon \). Therefore, \( |x-y| < \varepsilon + |y| < \varepsilon + |x|/2 < \varepsilon + \varepsilon/2 = (3/2) \varepsilon \). Hence \( \varepsilon \leq |x-y| \leq (3/2) |x-y| \). Therefore, the above integral is no larger than

\[ \int_{\varepsilon}^{(3/2) \varepsilon} B |z|^{-n} \, dz = B \int_{S_{n-1}} \int_{\varepsilon}^{(3/2) \varepsilon} \rho^{n-1} \, d\rho \, d\sigma = BC(n) \ln (3/2). \]

This establishes (30.4.47).

Now it remains to show (30.4.48), a statement about the Fourier transforms of \( K \varepsilon \). Fix \( \varepsilon \) and let \( y \neq 0 \) also be given.

\[ K_{\varepsilon R}(y) \equiv \begin{cases} K \varepsilon (y) & \text{if } |y| < R, \\ 0 & \text{if } |y| \geq R \end{cases} \]

where \( R > \frac{3\pi}{|y|} \) (The 3 here isn’t important. It just needs to be larger than 1.) Then

\[ |FK_{\varepsilon R}(y)| \leq \left| \int_{|x| < 3\pi |y|^{-1}} K \varepsilon (x) e^{-ix \cdot y} \, dx \right| + \left| \int_{|x| \geq R} K \varepsilon (x) e^{-ix \cdot y} \, dx \right|. \]
\[ A = B. \]  

Consider \( A \). By (30.4.47)

\[ \int_{\varepsilon < |x| < 3\pi|y|^{-1}} K_\varepsilon(x) \, dx = 0 \]

and so

\[ A = \int_{\varepsilon < |x| < 3\pi|y|^{-1}} K_\varepsilon(x) \left( e^{-ix \cdot y} - 1 \right) \, dx \]

Now

\[ |e^{-ix \cdot y} - 1| = |2 - 2 \cos(x \cdot y)|^{1/2} \leq 2 |x \cdot y| \leq 2 |x||y| \]

so, using polar coordinates, this expression is no larger than

\[ 2B \int_{\varepsilon < |x| < 3\pi|y|^{-1}} |x|^{-n} |x||y| \, dx \leq C(n) B |y| \int_{\varepsilon}^{3\pi/|y|} d\rho \leq BC(n). \]

Next, consider \( B \). This estimate is based on the trick which follows. Let

\[ z \equiv y \pi/|y|^2 \]

so that

\[ |z| = \pi/|y|, \quad z \cdot y = \pi. \]

Then

\[ \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-ix \cdot y} \, dx = \frac{1}{\pi} \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-ix \cdot y} \, dx \]

\[ = \frac{1}{2} \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-i(x+z) \cdot y} \, dx. \]  

(30.4.48)

Here is why. Note in the second of these integrals,

\[ \frac{1}{2} \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-i(x+z) \cdot y} \, dx \]

\[ = \frac{1}{2} \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-ix \cdot y} e^{-iz \cdot y} \, dx \]

\[ = \frac{1}{2} \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-ix \cdot y} e^{-i\pi} \, dx \]

\[ = \frac{1}{2} \int_{3\pi|y|^{-1} < |x| \leq R} K_\varepsilon(x) e^{-ix \cdot y} \, dx. \]
Then changing the variables in (30.4.48),

\[
\int_{3\pi|y|^{-1}<|x|\leq R} K_\epsilon(x) e^{-ix.y} dx
= \frac{1}{2} \int_{3\pi|y|^{-1}<|x|\leq R} K_\epsilon(x) e^{-ix.y} dx
- \frac{1}{2} \int_{3\pi|y|^{-1}<|x|\leq R} K_\epsilon(x-z) e^{-ix.y} dx.
\]

Thus

\[
\int_{3\pi|y|^{-1}<|x|\leq R} K_\epsilon(x) e^{-ix.y} dx = \\
\frac{1}{2} \int_{|x|\leq R} K_\epsilon(x) e^{-ix.y} dx - \frac{1}{2} \int_{|x-z|\leq R} K_\epsilon(x-z) e^{-ix.y} dx
+ \frac{1}{2} \int_{|x-z|>3\pi|y|^{-1}} K_\epsilon(x-z) e^{-ix.y} dx - \frac{1}{2} \int_{|x|>3\pi|y|^{-1}} K_\epsilon(x) e^{-ix.y} dx.
\]

(30.4.49)

Since $|z| = \pi/|y|$, it follows $|z| = \frac{\pi}{|y|} < \frac{3\pi}{|y|} < R$ and so the following picture describes the situation. In this picture, the radius of each ball equals either $R$ or $3\pi|y|^{-1}$ and each integral above is taken over one of the two balls in the picture, either the one centered at 0 or the one centered at $z$.

\[
\begin{array}{c}
\text{To begin with, consider the integrals which involve } K_\epsilon(x-z). \\
\int_{|x-z|\leq R} K_\epsilon(x-z) e^{-ix.y} dx \\
= \int_{|x|\leq R} K_\epsilon(x-z) e^{-ix.y} dx \\
- \int_{|x-z|>R, |x|<R} K_\epsilon(x-z) e^{-ix.y} dx \\
+ \int_{|x-z|<R, |x|>R} K_\epsilon(x-z) e^{-ix.y} dx.
\end{array}
\]

(30.4.50)
Look at the picture. Similarly,
\[
\int_{|x-z| \leq 3\pi|y|^{-1}} K_{\varepsilon}(x - z) e^{-i\mathbf{x} \cdot \mathbf{y}} \, dx = \int_{|x| \leq 3\pi|y|^{-1}} K_{\varepsilon}(x - z) e^{-i\mathbf{x} \cdot \mathbf{y}} \, dx - \int_{|x-z| > 3\pi|y|^{-1},|x| < 3\pi|y|^{-1}} K_{\varepsilon}(x - z) e^{-i\mathbf{x} \cdot \mathbf{y}} \, dx + \int_{|x-z| < 3\pi|y|^{-1},|x| > 3\pi|y|^{-1}} K_{\varepsilon}(x - z) e^{-i\mathbf{x} \cdot \mathbf{y}} \, dx. \tag{30.4.51}
\]

The last integral in (30.4.50) is taken over a set that is contained in
\[
B(0, R + |z|) \setminus B(0, R)
\]
illustrated in the following picture as the region between the small ball centered at \(0\) and the big ball which surrounds the two small balls.

\[\bullet \quad \bullet \quad z\]

and so this integral is dominated by
\[
B \left( \frac{1}{(R - |z|)^{n}} \right) \alpha(n) \left( (R + |z|)^{n} - R^{n} \right),
\]
an expression which converges to 0 as \(R \to \infty\). Similarly, the second integral on the right in (30.4.51) converges to zero as \(R \to \infty\). Now consider the last two integrals in (30.4.51). Letting \(3\pi|y|^{-1}\) play the role of \(R\) and using \(|z| = \pi/|y|\), these are each dominated by an expression of the form
\[
B \left( \frac{1}{(3\pi|y|^{-1} - |z|)^{n}} \right) \alpha(n) \left( (3\pi|y|^{-1} + |z|)^{n} - (3\pi|y|^{-1})^{n} \right) = B \left( \frac{1}{(3\pi|y|^{-1} - \pi|y|^{-1})^{n}} \right) \alpha(n) \cdot \left( (3\pi|y|^{-1} + \pi|y|^{-1})^{n} - (3\pi|y|^{-1})^{n} \right)
\]
\[ = \alpha (n) B \frac{|y|^n}{(2\pi)^n} \frac{1}{|y|^{n}} ((4\pi)^{n} - (3\pi)^{n}) = C (n) B. \]

Returning to \ref{30.4.49}, the terms involving \( x - y \) have now been estimated. Thus, collecting the terms which have not yet been estimated along with those that have,

\[ B = \left| \int_{|y|^{-1} < |x| \leq R} K_\varepsilon (x) e^{-ix \cdot y} dx \right| \]

\[ \leq \frac{1}{2} \left| \int_{|x| < R} K_\varepsilon (x) e^{-ix \cdot y} dx - \int_{|x| < R} K_\varepsilon (x - z) e^{-ix \cdot y} dx \right| \]

\[ + \int_{|x| < 3\pi |y|^{-1}} K_\varepsilon (x - z) e^{-ix \cdot y} dx - \int_{|x| < 3\pi |y|^{-1}} K_\varepsilon (x) e^{-ix \cdot y} dx \]

\[ + C (n) B + g (R) \]

where \( g (R) \to 0 \) as \( R \to \infty \). Using \( |z| = \pi / |y| \) again,

\[ B \leq \frac{1}{2} \int_{3|z| < |x| < R} |K_\varepsilon (x) - K_\varepsilon (x - z)| dx + C (n) B + g (R). \]

But the integral in the above is dominated by \( C (n) B \) by \ref{30.4.39} which was established earlier. Therefore, from \ref{30.4.39},

\[ |FK_\varepsilon R| \leq C (n) B + g (R) \]

where \( g (R) \to 0 \).

Now \( K_\varepsilon R \to K_\varepsilon \) in \( L^2 (\mathbb{R}^n) \) because

\[ \|K_\varepsilon R - K_\varepsilon\|_{L^2 (\mathbb{R}^n)} \leq B \int_{|x| > R} \frac{1}{|x|^{2n}} dx \]

\[ = B \int_{S^{n-1}} \int_{R}^{\infty} \frac{1}{\rho^{n+1}} d\rho d\sigma, \]

which converges to 0 as \( R \to \infty \) and so \( FK_\varepsilon R \to FK_\varepsilon \) in \( L^2 (\mathbb{R}^n) \) by Plancherel’s theorem. Therefore, by taking a subsequence, still denoted by \( R \), \( FK_\varepsilon R (y) \to FK_\varepsilon (y) \) a.e. which shows

\[ |FK_\varepsilon (y)| \leq C (n) B \text{ a.e.} \]

This proves the lemma.

**Corollary 30.4.4** Suppose \ref{30.4.39} \ref{30.4.40} \ref{30.4.41} hold. Then if \( g \in C^1_c (\mathbb{R}^n) \), \( K_\varepsilon \ast g \) converges uniformly and in \( L^p (\mathbb{R}^n) \) as \( \varepsilon \to 0 \).
Proof:

\[ K_\varepsilon g(x) \equiv \int K_\varepsilon(y) g(x-y) \, dy. \]

Let \( 0 < \eta < \varepsilon \). Then since \( g \in C_c^1(\mathbb{R}^n) \), there exists a constant, \( K \) such that \( K |u-v| \geq |g(u) - g(v)| \) for all \( u, v \in \mathbb{R}^n \).

\[ |K_\varepsilon g(x) - K_\eta g(x)| \leq BK \int_{\eta < |y| < \varepsilon} |y| \, dy = BK \int_{S^{n-1}} \int_\eta^\varepsilon \, d\sigma = C_n |\varepsilon - \eta|. \]

This proves the corollary.

**Theorem 30.4.5** Suppose \( \text{(30.4.40 - 30.4.42)} \). Then for \( K_\varepsilon \) given by \( \text{(30.4.43)} \) and \( p > 1 \), there exists a constant \( A(p,n,B) \) such that for all \( f \in L^p(\mathbb{R}^n) \),

\[ ||K_\varepsilon f||_p \leq A(p,n,B) ||f||_p. \] (30.4.52)

Also, for each \( f \in L^p(\mathbb{R}^n) \),

\[ Tf \equiv \lim_{\varepsilon \to 0} K_\varepsilon f \] (30.4.53)

exists in \( L^p(\mathbb{R}^n) \) and for all \( f \in L^p(\mathbb{R}^n) \),

\[ ||Tf||_p \leq A(p,n,B) ||f||_p. \] (30.4.54)

Thus \( T \) is a linear and continuous map defined on \( L^p(\mathbb{R}^n) \) for each \( p > 1 \).

**Proof:** From \( \text{(30.4.40)} \) it follows \( K_\varepsilon \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) where, as usual, \( 1/p + 1/p' = 1 \). By continuity of translation in \( L^p(\mathbb{R}^n) \), \( x \to K_\varepsilon f(x) \) is a continuous function. By Lemma \( \text{(30.4.3)} \), \( ||FK_\varepsilon||_\infty \leq C(n) B \) for all \( \varepsilon \). Therefore, by Lemma \( \text{(30.4.2)} \),

\[ ||K_\varepsilon g||_p \leq A(p,n,B) ||g||_p \]

for all \( g \in G \). Now let \( f \in L^p(\mathbb{R}^n) \) and \( g_k \to f \) in \( L^p(\mathbb{R}^n) \) where \( g_k \in G \). Then

\[ |K_\varepsilon f(x) - K_\varepsilon g_k(x)| \leq \int |K_\varepsilon(x-y)||g_k(y) - f(y)| \, dy \leq ||K_\varepsilon||_{p'} ||g_k - f||_p \]

which shows that \( K_\varepsilon g_k(x) \to K_\varepsilon f(x) \) pointwise and so by Fatou’s lemma,

\[ ||K_\varepsilon f||_p \leq \lim_{k \to \infty} \inf ||K_\varepsilon g_k||_p \leq \lim_{k \to \infty} \inf A(p,n,B) ||g_k||_p = A(p,n,B) ||f||_p. \]

This verifies \( \text{(30.4.52)} \).

To verify \( \text{(30.4.53)} \), let \( \delta > 0 \) be given and let

\[ f \in L^p(\mathbb{R}^n), g \in C_c^\infty(\mathbb{R}^n). \]
\[ |K_\varepsilon \ast f - K_\eta \ast f|_p \leq |K_\varepsilon \ast (f - g)|_p + |K_\varepsilon \ast g - K_\eta \ast g|_p \]
\[ + |K_\eta \ast (f - g)|_p \leq 2A(p, n, B) |f - g|_p + |K_\varepsilon \ast g - K_\eta \ast g|_p. \]

Choose \( g \) such that \( 2A(p, n, B) |f - g|_p \leq \delta/2 \). Then if \( \varepsilon, \eta \) are small enough, Corollary 30.4.4 implies the last term is also less than \( \delta/2 \). Thus, \( \lim_{\varepsilon \to 0} K_\varepsilon \ast f \) exists in \( L^p(\mathbb{R}^n) \). Let \( Tf \) be the element of \( L^p(\mathbb{R}^n) \) to which it converges. Then 30.4.5 follows and \( T \) is obviously linear because
\[ T(af + bg) = \lim_{\varepsilon \to 0} K_\varepsilon \ast (af + bg) = \lim_{\varepsilon \to 0} (aK_\varepsilon \ast f + bK_\varepsilon \ast g) = aTf + bTg. \]

This proves the theorem.

When do conditions 30.4.4 - 30.4.42 hold? It turns out this happens for \( K \) given by the following.
\[ K(x) \equiv \frac{\Omega (x)}{|x|^n}, \tag{30.4.55} \]
where
\[ \Omega (\lambda x) = \Omega (x) \text{ for all } \lambda > 0, \tag{30.4.56} \]
\[ \Omega \text{ is Lipschitz on } S^{n-1}, \]
\[ \int_{S^{n-1}} \Omega (x) d\sigma = 0. \tag{30.4.57} \]

**Theorem 30.4.6** For \( K \) given by 30.4.55 - 30.4.57, it follows there exists a constant \( B \) such that
\[ |K(x)| \leq B |x|^{-n}, \tag{30.4.58} \]
\[ \int_{a<|x|<b} K(x) dx = 0, \tag{30.4.59} \]
\[ \int_{|x|>2|y|} |K(x - y) - K(x)| dx \leq B. \tag{30.4.60} \]

Consequently, the conclusions of Theorem 30.4.5 hold also.

**Proof:** 30.4.58 is obvious. To verify 30.4.59,
\[ \int_{|x|<b} K(x) dx = \int_{a}^{b} \int_{S^{n-1}} \frac{\Omega (\rho \omega)}{\rho^n} \rho^{n-1} d\sigma d\rho \]
\[ = \int_{a}^{b} \frac{1}{\rho} \int_{S^{n-1}} \Omega (\omega) d\sigma d\rho = 0. \]

It remains to show 30.4.60,
\[ K(x - y) - K(x) = |x - y|^{-n} \left( \Omega \left( \frac{x - y}{|x - y|} \right) - \Omega \left( \frac{x}{|x|} \right) \right) + \Omega (x) \left( \frac{1}{|x - y|^n} - \frac{1}{|x|^n} \right). \tag{30.4.61} \]
where \( \text{Stein} \) was used to write \( \Omega \left( \frac{z}{|z|} \right) = \Omega \left( z \right) \). The first group of terms in (30.4.61) is dominated by

\[
|x - y|^n \operatorname{Lip}(\Omega) \left| \frac{x - y}{|x - y|} - \frac{x}{|x|} \right|
\]

and an estimate is required for \(|x| > 2|y|\). Since \(|x| > 2|y|\),

\[
|x - y|^n \leq (|x| - |y|)^n \leq \frac{2^n}{|x|^n}.
\]

Also

\[
\left| \frac{x - y}{|x - y|} - \frac{x}{|x|} \right| = \frac{(x - y)|x - x - y|}{|x||x - y|} \leq \frac{|x - y||x - x - y|}{|x|(|x|/2)}
\]

\[
= \frac{2}{|x|^2} |x - y||x - x - y| = \frac{2}{|x|^2} |x||x - y| - y|x|
\]

\[
\leq \frac{2}{|x|^2} |x| |x - y| + |y| |x| \leq \frac{2}{|x|^2} (|x| |x - y| + |y| |x|)
\]

\[
\leq \frac{4}{|x|^2} |x| |y| = 4\frac{|y|}{|x|}.
\]

Therefore,

\[
\int_{|x| > 2|y|} |x - y|^{-n} \left| \Omega \left( \frac{x - y}{|x - y|} \right) - \Omega \left( \frac{x}{|x|} \right) \right| dx
\]

\[
\leq 4 \left( 2^n \right) \int_{|x| > 2|y|} \frac{1}{|x|^n} |y| dx \operatorname{Lip}(\Omega)
\]

\[
= C \left( n, \operatorname{Lip} \Omega \right) \int_{|x| > 2|y|} \frac{|y|}{|x|^{n+1}} dx
\]

\[
= C \left( n, \operatorname{Lip} \Omega \right) \int_{|u| > 2|y|} \frac{1}{|u|^{n+1}} du. \tag{30.4.62}
\]

It remains to consider the second group of terms in (30.4.61) when \(|x| > 2|y|\).

\[
\left| \frac{1}{|x - y|^n} - \frac{1}{|x|^n} \right| = \left| \frac{|x|^n - |x - y|^n}{|x - y|^n |x|^n} \right|
\]

\[
\leq \frac{2^n}{|x|^{2n}} ||x|^n - |x - y|^n|
\]

\[
\leq \frac{2^n}{|x|^{2n}} |y| \left[ |x|^{n-1} + |x|^{n-2} |x - y| + \cdots + |x| |x - y|^{n-2} + |x - y|^{n-1} \right]
\]
\[ \leq \frac{2^n |y| C(n) |x|^{n-1}}{|x|^{2n}} = C(n) \frac{2^n |y|}{|x|^{n+1}}. \]

Thus
\[
\int_{|x|>2|y|} \left| \Omega(x) \left( \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right) \right| \, dx \\
\leq C(n) \int_{|x|>2|y|} \frac{|y|}{|x|^{n+1}} \, dx \\
\leq C(n) \int_{|u|>2} \frac{1}{|u|^{n+1}} \, du. \tag{30.4.63}
\]

From (30.4.62) and (30.4.63),
\[
\int_{|x|>2|y|} |K(x-y) - K(x)| \, dx \leq C(n, \text{Lip } \Omega).
\]

This proves the theorem.

### 30.5 Helmholtz Decompositions

It turns out that every vector field which has its components in \( L^p \) can be written as a sum of a gradient and a vector field which has zero divergence. This is a very remarkable result, especially when applied to vector fields which are only in \( L^p \).

Recall that for \( u \) a function of \( n \) variables, \( \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \).

**Definition 30.5.1**

Define
\[
\Phi(y) = \begin{cases} 
-\frac{1}{a_2} \ln |y|, & \text{if } n = 2, \\
\frac{1}{(n-2)a_{n-2}} |y|^{2-n}, & \text{if } n > 2.
\end{cases}
\]

where \( a_k \) denotes the area of the unit sphere, \( S^k \).

Then it is routine to verify \( \Delta \Phi = 0 \) away from 0. In fact, if \( n > 2 \),
\[
\Phi_{,ii}(y) = C_n \left[ \frac{1}{|y|^n} - n \frac{y_i^2}{|y|^{n+2}} \right], \quad \Phi_{,ij}(y) = C_n \frac{y_i y_j}{|y|^{n+2}}, \tag{30.5.64}
\]

while if \( n = 2 \),
\[
\Phi_{,22}(y) = C_2 \frac{y_1^2 - y_2^2}{(y_1^2 + y_2^2)^2}, \quad \Phi_{,11}(y) = C_2 \frac{y_2^2 - y_1^2}{(y_1^2 + y_2^2)^2}, \\
\Phi_{,ij}(y) = C_2 \frac{y_i y_j}{(y_1^2 + y_2^2)^2}.
\]

Also,
\[
\nabla \Phi(y) = \frac{-y}{a_{n-1} |y|^n}. \tag{30.5.65}
\]

In the above the subscripts following a comma denote partial derivatives.
Lemma 30.5.2 For \( n \geq 2 \)

\[
\Phi_{ij}(y) = \frac{\Omega_{ij}(y)}{|y|^n}
\]

where

\( \Omega_{ij} \) is Lipschitz continuous on \( S^{n-1} \),

\( \Omega_{ij}(\lambda y) = \Omega_{ij}(y) \),

for all \( \lambda > 0 \), and

\[
\int_{S^{n-1}} \Omega_{ij}(y) \, d\sigma = 0.
\]

Proof:

The case \( n = 2 \) is left to the reader. \( \text{Lemma 30.5.2} \) and \( \text{Lemma 30.5.3} \) are obvious from the above descriptions. It remains to verify \( \text{Lemma 30.5.4} \). If \( n \geq 3 \) and \( i \neq j \), then this formula is also clear from \( \text{Lemma 30.5.1} \). Thus consider the case when \( n \geq 3 \) and \( i = j \). By symmetry,

\[
I \equiv \int_{S^{n-1}} 1 - ny_i^2 \, d\sigma = \int_{S^{n-1}} 1 - ny_j^2 \, d\sigma.
\]

Hence

\[
nI = \sum_{i=1}^{n} \int_{S^{n-1}} 1 - ny_i^2 \, d\sigma = \int_{S^{n-1}} \left( n - n \sum_{i} y_i^2 \right) \, d\sigma
\]

\[
= \int_{S^{n-1}} (n - n) \, d\sigma = 0.
\]

This proves the lemma.

Let \( U \) be a bounded open set locally on one side of its boundary having Lipschitz boundary so the divergence theorem holds and let \( B = B(0,R) \) where

\[
B \supseteq U - U \equiv \{ x - y : x \in U, y \in U \}
\]

Let \( f \in C_c^\infty(U) \) and define for \( x \in U \),

\[
u(x) \equiv \int_B \Phi(y) f(x - y) \, dy = \int_U \Phi(x - y) f(y) \, dy.
\]

Let \( h(y) = f(x - y) \). Then since \( \Phi \) is in \( L^1(B) \),

\[
\Delta u(x) = \int_B \Phi(y) \Delta f(x - y) \, dy = \int_B \Phi(y) \Delta h(y) \, dy
\]

\[
= \int_{B \setminus B(0,\epsilon)} \nabla \cdot \left( \nabla h(y) \Phi(y) \right) - \nabla \Phi(y) \cdot \nabla h(y) \, dy
\]

\[
+ \int_{B(0,\epsilon)} \Phi(y) \Delta h(y) \, dy.
\]
The last term converges to 0 as $\varepsilon \to 0$ because $\Phi$ is in $L^1$ and $\Delta h$ is bounded. Since $\text{spt}(h) \subseteq B$, the divergence theorem implies

$$\Delta u(x) = -\int_{\partial B(0,\varepsilon)} \Phi(y) \nabla h(y) \cdot n \, d\sigma - \int_{B\setminus B(0,\varepsilon)} \nabla \Phi(y) \cdot \nabla h(y) \, dy + e(\varepsilon)$$

(30.5.69)

where here and below, $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. The first term in (30.5.69) converges to 0 as $\varepsilon \to 0$ because

$$\int_{\partial B(0,\varepsilon)} \Phi(y) \nabla h(y) \cdot n \, d\sigma \leq \begin{cases} C_{nh} \frac{1}{n-2} \varepsilon^{n-1} = C_{nh} \varepsilon & \text{if } n > 2 \\ C_{h} (\ln \varepsilon) \varepsilon & \text{if } n = 2 \end{cases}$$

and since $\Delta \Phi(y) = 0$,

$$\nabla \Phi(y) \cdot \nabla h(y) = \nabla \cdot (\nabla \Phi(y) h(y)).$$

Consequently

$$\Delta u(x) = -\int_{B\setminus B(0,\varepsilon)} \nabla \cdot (\nabla \Phi(y) h(y)) \, dy + e(\varepsilon).$$

Thus, by the divergence theorem, (30.5.69), and the definition of $h$ above,

$$\Delta u(x) = \int_{\partial B(0,\varepsilon)} f(x-y) \nabla \Phi(y) \cdot n \, d\sigma + e(\varepsilon)$$

$$= \int_{\partial B(0,\varepsilon)} f(x-y) \left(-\frac{y}{a_{n-1} |y|^{n-1}}\right) \left(-\frac{y}{|y|}\right) \, d\sigma + e(\varepsilon)$$

$$= -\int_{\partial B(0,\varepsilon)} f(x-y) \, d\sigma(y) \frac{1}{a_{n-1} \varepsilon^{n-1}} + e(\varepsilon).$$

Letting $\varepsilon \to 0$,

$$-\Delta u(x) = f(x).$$

This proves the following lemma.

**Lemma 30.5.3** Let $U$ be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and let $B \supseteq U - U$ where $B = B(0,R)$. Let $f \in C_0^\infty(U)$. Then for $x \in U$,

$$\int_{B} \Phi(y) f(x-y) \, dy = \int_{U} \Phi(x-y) f(y) \, dy,$$

and it follows that if $u$ is given by one of the above formulas, then for all $x \in U$,

$$-\Delta u(x) = f(x).$$

**Theorem 30.5.4** Let $f \in L^p(U)$. Then there exists $u \in L^p(U)$ whose weak derivatives are also in $L^p(U)$ such that in the sense of weak derivatives,

$$-\Delta u = f.$$
It is given by

\[ u(x) = \int_B \Phi(y) \tilde{f}(x - y) \, dy = \int_U \Phi(x - y) f(y) \, dy \]  

(30.5.70)

where \( \tilde{f} \) denotes the zero extension of \( f \) off of \( U \).

**Proof:** Let \( f \in L^p(U) \) and let \( f_k \in C_c^\infty(U) \), \( \|f_k - f\|_{L^p(U)} \to 0 \), and let \( u_k \) be given by (30.5.70) with \( f_k \) in place of \( f \). Then by Minkowski’s inequality,

\[
\|u - u_k\|_{L^p(U)} = \left( \int_U \left( \int_B \Phi(y) \left| \tilde{f}(x - y) - f_k(x - y) \right| \, dy \right)^p \, dx \right)^{1/p} 
\leq \left( \int_B |\Phi(y)| \left( \int_U \left| \tilde{f}(x - y) - f_k(x - y) \right|^p \, dx \right)^{1/p} \, dy \right) 
\leq \int_B |\Phi(y)| \, dy \|f - f_k\|_{L^p(U)} = C(B) \|f - f_k\|_{L^p(U)}
\]

and so \( u_k \to u \) in \( L^p(U) \). Also

\[
u_{k,i}(x) = \int_U \Phi_{i,j}(x - y) f_k(y) \, dy = \int_B f_k(x - y) \Phi_{i,j}(y) \, dy.
\]

Now let

\[
w_i = \int_B \tilde{f}(x - y) \Phi_{i,j}(y) \, dy. \tag{30.5.71}
\]

and since \( \Phi_{i,j} \in L^1(B) \), it follows from Minkowski’s inequality that

\[
\|u_{k,i} - w_i\|_{L^p(U)} \leq \left( \int_U \left( \int_B \left| f_k(x - y) - \tilde{f}(x - y) \right| |\Phi_{i,j}(y)| \, dy \right)^p \, dx \right)^{1/p} 
\leq \int_B |\Phi_{i,j}(y)| \left( \int_U \left| f_k(x - y) - \tilde{f}(x - y) \right|^p \, dx \right)^{1/p} \, dy 
\leq C(B) \|f_k - f\|_{L^p(U)}
\]

and so \( u_{k,i} \to w_i \) in \( L^p(U) \).

Now let \( \phi \in C_c^\infty(U) \). Then

\[
\int_U w_i \phi \, dx = \lim_{k \to \infty} \int_U u_{k,i} \phi \, dx = -\int_U u \phi \, dx.
\]

Thus \( u_i = w_i \in L^p(\mathbb{R}^n) \) and so if \( \phi \in C_c^\infty(U) \),

\[
\int_U f \phi \, dx = \lim_{k \to \infty} \int_U f_k \phi \, dx = \lim_{k \to \infty} \int_U \nabla u_{k,i} \cdot \nabla \phi \, dx = \int_U \nabla u \cdot \nabla \phi \, dx
\]

and so \(-\Delta u = f\) as claimed. This proves the theorem.
One could also ask whether the second weak partial derivatives of $u$ are in $L^p(U)$. This is where the theory of singular integrals is used. Recall from \[30.5.70\] and \[30.5.71\] along with the argument of the above lemma, that if $u$ is given by \[30.5.70\], then $u_{,i}$ is given by \[30.5.71\] which equals

$$
\int_U \Phi_{,i} (x - y) f (y) \, dy.
$$

**Lemma 30.5.5** Let $f \in L^p(U)$ and let

$$
w_i (x) = \int_U \Phi_{,i} (x - y) f (y) \, dy.
$$

Then $w_{i,j} \in L^p(U)$ for each $j = 1 \cdots n$ and the map $f \to w_{i,j}$ is continuous and linear on $L^p(U)$.

**Proof:** First let $f \in C_\infty^\infty(U)$. For such $f$,

$$
w_i (x) = \int_U \Phi_{,i} (x - y) f (y) \, dy = \int_{\mathbb{R}^n} \Phi_{,i} (x - y) f (y) \, dy
$$

and

$$
w_{i,j} (x) = \int_B \Phi_{,i} (y) f_{,j} (x - y) \, dy
$$

$$
= \int_{B \setminus B(0,\epsilon)} \Phi_{,i} (y) f_{,j} (x - y) \, dy + \int_{B(0,\epsilon)} \Phi_{,i} (y) f_{,j} (x - y) \, dy.
$$

The second term converges to 0 because $f_{,j}$ is bounded and by \[30.5.65\], $\Phi_{,i} \in L^1_{\text{loc}}$. Thus

$$
w_{i,j} (x) = \int_{B \setminus B(0,\epsilon)} \Phi_{,i} (y) f_{,j} (x - y) \, dy + e(\epsilon)
$$

$$
= \int_{B \setminus B(0,\epsilon)} - (\Phi_{,i} (y) f (x - y))_{,j} + \Phi_{,ij} (y) f (x - y) \, dy + e(\epsilon)
$$

where $e(\epsilon) \to 0$ as $\epsilon \to 0$. Using the divergence theorem, this yields

$$
w_{i,j} (x) = \int_{\partial B(0,\epsilon)} \Phi_{,i} (y) f (x - y) n_j \, d\sigma + \int_{B \setminus B(0,\epsilon)} \Phi_{,ij} (y) f (x - y) \, dy + e(\epsilon).
$$

Consider the first term on the right. This term equals, after letting $y = \epsilon z$,

$$
\epsilon^{n-1} \int_{\partial B(0,1)} \Phi_{,i} (\epsilon z) f (x - \epsilon z) n_j \, d\sigma = C_n \epsilon^{n-1} \int_{\partial B(0,1)} \epsilon^{1-n} z_i z_j f (x - \epsilon z) \, d\sigma (z)
$$

$$
= C_n \int_{\partial B(0,1)} z_i z_j f (x - \epsilon z) \, d\sigma (z)
$$
and this converges to 0 if \( i \neq j \) and it converges to

\[
C_n f (x) \int_{\mathcal{B}(0,1)} z_i^2 d\sigma (z)
\]

if \( i = j \). Thus

\[
w_{i,j} (x) = C_n \delta_{ij} f (x) + \int_{\mathcal{B} \setminus \mathcal{B}(0,\varepsilon)} \Phi_{ij} (y) f (x - y) dy + e (\varepsilon).
\]

Letting

\[
\Phi_{ij}^\varepsilon = \begin{cases} 0 & \text{if } |y| < \varepsilon, \\ \Phi_{ij} (y) & \text{if } |y| \geq \varepsilon,
\end{cases}
\]

it follows

\[
w_{i,j} (x) = C_n \delta_{ij} f (x) + \Phi_{ij}^\varepsilon f (x) + e (\varepsilon).
\]

By the theory of singular integrals, there exists a continuous linear map, \( K_{ij} \in \mathcal{L} (L^p (\mathbb{R}^n), L^p (\mathbb{R}^n)) \) such that

\[
K_{ij} f \equiv \lim_{\varepsilon \to 0} \Phi_{ij}^\varepsilon f.
\]

Therefore, letting \( \varepsilon \to 0 \),

\[
w_{i,j} = C_n \delta_{ij} f + K_{ij} \tilde{f}
\]

whenever \( f \in C_c^\infty (U) \).

Now let \( f \in L^p (U) \), let

\[
||f_k - f||_{L^p (U)} \to 0,
\]

where \( f_k \in C_c^\infty (U) \), and let

\[
w_{i}^{k} (x) = \int_{U} \Phi_{i} (x - y) f_k (y) dy.
\]

Then it follows as before that \( w_i^k \to w_i \) in \( L^p (U) \) and

\[
w_{i,j}^k = C_n \delta_{ij} f_k + K_{ij} \tilde{f}_k.
\]

Now let \( \phi \in C_c^\infty (U) \).

\[
w_{i,j} (\phi) = - \int_{U} w_i \phi_j dx = - \lim_{k \to \infty} \int_{U} w_i^k \phi_j dx
\]

\[
= \lim_{k \to \infty} \int_{U} w_{i,j}^k \phi dx = \lim_{k \to \infty} \int_{U} \left( C_n \delta_{ij} \tilde{f}_k + K_{ij} \tilde{f}_k \right) \phi dx
\]

\[
= \int_{U} \left( C_n \delta_{ij} \tilde{f} + K_{ij} \tilde{f} \right) \phi dx.
\]

It follows

\[
w_{i,j} = C_n \delta_{ij} \tilde{f} + K_{ij} \tilde{f}
\]

and this proves the lemma.
30.5. HELMHOLTZ DECOMPOSITIONS

Corollary 30.5.6 In the situation of Theorem 30.5.4, all weak derivatives of \( u \) of order 2 are in \( L^p(U) \) and also \( f \to u_{ij} \) is a continuous map.

Proof: 
\[
u_{ij}(x) = \int_U \Phi_{ij}(x-y) f(y) \, dy
\]
and so \( u_{ij} \in L^p(U) \) and \( f \to u_{ij} \) is continuous by Lemma 30.5.5.

With this preparation, it is possible to consider the Helmholtz decomposition. Let \( F \in L^p(U; \mathbb{R}^n) \) and define
\[
\phi(x) = \int_U \nabla \Phi(x-y) \cdot F(y) \, dy.
\]
(30.5.72)
Then by Lemma 30.5.5,
\[
\phi_{ij} = C_n \bar{F}_j + \sum_i K_{ij} \bar{F}_i \in L^p(\mathbb{R}^n)
\]
and the mapping \( F \to \nabla \phi \) is continuous from \( L^p(U; \mathbb{R}^n) \) to \( L^p(U; \mathbb{R}^n) \).

Now suppose \( F \in C^\infty_c(U; \mathbb{R}^n) \). Then
\[
\phi(x) = \int_U \sum_{i=1}^n -\frac{\partial}{\partial y^i} (\Phi(x-y) F_i(y)) + \Phi(x-y) \nabla \cdot F(y) \, dy
\]
and so by Lemma 30.5.3,
\[
\nabla \cdot \nabla \phi = \Delta \phi = -\nabla \cdot F.
\]

This continues to hold in the sense of weak derivatives if \( F \) is only in \( L^p(U; \mathbb{R}^n) \) because by Minkowski’s inequality and 30.5.72 the map \( F \to \phi \) is continuous. Also note that for \( F \in C^\infty_c(U; \mathbb{R}^n) \),
\[
\phi(x) = \int_B \Phi(y) \nabla \cdot F(x-y) \, dy.
\]

Next define \( \pi : L^p(U; \mathbb{R}^n) \to L^p(U; \mathbb{R}^n) \) by
\[
\pi F = -\nabla \phi, \quad \phi(x) = \int_U \nabla \Phi(x-y) \cdot F(y) \, dy.
\]

It was already shown that \( \pi \) is continuous, linear, and \( \nabla \cdot \pi F = \nabla \cdot F \). It is also true that \( \pi \) is a projection. To see this, let \( F \in C^\infty_c(U; \mathbb{R}^n) \). Then for \( B \) large enough,
\[
\pi^2 F(x) = -\nabla \int_B \Phi(z) \nabla \cdot \pi F(x-z) \, dz
\]
\[
= -\nabla \int_B \Phi(z) \nabla \int_B \Phi(w) \nabla \cdot F(x-z-w) \, dw \, dz
\]
\[
= -\nabla \int_B \Phi(z) \nabla \cdot F(x-z) \, dz = \pi F(x).
\]
Since $\pi$ is continuous and $C^\infty_c(U; \mathbb{R}^n)$ is dense in $L^p(U; \mathbb{R}^n)$, $\pi^2F = \pi F$ for all $F \in L^p(U; \mathbb{R}^n)$. This proves the following theorem which is the Helmholtz decomposition.

**Theorem 30.5.7** There exists a continuous projection

$$\pi : L^p(U; \mathbb{R}^n) \to L^p(U; \mathbb{R}^n)$$

such that $\pi F$ is a gradient and

$$\nabla \cdot (F - \pi F) = 0$$

in the sense of weak derivatives.

Note this theorem shows that any $L^p$ vector field is the sum of a gradient and a part which is divergence free, $F = F - \pi F + \pi F$. 
Chapter 31

Gelfand Triples And Related Stuff

Let $H$ be a separable real Hilbert space and let $V \subseteq H$ be a separable Banach space which is embedded continuously into $H$ and which is also dense in $H$. Then identifying $H$ and $H'$ you can write

$$V \subseteq H = H' \subseteq V'. $$

This is called a Gelfand triple. If $V$ is reflexive, you could conclude separability of $V$ from the separability of $H$. However, if $V$ is not reflexive, this might not happen. For example, you could take $V = L^\infty(0,1)$ and $H = L^2(0,1)$.

**Proposition 31.0.8** Suppose $V$ is reflexive and a subset of $H$ a separable Hilbert space with the inclusion map continuous. Suppose also that $V$ is dense in $H$. Then identifying $H$ and $H'$, it follows that $H$ is dense in $V'$ and $V$ is separable.

**Proof:** If $H$ is not dense in $V'$, then by the Hahn Banach theorem, there exists $\phi^{**} \in V''$ such that $\phi^{**}(H) = 0$ but $\phi^{**}(\phi^*) \neq 0$ for some $\phi^* \in V' \setminus H$. Since $V$ is reflexive there exists $v \in V$ such that $\phi^{***} = Jv$ for $J$ the standard mapping from $V$ to $V''$. Thus

$$\phi^{**}(h) \equiv \langle h, v \rangle \equiv (v, h)_H = 0$$

for all $h \in H$. Therefore, $v = 0$ and so $Jv = 0 = \phi^{**}$ which contradicts $\phi^{**}(\phi^*) \neq 0$. Therefore, $H$ is dense in $V'$. Now by Theorem 19.1.13 which says separability of the dual space implies separability of the space, it follows $V$ is separable as claimed. This proves the proposition.

From now on, it is assumed $V$ and $V'$ are both separable and that $H$ is dense in $V'$. This is summarized in the following definition.

**Definition 31.0.9** $V, H, V'$ will be called a Gelfand triple if $V, V'$ are separable, $V \subseteq H$ with the inclusion map continuous, $H = H'$, and $H = H'$ is dense in $V'$.
What about the Borel sets on $V$ and $H$?

**Proposition 31.0.10** Denote by $\mathcal{B}(X)$ the Borel sets of $X$ where $X$ is any separable Banach space. Then

$$\mathcal{B}(X) = \sigma(X').$$

Here $\sigma(X')$ is the smallest $\sigma$ algebra such that each $\phi \in X'$ is measurable. Also in the context of the above definition, $\mathcal{B}(V) = \sigma(i^* H')$ because $H'$ is dense in $V'$. Here $i^*$ is the restriction to $V$ so that $i^* h(v) \equiv h(v) \equiv (h, v)_H$ for all $v \in V$ and $\sigma(i^* H')$ denotes the smallest $\sigma$ algebra such that $i^* h$ is measurable for each $h \in H'$.

**Proof:** By Lemma 19.1.4 there exists a countable subset of the unit ball in $X'$

$$\{ \phi_n \}_{n=1}^{\infty} = D'$$

such that

$$||v||_X = \sup \{ |\phi(v)| : \phi \in D' \}.$$

Consider a closed ball $B(v_0, r)$ in $X$. This equals

$$\left\{ v \in X : \sup_n |\phi_n(v) - \phi_n(v_0)| \leq r \right\} = \cap_{n=1}^{\infty} \phi_n^{-1}(B(\phi_n(v_0), r))$$

and this last set is in $\sigma(D')$. Therefore, every closed ball is in $\sigma(D')$ which implies every open ball is also in $\sigma(D')$ since open balls are the countable union of closed balls. Since $X$ is separable, it follows every open set is the countable union of balls and so every open set is in $\sigma(D')$. It follows $\mathcal{B}(X) \subseteq \sigma(D') \subseteq \sigma(X')$. On the other hand, every $\phi \in X'$ is continuous and so it is Borel measurable. Hence $\sigma(X') \subseteq \mathcal{B}(X)$.

Now consider the last claim. From Lemma 19.1.4 and density of $H' = H$ in $V'$, it can be assumed $D' \subseteq H = H'$. Therefore, from the first part of the argument

$$\mathcal{B}(V) \subseteq \sigma(D') \subseteq \sigma(i^* H')$$

Also each $i^* h$ is continuous on $V$ so in fact, equality holds in the above because $\sigma(i^* H') \subseteq \mathcal{B}(V)$. This proves the proposition.

Next I want to verify that $V$ is not in $\mathcal{B}(H)$. This will be true if $V$ is reflexive. More generally, here is an interesting result.

**Proposition 31.0.11** Let $X \subseteq Y$, $X$ dense in $Y$ and suppose $X$, $Y$ are Banach spaces and that $X$ is reflexive. Then $X \in \mathcal{B}(Y)$.

**Proof:** Define the functional

$$\phi(x) = \begin{cases} ||x||_X & \text{if } x \in X \\ \infty & \text{if } x \in Y \setminus X \end{cases}$$

Then $\phi$ is lower semicontinuous on $Y$. Here is why. Suppose $(x, a) \notin \text{epi}(\phi)$ so that $a < \phi(x)$. I need to verify this situation persists for $(x, b)$ near $(x, a)$. If this is not so,
there exists $x_n \to x$ and $a_n \to a$ such that $a_n \geq \phi(x_n)$. If \( \liminf_{n \to \infty} \phi(x_n) < \infty \), then there exists a subsequence still denoted by $n$ such that \( \|x_n\|_X \) is bounded. Then by the Eberlein Smulian theorem, there exists a further subsequence such that $x_n$ converges weakly in $X$ to some $z$. Now since $X$ is dense in $Y$ it follows $Y'$ can be considered a subspace of $X'$ and so for $f \in Y'$

$$f(x_n) \to f(z), \ f(x_n) \to f(x)$$

and so \( f(z-x) = 0 \) for all $f \in Y'$ which requires $z = x$. Now $x \to \|x\|_X$ is convex and lower semicontinuous on $X$ so it follows from Corollary of

$$a = \lim \inf_{n \to \infty} a_n \geq \lim \inf_{n \to \infty} \phi(x_n) \geq \phi(x) > a$$

which is a contradiction. If \( \liminf_{n \to \infty} \phi(x_n) = \infty \), then

$$\infty > a = \lim \inf_{n \to \infty} a_n = \infty$$

another contradiction. Therefore, $\text{epi} \phi$ is closed and so $\phi$ is lower semicontinuous as claimed. Therefore,

$$X = Y \setminus \bigcap_{n=1}^{\infty} \phi^{-1}((n, \infty))$$

and since $\phi$ is lower semicontinuous, each $\phi^{-1}((n, \infty))$ is open. Hence $X$ is a Borel subset of $Y$. This proves the proposition.

### 31.1 An Unnatural Example

Recall Gelfand triples are of the form

$$V \subseteq H \subseteq V'$$

where $H$ is a Hilbert space and $V$ is a Banach space contained in $H$ and each of the above inclusions is continuous and each space is dense in the next one. The standard example of a Gelfand triple is $H^1_0(D) \subseteq L^2(D) \subseteq (H^1_0(D))^\prime$ with the convention that $L^2(D)$ is identified with its dual space. Thus for $f \in L^2(D)$, $f$ is considered as something in $(H^1_0(D))^\prime$ according to the rule

$$\langle f, \phi \rangle \equiv (f, \phi)_{L^2(D)}$$

This is a very pleasant thing to contemplate and it is natural and transparent. However, there are other ways to come up with a Gelfand triple which are much more perverse. The following is an example of such a thing along with an application. See [98] and references given there. I think this idea is due to Lions.

First consider the following situation.

$$X \overset{\theta}{\to} Y$$
where \( \theta \) is continuous, linear and one to one and \( X \) is a Banach space. Then 
\( \theta(X) \subseteq Y \) and you could define
\[
\| \theta x \|_{\theta(X)} \equiv \| x \|_X.
\]
Then \( \theta(X) \) can be considered the same thing as \( X \) because \( \theta \) preserves distances and all algebraic properties. Thus people write \( X \subseteq Y \) to save space. In the above simple example, it is obvious what \( \theta \) is. This is because the things in \( H_1^0 \) and things in \( L^2 \) are both functions defined on \( D \) and we can simply take \( \theta \) to be the identity map. However, you might have \( H \) be the dual space of something. Thus it consists of bounded linear transformations defined on some Banach space. Then it becomes necessary to specify the manner in which vectors in \( V \) can be considered as vectors of \( H \).

Let \( \infty > p \geq 2 \). Then letting \( D \) be a bounded open set, \( H_0^1(D) \) embeds continuously into \( L^p(D) \). That is
\[
\| \phi \|_{L^p'} \leq C \| \phi \|_{H_0^1}.
\]
Here \( \frac{1}{p'} + \frac{1}{p} = 1 \). Also note that an equivalent inner product on \( H_0^1(D) \) is
\[
(f,g)_{H_0^1} = \int_D \nabla f \cdot \nabla g dx
\]
Then with respect to this inner product, the Riesz map is given by \( -\Delta \).

\[
-\Delta : H_0^1(D) \rightarrow (H_0^1(D))'
\]
Thus a typical vector of \( (H_0^1(D))' \) is of the form \( -\Delta \phi \) where \( \phi \in H_0^1(D) \) and the following hold.
\[
(\phi,\psi)_{H_0^1} = (-\Delta \phi,\psi), (-\Delta \phi,-\Delta \psi)_{H_0^1'} = (\phi,\psi)_{H_0^1} = \langle -\Delta \psi,\phi \rangle
\]
The following is about the Gelfand triple
\[
V = L^p(D) \subseteq (H_0^1)' \subseteq (L^p(D))'
\]
**Lemma 31.1.1** It is possible to consider \( L^p(D) \equiv V \) as a dense subspace of \( (H_0^1)' \equiv H \) as follows. For \( f \in L^p(D) \) and \( \phi \in H_0^1(D) \),
\[
\langle f, \phi \rangle = \int_D f(x) \phi(x) dx
\]
One can also consider \( H \equiv (H_0^1)' \) as a dense subspace of \( (L^p(D))' \equiv V' \) as follows. For \( -\Delta \phi \in H \) and \( f \in L^p(D) \),
\[
\langle -\Delta \phi, f \rangle = (-\Delta \phi, f)_H \equiv (f, \phi)
\]
\(-\Delta \) maps \( H_0^1(D) \) to \( H \equiv (H_0^1)' \subseteq V' \). \(-\Delta \) can be extended to yield a map \(-\Delta_1 \) from \( L^p(D) \) to \( V' \).
31.1. AN UNNATURAL EXAMPLE

\[ H^1_0(D) \xrightarrow{\Delta} (H^1_0)' \]
\[ L^p(D) = V \xrightarrow{-\Delta} V' \]

**Proof:** First of all, note that by

\[ |\langle f, \phi \rangle| \leq ||f||_{L^p} ||\phi||_{L^p'} \leq C ||f||_{L^p} ||\phi||_{H^1_0} \]

and so it is certainly possible to consider \( L^p \subseteq H \equiv (H^1_0)' \) as just claimed. Now why can \( L^p(D) \) be considered dense in \( H \equiv (H^1_0)' \)? If it isn’t dense, then there exists \( \psi \in H^1_0(D) \), \( \psi \neq 0 \) such that

\[ (-\Delta \psi, f)_H = 0 \]

for all \( f \in L^p(D) \). However, the above would say that for all \( f \in L^p \),

\[ (-\Delta \psi, f)_H = \langle f, \psi \rangle = \int_{D} f \psi = 0 \]

But \( \psi \in L^{p'}(D) \) because \( H^1_0(D) \) embeds continuously into \( L^{p'}(D) \) and so the above holding for all \( f \in L^p(D) \) implies by the usual Riesz representation theorem that \( \psi = 0 \) contrary to the way \( \psi \) was chosen.

Now consider the next claim. For \( -\Delta \phi \in H \equiv (H^1_0)' \) and \( f \in L^p(D) \) and from the first part

\[ |\langle -\Delta \phi, f \rangle| \equiv |(-\Delta \phi, f)_H| = |\langle f, \phi \rangle| \leq C ||f||_{L^p} ||\phi||_{H^1_0(D)} \]

Thus \( -\Delta \phi \in H \) can be considered in \( (L^p(D))' \). Why should \( H \) be dense in \( (L^p(D))' \)? If it is not dense, then there exists \( g^* \in (L^p(D))' \) which is not the limit of vectors of \( H \). Then since \( L^p(D) \) is reflexive, an application of the Hahn Banach theorem shows there exists \( f \in L^p(D) \) such that

\[ \langle g^*, f \rangle_{(L^p(D))', L^p(D)} \neq 0, \langle -\Delta \phi, f \rangle_{(L^p(D))', L^p(D)} = 0 \quad (31.1.2) \]

for all \(-\Delta \phi \in H\). However, it was just shown \( H \) could be considered a subset of \((L^p(D))'\) in the manner described above. Therefore, the last equation in the above is of the form

\[ 0 = (-\Delta \phi, f)_H = \langle f, \phi \rangle = \int_{D} f \phi dx \]

and since this holds for all \( \phi \in H^1_0(D) \), it follows by density of \( H^1_0(D) \) in \( L^{p'}(D) \), that \( f = 0 \) and now this contradicts the inequality in \( 31.1.2 \).

Now \( \Delta \) is defined on \( H^1_0(D) \) and it delivers something in \( (H^1_0)' \equiv H \). Of course \( H^1_0(D) \) is dense in \( L^{p'}(D) \). Can \( \Delta \) be extended to all of \( L^{p'}(D) \)? The answer is yes
and it is more of the same given above. For \( \phi \in H^1_0(D) \), \( -\Delta \phi \in H \subseteq (L^p(D))' \). Then by the above, for \( \phi \in H^1_0(D) \) and \( f \in L^p(D) \),

\[
\langle -\Delta \phi, f \rangle \equiv \langle f, \phi \rangle \equiv \int_D f \phi dx
\]

\[
|\langle -\Delta \phi, f \rangle| \equiv |\langle f, \phi \rangle| \equiv \left| \int_D f \phi ds \right| \leq ||\phi||_{L^p(D)} ||f||_{L^p(D)}
\]

and so \( -\Delta \) is a continuous linear mapping defined on a dense subspace \( H^1_0(D) \) of \( L^p(D) \) and so this does indeed extend to a continuous linear map defined on all of \( L^p(D) \) given by the formula

\[
\langle -\Delta g, f \rangle \equiv \int_D fg dx
\]

This proves the lemma.

Thus letting \( V \equiv L^p(D) \), and \( H \equiv (H^1_0(D))' \), it follows \( V \subseteq H \subseteq V' \) is a Gelfand triple with the understanding of what it means for one space to be included in another described above. To emphasize the above, for \( -\Delta \phi \in H, f \in L^p \),

\[
\langle -\Delta \phi, f \rangle \equiv \langle -\Delta \phi, f \rangle_H \equiv \langle f, \phi \rangle \equiv \int_D f \phi dx
\]

More generally, for \( g \in L^{p'}(D), -\Delta g \in (L^p(D))' \) according to the rule

\[
\langle -\Delta g, f \rangle \equiv \int_D fg dx.
\]

With this example of a Gelfand triple, one can define a “porous medium operator” \( A : V \rightarrow V' \). Let \( \Psi \) be a real valued function defined on \( \mathbb{R} \) which satisfies

\[
\Psi \text{ is continuous} \quad (31.1.3)
\]

\[
(t - s) (\Psi(t) - \Psi(s)) \geq 0 \quad (31.1.4)
\]

There exists \( p \geq 2, p < \infty \) and \( \alpha \in (0, \infty) \) such that for all \( s \in \mathbb{R} \)

\[
s \Psi(s) \geq \alpha |s|^p - c \quad (31.1.5)
\]

There exist \( c_3, c_4 \in (0, \infty) \) such that for all \( s \in \mathbb{R} \)

\[
|\Psi(s)| \leq c_4 + c_3 |s|^{p-1} \quad (31.1.6)
\]

Note that \( 31.1.5 \) implies that if \( v \in L^p(D) \), then

\[
\int_D |\Psi(v)|^{p'} dx \leq C \int_D \left( 1 + |v|^{p(p-1)} \right) dx = C \int_D (1 + |v|^p) dx < \infty.
\]
Thus for \( v \in L^p(D) \), \( \Psi(v) \) is something you can do \( \Delta \) to and obtain something in \( V' \). The porous medium operator \( A : V \to V' \) is given as follows.

\[
\langle Av, w \rangle_{V', V} \equiv \langle \Delta \Psi(v), w \rangle_{V', V} \equiv -\int_D \Psi(v) \, w \, dx
\]

What are the properties of \( A \)?

\[
\langle A(u + \lambda v), w \rangle \equiv -\int_D \Psi(u + \lambda v) \, w \, dx
\]

and this is easily seen to be a continuous function of \( \lambda \). Thus \( A \) is Hemicontinuous.

\[
\langle A(u) - A(v), u - v \rangle \equiv -\int_D \Psi(u) (u - v) \, dx + \int_D \Psi(v) (u - v) \, dx \leq 0
\]

Thus \( -A \) is monotone. Also there is a coercivity estimate which is routine.

\[
\langle A(v), v \rangle \equiv -\int_D \Psi(v) \, v \leq \int_D c - \alpha |v|^p \, dx = C - \alpha \|v\|_V^p
\]

This operator also has a boundedness estimate.

\[
\|A(v)\|_{V'} \equiv \sup_{|w|_V \leq 1} |\langle A(v), w \rangle| \equiv \sup_{|w|_V \leq 1} \left| \int_D \Psi(v) \, w \right| \\
\leq \sup_{|w|_V \leq 1} \left( \int_D \left( c_4 + c_3 |v|^{p-1} \right) w \, dx \right) \\
\leq \left( \int_D C \left( 1 + |v|^p \right) \, dx \right)^{1/p'} \leq C + C \left( \int_D |v|^p \, dx \right)^{1/p'} \\
= C + C \|v\|_V^{p/p'} = C + C \|v\|_V^{p-1}.
\]

Since \( \Psi \) is continuous, it will also follow that \( A \) is \( B(V) \) measurable. Consider

\[
u \to \langle Au, w \rangle \equiv -\int_D \Psi(u) \, w \, dx
\]

for fixed \( w \in V \). Suppose \( u_n \to u \) in \( V \) and fix \( w \in L^\infty(D) \subseteq V \). Then it follows from an easy argument using the Vitali convergence theorem and the fact that from the estimates above

\[
\Psi(u_n) \to -\int_D \Psi(u) \, w \, dx
\]

is uniformly integrable that

\[
u \to -\int_D \Psi(u) \, w \, dx
\]
CHAPTER 31. GELFAND TRIPLES AND RELATED STUFF

is continuous. For general \( w \in L^p(D) \), let \( w_n \to w \) in \( L^p(D) \) where each \( w_n \) is in \( L^\infty(D) \). Then the function

\[
\Psi(u) \equiv \langle Au, w \rangle
\]

is the limit of the continuous functions

\[
u \to \int_D \Psi(u) \, w dx
\]

and so the function \( 31.1.7 \) is Borel measurable. Now by the Pettis theorem this shows \( A : V \to V' \) is \( B(V) \) measurable. This shows \( A \) is an example of an operator which satisfies some conditions which will be considered later.

31.2 Standard Techniques In Evolution Equations

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

**Lemma 31.2.1** Suppose \( g \in L^1([a,b];X) \) where \( X \) is a Banach space. Then if

\[
\int_a^b g(t) \phi(t) \, dt = 0 \quad \text{for all } \phi \in C^\infty_c(a,b),
\]

then \( g(t) = 0 \) a.e.

**Proof:** Let \( S \) be a measurable subset of \((a,b)\) and let \( K \subseteq S \subseteq V \subseteq (a,b) \) where \( K \) is compact, \( V \) is open and \( m(V \setminus K) < \varepsilon \). Let \( K \prec h \prec V \) as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging \( K \) slightly and convolving with a mollifier, it can be assumed \( h \in C^\infty_c(a,b) \). Then

\[
\left| \int_a^b \mathcal{X}_S(t) \, g(t) \, dt \right| = \left| \int_a^b (\mathcal{X}_S(t) - h(t)) \, g(t) \, dt \right|
\]

\[
\leq \int_a^b |\mathcal{X}_S(t) - h(t)| \, ||g(t)|| \, dt
\]

\[
\leq \int_{V \setminus K} ||g(t)|| \, dt.
\]

Now let \( K_n \subseteq S \subseteq V_n \) with \( m(V_n \setminus K_n) < 2^{-n} \). Then from the above,

\[
\left| \int_a^b \mathcal{X}_S(t) \, g(t) \, dt \right| \leq \int_a^b \mathcal{X}_{V_n \setminus K_n}(t) \, ||g(t)|| \, dt
\]

and the integrand of the last integral converges to 0 a.e. as \( n \to \infty \) because \( \sum_n m(V_n \setminus K_n) < \infty \). By the dominated convergence theorem, this last integral converges to 0. Therefore, whenever \( S \subseteq (a,b) \),

\[
\int_a^b \mathcal{X}_S(t) \, g(t) \, dt = 0.
\]
31.2. STANDARD TECHNIQUES IN EVOLUTION EQUATIONS

Since the endpoints have measure zero, it also follows that for any measurable \( S \), the above equation holds.

Now \( g \in L^1([a, b]; X) \) and so it is measurable. Therefore, \( g([a, b]) \) is separable. Let \( D \) be a countable dense subset and let \( E \) denote the set of linear combinations of the form \( \sum a_i d_i \) where \( a_i \) is a rational point of \( F \) and \( d_i \in D \). Thus \( E \) is countable. Denote by \( Y \) the closure of \( E \) in \( X \). Thus \( Y \) is a separable closed subspace of \( X \) which contains all the values of \( g \).

Now let \( S_n \equiv g^{-1}(B(y_n, ||y_n||/2)) \) where \( E = \{ y_n \}_{n=1}^{\infty} \). Therefore, \( \cup_n S_n = g^{-1}(X \setminus \{0\}) \). This follows because if \( x \in Y \) and \( x \neq 0 \), then in \( B(x, \frac{||x||}{4}) \) there is a point of \( E, y_n \). Therefore, \( ||y_n|| > \frac{3}{4} ||x|| \) and so \( \frac{||y_n||}{4} > \frac{3}{8} ||x|| > \frac{||x||}{4} \) so \( x \in B(y_n, ||y_n||/2) \). It follows that if each \( S_n \) has measure zero, then \( g(t) = 0 \) for a.e. \( t \). Suppose then that for some \( n \), the set, \( S_n \), has positive measure. Then from what was shown above,

\[
||y_n|| = \left| \frac{1}{m(S_n)} \int_{S_n} g(t) dt - y_n \right| = \left| \frac{1}{m(S_n)} \int_{S_n} g(t) dt - y_n dt \right| \\
\leq \frac{1}{m(S_n)} \int_{S_n} ||g(t) - y_n|| dt \leq \frac{1}{m(S_n)} \int_{S_n} ||y_n||/2 dt = ||y_n||/2
\]

and so \( y_n = 0 \) which implies \( S_n = \emptyset \), a contradiction to \( m(S_n) > 0 \). This contradiction shows each \( S_n \) has measure zero and so as just explained, \( g(t) = 0 \) a.e.

\[\square\]

**Definition 31.2.2** For \( f \in L^1(a, b; X) \), define an extension, \( \overline{f} \) defined on \([2a - b, 2b - a] = [a - (b - a), b + (b - a)]\) as follows.

\[
\overline{f}(t) = \begin{cases} f(t) & \text{if } t \in [a, b] \\ f(2a - t) & \text{if } t \in [2a - b, a) \\ f(2b - t) & \text{if } t \in [b, 2b - a] \end{cases}
\]

**Definition 31.2.3** Also if \( f \in L^p(a, b; X) \) and \( h > 0 \), define for \( t \in [a, b] \), \( f_h(t) = \overline{f}(t - h) \) for all \( h < b - a \). Thus the map \( f \to f_h \) is continuous and linear on \( L^p(a, b; X) \). It is continuous because

\[
\int_a^b ||f_h(t)||^p dt = \int_a^{a+h} ||f(2a - t + h)||^p dt + \int_{a+h}^{b+h} ||f(t)||^p dt = \int_a^{a+h} ||f(t)||^p dt + \int_a^b ||f(t)||^p dt \leq 2||f||_p^p.
\]

The following lemma is on continuity of translation in \( L^p(a, b; X) \).

**Lemma 31.2.4** Let \( \overline{f} \) be as defined in Definition 31.2.2. Then for \( f \in L^p(a, b; X) \) for \( p \in [1, \infty) \),

\[
\lim_{\delta \to 0} \int_a^b ||\overline{f}(t - \delta) - f(t)||_X^p dt = 0.
\]
Proof: Regarding the measure space as \((a, b)\) with Lebesgue measure, by regularity of the measure, there exists \(g \in C_c(a, b; X)\) such that \(\|f - g\|_p < \varepsilon\). Here the norm is the norm in \(L^p(a, b; X)\). Therefore,

\[
\|f_h - f\|_p \leq \|f_h - g_h\|_p + \|g_h - g\|_p + \|g - f\|_p
\leq \left(\frac{2^{1/p} + 1}{2^{1/p} + 1}\right) \|f - g\|_p + \|g_h - g\|_p
\leq \left(\frac{2^{1/p} + 1}{2^{1/p} + 1}\right) \varepsilon + \varepsilon
\]

whenever \(h\) is sufficiently small. This is because of the uniform continuity of \(g\). Therefore, since \(\varepsilon > 0\) is arbitrary, this proves the lemma.

Definition 31.2.5 Let \(f \in L^1(a, b; X)\). Then the distributional derivative in the sense of \(X\) valued distributions is given by

\[
f' \phi \equiv -\int_a^b f(t) \phi'(t) dt
\]

Then \(f' \in L^1(a, b; X)\) if there exists \(h \in L^1(a, b; X)\) such that for all \(\phi \in C_c^\infty(a, b)\),

\[
f' \phi = \int_a^b h(t) \phi(t) dt.
\]

Then \(f'\) is defined to equal \(h\). Here \(f\) and \(f'\) are considered as vector valued distributions in the same way as was done for scalar valued functions.

Lemma 31.2.6 The above definition is well defined.

Proof: Suppose both \(h\) and \(g\) work in the definition for \(f'\). Then for all \(\phi \in C_c^\infty(a, b)\),

\[
\int_a^b (h(t) - g(t)) \phi(t) dt = 0.
\]

Therefore, by Lemma 31.2.1, \(h(t) - g(t) = 0\) a.e. □

The other thing to notice about this is the following lemma. It follows immediately from the definition.

Lemma 31.2.7 Suppose \(f, f' \in L^1(a, b; X)\). Then if \([c, d] \subseteq [a, b]\), it follows that \((f|_{[c, d]})' = f'|_{[c, d]}\). This notation means the restriction to \([c, d]\).

Recall that in the case of scalar valued functions, if you had both \(f\) and its weak derivative, \(f'\) in \(L^1(a, b)\), then you were able to conclude that \(f\) is almost everywhere equal to a continuous function, still denoted by \(f\) and

\[
f(t) = f(a) + \int_a^t f'(s) ds.
\]
In particular, you can define \( f(a) \) to be the initial value of this continuous function. It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if \( f' = 0 \) where the derivative is taken in the sense of \( X \) valued distributions, then \( f \) equals a constant.

**Lemma 31.2.8** Suppose \( f \in L^1(a,b;X) \) and for all \( \phi \in C_c^\infty(a,b) \),

\[
\int_a^b f(t) \phi'(t) \, dt = 0.
\]

Then there exists a constant, \( a \in X \) such that \( f(t) = a \) a.e.

**Proof:** Let \( \phi_0 \in C_c^\infty(a,b) \), \( \int_a^b \phi_0(x) \, dx = 1 \) and define for \( \phi \in C_c^\infty(a,b) \)

\[
\psi_\phi(x) = \int_a^x \left[ \phi(t) - \left( \int_a^b \phi(y) \, dy \right) \phi_0(t) \right] dt.
\]

Then \( \psi_\phi \in C_c^\infty(a,b) \) and \( \psi_\phi' = \phi - \left( \int_a^b \phi(y) \, dy \right) \phi_0 \). Then

\[
\int_a^b f(t) \phi'(t) \, dt = \int_a^b f(t) \left( \psi_\phi'(t) + \left( \int_a^b \phi(y) \, dy \right) \phi_0(t) \right) dt
\]

\[
= 0 \text{ by assumption}
\]

\[
= \int_a^b f(t) \psi_\phi'(t) \, dt + \left( \int_a^b \phi(y) \, dy \right) \int_a^b f(t) \phi_0(t) \, dt
\]

\[
= \left( \int_a^b \left( \int_a^b f(t) \phi_0(t) \, dt \right) \phi(y) \, dy \right).
\]

It follows that for all \( \phi \in C_c^\infty(a,b) \),

\[
\int_a^b \left( f(y) - \left( \int_a^b f(t) \phi_0(t) \, dt \right) \right) \phi(y) \, dy = 0
\]

and so by Lemma 31.2.1,

\[
f(y) - \left( \int_a^b f(t) \phi_0(t) \, dt \right) = 0 \text{ a.e. } y \]

**Theorem 31.2.9** Suppose \( f, f' \) both are in \( L^1(a,b;X) \) where the derivative is taken in the sense of \( X \) valued distributions. Then there exists a unique point of \( X \), denoted by \( f(a) \) such that the following formula holds a.e. \( t \).

\[
f(t) = f(a) + \int_a^t f'(s) \, ds
\]
Proof:
\[ \int_a^b \left( f(t) - \int_a^t f'(s) \, ds \right) \phi'(t) \, dt = \int_a^b f(t) \phi'(t) \, dt - \int_a^b \int_a^t f'(s) \phi'(t) \, ds \, dt. \]

Now consider \( \int_a^b \int_a^t f'(s) \phi'(t) \, ds \, dt \). Let \( \Lambda \in X' \). Then it is routine from approximating \( f' \) with simple functions to verify
\[ \Lambda \left( \int_a^b \int_a^t f'(s) \phi'(t) \, ds \, dt \right) = \int_a^b \int_a^b \Lambda (f'(s)) \phi'(t) \, ds \, dt. \]

Now the ordinary Fubini theorem can be applied to obtain
\[ = \int_a^b \int_s^b \Lambda (f'(s)) \phi'(t) \, dt \, ds = \Lambda \left( \int_a^b \int_s^b f'(s) \phi'(t) \, dt \, ds \right). \]

Since \( X' \) separates the points of \( X \), it follows
\[ \int_a^b \int_a^t f'(s) \phi'(t) \, ds \, dt = \int_a^b \int_s^b f'(s) \phi'(t) \, dt \, ds. \]

Therefore,
\[ \int_a^b \left( f(t) - \int_a^t f'(s) \, ds \right) \phi'(t) \, dt \]
\[ = \int_a^b f(t) \phi'(t) \, dt - \int_a^b \int_s^b f'(s) \phi'(t) \, dt \, ds \]
\[ = \int_a^b f(t) \phi'(t) \, dt - \int_a^b f'(s) \int_s^b \phi'(t) \, dt \, ds \]
\[ = \int_a^b f(t) \phi'(t) \, dt + \int_a^b f'(s) \phi(s) \, ds = 0. \]

Therefore, by Lemma 31.2.8, there exists a constant, denoted as \( f(a) \) such that
\[ f(t) - \int_a^t f'(s) \, ds = f(a) \]

There is also a useful theorem about continuity of pointwise evaluation.

**Corollary 31.2.10** Let \( f, f' \in L^1(a,b; X) \) so that
\[ f(t) = f(0) + \int_0^t f'(s) \, ds \] (31.2.8)

where in this formula, \( t \to f(t) \) is the continuous representative of \( f \). Then there exists a constant \( C \) such that for each \( t \in [a,b] \),
\[ \| f(t) \|_X \leq C \left( \| f \|_{L^1(a,b; X)} + \| f' \|_{L^1(a,b; X)} \right) \]
31.2. STANDARD TECHNIQUES IN EVOLUTION EQUATIONS

**Proof:** From the integral equation

\[ f(t) = f(s) + \int_s^t f'(r) \, dr \]

\[ \|f(t)\|_X \leq \|f(s)\|_X + \left| \int_s^t \|f'(r)\|_X \, dr \right| \]

\[ \leq \|f(s)\|_X + \int_a^b \|f'(r)\|_X \, dr \]

and so, integrating both sides with respect to \( s \)

\[ (b - a) \|f(t)\|_X \leq \|f\|_{L^1(a,b;X)} + (b - a) \|f'\|_{L^1(a,b;X)} \]

and so

\[ \|f(t)\|_X \leq \left( \frac{1}{b - a} + 1 \right) \left( \|f\|_{L^1(a,b;X)} + \|f'\|_{L^1(a,b;X)} \right) \]

Let \( X \) be the space of functions \( f \in L^1(a, b; X) \) such that their weak derivatives \( f' \) are also in \( L^1(a, b; X) \). Then \( X \) is a Banach space with norm given by

\[ \|f\|_X \equiv \|f\|_{L^1(a,b;X)} + \|f'\|_{L^1(a,b;X)} \]

This is because the map \( f \rightarrow f' \) is a closed map. If \( f_n \rightarrow f \) in \( L^1(a, b; X) \) and \( f'_n \rightarrow \xi \) in \( L^1(a, b; X) \), then for \( \phi \in C_c^\infty(a, b) \),

\[ \int_a^b \xi \phi \, dt = \lim_{n \to \infty} \int_a^b f'_n \phi \, dt = \lim_{n \to \infty} - \int_a^b f_n \phi' \, dt = - \int_a^b f \phi' \, dt \]

showing that \( \xi = f' \). Thus if you have a Cauchy sequence in \( X, \{f_n\} \), then \( f_n \rightarrow f \) in \( L^1(a, b; X) \) and \( f'_n \rightarrow \xi \in L^1(a, b; X) \) for some \( \xi \). Hence \( f' = \xi \).

Then the above corollary says that pointwise evaluation is continuous as a map from \( X \) to \( X \). This is clearly a linear map. Also the formula obtained shows that in fact, this is continuous into \( C([a,b]; X) \).

\[ \|f\|_{C([a,b];X)} = \sup_{t \in [a,b]} \|f(t)\|_X \leq C \left( \|f\|_{L^1(a,b;X)} + \|f'\|_{L^1(a,b;X)} \right) = C \|f\|_X. \]

Now let \( \theta : X \rightarrow C([a,b]; X) \) be given by \( \theta f(t) \equiv f(t) \) where \( f(t) = f(0) + \int_0^t f'(s) \, ds \), \( f \) being the continuous representative of \( f \). Then \( \theta \) is continuous and linear. If \( \theta f \equiv f(t) \) so that it is pointwise evaluation at \( t \), then this \( \theta_t \) is also continuous and linear. Suppose \( X \) is also reflexive. It follows that if you have a sequence in \( X, \{f_n\} \) which is converging weakly to \( f \in X \), then you would also have \( \theta_t f_n = f_n(t) \rightarrow \theta_t f \equiv f(t) \) weakly in \( X \). If this is not so, then since \( X \) is reflexive, there is a subsequence, still denoted as \( f_n \) such that \( f_n(t) \rightarrow \xi \neq f(t) \). However, this says that \( (f, \xi) \) is in the weak closure of the graph of \( \theta_t \). Since this graph is strongly closed and convex, it is also weakly closed and hence \( \xi = \theta_t f \equiv f(t) \), a contradiction. This proves the following nice corollary.
Corollary 31.2.11 Suppose $f_n \to f$ weakly in $X$ where we assume also that $X$ is reflexive. Then $f_n (t) \to f(t)$ weakly in $X$.

The integration by parts formula is also important.

Corollary 31.2.12 Suppose $f, f' \in L^1 (a, b; X)$ and suppose $\phi \in C^1 ([a, b])$. Then the following integration by parts formula holds.

$$\int_a^b f(t) \phi'(t) \, dt = f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) \, dt.$$

Proof: From Theorem 31.2.9

$$\int_a^b f(t) \phi'(t) \, dt$$
$$= \int_a^b \left( f(a) + \int_a^t f'(s) \, ds \right) \phi'(t) \, dt$$
$$= f(a) (\phi(b) - \phi(a)) + \int_a^b \int_a^t f'(s) \, ds \phi'(t) \, dt$$
$$= f(a) (\phi(b) - \phi(a)) + \int_a^b f'(s) \int_s^b \phi'(t) \, dt \, ds$$
$$= f(a) (\phi(b) - \phi(a)) + \int_a^b f'(s) (\phi(b) - \phi(s)) \, ds$$
$$= f(a) (\phi(b) - \phi(a)) - \int_a^b f'(s) \phi(s) \, ds + (f(b) - f(a)) \phi(b)$$
$$= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(s) \phi(s) \, ds.$$

The interchange in order of integration is justified as in the proof of Theorem 31.2.9.

With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define $\overline{f}$ as in Definition 31.2.2.

Lemma 31.2.13 Let $\overline{f}$ be given in Definition 31.2.2 and suppose $f, f' \in L^1 (a, b; X)$. Then $\overline{f}, \overline{f}' \in L^1 (2a - b, 2b - a; X)$ also and

$$\overline{f}'(t) \equiv \begin{cases} f'(t) & \text{if } t \in [a, b] \\ -f'(2a - t) & \text{if } t \in [2a - b, a] \\ -f'(2b - t) & \text{if } t \in [b, 2b - a] \end{cases} \quad (31.2.9)$$

Proof: It is clear from the definition of $\overline{f}$ that $\overline{f} \in L^1 (2a - b, 2b - a; X)$ and that in fact

$$\|\overline{f}\|_{L^1 (2a - b, 2b - a; X)} \leq 3 \|f\|_{L^1 (a, b; X)} \quad (31.2.10)$$
Let $\phi \in C_c^\infty (2a - b, 2b - a)$. Then from the integration by parts formula,
\[
\int_{2a-b}^{2b-a} \bar{f}(t) \phi'(t) \, dt \\
= \int_a^b f(t) \phi'(t) \, dt + \int_{2b-a}^{2b-a} f(2b - t) \phi'(t) \, dt + \int_{2a-b}^a f(2a - t) \phi'(t) \, dt \\
= \int_a^b f(t) \phi'(t) \, dt + \int_a^b f(u) \phi'(2b - u) \, du + \int_{2a-b}^b f(2a - u) \phi'(t) \, du \\
= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) \, dt - f(b) \phi(b) + f(a) \phi(2b - a) \\
+ \int_a^b f'(u) \phi(2b - u) \, du - f(b) \phi(2a - b) \\
+ f(a) \phi(a) + \int_a^b f'(u) \phi(2a - u) \, du \\
= - \int_a^b f'(t) \phi(t) \, dt + \int_a^b f'(u) \phi(2b - u) \, du + \int_a^b f'(u) \phi(2a - u) \, du \\
= - \int_a^b f'(t) \phi(t) \, dt - \int_{2b-a}^{2b-a} f'(2b - t) \phi(t) \, dt - \int_{2a-b}^a f'(2a - t) \phi(t) \, dt \\
= - \int_{2a-b}^{2b-a} \bar{f}(t) \phi(t) \, dt
\]
where $\bar{f}'(t)$ is given in [12.4A].

**Definition 31.2.14** Let $V$ be a Banach space and let $H$ be a Hilbert space. (Typically $H = L^2(\Omega)$.) Suppose $V \subseteq H$ is dense in $H$ meaning that the closure in $H$ of $V$ gives $H$. Then it is often the case that $H$ is identified with its dual space, and then because of the density of $V$ in $H$, it is possible to write

$V \subseteq H = H' \subseteq V'$

When this is done, $H$ is called a pivot space. Another notation which is often used is $\langle f, g \rangle$ to denote $f(g)$ for $f \in V'$ and $g \in V$. This may also be written as $\langle f, g \rangle_{V', V}$. Another term is that $V \subseteq H = H' \subseteq V'$ is called a Gelfand triple.

The next theorem is an example of a trace theorem. In this theorem, $f \in L^p (0, T; V)$ while $f' \in L^p (0, T; V')$. It makes no sense to consider the initial values of $f$ in $V$ because it is not even continuous with values in $V$. However, because of the derivative of $f$ it will turn out that $f$ is continuous with values in a larger space and so it makes sense to consider initial values of $f$ in this other space. This other space is called a trace space.

**Theorem 31.2.15** Let $V$ and $H$ be a Banach space and Hilbert space as described in Definition [12.7A]. Suppose $f \in L^p (0, T; V)$ and $f' \in L^p (0, T; V')$. Then $f$ is
a.e. equal to a continuous function mapping \([0, T]\) to \(H\). Furthermore, there exists \(f(0) \in H\) such that
\[
\frac{1}{2} |f(t)|_H^2 - \frac{1}{2} |f(0)|_H^2 = \int_0^t \langle f'(s), f(s) \rangle \, ds,
\]
(31.2.11)
and for all \(t \in [0, T]\),
\[
\int_0^t f'(s) \, ds \in H,
\]
(31.2.12)
and for a.e. \(t \in [0, T]\),
\[
f(t) = f(0) + \int_0^t f'(s) \, ds \text{ in } H,
\]
(31.2.13)
Here \(f'\) is being taken in the sense of \(V'\) valued distributions and \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(p \geq 2\).

**Proof:** Let \(\Psi \in C_c^\infty (-T, 2T)\) satisfy \(\Psi (t) = 1\) if \(t \in [-T/2, 3T/2]\) and \(\Psi (t) \geq 0\). For \(t \in \mathbb{R}\), define
\[
\hat{f}(t) = \begin{cases} \tilde{f}(t) \Psi (t) & \text{if } t \in [-T, 2T] \\ 0 & \text{if } t \notin [-T, 2T] \end{cases}
\]
and
\[
f_n(t) \equiv \int_{-1/n}^{1/n} \hat{f}(t-s) \phi_n(s) \, ds \tag{31.2.14}
\]
where \(\phi_n\) is a mollifier having support in \((-1/n, 1/n)\). Then by Minkowski’s inequality
\[
\left\| f_n - \hat{f} \right\|_{L^p(\mathbb{R}; V)} = \left( \int_{\mathbb{R}} \left\| \int_{-1/n}^{1/n} \hat{f}(t-s) \phi_n(s) \, ds \right\|_V^p \, dt \right)^{1/p}
\]
\[
= \left( \int_{\mathbb{R}} \left[ \int_{-1/n}^{1/n} \left( \hat{f}(t) - \hat{f}(t-s) \right) \phi_n(s) \, ds \right]_V^p \, dt \right)^{1/p}
\]
\[
\leq \left( \int_{\mathbb{R}} \left[ \int_{-1/n}^{1/n} \left( \hat{f}(t) - \hat{f}(t-s) \right) \phi_n(s) \, ds \right]_V^p \, dt \right)^{1/p}
\]
\[
\leq \int_{-1/n}^{1/n} \phi_n(s) \left( \int_{\mathbb{R}} \left\| \hat{f}(t) - \hat{f}(t-s) \right\|_V^p \, dt \right)^{1/p} \, ds
\]
\[
\leq \int_{-1/n}^{1/n} \phi_n(s) \, ds = \varepsilon
\]
provided \(n\) is large enough. This follows from continuity of translation in \(L^p\) with Lebesgue measure. Since \(\varepsilon > 0\) is arbitrary, it follows \(f_n \to f\) in \(L^p (\mathbb{R}; V)\). Similarly,
\[ f_n \to f \text{ in } L^p(\mathbb{R}; H). \] This follows because \( p \geq 2 \) and the norm in \( V \) and norm in \( H \) are related by \( |x|_H \leq C||x||_V \) for some constant, \( C \). Now
\[
\tilde{f}(t) = \begin{cases} \Psi(t)f(t) & \text{if } t \in [0, T], \\ \Psi(t)(2T-t) & \text{if } t \in [T, 2T], \\ \Psi(t)f(-t) & \text{if } t \in [0, T], \\ 0 & \text{if } t \notin [-T, 2T]. \end{cases}
\]

An easy modification of the argument of Lemma \[\text{Lemma}\] yields
\[
\tilde{f}'(t) = \begin{cases} \Psi'(t)f(t) + \Psi'(y)f'(t) & \text{if } t \in [0, T], \\ \Psi'(t)(2T-t) - \Psi(t)f'(2T-t) & \text{if } t \in [T, 2T], \\ \Psi'(t)f(-t) - \Psi(t)f'(-t) & \text{if } t \in [-T, 0], \\ 0 & \text{if } t \notin [-T, 2T]. \end{cases}
\]

Recall
\[
f_n(t) = \int_{-1/n}^{1/n} \tilde{f}(t-s)\phi_n(s)\,ds = \int_{-1/n}^{1/n} \tilde{f}(t)\phi_n(t-s)\,ds.
\]

Therefore,
\[
f_n'(t) = \int_{-1/n}^{1/n} \tilde{f}'(t-s)\phi_n(s)\,ds = \int_{-1/n}^{1/n} \tilde{f}'(t)\phi_n(t-s)\,ds = \int_{-1/n}^{1/n} \tilde{f}'(t)\phi_n(t-s)\,ds
\]
and it follows from the first line above that \( f_n' \) is continuous with values in \( V \) for all \( t \in \mathbb{R} \). Also note that both \( f_n' \) and \( f_n \) equal zero if \( t \notin [-T, 2T] \) whenever \( n \) is large enough. Exactly similar reasoning to the above shows that \( f_n' \to \tilde{f}' \) in \( L^p(\mathbb{R}; V') \).

Now let \( \phi \in C_\infty^0(0, T) \).
\[
\int_{\mathbb{R}} |f_n(t)|_H^2 \phi'(t)\,dt = \int_{\mathbb{R}} (f_n(t), f_n(t))_H \phi'(t)\,dt \quad (31.2.15)
\]
\[
= -\int_{\mathbb{R}} 2 (f_n'(t), f_n(t)) \phi(t)\,dt = -\int_{\mathbb{R}} 2 (f_n'(t), f_n(t)) \phi(t)\,dt
\]

Now
\[
\left| \int_{\mathbb{R}} (f_n'(t), f_n(t)) \phi(t)\,dt - \int_{\mathbb{R}} (f'(t), f(t)) \phi(t)\,dt \right| \leq \int_{\mathbb{R}} (\|f_n'(t) - f'(t), f_n(t)\| + \|f'(t), f_n(t) - f(t)\|) \phi(t)\,dt.
\]
From the first part of this proof which showed that \( f_n \to \hat{f} \) in \( L^p (\mathbb{R}; V) \) and \( f'_n \to \hat{f}' \) in \( L^{p'} (\mathbb{R}; V') \), an application of H"older’s inequality shows the above converges to 0 as \( n \to \infty \). Therefore, passing to the limit as \( n \to \infty \) in the expression
\[
\int_{\mathbb{R}} \left| \hat{f} (t) \right|^2_H \phi' (t) \, dt = - \int_{\mathbb{R}} 2 \left\langle \hat{f}' (t), \hat{f} (t) \right\rangle \phi (t) \, dt
\]
which shows \( t \to \left| \hat{f} (t) \right|^2_H \) equals a continuous function a.e. and it also has a weak derivative equal to \( 2 \left\langle \hat{f}', \hat{f} \right\rangle \).

It remains to verify that \( \hat{f} \) is continuous on \([0, T]\). Of course \( \hat{f} = f \) on this interval. Let \( N \) be large enough that \( f_n (\cdot T) = 0 \) for all \( n > N \). Then for \( m, n > N \) and \( t \in [-T, 2T] \)
\[
\left| f_n (t) - f_m (t) \right|^2_H = 2 \int_{-T}^{T} (f'_n(s) - f'_m(s), f_n(s) - f_m(s)) \, ds \\
= 2 \int_{-T}^{T} (f'_n(s) - f'_m(s), f_n(s) - f_m(s))_{V', V} \, ds \\
\leq 2 \int \left( ||f'_n(s) - f'_m(s)||_{V'} ||f_n(s) - f_m(s)||_{V} \right) \, ds \\
\leq 2 ||f_n - f_m||_{L^{p'}(\mathbb{R}; V')} ||f_n - f_m||_{L^p(\mathbb{R}; V)}
\]
which shows from the above that \( \{f_n\} \) is uniformly Cauchy on \([-T, 2T]\) with values in \( H \). Therefore, there exists \( g \) a continuous function defined on \([-T, 2T]\) having values in \( H \) such that
\[
\lim_{n \to \infty} \max \{ |f_n (t) - g (t)|_H^2 ; t \in [-T, 2T] \} = 0.
\]
However, \( g = \hat{f} \) a.e. because \( f_n \) converges to \( f \) in \( L^p (0, T; V) \). Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that \( V \subseteq H = H' \subseteq V' \) and Theorem 31.2.2.1, there exists \( f (0) \in V' \) such that for a.e. \( t \),
\[
f (t) = f (0) + \int_{0}^{t} f' (s) \, ds \text{ in } V'
\]
Now \( g = f \) a.e. and \( g \) is continuous with values in \( H \) hence continuous with values in \( V' \) and so
\[
g (t) = f (0) + \int_{0}^{t} f' (s) \, ds \text{ in } V'
\]
for all \( t \). Since \( g \) is continuous with values in \( H \) it is continuous with values in \( V' \). Taking the limit as \( t \downarrow 0 \) in the above, \( g (a) = \lim_{t \to 0^+} g (t) = f (0) \), showing that \( f (0) \in H \). Therefore, for a.e. \( t \),
\[
f (t) = f (0) + \int_{0}^{t} f' (s) \, ds \text{ in } H, \int_{0}^{t} f' (s) \, ds \in H.
\]
31.3. An Important Formula

Note that if \( f \in L^p(0, T; V) \) and \( f' \in L^p(0, T; V') \), then you can consider the initial value of \( f \) and it will be in \( H \). What if you start with something in \( H \)? Is it an initial condition for a function \( f \in L^p(0, T; V) \) such that \( f' \in L^p(0, T; V') \)? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if \( V \) is a closed subspace of the Sobolev space, \( W^{1,p}(\Omega) \) which contains \( W^{1,2}_0(\Omega) \) for \( p \geq 2 \) and \( H = L^2(\Omega) \) the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

### 31.3 An Important Formula

It is not necessary to have \( p > 2 \) in order to do the sort of thing just described. First is an approximation theorem which says that a given function in \( L^p([0, T]; E) \) can be approximated by step functions.

**Lemma 31.3.1** Let \( \Phi : [0, T] \to E \) be Lebesgue measurable and suppose

\[
\Phi \in K \equiv L^p([0, T]; E), \ p \geq 1
\]

Then there exists a sequence of nested partitions, \( P_k \subseteq P_{k+1}, \)

\[
P_k \equiv \{t^k_0, \cdots , t^k_{m_k}\}
\]

such that the step functions given by

\[
\Phi^r_k(t) = \sum_{j=1}^{m_k} \Phi(t^k_j) \chi_{[t^k_{j-1}, t^k_j]}(t)
\]

\[
\Phi^l_k(t) = \sum_{j=1}^{m_k} \Phi(t^k_{j-1}) \chi_{[t^k_{j-1}, t^k_j]}(t)
\]

both converge to \( \Phi \) in \( K \) as \( k \to \infty \) and

\[
\lim_{k \to \infty} \max \{|t^k_j - t^k_{j+1}| : j \in \{0, \cdots , m_k\}\} = 0.
\]

In the formulas, define \( \Phi(0) = 0 \). The mesh points \( t^k_j \) \( j=0 \) can be chosen to miss a given set of measure zero.

Note that it would make no difference in terms of the conclusion of this lemma if you defined

\[
\Phi^t_k(t) = \sum_{j=1}^{m_k} \Phi(t^k_{j-1}) \chi_{[t^k_{j-1}, t^k_j]}(t)
\]

because the modified function equals the one given above off a countable subset of \( [0, T] \), the union of the mesh points.
Proof: For \( t \in \mathbb{R} \) let \( \gamma_n (t) \equiv k/2^n, \delta_n (t) \equiv (k + 1)/2^n \), where 
\[ t \in (k/2^n, (k + 1)/2^n], \]
and \( 2^{-n} < T/4 \). Also suppose \( \Phi \) is defined to equal 0 on \([0, T]^C \times \Omega \). There exists a set of measure zero \( N \) such that for \( \omega \notin N, t \to \| \Phi (t, \omega) \| \) is in \( L^p (\mathbb{R}) \). Therefore by continuity of translation, as \( n \to \infty \) it follows that for \( \omega \notin N, \) and \( t \in [0, T] \),
\[ \int_{\mathbb{R}} \| \Phi (\gamma_n (t) + s) - \Phi (t + s) \|_E^p ds \to 0 \]
The above is dominated by
\[ \int_{\mathbb{R}} 2^{p-1} \left( \| \Phi (s) \|_E^p + \| \Phi (s) \|_E^p \right) \chi_{[-2T, 2T]} (s) ds \]
\[ = \int_{-2T}^{2T} 2^{p-1} \left( \| \Phi (s) \|_E^p + \| \Phi (s) \|_E^p \right) ds < \infty \]
Consider
\[ \int_{-2T}^{2T} \left( \int_{\mathbb{R}} \| \Phi (\gamma_n (t) + s) - \Phi (t + s) \|_E^p ds \right) dt \]
By the dominated convergence theorem, this converges to 0 as \( n \to \infty \). Now Fubini. This yields
\[ \int_{\mathbb{R}} \int_{-2T}^{2T} \| \Phi (\gamma_n (t) + s) - \Phi (t + s) \|_E^p dt ds \]
Change the variables on the inside.
\[ \int_{\mathbb{R}} \int_{-2T+s}^{2T+s} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p dt ds \]
Now by definition, \( \Phi (t) \) vanishes if \( t \notin [0, T] \), thus the above reduces to
\[ \int_{\mathbb{R}} \int_{0}^{T} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p dt ds \]
\[ + \int_{\mathbb{R}} \int_{-2T+s}^{2T+s} \chi_{[0, T]^C} \| \Phi (\gamma_n (t - s) + s) \|_E^p dt ds \]
\[ = \int_{\mathbb{R}} \int_{0}^{T} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p dt ds \]
\[ + \int_{\mathbb{R}} \int_{-2T+s}^{2T+s} \chi_{[0, T]^C} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p dt ds \]
Also by definition, \( \gamma_n (t - s) + s \) is within \( 2^{-n} \) of \( t \) and so the integrand in the integral on the right equals 0 unless \( t \in [-2^{-n} - T, T + 2^{-n}] \subseteq [-2T, 2T] \). Thus the above reduces to
\[ \int_{\mathbb{R}} \int_{-2T}^{2T} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p dt ds. \]
This converges to 0 as \( n \to \infty \) as was shown above. Therefore,

\[
\int_0^T \int_0^T |\Phi (\gamma_n (t - s) + s) - \Phi (t)|_E^p \, dt \, ds
\]

also converges to 0 as \( n \to \infty \). The only problem is that \( \gamma_n (t - s) + s \geq t - 2^{-n} \) and so \( \gamma_n (t - s) + s \) could be less than 0 for \( t \in [0, 2^{-n}] \). Since this is an interval whose measure converges to 0 it follows

\[
\int_0^T \int_0^T \Phi (\gamma_n (t - s) + s) - \Phi (t) |_E^p \, dt \, ds
\]

converges to 0 as \( n \to \infty \). Let

\[
m_n (s) = \int_0^T \Phi (\gamma_n (t - s) + s) - \Phi (t) |_E^p \, dt
\]

Then letting \( \mu \) denote Lebesgue measure,

\[
\mu ([m_n (s) > \lambda]) \leq \frac{1}{\lambda} \int_0^T m_n (s) \, ds.
\]

It follows there exists a subsequence \( n_k \) such that

\[
\mu \left( \left[ m_{n_k} (s) > \frac{1}{k} \right] \right) < 2^{-k}
\]

Hence by the Borel Cantelli lemma, there exists a set of measure zero \( N \) such that for \( s \notin N \),

\[
m_{n_k} (s) \leq 1/k
\]

for all \( k \) sufficiently large. Pick such an \( s \). Then consider \( t \to \Phi (\gamma_{n_k} (t - s) + s) \). For \( n_k \), \( t \to (\gamma_{n_k} (t - s) + s) + \) has jumps at points of the form \( 0, s + l 2^{-n_k} \) where \( l \) is an integer. Thus \( P_{n_k} \) consists of points of \([0, T]\) which are of this form and these partitions are nested. Define \( \Phi_k^l (0) \equiv 0, \Phi_k^l (t) \equiv \Phi (\gamma_{n_k} (t - s) + s) \). Now suppose \( N_1 \) is a set of measure zero. Can \( s \) be chosen such that all jumps for all partitions occur off \( N_1 \)? Let \((a, b) \) be an interval contained in \([0, T]\). Let \( S_j \) be the points of \((a, b) \) which are translations of the measure zero set \( N_1 \) by \( t_j \) for some \( j \). Thus \( S_j \) has measure 0. Now pick \( s \in (a, b) \setminus \bigcup_j S_j \). To get the other sequence of step functions, the right step functions, just use a similar argument with \( \delta_n \) in place of \( \gamma_n \). Just apply the argument to a subsequence of \( n_k \) so that the same \( s \) can hold for both. □

**Theorem 31.3.2** Let \( V \subseteq H = H' \subseteq V' \) be a Gelfand triple and suppose \( Y \in L^p (0, T; V') \equiv K' \) and

\[
X (t) = X_0 + \int_0^t Y (s) \, ds \text{ in } V'
\]  

(31.3.16)
where \(X_0 \in H\), and it is known that \(X \in L^p(0, T, V) \equiv K\) for \(p > 1\). Then \(t \rightarrow X(t)\) is in \(C([0, T], H)\) and also

\[
\frac{1}{2}|X(t)|^2_H = \frac{1}{2}|X_0|^2_H + \int_0^t \langle Y(s), X(s) \rangle \, ds
\]

**Proof:** By Lemma 31.3.1, there exists a sequence of uniform partitions \(\{t^n_k\}_{k=0}^{m_n} = \mathcal{P}_n, \mathcal{P}_n \subseteq \mathcal{P}_{n+1}\), of \([0, T]\) such that the step functions

\[
\sum_{k=0}^{m_n-1} X(t^n_k) \chi_{(t^n_k, t^n_{k+1}]}(t) = X^l(t)
\]

\[
\sum_{k=0}^{m_n-1} X(t^n_{k+1}) \chi_{(t^n_k, t^n_{k+1}]}(t) = X^r(t)
\]

converge to \(X\) in \(K\) and in \(L^2([0, T], H)\).

**Lemma 31.3.3** Let \(s < t\). Then for \(X, Y\) satisfying 31.3.16

\[
|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du - |X(t) - X(s)|^2
\]

(31.3.17)

**Proof:** It follows from the following computations

\[
X(t) - X(s) = \int_s^t Y(u) \, du
\]

\[
-|X(t) - X(s)|^2 = -|X(t)|^2 + 2 \langle X(t), X(s) \rangle - |X(s)|^2
\]

\[
= -|X(t)|^2 + 2 \left( X(t), X(t) - \int_s^t Y(u) \, du \right) - |X(s)|^2
\]

\[
= -|X(t)|^2 + 2 |X(t)|^2 - 2 \left( \int_s^t Y(u) \, du, X(t) \right) - |X(s)|^2
\]

Hence

\[
|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du - |X(t) - X(s)|^2
\]

**Lemma 31.3.4** In the above situation,

\[
\sup_{t \in [0, T]} |X(t)|_H \leq C \left( \|Y\|_{K'}, \|X\|_K \right)
\]

Also, \(t \rightarrow X(t)\) is weakly continuous with values in \(H\).
31.3. AN IMPORTANT FORMULA

**Proof:** From the above formula applied to the \( k \)th partition of \([0, T]\) described above,

\[
|X(t_m)|^2 - |X_0|^2 = \sum_{j=0}^{m-1} |X(t_{j+1})|^2 - |X(t_j)|^2
\]

\[
= \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) \rangle \, du - |X(t_{j+1}) - X(t_j)|_H^2
\]

Thus, discarding the negative terms and denoting by \( P_k \) the \( k \)th of these partitions,

\[
\sup_{t_j \in P_k} |X(t_j)|_H^2 \leq |X_0|^2 + 2 \int_0^T \| Y(u) \|_V \| X_k^r(u) \|_V \, du
\]

\[
\leq |X_0|^2 + 2 \left( \int_0^T \| Y(u) \|_V^{p'} \, du \right)^{1/p'} \left( \int_0^T \| X_k^r(u) \|_V^p \, du \right)^{1/p} \leq C (\| Y \|_{K'}, \| X \|_K)
\]

because these partitions are chosen such that

\[
\lim_{k \to \infty} \left( \int_0^T \| X_k^r(u) \|_V^p \, du \right)^{1/p} = \left( \int_0^T \| X(u) \|_V^p \, du \right)^{1/p}
\]

and so these are bounded. This has shown that for the dense subset of \([0, T]\), \( D \equiv \cup_k P_k \),

\[
\sup_{t \in D} |X(t)| < C (\| Y \|_{K'}, \| X \|_K)
\]

Now let \( \{g_k\}_{k=1}^\infty \) be linearly independent vectors of \( V \) whose span is dense in \( V \). This is possible because \( V \) is separable. Then let \( \{e_j\}_{j=1}^\infty \) be an orthonormal basis for \( H \) such that \( e_k \in \text{span} \{g_1, \ldots, g_k\} \) and each \( g_k \in \text{span} \{e_1, \ldots, e_k\} \). This is done with the Gram Schmidt process. Then it follows that \( \text{span} \{\{e_k\}_{k=1}^\infty\} \) is dense in \( V \).

I claim

\[
|y|_H^2 = \sum_{j=1}^\infty |\langle y, e_j \rangle|_H^2.
\]

This is certainly true if \( y \in H \) because

\[
\langle y, e_j \rangle = (y, e_j)_H
\]
If \( y \not\in H \), then the series must diverge since otherwise, you could consider the infinite sum

\[
\sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \in H
\]

because

\[
\left| \sum_{j=p}^{q} \langle y, e_j \rangle e_j \right|^2 = \sum_{j=p}^{q} |\langle y, e_j \rangle|^2 \to 0 \text{ as } p, q \to \infty.
\]

Letting \( z = \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \), it follows that \( \langle y, e_j \rangle \) is the \( j^{th} \) Fourier coefficient of \( z \) and that

\[
\langle z - y, v \rangle = 0
\]

for all \( v \in \text{span}(\{e_k\}_{k=1}^{\infty}) \) which is dense in \( V \). Therefore, \( z = y \) in \( V' \) and so \( y \in H \).

It follows

\[
|X(t)|^2 = \sup_n \sum_{j=1}^{n} |\langle X(t), e_j \rangle|^2
\]

which is just the sup of continuous functions of \( t \). Therefore, \( t \to |X(t)|^2 \) is lower semicontinuous. It follows that for any \( t \), letting \( t_j \to t \) for \( t_j \in D \),

\[
|X(t)|^2 \leq \liminf_{j \to \infty} |X(t_j)|^2 \leq C (\|Y\|_{K'}, \|X\|_K)
\]

This proves the first claim of the lemma.

Consider now the claim that \( t \to X(t) \) is weakly continuous. Letting \( v \in V \),

\[
\lim_{t \to s} \langle X(t), v \rangle = \lim_{t \to s} \langle X(t), v \rangle = \langle X(s), v \rangle = \langle X(s), v \rangle
\]

Since it was shown that \( |X(t)| \) is bounded independent of \( t \), and since \( V \) is dense in \( H \), the claim follows. \( \blacksquare \)

Now

\[
- \sum_{j=0}^{m-1} |X(t_{j+1}) - X(t_j)|_H^2 = |X(t_m)|^2 - |X_0|^2 - \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X_k(u) \rangle \, du
\]

\[
= |X(t_m)|^2 - |X_0|^2 - 2 \int_{0}^{t_m} \langle Y(u), X_k(u) \rangle \, du
\]

Thus, since the partitions are nested, eventually \( |X(t_m)|^2 \) is constant for all \( k \) large enough and the integral term converges to

\[
\int_{0}^{t_m} \langle Y(u), X(u) \rangle \, du
\]

It follows that the term on the left does converge to something. It just remains to consider what it does converge to. However, from the equation solved by \( X \),

\[
X(t_{j+1}) - X(t_j) = \int_{t_j}^{t_{j+1}} Y(u) \, du
\]
Therefore, this term is dominated by an expression of the form
\[\sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y(u) \, du, X(t_{j+1}) - X(t_j) \right)\]
\[= \sum_{j=0}^{m_k-1} \left\langle \int_{t_j}^{t_{j+1}} Y(u) \, du, X(t_{j+1}) - X(t_j) \right\rangle\]
\[= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) - X(t_j) \rangle \, du\]
\[= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) \rangle - \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_j) \rangle \, du\]
\[= \int_0^T \langle Y(u), X^r(u) \rangle \, du - \int_0^T \langle Y(u), X^l(u) \rangle \, du\]

However, both \(X^r\) and \(X^l\) converge to \(X\) in \(K = L^p(0, T, V)\). Therefore, this term must converge to 0. Passing to a limit, it follows that for all \(t \in D\), the desired formula holds. Thus, for such \(t\),
\[|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(u), X(u) \rangle \, du\]

It remains to verify that this holds for all \(t\). Let \(t \notin D\) and let \(t(k) \in \mathcal{P}_k\) be the largest point of \(\mathcal{P}_k\) which is less than \(t\). Suppose \(t(m) \leq t(k)\) so that \(m \leq k\). Then
\[X(t(m)) = X_0 + \int_0^{t(m)} Y(s) \, ds,\]
a similar formula for \(X(t(k))\). Thus for \(t > t(m)\),
\[X(t) - X(t(m)) = \int_{t(m)}^t Y(s) \, ds\]

which is the same sort of thing already looked at except that it starts at \(t(m)\) rather than at 0 and \(X_0 = 0\). Therefore,
\[|X(t(k)) - X(t(m))|^2 = 2 \int_{t(m)}^{t(k)} \langle Y(s), X(s) - X(t(m)) \rangle \, ds\]

Thus, for \(m \leq k\),
\[\lim_{m,k \to \infty} \frac{|X(t(k)) - X(t(m))|^2}{2} = 0\]

Hence \(\{X(t(k))\}_{k=1}^{\infty}\) is a convergent sequence in \(H\). Does it converge to \(X(t)\)? Let \(\xi(t) \in H\) be what it does converge to. Let \(v \in V\). Then
\[\langle \xi(t), v \rangle = \lim_{k \to \infty} \langle X(t(k)), v \rangle = \lim_{k \to \infty} \langle X(t(k)), v \rangle = \langle X(t), v \rangle = (X(t), v)\]
because it is known that \( t \to X(t) \) is continuous into \( V' \) and it is also known that \( X(t) \in H \) and that the \( X(t) \) for \( t \in [0, T] \) are uniformly bounded. Therefore, since \( V \) is dense in \( H \), it follows that \( \xi(t) = X(t) \).

Now for every \( t \in D \), it was shown above that

\[
|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle \, ds
\]

Thus, using what was just shown, if \( t \notin D \) and \( t_k \to t \),

\[
|X(t)|^2 = \lim_{k \to \infty} |X(t_k)|^2 = \lim_{k \to \infty} \left( |X_0|^2 + 2 \int_0^{t_k} \langle Y(s), X(s) \rangle \, ds \right)
\]

\[
= |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle \, ds
\]

which proves the desired formula. From this it follows right away that \( t \to X(t) \) is continuous into \( H \) because it was just shown that \( t \to |X(t)| \) is continuous and \( t \to X(t) \) is weakly continuous. Since Hilbert space is uniformly convex, this implies the \( t \to X(t) \) is continuous. To see this in the special case of Hilbert space,

\[
|X(t) - X(s)|^2 = |X(t)|^2 - 2 \langle X(s), X(t) \rangle + |X(s)|^2
\]

Then \( \lim_{t \to s} \left( |X(t)|^2 - 2 \langle X(s), X(t) \rangle + |X(s)|^2 \right) = 0 \) by weak convergence of \( X(t) \) to \( X(s) \) and the convergence of \( |X(t)|^2 \) to \( |X(s)|^2 \).

### 31.4 The Implicit Case

The above theorem can be generalized to the case where the formula is of the form

\[
BX(t) = BX_0 + \int_0^t Y(s) \, ds
\]

This involves an operator \( B \in \mathcal{L}(W, W') \) and \( B \) satisfies

\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle
\]

for

\[
V \subseteq W, W' \subseteq V'
\]

Where \( V \) is dense in the Banach space \( W \). Before giving the theorem, here is a technical lemma. First is one which is not so technical.

**Lemma 31.4.1** Let \( V \) be a separable Banach space. Then there exists \( \{g_k\}_{k=1}^\infty \) which are linearly independent and whose span is dense in \( V \).
31.4. THE IMPLICIT CASE

Proof: Let \( \{f_k\} \) be a countable dense subset. Thus their span is dense. Delete \( f_{k_1} \) such that \( k_1 \) is the first index such that \( f_k \) is in the span of the other vectors. That is, it is the first which is a finite linear combination of the others. If no such vector exists, then you have what is wanted. Next delete \( f_{k_2} \) where \( k_2 \) is the next for which \( f_k \) is a linear combination of the others. Continue. The remaining vectors must be linearly independent. If not, there would be a first which is a linear combination of the others. Say \( f_m \). But the process would have eliminated it at the \( m^{th} \) step. \( \blacksquare \)

Lemma 31.4.2 Suppose \( V,W \) are separable Banach spaces such that \( V \) is dense in \( W \) and \( B \in \mathcal{L}(W,W') \) satisfies

\[
\langle Bx,x \rangle \geq 0, \quad \langle Bx,y \rangle = \langle By,x \rangle, \quad B \neq 0.
\]

Then there exists a countable set \( \{e_i\} \) of vectors in \( V \) such that

\[
\langle Be_i,e_j \rangle = \delta_{ij}
\]

and for each \( x \in W \),

\[
\langle Bx,x \rangle = \sum_{i=1}^{\infty} |\langle Bx,e_i \rangle|^2,
\]

and also

\[
Bx = \sum_{i=1}^{\infty} \langle Bx,e_i \rangle Be_i,
\]

the series converging in \( W' \). If \( B = B(\omega) \) and \( B \) is \( F \) measurable into \( \mathcal{L}(W,W') \) and if the \( e_i = e_i(\omega) \) are as described above, then these \( e_i \) are measurable into \( V \). If \( t \to B(t,\omega) \) is \( C^1([0,T],\mathcal{L}(W,W')) \) and if for each \( w \in W \),

\[
\langle B'(t,\omega)w,w \rangle \leq k_{w,\omega}(t) \langle B(t,\omega)w,w \rangle
\]

Where \( k_{w,\omega} \in L^1([0,T]) \), then the vectors \( e_i(t) \) can be chosen to also be right continuous functions of \( t \).

In the case of dependence on \( t \), the extra condition is trivial if \( \langle B(t,\omega)x,x \rangle \geq \delta \|w\|_W^2 \) for example. This includes the usual case of evolution equations where \( W = H = H' = W' \). It also includes the case where \( B \) does not depend on \( t \).

Proof: Let \( \{g_k\}_{k=1}^{\infty} \) be linearly independent vectors of \( V \) whose span is dense in \( V \). This is possible because \( V \) is separable. Thus, their span is also dense in \( W \). Let \( n_1 \) be the first index such that \( \langle Bg_{n_1},g_{n_1} \rangle \neq 0 \).

Claim: If there is no such index, then \( B = 0 \).

Proof of claim: First note that if there is no such first index, then if \( x = \sum_{i=1}^{k} a_i g_i \),

\[
|\langle Bx,x \rangle| = \left| \sum_{i \neq j} a_i a_j \langle Bg_i,g_j \rangle \right| \leq \sum_{i \neq j} |a_i| |a_j| |\langle Bg_i,g_j \rangle| \\
\leq \sum_{i \neq j} |a_i| |a_j| |\langle Bg_i,g_i \rangle|^{1/2} |\langle Bg_j,g_j \rangle|^{1/2} = 0
\]
Therefore, if \( x \) is given, you could take \( x_k \) in the span of \( \{g_1, \ldots, g_k\} \) such that \( \|x_k - x\|_W \to 0 \). Then

\[
\langle Bx, y \rangle = \lim_{k \to \infty} \langle Bx_k, y \rangle \leq \lim_{k \to \infty} \langle Bx_k, x_k \rangle^{1/2} \langle By, y \rangle^{1/2} = 0
\]

because \( \langle Bx_k, x_k \rangle \) is zero by what was just shown. Hence the conclusion of the lemma is trivially true. Just pick \( e_1 = g_1 \) and let \( \{e_1\} \) be your set of vectors.

Thus assume there is such a first index. Let

\[
e_1 = \frac{g_{n_1}}{\langle Bg_{n_1}, g_{n_1} \rangle^{1/2}}
\]

Then \( \langle Be_1, e_1 \rangle = 1 \). Now if you have constructed \( e_j \) for \( j \leq k \),

\[
e_j \in \text{span} \{g_{n_1}, \ldots, g_{n_k}\}, \quad \langle Be_i, e_j \rangle = \delta_{ij},
\]

\( g_{n_{j+1}} \) being the first in the list \( \{g_j\} \) for which

\[
\left\langle Bg_{n_{j+1}} - \sum_{i=1}^j \langle Bg_{n_{j+1}}, e_i \rangle Be_i, g_{n_{j+1}} - \sum_{i=1}^j \langle Bg_{n_{j+1}}, e_i \rangle e_i \right\rangle \neq 0,
\]

and

\[
\text{span} \{g_{n_1}, \ldots, g_{n_k}\} = \text{span} \{e_1, \ldots, e_k\},
\]

let \( g_{n_{k+1}} \) be such that \( g_{n_{k+1}} \) is the first in the list \( \{g_n\} n_{k+1} > n_k \) such that

\[
\left\langle Bg_{n_{k+1}} - \sum_{i=1}^k \langle Bg_{n_{k+1}}, e_i \rangle Be_i, g_{n_{k+1}} - \sum_{i=1}^k \langle Bg_{n_{k+1}}, e_i \rangle e_i \right\rangle \neq 0
\]

Note the difference between this and the Gram Schmidt process. Here you don’t necessarily use all of the \( g_k \) due to the possible degeneracy of \( B \).

**Claim:** If there is no such first \( g_{n_{k+1}} \), then \( B(\text{span} \{e_1, \ldots, e_k\}) = BW \) so in this case, \( \{Be_i\}_{i=1}^k \) is actually a basis for \( BW \).

**Proof:** To see this, note that if \( p \in (n_j, n_{j+1}) \), then by assumption,

\[
\left\langle B \left( g_p - \sum_{i=1}^j \langle Bg_p, e_i \rangle e_i \right), g_p - \sum_{i=1}^j \langle Bg_p, e_i \rangle e_i \right\rangle = 0
\]

Therefore,

\[
Bg_p = \sum_{i=1}^j \langle Bg_p, e_i \rangle Be_i
\]

Also, by assumption, if \( p > n_k \)

\[
\left\langle B \left( g_p - \sum_{i=1}^k \langle Bg_p, e_i \rangle e_i \right), g_p - \sum_{i=1}^k \langle Bg_p, e_i \rangle e_i \right\rangle = 0
\]
31.4. THE IMPLICIT CASE

so

$$B g_p = \sum_{i=1}^{k} \langle B g_p, e_i \rangle B e_i$$

which shows that span $$\left( \{ B g_j \}_{j=1}^{\infty} \right) \subseteq \text{span} \left( \{ B e_i \}_{i=1}^{k} \right)$$. If $$\sum_{i=1}^{k} c_i B e_i = 0$$, then for $$j \leq k$$,

$$0 = \sum_{i=1}^{k} c_i \langle B e_i, e_j \rangle = c_j$$

so $$\{ B e_i \}_{i=1}^{k}$$ is a basis for $$\text{span} \left( \{ B g_j \}_{j=1}^{\infty} \right) = B \left( \text{span} \left( \{ g_j \}_{j=1}^{\infty} \right) \right)$$. Hence if $$x \in W$$, then letting $$x_r \in \text{span} \left( \{ g_j \}_{j=1}^{\infty} \right)$$ with $$x_r \to x$$ in $$W$$, it follows

$$B x_r = \sum_{i=1}^{k} a_i B e_i = \sum_{i=1}^{k} \langle B x_r, e_i \rangle B e_i$$

Then passing to a limit, you get

$$B x = \sum_{i=1}^{k} \langle B x, e_i \rangle B e_i$$

Thus $$\{ B e_i \}_{i=1}^{k}$$ is a basis for $$BW$$. This proves the claim.

If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

If the process does not stop, let

$$e_{k+1} \equiv \frac{g_{n_{k+1}} - \sum_{i=1}^{k} \langle B g_{n_{k+1}}, e_i \rangle e_i}{\left( B \left( g_{n_{k+1}} - \sum_{i=1}^{k} \langle B g_{n_{k+1}}, e_i \rangle e_i \right), g_{n_{k+1}} - \sum_{i=1}^{k} \langle B g_{n_{k+1}}, e_i \rangle e_i \right)^{1/2}}$$

Thus, as in the usual argument for the Gram Schmidt process, $$\langle B e_i, e_j \rangle = \delta_{ij}$$ for $$i, j \leq k + 1$$. This is already known for $$i, j \leq k$$. Letting $$l \leq k$$, and using the orthogonality already shown,

$$\langle B e_{k+1}, e_l \rangle = C \left( B \left( g_{n_{k+1}} - \sum_{i=1}^{k} \langle B g_{n_{k+1}}, e_i \rangle e_i \right), e_l \right) = C \left( \langle B g_{k+1}, e_l \rangle - \langle B g_{n_{k+1}}, e_l \rangle \right) = 0$$

Consider

$$\left( B g_p - B \left( \sum_{i=1}^{k} \langle B g_p, e_i \rangle e_i \right), g_p - \sum_{i=1}^{k} \langle B g_p, e_i \rangle e_i \right)$$
If \( p \in (n_k, n_{k+1}) \), then the above equals zero which implies
\[
B g_p = \sum_{i=1}^{k} \langle B g_p, e_i \rangle B e_i
\]
On the other hand, suppose \( g_p = g_{n_k+1} \) for some \( n_{k+1} \) and so, from the construction, \( g_{n_k+1} = g_p \in \text{span} \{e_1, \ldots, e_{k+1}\} \) and therefore,
\[
g_p = \sum_{j=1}^{k+1} a_j e_j
\]
which requires easily that
\[
B g_p = \sum_{i=1}^{k+1} \langle B g_p, e_i \rangle B e_i,
\]
the above holding for all \( k \) large enough. To see this last claim, note that the coefficients of \( B g = \sum_{j=1}^{m} a_j B e_j \) are required to be \( a_j = \langle B g, e_j \rangle \) and from the construction, \( \langle B e_i, e_j \rangle = \delta_{ij} \). Thus if the upper limit is increased beyond what is needed, the new terms are all zero. It follows that for any \( x \in \text{span} \left( \{g_k\}_{k=1}^{\infty} \right) \),

\[
B x = \sum_{i=1}^{\infty} \langle B x, e_i \rangle B e_i \quad (31.4.18)
\]
because for all \( k \) large enough,
\[
B x = \sum_{i=1}^{k} \langle B x, e_i \rangle B e_i
\]
Also note that for such \( x \in \text{span} \left( \{g_j\}_{j=1}^{\infty} \right) \),
\[
\langle B x, x \rangle = \left\langle \sum_{i=1}^{k} \langle B x, e_i \rangle B e_i, x \right\rangle = \sum_{i=1}^{k} \langle B x, e_i \rangle \langle B x, e_i \rangle
\]
\[
= \sum_{i=1}^{k} |\langle B x, e_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle B x, e_i \rangle|^2
\]
Now for \( x \) arbitrary, let \( x_k \to x \) in \( W \) where \( x_k \in \text{span} \left( \{g_k\}_{k=1}^{\infty} \right) \). Then by Fatou’s lemma,
\[
\sum_{i=1}^{\infty} |\langle B x, e_i \rangle|^2 \leq \liminf_{k \to \infty} \sum_{i=1}^{\infty} |\langle B x_k, e_i \rangle|^2
\]
\[
= \liminf_{k \to \infty} \langle B x_k, x_k \rangle = \langle B x, x \rangle \quad (31.4.19)
\]
\[
\leq \|B x\|_W, \|x\|_W \leq \|B\| \|x\|_W^2
\]
31.4. THE IMPLICIT CASE

Thus the series on the left converges. Then also, from the above inequality,

\[
\left| \sum_{i=p}^{q} (Bx, e_i) Be_i, y \right| \leq \sum_{i=p}^{q} |(Bx, e_i)| |Be_i, y| \\
\leq \left( \sum_{i=p}^{q} |(Bx, e_i)|^2 \right)^{1/2} \left( \sum_{i=p}^{q} |(By, e_i)|^2 \right)^{1/2} \\
\leq \left( \sum_{i=p}^{q} |(Bx, e_i)|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |(By, e_i)|^2 \right)^{1/2} \\
\leq \left( \sum_{i=1}^{\infty} |(Bx, e_i)|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |(By, e_i)|^2 \right)^{1/2}
\]

By \(31.4.19\),

\[
\left( \sum_{i=1}^{\infty} |(Bx, e_i)|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |(By, e_i)|^2 \right)^{1/2} \leq \left( \sum_{i=p}^{q} |(Bx, e_i)|^2 \right)^{1/2} \|B\|^{1/2} \|y\|_{W'}
\]

It follows that

\[
\sum_{i=1}^{\infty} (Bx, e_i) Be_i \tag{31.4.20}
\]

converges in \(W'\) because it was just shown that

\[
\left\| \sum_{i=p}^{q} (Bx, e_i) Be_i \right\|_{W'} \leq \left( \sum_{i=p}^{q} |(Bx, e_i)|^2 \right)^{1/2} \|B\|^{1/2}
\]

and it was shown above that \(\sum_{i=1}^{\infty} |(Bx, e_i)|^2 < \infty\), so the partial sums of the series \(31.4.20\) are a Cauchy sequence in \(W'\). Also, the above estimate shows that for \(\|y\| = 1\),

\[
\left| \sum_{i=1}^{\infty} (Bx, e_i) Be_i, y \right| \leq \left( \sum_{i=1}^{\infty} |(By, e_i)|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |(Bx, e_i)|^2 \right)^{1/2} \\
\leq \left( \sum_{i=1}^{\infty} |(Bx, e_i)|^2 \right)^{1/2} \|B\|^{1/2}
\]

and so

\[
\left\| \sum_{i=1}^{\infty} (Bx, e_i) Be_i \right\|_{W'} \leq \left( \sum_{i=1}^{\infty} |(Bx, e_i)|^2 \right)^{1/2} \|B\|^{1/2} \tag{31.4.21}
\]
Now for \( x \) arbitrary, let \( x_k \in \text{span} \{g_j\}_{j=1}^\infty \) and \( x_k \to x \) in \( W \). Then for a fixed \( k \) large enough,

\[
\left\| Bx - \sum_{i=1}^\infty \langle Bx, e_i \rangle Be_i \right\| \leq \| Bx - Bx_k \|
\]

\[
+ \left\| Bx_k - \sum_{i=1}^\infty \langle Bx_k, e_i \rangle Be_i \right\| + \left\| \sum_{i=1}^\infty \langle Bx_k, e_i \rangle Be_i - \sum_{i=1}^\infty \langle Bx, e_i \rangle Be_i \left\|
\right.
\]

\[
\leq \varepsilon + \sum_{i=1}^\infty \langle B(x_k - x), e_i \rangle Be_i,
\]

the term

\[
\left\| Bx_k - \sum_{i=1}^\infty \langle Bx_k, e_i \rangle Be_i \right\|
\]

equaling 0 by 31.4.18. From 31.4.21 and 31.4.19,

\[
\leq \varepsilon + \| B \|^{1/2} \left( \sum_{i=1}^\infty \langle B(x_k - x), e_i \rangle^2 \right)^{1/2}
\]

\[
\leq \varepsilon + \| B \|^{1/2} \langle B(x_k - x), x_k - x \rangle^{1/2} < 2\varepsilon
\]

whenever \( k \) is large enough, the second inequality being implied by 31.4.15. Therefore,

\[
Bx = \sum_{i=1}^\infty \langle Bx, e_i \rangle Be_i
\]

in \( W' \). It follows that

\[
\langle Bx, x \rangle = \lim_{k \to \infty} \left( \sum_{i=1}^k \langle Bx, e_i \rangle Be_i, x \right) = \lim_{k \to \infty} \sum_{i=1}^k \langle Bx, e_i \rangle \sum_{i=1}^\infty \langle Bx, e_i \rangle^2 \equiv \sum_{i=1}^\infty \langle Bx, e_i \rangle^2
\]

Now consider the measurability assertion on the \( e_i \). Consider first \( e_1 \). Begin by considering \( n_1 (\omega) \)

\[
E_1^k \equiv \{ \omega : \langle B (\omega) g_k, g_k \rangle \neq 0 \} \cap \cap_{j<k} \{ \omega : \langle B (\omega) g_j, g_j \rangle = 0 \}
\]

As explained above, \( B (\omega) = 0 \), if and only if \( E_1^k = \emptyset \) for all \( k \). Also note that these \( E_k \) are disjoint and \( \mathcal{F} \) measurable. Then

\[
n_1 (\omega) \equiv \begin{cases} 
1 & \text{if } \omega \notin \cup_k E_1^k = \emptyset \\
0 & \text{if } \omega \in E_1^k
\end{cases}
\]

Then \( n_1 (\omega) \) is clearly measurable because it is constant on measurable sets. Then from the algorithm,

\[
e_1 (\omega) \equiv X_{\cup_k E_1^k} (\omega) \frac{g_{n_1 (\omega)}}{\langle B g_{n_1 (\omega)}, g_{n_1 (\omega)} \rangle^{1/2}}
\]
31.4. THE IMPLICIT CASE

Thus \( e_1(\omega) = 0 \) if \( \omega \notin \cup_k E^1_k \). Also \( e_1(\omega) \) is measurable because \( \omega \to n_1(\omega) \) is measurable. Thus \( e_1 \) has constant values on measurable sets. So suppose \( n_i(\omega) \) is measurable for \( i \leq m \). Then define \( E^{m+1}_p \equiv \)

\[
\{ \omega : \left< Bg_p - \sum_{i=1}^{m} \langle Bh \rangle, e_i \right> g_p - \sum_{i=1}^{m} \langle Bh \rangle, e_i \rangle \neq 0 \} \cap \{ \omega : n_m(\omega) < p \}
\]

\[
\cap_{n_m(\omega)<r<p} \{ \omega : \left< Bg_r - \sum_{i=1}^{m} \langle Bh \rangle, e_i \right> g_r - \sum_{i=1}^{m} \langle Bh \rangle, e_i \rangle = 0 \}
\]

As earlier, these sets \( \{ E^{m+1}_p \}_{p=1}^{\infty} \) are disjoint and measurable. As before, let \( n_{m+1}(\omega) = p \) where \( \omega \in E^{m+1}_p \). Then from the algorithm, \( e_{m+1}(\omega) \equiv \)

\[
\chi_{\cup_p E^{m+1}_p} \frac{g_{m+1}m(\omega) - \sum_{i=1}^{m} \langle Bh \rangle, e_i \rangle e_i}{D_m},
\]

where \( D_m = \left< B \left( g_{m+1}(\omega) - \sum_{i=1}^{m} \langle Bh \rangle, e_i \rangle e_i \right) \right> 1/2 \)

Thus the \( e_k(\omega) \) are all measurable into \( W \) thanks to the algorithm. However, they all have values in \( V \). Thus if \( \phi \in V' \), let \( \phi_n \to \phi \) in \( V' \) where \( \phi_n \in W' \).

\[
\langle \phi, e_k(\omega) \rangle_{V', V} = \lim_{n \to \infty} \langle \phi_n, e_k(\omega) \rangle_{V', V} = \lim_{n \to \infty} \langle \phi, e_k(\omega) \rangle_{W', W}
\]

which is the limit of measurable functions. By the Pettis theorem, this shows \( e_k \) is measurable into \( V \) also.

To verify the assertion on right continuity, the same kind of argument holds. We suppress the dependence on \( \omega \). Consider first \( e_1 \). Begin by considering \( n_1(t) \)

\[
E^1_k = \{ t : \left< B(t) g_k, g_k \right> \neq 0 \} \cap \cap_{j<k} \{ t : \left< B(t) g_j, g_j \right> = 0 \}
\]

As explained above, \( B(t) = 0 \), if and only if \( E^1_k = \emptyset \) for all \( k \). Also note that these \( E^1_k \) are disjoint. Then

\[
n_1(t) = \begin{cases} 
1 & \text{if } t \notin \cup_k E^1_k = \emptyset \\
k & \text{if } t \in E^1_k
\end{cases}
\]

If \( t \in E^1_k \), then from the definition, \( \left< B(t) g_k, g_k \right> \neq 0 \) and \( k \) is the first index for which this is nonzero. Let \( t_l \downarrow t \). Then by continuity, for all \( l \) large enough, \( \left< B(t_l) g_k, g_k \right> \neq 0 \). What of \( \left< B(t_l) g_j, g_j \right> \) for \( j < k \)? By assumption,

\[
\left< B'(t) g_j, g_j \right> \leq k_{g_j}(t) \left< B(t) g_j, g_j \right>
\]

and so, letting \( K_{g_j}(t) = \int_t^1 k_{g_j}(s) ds \),

\[
\frac{d}{dt} \left( e^{-K_{g_j}(t)} \left< B(t) g_j, g_j \right> \right) \leq 0
\]
CHAPTER 31. GELFAND TRIPLES AND RELATED STUFF

\[ e^{-K_{g_j}(t)} \langle B(t), g_j, g_j \rangle \leq e^{-K_{g_j}(t)} \langle B(t), g_j, g_j \rangle = 0 \]

Thus one obtains right continuity of \( t \to n_1(t) \) and for \( E^1_k \), there is an interval \([t, t + \delta] \subseteq E^1_k\). From the algorithm,

\[ e_1(t) \equiv \mathcal{X}_{\cup_k E^1_k}(t) \frac{g_{n_1(t)}}{\langle Bg_{n_1(t)}, g_{n_1(t)} \rangle^{1/2}} \]

Thus \( e_1(t) = 0 \) if \( t \notin \cup_k E^1_k \). Also \( e_1(t) \) is right continuous because \( t \to n_1(t) \) is. Thus \( e_1 \) has constant values on a small interval starting at \( t \). But what about \( t \notin \cup_k E^1_k \)? Why should it be right continuous there? If you have such a \( t \), then as explained above, \( B(t) = 0 \). Then letting \( s \) be arbitrary, \( s > t \) and \( x \in W \),

\[ \langle B'(s) x, x \rangle \leq k_x \langle B(s) x, x \rangle \]

and so as above,

\[ e^{-K_{x}(s)} \langle B(s) x, x \rangle \leq 0 \]

Thus this case reduces to having \( B(s) \equiv 0 \) for all \( s \geq t \) and there is nothing to prove. You have \( n_1(s) = 1 \) and \( e_1(s) = 0 \) for all \( s \geq t \).

Suppose \( t \rightarrow n_i(t) \) is right continuous for \( i \leq m \) and that \( e_i \) is also. Then define \( E^{m+1}_p \equiv \)

\[ \left\{ t : \left( Bg_p - \sum_{i=1}^{m} \langle Bg_p, e_i \rangle Be_i, g_p - \sum_{i=1}^{m} \langle Bg_p, e_i \rangle e_i \right) \neq 0 \right\} \cap \left\{ t : n_m(t) < p \right\} \cap_{n_m(t) < r < p} \left\{ t : \left( Bg_r - \sum_{i=1}^{m} \langle Bg_r, e_i \rangle Be_i, g_r - \sum_{i=1}^{m} \langle Bg_r, e_i \rangle e_i \right) = 0 \right\} \]

As earlier, these sets \( \{ E^{m+1}_p \}^\infty_{p=1} \) are disjoint. As before, let \( n_{m+1}(t) = p \) where \( t \in E^{m+1}_p \). Then by similar reasoning to the above, for small \( \delta, [t, t + \delta] \in E^{m+1}_p \) and \( n_{m+1}(s) = p \) for \( s \in [t, t + \delta] \). Then from the algorithm, \( e_{m+1}(t) \equiv \)

\[ \mathcal{X}_{\cup_p E^{m+1}_p}(t) \frac{g_{n_{m+1}(t)}}{D_m} \frac{\sum_{i=1}^{m} \langle Bg_{n_{m+1}(t)}, e_i(t) \rangle e_i(t)}{D_m^2} \]

where \( D_m = \)

\[ \left( B \left( g_{n_{m+1}(t)} - \sum_{i=1}^{m} \langle Bg_{n_{m+1}(t)}, e_i(t) \rangle e_i(t) \right) , g_{n_{m+1}(t)} - \sum_{i=1}^{m} \langle Bg_{n_{m+1}(t)}, e_i(t) \rangle e_i(t) \right)^{1/2} \]

and so is right continuous. What of \( t \notin \cup_p E^{m+1}_p \)? In this case, the process has terminated and what is desired has been found.

Then the main result in this section is the following integration by parts theorem.
31.4. THE IMPLICIT CASE

**Theorem 31.4.3** Let $V \subseteq W, W' \subseteq V'$ be separable Banach spaces, and let $Y \in L^p(0, T; V')$ and

$$Bu(t) = Bu_0 + \int_0^t Y(s) \, ds \text{ in } V', \quad u_0 \in W, Bu(t) = B(u(t)) \text{ for a.e. } t \quad (31.4.22)$$

As indicated, $Bu$ is the name of a function satisfying the above equation which satisfies $Bu(t) = B(u(t))$ for a.e. $t$. Thus $Y = (Bu)'$ as a weak derivative in the sense of $V'$ valued distributions. It is known that $u \in L^p(0, T, V)$ for $p > 1$. Then $t \to Bu(t)$ is continuous into $W'$ for $t$ off a set of measure zero $N$ and also there exists a continuous function $t \to \langle Bu, u \rangle(t)$ such that for all $t \notin N, \langle Bu, u \rangle(t) = \langle B(u(t)), u(t) \rangle, Bu(t) = B(u(t))$, and for all $t$,

$$\frac{1}{2} \langle Bu, u \rangle(t) = \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle Y(s), u(s) \rangle \, ds \quad (31.4.23)$$

Note that $\langle Bu, u \rangle(0) = \langle Bu_0, u_0 \rangle$.

**Proof:** By Lemma 31.4.4 there exists a sequence of partitions $\{t^n_k\}_{k=0}^{m_n} = P_n, P_n \subseteq P_{n+1}$, of $[0, T]$ such that the lengths of the sub intervals converge uniformly to 0 as $n \to \infty$ and the step functions

$$\sum_{k=0}^{m_n-1} u(t^n_k) \chi_{(t^n_k, t^n_{k+1}]}(t) \equiv u^l(t)$$

$$\sum_{k=0}^{m_n-1} u(t^n_k) \chi_{(t^n_k, t^n_{k+1}]}(t) \equiv u^r(t)$$

converge to $u$ in $L^p(0, T; V) \equiv K$. We assume that all of these partition points have empty intersection with the set of measure zero where $Bu(t) \neq B(u(t))$. Thus, at every partition point, $Bu(t_k) = B(u(t_k))$. As just mentioned, $L^p(0, T; V) \equiv K, L^p(0, T; V') = K'$.

**Lemma 31.4.4** Let $s < t$. Then for $u, Y$ satisfying (31.4.22)

$$\langle Bu(t), u(t) \rangle = \langle Bu(s), u(s) \rangle + 2 \int_s^t \langle Y(r), u(t) \rangle \, dr - \langle Bu(t) - Bu(s), u(t) - u(s) \rangle \quad (31.4.23)$$

**Proof:** It follows from the following computations

$$Bu(t) - Bu(s) = \int_s^t Y(r) \, dr$$

and so

$$2 \int_s^t \langle Y(r), u(t) \rangle \, dr - \langle Bu(t) - Bu(s), u(t) - u(s) \rangle$$
In the above situation, applied to the 31.4.4
Thus, discarding the negative terms and denoting by 
Also, 
Lemma 31.4.5 appears to work.
Note that in case 
Thus
Proof: From the above formula of Lemma , applied to the th partition of described above,
Thus, discarding the negative terms and denoting by the th of these partitions,

Lemma 31.4.5 In the above situation,

Also, is weakly continuous with values in where is the set of measure zero where \( Bu(t) \neq B(u(t)) \).

Proof: From the above formula of Lemma applied to the th partition of described above,

Thus, discarding the negative terms and denoting by the th of these partitions,
31.4. THE IMPLICIT CASE

\[
\langle Bu, u \rangle + 2 \left( \int_0^T \| Y(r) \|_{V'}^p \, dr \right)^{1/p'} \left( \int_0^T \| u_k(r) \|_{V'}^p \, dr \right)^{1/p} \\
\leq C (\| Y \|_{K'}, \| u \|_K)
\]

because these partitions are chosen such that

\[
\lim_{k \to \infty} \left( \int_0^T \| u_k(r) \|_V^p \, dr \right)^{1/p} = \left( \int_0^T \| u(r) \|_V^p \, dr \right)^{1/p}
\]

and so these are bounded. This has shown that for the dense subset of \([0, T], D \equiv \cup_k P_k, \sup_t \langle Bu(t), u(t) \rangle < C (\| Y \|_{K'}, \| u \|_K)

From Lemma 31.4.2 above, there exists \(\{e_i\} \subseteq V\) such that \(\langle Be_i, e_j \rangle = \delta_{ij}\) and for \(t \not\in N, \langle Bu(t), u(t) \rangle = \sum_{k=1}^{\infty} |\langle Bu(t), e_i \rangle|^2 = \sup_m \sum_{k=1}^m |\langle Bu(t), e_i \rangle|^2 \)

Thus, if \(s_n \to t, s_n \in D, \text{Fatou's lemma implies} \langle Bu(u(t), u(t)) = \sum_{k=1}^{\infty} |\langle Bu(t), e_i \rangle|^2 \\
\leq \lim_{n \to \infty} \inf_{n \to \infty} \sum_{k=1}^{\infty} |\langle Bu(s_n), e_i \rangle|^2 \leq C (\| Y \|_{K'}, \| u \|_K)
\]

and so

\[
\sup_{t \in N^C} \langle Bu, u \rangle (t) = \sup_{t \in N^C} \langle B (u(t)), u(t) \rangle \leq C (\| Y \|_{K'}, \| u \|_K)
\]

It only remains to verify the claim about weak continuity.

Consider now the claim that \(t \to Bu(t)\) is weakly continuous on \(N^C\). Letting \(v \in V, s \in N^C, \lim_{t \to s} \langle Bu(t), v \rangle = \langle Bu(s), v \rangle = \langle Bu(s), v \rangle \)

The limit follows from the formula 31.4.24 which implies \(t \to Bu(t)\) is continuous into \(V'\). Now for \(t \in N^C, \|

\[
\| Bu(t) \| = \sup_{\| v \| \leq 1} |\langle Bu(t), v \rangle| \leq \langle Bv, v \rangle^{1/2} \langle Bu(t), u(t) \rangle^{1/2}
\]

which was shown to be bounded for \(t, s \in N^C\). Now let \(w \in W\). Then

\[
|\langle Bu(t), w \rangle - \langle Bu(s), w \rangle| \leq |\langle Bu(t) - Bu(s), w - v \rangle| + |\langle Bu(t) - Bu(s), v \rangle|
\]

Then the first term is less than \(\varepsilon\) if \(v\) is close enough to \(w\) and the second converges to 0 so 31.4.24 holds for all \(v \in W\) and so this shows the weak continuity on \(N^C\).
Now pick $t \in D$, the union of all the mesh points. Then for all $k$ large enough, $t \in \mathcal{P}_k$. Say $t = t_m$. From Lemma 31.4.4,

$$- \sum_{j=0}^{m-1} \langle B(u(t_{j+1}) - u(t_j)) \rangle =$$

$$\langle Bu(t_m), u(t_m) \rangle - \langle Bu_0, u_0 \rangle - 2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u_r(r) \rangle dr$$

Thus, $\langle Bu(t_m), u(t_m) \rangle$ is constant for all $k$ large enough and the integral term converges to

$$\int_0^{t_m} \langle Y(r), u(r) \rangle dr$$

It follows that the term on the left does converge to something as $k \to \infty$. It just remains to consider what it does converge to. However, from the equation solved by $u$,

$$Bu(t_{j+1}) - Bu(t_j) = \int_{t_j}^{t_{j+1}} Y(r) dr$$

Therefore, this term is dominated by an expression of the form

$$\left| \sum_{j=0}^{m-1} \left( \int_{t_j}^{t_{j+1}} Y(r) dr, u(t_{j+1}) - u(t_j) \right) \right|$$

$$= \left| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_{j+1}) - u(t_j) \rangle \right|$$

$$= \left| \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_{j+1}) \rangle - \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_j) \rangle \right|$$

$$= \left| \int_0^{t_m} \langle Y(r), u^r(r) \rangle dr - \int_0^{t_m} \langle Y(r), u^l(r) \rangle dr \right|$$

$$\leq \int_0^T \left| \langle Y(r), u^r(r) - u^l(r) \rangle \right| dr$$

However, both $u^r$ and $u^l$ converge to $u$ in $K = L^p(0,T,V)$. Therefore, this term must converge to 0. Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$\langle Bu(t), u(t) \rangle = \langle Bu_0, u_0 \rangle + 2 \int_0^t \langle Y(r), u(r) \rangle dr$$
Then it was just shown that $Bu(t(k)) = Bu_0 + \int_{t(k)}^{t(m)} Y(s) \, ds$,

a similar formula for $u(t(k))$. Thus for $t > t(m)$,

$$Bu(t) - Bu(t(m)) = \int_{t(m)}^{t} Y(s) \, ds$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $u_0 = 0$. Therefore,

$$\langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle = 2\int_{t(m)}^{t(k)} \langle Y(s), u(s) - u(t(m)) \rangle \, ds$$

Thus, for $m \leq k$

$$\lim_{m,k \to \infty} \langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle = 0 \quad (31.4.25)$$

Hence $\{Bu(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W'$ because

$$\leq \langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle^{1/2} \langle By, y \rangle^{1/2}$$

$$\leq \langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle^{1/2} \|B\|^{1/2} \|y\|_{W}$$

Does it converge to $Bu(t)$? Let $\xi(t) \in W'$ be what it does converge to. Let $v \in V$. Then

$$\langle \xi(t), v \rangle = \lim_{k \to \infty} \langle Bu(t(k)), v \rangle = \lim_{k \to \infty} \langle Bu(t(k)), v \rangle = \langle Bu(t), v \rangle$$

because it is known that $t \to Bu(t)$ is continuous into $V'$. It is also known that for $t \in N^C$, $Bu(t) \in W' \subseteq V'$ and that the $Bu(t)$ for $t \in N^C$ are uniformly bounded in $W'$. Therefore, since $V$ is dense in $W$, it follows that $\xi(t) = Bu(t)$.

Now for every $t \in D$, it was shown above that

$$\langle Bu(t), u(t) \rangle = \langle Bu_0, u_0 \rangle + 2\int_{0}^{t} \langle Y(r), u(r) \rangle \, dr$$

Also it was just shown that $Bu(t(k)) \to Bu(t)$ for $t \notin N$. Then for $t \notin N$

$$\langle Bu(t(k)), u(t(k)) \rangle - \langle Bu(t), u(t) \rangle$$

$$\leq |\langle Bu(t(k)), u(t(k)) - u(t) \rangle| + |\langle Bu(t(k)) - Bu(t), u(t) \rangle|$$
Then the second term converges to 0. The first equals
\[ |\langle Bu(t(k)) - Bu(t), u(t(k)) \rangle| \]
\[ \leq \langle B(u(t(k)) - u(t)), u(t(k)) - u(t) \rangle^{1/2} \langle Bu(t(k)), u(t(k)) \rangle^{1/2} \]
From the above, this is dominated by an expression of the form
\[ \langle B(u(t(k)) - u(t)), u(t(k)) - u(t) \rangle^{1/2} C \]
Then using the lower semicontinuity of \( t \to \langle B(u(t(k)) - u(t)), u(t(k)) - u(t) \rangle \) on \( N^C \) which follows from the above, this is no larger than
\[ \lim_{m \to \infty} \inf \langle B(u(t(k)) - u(t(m))), u(t(k)) - u(t(m)) \rangle^{1/2} C < \varepsilon \]
provided \( k \) is large enough. This follows from [1.3.28]. Since \( \varepsilon \) is arbitrary, it follows that
\[ \lim_{k \to \infty} |\langle Bu(t(k)), u(t(k)) \rangle - \langle Bu(t), u(t) \rangle| = 0 \]
Then from the formula,
\[ \langle Bu(t), u(t) \rangle = \langle Bu_0, u_0 \rangle + 2 \int_0^t \langle Y(r), u(r) \rangle \, dr \]
valid for \( t \in D \), it follows that the same formula holds for all \( t \notin N \). Then define \( \langle Bu, u \rangle(t) \) to equal \( \langle Bu(t), u(t) \rangle \) off \( N \) and the right side for \( t \in N \). Thus \( t \to \langle Bu, u \rangle(t) \) is continuous and for all \( t \in [0, T] \),
\[ \langle Bu, u \rangle(t) = \langle Bu_0, u_0 \rangle + 2 \int_0^t \langle Y(r), u(r) \rangle \, dr \]
Also recall that \( t \to Bu(t) \) was shown to be weakly continuous into \( W' \) on \( N^C \). Then for \( t, s \in N^C \),
\[ \langle B(u(t) - u(s)), u(t) - u(s) \rangle \]
\[ = \langle Bu(t), u(t) \rangle - 2 \langle Bu(t), u(s) \rangle + \langle Bu(s), u(s) \rangle \]
From this, it follows that \( t \to Bu(t) \) is continuous into \( W' \) on \( N^C \) because \( \lim_{t \to s} \) of the right side gives 0 and so the same is true of the left. Hence,
\[ |\langle B(u(t) - u(s)), y \rangle| \leq \langle By, y \rangle^{1/2} \langle B(u(t) - u(s)), u(t) - u(s) \rangle^{1/2} \]
\[ \leq \|B\|^{1/2} \langle B(u(t) - u(s)), u(t) - u(s) \rangle^{1/2} \|y\| \]
so
\[ \|B(u(t) - u(s))\|_{W'} \leq \|B\|^{1/2} \langle B(u(t) - u(s)), u(t) - u(s) \rangle^{1/2} \]
which converges to 0 as \( t \to s \). ■
Consider the case that \( t \to B(u(t)) \) has a weak derivative, denoted as \((Bu)'(t)\) which is in \(L^{p'}(0,T;V')\). Then as shown above, there is a continuous function, denoted as \(Bu(t)\) which equals \(B(u(t))\) for a.e. \(t\)

\[
Bu(t) = Bu(0) + \int_0^t (Bu)'(s) \, ds
\]

Then the above theorem applies. Then one obtains the following corollary.

**Corollary 31.4.6** Let \( V \subseteq W, W' \subseteq V' \) be separable Banach spaces, and \( B \in \mathcal{L}(W,W') \) is nonnegative and self adjoint. Also suppose \( t \to B(u(t)) \) has a weak derivative \((Bu)' \in L^{p'}(0,T;V')\) for \( u \in L^p(0,T;V) \). Then there is a continuous function denoted as \(Bu(t)\) which equals \(B(u(t))\) a.e. \(t\). Say for \( t \notin N\). Suppose \(Bu(0) = Bu_0, u_0 \in W\). Then

\[
Bu(t) = Bu_0 + \int_0^t (Bu)'(s) \, ds \text{ in } V' \quad (31.4.26)
\]

Then \( t \to Bu(t) \) is in \( C(N^c, W') \) and also for such \(t\),

\[
\frac{1}{2} \langle Bu(t), u(t) \rangle = \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle (Bu)'(s), u(s) \rangle \, ds
\]

There exists a continuous function \( t \to \langle Bu, u \rangle(t) \) which equals the right side of the above for all \(t\) and equals \(\langle Bu(t), u(t) \rangle\) off \(N\). This also satisfies

\[
\sup_{t \in [0,T]} \langle Bu, u \rangle(t) \leq C \left( \left\| (Bu)' \right\|_{L^{p'}(0,T;V')}, \left\| u \right\|_{L^p(0,T;V)} \right)
\]

where we can take the right side to equal

\[
\langle Bu_0, u_0 \rangle + 2 \left\| (Bu)' \right\|_{L^{p'}(0,T;V')} \left\| u \right\|_{L^p(0,T;V)}
\]

This follows from the above theorem, in particular Lemma 31.3.8.

This also makes it easy to verify continuity of pointwise evaluation of \(Bu\). Let \(Lu = (Bu)'\).

\[
u = D(L) \equiv \left\{ u \in L^p(0,T;V) : L u \equiv (Bu)' \in L^{p'}(0,T,V') \right\}
\]

\[
\left\| u \right\|_X \equiv \left\| u \right\|_{L^p(0,T;V)} + \left\| Lu \right\|_{L^{p'}(0,T;V')}
\]

(31.4.27)

Since \(L\) is closed, this \(X\) is a Banach space.

Then the following theorem is obtained.

**Theorem 31.4.7** Say \((Bu)' \in L^{p'}(0,T;V')\) so

\[
Bu(t) = Bu(0) + \int_0^t (Bu)'(s) \, ds \text{ in } V'
\]

the map \(u \to Bu(t)\) is continuous as a map from \(X\) to \(V'\). Also, if \(Y\) denotes those \(f \in L^p([0,T];V)\) for which \(f' \in L^p([0,T];V)\), so that \(f\) has a representative such that \(f(t) = f(0) + \int_0^t f'(s) \, ds\), then if \(\|f\|_{Y} \equiv \|f\|_{L^p([0,T];V)} + \|f'\|_{L^p([0,T];V)}\), the map \(f \to f(t)\) is continuous.
**Proof:** First, why is \( u \to Bu(0) \) continuous? Say \( u, v \in X \) and say \( p \geq 2 \) first.

\[
Bu(t) - Bv(t) = Bu(0) - Bv(0) + \int_0^t (Bu)'(s) - (Bv)'(s) \, ds
\]

and so,

\[
\|Bu(0) - Bv(0)\|_{V'} \leq \|Bu(t) - Bv(t)\|_{V'} + \int_0^t \|(Bu)'(s) - (Bv)'(s)\|_{V'} \, ds
\]

then using the triangle inequality,

\[
\left( \int_0^T \|Bu(0) - Bv(0)\|_{V'}^p \, dt \right)^{1/p'} \leq \left( \int_0^T \|Bu(t) - Bv(t)\|_{V'}^p \, dt \right)^{1/p'}
\]

and so

\[
\|Bu(0) - Bv(0)\|_{V', T^{1/p'}} \leq \left( \|B\|_{L^{p'}([0,T];V')} + T^{1/p'} \|(Bu)' - (Bv)'\|_{L^{p'}([0,T];V')} \right)
\]

\[
\leq \left( \|B\|_{L^p([0,T];V)} + T^{1/p'} \|(Bu)' - (Bv)'\|_{L^{p'}([0,T];V')} \right)
\]

\[
\leq C(\|B\|, T) \|u - v\|_X
\]

Thus \( u \to Bu(0) \) is continuous into \( V' \). If \( p < 2 \), then you do something similar.

\[
\left( \int_0^T \|Bu(0) - Bv(0)\|_{V'}^p \, dt \right)^{1/p} \leq \left( \int_0^T \|Bu(t) - Bv(t)\|_{V'}^p \, dt \right)^{1/p} + \left( \int_0^T \|Bu(t) - Bv(t)\|_{V'}^p \, dt \right)^{1/p}
\]

\[
\leq \|B\|_{L^p([0,T];V')} + \int_0^T \left( \int_0^T \|(Bu)'(s) - (Bv)'(s)\|_{V'}^p \, dt \right)^{1/p} \, ds
\]

\[
\leq \|B\|_{L^p([0,T];V')} + T^{1/p} \int_0^T \|(Bu)'(s) - (Bv)'(s)\|_{V'} \, ds
\]

\[
\leq \|B\|_{L^p([0,T];V')} + CT^{1/p} \|(Bu)' - (Bv)'\|_{L^{p'}([0,T];V')}^p
\]

Thus

\[
\|Bu(0) - Bv(0)\|_{V', T^{1/p'}} \leq \|B\|_{L^p([0,T];V')} + C(T) \|(Bu)' - (Bv)'\|_{L^{p'}([0,T];V')}^p \leq C(\|B\|, T) \|u - v\|_X.
\]
However, one could just as easily have done this for an arbitrary $s < T$ by repeating the argument for

$$Bu (t) = Bu (s) + \int_s^t (Bu)' (r) \, dr$$

Thus this mapping is certainly continuous into $V'$. The last assertion is similar. You just use $f$ instead of $Bu$ and make easy modifications in the argument. It is all happening in one space in the second case. ■

For $u \in X$ defined above,

$$Bu (t) = Bu (0) + \int_0^t (Bu)' (s) \, ds,$$

and also

$$\frac{1}{2} \langle Bu (t), u (t) \rangle = \frac{1}{2} \langle Bu, u \rangle (0) + \int_0^t \langle (Bu)' (s), u (s) \rangle \, ds$$

This follows from a similar argument given above, (Note we write $\langle Bu, u \rangle (0)$ instead of $\langle Bu_0, u_0 \rangle$ since no $u_0$ is mentioned. One could also use the above by considering the problem on $[s, t]$ where $s$ is not in the exceptional set where it makes a difference between writing $Bu (s)$ and $B (u (s))$. Then you would get the above with $0$ replaced with $s$ and then let $s \to 0$ to finally obtain the above displayed formula. ) and

$$\sup_{t \in [0,T]} \langle Bu, u \rangle (t) \leq C \left( \| (Bu)' \|_{L^p (0,T,V')} , \| u \|_{L^p (0,T,V')} \right) = C \left( \| u \|_X \right)$$

where $X$ was defined in 31.4.27, then

$$\sup_{t \in [0,T]} \left\langle B \frac{u}{\| u \|_X} , \frac{u}{\| u \|_X} \right\rangle (t) \leq C \left( 1 \right) = C$$

and so

$$\sup_{t \in [0,T]} \langle Bu, u \rangle (t) \leq C \| u \|_X^2$$

Now define for $u, v \in X$

$$\langle Bu, v \rangle (t) \equiv \frac{1}{2} \left[ \langle B (u + v) , u + v \rangle (t) - (\langle Bu, u \rangle (t) + \langle Bv, v \rangle (t)) \right]$$

and so for a.e. $t$, $\langle Bu, v \rangle (t) = \langle B (u (t)) , v (t) \rangle$ and $t \to \langle Bu, v \rangle (t)$ is continous. Also, there must exist $C$ such that for all $u, v$ and $t \in [0, T]$

$$| \langle Bu, v \rangle (t) | \leq C \| u \|_X \| v \|_X$$

If this is not so, then you could get $u_n, v_n$ having norm equal to 1 in $X$ such that

$$\sup_{t \in [0,T]} | \langle Bu_n, v_n \rangle (t) | > n$$
But then, letting $t_n$ be a point where $|\langle Bu_n, v_n \rangle(t_n)| > n$, 
\[ n < |\langle Bu_n, v_n \rangle(t_n)| \leq \frac{1}{2} \left[ C \left( \|u_n + v_n\|_X^2 + \|u\|_X^2 + \|v\|_X^2 \right) \right] = \frac{C}{2} (4 + 1 + 1) = 3C \]
which is clearly a contradiction. It follows that one can define $K : X \to X'$ as follows.
\[ \langle Ku, v \rangle \equiv \int_0^T \langle Lu, v \rangle \, ds + \langle Bu, v \rangle(0) \]
Thus $K$ is linear and continuous. In addition,
\[ \langle Ku, u \rangle = \frac{1}{2} \left[ \langle Bu, u \rangle(T) + \langle Bu, u \rangle(0) \right] \]
To see this, Corollary 31.4.6 implies
\[ \frac{1}{2} \langle Bu, u \rangle(T) = \frac{1}{2} \langle Bu, u \rangle(0) + \int_0^T \langle (Bu)'(s), u(s) \rangle \, ds \]
and so
\[ \frac{1}{2} \langle Bu, u \rangle(T) + \langle Bu, u \rangle(0) \]
\[ = \frac{1}{2} \langle Bu, u \rangle(0) + \int_0^T \langle (Bu)'(s), u(s) \rangle \, ds + \langle Bu, u \rangle(0) \]
and so, this yields
\[ \int_0^T \langle (Bu)'(s), u(s) \rangle \, ds + \langle Bu, u \rangle(0) = \langle Ku, u \rangle = \frac{1}{2} \left[ \langle Bu, u \rangle(T) + \langle Bu, u \rangle(0) \right] \]
as claimed. This proves most of the following.

**Proposition 31.4.8** Let
\[ X = \left\{ u \in L^p(0, T; V) \equiv V : Lu \equiv (Bu)' \in L^{p'}(0, T, V') \right\} \]
where $V$ is a reflexive Banach space. Let a norm on $X$ be given by
\[ \|u\|_X \equiv \|u\|_V + \|Lu\|_{V'} \]
Then there is a continuous function $t \to \langle Bu, v \rangle(t)$ such that $\langle Bu, v \rangle(t) = \langle (Bu)(t), v(t) \rangle$ a.e. $t$ such that
\[ \sup_{t \in [0, T]} |\langle Bu, v \rangle(t)| \leq C \|u\|_X \|v\|_X \]
and if $K : X \to X'$
\[ \langle Ku, v \rangle \equiv \int_0^T \langle Lu, v \rangle \, ds + \langle Bu, v \rangle(0) \]
31.5. THE IMPLICIT CASE, $B = B (T)$

Then $K$ is continuous and linear and

$$
\langle Ku, u \rangle = \frac{1}{2} [ \langle Bu, u \rangle \left( T \right) + \langle Bu, u \rangle \left( 0 \right) ]
$$

If $u \in X$ and $Bu \left( 0 \right) = 0$ then there exists a sequence $\{u_n\}$ such that $\|u_n - u\|_X \to 0$ but $u_n \left( t \right) = 0$ for all $t$ close to 0.

**Proof:** It only remains to verify the last assertion. Let $\psi_n$ be increasing and piecewise linear such that $\psi_n \left( t \right) = 1$ for $t \geq 2/n$ and equals 0 on $[0, 1/n]$. Then clearly $\psi_n u \to u$ in $V$.

$$
(B(\psi_n u))' = \psi_n' Bu + \psi_n (Bu)'
$$

The second term converges to $(Bu)'$ in $V'$. It remains to consider the first term.

$$
\int_0^T \|\psi_n' Bu\|_{V'}^p \, dt \leq n \int_0^{2/n} \left\| \int_0^t (Bu)' \, ds \right\|_{V'}^p \, dt
$$

$$
\leq n \int_0^{2/n} t^{p'-1} \left\| (Bu)' \right\|_{V'}^{p'} \, ds \, dt \leq \int_0^{2/n} \left\| (Bu)' \right\|_{V'}^{p'} \, ds \frac{1}{p} \left( \frac{2}{n} \right)^{p'}
$$

Since $p' > 1$, this converges to 0. ■

Note that, by convolving with a mollifier, we could assume each $u_n$ is also smooth. In addition to this, we can draw a similar conclusion at the right endpoint. That is, if $Bu \left( T \right) = 0$ there is a sequence $\{u_n\} \subseteq X$ where $u_n \left( t \right) = 0$ for $t$ near $T$ which converges to $u$ in $X$.

31.5 The Implicit Case, $B = B (t)$

The above theorem can be generalized to the case where the formula is of the form

$$
BX \left( t \right) = BX_0 + \int_0^t Y \left( s \right) \, ds
$$

This involves an operator $B \left( t \right) \in \mathcal{L} (W, W')$ and $B \left( t \right)$ satisfies

$$
\langle B \left( t \right) x, x \rangle \geq 0, \quad \langle B \left( t \right) x, y \rangle = \langle B \left( t \right) y, x \rangle
$$

for

$$
V \subseteq W, W' \subseteq V'
$$

Where we assume $t \to B \left( t \right)$ is in $C^1 \left( [0, T]; \mathcal{L} (W, W') \right)$ and $V$ is dense in the Banach space $W$.

Then the main result in this section is the following integration by parts theorem.
Theorem 31.5.1 Let $V \subseteq W, W' \subseteq V'$ be separable Banach spaces, and let $Y \in L^p(0, T; V')$ and

$$Bu(t) = Bu_0 + \int_0^t Y(s) \, ds \text{ in } V', \quad u_0 \in W, Bu(t) = B(t)(u(t)) \text{ for a.e. } t$$

(31.5.28)

As indicated, $Bu$ is the name of a function satisfying the above equation which satisfies $Bu(t) = B(t)(u(t))$ for a.e. $t$. Thus $Y = (Bu)'$ as a weak derivative in the sense of $V'$ valued distributions. Suppose that $u \in L^p([0, T], V)$ and $(s, t) \rightarrow B'(s)u(t)$ is bounded in $V'$ in case $p > 2$. (If $B(t)$ is constant in $t$ this is obvious.) In the case where $p \geq 2$, it is enough to assume $B' \in C^1([0, T]; L(W, W'))$. Then $t \rightarrow Bu(t)$ is continuous into $W'$ for $t$ off a set of measure zero $N$ and also there exists a continuous function $t \rightarrow \langle Bu, u \rangle(t)$ such that for all $t \notin N, \langle Bu, u \rangle(t) = \langle B(u(t)), u(t) \rangle, Bu(t) = B(t)(u(t))$, and for all $t$,

$$\frac{1}{2}\langle Bu, u \rangle(t) + \frac{1}{2} \int_0^t \langle B' u, u \rangle \, ds = \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle Y(s), u(s) \rangle \, ds$$

Proof: By Lemma 31.3.1 there exists a sequence of partitions $\{t_k^n\}_{k=0}^{m_n} = \mathcal{P}_n, \mathcal{P}_n \subseteq \mathcal{P}_{n+1},$ of $[0, T]$ such that the lengths of the sub intervals converge uniformly to 0 as $n \rightarrow \infty$ and the step functions

$$\sum_{k=0}^{m_n-1} u(t_k^n) \chi_{(t_k^n, t_{k+1}^n)}(t) = u_n^l(t)$$

$$\sum_{k=0}^{m_n-1} u(t_{k+1}^n) \chi_{(t_k^n, t_{k+1}^n)}(t) = u_n^r(t)$$

converge to $u$ in $L^p(0, T; V) = K$. We assume that all of these partition points have empty intersection with the set of measure zero where $Bu(t) \neq B(t)(u(t))$. Thus, at every partition point, $Bu(t_k) = B(t_k)(u(t_k))$. As just mentioned, $L^p(0, T; V) = K$, $L^p(0, T; V') = K'$. Taking a subsequence, we can have

$$\|u_n^l - u\|_K + \|u_n^r - u\|_K + \|Bu_n^l - Bu\|_{K'} + \|Bu_n^r - Bu\|_{K'} + \|B'u_n^l - B'u\|_{L^2([0, T], W')} + \|B'u_n^r - B'u\|_{L^2([0, T], W')} < 2^{-n}$$

(31.5.29)

and so, we can assume that a.e. convergence also takes place for $Bu_n^l, Bu_n^r, B'u_n^l, B'u_n^r, u_n^l, u_n^r$. Is $Bu(0) = B(0)u_0$? The integral equation gives this it seems. To save notation, $B(0) u_0$ will be written as $Bu_0$. This is not inconsistent because $t \rightarrow B(t)u_0$ is continuous and its value at 0 is $B(0)u_0$.

Lemma 31.5.2 Let $s < t$. Then for $u, Y$ satisfying

$$\langle Bu(t), u(t) \rangle - \langle Bu(s), u(s) \rangle + \langle (B(t) - B(s))u(s), u(t) \rangle$$

CHAPTER 31. GELFAND TRIPLES AND RELATED STUFF
31.5. THE IMPLICIT CASE, \( B = B(T) \)

\[ + \langle (B(t) - B(s)) u(s), u(t) - u(s) \rangle = 2 \int_s^t \langle Y(r), u(t) \rangle \, dr \]

\[ - \langle B(t) u(t) - B(t) u(s), u(t) - u(s) \rangle \]  \hspace{1cm} (31.5.30)

**Proof:** It follows from the following computations

\[
B(t) u(t) - B(s) u(s) = \int_s^t Y(r) \, dr
\]

and so

\[
2 \int_s^t \langle Y(r), u(t) \rangle \, dr - \langle B(t) u(t) - B(s) u(s), u(t) - u(s) \rangle
\]

\[
= 2 \left( \int_s^t Y(r) \, dr, u(t) \right) - \langle B(t) u(t) - B(s) u(s), u(t) - u(s) \rangle
\]

\[
= 2 \langle B(t) u(t) - B(s) u(s), u(t) \rangle - \langle B(t) u(t) - B(s) u(s), u(t) - u(s) \rangle
\]

\[
= 2 \langle B(t) u(t) - B(s) u(s), u(t) \rangle - 2 \langle B(s) u(s), u(t) \rangle - \langle B(t) u(t), u(t) \rangle
\]

\[
+ \langle B(t) u(t), u(s) \rangle + \langle B(s) u(s), u(t) \rangle - \langle B(s) u(s), u(s) \rangle
\]

\[
= \langle B(t) u(t), u(t) \rangle - \langle B(s) u(s), u(s) \rangle
\]

\[
+ \langle B(t) - B(s) u(s), u(t) \rangle
\]

Thus

\[
\langle Bu(t), u(t) \rangle - \langle Bu(s), u(s) \rangle + \langle (B(t) - B(s)) u(s), u(t) \rangle
\]

\[= 2 \int_s^t \langle Y(r), u(t) \rangle \, dr - \langle B(t) u(t) - B(s) u(s), u(t) - u(s) \rangle \]

Now consider the last term. It equals

\[
\langle B(t) u(t) - (B(s) - B(t) + B(t)) u(s), u(t) - u(s) \rangle
\]

\[
= \langle B(t) u(t) - ((B(s) - B(t)) u(s) + B(t) u(s)), u(t) - u(s) \rangle
\]

\[
= \langle B(t) u(t) - B(t) u(s), u(t) - u(s) \rangle + \langle (B(t) - B(s)) u(s), u(t) - u(s) \rangle
\]

It follows that

\[
\langle Bu(t), u(t) \rangle - \langle Bu(s), u(s) \rangle + \langle (B(t) - B(s)) u(s), u(t) \rangle
\]

\[+ \langle (B(t) - B(s)) u(s), u(t) - u(s) \rangle \]
\[ = 2 \int_s^t \langle Y(r) , u(t) \rangle \, dr - \langle B(t) u(t) - B(t) u(s) , u(t) - u(s) \rangle \]

Of course this computation is under the assumption that neither \( s, t \) are in the exceptional set off which \( B(t) u(t) = Bu(t) \). In case \( s = 0 \) the same formula holds except you need to replace \( u(s) \) with \( u_0 \) and \( Bu(s) \) with \( Bu(0) \).

It is good to emphasize part of the above.

\[ \langle (B(t) u(t) - B(t) u(s) , u(t) - u(s)) - \langle B(t) u(t) - B(s) u(s) , u(t) - u(s) \rangle \]

\[ = \langle (B(s) - B(t)) u(s) , u(t) - u(s) \rangle \]

**Lemma 31.5.3** Let the partitions \( P_k \) be as above such that \( 31.5.29 \), \( P_k = \{ t_j^k \}_{j=0}^{m_k} \). Then for any \( m \leq m_k \),

\[
\sum_{j=0}^{m-1} \langle B(t_{j+1}^k) u(t_{j+1}^k) - B(t_j^k) u(t_j^k) , u(t_{j+1}^k) - u(t_j^k) \rangle -
\sum_{j=0}^{m-1} \langle B(t_{j+1}^k) u(t_{j+1}^k) - B(t_j^k) u(t_j^k) , u(t_{j+1}^k) - u(t_j^k) \rangle = \varepsilon^m(k)
\]

where \( \lim_{k \to \infty} \varepsilon^m(k) = 0 \). Here

\[ \varepsilon^m(k) = \sum_{j=0}^{m-1} \langle (B(t_j^k) - B(t_{j+1}^k)) u(t_j^k) , u(t_{j+1}^k) - u(t_j^k) \rangle \]

**Proof:** From the above lemma, the absolute value of the left side is no larger than

\[
\sum_{j=0}^{m-1} \left| \langle (B(t_j^k) - B(t_{j+1}^k)) u(t_j^k) , u(t_{j+1}^k) - u(t_j^k) \rangle \right|
\]

\[
\leq \sum_{j=0}^{m-1} \int_{t_j^k}^{t_{j+1}^k} \| B'(\tau) u(t_j^k) \|_{W} \, d\tau \| u(t_{j+1}^k) - u(t_j^k) \|_{W} \quad (31.5.31)
\]
In case \( p \geq 2 \) then for \( C \geq \max_s \| B' (s) \|_{L^p (W, W')} \),
\[
\begin{align*}
&\leq C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \| u^l (\tau) \|_W \| u^l_k (\tau) - u^l (\tau) \|_W d\tau \\
&= C \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} X_{[t_j, t_{j+1}]} (\tau) \| u^l_k (\tau) \|_W \| u^l_k (\tau) - u^l (\tau) \|_W d\tau \\
&= C \int_0^{t_m} \sum_{j=0}^{m-1} X_{[t_j, t_{j+1}]} (\tau) \| u^l_k (\tau) \|_W \| u^l_k (\tau) - u^l (\tau) \|_W d\tau \\
&\leq C \int_0^{t_m} \| u^l (\tau) \|_W \| u^l_k (\tau) - u^l k (\tau) \|_W d\tau \\
&\leq \hat C (2)^{-k}
\end{align*}
\]
by (4.13). In case \( p < 2 \), then from assumption and (4.13), the absolute value of
the left side is no larger than
\[
\begin{align*}
&\frac{1}{\lambda^k} \sum_{j=0}^{m-1} C \{ (t_{j+1}^k - t_j^k ) \| u (t_{j+1}^k ) - u (t_j^k ) \|_W \\
&= C \int_0^{t_m} \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} X_{[t_j, t_{j+1}]} (s) \| u^l_k (s) - u^l_k (s) \|_W \\
&= C \int_0^{t_m} \| u^l_k (s) - u^l_k (s) \|_W
\end{align*}
\]
which converges to 0 as \( k \to \infty \) thanks to (4.13).

Lemma 31.5.4 In the above situation,
\[
\sup_{t \in N' } \langle Bu (t) , u (t) \rangle + \int_0^T \langle B' u , u \rangle ds \leq C (\| Y \|_{K' ,} , \| u \|_K)
\]
Also, \( t \to Bu (t) \) is weakly continuous with values in \( W' \) on \( N' \) where \( N \) is a set
of measure zero including the set where \( Bu (t) \neq B (t) (u (t)) \).

Proof: From the above formula of Lemma 4.13\( \text{applied to the } k^{th} \) partition
of \([0, T]\) described above,
\[
\langle Bu (t_m) , u (t_m) \rangle - \langle Bu_0 , u_0 \rangle + \sum_{j=0}^{m-1} \langle (B (t_{j+1}) - B (t_j)) u (t_j) , u (t_{j+1}) \rangle \\
* \sum_{j=0}^{m-1} \langle (B (t_{j+1}) - B (t_j)) u (t_j) , u (t_{j+1}) - u (t_j) \rangle
\]
\[= \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_{j+1}) \rangle \, dr - \langle B(t_{j+1}) u(t_{j+1}) - B(t_j) u(t_j), u(t_{j+1}) - u(t_j) \rangle \]  

(31.5.32)

Consider the third term on the left,

\[
\int_0^{t_m} \left( \sum_{j=0}^{m-1} \langle B(t_{j+1}) - B(t_j) \rangle u(t_{j+1}) \right)
\]

\[
= \int_0^{t_m} \left( \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} X(t_k, t_{k+1}) \right) \frac{B(t_{j+1}) - B(t_j)}{t_{j+1} - t_j} u_n(t) \, dt
\]

Using a simple approximate identity argument and the assumption that \( t \to B(t) \) is in \( C^1([0, T], \mathcal{L}(W, W')) \),

\[
\sum_{j=0}^{m-1} \sum_{k=0}^{m-1} X(t_k, t_{k+1}) \frac{B(t_{j+1}) - B(t_j)}{t_{j+1} - t_j} u_n(t) \to B'(t)
\]

uniformly on \( (0, T] \). Then \( \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} X(t_k, t_{k+1}) \frac{B(t_{j+1}) - B(t_j)}{t_{j+1} - t_j} u_n(t) \to B'u \) strongly in \( L^2([0, T], W') \) while \( u_n \to u \) strongly in \( L^2([0, T]; W) \). It follows that the third term on the left is

\[
\varepsilon(k) + 2 \int_0^T \langle B'u, u \rangle \, ds, \quad \varepsilon(k) \to 0.
\]

whenever \( n \) is sufficiently large. Also, \( T \) could be replaced with \( t_j \) for any of the mesh points.

Next consider the term labelled \(*\). From Lemma [Lemma], it is of the form \( \varepsilon_m(k) \) where \( \lim_{k \to \infty} \varepsilon_m(k) = 0 \). Thus [Lemma] reduces to

\[
\langle Bu(t_m), u(t_m) \rangle - \langle Bu_0, u_0 \rangle + \int_0^{t_m} \langle B'u, u \rangle \, ds = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(r), u_k(r) \rangle \, dr
\]

\[
- \sum_{j=0}^{m-1} \langle B(t_{j+1}) u(t_{j+1}) - B(t_j) u(t_j), u(t_{j+1}) - u(t_j) \rangle + \varepsilon(k)
\]

where \( t_m \in \mathcal{P}_k \).

Thus, discarding the negative terms which occur at the end and denoting by \( \mathcal{P}_k \) the \( k^{th} \) of these partitions,

\[
\sup_{t_j \in \mathcal{P}_k} \langle Bu(t_j), u(t_j) \rangle + \int_0^T \langle B'u, u \rangle \, ds \leq \langle Bu_0, u_0 \rangle + 2 \int_0^T \| Y(r) \| \| u_k(r) \| \, dr + \varepsilon
\]

\[
\leq \langle Bu_0, u_0 \rangle + 2 \int_0^T \| Y(r) \| \| u_k(r) \| \, dr + \varepsilon
\]
31.5. THE IMPLICIT CASE, \( B = B(T) \)

\[
\leq \langle B u_0, u_0 \rangle + 2 \left( \int_0^T \| Y(r) \|_{V'}^p \, dr \right)^{1/p'} \left( \int_0^T \| u_k^r(r) \|_{V'}^p \, dr \right)^{1/p} + \varepsilon
\]

\[
\leq C (\| Y \|_{K'}, \| u \|_K) + \varepsilon
\]

whenever \( k \) is large enough because these partitions are chosen such that

\[
\lim_{k \to \infty} \left( \int_0^T \| u_k^r(r) \|_{V'}^p \, dr \right)^{1/p} = \left( \int_0^T \| u(r) \|_{V'}^p \, dr \right)^{1/p}
\]

and so these are bounded. This has shown that for the dense subset of \([0, T]\),

\[
\sup_{t \in D} \langle B(t) u(t), u(t) \rangle + \int_0^T \langle B'u, u \rangle \, ds < C (\| Y \|_{K'}, \| u \|_K) + \varepsilon
\]

However, \( \varepsilon \) was arbitrary and the partitions are nested. Hence the above holds for all \( \varepsilon \) and so

\[
\sup_{t \in D} \langle B(t) u(t), u(t) \rangle + \int_0^T \langle B'u, u \rangle \, ds < C (\| Y \|_{K'}, \| u \|_K)
\]

By the integral equation, there is a set of measure zero including all the earlier sets of measure zero \( N \) such that for \( t \notin N, u^*_n(t), u^*_n(t) \to u(t) \) pointwise in \( V \). Also, \( B(t) u^*_n(t) \to Bu(t) \) in \( V' \). This last can be obtained from the integral equation solved, \( t \to Bu(t) \) is continuous into \( V' \). Then let \( t \notin N \). We have \( u^*_n(t) \to u(t) \) in \( V \). Now \( B(t) u^*_n(t) = B(t) u(s_n) \) where \( s_n \in D \) and \( s_n \to t \). Then \( Bu(t) = B(t) u(t) \) and

\[
\| B(s_n) u(s_n) - B(t) u(t) \|_{V'} \leq \| B(s_n) - B(t) \| u(s_n) \|_{V'} + \| B(t) (u(s_n) - u(t)) \|_{V'}
\]

\[
\leq C_t \| B(s_n) - B(t) \| + \| u(s_n) - u(t) \|_{V'}
\]

where \( C_t \) is a constant which comes because \( u(s_n) \to u(t) \) in \( V \) and so is bounded. The constant \( C \) is just \( \max_{\| B(t) \|} \). Then, since the two terms on the right converge to 0 as \( n \to \infty \), it follows that as \( s_n \to t, B(s_n) u(s_n) \to B(t) u(t) = Bu(t) \) in \( V' \) while \( u(s_n) \to u(t) \) in \( V \). It follows that for \( t \notin N \),

\[
\langle Bu(t), u(t) \rangle + \int_0^T \langle B'u, u \rangle \, ds = \lim_{n \to \infty} (Bu(s_n), u(s_n)) + \int_0^T \langle B'u, u \rangle \, ds \leq C (\| Y \|_{K'}, \| u \|_K)
\]

Hence,

\[
\sup_{t \notin N} \langle Bu(t), u(t) \rangle + \int_0^T \langle B'u, u \rangle \, ds \leq C (\| Y \|_{K'}, \| u \|_K)
\]

It only remains to verify the claim about weak continuity.

Consider now the claim that \( t \to Bu(t) \) is weakly continuous on \( N^C \). Letting \( v \in V, s \in N^C \),

\[
\lim_{t \to s} \langle Bu(t), v \rangle = \langle Bu(s), v \rangle = \langle Bu(s), v \rangle
\]

(31.5.34)
The limit follows from the formula \(31.5.28\) which implies \(t \to Bu(t)\) is continuous into \(V'\). Now for \(t \in N^C\),

\[
\|Bu(t)\|_{V'} = \sup_{\|v\|_{V'} \leq 1} |\langle Bu(t), v \rangle| \leq \langle Bu, v \rangle^{1/2} \langle Bu(t), u(t) \rangle^{1/2}
\]

\[
\leq \left( C(\|Y\|_K, \|u\|_K) - \int_0^T \langle B'u, u \rangle \, ds \right)
\]

\[
\sup_{t \notin N} \|Bu(t)\|_{V'} \leq \left( C(\|Y\|_K, \|u\|_K) - \int_0^T \langle B'u, u \rangle \, ds \right)
\]

Now let \(w \in W\). Then

\[
|\langle Bu(t), w \rangle - \langle Bu(s), w \rangle| \leq |\langle Bu(t) - Bu(s), w - v \rangle| + |\langle Bu(s), v \rangle|_{V', V}
\]

Then the first term is less than \(\varepsilon\) if \(v\) is close enough to \(w\) and the second converges to 0 by continuity of \(t \to Bu(t)\) which comes from the integral equation, so \(31.5.28\) holds for all \(v \in W\) and so this shows the weak continuity of \(t \to Bu(t)\) on \(N^C\). \(\blacksquare\)

Now pick \(t \in D\), the union of all the mesh points. Then for all \(k\) large enough, \(t \in \mathcal{T}_k\). Say \(t = t_m\). From

\[
\langle Bu(t_m), u(t_m) \rangle - \langle Bu_0, u_0 \rangle + \int_0^{t_m} \langle B'u, u \rangle \, ds = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(r), u^r(r) \rangle \, dr + \varepsilon(k)
\]

\[
- \sum_{j=0}^{m-1} \langle B(t_{j+1})u(t_{j+1}) - B(t_j)u(t_j), u(t_{j+1}) - u(t_j) \rangle
\]

(31.5.35)

where \(\varepsilon(k) \to 0\). By Lemma \(31.5.36\), you can modify \(\varepsilon(k)\) and write this in the form

\[
\langle Bu(t_m), u(t_m) \rangle - \langle Bu_0, u_0 \rangle + \int_0^{t_m} \langle B'u, u \rangle \, ds = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(r), u^r(r) \rangle \, dr + \varepsilon(k)
\]

\[
- \sum_{j=0}^{m-1} \langle B(t_{j+1})u(t_{j+1}) - B(t_j)u(t_j), u(t_{j+1}) - u(t_j) \rangle
\]

(31.5.36)

Thus, \(\langle Bu(t_m), u(t_m) \rangle\) is constant for all \(k\) large enough and the integral term on the right converges as \(k \to \infty\) to

\[
\int_0^{t_m} \langle Y(r), u(r) \rangle \, dr
\]

It follows that the last term on the right does converge to something as \(k \to \infty\). It just remains to consider what it does converge to. However, from the equation solved by \(u\),

\[
Bu(t_{j+1}) - Bu(t_j) = \int_{t_j}^{t_{j+1}} Y(r) \, dr
\]
31.5. THE IMPLICIT CASE, $B = B(T)$

Therefore, this term is dominated by an expression of the form

$$\left| \sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y(r) \, dr, u(t_{j+1}) - u(t_j) \right) \right|$$

$$= \left| \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_{j+1}) - u(t_j) \rangle \, dr \right|$$

$$= \left| \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_{j+1}) \rangle - \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y(r), u(t_j) \rangle \right|$$

$$= \left| \int_{t^m}^{T} \langle Y(r), u^r(r) \rangle \, dr - \int_{t^m}^{t^k} \langle Y(r), u^l(r) \rangle \, dr \right|$$

$$\leq \int_{t^m}^{T} \left| \langle Y(r), u^r(r) - u^l(r) \rangle \right| \, dr$$

However, both $u^r$ and $u^l$ converge to $u$ in $K = L^p(0, T, V)$. Therefore, this term must converge to 0. Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$\langle Bu(t), u(t) \rangle + \int_0^t \langle B'u, u \rangle \, dr = \langle Bu_0, u_0 \rangle + 2 \int_0^t \langle Y(r), u(r) \rangle \, dr$$

It remains to verify that this holds for all $t \notin N$. Let $t \in N^C \setminus D$ and let $t(k) \in P_k$ be the largest point of $P_k$ which is less than $t$. Suppose $t(m) \leq t(k)$ so that $m \leq k$. Then

$$Bu(t(m)) = Bu_0 + \int_0^{t(m)} Y(s) \, ds,$$

a similar formula for $u(t(k))$. Thus for $t > t(m)$,

$$Bu(t) - Bu(t(m)) = \int_{t(m)}^t Y(s) \, ds$$

which is the same sort of thing already looked at except that it starts at $t(m)$ rather than at 0 and $u_0 = 0$. Therefore,

$$\langle B(u(t(k)) - u(t(m))) \rangle,$$

$$\langle Bu(t(k)) - Bu(t(m)) \rangle$$

$$= 2 \int_{t(m)}^{t(k)} \langle Y(s), u(s) - u(t(m)) \rangle \, ds$$
Thus, for $m \leq k$

$$\lim_{m,k \to \infty} \langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle = 0 \quad (31.5.37)$$

Hence $\{Bu(t(k))\}_{k=1}^{\infty}$ is a convergent sequence in $W'$ because

$$|\langle B(u(t(k)) - u(t(m)), y \rangle| \leq \langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle^{1/2} \langle By, y \rangle^{1/2}$$

$$\leq \langle B(u(t(k)) - u(t(m)), u(t(k)) - u(t(m)) \rangle^{1/2} \|B\|^{1/2} \|y\|_W$$

Does it converge to $Bu(t)$? Let $\xi(t) \in W'$ be what it does converge to. Let $v \in V$. Then

$$\langle \xi(t), v \rangle = \lim_{k \to \infty} \langle Bu(t(k)), v \rangle = \lim_{k \to \infty} \langle Bu(t(k)), v \rangle = \langle Bu(t), v \rangle$$

because it is known that $t \to Bu(t)$ is continuous into $V'$. It is also known that for $t \in NC$, $Bu(t) \in W' \subseteq V'$ and that the $Bu(t)$ for $t \in NC$ are uniformly bounded in $W'$. Therefore, since $V$ is dense in $W$, it follows that $\xi(t) = Bu(t)$.

Now for every $t \in D$, it was shown above that

$$\langle Bu(t), u(t) \rangle + \int_0^t \langle B'u, u \rangle \, dr = \langle Bu_0, u_0 \rangle + 2 \int_0^t \langle Y(r), u(r) \rangle \, dr$$

Also it was just shown that $Bu(t(k)) \to Bu(t)$ for $t \notin N$. Then for $t \notin N$

$$|\langle Bu(t(k)), u(t(k)) \rangle - \langle Bu(t), u(t) \rangle| \leq |\langle B(t(k)) u(t(k)), u(t(k)) - u(t) \rangle| + |\langle Bu(t(k)) - Bu(t), u(t) \rangle|$$

Then the second term converges to 0. The first equals

$$|\langle B(t(k)) u(t(k)) - B(t(k)) u(t), u(t(k)) \rangle|$$

$$\leq \langle B(t(k)) u(t(k)) - u(t), u(t(k)) - u(t) \rangle^{1/2} \langle Bu(t(k)), u(t(k)) \rangle^{1/2}$$

From the above, this is dominated by an expression of the form

$$\langle B(t(k)) u(t(k)) - u(t), u(t(k)) - u(t) \rangle^{1/2} C$$

Then from the choice of $N$ and the pointwise convergence of $u_n^t$ to $u$ off $N$ the above converges to 0 for each $t \notin N$. It follows that

$$\lim_{k \to \infty} |\langle Bu(t(k)), u(t(k)) \rangle - \langle Bu(t), u(t) \rangle| = 0$$

Then from the formula,

$$\langle Bu(t), u(t) \rangle = \langle Bu_0, u_0 \rangle + 2 \int_0^t \langle Y(r), u(r) \rangle \, dr - \int_0^t \langle B'u, u \rangle \, dr$$
Corollary 31.5.5 Let $V \subseteq W, W' \subseteq V'$ be separable Banach spaces, and $B(t) \in \mathcal{L}(W, W')$ is nonnegative and self-adjoint, $B \in C^1([0, T]; W')$. Also suppose $t \to B(u(t))$ has a weak derivative $(Bu)' \in L^p(0, T; V')$ for $u \in L^p([0, T]; V) \cap L^2([0, T]; W)$. Then the above theorem applies. Then one obtains the following corollary.
Then there is a continuous function denoted as \( B_u(t) \) which equals \( B(t)(u(t)) \) a.e. \( t \). Say for \( t \notin N \). Suppose \( B_u(0) = B_{u_0}, \ u_0 \in W \). Then

\[
B_u(t) = B_{u_0} + \int_0^t (Bu)'(s)\, ds \quad \text{in} \quad V' \tag{31.5.38}
\]

Then \( t \to B_u(t) \) is in \( C(N, W') \) and also for such \( t \),

\[
\frac{1}{2} \langle Bu(t), u(t) \rangle + \frac{1}{2} \int_0^t \langle B'(s) u(s), u(s) \rangle\, ds = \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle (Bu)'(s), u(s) \rangle\, ds
\]

There exists a continuous function \( t \to \langle Bu, u \rangle(t) \) which equals the right side of the above for all \( t \) and equals \( \langle B(t) u(t), u(t) \rangle \) off \( N \). This satisfies

\[
\sup_{t \in [0,T]} \langle Bu, u \rangle(t) \leq C(\|Y\|_{K'}, \|u\|_K)
\]

In particular, this last inequality follows from Lemma \[31.9.4\] and the assumption that \( B' \) is bounded.

Note how if everything is nice and smooth, this integration by parts formula is what you would be expected to get. To see this, assume \( u \) is smooth and formally work on the right side.

\[
\frac{d}{dt} \langle Bu, u \rangle = \langle (Bu)', u \rangle + \langle Bu, u' \rangle
\]

\[
= \langle (Bu)', u \rangle + \langle Bu', u \rangle
\]

\[
= 2 \langle (Bu)', u \rangle - \langle B'u, u \rangle
\]

Thus

\[
\frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle (Bu)'(s), u(s) \rangle\, ds
\]

\[
= \frac{1}{2} \langle Bu_0, u_0 \rangle + \frac{1}{2} \left[ \int_0^t \frac{d}{ds} \langle Bu, u \rangle\, ds + \int_0^t \langle B'u, u \rangle\, ds \right]
\]

\[
= \frac{1}{2} \langle Bu(t), u(t) \rangle + \frac{1}{2} \int_0^t \langle B'u, u \rangle\, ds
\]

which equals the left side.

A related topic is the continuity of pointwise evaluation of \( Bu \). Let \( Lu = (Bu)' \).

\[
u = D(L) \equiv \left\{ u \in L^p(0,T; V) : Lu \in L^{p'}(0,T,V') \right\}
\]

\[
\|u\|_X = \|u\|_{L^p(0,T;V)} + \|Lu\|_{L^{p'}(0,T,V')}
\]

Since \( L \) is closed, this \( X \) is a Banach space. Then the following theorem is obtained.
Theorem 31.5.6 In the above corollary, the map \( u \to Bu(t) \) is continuous as a map from \( X \) to \( V' \). Also if \( Y \) denotes those \( f \in L^p([0,T];V) \) for which \( f' \in L^p([0,T];V') \), so that \( f \) has a representative such that \( f(t) = f(0) + \int_0^t f'(s) \, ds \), then if \( \|f\|_Y = \|f\|_{L^p([0,T];V)} + \|f'\|_{L^p([0,T];V')} \) the map \( f \to f(t) \) is continuous.

**Proof:** First, why is \( u \to Bu(0) \) continuous? Say \( u,v \in X \) and say \( p \geq 2 \) first.

\[
Bu(t) - Bv(t) = Bu(0) - Bv(0) + \int_0^t (Bu)'(s) - (Bv)'(s) \, ds
\]

and so,

\[
\left( \int_0^T \|Bu(0) - Bv(0)\|_{V'}^{p'} \, dt \right)^{1/p'} \leq \left( \int_0^T \|Bu(t) - Bv(t)\|_{V'}^{p'} \, dt \right)^{1/p'}
\]

and so

\[
\|Bu(0) - Bv(0)\|_{V',T^{1/p'}} \leq \left( \|B\| \|u - v\|_{L^{p'}([0,T];V')} + T^{1/p'} \|(Bu)' - (Bv)'\|_{L^{p'}([0,T];V')} \right)
\]

\[
\leq C(\|B\|,T) \|u - v\|_X.
\]

Thus \( u \to Bu(0) \) is continuous into \( V' \). If \( p < 2 \), then you do something similar.

\[
\left( \int_0^T \|Bu(0) - Bv(0)\|_{V'}^{p'} \, dt \right)^{1/p'} \leq \left( \int_0^T \|Bu(t) - Bv(t)\|_{V'}^{p'} \, dt \right)^{1/p'}
\]

\[
+ \left( \int_0^T \|\int_s^t (Bu)'(s) - (Bv)'(s) \, ds \|_{V'}^{p} \, dt \right)^{1/p}
\]

\[
\|Bu(0) - Bv(0)\|_{V',T^{1/p}} \leq \|B\| \|u - v\|_{L^p} + C(T) \|(Bu)' - (Bv)'\|_{L^{p'}([0,T];V')}
\]

\[
\leq C(\|B\|,T) \|u - v\|_X.
\]

However, one could just as easily have done this for an arbitrary \( s < T \) by repeating the argument for

\[
Bu(t) = Bu(s) + \int_s^t (Bu)'(r) \, dr
\]

Thus this mapping is certainly continuous into \( V' \). The last assertion is similar. ■
31.6 Another Approach

The above approach is pretty interesting, but there is a quicker way to do it discussed in this section. I am also including the case where the operator $B$ is actually a function of $t$. I have never had a reason to use this level of generality, but it is here if it is of any interest. Also, this is presented in the context of complex Banach spaces. In addition, it is shown that by including $i^*$ in various formulas, you don’t need to have $V$ dense in $W$. Of course, this is typically not of any interest, but for the sake of generality, it is included. The approach is due to Lions. It is assumed for convenience that $p \geq 2$. This was apparently not needed in the last section. It may be that this approach can also be generalized to not require this.

Let $B(t) \in \mathcal{L}(W, W')$ satisfy
\[
\langle B(t)u, v \rangle = \langle B(t)v, u \rangle, \ u, v \in W
\]
\[
\langle B(t)u, u \rangle \geq 0
\]
\[
B(t) = B(0) + \int_0^t B'(s) \, ds
\]
where $B' \in L^\infty(0, T; \mathcal{L}(W, W'))$. Here $W$ is a Banach space such that $V \subseteq W$. Also $V_I \equiv L^p(I; V)$ and $W_I \equiv L^2(I; W)$.

Now let $I = [a, b]$ and $c < a < b < d$. Here and in what follows $\phi_n(t) = n\phi(nt)$ where $\phi \geq 0, \phi \in C_0^\infty(-1, 1)$, and $\int \phi dt = 1$. The following proposition is known and the essential features of its proof may be found in [83]. We give a proof for the convenience of the reader.

**Proposition 31.6.1** Suppose $D(t) \in \mathcal{L}(W, W')$ and $D(t) = 0$ if $t \notin (c, d)$. Suppose also that
\[
D(t) = \int_c^t D'(s) \, ds, \ D' \in L^\infty(c, d; \mathcal{L}(W, W')).
\]
For $u \in W_I$ and $a - n^{-1} > c$, $b + n^{-1} < d$, define
\[
T_n u = (D(u \ast \phi_n))' - ((Du) \ast \phi_n)'
\]
where we let $u = 0$ off $I$. Then
\[
\|T_n u\|_{W'_I} \to 0
\]

**Proof:** First, we show that $\|T_n\|$ is uniformly bounded. Letting $w = 0$ off $I$,
\[
\|T_n u, w\| = \left| \int_\mathbb{R} \langle D'(t) \int_\mathbb{R} u(s) \phi_n(t-s) \, ds, w(t) \rangle dt \right|
\]
\[
+ \left| \int_\mathbb{R} \langle (D(t) - D(s)) u(s) \phi_n(t-s) \, ds, w(t) \rangle dt \right|
\]
\[
\leq C \|u\|_{W_I} \|w\|_{W_I} + \int_\mathbb{R} \|D(t) - D(s)\| \|u(s)\|^2 \|\phi'(n(t-s))\| \|w(t)\| \, ds \, dt
\]
\[31.6. \text{ANOTHER APPROACH}\]

\[
\leq C \|u\|_{W_I} \|w\|_{W_I} + \int_{\mathbb{R}} \int_{-1}^{1} \left| D(t) - D\left(t - \frac{r}{n}\right) \right| \left| u\left(t - \frac{r}{n}\right) \right| n^2 |\phi'(r)| \|w(t)\| \frac{1}{n} dr dt
\]

\[
\leq C \|u\|_{W_I} \|w\|_{W_I} + C \int_{-1}^{1} \left| u\left(t - \frac{r}{n}\right) \right| \|w(t)\|_W dt dr
\]

\[
\leq C \|u\|_{W_I} \|w\|_{W_I}.
\]

Where \(C\) is a positive constant independent of \(n\) and \(u\). Thus \(\|T_n\|\) is bounded independent of \(n\).

Next let \(u \in C_0^\infty (I; V)\), a dense subset of \(W_I\). Then a little computation shows

\[
\langle T_n u, w \rangle_{W_I} \leq C (\phi) \int_a^b \int_{-1}^{1} \left| D'(t) - D'\left(t - \frac{r}{n}\right) \right| \left| u\left(t - \frac{r}{n}\right) \right| \|w(t)\|_W dr dt + C (\phi) \int_a^b \int_{-1}^{1} \left| u'(t - \frac{r}{n}) \right| \|w(t)\|_W dr dt
\]

\[
\equiv A + B.
\]

Now

\[
B \leq C (\phi, D') n^{-1/2} \|u''\|_W \|w\|_{W_I}.
\]

Since \(u\) is bounded,

\[
A \leq C (\phi, u) \int_a^b \int_{-1}^{1} \left| D'(t) - D'\left(t - \frac{r}{n}\right) \right| \|w(t)\|_W dr dt \leq C (\phi, u) \int_a^b \|w(t)\|_W n \int_{t-n}^{t+n-1} \|D'(t) - D'\left(s\right)\| ds dt
\]

By Holder’s inequality, this is no larger than

\[
C (\phi, u) \left( \int_a^b \left( n \int_{t-n}^{t+n-1} \|D'(t) - D'\left(s\right)\| ds \right)^2 dt \right)^{1/2} \|w\|_{W_I}.
\]

If \(t\) is a Lebesgue point,

\[
n \int_{t-n}^{t+n-1} \|D'(t) - D'\left(s\right)\| ds \to 0
\]

and also

\[
n \int_{t-n}^{t+n-1} \|D'(t) - D'\left(s\right)\| ds \leq 4 \|D'\|_\infty
\]
so the dominated convergence theorem implies
\[
\int_a^b \left( n \int_{t-n^{-1}}^{t+n^{-1}} ||D'(t) - D'(s)|| ds \right)^2 dt \to 0.
\]
Hence
\[
||T_n u||_{W'} \leq C(\phi, u, D') \left( n^{-1/2} + \left( \int_a^b (n \int_{t-n^{-1}}^{t+n^{-1}} ||D'(t) - D'(s)|| ds)^2 dt \right)^{1/2} \right)
\]
and so \( T_n u \to 0 \) for all \( u \) in the dense subset, \( C_0^\infty (I; V) \).

We have also the following simple corollary.

**Corollary 31.6.2** In the situation of Proposition 31.6.1, \( ||(i^* D (u \ast \phi_n))' - ((i^* D u) \ast \phi_n)'||_{V'} \to 0 \)
where \( i \) is the inclusion map of \( V \) into \( W \).

For \( f \in L^1 (a, b; V') \) we define \( f' \) in the sense of \( V' \) valued distributions as follows. For \( \phi \in C_0^\infty (a, b) \),
\[
f'(\phi) \equiv -\int_a^b f(t) \phi'(t) dt.
\]
We say \( f' \in L^1 (a, b; V') \) if there exists \( g \in L^1 (a, b; V') \), necessarily unique, such that for all \( \phi \in C_0^\infty (a, b) \),
\[
\int_a^b g(t) \phi(t) dt = f'(\phi).
\]
To save on notation, we let \( V \equiv V_{[0,T]} \) and \( W \equiv W_{[0,T]} \). Define
\[
D(L) \equiv \{ u \in V : (i^* Bu)' \in V' \}, \quad L u \equiv (i^* Bu)' \text{ for } u \in D(L).
\]
(31.6.44) (31.6.45)

Note that for \( u \in D(L) \), it is automatically the case that \( i^* Bu \in V' \).

**Lemma 31.6.3** \( L \) is a closed operator.

We define
\[
X \equiv D(L), \quad ||u||_X \equiv ||Lu||_{V'} + ||u||_{V'}.
\]
Then \( X \) is isometric to a closed subspace of a product of reflexive Banach spaces and so \( X \) is reflexive by Lemma [15.3.11].

**Theorem 31.6.4** Let \( p \geq 2 \) in what follows. For \( u, v \in X \), the following hold.
ANOTHER APPROACH

1. \( t \to (B(t)u(t),v(t))_{W',W} \) equals an absolutely continuous function a.e., denoted by \( (Bu,v)(\cdot) \).

2. \( \Re (Lu(t),u(t)) = \frac{1}{2} \left[ (Bu,u)'(t) + (B'(t)u(t),u(t)) \right] \) a.e. \( t \)

3. \( |(Bu,v)(t)| \leq C \|u\|_X \|v\|_X \) for some \( C > 0 \) and for all \( t \in [0,T] \).

4. \( t \to B(t)u(t) \) equals a function in \( C(0,T;W') \) a.e., denoted by \( Bu(\cdot) \).

5. \( \sup \{|(Bu,v)(t)|_{W'}, t \in [0,T]\} \leq C \|u\|_X \) for some \( C > 0 \).

6. \( K \) is linear, continuous and weakly continuous.

7. \( \Re(Ku,u) = \frac{1}{2}[(Bu,u)(T) + (Bu,u)(0)] + \frac{1}{2} \int_0^T (B'(t)u(t),u(t))dt \).

8. If \( Bu(0) = 0 \), for \( u \in X \), there exists \( u_n \to u \) in \( X \) such that \( u_n(t) \) is 0 near 0. A similar conclusion could be deduced at \( T \) if \( Bu(T) = 0 \).

**Proof:** For \( h \) a function defined on \([0,T]\), let \( h_1 \) be even, \( 2T \) periodic, and \( h_1(t) = h(t) \) for all \( t \in [0,T] \). Let \( C(\cdot) \in C^\infty_0(-T,2T), C(t) \in [0,1], C(t) = 1 \) on \([0,T]\).

Let \( \tilde{B}(t) = C(t)B_1(t) \) for all \( t \in \mathbb{R} \) and define

\[
\tilde{u}(t) = \begin{cases} 
  u_1(t), & t \in [-T,2T] \\
  0, & t \notin [-T,2T]
\end{cases}
\]
Now let \( u \in X \). Then

\[
\left( i^* \hat{B}u \right)'(t) = \begin{cases} 
0, & t < -T \\
C'(t) (i^* Bu)(-t) - C(t) (i^* Bu)'(-t), & t \in [-T, 0] \\
(i^* Bu)'(t), & t \in [0, T] \\
C'(t) (i^* Bu)(2T - t) - C(t) (i^* Bu)'(2T - t), & t \in [T, 2T] \\
0, & t > 2T 
\end{cases}
\quad (31.6.46)
\]

Thus, if \( I \supseteq [-T, 2T] \), then \( \left( \hat{B}u \right)' \in \mathcal{V}' \). Defining \( u_n \equiv \hat{u} \ast \phi_n \), then for a.e. \( t \),

\[
\text{Re} \left( \left( i^* \hat{B}u_n \right)'(t), u_n(t) \right) = \frac{1}{2} \left[ \langle \hat{B}u_n, u_n \rangle'(t) + \langle \hat{B}'(t) u_n(t), u_n(t) \rangle \right]. \quad (31.6.47)
\]

From Corollary 31.6.4 and Proposition 31.6.2, the following holds in \( \mathcal{V}'_{[-T,2T]} \).

\[
\lim_{n \to \infty} \left( i^* \hat{B}u_n \right)' = \lim_{n \to \infty} \left( i^* \hat{B} (\hat{u} \ast \phi_n) \right)' = \lim_{n \to \infty} \left( (i^* \hat{B}u) \ast \phi_n \right)' = \lim_{n \to \infty} (i^* \hat{B}u)' \ast \phi_n = (i^* \hat{B}u)' 
\]

Where the second equality follows from Corollary 31.6.4, the third follows from the pointwise a.e. equality of \( (i^* \hat{B}u) \ast \phi_n \)' and \( (i^* \hat{B}u)' \ast \phi_n \), while the fourth follows from 31.6.48 and standard properties of convolutions.

By choosing a subsequence we can use 31.6.24 to obtain

\[
u_n \to u \text{ a.e. and in } \mathcal{V} \quad (31.6.49)
\]

\[
\left( i^* \hat{B}u_n \right)' \to (i^* Bu)' \text{ a.e. and in } \mathcal{V}'.
\]

From 31.6.24

\[
\text{Re} \left( \left( i^* \hat{B}u_n \right)'(t), u_n(t) \right) \to \text{Re} \langle (i^* Bu)'(t), u(t) \rangle \text{ a.e. } t \in [0, T] \quad (31.6.50)
\]

\[
\langle B'(t) u_n(t), u_n(t) \rangle \to \langle B'(t) u(t), u(t) \rangle \text{ a.e. } t \in [0, T]. \quad (31.6.51)
\]

If \( g \in L^\infty(0,T) \),

\[
\lim_{n \to \infty} \int_0^T g(t) \langle \left( i^* \hat{B}u_n \right)'(t), u_n(t) \rangle dt = \lim_{n \to \infty} \langle \left( i^* \hat{B}u_n \right)', gu_n(t) \rangle
\]

\[
= \langle (i^* Bu)', gu \rangle = \int_0^T g(t) \langle (i^* Bu)'(t), u(t) \rangle dt.
\]
Thus we have the following weak convergence:

\[
\text{Re} \langle (i^* \tilde{B}u_n)' , u_n \rangle \rightarrow \text{Re} \langle (i^* Bu)' , u \rangle \text{ in } L^1 (0,T).
\]

Similarly,

\[
\langle B'u_n, u_n \rangle \rightarrow \langle B'u, u \rangle \text{ in } L^1 (0,T)
\]

It follows from \textbf{4.6} that

\[
\langle \tilde{B}u_n , u_n \rangle' (\cdot) \text{ converges a.e. and weakly in } L^1 (0,T).
\]

\[
\langle \tilde{B}u_n , u_n \rangle (\cdot) \text{ converges a.e. and strongly in } L^1 (0,T) \text{ to } \langle Bu , u \rangle (\cdot).
\]

Therefore, \( \langle Bu_n , u_n \rangle' (\cdot) \) converges a.e. and weakly in \( L^1 (0,T) \) to \( \langle Bu , u \rangle' (\cdot) \). Since \( \langle Bu , u \rangle \) and \( \langle Bu , u \rangle' \) are both in \( L^1 (0,T) \), this proves part \( \ddagger \) in the case where \( v = u \). This also establishes formula \( \ddagger \). To get \( \ddagger \) for \( u \neq v \), apply what was just shown to

\[
\langle B(t)(u(t) + v(t)) , u(t) + v(t) \rangle.
\]

Next let \( t \in [0,T] \) and use \textbf{4.6} to write

\[
\langle \tilde{B}u_n , u_n \rangle (t) = 2 \text{Re} \int_{-T}^{t} \bigl( (i^* \tilde{B}u_n)' (s) , u_n (s) \bigr) ds - 2 \text{Re} \int_{-T}^{t} \langle \tilde{B}'(s) u_n (s) , u_n (s) \rangle ds. \tag{31.6.52}
\]

Using \textbf{4.6}, we let \( n \to \infty \) in \textbf{4.6} and obtain

\[
\langle \tilde{B}u , u \rangle (t) = 2 \text{Re} \int_{-T}^{t} \bigl( (i^* \tilde{B}u)' (s) , \tilde{u} (s) \bigr) ds - 2 \text{Re} \int_{-T}^{t} \langle \tilde{B}'(s) \tilde{u} (s) , \tilde{u} (s) \rangle ds. \tag{31.6.53}
\]

Hence from \textbf{4.6},

\[
\langle \tilde{B}u , u \rangle (t) \leq C \left[ \| (i^* \tilde{B}u) \|_{V_{L^2 (0,T)}} \| \tilde{u} \|_{V_{L^2 (0,T)}} + \| \tilde{u} \|^2_{L^2 (0,T)} \right]
\]

\[
\leq C \left[ \| Lu \|^2_{V_{2 (0,T)}} + \| u \|^2_{V_{2 (0,T)}} \right] \leq C ||u||^2_X. \tag{31.6.54}
\]

This verifies \( \ddagger \) in the case \( u = v \). To obtain the general case,

\[
\langle (Bu , v) (t) \rangle \leq \langle Bu , u \rangle^{1/2} (t) \langle Bv , v \rangle^{1/2} (t) \leq C ||u||_X ||v||_X.
\]

To verify \( \ddagger \), use \textbf{4.1.7} to write for \( t \in [0,T] \) and \( I = [-T,2T] \),

\[
\left| \langle \tilde{B}u_n (t) - \tilde{B}u_m (t) , u_n (t) - u_m (t) \rangle \right|
\]

\[
\leq 2 \int_{-T}^{2T} \left| (i^* \tilde{B} (u_n - u_m))' (s) , u_n (s) - u_m (s) \right| ds
\]
follows easily from the first five parts. It remains to get 

\[ C \left\| \left( i^* \tilde{B}_n \right)' - \left( i^* \tilde{B}_m \right)' \right\|_{\mathcal{W}'} \left\| u_n - u_m \right\|_{\mathcal{W}_j} + \left\| u_n - u_m \right\|^2 \right\| = E_{nm}. \]

Then from Proposition 31.6.5, \( \lim_{n,m \to \infty} E_{nm} = 0 \) and so, for \( t \in [0, T] \),

\[ \left| \langle \tilde{B}_n (t) - \tilde{B}_m (t), w \rangle \right| \leq E_{nm}^{1/2} (B (t) w, w)^{1/2} \leq C E_{nm}^{1/2} \left\| w \right\|_{\mathcal{W}}. \]

It follows that \( \tilde{B}_n (\cdot) \) is uniformly Cauchy in \( C (0, T; W') \) and so it converges to \( z \in C (0, T; W') \). But \( \tilde{B}_n \) converges in \( L^2 (0, T; W') \) to \( B \cdot \). Therefore \( B (t) u (t) = z (t) \) a.e. Letting \( B \cdot = z (\cdot) \), this shows 2. Formula 1 follows from 2 and the following argument.

\[ \left| \langle Bu, w \rangle \right| \leq \left\| Bu \right\|_{X} \left\| w \right\|_{W} \leq C \left\| Bu \right\| \left\| w \right\|_{W}. \]

Assertion 1 follows easily from the first five parts. It remains to get 2.

\[
\text{Re} \langle Ku, u \rangle = \int_0^T \text{Re} \langle Lu, u \rangle dt + \langle Bu, u \rangle (0)
\]

\[
= \int_0^T \frac{1}{2} \left[ \langle Bu, u \rangle' (t) + \langle B' (t) u (t), u (t) \rangle \right] dt + \langle Bu, u \rangle (0)
\]

\[
= \frac{1}{2} \langle Bu, u \rangle (T) + \frac{1}{2} \langle Bu, u \rangle (0) + \frac{1}{2} \int_0^T \langle B' (t) u (t), u (t) \rangle dt
\]

It only remains to verify the last assertion. Let \( \psi_n \) be increasing and piecewise linear such that \( \psi_n (t) = 1 \) for \( t \geq 2/n \) and equals 0 on \( [0, 1/n] \). Then clearly \( \psi_n u \to u \) in \( \mathcal{W} \). Also

\[ (B (\psi_n u)') = \psi_n' Bu + \psi_n (Bu)'+ \]

The latter term converges to \( (Bu)' \) in \( \mathcal{V}' \). Now consider the first term.

\[
\int_0^T \left\| \psi_n' Bu \right\|_{\mathcal{V}'}^p dt \leq \int_0^{2/n} \left\| (Bu)' \right\|_{\mathcal{V}}^p dt
\]

\[
\leq n \int_0^{2/n} t^{p-1} \int_0^t \left\| (Bu)' \right\|_{\mathcal{V}}^p ds dt \leq \int_0^{2/n} \left\| (Bu)' \right\|_{\mathcal{V}}^p ds \frac{1}{p} (2/n)^{p'} n
\]

Since \( p' > 1 \), this converges to 0.

**Corollary 31.6.5** If \( Bu (0) = 0 \) for \( u \in X \), then \( \langle Bu, u \rangle (0) = 0 \). The converse is also true. An analogous result will hold with 0 replaced with \( T \).

**Proof:** Let \( u_n \to u \) in \( X \) with \( u_n (t) = 0 \) for all \( t \) close enough to 0. For \( t \) off a set of measure zero consisting of the union of sets of measure zero corresponding to \( u_n \) and \( u \),

\[ \langle Bu_n, u_n \rangle (t) = \langle B (t) u_n (t), u_n (t) \rangle, \langle Bu, u \rangle (t) = \langle B (t) u (t), u (t) \rangle, \]

\[ \langle Bu_n, u \rangle (t) = \langle B (t) u_n (t), u \rangle, \langle Bu, u_n \rangle (t) = \langle B (t) u (t), u_n (t) \rangle. \]
31.7. SOME IMBEDDING THEOREMS

\[ \langle B(u - u_n), u \rangle(t) = \langle B(t)(u(t) - u_n(t)), u(t) \rangle \]
\[ \langle Bu_n, u - u_n \rangle(t) = \langle B(t)u_n(t), u(t) - u_n(t) \rangle \]

Then, considering such \( t \),
\[ \langle Bu(t), u(t) \rangle(t) - \langle B(t)u_n(t), u_n(t) \rangle = \langle B(t)(u(t) - u_n(t)), u(t) \rangle + \langle B(t)u_n(t), u(t) - u_n(t) \rangle \]

Hence from Theorem 31.6.4,
\[ |\langle B(t)u(t), u(t) \rangle - \langle B(t)u_n(t), u_n(t) \rangle| \leq C||u - u_n||_X(||u||_X + ||u_n||_X) \]

Thus if \( n \) is sufficiently large,
\[ |\langle B(t)u(t), u(t) \rangle - \langle B(t)u_n(t), u_n(t) \rangle| < \varepsilon \]

So let \( n \) be fixed and this large and now let \( t_k \to 0 \) to obtain \( \langle B(t_k)u_n(t_k), u_n(t_k) \rangle = 0 \) for \( k \) large enough. Hence
\[ \langle Bu, u \rangle(0) = \lim_{k \to \infty} \langle B(t_k)u(t_k), u(t_k) \rangle < \varepsilon \]

Since \( \varepsilon \) is arbitrary, \( \langle Bu, u \rangle(0) = 0 \).

Next suppose \( \langle Bu, u \rangle(0) = 0 \). Then letting \( v \in X \), with \( v \) smooth,
\[ \langle Bu(0), v(0) \rangle = \langle Bu, v \rangle(0) = \langle Bu, u \rangle^{1/2}(0) \langle Bu, v \rangle^{1/2}(0) = 0 \]

and it follows that \( Bu(0) = 0 \).

31.7 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling's lemma.

**Definition 31.7.1** Let \( E, W \) be Banach spaces such that \( E \subseteq W \) and the injection map from \( E \) into \( W \) is continuous. The injection map is said to be compact if every bounded set in \( E \) has compact closure in \( W \). In other words, if a sequence is bounded in \( E \) it has a convergent subsequence converging in \( W \). This is also referred to by saying that bounded sets in \( E \) are precompact in \( W \).

**Theorem 31.7.2** Let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Then for every \( \varepsilon > 0 \) there exists a constant, \( C_\varepsilon \) such that for all \( u \in E \),
\[ ||u||_W \leq \varepsilon ||u||_E + C_\varepsilon ||u||_X \]
Proof: Suppose not. Then there exists $\varepsilon > 0$ and for each $n \in \mathbb{N}$, $u_n$ such that

$$||u_n||_W > \varepsilon ||u_n||_E + n ||u_n||_X$$

Now let $v_n = u_n / ||u_n||_E$. Therefore, $||v_n||_E = 1$ and

$$||v_n||_W > \varepsilon + n ||v_n||_X$$

It follows there exists a subsequence, still denoted by $v_n$ such that $v_n$ converges to $v$ in $W$. However, the above inequality shows that $||v_n||_X \to 0$. Therefore, $v = 0$. But then the above inequality would imply that $||v_n||_W > \varepsilon$ and passing to the limit yields $0 > \varepsilon$, a contradiction.

Definition 31.7.3 Define $C([a,b] ; X)$ the space of functions continuous at every point of $[a,b]$ having values in $X$.

You should verify that this is a Banach space with norm

$$||u||_{\infty,X} = \max \{|u_{n_k}(t) - u(t)| : t \in [a,b]\}.$$ 

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. It is like a well known result due to Simon [106]. It is an appropriate generalization when you do not have weak derivatives.

Theorem 31.7.4 Let $q > 1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$\{ u \text{ such that } ||u(t)||_E \leq R \text{ for all } t \in [a,b], \text{ and } ||u(s) - u(t)||_X \leq R|t - s|^{1/q} \}.$$ 

Thus $S$ is bounded in $L^\infty(a,b,E)$ and in addition, the functions are uniformly Holder continuous into $X$. Then $S \subseteq C([a,b] ; W)$ and if $\{u_n\} \subseteq S$, there exists a subsequence, $\{u_{n_k}\}$ which converges to a function $u \in C([a,b] ; W)$ in the following way.

$$\lim_{k \to \infty} ||u_{n_k} - u||_{\infty,W} = 0.$$ 

Proof: First consider the issue of $S$ being a subset of $C([a,b] ; W)$. Let $\varepsilon > 0$ be given. Then by Theorem 31.7.2, there exists a constant, $C_\varepsilon$ such that for all $u \in W$,

$$||u||_W \leq \frac{\varepsilon}{6R} ||u||_E + C_\varepsilon ||u||_X.$$ 

Therefore, for all $u \in S$,

$$||u(t) - u(s)||_W \leq \frac{\varepsilon}{6R} ||u(t) - u(s)||_E + C_\varepsilon ||u(t) - u(s)||_X \leq \frac{\varepsilon}{6R} (||u(t)||_E + ||u(s)||_E) + C_\varepsilon ||u(t) - u(s)||_X \leq \frac{\varepsilon}{3} + C_\varepsilon R|t - s|^{1/q}. \quad (31.7.56)$$
31.7. SOME IMBEDDING THEOREMS

Since \( \varepsilon \) is arbitrary, it follows \( u \in C ([a, b]; W) \).

Let \( D = \mathbb{Q} \cap [a, b] \) so \( D \) is a countable dense subset of \([a, b]\). Let \( D = \{ t_n \}_{n=1}^{\infty} \).

By compactness of the embedding of \( E \) into \( W \), there exists a subsequence \( u_{(n,1)} \) such that as \( n \to \infty \), \( u_{(n,1)} (t_1) \) converges to a point in \( W \). Now take a subsequence of this, called \((n,2)\) such that as \( n \to \infty \), \( u_{(n,2)} (t_2) \) converges to a point in \( W \). It follows that \( u_{(n,2)} (t_1) \) also converges to a point of \( W \). Continue this way. Now consider the diagonal sequence, \( u_k \equiv u_{(k,k)} \). This sequence is a subsequence of \( u_{(n,l)} \) whenever \( k > l \). Therefore, \( u_k (t_j) \) converges for all \( t_j \in D \).

**Claim:** Let \( \{ u_k \} \) be as just defined, converging at every point of \( D = [a, b] \) with \( \{ u_k \} \) converging at every point of \([a, b] \). Let \( \{ u_k \} \) converges at every point of \([a, b] \).

**Proof of Claim:** Let \( \varepsilon > 0 \) be given. Let \( t \in [a, b] \). Pick \( t_m \in D \cap [a, b] \) such that \( |t - t_m| < \varepsilon/3 \). Then there exists \( N \) such that if \( l, n > N \), then \( ||u_l (t_m) - u_n (t_m)||_W < \varepsilon/3 \). It follows that for \( l, n > N \),

\[
||u_l (t) - u_n (t)||_W \leq ||u_l (t) - u_l (t_m)||_W + ||u_l (t_m) - u_n (t_m)||_W + ||u_n (t_m) - u_n (t)||_W \\
\leq \frac{2\varepsilon}{3} + \frac{2\varepsilon}{3} < 2\varepsilon
\]

Since \( \varepsilon \) was arbitrary, this shows \( \{ u_k (t) \}_{k=1}^{\infty} \) is a Cauchy sequence. Since \( W \) is complete, this shows this sequence converges.

Now let \( t \in [a, b] \), it was just shown that if \( \varepsilon > 0 \) there exists \( N_\varepsilon \) such that if \( n, m > N_\varepsilon \), then

\[
||u_n (t) - u_m (t)||_W < \frac{\varepsilon}{3}.
\]

Now let \( s \neq t \). Then

\[
||u_n (s) - u_m (s)||_W \leq ||u_n (s) - u_n (t)||_W + ||u_n (t) - u_m (t)||_W + ||u_m (t) - u_m (s)||_W
\]

From (31.7.57)

\[
||u_n (s) - u_m (s)||_W \leq 2 \left( \frac{\varepsilon}{3} + C_\varepsilon R |t - s|^{1/q} \right) + ||u_n (t) - u_m (t)||_W
\]

and so it follows that if \( \delta \) is sufficiently small and \( s \in B (t, \delta) \), then when \( n, m > N_\varepsilon \)

\[
||u_n (s) - u_m (s)|| < \varepsilon.
\]

Since \([a, b] \) is compact, there are finitely many of these balls, \( \{ B (t_i, \delta) \}_{i=1}^{p} \), such that for \( s \in B (t_i, \delta) \) and \( n, m > N_\varepsilon \), the above inequality holds. Let \( N > \max \{ N_\varepsilon, \cdots, N_p \} \). Then if \( m, n > N \) and \( s \in [a, b] \) is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

**Claim:** Let \( \varepsilon > 0 \) be given. Then there exists \( N \) such that if \( m, n > N \), then

\[
||u_n - u_m||_{W, \infty} < \varepsilon.
\]

Now let \( u (t) = \lim_{k \to \infty} u_k (t) \).

\[
||u (t) - u (s)||_W \leq ||u (t) - u_n (t)||_W + ||u_n (t) - u_n (s)||_W + ||u_n (s) - u (s)||_W
\]

(31.7.57)
Let $N$ be in the above claim and fix $n > N$. Then
$$\|u(t) - u_n(t)\|_W = \lim_{m \to \infty} \|u_m(t) - u_n(t)\|_W \leq \varepsilon$$
and similarly, $\|u_n(s) - u(s)\|_W \leq \varepsilon$. Then if $|t - s|$ is small enough, the middle term in $31.7.56$ shows the middle term in $31.7.57$ is also smaller than $\varepsilon$. Therefore, if $|t - s|$ is small enough,
$$\|u(t) - u(s)\|_W < 3\varepsilon.$$ Thus $u$ is continuous. Finally, let $N$ be as in the above claim. Then letting $m, n > N$, it follows that for all $t \in [a, b],$
$$\|u_m(t) - u_n(t)\|_W < \varepsilon.$$ Therefore, letting $m \to \infty$, it follows that for all $t \in [a, b],$
$$\|u(t) - u_n(t)\|_W \leq \varepsilon.$$ and so $\|u - u_n\|_{\infty, W} \leq \varepsilon$. \[\square\]

Here is an interesting corollary. Recall that for $E$ a Banach space $C^{0,\alpha}([0, T], E)$ is the space of continuous functions $u$ from $[0, T]$ to $E$ such that
$$\|u\|_{\alpha, E} \equiv \|u\|_{\infty, E} + \rho_{\alpha, E}(u) < \infty$$
where here
$$\rho_{\alpha, E}(u) \equiv \sup_{t \neq s} \frac{\|u(t) - u(s)\|_E}{|t - s|^\alpha}$$

**Corollary 31.7.5** Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Then if $\gamma > \alpha$, the embedding of $C^{0,\gamma}([0, T], E)$ into $C^{0,\alpha}([0, T], X)$ is compact.

**Proof:** Let $\phi \in C^{0,\gamma}([0, T], E)$
$$\frac{\|\phi(t) - \phi(s)\|_X}{|t - s|^\gamma} \leq \left( \frac{\|\phi(t) - \phi(s)\|_W}{|t - s|^\gamma} \right)^{\alpha/\gamma} \|\phi(t) - \phi(s)\|_W^{1 - (\alpha/\gamma)} \leq \rho_{\gamma, E}(\phi) \|\phi(t) - \phi(s)\|_W^{1 - (\alpha/\gamma)}$$

Now suppose $\{u_n\}$ is a bounded sequence in $C^{0,\gamma}([0, T], E)$. By Theorem 31.7.4 above, there is a subsequence still called $\{u_n\}$ which converges in $C^{0}([0, T], W)$. Thus from the above inequality
$$\|u_n(t) - u_m(t) - (u_n(s) - u_m(s))\|_X \leq \rho_{\gamma, E}(u_n - u_m) \|u_n(t) - u_m(t) - (u_n(s) - u_m(s))\|_W^{1 - (\alpha/\gamma)} \leq C(\{u_n\}) \left( 2 \|u_n - u_m\|_{\infty, W} \right)^{1 - (\alpha/\gamma)}$$
which converges to 0 as \( n, m \to \infty \). Thus

\[
\rho_{\alpha,X} (u_n - u_m) \to 0 \quad \text{as} \quad n, m \to \infty
\]

Also \( \|u_n - u_m\|_{\infty,X} \to 0 \) as \( n, m \to \infty \) so this is a Cauchy sequence in \( C^{0,\alpha}([0,T], X) \).

The next theorem is a well known result probably due to Lions, Temam, or Aubin.

**Theorem 31.7.6** Let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Let \( p \geq 1 \), let \( q > 1 \), and define

\[
S \equiv \{ u \in L^p([a,b];E) : \text{for some } C, \|u(t) - u(s)\|_X \leq C|t - s|^{1/q} \}
\]

and \( \|u\|_{L^p([a,b];E)} \leq R \).

Thus \( S \) is bounded in \( L^p([a,b];E) \) and Holder continuous into \( X \). Then \( S \) is precompact in \( L^p([a,b];W) \). This means that if \( \{u_n\}_{n=1}^\infty \subseteq S \), it has a subsequence \( \{u_{nk}\} \) which converges in \( L^p([a,b];W) \).

**Proof:** By Proposition 31.7.4 on Page 410 it suffices to show \( S \) has an \( \eta \) net in \( L^p([a,b];W) \) for each \( \eta > 0 \).

If not, there exists \( \eta > 0 \) and a sequence \( \{u_n\} \subseteq S \), such that

\[
\|u_n - u_m\| \geq \eta
\]

for all \( n \neq m \) and the norm refers to \( L^p([a,b];W) \). Let

\[
a = t_0 < t_1 < \cdots < t_k = b, \quad t_i - t_{i-1} = (b - a)/k.
\]

Now define

\[
\bar{u}_n (t) \equiv \sum_{i=1}^k \bar{u}_{n,i} \chi_{[t_{i-1},t_i)} (t), \quad \bar{u}_{n,i} \equiv \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n (s) \, ds.
\]

The idea is to show that \( \bar{u}_n \) approximates \( u_n \) well and then to argue that a subsequence of the \( \{\bar{u}_n\} \) is a Cauchy sequence yielding a contradiction to (31.7.58).

Therefore,

\[
\begin{align*}
\|u_n - \bar{u}_n\| & \equiv \sum_{i=1}^k u_n (t) \chi_{[t_{i-1},t_i)} (t) - \bar{u}_{n,i} \chi_{[t_{i-1},t_i)} (t) \\
& = \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n (t) \, ds \chi_{[t_{i-1},t_i)} (t) - \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n (s) \, ds \chi_{[t_{i-1},t_i)} (t) \\
& = \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n (t) - u_n (s)) \, ds \chi_{[t_{i-1},t_i)} (t).
\end{align*}
\]
It follows from Jensen’s inequality that

\[ \| u_n ( t ) - \bar{u}_n ( s ) \|_W^p \]

\[ = \sum_{i=1}^{k} \left\| \frac{t_i - t_{i-1}}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n ( t ) - u_n ( s )) \, ds \right\|_W^p x_{(t_{i-1}, t_i)} ( t ) \]

\[ \leq \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n ( t ) - u_n ( s ) \|_W^p \, ds x_{(t_{i-1}, t_i)} ( t ) \]

and so

\[ \int_a^b \| (u_n ( t ) - \bar{u}_n ( s )) \|_W^p \, ds \]

\[ \leq \int_a^b \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n ( t ) - u_n ( s ) \|_W^p \, ds x_{(t_{i-1}, t_i)} ( t ) \, dt \]

\[ = \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \| u_n ( t ) - u_n ( s ) \|_W^p \, ds \, dt. \quad (31.7.59) \]

From Theorem 31.7.58, if \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that

\[ \| u_n ( t ) - u_n ( s ) \|_W^p \leq \varepsilon \| u_n ( t ) - u_n ( s ) \|_X^p + C_\varepsilon \| u_n ( t ) - u_n ( s ) \|_X^p \]

\[ \leq 2^{p-1} \varepsilon (\| u_n ( t ) \|_W^p + \| u_n ( s ) \|_W^p) + C_\varepsilon | t - s |^{p/q} \]

This is substituted in to (31.7.59) to obtain

\[ \int_a^b \| (u_n ( t ) - \bar{u}_n ( s )) \|_W^p \, ds \leq \]

\[ \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \left( 2^{p-1} \varepsilon (\| u_n ( t ) \|_W^p + \| u_n ( s ) \|_W^p) + C_\varepsilon | t - s |^{p/q} \right) \, ds \, dt \]

\[ = \sum_{i=1}^{k} 2^{p} \varepsilon \int_a^b \| u_n ( t ) \|_W^p \, dt + C_\varepsilon \sum_{i=1}^{k} \frac{1}{(t_i - t_{i-1})} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} | t - s |^{p/q} \, ds \, dt \]

\[ \leq 2^{p} \varepsilon \int_a^b \| u_n ( t ) \|_W^p \, dt + C_\varepsilon \sum_{i=1}^{k} \frac{1}{(t_i - t_{i-1})} (t_i - t_{i-1})^{p/q} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \, ds \, dt \]

\[ = 2^{p} \varepsilon \int_a^b \| u_n ( t ) \|_W^p \, dt + C_\varepsilon \sum_{i=1}^{k} \frac{1}{(t_i - t_{i-1})^2} (t_i - t_{i-1})^{p/q} (t_i - t_{i-1})^2 \]

\[ \leq 2^{p} \varepsilon R^p + C_\varepsilon \sum_{i=1}^{k} (t_i - t_{i-1})^{1+p/q} = 2^{p} \varepsilon R^p + C_\varepsilon \left( \frac{b - a}{k} \right)^{1+p/q}. \]
31.7. SOME IMBEDDING THEOREMS

Taking $\varepsilon$ so small that $2^p \varepsilon R^p < \eta^p / 8^p$ and then choosing $k$ sufficiently large, it follows

$$\|u_n - u_m\|_{L^p([a,b];W)} < \frac{\eta}{4}.$$ 

Thus $k$ is fixed and $u_n$ at a step function with $k$ steps having values in $E$. Now use compactness of the embedding of $E$ into $W$ to obtain a subsequence such that $\{u_n\}$ is Cauchy in $L^p(a,b;W)$ and use this to contradict \textit{\textbf{117.48}}. The details follow.

Suppose $u_n(t) = \sum_{i=1}^{k} u^i_n \chi_{[t_{i-1}, t_i]}(t)$. Thus

$$\|u_n(t)\|_{E} = \sum_{i=1}^{k} \|u^i_n\|_{E} \chi_{[t_{i-1}, t_i]}(t)$$

and so

$$R \geq \int_a^b \|u_n(t)\|_{E}^p \, dt = \frac{T}{k} \sum_{i=1}^{k} \|u^i_n\|_{E}^p$$

Therefore, the $\{u_n\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k}$ is a Cauchy sequence in $L^p(a,b;W)$. You simply get a subsequence such that $u^i_{n_k}$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n$,

$$\|u_n - u_m\|_{L^p(a,b;W)} \leq \|u_n - u_m\|_{L^p(a,b;W)} + \|u_n - u_m\|_{L^p(a,b;W)} + \frac{\eta}{4} < \frac{\eta}{4}$$

provided $m,n$ are large enough, contradicting \textit{\textbf{117.48}}.

You can give a different version of the above to include the case where there is, instead of a Holder condition, a bound on $u^i$ for $u \in S$. It is stated next. See \textit{\textbf{118}}.

\textbf{Corollary 31.7.7} Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q > 1$, and define

$$S = \{ u \in L^p([a,b];E) : \text{for some } C, \|u(t) - u(s)\|_{X} \leq C |t - s|^{1/q} \} \text{ and } \|u\|_{L^p([a,b];E)} \leq R \}.$$

Thus $S$ is bounded in $L^p([a,b];E)$ and Holder continuous into $X$. Then $S$ is pre-compact in $L^p([a,b];W)$. This means that if $\{u_n\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\{u_{n_k}\}$ which converges in $L^p([a,b];W)$. The same conclusion can be drawn if it is known instead of the Holder condition that $\|u^i\|_{L^1([a,b];X)}$ is bounded.

\textbf{Proof:} The first part is Theorem \textit{\textbf{31.7.8}}. Therefore, we just prove the new stuff which involves a bound on the $L^1$ norm of the derivative. By Proposition \textit{\textbf{119.40}} on Page \textit{\textbf{189}} it suffices to show $S$ has an $\eta$ net in $L^p([a,b];W)$ for each $\eta > 0$.

If not, there exists $\eta > 0$ and a sequence $\{u_n\} \subseteq S$, such that

$$\|u_n - u_m\| \geq \eta \quad (31.7.60)$$
for all $n \neq m$ and the norm refers to $L^p ([a, b]; \mathcal{W})$. Let

$$a = t_0 < t_1 < \cdots < t_k = b, \ t_i - t_{i-1} = \left( b - a \right) / k.$$  

Now define

$$\pi_n(t) \equiv \sum_{i=1}^{k} \pi_n, \mathcal{X}_{[t_{i-1}, t_i)} (t), \quad \pi_{n, i} \equiv \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(s) \, ds.$$  

The idea is to show that $\pi_n$ approximates $u_n$ well and then to argue that a subsequence of the $\{\pi_n\}$ is a Cauchy sequence yielding a contradiction to (31.7.61).

Therefore,

$$u_n(t) - \pi_n(t) = \sum_{i=1}^{k} u_n(t, \mathcal{X}_{[t_{i-1}, t_i)} (t) - \sum_{i=1}^{k} \pi_{n, i} \mathcal{X}_{[t_{i-1}, t_i)} (t) \equiv \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \mathcal{X}_{[t_{i-1}, t_i)} (t).$$  

It follows from Jensen’s inequality that

$$\left\| u_n(t) - \pi_n(t) \right\|_{W}^{p} = \sum_{i=1}^{k} \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \mathcal{X}_{[t_{i-1}, t_i)} (t) \right\|_{W}^{p}.$$  

And so

$$\int_{0}^{T} \left\| u_n(t) - \pi_n(t) \right\|_{W}^{p} \, dt = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \right\|_{W}^{p} \, dt$$

$$\leq \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \varepsilon \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \right\|_{E}^{p} \, dt$$

$$+ C \varepsilon \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \right\|_{X}^{p} \, dt \quad (31.7.61)$$

Consider the second of these. It equals

$$C \varepsilon \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(t) \, dt \right\|_{X}^{p} \, dt$$
31.7. SOME IMBEDDING THEOREMS

This is no larger than

$$\leq C_\varepsilon \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left( \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n'(\tau) \|_X d\tau ds \right)^p dt$$

$$= C_\varepsilon \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left( \int_{t_{i-1}}^{t_i} \| u_n' (\tau) \|_X d\tau \right)^p dt$$

$$= C_\varepsilon \sum_{i=1}^{k} \left( \frac{(t_i - t_{i-1})^{1/p}}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n' (\tau) \|_X d\tau \right)^p$$

Since \( p \geq 1 \),

$$\leq C_\varepsilon \left( \sum_{i=1}^{k} (t_i - t_{i-1})^{1/p} \int_{t_{i-1}}^{t_i} \| u_n' (\tau) \|_X d\tau \right)^p$$

$$\leq \frac{C_\varepsilon (b-a)}{k} \left( \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \| u_n' (\tau) \|_X d\tau \right)^p$$

$$= \frac{C_\varepsilon (b-a)}{k} \left( \| u_n' \|_{L^1([a,b],X)} \right)^p < \frac{\eta^p}{10^p}$$

if \( k \) is chosen large enough. Now consider the first in (31.7.61) By Jensen’s inequality

$$\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \varepsilon \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) ds \right\|_E^p dt \leq$$

$$\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \varepsilon \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n(t) - u_n(s) \|_E^p ds dt \right\|$$

$$\leq \varepsilon 2^{p-1} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n(t) \|_E^p + \| u_n(s) \|_E^p ds dt \right\|$$

$$= 2\varepsilon 2^{p-1} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \| u_n(t) \|_E^p dt = \varepsilon (2) \left( 2^{p-1} \right) \| u_n \|_{L^p([a,b],E)} \leq M\varepsilon$$

Now pick \( \varepsilon \) sufficiently small that \( M\varepsilon < \frac{\eta^p}{10^p} \) and then \( k \) large enough that the second term in (31.7.61) is also less than \( \eta^p/10^p \). Then it will follow that

$$\| \bar{u}_n - u_n \|_{L^p([a,b],W)} < \left( \frac{2\eta^p}{10^p} \right)^{1/p} = 2^{1/p} \frac{\eta}{10} \leq \frac{\eta}{5}$$

Thus \( k \) is fixed and \( \bar{u}_n \) at a step function with \( k \) steps having values in \( E \). Now use compactness of the embedding of \( E \) into \( W \) to obtain a subsequence such that
\{u_n\} is Cauchy in $L^p([a, b]; W)$ and use this to contradict 31.7.60. The details follow.

Suppose $u_n(t) = \sum_{i=1}^{k} u_i^n X_{[t_{i-1}, t_i)}(t)$. Thus
\[
\|u_n(t)\|_E = \sum_{i=1}^{k} \|u_i^n\|_E X_{[t_{i-1}, t_i)}(t)
\]
and so
\[
R \geq \int_a^b \|u_n(t)\|_E^p dt = T \sum_{i=1}^{k} \|u_i^n\|_E^p
\]
Therefore, the $\{u_i^n\}$ are all bounded. It follows that after taking subsequences $k$ times there exists a subsequence $\{u_n^k\}$ such that $u_n^k$ is a Cauchy sequence in $L^p([a, b]; W)$. You simply get a subsequence such that $u_i^m$ is a Cauchy sequence in $W$ for each $i$. Then denoting this subsequence by $n_k$,
\[
\|u_n - u_m\|_{L^p(a, b; W)} \leq \|u_n - u_n^k\|_{L^p(a, b; W)} + \|u_n^k - u_m\|_{L^p(a, b; W)} + \|u_m - u_m^k\|_{L^p(a, b; W)} + \frac{\eta}{4}
\]
provided $m, n$ are large enough, contradicting 31.7.60.  

### 31.8 Some Evolution Inclusions

Let $H$ be a Hilbert space and let $\mathcal{H}$ denote $L^2(0, T; H)$. Here will be an application to an evolution equation having values in $\mathcal{H}$. It will always be the case that $H = H'$ so this is the simplest sort of a Gelfand triple, $V = H = H' = V'$. First is given a maximal monotone operator.

**Definition 31.8.1** Let $D(L) \equiv \{u \in \mathcal{H} such that u' \in \mathcal{H} and u(0) = u_0\}$. Then for $u \in D(L)$, $Lu \equiv u'$.

Note that $L$ is not linear.

**Lemma 31.8.2** For $L$ as just defined, $L$ is maximal monotone $L: \mathcal{H} \to \mathcal{H}$.

**Proof:** To show it is maximal monotone, it suffices to verify that $L + I$ is onto. This is by Theorem 23.7.13 on Page 863. Thus consider the equation
\[
u' + u = f, \ u(0) = u_0
\]
Is there a solution? Of course there is and it equals
\[
u(t) = e^{-t}u_0 + \int_0^t e^{-(t-s)} f(s) \, ds
\]
by the usual application of integrating factors and so forth.  

Then with this, the following is from Theorem 31.7.13 on Page 863. This is a well known result found in Brezis.
31.8. SOME EVOLUTION INCLUSIONS

**Theorem 31.8.3** Let \( u_0 \in D(\phi) \) where \( \phi : H \to [0, \infty] \) is proper, lower semicontinuous, and convex. Also let \( f \in H \) be given and \( u_0 \in D(\phi) \). Then there exists a solution \( u \) to the evolution initial value problem,

\[
  u'(t) + \partial \phi (u(t)) \ni f(t) \quad \text{a.e. in } H, \quad u(0) = u_0
\]

This solution satisfies \( u(t) \in D(\partial \phi) \) for a.e. \( t \) there exists \( z \in H \) such that \( z(t) \in \partial \phi (u(t)) \) for a.e. \( t \) such that the inclusion is an equation with \( \partial \phi (u(t)) \) replaced with \( z(t) \).

**Proof:** Define a function \( \Phi : H \to \mathbb{R} \)

\[
  \Phi (u) \equiv \int_0^T \phi (u) \, dt
\]

There are no measurability issues because \( \phi \) is lower semicontinuous and so the composition \( \phi (u) \) will be appropriately measurable. Then this is clearly convex. It is proper because \( \Phi(u_0) = \phi(u_0)T \) so \( u_0 \in D(\Phi) \). If \( u_n \to u \) in \( H \), does it follow that

\[
  \lim \inf_{n \to \infty} \Phi (u_n) \geq \Phi (u)
\]

Suppose not so \( \Phi (u) > \lim \inf_{n \to \infty} \Phi (u_n) \). Then choosing a subsequence such that

\[
  u_n \to u \text{ pointwise a.e.,}
\]

\[
  \Phi (u) \geq \lim \inf_{n \to \infty} \Phi (u_n) \equiv \lim \inf_{n \to \infty} \int_0^T \phi (u_n) \, dt
\]

\[
  \geq \int_0^T \lim \inf_{n \to \infty} \phi (u_n) \, dt = \int_0^T \phi (u) \, dt = \Phi (u)
\]

which is a contradiction. Thus \( \Phi \) is also lower semicontinuous.

The constant function \( u \equiv u_0 \) is in \( D(L) \cap D(\Phi) \). To use Theorem [124] on Page 902, it is required to show that

\[
  \Phi (J_\lambda u) \leq \Phi (u) + C\lambda
\]

In this case, the duality map is just the identity map. Hence \( J_\lambda u \) is the solution to

\[
  0 = (J_\lambda u - u) + \lambda L (J_\lambda u)
\]

Hence letting \( J_\lambda u \) be denoted by \( u_\lambda \), it follows that \( u_\lambda \) would be the solution to

\[
  \lambda u_\lambda' + u_\lambda = u, \quad u_\lambda (0) = u_0
\]

Using the usual integrating factor procedure, it follows that

\[
  u_\lambda (t) = e^{-(1/\lambda)t} u_0 + \int_0^t e^{-(1/\lambda)(t-s)} \frac{1}{\lambda} u (s) \, ds
\]
Note that
\[ e^{-\lambda t} + \int_0^t \frac{1}{\lambda} e^{-\lambda (t-s)} ds = 1 \]

Thus by Jensen’s inequality,
\[ \phi(u_\lambda(t)) \leq e^{-\lambda t} \phi(u_0) + \int_0^t e^{-\lambda (t-s)} \frac{1}{\lambda} \phi(u(s)) ds \]

Then
\[ \Phi(u_\lambda) \leq \int_0^T e^{-\lambda t} \phi(u_0) dt + \int_0^T \int_0^t e^{-\lambda (t-s)} \frac{1}{\lambda} \phi(u(s)) dsdt \]
\[ = \int_0^T e^{-\lambda t} \phi(u_0) dt + \int_0^T \phi(u(s)) \int_s^T e^{-\lambda (t-s)} \frac{1}{\lambda} dt ds \]

Now \( \int_s^T e^{-\lambda (t-s)} \frac{1}{\lambda} dt = 1 - e^{\frac{\lambda}{\lambda} t - \frac{1}{2} T} < 1 \) and since \( \phi \geq 0 \), this shows that
\[ \Phi(u_\lambda) \leq \phi(u_0) \int_0^\infty e^{-\lambda t} dt + \int_0^T \phi(u(s)) dt \]
so
\[ \Phi(u_\lambda) \leq \phi(u_0) \lambda + \Phi(u) \]

It follows that the conditions of Theorem 33.1 on Page 1412 are satisfied and so \( L + \partial \Phi \) is maximal monotone. Thus if \( f \in \mathcal{H} \), there exists \( u \in D(L) \cap D(\partial \Phi) \) such that for each \( v \in \mathcal{H} \) there exists a solution \( u_v \) to

\[ u_v' + z_v + u_v = f + v, \quad u_v(0) = u_0 \in D(\phi) \]

where \( z_v \in \partial \Phi(u_v) \). I will show that \( v \to u_v \) has a fixed point. Note that if \( z_1 \in \partial \Phi(u_1) \), then
\[ \int_{t-h}^t (z_1 - z_2, u_1 - u_2) ds \geq 0, \text{ any } h \leq t \]

To see this, you could simply let \( u_2 = u_1 \) off \([t-h, t]\) and pick \( z_1 = z_2 \) also on this set. Also, you can conclude that \( u \in \partial \Phi \) implies \( u(t) \in \partial \phi \) for a.e. \( t \). I show this now. Let \([a, b] \in \mathcal{G}(\partial \phi) \). Then as just noted, for \([u, z] \in \mathcal{G}(\partial \Phi) \),
\[ \int_{t-h}^t (z - b, u - a) ds \geq 0 \]

Then by the fundamental theorem of calculus, for a.e. \( t, (z(t) - b, u(t) - a) \geq 0 \) a.e. Letting \( \{[a_i, b_i]\}_{i=1}^\infty \) be a dense subset of \( \mathcal{G}(\partial \phi) \), one can take the union of countably many sets of measure zero, one for each \([a_i, b_i]\) and conclude that off this set of measure zero, \( (z(t) - b_i, u(t) - a_i) \geq 0 \) for all \( i \). Hence this is also true for all \([a, b] \in \mathcal{G}(\partial \phi) \) and so \( z(t) \in \partial \phi(u(t)) \) for a.e. \( t \).
Then if you have \( v_i, i = 1, 2 \)

\[
Lu_{v_1} - Lu_{v_2} + z_{v_1} - z_{v_2} + u_{v_1} - u_{v_2} = v_1 - v_2
\]

Then taking inner products with \( u_{v_1} - u_{v_2} \) and integrating up to \( t \),

\[
\frac{1}{2} |(u_{v_1} - u_{v_2})(t)|^2_H + \int_0^t (z_{v_1} - z_{v_2}, u_{v_1} - u_{v_2}) dt + \int_0^t |u_{v_1} - u_{v_2}|^2 ds
\]

\[
\leq \frac{1}{2} \int_0^t |v_1 - v_2|^2 ds + \frac{1}{2} \int_0^t |u_{v_1} - u_{v_2}|^2 ds
\]

Now by monotonicity of \( \phi \) and the above,

\[
|(u_{v_1} - u_{v_2})(t)|^2_H \leq \int_0^t |v_1 - v_2|^2 ds
\]

which shows that a high enough power of the mapping \( v \to u_v \) is a contraction map on \( H \) and so there exists a unique fixed point \( u \). Thus \( u_u = u \) and so

\[
\begin{align*}
u' + z + u &= f + u, \quad u_v(0) = u_0 \in D(\phi), \quad z(t) \in \partial\phi(t) \text{ a.e.}
\end{align*}
\]

and so

\[
\begin{align*}
u' + z &= f \text{ in } H, \quad u(t) \in D(\partial\phi) \text{ a.e., } z(t) \in \partial\phi(u(t)) \text{ a.e., } u' \in H, \text{ and } u(0) = u_0.
\end{align*}
\]

Note that in the above, the initial condition only needs to be in \( D(\phi) \), not in the smaller \( D(\partial\phi) \), although the solution is in \( D(\partial\phi) \) for a.e. \( t \). Also note that \( f \) has no smoothness. It only is in \( H \). This is really a nice result.
Chapter 32

Maximal Monotone Operators, Hilbert Space

32.1 Basic Theory

Here is provided a short introduction to some of the most important properties of maximal monotone operators in Hilbert space. The following definition describes them. It is more specialized than the earlier material on maximal monotone operators from a Banach space to its dual and therefore, better results can be obtained. More on this can be read in [23] and [105].

Definition 32.1.1 Let $H$ be a real Hilbert space and let $A : D(A) \to \mathcal{P}(H)$ have the following properties.

1. For each $y \in H$ there exists $x \in D(A)$ such that $y \in x + Ax$.
2. $A$ is monotone. That is, if $z \in Ax$ and $w \in Ay$ then
   \[(z - w, x - y) \geq 0\]

Such an operator is called a maximal monotone operator.

It turns out that whenever $A$ is maximal monotone, so is $\lambda A$ for all $\lambda > 0$.

Lemma 32.1.2 Suppose $A$ is maximal monotone. Then so is $\lambda A$. Also $J_\lambda \equiv (I + \lambda A)^{-1}$ makes sense for each $\lambda > 0$ and is Lipschitz continuous.

Proof: To begin with consider $(I + A)^{-1}$. Suppose
\[x_1, x_2 \in (I + A)^{-1}(y)\]
Then $y \in (I + A)x_i$ and so $y - x_i \in Ax_i$. By monotonicity
\[(y - x_i - (y - x_2), x_1 - x_2) \geq 0\]
and so 

\[ 0 \geq |x_1 - x_2|^2 \]

which shows \( J_1 \equiv (I + A)^{-1} \) makes sense. In fact this is Lipschitz with Lipschitz constant 1. Here is why. \( x \in (I + A)J_1x \) and \( y \in (I + A)J_1y \). Then

\[ x - J_1x \in AJ_1x, \ y - J_1y \in AJ_1y \]

and so by monotonicity

\[ 0 \leq (x - J_1x - (y - J_1y), J_1x - J_1y) \]

which yields

\[ |J_1x - J_1y|^2 \leq (x - y, J_1x - J_1y) \leq |x - y||J_1x - J_1y| \]

which yields the result.

Next consider the claim that \( \lambda A \) is maximal monotone. The monotone part is immediate. The only thing in question is whether \( I + \lambda A \) is onto. Let \( r \in (-1, 1) \) and pick \( f \in H \). Consider solving the equation for \( u \)

\[ (1 + r)u + Au \ni (1 + r)f \]

This is equivalent to finding \( u \) such that

\[ (I + A)u \ni (1 + r)f - ru \]

or in other words finding \( u \) such that

\[ u = J_1((1 + r)f - ru) \]

However, if

\[ Tu \equiv J_1((1 + r)f - ru), \]

then since \( |r| < 1, T \) is a contraction mapping and so there exists a unique solution to 32.1.1. Thus

\[ u + \frac{1}{1 + r}Au \ni f \]

It follows for any \( |r| < 1, (1 + r)^{-1}A \) is maximal monotone. This takes care of all \( \lambda \in \left( \frac{1}{2}, \infty \right) \). Now do the same thing for \( (2/3)A \) to get the result for all \( \lambda \in \left( \left( \frac{2}{3} \right) \left( \frac{1}{2} \right), \infty \right) \). Now apply the same argument to \((2/3)^2A\) to get the result for all \( \lambda \in \left( \left( \frac{2}{3} \right)^2 \left( \frac{1}{2} \right), \infty \right) \). Next consider the same argument to \((2/3)^3A\) to get the desired result for all \( \lambda \in \left( \left( \frac{2}{3} \right)^3 \left( \frac{1}{2} \right), \infty \right) \). Continuing this way shows \( \lambda A \) is maximal monotone for all \( \lambda > 0 \). Also from the first part of the proof \((I + \lambda A)^{-1}\) is Lipschitz continuous with Lipschitz constant 1. This proves the lemma.
32.1. BASIC THEORY

A maximal monotone operator can be approximated with a Lipschitz continuous operator which is also monotone and has certain salubrious properties. This operator is called the Yosida approximation and as in the case of linear operators it is obtained by formally considering

$$\frac{A}{1 + \lambda A}$$

If you do the division formally you get the definition for $A_\lambda$,

$$A_\lambda x \equiv \frac{1}{\lambda} x - \frac{1}{\lambda} J_\lambda x$$

(32.1.2)

where $J_\lambda = (I + \lambda A)^{-1}$ as above. It is obvious that $A_\lambda$ is Lipschitz continuous with Lipschitz constant no more than $2/\lambda$. Actually you can show $1/\lambda$ also works but this is not important here.

Lemma 32.1.3 $A_\lambda x \in AJ_\lambda x$ and $|A_\lambda x| \leq |y|$ for all $y \in Ax$ whenever $x \in D(A)$. Also $A_\lambda$ is monotone.

Proof: Consider the first claim. From the definition,

$$A_\lambda x \equiv \frac{1}{\lambda} x - \frac{1}{\lambda} J_\lambda x$$

Is

$$\frac{1}{\lambda} x - \frac{1}{\lambda} J_\lambda x \in AJ_\lambda x?$$

Is

$$x - J_\lambda x \in \lambda AJ_\lambda x?$$

Is

$$x \in J_\lambda x + \lambda AJ_\lambda x?$$

Is

$$x \in (I + \lambda A) J_\lambda x?$$

Certainly so. This is how $J_\lambda$ is defined.

Now consider the second claim. Let $y \in Ax$ for some $x \in D(A)$. Then by monotonicity and what was just shown

$$0 \leq (A_\lambda x - y, J_\lambda x - x) = -\lambda (A_\lambda x - y, A_\lambda x)$$

and so

$$|A_\lambda x|^2 \leq (y, A_\lambda x) \leq |y| |A_\lambda x|$$

Finally, to show $A_\lambda$ is monotone,

$$(A_\lambda x - A_\lambda y, x - y) =$$

$$\left( \frac{1}{\lambda} x - \frac{1}{\lambda} J_\lambda x - \left( \frac{1}{\lambda} y - \frac{1}{\lambda} J_\lambda y \right), x - y \right)$$
\[
\begin{align*}
= & \frac{1}{\lambda} |x - y|^2 - \frac{1}{\lambda} (J_\lambda x - J_\lambda y, x - y) \\
\geq & \frac{1}{\lambda} |x - y|^2 - \frac{1}{\lambda} |x - y| |J_\lambda x - J_\lambda y| \\
\geq & \frac{1}{\lambda} |x - y|^2 - \frac{1}{\lambda} |x - y|^2 = 0
\end{align*}
\]

and this proves the lemma.

**Proposition 32.1.4** Suppose \( D(A) \) is dense in \( H \). Then for all \( x \in H \),

\[ |J_\lambda x - x| \to 0 \]

**Proof:** From the above, if \( u \in D(A) \) and \( y \in Au \), then

\[ \left| \frac{1}{\lambda} u - \frac{1}{\lambda} J_\lambda u \right| \leq |y| \]

Hence \( J_\lambda u \to u \). Now for \( x \) arbitrary,

\[ |J_\lambda x - x| \leq |J_\lambda x - J_\lambda u| + |J_\lambda u - u| + |u - x| \]

\[ < 2\varepsilon + |J_\lambda u - u| \]

where the last term converges to 0 as \( \lambda \to 0 \). Since \( \varepsilon \) is arbitrary, this shows the proposition.

Thus in the case where \( D(A) \) is dense, if you have

\[ x \in \varepsilon Ax + x_\varepsilon \]

so that \( x_\varepsilon = J_\varepsilon x \), then \( |x - x_\varepsilon| \to 0 \).

The next lemma gives a way to determine whether a pair \([x, y]\) is in the graph of \( A \) defined as

\[ \{(x, y) : y \in Ax\} \equiv \mathcal{G}(A) \]

Here I am writing \([\cdot, \cdot]\) rather than \((\cdot, \cdot)\) to avoid confusion with the inner product. It is the conclusion of this lemma which accounts for the use of the term “maximal”. It essentially says there is no larger monotone graph which includes the one for \( A \).

**Lemma 32.1.5** Suppose \((y_1 - y, x_1 - x) \geq 0\) for all \([x, y] \in \mathcal{G}(A)\) where \( A \) is maximal monotone. Then \( x_1 \in D(A) \) and \( y_1 \in Ax_1 \). Also if \([x_k, y_k] \in \mathcal{G}(A)\) and \( x_k \to x, y_k \to y \) where the half arrow denotes weak convergence, then \([x, y] \in \mathcal{G}(A)\).

**Proof:** I want to show \( y_1 \in Ax_1 \) or in other words I want to show

\[ x_1 + \lambda y_1 \in x_1 + \lambda Ax_1 \]

or in other words

\[ J_\lambda (x_1 + \lambda y_1) = x_1. \]

This is the motivation for the following argument.
32.1. BASIC THEORY

From Lemma 32.1.3, \( A_\lambda (x_1 + \lambda y_1) \in AJ_\lambda (x_1 + \lambda y_1) \) and so by the above assumption

\[
0 \leq (y_1 - A_\lambda (x_1 + \lambda y_1), x_1 - J_\lambda (x_1 + \lambda y_1))
\]

\[
= \left( y_1 - \left( \frac{1}{\lambda} (x_1 + \lambda y_1) - \frac{1}{\lambda} J_\lambda (x_1 + \lambda y_1) \right), x_1 - J_\lambda (x_1 + \lambda y_1) \right)
\]

\[
= \left( \left( -\frac{1}{\lambda} x_1 + \frac{1}{\lambda} J_\lambda (x_1 + \lambda y_1) \right), x_1 - J_\lambda (x_1 + \lambda y_1) \right)
\]

\[
= \frac{1}{\lambda} (x_1 - J_\lambda (x_1 + \lambda y_1), x_1 - J_\lambda (x_1 + \lambda y_1))
\]

which requires

\[
x_1 = J_\lambda (x_1 + \lambda y_1)
\]

and this says \( x_1 \in D (A) \) because \( J_\lambda \) maps into \( D (A) \). Also it says

\[
x_1 + Ax \ni x_1 + \lambda y_1
\]

This makes the last claim pretty easy. Suppose \( x_k \to x \) where \( x_k \in D (A) \) and that \( y_k \in Ax_k \) and \( y_k \rightharpoonup y \). I need to verify \( y = Ax \) and \( x \in D (A) \). Let 

\[ [u, v] \in G (A). \]

Then

\[
(y - v, x - u) = \lim_{k \to \infty} (y_k - v, x_k - u) \geq 0
\]

and so, by the first part, \( x \in D (A) \) and \( y \in Ax \). Why does that limit hold? It is because

\[
|((y - v, x - u) - (y_k - v, x_k - u)|
\]

\[
\leq |(y - v, x - u) - (y_k - v, x - u)| + |(y - v, x - x)|
\]

The second term is no larger than

\[
|y_k - v| |x_k - x|
\]

which converges to 0 since \( y_k \) is weakly convergent, hence bounded. The first term converges to 0 because of the assumption that \( y_k \) converges weakly to \( y \). This proves the lemma.

What about the sum of maximal monotone operators? This might not be maximal monotone but what you can say is the following.

Proposition 32.1.6 Let \( A \) be maximal monotone and let \( B \) be Lipschitz and monotone. Then \( A + B \) is maximal monotone.

**Proof:** First suppose \( B \) has a Lipschitz constant less than 1. The monotonicity is obvious. I need to show that for any \( y \) there exists \( x \in D (A) \) such that

\[
y \in x + Bx + Ax
\]
CHAPTER 32. MAXIMAL MONOTONE OPERATORS, HILBERT SPACE

This happens if and only if
\[ y - Bx \in (I + A)x \]
if and only if
\[ x = (I + A)^{-1}(y - Bx). \]
Let
\[ Tx \equiv (I + A)^{-1}(y - Bx) \]
Then \( T \) is clearly a contraction mapping because \((I + A)^{-1}\) is Lipschitz with Lipschitz constant 1. Therefore, there exists a unique fixed point and this shows \( A + B \) is maximal monotone. Now the same argument applied to \( A + B \) shows that \( A + 2B \) is maximal monotone. Continuing this way \( A + nB \) is maximal monotone. Now for arbitrary \( B \) let \( n \) be large enough that \( n^{-1}B \) has Lipschitz constant less than 1. Then as just explained, \( A + n(n^{-1}B) = A + B \) is maximal monotone. This proves the proposition.

The following is a useful result for determining conditions under which \( A + B \) is maximal monotone or more particularly whether a given \( y \) is in \((I + A + B)(H)\) where \( A, B \) are both maximal monotone.

**Theorem 32.1.7** Let \( A \) and \( B \) be maximal monotone, let
\[ y \in x_\lambda + B_\lambda x_\lambda + Ax_\lambda, \]
and suppose \( B_\lambda x_\lambda \) is bounded independent of \( \lambda \). Then there exists \( x \in D(A) \cap D(B) \) such that \( y = x + Ax + Bx \).

**Proof:** First of all, it follows from Proposition 32.1.6 that there exists a unique \( x_\lambda \).

Note
\[ y - x_\lambda - B_\lambda x_\lambda \in Ax_\lambda \]
\[ y - x_\mu - B_\mu x_\mu \in Ax_\mu \]
and so by monotonicity of \( A \),
\[ (x_\mu - x_\lambda + B_\mu x_\mu - B_\lambda x_\lambda, x_\lambda - x_\mu) \geq 0 \]
and so
\[ |x_\lambda - x_\mu|^2 \leq (B_\mu x_\mu - B_\lambda x_\lambda, x_\lambda - x_\mu) = -(B_\lambda x_\lambda - B_\mu x_\mu, x_\lambda - x_\mu) \]  \( (32.1.3) \)

I want to write as many things as possible in terms of the \( B_\lambda \) and \( B_\mu \). Denote as \( J_\lambda (B) \) the operator \((I + \lambda B)^{-1}\). Then
\[ B_\lambda x_\lambda = \frac{1}{\lambda}(x_\lambda - J_\lambda (B)x_\lambda) \]
and so
\[ x_\lambda = \lambda B_\lambda x_\lambda + J_\lambda (B)x_\lambda \]
32.1. BASIC THEORY

Thus \(|x_\lambda - x_\mu|^2 = \)

\[-(B_\lambda x_\lambda - B_\mu x_\mu, \lambda B_\lambda x_\lambda + J_\lambda (B) x_\lambda - (\mu B_\mu x_\mu + J_\mu (B) x_\mu)) \]

\[= -(B_\lambda x_\lambda - B_\mu x_\mu, \lambda B_\lambda x_\lambda - \mu B_\mu x_\mu) + (B_\mu x_\mu - B_\lambda x_\lambda, J_\lambda (B) x_\lambda - J_\mu (B) x_\mu) \]

\[= -(B_\lambda x_\lambda - B_\mu x_\mu, \lambda B_\lambda x_\lambda - \lambda B_\mu x_\mu) - (B_\lambda x_\lambda - B_\mu x_\mu, (\lambda - \mu) B_\mu x_\mu) \]

Now recall \(B_\mu x \in BJ_\mu (B) x\). Then by monotonicity the first and last terms to the right of the equal sign in the above are negative. Therefore,

\[|x_\lambda - x_\mu|^2 \leq |(B_\lambda x_\lambda - B_\mu x_\mu, (\lambda - \mu) B_\mu x_\mu)| \leq C |\lambda - \mu| \]

where \(C\) is some constant which comes from the assumption the \(B_\lambda x_\lambda\) are bounded.

Therefore, letting \(\lambda\) denote a sequence converging to 0 it follows

\[\lim_{\lambda \to 0} x_\lambda = x_1 \in H \]

for some \(x\), the convergence being strong convergence. Also taking a further subsequence and using weak compactness it can be assumed

\[B_\lambda x_\lambda \rightharpoonup z_1 \]

where this time the convergence is weak. Taking another subsequence, it can also be assumed

\[y - x_\lambda - B_\lambda x_\lambda \rightharpoonup z_2 \quad (32.1.4)\]

the convergence being weak convergence. Recall \(B_\lambda x_\lambda \in BJ_\lambda (B) x_\lambda\) and also note that by assumption there is a constant \(C\) independent of \(\lambda\) such that

\[C \geq |B_\lambda x_\lambda| \geq \frac{1}{\lambda} (x_\lambda - J_\lambda (B) x) \]

which shows

\[J_\lambda (B) x_\lambda \to x_1 \]

also. Now it follows from Lemma \(32.1.5\) that \(x_1 \in D (B)\) and \(z_1 \in Bx_1\). Recall

\[y - x_\lambda - B_\lambda x_\lambda \in Ax_\lambda \]

and so by the same lemma again,

\[x_1 \in D (A), \; z_2 \in Ax_1 \]

By \(32.1.6\) it follows

\[y - x_1 - z_1 = z_2 \in Ax_1 \]

Thus

\[y = x_1 + z_1 + z_2 \in x_1 + Bx_1 + Ax_1 \]

and this proves the theorem.
CHAPTER 32. MAXIMAL MONOTONE OPERATORS, HILBERT SPACE

32.2 Evolution Inclusions

One of the interesting things about maximal monotone operators is the concept of evolution inclusions. To facilitate this, here is a little lemma.

Lemma 32.2.1 Let $f : [0, T] \to \mathbb{R}$ be continuous and suppose

$$D^+ f(t) \equiv \limsup_{h \to 0^+} \frac{f(t + h) - f(t)}{h} < g(t)$$

where $g$ is a continuous function. Then

$$f(t) - f(0) \leq \int_0^t g(s) \, ds.$$  

Proof: Suppose this is not so. Then let

$$S \equiv \{ t \in [0, T] : f(t) - f(0) > \int_0^t g(s) \, ds \}$$

and it would follow that $S \neq \emptyset$. Let $a = \inf S$. Then there exists a decreasing sequence $h_n \to 0$ such that

$$f(a + h_n) - f(0) > \int_a^{a + h_n} g(s) \, ds$$  

(32.2.5)

First suppose $a = 0$. Then dividing by $h_n$ and taking the limit,

$$g(0) > D^+ f(0) \geq g(0),$$

a contradiction. Therefore, assume $a > 0$. Then by continuity

$$f(a) - f(0) \geq \int_0^a g(s) \, ds$$

If strict inequality holds, then $a \neq \inf S$. It follows

$$f(a) - f(0) = \int_0^a g(s) \, ds$$

and so from

$$\frac{f(a + h_n) - f(a)}{h_n} > \frac{1}{h_n} \int_a^{a + h_n} g(s) \, ds.$$  

Then doing $\limsup_{n \to \infty}$ to both sides,

$$g(a) > D^+ f(a) \geq g(a)$$

the same sort of contradiction obtained earlier. Thus $S = \emptyset$ and this proves the lemma.

The following is the main result.
Theorem 32.2.2

Let $H$ be a Hilbert space and let $A$ be a maximal monotone operator as described above. Let $f : [0, T] \to H$ be continuous such that $f' \in L^2(0, T; H)$. Then there exists a unique solution to the evolution inclusion

$$y' + Ay \ni f, \quad y(0) = y_0 \in D(A)$$

Here $y'$ exists a.e., $y(t) \in D(A)$ a.e., $y$ is continuous.

Proof: Let $y_{\lambda}$ be the solution to

$$y'_{\lambda} + A_{\lambda} y_{\lambda} = f, \quad y_{\lambda}(0) = y_0$$

I will base the entire proof on estimating the solutions to the corresponding integral equation

$$y_{\lambda}(t) - y_0 + \int_0^t A_{\lambda} y_{\lambda}(s) \, ds = \int_0^t f(s) \, ds \quad (32.2.6)$$

Let $h, k$ be small positive numbers. Then

$$y_{\lambda}(t + h) - y_{\lambda}(t) + \int_t^{t+h} A_{\lambda} y_{\lambda}(s) \, ds = \int_t^{t+h} f(s) \, ds \quad (32.2.7)$$

Next consider the difference operator

$$D_k g(t) = \frac{g(t+k) - g(t)}{k}$$

Do this $D_k$ to both sides of (32.2.7) where $k < h$. This gives

$$D_k (y_{\lambda}(t + h) - y_{\lambda}(t)) + \frac{1}{k} \left( \int_{t+h}^{t+h+k} A_{\lambda} y_{\lambda}(s) \, ds - \int_t^{t+k} A_{\lambda} y_{\lambda}(s) \, ds \right)$$

$$= \frac{1}{k} \left( \int_{t+h}^{t+h+k} f(s) \, ds - \int_t^{t+k} f(s) \, ds \right) \quad (32.2.8)$$

Now multiply both sides by $y_{\lambda}(t + h + k) - y_{\lambda}(t + k)$. Consider the first term. To simplify the ideas consider instead

$$\langle D_k g(t), g(t + k) \rangle = \frac{1}{k} \left( |g(t + k)|^2 - (g(t), g(t + k)) \right)$$

$$\ge \frac{1}{k} \left( |g(t + k)|^2 - |g(t)||g(t + h)| \right)$$

$$\ge \frac{1}{k} \left( \frac{1}{2} |g(t + k)|^2 - \frac{1}{2} |g(t)|^2 \right) \quad (32.2.9)$$

Then applying this simple observation to (32.2.8)

$$\frac{1}{2} \frac{1}{k} \left( |y_{\lambda}(t + h + k) - y_{\lambda}(t + k)|^2 - |y_{\lambda}(t + h) - y_{\lambda}(t)|^2 \right) + \ldots$$
\[ \left( \frac{1}{k} \left( \int_{t+h}^{t+h+k} A_\lambda y_\lambda(s) \, ds - \int_{t}^{t+k} A_\lambda y_\lambda(s) \, ds \right), y_\lambda(t + h + k) - y_\lambda(t + k) \right) + \]
\[ \leq \left( \frac{1}{k} \left( \int_{t+h}^{t+h+k} f(s) \, ds - \int_{t}^{t+k} f(s) \, ds \right), y_\lambda(t + h + k) - y_\lambda(t + k) \right) \]

Taking \( \limsup_{k \to 0} \) of both sides yields
\[ \frac{1}{2} D^+ \left( |y_\lambda(t + h) - y_\lambda(t)|^2 \right) + (A_\lambda y_\lambda(t + h) - A_\lambda y_\lambda(t), y_\lambda(t + h) - y_\lambda(t)) \]
\[ \leq (f(t + h) - f(t), y_\lambda(t + h) - y_\lambda(t)) \]

Now recall that \( A_\lambda \) is monotone. Therefore,
\[ D^+ \left( |y_\lambda(t + h) - y_\lambda(t)|^2 \right) \leq |f(t + h) - f(t)|^2 + |y_\lambda(t + h) - y_\lambda(t)|^2 \]

From Lemma 32.2.1 it follows that for all \( \varepsilon > 0, \)
\[ |y_\lambda(t + h) - y_\lambda(t)|^2 - |y_\lambda(h) - y_0|^2 \]
\[ \leq \int_0^t |f(s + h) - f(s)|^2 \, ds + \int_0^t |y_\lambda(s + h) - y_\lambda(s)|^2 \, ds + \varepsilon t \]

and so since \( \varepsilon \) is arbitrary, the term \( \varepsilon t \) can be eliminated. By Gronwall’s inequality,
\[ |y_\lambda(t + h) - y_\lambda(t)|^2 \leq e^t \left( |y_\lambda(h) - y_0|^2 + \int_0^t |f(s + h) - f(s)|^2 \, ds \right). \] (32.2.10)

The last integral equals
\[ \int_0^t \left| \int_s^{s+h} f'(r) \, dr \right|^2 \, ds \leq \int_0^t h \int_s^{s+h} |f'(r)|^2 \, dr \, ds \]
\[ = h \left[ \int_0^h \int_0^r |f'(r)|^2 \, ds \, dr + \int_0^h \int_{r-h}^r |f'(r)|^2 \, ds \, dr + \int_t^{t+h} \int_{r-h}^{r-h} |f'(r)|^2 \, ds \, dr \right] \]
\[ \leq h^2 \int_0^{t+h} |f'(r)|^2 \, dr \]

and now it follows that for all \( t + h < T, \)
\[ \frac{|y_\lambda(t + h) - y_\lambda(t)|^2}{h} \leq e^T \left( \frac{|y_\lambda(h) - y_0|^2}{h} + \|f'\|^2_{L^2(0,T;H)} \right). \] (32.2.11)

Now return to 32.2.11
\[ \left| y_\lambda(h) - y_0 \right| \leq \frac{1}{h} \int_0^h A_\lambda y_\lambda(s) \, ds + \frac{1}{h} \int_0^h f(s) \, ds \]
Then taking lim sup$ _{h \to 0}$ of both sides
\[
\limsup_{h \to 0} \left| \frac{y_\lambda (h) - y_0}{h} \right| \leq |A_\lambda y_0| + |f(0)|
\]

From Lemma 32.2.13 $|A_\lambda y_0| \leq |a|$ for all $a \in A y_0$. This is where $y_0 \in D(A)$ is used. Thus from 32.2.14 there exists a constant $C$ independent of $t$ and $h$ and $\lambda$ such that
\[
\left| \frac{y_\lambda (t + h) - y_\lambda (t)}{h} \right|^2 \leq C
\]

From the estimate just obtained and 32.2.14, this implies
\[
\frac{y_\lambda (t + h) - y_\lambda (t)}{h} + \frac{1}{h} \int_t^{t+h} A_\lambda y_\lambda (s) \, ds = \frac{1}{h} \int_t^{t+h} f(s) \, ds \tag{32.2.12}
\]

Now letting $h \to 0$, it follows that for all $t \in [0, T)$, there exists a constant $C$ independent of $t, \lambda$ such that
\[
|A_\lambda y_\lambda (t)| \leq C. \tag{32.2.13}
\]

This is a very nice estimate. The next task is to show uniform convergence of the $y_\lambda$ as $\lambda \to 0$. From 32.2.15
\[
(D_h (y_\lambda (t) - y_\mu (t)), y_\lambda (t + h) - y_\mu (t + h)) +
\left( \frac{1}{h} \int_t^{t+h} (A_\lambda y_\lambda (s) - A_\mu y_\mu (s)) \, ds, y_\lambda (t + h) - y_\mu (t + h) \right) = 0
\]

Then from the argument in 32.2.16,
\[
\frac{1}{h} \int_t^{t+h} (A_\lambda y_\lambda (s) - A_\mu y_\mu (s)) \, ds, y_\lambda (t + h) - y_\mu (t + h) \right) \leq 0
\]

Now take lim sup$ _{h \to 0}$ to obtain
\[
\frac{1}{2} D^+ |y_\lambda (t) - y_\mu (t)|^2 + (A_\lambda y_\lambda (t) - A_\mu y_\mu (t), y_\lambda (t) - y_\mu (t)) \leq 0
\]

Using the definition of $A_\lambda$ this equals
\[
\frac{1}{2} D^+ |y_\lambda (t) - y_\mu (t)|^2 +
(A_\lambda y_\lambda (t) - A_\mu y_\mu (t), \lambda A_\lambda y_\lambda (t) + J_\lambda y_\lambda (t) - (\mu A_\mu y_\mu (t) + J_\mu y_\mu (t))) \leq 0
\]

Now this last term splits into the following sum
\[
(A_\lambda y_\lambda (t) - A_\mu y_\mu (t), \lambda A_\lambda y_\lambda (t) - \mu A_\mu y_\mu (t))
+ (A_\lambda y_\lambda (t) - A_\mu y_\mu (t), J_\lambda y_\lambda (t) - J_\mu y_\mu (t))
\]
By Lemma 32.1.3, the second of these terms is nonnegative. Also from the estimate 32.2.13, the first term converges to 0 uniformly in \( t \) as \( \lambda, \mu \to 0 \). Then by Lemma 32.2.1 it follows that if \( \lambda \) is any sequence converging to 0, \( y_\lambda (t) \) is uniformly Cauchy. Let

\[
y(t) = \lim_{\lambda \to 0} y_\lambda (t).
\]

Thus \( y \) is continuous because it is the uniform limit of continuous functions. Since \( A_\lambda y_\lambda (t) \) is uniformly bounded, it also follows

\[
y(t) = \lim_{\lambda \to 0} J_\lambda y_\lambda (t)
\]

uniformly in \( t \). (32.2.14)

Taking a further subsequence, you can assume

\[
A_\lambda y_\lambda \rightharpoonup z \text{ weak }^* \text{ in } L^\infty (0, T; H).
\]

(32.2.15)

Thus \( z \in L^\infty (0, T; H) \). Recall \( A_\lambda y_\lambda \in AJ_\lambda y_\lambda \).

Now \( A \) can be considered a maximal monotone operator on \( L^2 (0, T; H) \) according to the rule

\[
Ay(t) \equiv A(y(t))
\]

where

\[
D(A) \equiv \{ f \in L^2 (0, T; H) : f(t) \in D(A) \text{ a.e. } t \}
\]

By Lemma 32.1.5 applied to \( A \) considered as a maximal monotone operator on \( L^2 (0, T; H) \) and using 32.2.13 and 32.2.14, it follows \( y(t) \in D(A) \text{ a.e. } t \) and \( z(t) \in Ay(t) \text{ a.e. } t \). Then passing to the limit in 32.2.14 yields

\[
y(t) - y_0 + \int_0^t z(s) \, ds = \int_0^t f(s) \, ds.
\]

(32.2.16)

Then by fundamental theorem of calculus, \( y'(t) \) exists a.e. \( t \) and

\[
y' + z = f, \; y(0) = y_0
\]

where \( z(t) \in Ay(t) \text{ a.e. } \)

It remains to verify uniqueness. Suppose \([y_1, z_1]\) is another pair which works. Then from 32.2.14,

\[
y(t) - y_1(t) + \int_0^t (z(r) - z_1(r)) \, dr = 0
\]

\[
y(s) - y_1(s) + \int_0^s (z(r) - z_1(r)) \, dr = 0
\]

Therefore for \( s < t \),

\[
y(t) - y_1(t) - (y(s) - y_1(s)) = \int_s^t (z(r) - z_1(r)) \, dr
\]
32.3. SUBGRADIENTS

and so

\[ ||y(t) - y_1(t)|| - |y(s) - y_1(s)|| \leq K |s - t| \]

for some \( K \) depending on \( ||z||_{L^\infty} , ||z_1||_{L^\infty} \). Since \( y, y_1 \) are bounded, it follows that \( t \to ||y(t) - y_1(t)||^2 \) is also Lipschitz. Therefore by Corollary 24.4.3, it is the integral of its derivative which exists a.e. So what is this derivative? As before,

\[
(D_h (y(t) - y_1(t)), y(t + h) - y_1(t + h))
+ \left( \frac{1}{h} \int_t^{t+h} (z(s) - z_1(s)) \ ds, y(t + h) - y_1(t + h) \right) = 0
\]

and so

\[
\frac{1}{h} \left( \frac{|y(t + h) - y_1(t + h)|^2}{2} - \frac{|y(t) - y_1(t)|^2}{2} \right)
+ \left( \frac{1}{h} \int_t^{t+h} (z(s) - z_1(s)) \ ds, y(t + h) - y_1(t + h) \right) \leq 0
\]

Then taking \( \lim_{h \to 0} \) it follows that for a.e. \( t \) (Lebesgue points of \( z - z_1 \) intersected with the points where \( ||y - y_1||^2 \) has a derivative)

\[
\frac{1}{2} \frac{d}{dt} |y(t) - y_1(t)|^2 + (z(t) - z_1(t), y(t) - y_1(t)) \leq 0
\]

Thus for a.e. \( t \),

\[
\frac{d}{dt} |y(t) - y_1(t)|^2 \leq 0
\]

and so

\[
|y(t) - y_1(t)|^2 - |y_0 - y_0|^2 = \int_0^t \frac{d}{dt} |y(s) - y_1(s)|^2 \ ds \leq 0.
\]

This proves the theorem.

32.3 Subgradients

32.3.1 General Results

**Definition 32.3.1** Let \( X \) be a real locally convex topological vector space. For \( x \in X \), \( \delta \phi(x) \subseteq X' \), possibly \( \emptyset \). This subset of \( X' \) is defined by \( y^* \in \delta \phi(x) \) means for all \( z \in X \),

\[
y^*(z - x) \leq \phi(z) - \phi(x).
\]

Also \( x \in \delta y^* (y^*) \) means that for all \( z^* \in X' \),

\[
(z^* - y^*) (x) \leq \phi^* (z^*) - \phi^* (y^*).
\]

We define \( \text{dom}(\delta \phi) \equiv \{ x : \delta \phi(x) \neq \emptyset \} \).
The subgradient is an attempt to generalize the derivative. For example, a function may have a subgradient but fail to be differentiable at some point. A good example is \( f(x) = |x| \). At \( x = 0 \), this function fails to have a derivative but it does have a subgradient. In fact, \( \delta f(0) = [-1, 1] \).

To begin with consider the question of existence of the subgradient of a convex function. There is a very simple criterion for existence. It is essentially that the subgradient is nonempty at every point of the interior of the domain of \( \phi \). First recall Lemma 16.2.15 which says the interior of a convex set is convex and if nonempty, then every point of the convex set can be obtained as the limit of a sequence of points of the interior.

**Theorem 32.3.2** Let \( \phi : X \to (-\infty, \infty] \) be convex and suppose for some \( u \in \text{dom} (\phi) \), \( \phi \) is continuous. Then \( \delta \phi (x) \neq \emptyset \) for all \( x \in \text{int} (\text{dom} (\phi)) \). Thus

\[
\text{dom} (\delta \phi) \supseteq \text{int} (\text{dom} (\phi)).
\]

**Proof:** Let \( x_0 \in \text{int} (\text{dom} (\phi)) \) and let

\[
A \equiv \{(x_0, \phi (x_0))\}, B \equiv \text{epi} (\phi) \cap X \times \mathbb{R}.
\]

Then \( A \) and \( B \) are both nonempty and convex. Recall epi (\( \phi \)) can contain a point like \( (x, \infty) \). Since \( \phi \) is continuous at \( u \in \text{dom} (\phi) \),

\[
(u, \phi (u) + 1) \in \text{int} (\text{epi} \phi \cap X \times \mathbb{R}).
\]

Thus \( \text{int} (B) \neq \emptyset \) and also \( \text{int} (B) \cap A = \emptyset \). By Lemma 16.2.15 \( \text{int} (B) \) is convex and so by Theorem 16.2.14 there exists \( x^* \in X' \) and \( \beta \in \mathbb{R} \) such that

\[
(x^*, \beta) \neq (0, 0) \tag{32.3.17}
\]

and for all \( (x, a) \in \text{int} B \),

\[
x^* (x) + \beta a > x^* (x_0) + \beta \phi (x_0). \tag{32.3.18}
\]

From Lemma 16.2.14, whenever \( x \in \text{dom} (\phi) \),

\[
x^* (x) + \beta \phi (x) \geq x^* (x_0) + \beta \phi (x_0).
\]

If \( \beta = 0 \), this would mean \( x^* (x - x_0) \geq 0 \) for all \( x \in \text{dom} (\phi) \). Since \( x_0 \in \text{int} (\text{dom} (\phi)) \), this implies \( x^* = 0 \), contradicting 16.2.14. If \( \beta < 0 \), apply 16.2.14 to the case when \( a = \phi (x_0) + 1 \) and \( x = x_0 \) to obtain a contradiction. It follows \( \beta > 0 \) and so

\[
\phi (x) - \phi (x_0) \geq -\frac{x^*}{\beta} (x - x_0)
\]

which says \( -x^*/\beta \in \delta \phi (x_0) \). This proves the theorem.

**Definition 32.3.3** Let \( \phi : X \to (-\infty, \infty] \) be some function, not necessarily convex but satisfying \( \phi (y) < \infty \) for some \( y \in X \). Define \( \phi^* : X' \to (-\infty, \infty] \) by

\[
\phi^* (x^*) \equiv \sup \{x^* (y) - \phi (y) : y \in X\}.
\]
This function, \( \phi^* \), defined above, is called the conjugate function of \( \phi \) or the polar of \( \phi \). Note \( \phi^* (x^*) \neq -\infty \) because \( \phi (y) < \infty \) for some \( y \).

**Theorem 32.3.4** Let \( X \) be a real Banach space. Then \( \phi^* \) is convex and l.s.c.

**Proof:** Let \( \lambda \in [0,1] \). Then

\[
\phi^* (\lambda x^* + (1 - \lambda) y^*) = \sup \{ (\lambda x^* + (1 - \lambda) y^*) (y) - \phi (y) : y \in X \}
\]

\[
\leq \lambda \phi^* (x^*) + (1 - \lambda) \phi^* (y^*).
\]

It remains to show the function is l.s.c. Consider \( f_y (x^*) \equiv x^* (y) - \phi (y) \). Then \( f_y \) is obviously convex. Also to say that \((x, \alpha) \in \text{epi} (\phi^*)\) is to say that \( \alpha \geq x^* (y) - \phi (y) \) for all \( y \). Thus

\[
\text{epi} (\phi^*) = \bigcap_{y \in X} \text{epi} (f_y).
\]

Therefore, if \( \text{epi} (f_y) \) is closed, this will prove the theorem. If \((x^*, a) \notin \text{epi} (f_y)\), then \( a < x^* (y) - \phi (y) \) and, by continuity, for \( b \) close enough to \( a \) and \( y^* \) close enough to \( x^* \) then

\[
b < y^* (y) - \phi (y), \ (y^*, b) \notin \text{epi} (f_y)
\]

Thus \( \text{epi} (f_y) \) is closed. □

Note this theorem holds with no change in the proof if \( X \) is only a locally convex topological vector space and \( X' \) is given the weak * topology.

**Definition 32.3.5** We define \( \phi^{**} \) on \( X \) by

\[
\phi^{**} (x) \equiv \sup \{ x^* (x) - \phi^* (x^*) : x^* \in X' \}.
\]

The following lemma comes from separation theorems. First is a simple observation.

**Observation 32.3.6** \( f \in (X \times \mathbb{R})' \) if and only if there exists \( x^* \in X' \) and \( \alpha \in \mathbb{R} \) such that \( f(x, \lambda) = x^* (x) + \lambda \alpha \). To get \( x^* \), you can simply define \( x^* (x) \equiv f(x, 0) \) and to get \( \alpha \) you just let \( \alpha \lambda \equiv f(0, \lambda) \). Why does such an \( \alpha \) exist? You know that \( f(0, a \lambda + b \delta) = a f(0, \alpha) + b f(0, \delta) \) and so in fact \( \lambda \to f(0, \lambda) \) satisfies the Cauchy functional equation \( g(x + y) = g(x) + g(y) \) and is continuous so there is only one thing it can be and that is \( f(0, \lambda) = \alpha \lambda \) for some \( \alpha \).

This picture illustrates the conclusion of the following lemma.
Lemma 32.3.7 Let $\phi : X \to (-\infty, \infty]$ be convex and lower semicontinuous and $\phi(x) < \infty$ for some $x$. (proper). Then if $\beta < \phi(x_0)$ so that $(x_0, \beta)$ is not in $\text{epi}\left(\phi\right)$, it follows that there exists $\delta > 0$ and $z^* \in X'$ such that for all $y$,

$$z^* (y - x_0) + \beta + \delta < \phi(y), \text{ all } y \in X$$

**Proof:** Let $C = \text{epi}(\phi) \cap (X \times \mathbb{R})$. Then $C$ is a closed convex nonempty set and it does not contain the point $(x_0, \beta)$. Let $\hat{\beta} > \beta$ be slightly larger so that also $(x_0, \hat{\beta}) \notin C$. Thus there exists $y^* \in X'$ and $\alpha \in \mathbb{R}$ such that for some $\hat{c}$, and all $y \in X$,

$$y^* (x_0) + \alpha \hat{\beta} > \hat{c} > y^* (y) + \alpha \phi(y)$$

for all $y \in X$. Now you can’t have $\alpha \geq 0$ because

$$\alpha \left( \hat{\beta} - \phi(y) \right) > y^* (y - x_0)$$

and you can let $y = x_0$ to have

$$\alpha \left( \frac{\hat{c} - \phi(x_0)}{\beta - \phi(x_0)} \right) > 0$$

Hence $\alpha < 0$ and so, dividing by it yields that for all $y \in X$,

$$x^* (x_0) + \hat{\beta} < c < x^* (y) + \phi(y)$$

where $x^* = y^*/\alpha, \hat{c}/\alpha \equiv c$. Then

$$(-x^*) (y - x_0) + \beta + \left( \hat{\beta} - \beta \right) < c - x^* (y) < \phi(y)$$

$$(-x^*) (y - x_0) + \beta + \delta < \phi(y), \delta \equiv \hat{\beta} - \beta$$

Let $z^* = -x^*$. $\blacksquare$

Theorem 32.3.8 $\phi^{**}(x) \leq \phi(x)$ for all $x$ and if $\phi$ is convex and l.s.c., $\phi^{**}(x) = \phi(x)$ for all $x \in X$.

**Proof:**

$$\phi^{**}(x) \equiv \sup \left\{ x^* (x) - \sup \{x^* (y) - \phi(y) : y \in X \} : x^* \in X' \right\}$$

$$\leq \sup \{x^* (x) - (x^* (x) - \phi(x))\} = \phi(x).$$

Next suppose $\phi$ is convex and l.s.c. If $\phi^{**}(x_0) < \phi(x_0)$, then using Lemma 32.3.7, there exists $x_0^*, \delta > 0$ such that for all $y \in X$,

$$(x_0^*) (y - x_0) + \phi^{**}(x_0) + \delta < \phi(y)$$
32.3. SUBGRADIENTS

\[ x_0^*(y) - \phi(y) + \delta < x_0^*(x_0) - \phi^{**}(x_0) \]

Thus, since this holds for all \( y \),

\[
\begin{align*}
\phi^*(x_0^*) + \delta &\leq x_0^*(x_0) - \phi^{**}(x_0) \\
\phi^{**}(x_0) + \delta &\leq x_0^*(x_0) - \phi^*(x_0^*)
\end{align*}
\]

Then

\[
\phi^{**}(x_0) \equiv \sup \{ x^*(x_0) - \phi^*(x^*), x^* \in X' \} 
\geq x_0^*(x_0) - \phi^*(x_0^*) \geq \phi^{**}(x_0) + \delta
\]
a contradiction. ■

The following corollary is descriptive of the situation just discussed. It says that to find \( \text{epi}(\phi^{**}) \) it suffices to take the intersection of all closed convex sets which contain \( \text{epi}(\phi) \).

**Corollary 32.3.9** \( \text{epi}(\phi^{**}) \) is the smallest closed convex set containing \( \text{epi}(\phi) \).

**Proof:** \( \text{epi}(\phi^{**}) \supseteq \text{epi}(\phi) \) from Theorem 32.3.8. Also \( \text{epi}(\phi^{**}) \) is closed by the proof of Theorem 32.3.4. Suppose \( \text{epi}(\phi) \subseteq K \subseteq \text{epi}(\phi^{**}) \) and \( K \) is convex and closed. Let

\[ \psi(x) \equiv \min \{ a : (x, a) \in K \}. \]

(\( \{ a : (x, a) \in K \} \) is a closed subset of \( (-\infty, \infty) \) so the minimum exists.) \( \psi \) is also a convex function with \( \text{epi}(\psi) = K \). To see \( \psi \) is convex, let \( \lambda \in [0, 1] \). Then, by the convexity of \( K \),

\[
\lambda (x, \psi(x)) + (1 - \lambda) (y, \psi(y)) = (\lambda x + (1 - \lambda) y, \lambda \psi(x) + (1 - \lambda) \psi(y)) \in K.
\]

It follows from the definition of \( \psi \) that

\[
\psi(\lambda x + (1 - \lambda) y) \leq \lambda \psi(x) + (1 - \lambda) \psi(y).
\]

Then

\[ \phi^{**} \leq \psi \leq \phi \]

and so from the definitions,

\[ \phi^{***} \geq \psi^* \geq \phi^* \]

which implies from the definitions and Theorem 32.3.8 that

\[ \phi^{**} = \phi^{****} \leq \psi^{**} = \psi \leq \phi^{**}. \]

Therefore, \( \psi = \phi^{**} \) and \( \text{epi}(\phi^{**}) \) is the smallest closed convex set containing \( \text{epi}(\phi) \) as claimed. ■

There is an interesting symmetry which relates \( \delta \phi, \delta \phi^*, \phi, \) and \( \phi^* \).
**Theorem 32.3.10** Suppose $\phi$ is convex, l.s.c. (lower semicontinuous or in other words having a closed epigraph), and proper. Then

$$ y^* \in \delta \phi (x) \text{ if and only if } x \in \delta \phi^* (y^*) $$

where this last expression means

$$ (z^* - y^*) (x) \leq \phi^* (z^*) - \phi^* (y^*) $$

for all $z^*$ and in this case,

$$ y^* (x) = \phi^* (y^*) + \phi (x). $$

**Proof:** If $y^* \in \delta \phi (x)$ then $y^* (z - x) \leq \phi (z) - \phi (x)$ and so

$$ y^* (z) - \phi (z) \leq y^* (x) - \phi (x) $$

for all $z \in X$. Therefore,

$$ \phi^* (y^*) \leq y^* (x) - \phi (x) \leq \phi^* (y^*). $$

Hence

$$ y^* (x) = \phi^* (y^*) + \phi (x). \quad (32.3.19) $$

Now if $z^* \in X'$ is arbitrary, (32.3.19) shows

$$ (z^* - y^*) (x) = z^* (x) - y^* (x) = z^* (x) - \phi (x) - \phi^* (y^*) \leq \phi^* (z^*) - \phi^* (y^*) $$

and this shows $x \in \delta \phi^* (y^*)$.

Now suppose $x \in \delta \phi^* (y^*)$. Then for $z^* \in X'$,

$$ (z^* - y^*) (x) \leq \phi^* (z^*) - \phi^* (y^*) $$

so

$$ z^* (x) - \phi^* (z^*) \leq y^* (x) - \phi^* (y^*) $$

and so, taking sup over all $z^*$, and using Theorem 32.3.8,

$$ \phi^{**} (x) = \phi (x) \leq y^* (x) - \phi^* (y^*) \leq \phi^{**} (x). $$

Thus

$$ y^* (x) = \phi^* (y^*) + \phi^{**} (x) = \phi^* (y^*) + \phi (x) \geq y^* (z) - \phi (z) + \phi (x) $$

for all $z \in X$ and this implies for all $z \in X$,

$$ \phi (z) - \phi (x) \geq y^* (z - x) $$

so $y^* \in \delta \phi (x)$ and this proves the theorem.
Definition 32.3.11 If $X$ is a Banach space define $u \in W^{1, p}([0, T]; X)$ if there exists $g \in L^p ([0, T]; X)$ such that

$$ u(t) = u(0) + \int_0^t g(s) \, ds $$

When this occurs define $u'(\cdot) \equiv g(\cdot)$. As usual, $p > 1$.

The next Lemma is quite interesting for its own sake but it is also used in the next theorem.

Lemma 32.3.12 Suppose $g \in L^p (0, T; X)$. Then as $h \to 0$,

$$ \frac{1}{h} \int_{(\cdot)+h}^{(\cdot)+h} g(s) \, ds \chi_{[0, T-h]}(\cdot) \to g $$

in $L^p ([0, T]; X)$.

Proof: Let

$$ \tilde{g} (u) \equiv \begin{cases} g(u) & \text{if } u \in [0, T] \\ 0 & \text{if } u \notin [0, T] \end{cases}, \quad \phi_h (r) \equiv \frac{1}{h} \chi_{[0-h, 0]} (r). $$

Thus $\tilde{g} \in L^p (\mathbb{R}; X)$ and

$$ \tilde{g} * \phi_h (t) \equiv \int_{\mathbb{R}} \tilde{g}(t-s) \phi_h (s) \, ds. $$

Then

$$ ||\tilde{g} * \phi_h - \tilde{g}||_{L^p (\mathbb{R}; X)} \leq \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} ||\tilde{g}(t) - \tilde{g}(t-s)||_X \phi_h (s) \, ds \right)^p \, dt \right)^{1/p} $$

which by Minkowski’s inequality for integrals is no larger than

$$ \leq \int_{\mathbb{R}} \phi_h (s) \left( \int_{\mathbb{R}} ||\tilde{g}(t) - \tilde{g}(t-s)||_X^p \, dt \right)^{1/p} \, ds $$

$$ = \frac{1}{h} \int_{-h}^0 \left( \int_{\mathbb{R}} ||\tilde{g}(t) - \tilde{g}(t-s)||_X^p \, dt \right)^{1/p} \, ds < \frac{1}{h} \int_{-h}^0 \varepsilon \, ds = \varepsilon $$

whenever $h$ is small enough. This follows from continuity of translation in $L^p (\mathbb{R}; X)$, a consequence of the regularity of the measure. Thus, $\tilde{g} * \phi_h \to \tilde{g}$ in $L^p (\mathbb{R}; X)$. Now

$$ \tilde{g} * \phi_h (t) - \frac{1}{h} \int_t^{t+h} g(s) \, ds \chi_{[0, T-h]} (t) $$

$$ = \begin{cases} 0 & \text{if } t \in [0, T-h] \\ \frac{1}{h} \int_t^{t+h} \tilde{g}(u) \, du & \text{if } t \notin [0, T-h] \end{cases} $$
and therefore,
\[
\left\| \tilde{g} * \phi_h (\cdot) - \frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) \, ds\mathcal{X}_{[0,T-h]} (\cdot) \right\|_{L^p(\mathbb{R};X)} = \\
\left( \int_{-h}^{0} \left| \int_{t}^{t+h} \tilde{g} (u) \, du \right|^p \, dt \right)^{1/p} + \left( \int_{T-h}^{T} \left| \int_{t}^{t+h} \tilde{g} (u) \, du \right|^p \, dt \right)^{1/p} \\
\leq \frac{1}{h} \left( \int_{-h}^{0} \left( \int_{-h}^{h} \| \tilde{g} (u) \|_X \, du \right)^p \, dt \right)^{1/p} \\
\quad + \frac{1}{h} \left( \int_{T-h}^{T} \left( \int_{T-h}^{T+h} \| \tilde{g} (u) \|_X \, du \right)^p \, dt \right)^{1/p}
\]
which by Minkowski’s inequality for integrals is no larger than
\[
\leq \frac{1}{h} \int_{-h}^{h} \left( \int_{-h}^{h} \| \tilde{g} (u) \|_X \, du \right)^{1/p} \, du + \frac{1}{h} \int_{T-h}^{T} \left( \int_{T-h}^{T+h} \| \tilde{g} (u) \|_X \, du \right)^{1/p} \, du \\
\leq \frac{1}{h} \int_{-h}^{h} \varepsilon \, du + \frac{1}{h} \int_{T-h}^{T} \varepsilon \, du = 4\varepsilon
\]
whenever \( h \) is small enough because of the fact that \( \| \tilde{g} \|^p_X \in L^1(\mathbb{R};X) \). Since \( \varepsilon \) is arbitrary, this shows
\[
\left\| \tilde{g} * \phi_h (\cdot) - \frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) \, ds\mathcal{X}_{[0,T-h]} (\cdot) \right\|_{L^p(\mathbb{R};X)} \to 0
\]
and also, it was shown above that
\[
\| \tilde{g} * \phi_h (\cdot) - \tilde{g} \|_{L^p(\mathbb{R};X)} \to 0
\]
It follows that
\[
\frac{1}{h} \int_{(\cdot)}^{(\cdot)+h} g(s) \, ds\mathcal{X}_{[0,T-h]} (\cdot) \to \tilde{g}
\]
in \( L^p(\mathbb{R};X) \) and consequently in \( L^p([0,T];X) \) as well. But \( \tilde{g} = g \) on \([0,T] \).

The following theorem is a form of the chain rule in which the derivative is replaced by the subgradient.

**Theorem 32.3.13** Suppose \( u \in W^{1,p}([0,T];X) \), \( z \in L^{p'}([0,T];X') \), and \( z(t) \in \delta \phi (u(t)) \) a.e \( t \in [0,T] \). Then the function, \( t \to \phi(u(t)) \) is in \( L^{1} (0,T) \) and its weak derivative equals \( \langle z,u' \rangle \). In particular,
\[
\phi(u(t)) - \phi(u(0)) = \int_{0}^{t} \langle z(s),u'(s) \rangle \, ds
\]
32.3. SUBGRADIENTS

**Proof:** Modify \( u \) on a set of measure zero such that \( \delta \phi \left( u (t) \right) \neq \emptyset \) for all \( t \). Next modify \( z \) on a set of measure zero such that for \( \tilde{u} \) and \( \tilde{z} \) the modified functions, \( \tilde{z} (t) \in \delta \phi \left( \tilde{u} (t) \right) \) for all \( t \). First I claim \( t \rightarrow \phi \left( \tilde{u} (t) \right) \) is in \( L^1 \left( 0, T \right) \). Pick \( t_0 \in [0, T] \) and let

\[
\tilde{z} (t_0) \in \delta \phi \left( \tilde{u} (t_0) \right).
\]

Then for \( t \in [0, T] \),

\[
\langle \tilde{z} (t_0), \tilde{u} (t) - \tilde{u} (t_0) \rangle + \phi (\tilde{u} (t_0)) \leq \phi (\tilde{u} (t)) \leq \langle \tilde{z} (t), \tilde{u} (t) - \tilde{u} (t_0) \rangle + \phi (\tilde{u} (t_0))
\]

(32.3.20)

Then Exercise 32.3.21 shows \( t \rightarrow \phi \left( \tilde{u} (t) \right) \) is in \( L^1 \left( 0, T \right) \) since \( \tilde{z} \in L^p \left( [0, T]; X' \right) \) and \( \tilde{u} \in L^p \left( [0, T]; X \right) \). Also, for \( t \in [0, T - h] \),

\[
\left\langle \mathcal{X}_{[0, T-h]} (t) \tilde{z} (t), \frac{\tilde{u} (t + h) - \tilde{u} (t)}{h} \right\rangle \leq \mathcal{X}_{[0, T-h]} (t) \frac{\phi (\tilde{u} (t + h)) - \phi (\tilde{u} (t))}{h}
\]

\[
\leq \left\langle \mathcal{X}_{[0, T-h]} (t) \tilde{z} (t + h), \frac{\tilde{u} (t + h) - \tilde{u} (t)}{h} \right\rangle
\]

Now \( \mathcal{X}_{[0, T-h]} (\cdot) \tilde{z} (\cdot + h) \rightarrow z (\cdot) \) in \( L^{p'} \left( 0, T; X' \right) \) by continuity of translation. Also,

\[
\mathcal{X}_{[0, T-h]} (\cdot) \frac{\tilde{u} (\cdot + h) - \tilde{u} (\cdot)}{h} = \mathcal{X}_{[0, T-h]} (\cdot) \frac{u (\cdot + h) - u (\cdot)}{h}
\]

\[
= \mathcal{X}_{[0, T-h]} (\cdot) \frac{1}{h} \int_{\cdot}^{(\cdot) + h} u' (s) \, ds
\]

in \( L^p \left( 0, T; X \right) \) and so by Lemma 32.3.22

\[
\mathcal{X}_{[0, T-h]} (\cdot) \left[ \phi \left( \tilde{u} (\cdot + h) \right) - \phi \left( \tilde{u} (\cdot) \right) \right] \rightarrow \langle z, u' \rangle
\]

in \( L^1 \left( 0, T \right) \).

It follows from the definition of weak derivatives that in the sense of weak derivatives,

\[
\frac{d}{dt} \left( \phi \left( u (\cdot) \right) \right) = \langle z, u' \rangle \in L^1 \left( 0, T \right).
\]

Note that by Theorem 42.3.4, this implies that for a.e. \( t \in [0, T] \), \( \phi \left( u (t) \right) \) is equal to a continuous function, \( \phi \circ u \), and that

\[
(\phi \circ u) (t) - (\phi \circ u) (0) = \int_0^t \langle z (s), u' (s) \rangle \, ds.
\]

There are other rules of calculus which have a generalization to subgradients. The following theorem is on such a generalization. It generalizes the theorem which states that the derivative of a sum equals the sum of the derivatives.
Theorem 32.3.14 Let $\phi_1$ and $\phi_2$ be convex, l.s.c. and proper having values in $(-\infty, \infty)$. Then

$$
\delta (\lambda \phi_1) (x) = \lambda \delta \phi_1 (x), \quad \delta (\phi_1 + \phi_2) (x) \supseteq \delta \phi_1 (x) + \delta \phi_2 (x) \quad (32.3.21)
$$

if $\lambda > 0$. If there exists $\tau \in \text{dom}(\phi_1) \cap \text{dom}(\phi_2)$ and $\phi_1$ is continuous at $\tau$ then for all $x \in X$,

$$
\delta (\phi_1 + \phi_2) (x) = \delta \phi_1 (x) + \delta \phi_2 (x). \quad (32.3.22)
$$

\textbf{Proof:} It is obvious so we only need to show (32.3.22). Suppose $\tau$ is as described. It is clear, therefore, that (32.3.22) holds whenever $x \notin \text{dom}(\phi_1) \cap \text{dom}(\phi_2)$ since then both sides equal $\emptyset$. Therefore, assume

$$
x \in \text{dom}(\phi_1) \cap \text{dom}(\phi_2)
$$

in what follows. Let $x^* \in \delta (\phi_1 + \phi_2) (x)$. Is $x^*$ the sum of an element of $\delta \phi_1 (x)$ and $\delta \phi_2 (x)$? Does there exist $x_1^*$ and $x_2^*$ such that for every $y$,

$$
x^* (y - x) = x_1^* (y - x) + x_2^* (y - x) \leq \phi_1 (y) - \phi_1 (x) + \phi_2 (y) - \phi_2 (x) ?
$$

If so, then

$$
\phi_1 (y) - \phi_1 (x) - x^* (y - x) \geq \phi_2 (x) - \phi_2 (y).
$$

Define

$$
C_1 \equiv \{(y, a) \in X \times \mathbb{R} : \phi_1 (y) - \phi_1 (x) - x^* (y - x) \leq a\},
$$

$$
C_2 \equiv \{(y, a) \in X \times \mathbb{R} : a \leq \phi_2 (x) - \phi_2 (y)\}.
$$

I will show int $(C_1) \cap C_2 = \emptyset$ and then by Theorem 32.3.13 there exists something interesting.

Both $C_1$ and $C_2$ are convex and nonempty. $C_1$ is nonempty because it contains $(\tau, \phi_1 (\tau) - \phi_1 (x) - x^* (\tau - x))$ since

$$
\phi_1 (\tau) - \phi_1 (x) - x^* (\tau - x) \leq \phi_1 (\tau) - \phi_1 (x) - x^* (\tau - x)
$$

$C_2$ is also nonempty because it contains $(\tau, \phi_2 (\tau) - \phi_2 (\tau))$ since

$$
\phi_2 (\tau) - \phi_2 (\tau) \leq \phi_2 (\tau) - \phi_2 (\tau)
$$

In addition to this,

$$
(\tau, \phi_1 (\tau) - x^* (\tau - x) - \phi_1 (x) + 1) \in \text{int} (C_1)
$$

due to the assumed continuity of $\phi_1$ at $\tau$ and so int $(C_1) \neq \emptyset$. If $(y, a) \in \text{int} (C_1)$ then

$$
\phi_1 (y) - x^* (y - x) - \phi_1 (x) \leq a - \varepsilon
$$

whenever $\varepsilon$ is small enough. Therefore, if $(y, a)$ is also in $C_2$, the assumption that $x^* \in \delta (\phi_1 + \phi_2) (x)$ implies

$$
a - \varepsilon \geq \phi_1 (y) - x^* (y - x) - \phi_1 (x) \geq \phi_2 (x) - \phi_2 (y) \geq a,
$$

1244

CHAPTER 32. MAXIMAL MONOTONE OPERATORS, HILBERT SPACE
32.3. SUBGRADIENTS

a contradiction. Therefore \( \text{int} (C_1) \cap C_2 = \emptyset \) and so by Theorem 16.2.14, there exists \((w^*, \beta) \in X' \times \mathbb{R}\) with

\[
(w^*, \beta) \neq (0, 0),
\]

and

\[
w^* (y) + \beta a \geq w^* (y_1) + \beta a_1,
\]

whenever \((y, a) \in C_1\) and \((y_1, a_1) \in C_2\).

**Claim:** \( \beta > 0 \).

**Proof of claim:** If \( \beta < 0 \) let

\[
a = \phi_1 (\overline{x}) - x^* (\overline{x} - x) - \phi_1 (x) + 1,
\]

\[
a_1 = \phi_2 (x) - \phi_2 (\overline{x}), \text{ and } y = y_1 = \overline{x}.
\]

Then from (32.3.23)

\[
\beta (\phi_1 (\overline{x}) - x^* (\overline{x} - x) - \phi_1 (x) + 1) \geq \beta (\phi_2 (x) - \phi_2 (\overline{x})).
\]

Dividing by \( \beta \) yields

\[
\phi_1 (\overline{x}) - x^* (\overline{x} - x) - \phi_1 (x) + 1 \leq \phi_2 (x) - \phi_2 (\overline{x})
\]

and so

\[
\phi_1 (\overline{x}) + \phi_2 (\overline{x}) - (\phi_1 (x) + \phi_2 (x)) + 1 \leq x^* (\overline{x} - x)
\]

\[
\leq \phi_1 (\overline{x}) + \phi_2 (\overline{x}) - (\phi_1 (x) + \phi_2 (x)),
\]

a contradiction. Therefore, \( \beta \geq 0 \).

Now suppose \( \beta = 0 \). Letting

\[
a = \phi_1 (\overline{x}) - x^* (\overline{x} - x) - \phi_1 (x) + 1,
\]

\[
(\overline{x}, a) \in \text{int} (C_1),
\]

and so there exists an open set \( U \) containing \( 0 \) and \( \eta > 0 \) such that

\[
\overline{x} + U \times (a - \eta, a + \eta) \subseteq C_1.
\]

Therefore, (32.3.24) applied to \((\overline{x} + z, a) \in C_1\) and \((\overline{x}, \phi_2 (x) - \phi_2 (\overline{x})) \in C_2\) for \( z \in U \) yields

\[
w^* (\overline{x} + z) \geq w^* (\overline{x})
\]

for all \( z \in U \). Hence \( w^* (z) = 0 \) on \( U \) which implies \( w^* = 0 \), contradicting (32.3.24). This proves the claim.

Now with the claim, it follows \( \beta > 0 \) and so, letting \( z^* = w^* / \beta \), (32.3.24) and Lemma 16.2.15 implies

\[
z^* (y) + a \geq z^* (y_1) + a_1
\]

whenever \((y, a) \in C_1\) and \((y_1, a_1) \in C_2\). In particular,

\[
(y, \phi_1 (y) - \phi_1 (x) - x^* (y - x)) \in C_1
\]

(32.3.25)
because
\[ \phi_1(y) - \phi_1(x) - x^*(y - x) \leq \phi_1(y) - x^*(y - x) - \phi_1(x) \]
and
\[ (y_1, \phi_2(x) - \phi_2(y_1)) \in C_2. \quad (32.3.27) \]
by similar reasoning so letting \( y = x \),
\[
z^*(x) + \left( \phi_1(x) - x^*(x - x) - \phi_1(x) \right) = 0 \geq z^*(y_1) + \phi_2(x) - \phi_2(y_1).
\]
Therefore,
\[ z^*(y_1 - x) \leq \phi_2(y_1) - \phi_2(x) \]
for all \( y_1 \) and so \( z^* \in \delta \phi_2(x) \). Now let \( y_1 = x \) in (32.3.27) and using (32.3.25) and (32.3.26), it follows
\[
z^*(y) + \phi_1(y) - x^*(y - x) - \phi_1(x) \geq z^*(x)
\]
\[ \phi_1(y) - \phi_1(x) \geq x^*(y - x) - z^*(y - x) \]
and so \( x^* - z^* \in \delta \phi_1(x) \) so \( x^* = z^* + (x^* - z^*) \in \delta \phi_2(x) + \delta \phi_1(x) \) and this proves the theorem.

Next is a very important example known as the duality map from a Banach space to its dual space. Before doing this, consider a Hilbert space \( H \). Define a map \( R \) from \( H \) to \( H' \), called the Riesz map, by the rule
\[ R(x)(y) \equiv (y, x). \]
By the Riesz representation theorem, this map is onto and one to one with the properties
\[ R(x)(x) = ||x||^2, \text{ and } ||Rx||^2 = ||x||^2. \]
The duality map from a Banach space to its dual is an attempt to generalize this notion of Riesz map to an arbitrary Banach space.

**Definition 32.3.15** For \( X \) a Banach space define \( F : X \to \mathcal{P}(X') \) by
\[
F(x) \equiv \left\{ x^* \in X' : x^*(x) = ||x||^2, ||x^*|| \leq ||x|| \right\}.
\]
(32.3.28)

**Lemma 32.3.16** With \( F(x) \) defined as above, it follows that
\[
F(x) = \left\{ x^* \in X' : x^*(x) = ||x||^2, ||x^*|| = ||x|| \right\}
\]
and \( F(x) \) is a closed, nonempty, convex subset of \( X' \).
Proof: If \( x^* \) is in the set described in \(32.3.28\),
\[
x^* \left( \frac{x}{||x||} \right) = ||x||
\]
and so \( ||x^*|| \geq ||x|| \). Therefore
\[
x^* \in \left\{ x^* \in X' : x^* (x) = ||x||^2, ||x^*|| = ||x|| \right\}.
\]
This shows this set and the set of \(32.3.28\) are equal. It is also clear the set of \(32.3.28\) is closed and convex. It only remains to show this set is nonempty.

Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(\alpha x) = \alpha ||x||^2 \). Then the norm of \( f \) on \( \mathbb{R} \) is \( ||f|| \) and \( f(x) = ||x||^2 \). By the Hahn Banach theorem, \( f \) has an extension to all of \( X \), and this extension is in the set of \(32.3.28\), showing this set is nonempty as required.

The next theorem shows this duality map is the subgradient of \( \frac{1}{2} ||x||^2 \).

**Theorem 32.3.17** For \( X \) a real Banach space, let \( \phi(x) \equiv \frac{1}{2} ||x||^2 \). Then \( F(x) = \delta \phi(x) \).

Proof: Let \( x^* \in F(x) \). Then
\[
\langle x^*, y-x \rangle = \langle x^*, y \rangle - \langle x^*, x \rangle \\
\leq ||x|| ||y|| - ||x||^2 \leq \frac{1}{2} ||y||^2 - \frac{1}{2} ||x||^2.
\]
This shows \( F(x) \subseteq \delta \phi(x) \).

Now let \( x^* \in \delta \phi(x) \). Then for all \( t \in \mathbb{R} \),
\[
\langle x^*, ty \rangle = \langle x^*, (t + x) \rangle - \langle x^*, x \rangle \leq \frac{1}{2} \left( ||x + ty||^2 - ||x||^2 \right). \tag{32.3.29}
\]
Now if \( t > 0 \), divide both sides by \( t \). This yields
\[
\langle x^*, y \rangle \leq \frac{1}{2} t \left( ||x|| + t ||y|| \right)^2 - ||x||^2
\]
\[
= \frac{1}{2} \left( 2t ||x|| ||y|| + t^2 ||y||^2 \right)
\]
Letting \( t \rightarrow 0 \),
\[
\langle x^*, y \rangle \leq ||x|| ||y||. \tag{32.3.30}
\]
Next suppose \( t = -s \), where \( s > 0 \) in \(32.3.66\). Then, since when you divide by a negative, you reverse the inequality, for \( s > 0 \)
\[
\langle x^*, y \rangle \geq \frac{1}{2s} \left[ ||x||^2 - ||x - sy||^2 \right] \geq \frac{1}{2s} \left[ ||x - sy||^2 - 2 ||x - sy|| ||sy|| + ||sy||^2 - ||x - sy||^2 \right]. \tag{32.3.31}
\]
Taking a limit as \( s \to 0 \) yields

\[
\langle x^*, y \rangle \geq -||x|| ||y||.
\]  
(32.3.33)

It follows from (32.3.33) and (32.3.30) that

\[
|\langle x^*, y \rangle| \leq ||x|| ||y||
\]
and that, therefore, \( ||x^*|| \leq ||x|| \) and \( ||x^*, x|| \leq ||x||^2 \). Now return to (32.3.32) and let \( y = x \). Then

\[
\langle x^*, x \rangle \geq \frac{1}{2s} \left[ -2 ||x - sx|| ||sy|| + ||sy||^2 \right]
\]
(32.3.32)

Letting \( s \to 1 \),

\[
\langle x^*, x \rangle \geq ||x||^2.
\]

Since it was already shown that \( |\langle x^*, x \rangle| \leq ||x||^2 \), this shows \( \langle x^*, x \rangle = ||x||^2 \) and also \( ||x^*|| \leq ||x|| \). Thus

\[
||x^*|| \geq \frac{\langle x^*, x \rangle}{||x||} = ||x||
\]
so in fact \( x^* \in F(x) \). \( \blacksquare \)

The next result gives conditions under which the subgradient is onto. This means that if \( y^* \in X' \), then there exists \( x \in X \) such that \( y^* \in \partial \phi (x) \).

**Theorem 32.3.18** Suppose \( X \) is a reflexive Banach space and suppose \( \phi : X \to (-\infty, \infty] \) is convex, proper, l.s.c., and for all \( y^* \in X' \), \( x \to \phi (x) - y^* (x) \) is coercive. Then \( \partial \phi \) is onto.

**Proof:** The function \( x \to \phi (x) - y^* (x) \equiv \psi (x) \) is convex, proper, l.s.c., and coercive. Let

\[
\lambda \equiv \inf \{ \phi (x) - y^* (x) : x \in X \}
\]
and let \( \{x_n\} \) be a minimizing sequence satisfying

\[
\lambda = \lim_{n \to \infty} \phi (x_n) - y^* (x_n)
\]

By coercivity,

\[
\lim_{||x|| \to \infty} \phi (x) - y^* (x) = \infty
\]

and so this minimizing sequence is bounded. By the Eberlein Smulian theorem, Theorem 15.5.12, there is a weakly convergent subsequence \( x_{n_k} \to x \). By Theorem 16.2.11, \( \phi \) is also weakly lower semicontinuous. Therefore,

\[
\lambda = \phi (x) - y^* (x) \leq \lim_{k \to \infty} \inf \phi (x_{n_k}) - y^* (x_{n_k}) = \lambda
\]
so there exists \( x \) which minimizes \( x \to \phi(x) - y^*(x) \equiv \psi(x) \). Therefore, \( 0 \in \delta \psi(x) \) because
\[
\psi(y) - \psi(x) \geq 0 = 0(y-x)
\]
by Theorem 32.3.14, \( 0 \in \delta \psi(x) = \delta \phi(x) - y^* \) and this proves the theorem.

**Corollary 32.3.19** Suppose \( X \) is a reflexive Banach space and \( \phi : X \to (-\infty, \infty] \) is convex, proper, and l.s.c. Then for each \( y^* \in X' \) there exist \( x \in X \), \( x^*_1 \in F(x) \), and \( x^*_2 \in \delta \phi(x) \) such that
\[
y^* = x^*_1 + x^*_2.
\]

**Proof:** Apply Theorem 32.3.18 to the convex function \( \frac{1}{2} ||x||^2 + \phi(x) \) and use Theorems 32.3.14 and 32.3.17.

### 32.3.2 Hilbert Space

In this section the subgradients are of a slightly different form and defined on a subset of \( H \), a real Hilbert space. In Hilbert space the duality map is just the Riesz map defined earlier by
\[
Rx(y) \equiv (y,x).
\]

**Definition 32.3.20** \( \text{dom} (\partial \phi) \equiv \text{dom} (\delta \phi) \) and for \( x \in \text{dom} (\partial \phi) \),
\[
\partial \phi(x) \equiv R^{-1} \delta \phi(x).
\]

Thus \( y \in \partial \phi(x) \) if and only if for all \( z \in H \),
\[
Ry(z-x) = (y, z-x) \leq \phi(z) - \phi(x).
\]

Recall the definition of a maximal monotone operator.

**Definition 32.3.21** A mapping \( A : D(A) \subseteq H \to \mathcal{P}(H) \) is called monotone if whenever \( y_i \in Ax_i \),
\[
(y_1 - y_2, x_1 - x_2) \geq 0.
\]

A monotone map is called maximal monotone if whenever \( z \in H \), there exists \( x \in D(A) \) and \( y \in A(x) \) such that \( z = y + x \). Put more simply, \( I + A \) maps \( D(A) \) onto \( H \).

The following lemma states, among other things, that when \( \phi \) is a convex, proper, l.s.c. function defined on a Hilbert space, \( \partial \phi \) is maximal monotone.

**Lemma 32.3.22** If \( \phi \) is a convex, proper, l.s.c. function defined on a Hilbert space, then \( \partial \phi \) is maximal monotone and \( (I + \partial \phi)^{-1} \) is a Lipschitz continuous map from \( H \) to \( \text{dom}(\partial \phi) \) having Lipschitz constant 1.
Chapter 32. Maximal Monotone Operators, Hilbert Space

Proof: Let \( y \in H \). Then \( Ry \in H' \) and by Corollary 32.3.19, there exists \( x \in \text{dom } (\delta \phi) \) such that \( Rx + \delta \phi (x) \ni Ry \). Multiplying by \( R^{-1} \) we see \( y \in x + \partial \phi (x) \). This shows \( I + \partial \phi \) is onto. If \( y_i \in \partial \phi (x_i) \), then \( Ry_i \in \delta \phi (x_i) \) and so by the definition of subgradients,

\[
\begin{align*}
(y_1 - y_2, x_1 - x_2) &= R(y_1 - y_2)(x_1 - x_2) \\
&= Ry_1 (x_1 - x_2) - Ry_2 (x_1 - x_2) \\
&\geq \phi (x_1) - \phi (x_2) - (\phi (x_1) - \phi (x_2)) = 0
\end{align*}
\]

showing \( \partial \phi \) is monotone. Now suppose \( x_i \in (I + \partial \phi)^{-1} (y) \). Then \( y - x_i \in \partial \phi (x_i) \) and by monotonicity of \( \partial \phi \),

\[-|x_1 - x_2|^2 = (y - x_1 - (y - x_2), x_1 - x_2) \geq 0\]

and so \( x_1 = x_2 \). Thus \( (I + \partial \phi)^{-1} \) is well defined. If \( x_i = (I + \partial \phi)^{-1} (y_i) \), then by

the monotonicity of \( \partial \phi \),

\[
(y_1 - x_1 - (y_1 - x_2), x_1 - x_2) \geq 0
\]

and so

\[
|y_1 - y_2| |x_1 - x_2| \geq |x_1 - x_2|^2
\]

which shows

\[
|(I + \partial \phi)^{-1} (y_1) - (I + \partial \phi)^{-1} (y_2)| \leq |y_1 - y_2|.
\]

This proves the lemma.

Here is another proof.

Lemma 32.3.23 Let \( \phi \) be convex, proper and lower semicontinuous on \( X \) a reflexive Banach space having strictly convex norm, then for each \( \alpha > 0 \),

\[
I + \alpha \partial \phi
\]

is onto.

Proof: By separation theorems applied to the epigraph of \( \phi \), and since \( \phi \) is proper, there exists \( w^* \) such that

\[
(w^*, x) + b \leq \alpha \phi (x)
\]

for all \( x \). Pick \( y \in H \). Then consider

\[
\frac{1}{2} |y - x|^2 + \alpha \phi (x)
\]

This functional of \( x \) is bounded below by

\[
\frac{1}{2} |y - x|^2 + (w^*, x) + b
\]
Thus it is clearly coercive. Hence any minimizing sequence has a weakly convergent subsequence. It follows from lower semicontinuity that there exists $x_0$ which minimizes this functional. Hence, if $z \neq x_0$,

$$0 \leq \frac{1}{2} |y - z|^2 + \alpha \phi(z) - \left(\frac{1}{2} |y - x_0|^2 + \alpha \phi(x_0)\right)$$

Then writing $|y - z|^2 = |y - x_0|^2 + |z - x_0|^2 - 2(y - x_0, z - x_0)$,

$$= \frac{1}{2} |y - x_0|^2 + \frac{1}{2} |z - x_0|^2 - (y - x_0, z - x_0) + \alpha \phi(z) - \frac{1}{2} |y - x_0|^2 - \alpha \phi(x_0)$$

$$= \frac{1}{2} |z - x_0|^2 - (y - x_0, z - x_0) + \alpha \phi(z) - \alpha \phi(x_0)$$

Thus, letting $z$ be replaced with $x_0 + t(z - x_0)$ for small positive $t$,

$$t (y - x_0, z - x_0) \leq \frac{t^2}{2} |z - x_0|^2 + \alpha \phi(x_0 + t(z - x_0)) - \alpha \phi(x_0)$$

Using convexity of $\phi$,

$$\leq \frac{t^2}{2} |z - x_0|^2 + t \alpha \phi(z) - t \alpha \phi(x_0)$$

Divide by $t$ and let $t \to 0$ to obtain that

$$(y - x_0, z - x_0) \leq \alpha \phi(z) - \alpha \phi(x_0)$$

and so

$$y - x_0 \in \partial (\alpha \phi(x_0))$$

Thus $y = x_0 + \alpha \partial \phi(x_0)$ because $\partial (\alpha \phi) = \alpha \partial \phi$. ■

There is a really amazing theorem, Moreau’s theorem. It is in [23], [12] and [105]. It involves approximating a convex function with one which is differentiable.

**Theorem 32.3.24** Let $\phi$ be a convex lower semicontinuous proper function defined on $H$. Define

$$\phi_\lambda (x) \equiv \min_{y \in H} \left( \frac{1}{2\lambda} |x - y|^2 + \phi(y) \right)$$

Then the function is well defined, convex, Frechet differentiable, and for all $x \in H$,

$$\lim_{\lambda \to 0} \phi_\lambda(x) = \phi(x),$$

$\phi_\lambda(x)$ increasing as $\lambda$ decreases. In addition,

$$\phi_\lambda(x) = \frac{1}{2\lambda} |x - J_\lambda x|^2 + \phi(J_\lambda(x))$$
where \( J_\lambda x \equiv (I + \lambda \partial \phi)^{-1}(x) \). The Fréchet derivative at \( x \) equals \( A_\lambda x \) where

\[
A_\lambda = \frac{1}{\lambda} - \frac{1}{\lambda} (I + \lambda \partial \phi)^{-1} = \frac{1}{\lambda} - \frac{1}{\lambda} J_\lambda
\]

Also, there is an interesting relation between the domain of \( \phi \) and the domain of \( \partial \phi \)

\[
D(\partial \phi) \subseteq D(\phi) \subseteq \overline{D(\partial \phi)}
\]

**Proof:** First of all, why does the minimum take place? By the convexity, closed epigraph, and assumption that \( \phi \) is proper, separation theorems apply and one can say that there exists \( z^* \) such that for all \( y \in H \),

\[
\frac{1}{2\lambda} |x-y|^2 + \phi(y) \geq \frac{1}{2\lambda} |x-y|^2 + \langle z^*, y \rangle + c \quad (32.3.34)
\]

It follows easily that a minimizing sequence is bounded and so from lower semicontinuity which implies weak lower semicontinuity, there exists \( y_x \) such that

\[
\min_{y \in H} \left( \frac{1}{2\lambda} |x-y|^2 + \phi(y) \right) = \left( \frac{1}{2\lambda} |x-y_x|^2 + \phi(y_x) \right)
\]

Why is \( \phi_\lambda \) convex? For \( \theta \in [0, 1] \),

\[
\phi_\lambda (\theta x + (1-\theta) z) = \frac{1}{2\lambda} |\theta x + (1-\theta) z - y_{\theta x + (1-\theta) z}|^2 + \phi(y_{\theta x + (1-\theta) z})
\]

\[
\leq \frac{1}{2\lambda} |\theta x + (1-\theta) z - (\theta y_x + (1-\theta) y_z)|^2 + \phi(\theta y_x + (1-\theta) y_z)
\]

\[
\leq \frac{\theta}{2\lambda} |x-y_x|^2 + \frac{1-\theta}{2\lambda} |z-y_z|^2 + \theta \phi(y_x) + (1-\theta) \phi(y_z)
\]

\[
= \theta \phi_\lambda(x) + (1-\theta) \phi_\lambda(z)
\]

So is there a formula for \( y_x \)? Since it involves minimization of the functional, it follows as in Lemma 32.3.23 that

\[
\frac{1}{\lambda} (x-y_x) \in \partial \phi(y_x)
\]

Thus

\[
x \in y_x + \lambda \partial \phi(y_x)
\]

and so

\[
y_x = J_\lambda x.
\]

Thus

\[
\phi_\lambda(x) = \frac{1}{2\lambda} |x-J_\lambda x|^2 + \phi(J_\lambda x) = \frac{\lambda}{2} |A_\lambda x|^2 + \phi(J_\lambda x)
\]

Note that \( J_\lambda x \in D(\partial \phi) \) and so it must also be in \( D(\phi) \). Now also

\[
A_\lambda x \equiv \frac{x}{\lambda} - \frac{1}{\lambda} J_\lambda x \in \partial \phi(J_\lambda x)
\]
This is so if and only if
\[ x \in J_\lambda x + \lambda \partial \phi (J_\lambda x) = (I + \lambda \partial \phi) (J_\lambda x) = (I + \lambda \partial \phi) \left((I + \lambda \partial \phi)^{-1} x\right) \]
which is clearly true by definition.

Next consider the claim about differentiability.

\[
\phi_\lambda (y) - \phi_\lambda (x) = \frac{\lambda}{2} |A_\lambda y|^2 + \phi (J_\lambda y) - \left(\frac{\lambda}{2} |A_\lambda x|^2 + \phi (J_\lambda x)\right)
\]
\[
\geq \frac{\lambda}{2} \left(|A_\lambda y|^2 - |A_\lambda x|^2\right) + \phi (J_\lambda y) - \phi (J_\lambda x)
\]
\[
\geq \frac{\lambda}{2} \left(|A_\lambda y|^2 - |A_\lambda x|^2\right) + \lambda (A_\lambda x, y - x) + \lambda (A_\lambda x, A_\lambda x - A_\lambda y)
\]
\[
\geq \frac{\lambda}{2} \left(|A_\lambda y|^2 - |A_\lambda x|^2\right) + \lambda |A_\lambda x|^2 - \frac{\lambda}{2} |A_\lambda x|^2 - \frac{\lambda}{2} |A_\lambda y|^2 + (A_\lambda x, y - x)
\]
\[
= (A_\lambda x, y - x) = (A_\lambda x - A_\lambda y, y - x) + (A_\lambda y, y - x)
\]

(32.3.35)

Then it follows that

\[-(A_\lambda x - A_\lambda y, y - x) \geq \phi_\lambda (x) - \phi_\lambda (y) - (A_\lambda y, x - y)\]

However, \( A_\lambda \) is Lipschitz continuous with constant \( 1/\lambda \) and so

\[
\frac{1}{\lambda} |x - y|^2 \geq \phi_\lambda (x) - \phi_\lambda (y) - (A_\lambda y, x - y)
\]

(32.3.36)

Then switching \( x, y \) in the equation (32.3.36),

\[
\frac{1}{\lambda} |x - y|^2 \geq \phi_\lambda (y) - \phi_\lambda (x) - (A_\lambda x, y - x)
\]

(32.3.37)

But also that term on the end in (32.3.36) equals \( (A_\lambda y, y - x) \geq (A_\lambda x, y - x) \) and so it is also the case that

\[
\frac{1}{\lambda} |x - y|^2 \geq \phi_\lambda (x) - \phi_\lambda (y) + (A_\lambda x, y - x)
\]
\[
= - (\phi_\lambda (y) - \phi_\lambda (x) - (A_\lambda x, y - x))
\]

(32.3.38)

From (32.3.36) and (32.3.37) it follows that

\[
\frac{1}{\lambda} |x - y|^2 \geq |\phi_\lambda (y) - \phi_\lambda (x) - (A_\lambda x, y - x)|
\]
which shows that $D\phi_\lambda(x) = A_\lambda x$. This proves the differentiability part.

Next recall that for any maximal monotone operator $A$, if you have $x \in D(A)$, then

$$\lim_{\lambda \to 0} J_\lambda x = x$$

Recall why this was so. If $x \in D(A)$, then

$$x - J_\lambda x \in \lambda Ax$$

and so, $|x - J_\lambda x| \to 0$ as $\lambda \to 0$. If $x$ is only in $D(A)$, it also works because for $y \in D(A)$

$$|x - J_\lambda x| \leq |x - y| + |y - J_\lambda y| + |J_\lambda y - J_\lambda x| \leq 2|x - y| + |y - J_\lambda y|$$

If $\varepsilon$ is given, simply pick $|y - x| < \varepsilon/2$ and then

$$|x - J_\lambda x| \leq \varepsilon + |y - J_\lambda y|$$

and the last converges to 0. Therefore, $J_\lambda x \to x$ on $D(A)$.

Returning to the proof of the theorem, if $x \in D(\partial \phi)$ then recall that

$$\phi_\lambda(x) = \frac{1}{2\lambda} |x - J_\lambda x|^2 + \phi(J_\lambda x)$$

and so,

$$\liminf_{\lambda \to 0} \phi_\lambda(x) \geq \liminf_{\lambda \to 0} \phi(J_\lambda x) \geq \phi(x) \geq \limsup_{\lambda \to 0} \phi_\lambda(x)$$

which shows the desired result in case $x \in D(\partial \phi)$. Now consider the case where $x \notin D(\partial \phi)$. In this case, there is a positive lower bound $\delta$ to $|x - J_\lambda x|$ because each $J_\lambda x \in D(\partial \phi)$. Then from the definition and what was shown above,

$$\phi_\lambda(x) = \frac{\lambda}{2} |A_\lambda x|^2 + \phi(J_\lambda x) \geq \frac{\lambda}{2} |A_\lambda x|^2 + (z^*, J_\lambda x) + c$$

$$\geq \frac{\lambda}{2} |A_\lambda x|^2 + (z^*, J_\lambda x - x) + (z^*, x) + c$$

$$\geq \frac{1}{2} |A_\lambda x| |x - J_\lambda x| - |z^*||J_\lambda x - x| - |z^*||x| + c$$

$$\geq \frac{1}{2} (|A_\lambda x| - |z^*|) \delta - |z^*||x| + c$$

$$\geq \frac{1}{2} \left( \frac{\delta}{\lambda} - |z^*| \right) \delta - |z^*||x| + c$$

Hence $\phi_\lambda(x) \to \infty$ and since $\phi(x) \geq \phi_\lambda(x)$ by construction, it follows that $\phi(x) = \infty$. The construction of $\phi_\lambda$ also shows that as $\lambda$ decreases, $\phi_\lambda(x)$ increases.

Note that the last part of the argument shows that if $x \notin D(\partial \phi)$, then $x \notin D(\partial \phi)$. Hence this shows that

$$D(\partial \phi) \subseteq D(\phi) \subseteq D(\partial \phi) \quad \blacksquare$$
32.4 A Perturbation Theorem

In this section is a simple perturbation theorem found in [23] and [105].

Recall that for a maximal monotone operator, $B\lambda$, the Yosida approximation, $B\lambda x$, is defined by

$$B\lambda x \equiv \frac{1}{\lambda} (x - J\lambda x), \quad J\lambda x \equiv (I + \lambda B)^{-1} x.$$  

This follows from Theorem 32.1.7 on Page 1228.

**Theorem 32.4.1** Let $A$ and $B$ be maximal monotone operators and let $x\\lambda$ be the solution to

$$y \in x\\lambda + B\lambda x\\lambda + Ax.$$  

Then $y \in x + Bx + Ax$ for some $x \in D(A) \cap D(B)$ if $B\lambda x$ is bounded independent of $\lambda$.

The following is the perturbation theorem of this section. See [23] and [105].

**Theorem 32.4.2** Let $H$ be a real Hilbert space and let $\Phi$ be non-negative, convex, proper, and lower semicontinuous. Suppose also that $A$ is a maximal monotone operator and there exists $\xi \in D(A) \cap D(\Phi)$.  

(32.4.39)

Suppose also that for $J\lambda x \equiv (I + \lambda A)^{-1} x$,

$$\Phi(J\lambda x) \leq \Phi(x) + C \lambda$$  

(32.4.40)

Then $A + \partial \Phi$ is maximal monotone.

**Proof:** Letting $A\lambda$ be the Yosida approximation of $A$,

$$A\lambda x = \frac{1}{\lambda} (x - J\lambda x),$$

and letting $y \in H$, it follows from the Hilbert space version of Proposition 32.1.6 there exists $x\lambda \in H$ such that

$$y \in x\lambda + A\lambda x\lambda + \partial \Phi(x\lambda).$$

Consequently,  

$$y - x\lambda - A\lambda x\lambda \in \partial \Phi(x\lambda)$$  

(32.4.41)

and so  

$$(y - x\lambda - A\lambda x\lambda, J\lambda x\lambda - x\lambda) \leq \Phi(J\lambda x\lambda) - \Phi(x\lambda) \leq C \lambda$$  

(32.4.42)

which implies

$$-(y - x\lambda - A\lambda x\lambda, A\lambda x\lambda) = |A\lambda x\lambda|^2 - |y - x\lambda||A\lambda x\lambda| \leq C.$$  

(32.4.43)
By\textsuperscript{32.4.41} and monotonicity of $A_{\lambda}$,
\[
\Phi (\xi) - \Phi (x_{\lambda}) \geq \left( y - x_{\lambda} - A_{\lambda}x_{\lambda}, \xi - x_{\lambda} \right) \\
= (y - x_{\lambda}, \xi - x_{\lambda}) - (A_{\lambda}x_{\lambda}, \xi - x_{\lambda}) \\
\geq (y - \xi, \xi - x_{\lambda}) + |\xi - x_{\lambda}|^2 - (A_{\lambda}\xi, \xi - x_{\lambda}) \\
= |\xi - x_{\lambda}|^2 + (y - \xi - A_{\lambda}\xi, \xi - x_{\lambda}) \\
\geq |\xi - x_{\lambda}|^2 - C_{\xi y} |\xi - x_{\lambda}|
\]
where $C_{\xi y}$ depends on $\xi$ and $y$ but is independent of $\lambda$ because of the assumption that $\xi \in D (A) \cap D (\Phi)$ and Lemma\textsuperscript{32.4.42} which gives a bound on $|A_{\lambda}\xi|$ in terms of $|y|$ for $y \in Ax$. Therefore, there exist constants, $C_1$ and $C_2$, depending on $\xi$ and $y$ but not on $\lambda$ such that
\[
\Phi (\xi) \geq \Phi (x_{\lambda}) + |x_{\lambda}|^2 - C_1 |x_{\lambda}| - C_2.
\]
Since $\Phi \geq 0$, this shows that $|x_{\lambda}|$ is bounded independent of $\lambda$.
\[
2 \left( \Phi (\xi) + C_2 + \frac{C_1^2}{2} \right) \geq \Phi (x_{\lambda}) + |x_{\lambda}|^2.
\]
This shows $|x_{\lambda}|$ is bounded independent of $\lambda$. Therefore, by\textsuperscript{32.4.41} $|A_{\lambda}x_{\lambda}|$ is bounded independent of $\lambda$. By Theorem\textsuperscript{32.4.43} this shows there exists $x \in D (\partial \Phi) \cap D (A)$ such that
\[
y \in Ax + \partial \Phi (x) + x
\]
and so $A + \partial \Phi$ is maximal monotone since $y \in H$ was arbitrary. \hfill \qed

### 32.5 An Evolution Inclusion

In this section is a theorem on existence and uniqueness for the initial value problem
\[
x' + \partial \phi (x) \ni f, \quad x (0) = x_0.
\]
Suppose $\phi$ is a mapping from $H$ to $[0, \infty]$ which satisfy the following axioms.
\[
\phi \text{ is convex and lower semicontinuous, and proper,} \quad (32.5.44)
\]
\textbf{Lemma 32.5.1} For $x \in L^2 (0, T; H), t \rightarrow \phi (x)$ is measurable.
32.5. AN EVOLUTION INCLUSION

**Proof:** This follows because $\phi$ is Borel measurable and so $\phi \circ x$ is also measurable.

Now define the following function $\Phi$, on the Hilbert space, $L^2(0, T; H)$.

$$
\Phi(x) \equiv \begin{cases} 
\int_0^T \phi(x(t)) \, dt & \text{if } x(t) \in D \text{ for a.e. } t \\
+\infty & \text{otherwise}
\end{cases}
$$

(32.5.45)

**Lemma 32.5.2** $\Phi$ is convex, nonnegative, and lower semicontinuous on $L^2(0, T; H)$.

**Proof:** Since $\phi$ is nonnegative and convex, it follows that $\Phi$ is also nonnegative and convex. It remains to verify lower semicontinuity. Suppose, $x_n \to x$ in $L^2(0, T; H)$ and let

$$
\lambda = \lim_{n \to \infty} \inf \Phi(x_n).
$$

Is $\lambda \geq \Phi(x)$? Then it suffices to assume $\lambda < \infty$. Suppose not. Then $\lambda < \Phi(x)$. Taking a subsequence, we can have $\lambda = \lim_{n \to \infty} \Phi(x_n)$ and we can take a further subsequence for which convergence of $x_n$ to $x$ is pointwise a.e. Then

$$
\lambda < \Phi(x) \leq \int_0^T \phi(x(t)) \, dt \leq \int_0^T \lim_{n \to \infty} \phi(x_n(t)) \, dt \\
\leq \lim_{n \to \infty} \inf \int_0^T \phi(x_n(t)) \, dt = \lim_{n \to \infty} \Phi(x_n) = \lambda
$$

which is a contradiction. ■

Define

$$
D(L) \equiv \{ x \in L^2(0, T; H) : \text{such that} \} \\
x(t) = x_0 + \int_0^t x'(s) \, ds \text{ where } x' \in L^2(0, T; H) \}
$$

(32.5.46)

and for $x \in D(L)$,

$$
Lx \equiv x'.
$$

Then $L$ is maximal monotone. To see this, consider the equation

$$
\lambda x' + x = z, \ x(0) = x_0
$$

It clearly has a solution so $\lambda L + I$ is onto. In fact, the solution is

$$
x = e^{-t} x_0 + \frac{1}{\lambda} e^{-t} \int_0^t e^{s} z(s) \, ds
$$

Also,

$$
(Lx - Ly, x - y)_{L^2(0, T; H)} = \int_0^T (x' - y', x - y)_H \, dt
$$
\[ \begin{align*}
&= \int_0^T \left( x'(t) - y'(t) , \int_0^t x'(s) - y'(s) \, ds \right) \, dt \\
&= \frac{1}{2} \int_0^T \frac{d}{dt} \left( \left| \int_0^t x'(s) - y'(s) \, ds \right|^2 \right) \, dt \\
&= \left| \int_0^T x'(s) - y'(s) \, ds \right|^2_H \geq 0
\end{align*} \]

Thus we have the following lemma.

**Lemma 32.5.3** \( L \) is maximal monotone and if \( z \in L^2(0,T;H) \), then \( J_\lambda z \) is given by

\[ J_\lambda [z](t) \equiv (I + \lambda L)^{-1}([z])(t) = e^{-\lambda t} x_0 + \frac{1}{\lambda} \int_0^t e^{\lambda s} z(s) \, ds. \] (32.5.47)

The main theorem is the following.

**Theorem 32.5.4** Let \( x_0 \in D \equiv D(\phi) \). Then \( L + \partial \Phi \) is maximal monotone so there exists a unique solution to

\[ Lx + x + \partial \Phi(x) \ni f \] (32.5.48)

for every \( f \in L^2(0,T;H) \). Thus there exists \( x \in L^2(0,T;H) \) such that \( x' \in L^2(0,T;H), x(0) = x_0 \in D(\phi) \), and

\[ x' + x + \partial \Phi(x) \ni f, \ x(0) = x_0 \]

**Proof:** This is from Theorem 32.4.2. Since \( x_0 \in D \), it follows that \( \phi(x_0) < \infty \).

Let \( z \in D(\Phi) \), the effective domain of \( \Phi \). Then \( \int_0^T \phi(z(t)) \, dt < \infty \), so by convexity of \( \phi \) and

\[ \phi(J_\lambda z(t)) \leq e^{-\lambda t} \phi(x_0) + \frac{1}{\lambda} \int_0^t e^{\lambda s} \phi(z(s)) \, ds. \] (32.5.49)

Then

\[ \Phi(J_\lambda z) = \int_0^T \phi(J_\lambda z(t)) \, dt \leq \phi(x_0) \lambda + \int_0^T \frac{1}{\lambda} \int_0^t e^{-(t-s)/\lambda} \phi(z(s)) \, ds \, dt \\
\leq \lambda \phi(x_0) + \frac{1}{\lambda} \int_0^T \phi(z(s)) \int_s^T e^{-(t-s)/\lambda} \, dt \, ds \\
\leq \lambda \phi(x_0) + \left( \int_0^T \phi(z(s)) \, ds \right) \frac{1}{\lambda} \int_0^\infty e^{-t/\lambda} \, dt \\
= \phi(x_0) \lambda + \int_0^T \phi(z(s)) \, ds \\
= \phi(x_0) \lambda + \Phi(z) \]
The conditions of Theorem \[32.4.2\] are satisfied. This proves \( L + \partial \Phi \) is maximal monotonous on \( L^2(0, T; H) \) and consequently there exists a unique solution to the differential inclusion of the theorem.

Then the main result is the following.

**Theorem 32.5.5** Let \( f \in L^2(0, T; H) \) and \( x_0 \in D \). Let \( \phi \) be as described above, a lower semicontinuous convex proper function defined on \( H \). Then there exists a unique solution \( x \in L^2(0, T; H) \), \( x' \in L^2(0, T; H) \), to

\[
x' + \partial \Phi (x) \ni f \text{ in } L^2(0, T; H), \ x(0) = x_0
\]

This satisfies the pointwise condition

\[
x'(t) + \partial \phi (x(t)) \ni f(t) \text{ for a.e. } t, \ x(0) = x_0
\]

**Proof:** From Theorem \[32.5.4\], there exists a unique solution to

\[
x' + \partial \Phi (x) + x \ni f + v \text{ in } L^2(0, T; H), \ x(0) = x_0
\]

whenever \( v \in L^2(0, T; H) \). Then a simple argument based on fundamental theorem of calculus implies that for a.e. \( t \),

\[
x'(t) + \partial \phi (x(t)) + x(t) \ni f(t) + v(t)
\]

Then for given \( v, u \) one can act on \( x_v(t) - x_u(t) \) and integrate. This yields

\[
\frac{1}{2} |x_v(t) - x_u(t)|_H^2 + \int_0^t |x_v - x_u|^2 ds \leq \int_0^t |v(s) - u(s)|_H^2 ds
\]

It follows that a sufficiently high power of the mapping \( u \rightarrow x_u \) is a contraction map on \( L^2(0, T; H) \) and so there exists a unique fixed point \( v \) in \( L^2(0, T; H) \). Thus \( x_v = v \) and so

\[
v' + \partial \Phi (v) \ni f \text{ in } L^2(0, T; H), \ v(0) = x_0
\]

---

### 32.6 A More Complicated Perturbation Theorem

In this section is a simple perturbation theorem which is a small generalization of one found in [23] and [105].

Recall that for \( B \) a maximal monotone operator, \( B_\lambda \), the Yosida approximation, is defined by

\[
B_\lambda x \equiv \frac{1}{\lambda} (x - J_\lambda x), \ J_\lambda x \equiv (I + \lambda B)^{-1} x.
\]

This follows from Theorem \[32.1.7\] on Page 125.

**Theorem 32.6.1** Let \( A \) and \( B \) be maximal monotone operators and let \( x_\lambda \) be the solution to

\[
y \in x_\lambda + B_\lambda x_\lambda + Ax_\lambda.
\]

Then \( y \in x + Bx + Ax \) for some \( x \in D(A) \cap D(B) \) if \( B_\lambda x_\lambda \) is bounded independent of \( \lambda \).
The following is the perturbation theorem of this section. It generalizes a well-known result in [23] and [105].

**Theorem 32.6.2** Let $H$ be a real Hilbert space and let $\Phi$ be non negative, convex, proper, and lower semicontinuous. Suppose also that $A$ is a maximal monotone operator and there exists $\xi \in D(A) \cap D(\Phi)$. (32.6.50)

Suppose also that for $J_\lambda x \equiv (I + \lambda A)^{-1} x$,

$$\Phi(J_\lambda x) \leq \Phi(x) + C(x) \lambda$$

where for some constants, $K_1, K_2$,

$$K_2 + K_1 (\Phi(x) + |x|^2) \geq C(x).$$

Then $A + \partial \Phi$ is maximal monotone.

**Proof:** Letting $A_\lambda$ be the Yosida approximation of $A$,

$$A_\lambda x = \frac{1}{\lambda} (x - J_\lambda x),$$

and letting $y \in H$, it follows from the Hilbert space version of Proposition 32.1.6 there exists $x_\lambda \in H$ such that

$$y \in x_\lambda + A_\lambda x_\lambda + \partial \Phi(x_\lambda).$$

Consequently,

$$y - x_\lambda - A_\lambda x_\lambda \in \partial \Phi(x_\lambda)$$

and so

$$(y - x_\lambda - A_\lambda x_\lambda, J_\lambda x_\lambda - x_\lambda) \leq \Phi(J_\lambda x_\lambda) - \Phi(x_\lambda) \leq C(x_\lambda) \lambda$$

which implies

$$-(y - x_\lambda - A_\lambda x_\lambda, A_\lambda x_\lambda) \leq C(x_\lambda).$$

I claim $\{C(x_\lambda)\}$ and $\{|x_\lambda|\}$ are bounded independent of $\lambda$.

By boundedness and monotonicity of $A_\lambda$,

$$\Phi(\xi) - \Phi(x_\lambda) \geq (y - x_\lambda - A_\lambda x_\lambda, \xi - x_\lambda)$$

$$\geq (y - x_\lambda, \xi - x_\lambda) - (A_\lambda x_\lambda, \xi - x_\lambda)$$

$$\geq (y - x_\lambda, \xi - x_\lambda) - (A_\lambda \xi, \xi - x_\lambda)$$

$$\geq (y - \xi, \xi - x_\lambda) + |\xi - x_\lambda|^2 - (A_\lambda \xi, \xi - x_\lambda)$$

$$\geq |\xi - x_\lambda|^2 - C_{\xi y} |\xi - x_\lambda|$$
where $C_{\xi y}$ depends on $\xi$ and $y$ but is independent of $\lambda$ because of the assumption that $\xi \in D(A) \cap D(\Phi)$ and Lemma 32.1.3 which gives a bound on $|A_\lambda \xi|$ in terms of $|y|$ for $y \in Ax$. Therefore, there exist constants, $C_1$ and $C_2$, depending on $\xi$ and $y$ but not on $\lambda$ such that

$$\Phi(\xi) \geq \Phi(x_\lambda) + |x_\lambda|^2 - C_1 |x_\lambda| - C_2.$$ 

Since $\Phi \geq 0$,

$$2 \left( \Phi(\xi) + C_2 + \frac{C_1^2}{2} \right) \geq \Phi(x_\lambda) + |x_\lambda|^2.$$

This shows $|x_\lambda|$ is bounded independent of $\lambda$. Therefore, by 32.6.52

$$K_2 + 2K_1 \left( \Phi(\xi) + C_2 + \frac{C_1^2}{2} \right) \geq K_2 + K_1 \left( \Phi(x_\lambda) + |x_\lambda|^2 \right) \geq C(x_\lambda),$$

showing that both $|x_\lambda|$ and $C(x_\lambda)$ are bounded independent of $\lambda$. Therefore, from 32.6.5, it follows $A_\lambda x_\lambda$ is bounded independent of $\lambda$. By Theorem 32.6.1 this shows there exists $x \in D(\partial \Phi) \cap D(A)$ such that

$$y \in Ax + \partial \Phi(x) + x$$

and so $A + \partial \Phi$ is maximal monotone since $y \in H$ was arbitrary. This proves the theorem.

### 32.7 An Evolution Inclusion

In this section is a theorem on existence and uniqueness for the initial value problem

$$x' + \partial_2 \phi(t, x) \ni f, \quad x(0) = x_0.$$

Suppose $\{\phi(t, \cdot)\}_{t \in [0, T]}$ is a family of functions mapping $H$ to $[0, \infty]$ which satisfy the following axioms.

$$\phi(t, \cdot) \text{ is convex and lower semicontinuous}, \quad (32.7.56)$$

$$D(\phi(t, \cdot)) = D, \quad \text{independent of } t \in [0, T], \quad (32.7.57)$$

There exists a constant, $K$, such that for all $x \in D$,

$$|\phi(t, x) - \phi(s, x)| \leq K \left( \phi(r, x) + |x|^2 + 1 \right) |t - s| \quad (32.7.58)$$

for all $r \in [0, T]$.

**Lemma 32.7.1** Under the conditions, 32.7.56 - 32.7.58, $\phi : H \times [0, T] \to [0, \infty]$ is lower semicontinuous.
Proof: Let \((x_n, t_n) \to (x, t)\) and let \(\lambda \equiv \liminf_{n \to \infty} \phi(t_n, x_n)\). Is \(\phi(t, x) \leq \lambda\)?

It suffices to assume \(\lambda < \infty\) and by taking a subsequence, \(x_n \in D\) for all \(n\) and 

\[\phi(t_n, x_n) \to \lambda.\]

Then

\[
\liminf_{n \to \infty} \phi(t_n, x_n) = \liminf_{n \to \infty} [\phi(t_n, x_n) - \phi(t, x_n) + \phi(t, x_n)].
\] (32.7.59)

Now

\[
\limsup_{n \to \infty} |\phi(t_n, x_n) - \phi(t, x_n)| \leq \limsup_{n \to \infty} K \left(\phi(t_n, x_n) + |x_n|^2 + 1\right) |t_n - t| = 0.
\]

Therefore, from 32.7.59

\[
\lambda = \liminf_{n \to \infty} \phi(t_n, x_n) = \liminf_{n \to \infty} \phi(t, x_n) \geq \phi(t, x)
\]

because of the assumption that \(\phi(t, \cdot)\) is lower semicontinuous. This proves the lemma.

In all that follows \([x]\) is an element of \(L^2(0, T; H)\). Thus \([x]\) is the equivalence class of measurable square integrable functions which equal \(x\) a.e. This seems a little fussy but since the existence results are based on surjectivity theorems and the Hilbert space they apply to is \(L^2(0, T; H)\), it seems best to emphasize the equivalence classes of functions by using this notation, at least while proving theorems on existence and uniqueness.

Corollary 32.7.2 For \([x] \in L^2(0, T; H), t \to \phi(t, x(t))\) is measurable.

Proof: This follows because, due to Lemma 32.7.1, \(\phi\) is Borel measurable and so \(\phi \circ x\) is also measurable.

Now define the following function, \(\Phi\), on the Hilbert space, \(L^2(0, T; H)\).

\[
\Phi([x]) = \begin{cases} 
\int_0^T \phi(t, x(t)) \, dt & \text{if } x(t) \in D \text{ for all } t \text{ for some } x(\cdot) \in [x] \\
+\infty & \text{otherwise} 
\end{cases}
\] (32.7.60)

Note that since the functions \(\phi(t, \cdot)\) are proper, the top condition is equivalent to the condition

\[\int_0^T \phi(t, x(t)) \, dt \text{ if } x(t) \in D \text{ a.e. for all } x(\cdot) \in [x].\]

Lemma 32.7.3 \(\Phi\) is convex, nonnegative, and lower semicontinuous on \(L^2(0, T; H)\).
32.7. AN EVOLUTION INCLUSION

Proof: Since each \( \phi(t, \cdot) \) is nonnegative and convex, it follows that \( \Phi \) is also nonnegative and convex. It remains to verify lower semicontinuity. Suppose, \([x_n] \to [x]\) in \( L^2(0, T; H) \) and let

\[
\lambda = \lim_{n \to \infty} \inf \Phi([x_n]).
\]

Is \( \lambda \geq \Phi([x]) \)? It suffices to assume \( \lambda < \infty \), \( x_n(t) \in D \) for all \( t \), and \( x_n(t) \to x(t) \) a.e. say for \( t \notin N \) where \( N \) has measure zero. Let

\[
\bar{x}(t) = \begin{cases} x(t) & \text{if } t \notin N \\ x_1(t) & \text{if } t \in N \end{cases}
\]

Then \([\bar{x}] = [x]\) and \( \bar{x}(t) \in D \) for all \( t \). Then by pointwise convergence and Fatou’s lemma,

\[
\Phi([x]) = \Phi([\bar{x}]) = \int_0^T \phi(t, \bar{x}(t)) \, dt \leq \int_0^T \lim_{n \to \infty} \phi(t, x_n(t)) \, dt
\]

\[
\leq \lim_{n \to \infty} \inf \int_0^T \phi(t, x_n(t)) \, dt = \lim_{n \to \infty} \Phi([x_n]) \equiv \lambda.
\]

This proves the lemma.

Define

\[
D(L) \equiv \{ [x] \in L^2(0, T; H) : \text{for some } x \in [x] \text{ such that}
\]

\[
x(t) = x_0 + \int_0^t x'(s) \, ds \text{ where } [x'] \in L^2(0, T; H) \}
\quad (32.7.61)
\]

and for \([x] \in D(L)\),

\[
L[x] \equiv [x']
\]

The following lemma is easily obtained.

Lemma 32.7.4 \( L \) is maximal monotone and if \([z] \in L^2(0, T; H)\), then the equivalence class, \([J_\lambda [z]]\) is determined by the function,

\[
J_\lambda [z](t) \equiv (I + \lambda L)^{-1} ([z]) (t) = e^{\frac{-\lambda t}{\epsilon}} x_0 + \frac{1}{\lambda} e^{\frac{-\lambda t}{\epsilon}} \int_0^t e^{\frac{\lambda s}{\epsilon}} z(s) \, ds.
\quad (32.7.62)
\]

The main theorem is the following.

Theorem 32.7.5 Let \( x_0 \in D \). Then \( L + \partial \Phi \) is maximal monotone so there exists a unique solution to

\[
L[x] + [x] + \partial \Phi([x]) \ni [f]
\quad (32.7.63)
\]

for every \([f] \in L^2(0, T; H)\).
Proof: This is from Theorem 32.4. Since \( x_0 \in D \), it follows from 32.4.5 that \( \phi(t, x_0) \) is bounded.

Let \( [z] \in D(\Phi) \), the effective domain of \( \Phi \). Then there exists \( z \in [z] \) such that \( z(t) \in D \) for all \( t \), and \( \int_0^T \phi(t, z(t)) \, dt < \infty \), so by convexity of \( \phi(t, \cdot) \) and 32.4.6,

\[
\phi(t, J_\lambda [z](t)) \leq e^{\frac{\lambda}{2} t} \phi(t, x_0) + \frac{1}{\lambda} e^{\frac{\lambda}{2} t} \int_0^t e^{\frac{\lambda}{2} s} \phi(t, z(s)) \, ds. \tag{32.7.64}
\]

Now the first term in 32.7.63 is bounded so consider the second. The integral in this term is of the form

\[
\int_0^t e^{\frac{\lambda}{2} s} \phi(s, z(s)) \, ds + \int_0^t e^{\frac{\lambda}{2} t} (\phi(t, z(s)) - \phi(s, z(s))) \, ds. \tag{32.7.65}
\]

Since \([z] \in D(\Phi)\), \( \phi(s, z(s)) \) is finite for all \( s \) and also the first integral in 32.7.67 is bounded and so the first term in 32.7.66 is dominated by

\[
C \lambda \int_0^t K (1 + \phi(s, z(s)) + |z(s)|^2) |t - s| \, ds < \infty.
\]

This shows \( \phi(t, J_\lambda [z](t)) \) is finite for all \( t \) and so \( \Phi([J_\lambda [z]]) \) is given by the top line of 32.7.66. Therefore, by convexity of \( \phi(t, \cdot) \) and Jensen’s inequality,

\[
\Phi([J_\lambda [z]]) = \int_0^T \phi(t, e^{\frac{\lambda}{2} t} x_0 + \frac{1}{\lambda} e^{\frac{\lambda}{2} t} \int_0^t e^{\frac{\lambda}{2} s} \phi(t, z(s)) \, ds) \, dt
\leq \int_0^T \left( e^{\frac{\lambda}{2} t} \phi(t, x_0) + \frac{1}{\lambda} e^{\frac{\lambda}{2} t} \int_0^t e^{\frac{\lambda}{2} s} \phi(t, z(s)) \, ds \right) \, dt
= \int_0^T e^{\frac{\lambda}{2} t} \phi(t, x_0) \, dt + \int_0^T \frac{1}{\lambda} e^{\frac{\lambda}{2} t} \int_0^t e^{\frac{\lambda}{2} s} \phi(s, z(s)) \, ds dt
+ \int_0^T \frac{1}{\lambda} e^{\frac{\lambda}{2} t} \int_0^t e^{\frac{\lambda}{2} s} (\phi(t, z(s)) - \phi(s, z(s))) \, ds dt. \tag{32.7.66}
\]

By 32.7.65 the last term in 32.7.66 is dominated by

\[
\int_0^T \int_0^t e^{-\frac{\lambda}{2} (t-s)} K (1 + \phi(s, z(s)) + |z(s)|^2) \, |t - s| \, ds dt =
\int_0^T \int_s^T e^{-\frac{\lambda}{2} (t-s)} \frac{t - s}{\lambda} dt K (1 + \phi(s, z(s)) + |z(s)|^2) \, ds
\leq C \lambda + C \lambda \left( \Phi([z]) + ||z||^2 \right). \tag{32.7.67}
\]

for some constant, \( C \). From 32.7.65, \( \phi(t, x_0) \) is bounded and so the first term in 32.7.66 is dominated by an expression of the form \( C \lambda \). Now consider the middle term of 32.7.66. Since \( \phi \) is nonnegative,

\[
\int_0^T \frac{1}{\lambda} e^{\frac{\lambda}{2} t} \int_0^t e^{\frac{\lambda}{2} s} \phi(s, z(s)) \, ds dt = \int_0^T \int_s^T \frac{1}{\lambda} e^{-\frac{\lambda}{2} (t-s)} dt \phi(s, z(s)) \, ds
\]
32.7. AN EVOLUTION INCLUSION

\[ \leq \int_0^T \int_0^\infty e^{-u}du \phi(s, z(s)) \, ds = \Phi([z]). \]  
(32.7.68)

It follows
\[ \Phi([J_\lambda [z]]) \leq \Phi([z]) + C\lambda + C\lambda \left( \Phi([z]) + |[z]|^2 \right). \]

The conditions of Theorem 32.7.69 are satisfied with \( K_1 = K_2 = C \). This proves \( L + \partial \Phi \) is maximal monotone on \( L^2(0, T; H) \) and consequently there exists a unique solution to the differential inclusion of the theorem.

Of course it is desirable to be able to say that \( [y] \in \partial \Phi([x]) \) if and only if \( y(t) \in \partial \Phi(t, x(t)) \) for some \( t \in [x] \). To obtain this, here are two more assumptions. For all \( x \in H \),
\[ t \to J_1(t) x \text{ is measurable}, \]
(32.7.69)
where \( J_1(t) x \) is the solution, \( y \), to \( y(t) + \partial \Phi(t, y(t)) \ni x \), and there exists \( \xi \in L^2(0, T; H) \) such that
\[ [J_1(\cdot) \xi] \in L^2(0, T; H). \]
(32.7.70)

**Lemma 32.7.6** If \( \overline{\text{32.7.69}} \) and \( \overline{\text{32.7.70}} \) hold, and if \( [y] \in L^2(0, T; H) \), then \( [y] \in \partial \Phi([x]) \) if and only if there exists \( x \in [x] \) such that \( \partial \Phi(t, x(t)) \neq \emptyset \) for all \( t \) and \( y(t) \in \partial \Phi(t, x(t)) \) a.e.

**Proof:** First suppose \( y(t) \in \partial \Phi(t, x(t)) \) a.e. and \( \partial \Phi(t, x(t)) \neq \emptyset \) for all \( t \) where \( x \in [x] \). Then for all \([w] \in L^2(0, T; H)\),
\[ ([y],[w])_{L^2(0,T;H)} \equiv \int_0^T (y(t), w(t))_H \, dt \]
\[ \leq \int_0^T \phi(t, x(t) + w(t)) \, dt - \int_0^T \phi(t, x(t)) \, dt \leq \Phi([x] + [w]) - \Phi([x]). \]

To prove the converse, define \( A : D(\partial \Phi) \to \mathcal{P}(L^2(0, T; H)) \) as follows.
\[ [y] \in A[x] \text{ if and only if for some } x \in [x], \]
\[ \partial \Phi(t, x(t)) \neq \emptyset \text{ for all } t \text{ and } y(t) \in \partial \Phi(t, x(t)) \text{ a.e. } t. \]

It follows \( A \) is monotone. I will show \( A \) is maximal monotone. From the first part of the proof, the graph of \( A \) is contained in the graph of \( \partial \Phi \). Since \( A \) is maximal, this will imply \( A = \partial \Phi \) and prove the lemma.

It remains to show \( A \) is maximal monotone. By \( \overline{\text{32.7.69}} \) for each \( x \in H \), \( J_1(t) x \) is measurable. Now from \( \overline{\text{32.7.70}} \) and using the fact that \( J_1(t) \) is a contraction,
\[ |J_1(t) x - J_1(t) \xi(t)| \leq |x - \xi(t)| \]
and so \([J_1(\cdot)x] \in L^2(0, T; H) \). Now if
\[ s(t) = \sum_{i=1}^n X_{E_i} (t) x_i \]
is a simple function,

$$J_1(t) s(t) = \sum_{i=1}^{n} \mathcal{K}_{E_i}(t) J_1(t) x,$$

and $[J_1(\cdot) s]$ is in $L^2(0,T;H)$. If $[f] \in L^2(0,T;H)$ is arbitrary, take a sequence of simple functions, $s_n$ converging to $f$ pointwise and $[s_n] \rightarrow [f]$ in $L^2(0,T;H)$. Then

$$|J_1(t) s_n(t) - J_1(t) f(t)| \leq |s_n(t) - f(t)|$$

and it follows $J_1(t) s_n(t)$ converges pointwise to $J_1(t) f(t)$ showing that $t \rightarrow J_1(t) f(t)$ is measurable. Now the equivalence class of functions equal to this one a.e. is in $L^2(0,T;H)$ by Fatou’s lemma and the assumption that the simple functions, $s_n$ converge in $L^2(0,T;H)$. This shows $A$ is maximal and proves the lemma.

Conditions (32.7.69) and (32.7.70) are just what is needed to obtain the conclusion of Lemma 32.7.7 but it may not be clear how to verify these conditions easily. The following lemma gives sufficient conditions which are easy to verify which imply (32.7.69) and (32.7.70).

**Lemma 32.7.7** Suppose there exists $[\xi] \in L^2(0,T;H)$ such that

$$J_1(t) \xi(t), \phi(t, J_1(t) \xi(t))$$

are bounded independent of $t \in [0,T]$ and $t \rightarrow J_1(t) \xi(t)$ is measurable. Then the conclusion of Lemma 32.7.64 holds.

**Proof:** Let $y(t) = J_1(t) \xi(t)$. Thus

$$y(t) + \partial_2 \phi(t, y(t)) \ni \xi(t).$$

Now suppose $x \in H$ and let

$$x(s) + z(s) = x \tag{32.7.71}$$

where $z(s) \in \partial_2 \phi(s, x(s))$, so $x(s) = J_1(s) x$. Take the inner product of both sides with $x(s) - y(s)$ to obtain

$$(x(s), x(s) - y(s))_H + (z(s), x(s) - y(s))_H = (x, x(s) - y(s))_H$$

and therefore,

$$\frac{1}{2} |x(s)|^2_H - \frac{1}{2} |y(s)|^2_H \leq \phi(s, y(s)) - \phi(s, x(s))$$

$$+ |x|_H |x(s)|_H + |x|_H |y(s)|_H \leq \frac{1}{4} |x(s)|^2_H + c|x|^2_H + \frac{1}{2} |y(s)|^2_H$$

$$+ \phi(s, y(s)) - \phi(s, x(s)).$$

Consequently,

$$\phi(s, x(s)) + \frac{1}{4} |x(s)|^2_H \leq |y(s)|^2_H + c|x|^2_H + \phi(s, y(s)) < C \tag{32.7.72}$$
a constant depending on $x$. Replacing $s$ with $t$ in (32.7.71) and subtracting yields

$$x(t) - x(s) + z(t) - z(s) = 0.$$  

Now taking the inner product of this with $x(t) - x(s)$ it follows from (32.7.58),

$$|x(s) - x(t)|_H^2 = (z(s) - z(t), x(t) - x(s))_H,$$

$$\leq \phi(s, x(t)) - \phi(s, x(s)) + \phi(t, x(s)) - \phi(t, x(t))$$

$$\leq \left(K \left(\phi(t, x(t)) + |x(t)|_H^2 + 1\right) + K \left(\phi(s, x(s)) + |x(s)|_H^2 + 1\right)\right)|t - s|$$

which shows by (32.7.72) that $x(t)$ is Lipschitz continuous and is therefore measurable which verifies (32.7.69). The assumptions of the lemma include (32.7.70). It follows the conclusion of Lemma 32.7.6 holds.

**Remark 32.7.8** Note that if $\phi(t, \cdot)$ has a minimum at $\xi(t)$ and if $t \to \xi(t)$ and $t \to \phi(t, \xi(t))$ are bounded and measurable, then

$$\xi(t) + 0 = \xi(t)$$

and $0 \in \partial_2 \phi(t, \xi(t))$. Therefore, in this case $J_1(t) \xi(t) = \xi(t)$ and so the hypotheses of Lemma 32.7.7 hold.

**Corollary 32.7.9** Assume (32.7.56)-(32.7.58) and (32.7.69). Let $x_0 \in D$ and let $[f] \in L^2(0, T; H)$. Then there exists a unique function, $x$, satisfying

$$[x] \text{ and } [x'] \text{ are in } L^2(0, T; H)$$

which is a solution to

$$x' + \partial_2 \phi(t, x) \ni f \text{ a.e., } x(0) = x_0, x(t) = x_0 + \int_0^t x'(s) \, ds. \quad (32.7.73)$$

**Proof:** Let $[v] \in L^2(0, T; H)$ and let $[x]$ be the unique solution to

$$L[x] + [x] + \partial \Phi([x]) \ni [f] + [v]. \quad (32.7.74)$$

Letting $[x_i]$ be the solution corresponding to (32.7.74) in which $v$ is replaced with $v_i$, and $x_i \in [x_i]$ is such that

$$x_i(t) = x_0 + \int_0^t x_i'(s) \, ds, \quad i = 1, 2,$$

from Lemma 32.7.8 and 32.7.10 that for each $t \in [0, T]$,

$$\frac{1}{2} |x_1(t) - x_2(t)|_H^2 + \frac{1}{2} \int_0^t |x_1 - x_2|^2_H \, ds \leq \frac{1}{2} \int_0^t |v_1(s) - v_2(s)|_H^2 \, ds$$
and so
\[ |x_1(t) - x_2(t)|^2_H \leq \int_0^t |v_1(s) - v_2(s)|^2_H \, ds. \]

Now define a mapping, \( \Lambda : L^2(0,T;H) \to L^2(0,T;H) \) by \( \Lambda [v] = [x] \) where \([x]\) is the solution to \(32.7.74\). Then, if \([v_i]\) is in \(L^2(0,T;H)\) and \([x_i]\) is the corresponding solution to \(32.7.74\),
\[ ||\Lambda [v_1] - \Lambda [v_2]||_{L^2(0,t;H)} \equiv \int_0^t |x_1(s) - x_2(s)|^2_H \, ds \leq \int_0^t \int_0^t |v_1(r) - v_2(r)|^2_H \, dr \, ds. \]

Iterating this inequality, by replacing \( \Lambda \) with \( \Lambda^k \), it follows that for all \( k \) large enough, \( \Lambda^k \) is a contraction map on \( L^2(0,T;H) \). Thus there exists a unique fixed point for \( \Lambda, [x] \). Thus
\[ L [x] + [x] + \partial \Phi ([x]) \ni [f] + [x]. \]

Let \( x \in [x] \) be such that
\[ x(t) = x_0 + \int_0^t x'(s) \, ds. \]

By Lemma \(32.7.6\),
\[ x' + x + \partial_2 \phi (t, x) \ni f + x \]

This function, \( x(\cdot) \) is the unique solution to \(32.7.74\) because if \( x_1 \) is another solution, then \([x_1] = [x]\) and since both functions are continuous, they must coincide. This proves the corollary.
Part III

Sobolev Spaces
Chapter 33

Weak Derivatives

33.1 Weak * Convergence

A very important sort of convergence in applications of functional analysis is the concept of weak or weak * convergence. It is important because it allows you to assert the existence of a convergent subsequence of a given bounded sequence. The only problem is the convergence is very weak so it does not tell you as much as you would like. Nevertheless, it is a very useful concept. The big theorems in the subject are the Eberlein Smulian theorem and the Banach Alaoglu theorem about the weak or weak * compactness of the closed unit balls in either a Banach space or its dual space. These theorems are proved in Yosida [115]. Here I will present a special case which turns out to be by far the most important in applications and it is not hard to get from the Riesz representation theorem for $L^p$. First I define weak and weak * convergence.

Definition 33.1.1 Let $X'$ be the dual of a Banach space $X$ and let $\{x^*_n\}$ be a sequence of elements of $X'$. Then $x^*_n$ converges weak * to $x^*$ if and only if for all $x \in X$,

$$\lim_{n \to \infty} x^*_n(x) = x^*(x).$$

A sequence in $X$, $\{x_n\}$ converges weakly to $x \in X$ if and only if for all $x^* \in X'$

$$\lim_{n \to \infty} x^*(x_n) = x^*(x).$$

The main result is contained in the following lemma.

Lemma 33.1.2 Let $X'$ be the dual of a Banach space, $X$ and suppose $X$ is separable. Then if $\{x^*_n\}$ is a bounded sequence in $X'$, there exists a weak * convergent subsequence.

Proof: Let $D$ be a dense countable set in $X$. Then the sequence, $\{x^*_n(x)\}$ is bounded for all $x$ and in particular for all $x \in D$. Use the Cantor diagonal process to
obtain a subsequence, still denoted by \( n \) such that \( x_n^* (d) \) converges for each \( d \in D \). Now let \( x \in X \) be completely arbitrary. In fact \( \{ x_n^*(x) \} \) is a Cauchy sequence. Let \( \varepsilon > 0 \) be given and pick \( d \in D \) such that for all \( n \)

\[
|x_n^*(x) - x_n^*(d)| < \frac{\varepsilon}{3}.
\]

This is possible because \( D \) is dense. By the first part of the proof, there exists \( N_\varepsilon \) such that for all \( m, n > N_\varepsilon \),

\[
|x_n^*(d) - x_m^*(d)| < \frac{\varepsilon}{3}.
\]

Then for such \( m, n \),

\[
|x_n^*(x) - x_m^*(x)| \leq |x_n^*(x) - x_n^*(d)| + |x_n^*(d) - x_m^*(d)| + |x_m^*(d) - x_m^*(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this shows \( \{ x_n^*(x) \} \) is a Cauchy sequence for all \( x \in X \).

Now define \( f(x) \equiv \lim_{n \to \infty} x_n^* (x) \). Since each \( x_n^* \) is linear, it follows \( f \) is also linear. In addition to this,

\[
|f(x)| = \lim_{n \to \infty} |x_n^*(x)| \leq K \|x\|
\]

where \( K \) is some constant which is larger than all the norms of the \( x_n^* \). Such a constant exists because the sequence, \( \{ x_n^* \} \) was bounded. This proves the lemma.

The lemma implies the following important theorem.

**Theorem 33.1.3** Let \( \Omega \) be a measurable subset of \( \mathbb{R}^n \) and let \( \{ f_k \} \) be a bounded sequence in \( L^p(\Omega) \) where \( 1 < p \leq \infty \). Then there exists a weak * convergent subsequence.

**Proof:** Since \( L^p(\Omega) \) is separable, this follows from the Riesz representation theorem.

Note that from the Riesz representation theorem, it follows that if \( p < \infty \), then the sequence converges weakly.

### 33.2 Test Functions And Weak Derivatives

In elementary courses in mathematics, functions are often thought of as things which have a formula associated with them and it is the formula which receives the most attention. For example, in beginning calculus courses the derivative of a function is defined as the limit of a difference quotient. You start with one function which tends to be identified with a formula and, by taking a limit, you get another formula for the derivative. A jump in abstraction occurs as soon as you encounter the derivative of a function of \( n \) variables where the derivative is defined as a certain linear transformation which is determined not by a formula but by what it does to
vectors. When this is understood, it reduces to the usual idea in one dimension. The
idea of weak partial derivatives goes further in the direction of defining something
in terms of what it does rather than by a formula, and extra generality is obtained
when it is used. In particular, it is possible to differentiate almost anything if
the notion of what is meant by the derivative is sufficiently weak. This has the
advantage of allowing the consideration of the weak partial derivative of a function
without having to agonize over the important question of existence but it has the
disadvantage of not being able to say much about the derivative. Nevertheless, it
is the idea of weak partial derivatives which makes it possible to use functional
analytic techniques in the study of partial differential equations and it is shown in
this chapter that the concept of weak derivative is useful for unifying the discussion
of some very important theorems. Certain things which should be true are.

Let $\Omega \subseteq \mathbb{R}^n$. A distribution on $\Omega$ is defined to be a linear functional on $C^\infty_c (\Omega)$,
called the space of test functions. The space of all such linear functionals will be
denoted by $D^* (\Omega)$. Actually, more is sometimes done here. One imposes a topology
on $C^\infty_c (\Omega)$ making it into a topological vector space, and when this has been done,
$D' (\Omega)$ is defined as the dual space of this topological vector space. To see this,
consult the book by Yosida or the book by Rudin.

**Example:** The space $L^1_{\text{loc}} (\Omega)$ may be considered as a subset of $D^* (\Omega)$ as follows.

$$ f (\phi) \equiv \int_\Omega f (x) \phi (x) \, dx $$

for all $\phi \in C^\infty_c (\Omega)$. Recall that $f \in L^1_{\text{loc}} (\Omega)$ if $f|_K \in L^1 (\Omega)$ whenever $K$ is
compact.

**Example:** $\delta_x \in D^* (\Omega)$ where $\delta_x (\phi) \equiv \phi (x)$.

It will be observed from the above two examples and a little thought that $D^* (\Omega)$
is truly enormous. The derivative of a distribution will be defined in such a way that
it agrees with the usual notion of a derivative on those distributions which are also
continuously differentiable functions. With this in mind, let $f$ be the restriction to
$\Omega$ of a smooth function defined on $\mathbb{R}^n$. Then $D_x f$ makes sense and for $\phi \in C^\infty_c (\Omega)$

$$ D_x f (\phi) \equiv \int_\Omega D_x f (x) \phi (x) \, dx = - \int_\Omega f D_x \phi \, dx = - f (D_x \phi). $$

This motivates the following definition.

**Definition 33.2.1** For $T \in D^* (\Omega)$

$$ D_x T (\phi) \equiv -T (D_x \phi). $$

Of course one can continue taking derivatives indefinitely. Thus,

$$ D_{x_i x_j} T \equiv D_{x_i} (D_{x_j} T) $$

and it is clear that all mixed partial derivatives are equal because this holds for
the functions in $C^\infty_c (\Omega)$. In this weak sense, the derivative of almost anything
exists, even functions that may be discontinuous everywhere. However the notion of “derivative” is very weak, hence the name, “weak derivatives”.

**Example:** Let $\Omega = \mathbb{R}$ and let

$$H(x) \equiv \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then

$$DH(\phi) = -\int H(x) \phi'(x) \, dx = \phi(0) = \delta_0(\phi).$$

Note that in this example, $DH$ is not a function.

What happens when $Df$ is a function?

**Theorem 33.2.2** Let $\Omega = (a,b)$ and suppose that $f$ and $Df$ are both in $L^1(a,b)$. Then $f$ is equal to a continuous function a.e., still denoted by $f$ and

$$f(x) = f(a) + \int_a^x Df(t) \, dt.$$

In proving Theorem 33.2.2 the following lemma is useful.

**Lemma 33.2.3** Let $T \in \mathcal{D}'(a,b)$ and suppose $DT = 0$. Then there exists a constant $C$ such that

$$T(\phi) = \int_a^b C\phi \, dx.$$**

**Proof:** $T(D\phi) = 0$ for all $\phi \in C_c^\infty(a,b)$ from the definition of $DT = 0$. Let

$$\phi_0 \in C_c^\infty(a,b), \quad \int_a^b \phi_0(x) \, dx = 1,$$

and let

$$\psi_\phi(x) = \int_a^x [\phi(t) - \left(\int_a^b \phi(y) \, dy\right) \phi_0(t)] \, dt$$

for $\phi \in C_c^\infty(a,b)$. Thus $\psi_\phi \in C_c^\infty(a,b)$ and

$$D\psi_\phi = \phi - \left(\int_a^b \phi(y) \, dy\right) \phi_0.$$**

Therefore,

$$\phi = D\psi_\phi + \left(\int_a^b \phi(y) \, dy\right) \phi_0$$

and so

$$T(\phi) = T(D\psi_\phi) + \left(\int_a^b \phi(y) \, dy\right) T(\phi_0) = \int_a^b T(\phi_0) \, \phi(y) \, dy.$$
Let $C = T\phi_0$. This proves the lemma.

**Proof of Theorem 33.2.2** Since $f$ and $Df$ are both in $L^1(a,b)$,

$$Df(\phi) - \int_a^b Df(x)\phi(x) \, dx = 0.$$  

Consider

$$f(\cdot) - \int_a^b Df(t) \, dt$$

and let $\phi \in C_\infty^c(a,b)$.

$$D\left(f(\cdot) - \int_a^b Df(t) \, dt\right)(\phi)$$

$$= -\int_a^b f(x)\phi'(x) \, dx + \int_a^b \left(\int_a^x Df(t) \, dt\right)\phi'(x) \, dx$$

$$= Df(\phi) + \int_a^b \int_t^b Df(t)\phi'(x) \, dx \, dt$$

$$= Df(\phi) - \int_a^b Df(t)\phi(t) \, dt = 0.$$  

By Lemma 33.2.3, there exists a constant, $C$, such that

$$\left(f(\cdot) - \int_a^b Df(t) \, dt\right)(\phi) = \int_a^b C\phi(x) \, dx$$

for all $\phi \in C_\infty^c(a,b)$. Thus

$$\int_a^b \left(\left(f(x) - \int_a^x Df(t) \, dt\right) - C\right)\phi(x) \, dx = 0$$

for all $\phi \in C_\infty^c(a,b)$. It follows from Lemma 33.3.3 in the next section that

$$f(x) - \int_a^x Df(t) \, dt - C = 0 \text{ a.e. } x.$$  

Thus let $f(a) = C$ and write

$$f(x) = f(a) + \int_a^x Df(t) \, dt.$$  

This proves Theorem 33.2.2.

Theorem 33.2.2 says that

$$f(x) = f(a) + \int_a^x Df(t) \, dt.$$
CHAPTER 33. WEAK DERIVATIVES
whenever it makes sense to write \( \int_a^x Df (t) \, dt \), if \( Df \) is interpreted as a weak derivative. Somehow, this is the way it ought to be. It follows from the fundamental theorem of calculus that \( f' (x) \) exists for a.e. \( x \) where the derivative is taken in the sense of a limit of difference quotients and \( f' (x) = Df (x) \). This raises an interesting question. Suppose \( f \) is continuous on \([a, b] \) and \( f' (x) \) exists in the classical sense for a.e. \( x \). Does it follow that
\[
 f(x) = f(a) + \int_a^x f'(t) \, dt. 
\]
The answer is no. To see an example, consider Problem 4 on Page 955 which gives an example of a function which is continuous on \([0, 1] \), has a zero derivative for a.e. \( x \) but climbs from 0 to 1 on \([0, 1] \). Thus this function is not recovered from integrating its classical derivative.

In summary, if the notion of weak derivative is used, one can at least give meaning to the derivative of almost anything, the mixed partial derivatives are always equal, and, in one dimension, one can recover the function from integrating its derivative. None of these claims are true for the classical derivative. Thus weak derivatives are convenient and rule out pathologies.

33.3 Weak Derivatives In \( L^p_{loc} \)

**Definition 33.3.1** Let \( U \) be an open set in \( \mathbb{R}^n \). \( f \in L^p_{loc}(U) \) if \( f|_K \in L^p \) whenever \( K \) is a compact subset of \( U \).

**Definition 33.3.2** For \( \alpha = (k_1, \ldots, k_n) \) where the \( k_i \) are nonnegative integers, define
\[
|\alpha| = \sum_{i=1}^n |k_i|, \quad D^\alpha f (x) = \frac{\partial^{|\alpha|} f (x)}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}.
\]
Also define \( \phi_k \) to be a mollifier if \( \text{spt} (\phi_k) \subseteq B (0, \frac{1}{k}) \), \( \phi_k \geq 0 \), \( \int \phi_k \, dx = 1 \), and \( \phi_k \in C^\infty_c (B (0, \frac{1}{k})) \). In the case a Greek letter like \( \delta \) or \( \varepsilon \) is used as a subscript, it will mean \( \text{spt} (\phi_\delta) \subseteq B (0, \delta) \), \( \phi_\delta \geq 0 \), \( \int \phi_\delta \, dx = 1 \), and \( \phi_\delta \in C^\infty_c (B (0, \delta)) \). You can always get a mollifier by letting \( \phi \geq 0 \), \( \phi \in C^\infty_c (B (0, 1)) \), \( \int \phi \, dx = 1 \), and then defining \( \phi_k (x) \equiv k^n \phi (kx) \) or in the case of a Greek subscript, \( \phi_\delta (x) = \frac{1}{\delta^n} \phi (\frac{x}{\delta}) \).

Consider the case where \( u \) and \( D^\alpha u \) for \( |\alpha| = 1 \) are each in \( L^p_{loc} (\mathbb{R}^n) \). The next lemma is the one alluded to in the proof of Theorem 33.2.2.

**Lemma 33.3.3** Suppose \( f \in L^1_{loc} (U) \) and suppose
\[
\int f \phi \, dx = 0
\]
for all \( \phi \in C^\infty_c (U) \). Then \( f (x) = 0 \) a.e. \( x \).
Proof: Without loss of generality \( f \) is real valued. Let

\[ E \equiv \{ x : f(x) > \varepsilon \} \]

and let

\[ E_m = E \cap B(0,m). \]

Is \( m(E_m) = 0? \) If not, there exists an open set, \( V \), and a compact set \( K \) satisfying

\[ K \subseteq E_m \subseteq V \subseteq B(0,m), \quad m(V \setminus K) < 4^{-1}m(E_m), \]

\[ \int_{V \setminus K} |f| dx < \varepsilon 4^{-1}m(E_m). \]

Let \( H \) and \( W \) be open sets satisfying

\[ K \subseteq H \subseteq \overline{H} \subseteq W \subseteq \overline{W} \subseteq V \]

and let

\[ \overline{H} \preceq g \preceq W \]

where the symbol, \( \preceq \), in the above implies \( \text{spt} (g) \subseteq W \), \( g \) has all values in \([0,1]\), and \( g(x) = 1 \) on \( \overline{H} \). Then let \( \phi_\delta \) be a mollifier and let \( h \equiv g \ast \phi_\delta \) for \( \delta \) small enough that

\[ K \preceq h \preceq V. \]

Thus

\[ 0 = \int fhdx = \int_K f dx + \int_{V \setminus K} fhdx \geq \varepsilon m(K) - \varepsilon 4^{-1}m(E_m) \geq \varepsilon (m(E_m) - 4^{-1}m(E_m)) - \varepsilon 4^{-1}m(E_m) \geq 2^{-1} \varepsilon m(E_m). \]

Therefore, \( m(E_m) = 0 \), a contradiction. Thus

\[ m(E) \leq \sum_{m=1}^{\infty} m(E_m) = 0 \]

and so, since \( \varepsilon > 0 \) is arbitrary,

\[ m(\{ x : f(x) > 0 \}) = 0. \]

Similarly \( m(\{ x : f(x) < 0 \}) = 0 \). This proves the lemma.

This lemma allows the following definition.

**Definition 33.3.4** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( u \in L^1_{\text{loc}}(U) \). Then \( D^\alpha u \in L^1_{\text{loc}}(U) \) if there exists a function \( g \in L^1_{\text{loc}}(U) \), necessarily unique by Lemma \ref{lem:weak_derivative}, such that for all \( \phi \in C_0^\infty(U) \),

\[ \int_U g \phi dx = D^\alpha u (\phi) \equiv \int_U (-1)^{|\alpha|} u (D^\alpha \phi) dx. \]

Then \( D^\alpha u \) is defined to equal \( g \) when this occurs.

---

**Proof:** Without loss of generality \( f \) is real valued. Let

\[ E \equiv \{ x : f(x) > \varepsilon \} \]

and let

\[ E_m = E \cap B(0,m). \]

Is \( m(E_m) = 0? \) If not, there exists an open set, \( V \), and a compact set \( K \) satisfying

\[ K \subseteq E_m \subseteq V \subseteq B(0,m), \quad m(V \setminus K) < 4^{-1}m(E_m), \]

\[ \int_{V \setminus K} |f| dx < \varepsilon 4^{-1}m(E_m). \]

Let \( H \) and \( W \) be open sets satisfying

\[ K \subseteq H \subseteq \overline{H} \subseteq W \subseteq \overline{W} \subseteq V \]

and let

\[ \overline{H} \preceq g \preceq W \]

where the symbol, \( \preceq \), in the above implies \( \text{spt} (g) \subseteq W \), \( g \) has all values in \([0,1]\), and \( g(x) = 1 \) on \( \overline{H} \). Then let \( \phi_\delta \) be a mollifier and let \( h \equiv g \ast \phi_\delta \) for \( \delta \) small enough that

\[ K \preceq h \preceq V. \]

Thus

\[ 0 = \int fhdx = \int_K f dx + \int_{V \setminus K} fhdx \geq \varepsilon m(K) - \varepsilon 4^{-1}m(E_m) \geq \varepsilon (m(E_m) - 4^{-1}m(E_m)) - \varepsilon 4^{-1}m(E_m) \geq 2^{-1} \varepsilon m(E_m). \]

Therefore, \( m(E_m) = 0 \), a contradiction. Thus

\[ m(E) \leq \sum_{m=1}^{\infty} m(E_m) = 0 \]

and so, since \( \varepsilon > 0 \) is arbitrary,

\[ m(\{ x : f(x) > 0 \}) = 0. \]

Similarly \( m(\{ x : f(x) < 0 \}) = 0 \). This proves the lemma.

This lemma allows the following definition.

**Definition 33.3.4** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( u \in L^1_{\text{loc}}(U) \). Then \( D^\alpha u \in L^1_{\text{loc}}(U) \) if there exists a function \( g \in L^1_{\text{loc}}(U) \), necessarily unique by Lemma \ref{lem:weak_derivative}, such that for all \( \phi \in C_0^\infty(U) \),

\[ \int_U g \phi dx = D^\alpha u (\phi) \equiv \int_U (-1)^{|\alpha|} u (D^\alpha \phi) dx. \]

Then \( D^\alpha u \) is defined to equal \( g \) when this occurs.
Lemma 33.3.5 Let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and suppose $u, u_i \in L^1_{\text{loc}}(\mathbb{R}^n)$, where the subscript on the $u$ following the comma denotes the $i$th weak partial derivative. Then if $\phi_\varepsilon$ is a mollifier and $u_\varepsilon \equiv u * \phi_\varepsilon$, it follows $u_{\varepsilon,i} \equiv u,i * \phi_\varepsilon$.

Proof: If $\psi \in C^\infty(\mathbb{R}^n)$, then
\[
\int u(x - y) \psi_i(x) \, dx = \int u(z) \psi_i(z + y) \, dz
\]
\[= - \int u,i(z) \psi(z + y) \, dz
\]
\[= - \int u,i(x - y) \psi(x) \, dx.
\]
Therefore,
\[
u_{\varepsilon,i}(\psi) = - \int u_{\varepsilon} \psi_i = - \int \int u(x - y) \phi_\varepsilon(y) \psi_i(x) \, dy \, dx
\]
\[= - \int \int u(x - y) \phi_\varepsilon(y) \psi_i(x) \, dx \, dy
\]
\[= \int \int u,i(x - y) \psi(x) \phi_\varepsilon(y) \, dx \, dy
\]
\[= \int u,i * \phi_\varepsilon(x) \psi(x) \, dx.
\]

The technical questions about product measurability in the use of Fubini’s theorem may be resolved by picking a Borel measurable representative for $u$. This proves the lemma.

What about the product rule? Does it have some form in the context of weak derivatives?

Lemma 33.3.6 Let $U$ be an open set, $\psi \in C^\infty(U)$ and suppose $u, u_i \in L^p_{\text{loc}}(U)$. Then $(u\psi)_i$ and $u \psi$ are in $L^p_{\text{loc}}(U)$ and
\[(u\psi)_i = u_i \psi + u \psi_i.
\]

Proof: Let $\phi \in C^\infty(U)$ then
\[(u\psi)_i(\phi) = - \int_U u\phi \, dx
\]
\[= - \int_U (u \phi)_i - u_i \phi \, dx
\]
\[= \int_U (u_i \psi \phi + u \psi_i \phi) \, dx
\]
\[= \int_U (u_i \psi + u \psi_i) \phi \, dx
\]

This proves the lemma.
Recall the notation for the gradient of a function.

\[ \nabla u(x) \equiv (u_1(x) \cdots u_n(x))^T \]

thus

\[ Du(x) v = \nabla u(x) \cdot v. \]

### 33.4 Morrey’s Inequality

The following inequality will be called Morrey’s inequality. It relates an expression which is given pointwise to an integral of the \( p^{th} \) power of the derivative.

**Lemma 33.4.1** Let \( u \in C^1(\mathbb{R}^n) \) and \( p > n \). Then there exists a constant, \( C \), depending only on \( n \) such that for any \( x, y \in \mathbb{R}^n \),

\[
|u(x) - u(y)| \leq C \left( \int_{B(x,2|y-x|)} |\nabla u(z)|^p dz \right)^{1/p} \left( |y - x|^{(1-n/p)} \right). \tag{33.4.1}
\]

**Proof:** In the argument \( C \) will be a generic constant which depends on \( n \). Consider the following picture.

This is a picture of two balls of radius \( r \) in \( \mathbb{R}^n \), \( U \) and \( V \) having centers at \( x \) and \( y \) respectively, which intersect in the set, \( W \). The center of \( U \) is on the boundary of \( V \) and the center of \( V \) is on the boundary of \( U \) as shown in the picture. There exists a constant, \( C \), independent of \( r \) depending only on \( n \) such that

\[
\frac{m(W)}{m(U)} = \frac{m(W)}{m(V)} = C.
\]

You could compute this constant if you desired but it is not important here.

Define the average of a function over a set, \( E \subseteq \mathbb{R}^n \) as follows.

\[
\int_E f dx \equiv \frac{1}{m(E)} \int_E f dx.
\]
Then
\[
|u(x) - u(y)| = \int_{W} |u(x) - u(y)| \, dz
\]
\[
\leq \int_{W} |u(x) - u(z)| \, dz + \int_{W} |u(z) - u(y)| \, dz
\]
\[
= \frac{C}{m(U)} \left[ \int_{W} |u(x) - u(z)| \, dz + \int_{W} |u(z) - u(y)| \, dz \right]
\]
\[
\leq C \left[ \int_{U} |u(x) - u(z)| \, dz + \int_{V} |u(y) - u(z)| \, dz \right]
\]

Now consider these two terms. Using spherical coordinates and letting \( U_0 \) denote the ball of the same radius as \( U \) but with center at \( 0 \),

\[
\int_{U} |u(x) - u(z)| \, dz
\]
\[
= \frac{1}{m(U_0)} \int_{U_0} |u(x) - u(z + x)| \, dz
\]
\[
= \frac{1}{m(U_0)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} |u(x) - u(\rho w + x)| \, d\sigma(w) \, d\rho
\]
\[
\leq \frac{1}{m(U_0)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} \int_{0}^{\rho} |\nabla u(x + tw)| \, dtd\sigma \, d\rho
\]
\[
\leq \frac{1}{m(U_0)} \int_{0}^{r} \rho^{n-1} \int_{S^{n-1}} \int_{0}^{\rho} |\nabla u(x + tw)| \, dtd\sigma \, d\rho
\]
\[
\leq C \int_{0}^{r} \int_{S^{n-1}} \int_{0}^{\rho} |\nabla u(x + tw)| \, dtd\sigma \, d\rho
\]
\[
= C \int_{0}^{r} \int_{S^{n-1}} \int_{0}^{\rho} |\nabla u(x + tw)| \frac{1}{t^{n-1}} \, dtd\sigma \, d\rho
\]
\[
= C \int_{S^{n-1}} \int_{0}^{r} |\nabla u(x + tw)| \frac{1}{t^{n-1}} \, dtd\sigma
\]
\[
= C \int_{U_0} \frac{|\nabla u(x + z)|}{|z|^{n-1}} \, dz
\]
\[
\leq C \left( \int_{U_0} |\nabla u(x + z)|^p \, dz \right)^{1/p} \left( \int_{U_0} |z|^{p' - np'} \right)^{1/p'}
\]
\[
= C \left( \int_{U_0} |\nabla u(z)|^p \, dz \right)^{1/p} \left( \int_{S^{n-1}} \int_{0}^{r} \rho^{p' - np'} \rho^{n-1} \, d\rho \, d\sigma \right)^{(p-1)/p}
\]
\[
= C \left( \int_{U_0} |\nabla u(z)|^p \, dz \right)^{1/p} \left( \int_{S^{n-1}} \int_{0}^{r} \frac{1}{\rho^{n-1}} \, d\rho \, d\sigma \right)^{(p-1)/p}
\]
33.4. MORREY’S INEQUALITY

\[
= C \left( \frac{p-1}{p-n} \right)^{(p-1)/p} \left( \int_{U} |\nabla u(z)|^p \, dz \right)^{1/p} r^{1-\frac{n}{p}}
\]

Similarly,

\[
\int_{V} |u(y) - u(z)| \, dz \leq C \left( \frac{p-1}{p-n} \right)^{(p-1)/p} \left( \int_{V} |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1-\frac{n}{p}}
\]

Therefore,

\[
|u(x) - u(y)| \leq C \left( \frac{p-1}{p-n} \right)^{(p-1)/p} \left( \int_{B(x,2|x-y|)} |\nabla u(z)|^p \, dz \right)^{1/p} |x - y|^{1-\frac{n}{p}}
\]

because \( B(x,2|x-y|) \supseteq V \cup U \). This proves the lemma.

The following corollary is also interesting

**Corollary 33.4.2** Suppose \( u \in C^1(\mathbb{R}^n) \). Then

\[
|u(y) - u(x) - \nabla u(x) \cdot (y - x)|
\]

\[
\leq C \left( \frac{1}{m(B(x,2|x-y|))} \right) \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x - y|. \tag{33.4.2}
\]

**Proof:** This follows easily from letting \( g(y) \equiv u(y) - u(x) - \nabla u(x) \cdot (y - x) \).

Then \( g \in C^1(\mathbb{R}^n) \), \( g(x) = 0 \), and \( \nabla g(z) = \nabla u(z) - \nabla u(x) \). From Lemma [53.17],

\[
|u(y) - u(x) - \nabla u(x) \cdot (y - x)|
\]

\[
= |g(y)| = |g(y) - g(x)|
\]

\[
\leq C \left( \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x - y|^{1-\frac{n}{p}}
\]

\[
= C \left( \frac{1}{m(B(x,2|x-y|))} \right) \int_{B(x,2|x-y|)} |\nabla u(z) - \nabla u(x)|^p \, dz \right)^{1/p} |x - y|.
\]

This proves the corollary.

It may be interesting at this point to recall the definition of differentiability on Page [22]. If you knew the above inequality held for \( \nabla u \) having components in \( L^1_{loc}(\mathbb{R}^n) \), then at Lebesgue points of \( \nabla u \), the above would imply \( Du(x) \) exists. This is exactly the approach taken below.
33.5 Rademacher’s Theorem

The inequality of Corollary 33.4.2 can be extended to the case where \( u \) and \( u_i \) are in \( L^p_{\text{loc}}(\mathbb{R}^n) \) for \( p > n \). This leads to an elegant proof of the differentiability a.e. of a Lipschitz continuous function as well as a more general theorem.

**Theorem 33.5.1** Suppose \( u \) and all its weak partial derivatives, \( u_i \), are in \( L^p_{\text{loc}}(\mathbb{R}^n) \). Then there exists a set of measure zero, \( E \) such that if \( x, y \not\in E \) then inequalities 33.4.2 and 33.4.1 are both valid. Furthermore, \( u \) equals a continuous function a.e.

**Proof:** Let \( u \in L^p_{\text{loc}}(\mathbb{R}^n) \) and \( \psi_k \in \mathcal{C}_c^\infty(\mathbb{R}^n) \), \( \psi_k \geq 0 \), and \( \psi_k(z) = 1 \) for all \( z \in B(0,k) \). Then it is routine to verify that 

\[
(u\psi_k, (u\psi_k)_i) \in L^p(\mathbb{R}^n).
\]

Here is why:

\[
(u\psi_k)_i = u\psi_k + u_{\psi_k,i}
\]

which shows 

\[
(u\psi_k) = u\psi_k + u_{\psi_k,i}
\]

as expected.

Let \( \phi \) be a mollifier and consider

\[
(u\psi_k)_\varepsilon = u\psi_k \ast \phi_\varepsilon.
\]

By Lemma 33.3.5 on Page 1278,

\[
(u\psi_k)_{\varepsilon,i} = (u\psi_k)_i \ast \phi_\varepsilon.
\]

Therefore 

\[
(u\psi_k)_{\varepsilon,i} \to (u\psi_k)_i \text{ in } L^p(\mathbb{R}^n)
\]

and

\[
(u\psi_k)_\varepsilon \to u\psi_k \text{ in } L^p(\mathbb{R}^n)
\]

as \( \varepsilon \to 0 \). By 33.5.3, there exists a subsequence \( \varepsilon \to 0 \) such that for \( |z| < k \) and for each \( i = 1, 2, \cdots, n \)

\[
(u\psi_k)_{\varepsilon,i}(z) \to (u\psi_k)_i(z) = u_i(z) \text{ a.e.}
\]
Denoting the exceptional set by $E_k$, let
\[ x, y \notin \bigcup_{k=1}^{\infty} E_k \equiv E \]
and let $k$ be so large that
\[ B(0, k) \supseteq B(x, 2|x - y|) . \]
Then by (33.4.1) and for $x, y \notin E$,
\[
| (u \psi_k \epsilon \zeta)(x) - (u \psi_k \epsilon \zeta)(y) | \leq C \left( \frac{1}{m(B(x, 2|x - y|))} \int_{B(x, 2|x - y|)} |\nabla (u \psi_k \epsilon \zeta)(z) - \nabla (u \psi_k \epsilon \zeta)(x)| p \, dz \right)^{1/p} |x - y|^{(1-n/p)}
\]
where $C$ depends only on $n$. Similarly, by (33.4.2),
\[
| (u \psi_k \epsilon \zeta)(x) - (u \psi_k \epsilon \zeta)(y) - \nabla (u \psi_k \epsilon \zeta)(x) \cdot (y - x) | \leq C \left( \frac{1}{m(B(x, 2|x - y|))} \int_{B(x, 2|x - y|)} |\nabla u - \nabla u(x)| p \, dz \right)^{1/p} |x - y|.
\]
Now by (33.5.5) and (33.5.3) passing to the limit as $\epsilon \to 0$ yields
\[
|u(x) - u(y)| \leq C \left( \int_{B(x, 2|x - y|)} |\nabla u|^p \, dz \right)^{1/p} |x - y|^{(1-n/p)} \quad (33.5.6)
\]
and
\[
|u(y) - u(x) - \nabla u(x) \cdot (y - x) | \leq C \left( \frac{1}{m(B(x, 2|x - y|))} \int_{B(x, 2|x - y|)} |\nabla u(z) - \nabla u(x)| p \, dz \right)^{1/p} |x - y| \quad (33.5.7)
\]
Redefining $u$ on the set of measure zero, $E$ yields (33.5.6) for all $x, y$. This proves the theorem.

**Corollary 33.5.2** Let $u, u_i \in L^p_{loc}(\mathbb{R}^n)$ for $i = 1, \cdots, n$ and $p > n$. Then the representative of $u$ described in Theorem 33.5.1 is differentiable a.e.

**Proof:** From Theorem 33.5.1
\[
|u(y) - u(x) - \nabla u(x) \cdot (y - x) | \leq C \left( \frac{1}{m(B(x, 2|x - y|))} \int_{B(x, 2|x - y|)} |\nabla u(z) - \nabla u(x)| p \, dz \right)^{1/p} |x - y| \quad (33.5.8)
\]
and at every Lebesgue point, \( x \) of \( \nabla u \)

\[
\lim_{y \to x} \left( \frac{1}{m(B(x, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(x, 2|\mathbf{x} - \mathbf{y}|)} |\nabla u(z) - \nabla u(x)|^p dz \right)^{1/p} = 0
\]

and so at each of these points,

\[
\lim_{y \to x} \frac{|u(y) - u(x) - \nabla u(x) \cdot (\mathbf{y} - \mathbf{x})|}{|\mathbf{x} - \mathbf{y}|} = 0
\]

which says that \( u \) is differentiable at \( x \) and \( Du(x)(\mathbf{v}) = \nabla u(x) \cdot \mathbf{v} \). See Page 2124. This proves the corollary.

**Definition 33.5.3** Now suppose \( u \) is Lipschitz on \( \mathbb{R}^n \),

\[
|u(x) - u(y)| \leq K|\mathbf{x} - \mathbf{y}|
\]

for some constant \( K \). Define \( \text{Lip}(u) \) as the smallest value of \( K \) that works in this inequality.

The following corollary is known as Rademacher’s theorem. It states that every Lipschitz function is differentiable a.e.

**Corollary 33.5.4** If \( u \) is Lipschitz continuous then \( u \) is differentiable a.e. and \( ||u_{,i}||_{\infty} \leq \text{Lip}(u) \).

**Proof:** This is done by showing that Lipschitz continuous functions have weak derivatives in \( L^\infty(\mathbb{R}^n) \) and then using the previous results. Let

\[
D_{\mathbf{e}_i}^h u(x) \equiv h^{-1} [u(x + he_i) - u(x)].
\]

Then \( D_{\mathbf{e}_i}^h u \) is bounded in \( L^\infty(\mathbb{R}^n) \) and

\[
||D_{\mathbf{e}_i}^h u||_{\infty} \leq \text{Lip}(u).
\]

It follows that \( D_{\mathbf{e}_i}^h u \) is contained in a ball in \( L^\infty(\mathbb{R}^n) \), the dual space of \( L^1(\mathbb{R}^n) \). By Theorem 33.1.3 on Page 1274, there is a subsequence \( h \to 0 \) such that

\[
D_{\mathbf{e}_i}^h u \rightharpoonup w, \quad ||w||_{\infty} \leq \text{Lip}(u)
\]

where the convergence takes place in the weak * topology of \( L^\infty(\mathbb{R}^n) \). Let \( \phi \in C_c^{\infty}(\mathbb{R}^n) \). Then

\[
\int w\phi dx = \lim_{h \to 0} \int D_{\mathbf{e}_i}^h u\phi dx
\]

\[
= \lim_{h \to 0} \int u(x) \frac{(\phi(x - he_i) - \phi(x))}{h} dx
\]

\[
= - \int u(x) \phi_{,i}(x) dx.
\]

Thus \( w = u_{,i} \) and \( u_{,i} \in L^\infty(\mathbb{R}^n) \) for each \( i \). Hence \( u, u_{,i} \in L^p_{\text{loc}}(\mathbb{R}^n) \) for all \( p > n \) and so \( u \) is differentiable a.e. by Corollary 33.5.2. This proves the corollary.
33.6 Change Of Variables Formula Lipschitz Maps

With Rademacher’s theorem, one can give a general change of variables formula involving Lipschitz maps. First here is an elementary estimate.

Lemma 33.6.1 Suppose $V$ is an $n - 1$ dimensional subspace of $\mathbb{R}^n$ and $K$ is a compact subset of $V$. Then letting

$$K_\varepsilon \equiv \bigcup_{x \in K} B(x, \varepsilon) = K + B(0, \varepsilon),$$

it follows that

$$m_n(K_\varepsilon) \leq 2^n \varepsilon (\text{diam}(K) + \varepsilon)^{n-1}.$$

Proof: Let an orthonormal basis for $V$ be $\{v_1, \cdots, v_{n-1}\}$ and let $\{v_1, \cdots, v_{n-1}, v_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Now define a linear transformation, $Q$ by $Qv_i = e_i$. Thus $QQ^* = Q^*Q = I$ and $Q$ preserves all distances because

$$\left| Q \sum a_i e_i \right|^2 = \sum a_i v_i^2 = \sum |a_i|^2 = \left| \sum a_i e_i \right|^2.$$

Letting $k_0 \in K$, it follows $K \subseteq B(k_0, \text{diam}(K))$ and so,

$$QK \subseteq B^{n-1}(Qk_0, \text{diam}(QK)) = B^{n-1}(Qk_0, \text{diam}(K))$$

where $B^{n-1}$ refers to the ball taken with respect to the usual norm in $\mathbb{R}^{n-1}$. Every point of $K_\varepsilon$ is within $\varepsilon$ of some point of $K$ and so it follows that every point of $QK_\varepsilon$ is within $\varepsilon$ of some point of $QK$. Therefore,

$$QK_\varepsilon \subseteq B^{n-1}(Qk_0, \text{diam}(QK) + \varepsilon) \times (-\varepsilon, \varepsilon),$$

To see this, let $x \in QK_\varepsilon$. Then there exists $k \in QK$ such that $|k - x| < \varepsilon$. Therefore, $|(x_1, \cdots, x_{n-1}) - (k_1, \cdots, k_{n-1})| < \varepsilon$ and $|x_n - k_n| < \varepsilon$ and so $x$ is contained in the set on the right in the above inclusion because $k_n = 0$. However, the measure of the set on the right is smaller than

$$[2(\text{diam}(QK) + \varepsilon)]^{n-1} (2\varepsilon) = 2^n [(\text{diam}(K) + \varepsilon)]^{n-1} \varepsilon.$$

This proves the lemma.

Next is the definition of a point of density. This is sort of like an interior point but not as good.

Definition 33.6.2 Let $E$ be a Lebesgue measurable set. $x \in E$ is a point of density if

$$\lim_{r \to 0} \frac{m(E \cap B(x, r))}{m(B(x, r))} = 1.$$
You see that if $x$ were an interior point of $E$, then this limit will equal 1. However, it is sometimes the case that the limit equals 1 even when $x$ is not an interior point. In fact, these points of density make sense even for sets that have empty interior.

**Lemma 33.6.3** Let $E$ be a Lebesgue measurable set. Then there exists a set of measure zero, $N$, such that if $x \in E \setminus N$, then $x$ is a point of density of $E$.

**Proof:** Consider the function, $f(x) = \mathcal{X}_E(x)$. This function is in $L^1_{\text{loc}}(\mathbb{R}^n)$. Let $N^C$ denote the Lebesgue points of $f$. Then for $x \in E \setminus N$,

$$1 = \mathcal{X}_E(x) = \lim_{r \to 0} \frac{1}{m_n(B(x,r))} \int_{B(x,r)} \mathcal{X}_E(y) \, dm_n$$

$$= \lim_{r \to 0} \frac{m_n(B(x,r) \cap E)}{m_n(B(x,r))}.$$

In this section, $\Omega$ will be a Lebesgue measurable set in $\mathbb{R}^n$ and $h: \Omega \to \mathbb{R}^n$ will be Lipschitz. Recall the following definition and theorems. See Page 11.4.2 for the proofs and more discussion.

**Definition 33.6.4** Let $F$ be a collection of balls that cover a set, $E$, which have the property that if $x \in E$ and $\epsilon > 0$, then there exists $B \in F$, diameter of $B < \epsilon$ and $x \in B$. Such a collection covers $E$ in the sense of Vitali.

**Theorem 33.6.5** Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by $m_n$, n dimensional Lebesgue measure, and let $F$, be a collection of closed balls of bounded radii such that $F$ covers $E$ in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m_n}(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.

Now this theorem implies a simple lemma which is what will be used.

**Lemma 33.6.6** Let $V$ be an open set in $\mathbb{R}^r$, $m_r(V) < \infty$. Then there exists a sequence of disjoint open balls $\{B_i\}$ having radii less than $\delta$ and a set of measure 0, $T$, such that

$$V = (\bigcup_{i=1}^\infty B_i) \cup T.$$

As in the proof of the change of variables theorem given earlier, the first step is to show that $h$ maps Lebesgue measurable sets to Lebesgue measurable sets. In showing this the key result is the next lemma which states that $h$ maps sets of measure zero to sets of measure zero.

**Lemma 33.6.7** If $m_n(T) = 0$ then $m_n(h(T)) = 0$.

**Proof:** Let $V$ be an open set containing $T$ whose measure is less than $\epsilon$. Now using the Vitali covering theorem, there exists a sequence of disjoint balls $\{B_i\}$,
\( B_i = B(x_i, r_i) \) which are contained in \( V \) such that the sequence of enlarged balls, \( \{\tilde{B}_i\} \), having the same center but 5 times the radius, covers \( T \). Then

\[
m_n(h(T)) \leq m_n \left( h \left( \bigcup_{i=1}^{\infty} \tilde{B}_i \right) \right)
\]

\[
\leq \sum_{i=1}^{\infty} m_n \left( h(\tilde{B}_i) \right)
\]

\[
\leq \sum_{i=1}^{\infty} \alpha(n) (\text{Lip}(h))^{n^5 r_i^n} = 5^n (\text{Lip}(h))^{n} \sum_{i=1}^{\infty} m_n(B_i)
\]

\[
\leq (\text{Lip}(h))^{n^5} m_n(V) \leq \varepsilon (\text{Lip}(h))^{n^5}.
\]

Since \( \varepsilon \) is arbitrary, this proves the lemma.

With the conclusion of this lemma, the next lemma is fairly easy to obtain.

**Lemma 33.6.8**  
If \( A \) is Lebesgue measurable, then \( h(A) \) is \( m_n \) measurable. Furthermore,

\[
m_n(h(A)) \leq (\text{Lip}(h))^{n} m_n(A).
\]  
(33.6.9)

**Proof:** Let \( A_k = A \cap B(0,k), k \in \mathbb{N} \). Let \( V \supseteq A_k \) and let \( m_n(V) < \infty \). By Lemma 33.6.6, there is a sequence of disjoint balls \( \{B_i\} \) and a set of measure 0, \( T \), such that

\[
V = \bigcup_{i=1}^{\infty} B_i \cup T, \ B_i = B(x_i, r_i).
\]

By Lemma 33.6.6,

\[
\overline{m}_n(h(A_k)) \leq \overline{m}_n(h(V))
\]

\[
\leq \overline{m}_n(h(\bigcup_{i=1}^{\infty} B_i)) + \overline{m}_n(h(T)) = \overline{m}_n(h(\bigcup_{i=1}^{\infty} B_i))
\]

\[
\leq \sum_{i=1}^{\infty} \overline{m}_n(h(B_i)) \leq \sum_{i=1}^{\infty} \overline{m}_n(B(h(x_i), \text{Lip}(h) r_i))
\]

\[
\leq \sum_{i=1}^{\infty} \alpha(n) (\text{Lip}(h) r_i)^n = (\text{Lip}(h))^n \sum_{i=1}^{\infty} m_n(B_i) = (\text{Lip}(h))^n m_n(V).
\]

Therefore,

\[
\overline{m}_n(h(A_k)) \leq (\text{Lip}(h))^n m_n(V).
\]

Since \( V \) is an arbitrary open set containing \( A_k \), it follows from regularity of Lebesgue measure that

\[
\overline{m}_n(h(A_k)) \leq (\text{Lip}(h))^n m_n(A_k).
\]  
(33.6.10)

Now let \( k \to \infty \) to obtain 33.6.9. This proves the formula. It remains to show \( h(A) \) is measurable.
By inner regularity of Lebesgue measure, there exists a set, \( F \), which is the countable union of compact sets and a set \( T \) with \( m_n(T) = 0 \) such that 
\[
F \cup T = A_k.
\]

Then \( h(F) \subseteq h(A_k) \subseteq h(F) \cup h(T) \). By continuity of \( h \), \( h(F) \) is a countable union of compact sets and so it is Borel. By Lemma 33.6.10 with \( T \) in place of \( A_k \), 
\[
m_n(h(T)) = 0
\]
and so \( h(T) \) is \( m_n \) measurable. Therefore, \( h(A_k) \) is \( m_n \) measurable because \( m_n \) is a complete measure and this exhibits \( h(A_k) \) between two \( m_n \) measurable sets whose difference has measure 0. Now 
\[
h(A) = \bigcup_{k=1}^{\infty} h(A_k)
\]
so \( h(A) \) is also \( m_n \) measurable and this proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [102], will be important for the following arguments. The idea is that if a continuous function mapping a ball in \( \mathbb{R}^k \) to \( \mathbb{R}^k \) doesn’t move any point very much, then the image of the ball must contain a slightly smaller ball.

**Lemma 33.6.9** Let \( B = B(0, r) \), a ball in \( \mathbb{R}^k \) and let \( F : \overline{B} \to \mathbb{R}^k \) be continuous and suppose for some \( \varepsilon < 1 \), 
\[
|F(v) - v| < \varepsilon r
\]
for all \( v \in \overline{B} \). Then 
\[
F(\overline{B}) \supseteq \overline{B}(0, r(1 - \varepsilon)).
\]

**Proof:** Suppose \( a \in \overline{B}(0, r(1 - \varepsilon)) \setminus F(\overline{B}) \) and let 
\[
G(v) = \frac{r(a - F(v))}{|a - F(v)|}.
\]

Then by the Brouwer fixed point theorem, \( G(v) = v \) for some \( v \in \overline{B} \). Using the formula for \( G \), it follows \( |v| = r \). Taking the inner product with \( v \), 
\[
(G(v), v) = |v|^2 = r^2 = \frac{r}{|a - F(v)|} (a - F(v), v)
\]
\[
= \frac{r}{|a - F(v)|} (a - v + v - F(v), v)
\]
\[
= \frac{r}{|a - F(v)|} [(a, v) + (v - F(v), v)]
\]
\[
= \frac{r}{|a - F(v)|} [r^2 (1 - \varepsilon) - r^2 + r^2 \varepsilon]
\]
\[
= 0,
\]
33.6. CHANGE OF VARIABLES FORMULA LIPSCHITZ MAPS

a contradiction. Therefore, \( B(0, r(1 - \varepsilon)) \setminus F(B) = \emptyset \) and this proves the lemma.

Now let \( \Omega \) be a Lebesgue measurable set and suppose \( h : \mathbb{R}^n \to \mathbb{R}^n \) is Lipschitz continuous and one to one on \( \Omega \). Let

\[
N \equiv \{ x \in \Omega : D h(x) \text{ does not exist} \} \tag{33.6.11}
\]

\[
S \equiv \{ x \in \Omega \setminus N : D h(x)^{-1} \text{ does not exist} \} \tag{33.6.12}
\]

**Lemma 33.6.10** Let \( x \in \Omega \setminus (S \cup N) \). Then if \( \varepsilon \in (0, 1) \) the following hold for all \( r \) small enough.

\[
m_n \left( h \left( B(x, r) \right) \right) \geq m_n \left( D h(x) B(0, r(1 - \varepsilon)) \right), \tag{33.6.13}
\]

\[
h \left( B(x, r) \right) \subseteq h(x) + D h(x) B(0, r(1 + \varepsilon)), \tag{33.6.14}
\]

\[
m_n \left( h \left( B(x, r) \right) \right) \leq m_n \left( D h(x) B(0, r(1 + \varepsilon)) \right) \tag{33.6.15}
\]

If \( x \in \Omega \setminus (S \cup N) \) is also a point of density of \( \Omega \), then

\[
\lim_{r \to 0} \frac{m_n(h(B(x, r) \cap \Omega))}{m_n(h(B(x, r)))} = 1. \tag{33.6.16}
\]

If \( x \in \Omega \setminus N \), then

\[
|\det D h(x)| = \lim_{r \to 0} \frac{m_n(h(B(x, r)))}{m_n(B(x, r))} \text{ a.e.} \tag{33.6.17}
\]

**Proof:** Since \( D h(x)^{-1} \) exists,

\[
h(x + v) = h(x) + D h(x) v + o(|v|) \tag{33.6.18}
\]

\[
= h(x) + D h(x) \left( v + D h(x)^{-1} o(|v|) \right) \tag{33.6.19}
\]

Consequently, when \( r \) is small enough, holds. Therefore, holds. From Lemma 33.6.14 and the assumption that \( D h(x)^{-1} \) exists,

\[
D h(x)^{-1} h(x + v) - D h(x)^{-1} h(x) = o(|v|). \tag{33.6.20}
\]

Letting

\[
F(v) = D h(x)^{-1} h(x + v) - D h(x)^{-1} h(x),
\]

apply Lemma 33.6.14 to conclude that for \( r \) small enough, whenever \( |v| < r \),

\[
D h(x)^{-1} h(x + v) - D h(x)^{-1} h(x) \supseteq B(0, (1 - \varepsilon) r).
\]

Therefore,

\[
h \left( B(x, r) \right) \supseteq h(x) + D h(x) B(0, (1 - \varepsilon) r)
\]
which implies
\[ m_n \left( h \left( B(x, r) \right) \right) \geq m_n (Dh(x) B(0, r(1-\varepsilon))) \]
which shows 33.6.13.

Now suppose that \( x \) is a point of density of \( \Omega \) as well as being a point where \( Dh(x)^{-1} \) and \( Dh(x) \) exist. Then whenever \( r \) is small enough,
\[ 1 - \varepsilon < \frac{m_n (h (B(x, r) \cap \Omega))}{m_n (h (B(x, r)))} \leq 1 \]
and so
\[ 1 - \varepsilon < \frac{m_n (h (B(x, r) \cap \Omega))}{m_n (h (B(x, r)))} + \frac{m_n (h (B(x, r) \cap \Omega))}{m_n (h (B(x, r)))} \]
which implies
\[ m_n (B(x, r) \setminus \Omega) < \varepsilon \alpha (n) r^n. \] (33.6.21)

Then for such \( r \),
\[ 1 \geq \frac{m_n (h (B(x, r) \cap \Omega))}{m_n (h (B(x, r)))} \geq \frac{m_n (h (B(x, r))) - m_n (h (B(x, r) \setminus \Omega))}{m_n (h (B(x, r)))}. \]
From Lemma 33.6.8, 33.6.21, and 33.6.13, this is no larger than
\[ 1 - \frac{\text{Lip} (h)^n \varepsilon \alpha (n) r^n}{m_n (Dh(x) B(0, r(1-\varepsilon)))}. \]
By the theorem on the change of variables for a linear map, this expression equals
\[ 1 - \frac{\text{Lip} (h)^n \varepsilon \alpha (n) r^n}{|\det (Dh(x))| r^n \alpha (n) (1-\varepsilon)^n} \equiv 1 - g(\varepsilon) \]
where \( \lim_{\varepsilon \to 0} g(\varepsilon) = 0 \). Then for all \( r \) small enough,
\[ 1 \geq \frac{m_n (h (B(x, r) \cap \Omega))}{m_n (h (B(x, r)))} \geq 1 - g(\varepsilon) \]
which shows 33.6.14 since \( \varepsilon \) is arbitrary. It remains to verify 33.6.14.

In case \( x \in S \), for small \(|v|\),
\[ h(x + v) = h(x) + Dh(x) v + o(|v|) \]
where \(|o(|v|)| < \varepsilon |v| \). Therefore, for small enough \( r \),
\[ h(B(x, r)) - h(x) \subseteq K + B(0, r\varepsilon) \]
33.6. CHANGE OF VARIABLES FORMULA LIPSCHITZ MAPS

where $K$ is a compact subset of an $n-1$ dimensional subspace contained in $Dh(x)(\mathbb{R}^n)$ which has diameter no more than $2 ||Dh(x)|| r$. By Lemma 33.6.1 on Page 1285

$$m_n(h(B(x,r))) = m_n(h(B(x,r)) - h(x)) \leq 2^n \varepsilon r (2 ||Dh(x)|| r + r \varepsilon)^{n-1}$$

and so, in this case, letting $r$ be small enough,

$$\frac{m_n(h(B(x,r)))}{m_n(B(x,r))} \leq \frac{2^n \varepsilon r (2 ||Dh(x)|| r + r \varepsilon)^{n-1}}{\alpha(n) r^n} \leq C \varepsilon.$$

Since $\varepsilon$ is arbitrary, the limit as $r \to 0$ of this quotient equals 0.

If $x \not\in S$, use 33.6.13 - 33.6.15 along with the change of variables formula for linear maps. This proves the Lemma.

Since $h$ is one to one, there exists a measure, $\mu$, defined by

$$\mu(E) \equiv m_n(h(E))$$
on the Lebesgue measurable subsets of $\Omega$. By Lemma 33.6.8 $\mu \ll m_n$ and so by the Radon Nikodym theorem, there exists a nonnegative function, $J(x)$ in $L^1_{loc}(\mathbb{R}^n)$ such that whenever $E$ is Lebesgue measurable,

$$\mu(E) = m_n(h(E \cap \Omega)) = \int_{E \cap \Omega} J(x) \, dm_n. \quad (33.6.22)$$

Extend $J$ to equal zero off $\Omega$.

**Lemma 33.6.11** The function, $J(x)$ equals $|\det Dh(x)|$ a.e.

**Proof:** Define

$$Q \equiv \{ x \in \Omega : \text{x is not a point of density of } \Omega \} \cup N \cup \{ x \in \Omega : \text{x is not a Lebesgue point of } J \}.$$

Then $Q$ is a set of measure zero and if $x \not\in Q$, then by 33.4 we have

$$\lim_{r \to 0} \frac{m_n(h(B(x,r)))}{m_n(B(x,r))} = \frac{m_n(h(B(x,r)))}{m_n(h(B(x,r) \cap \Omega))} \frac{m_n(h(B(x,r) \cap \Omega))}{m_n(B(x,r))} \frac{1}{m_n(B(x,r))} \int_{B(x,r) \cap \Omega} J(y) \, dm_n$$

the last equality because $J$ was extended to be zero off $\Omega$. This proves the lemma.

Here is the change of variables formula for Lipschitz mappings. It is a special case of the area formula.
CHAPTER 33. WEAK DERIVATIVES

**Theorem 33.6.12** Let $\Omega$ be a Lebesgue measurable set, let $f \geq 0$ be Lebesgue measurable. Then for $h$ a Lipschitz mapping defined on $\mathbb{R}^n$ which is one to one on $\Omega$,

$$\int_{h(\Omega)} f(y) \, dm_n = \int_{\Omega} f(h(x)) |\det Dh(x)| \, dm_n. \quad (33.6.23)$$

**Proof:** Let $F$ be a Borel set. It follows that $h^{-1}(F)$ is a Lebesgue measurable set. Therefore, by (33.6.22),

$$m_n \left( h \left( h^{-1}(F) \cap \Omega \right) \right) = \int_{h(\Omega)} \mathcal{X}_F(y) \, dm_n = \int_{\Omega} \mathcal{X}_{h^{-1}(F)}(x) J(x) \, dm_n$$

$$= \int_{\Omega} \mathcal{X}_F(h(x)) J(x) \, dm_n.$$

What if $F$ is only Lebesgue measurable? Note there are no measurability problems with the above expression because $x \rightarrow \mathcal{X}_F(h(x))$ is Borel measurable due to the assumption that $h$ is continuous while $J$ is given to be Lebesgue measurable. However, if $F$ is Lebesgue measurable, not necessarily Borel measurable, then it is no longer clear that $x \rightarrow \mathcal{X}_F(h(x))$ is measurable. In fact this is not always even true. However, $x \rightarrow \mathcal{X}_F(h(x)) J(x)$ is measurable and (33.6.24) holds.

Let $F$ be Lebesgue measurable. Then by inner regularity, $F = H \cup N$ where $N$ has measure zero, $H$ is the countable union of compact sets so it is a Borel set, and $H \cap N = \emptyset$. Therefore, letting $N'$ denote a Borel set of measure zero which contains $N$,

$$b(x) \equiv \mathcal{X}_H(h(x)) J(x) \leq \mathcal{X}_F(h(x)) J(x)$$

$$= \mathcal{X}_H(h(x)) J(x) + \mathcal{X}_N(h(x)) J(x)$$

$$\leq \mathcal{X}_H(h(x)) J(x) + \mathcal{X}_{N'}(h(x)) J(x) \equiv u(x)$$

Now since $N'$ is Borel,

$$\int_{\Omega} (u(x) - b(x)) \, dm_n = \int_{\Omega} \mathcal{X}_{N'}(h(x)) J(x) \, dm_n$$

$$= m_n \left( h \left( h^{-1} \left( N' \cap \Omega \right) \right) \right) = m_n \left( N' \cap h \left( \Omega \right) \right) = 0$$

and this shows $\mathcal{X}_H(h(x)) J(x) = \mathcal{X}_F(h(x)) J(x)$ except on a set of measure zero. By completeness of Lebesgue measure, it follows $x \rightarrow \mathcal{X}_F(h(x)) J(x)$ is Lebesgue measurable and also since $h$ maps sets of measure zero to sets of measure zero,

$$\int_{\Omega} \mathcal{X}_F(h(x)) J(x) \, dm_n = \int_{\Omega} \mathcal{X}_H(h(x)) J(x) \, dm_n$$

$$= \int_{h(\Omega)} \mathcal{X}_H(y) \, dm_n$$

$$= \int_{h(\Omega)} \mathcal{X}_F(y) \, dm_n.$$
33.6. CHANGE OF VARIABLES FORMULA LIPSCHITZ MAPS

It follows that if $s$ is any nonnegative Lebesgue measurable simple function,

$$\int_{\Omega} s(h(x)) \, J(x) \, dm_n = \int_{h(\Omega)} s(y) \, dm_n \quad (33.6.25)$$

and now, if $f \geq 0$ is Lebesgue measurable, let $s_k$ be an increasing sequence of Lebesgue measurable simple functions converging pointwise to $f$. Then since $33.6.25$ holds for $s_k$, the monotone convergence theorem applies and yields $33.6.23$. This proves the theorem.

It turns out that a Lipschitz function defined on some subset of $\mathbb{R}^n$ always has a Lipschitz extension to all of $\mathbb{R}^n$. The next theorem gives a proof of this. For more on this sort of theorem see Federer [17]. He gives a better but harder theorem than what follows.

**Theorem 33.6.13** If $h : \Omega \to \mathbb{R}^m$ is Lipschitz, then there exists $\overline{h} : \mathbb{R}^n \to \mathbb{R}^m$ which extends $h$ and is also Lipschitz.

**Proof:** It suffices to assume $m = 1$ because if this is shown, it may be applied to the components of $h$ to get the desired result. Suppose

$$|h(x) - h(y)| \leq K|x - y|. \quad (33.6.26)$$

Define

$$\overline{h}(x) \equiv \inf \{h(w) + K|x - w| : w \in \Omega\}. \quad (33.6.27)$$

If $x \in \Omega$, then for all $w \in \Omega$,

$$h(w) + K|x - w| \geq h(x)$$

by $33.6.26$. This shows $h(x) \leq \overline{h}(x)$. But also you could take $w = x$ in $33.6.26$ which yields $\overline{h}(x) \leq h(x)$. Therefore $\overline{h}(x) = h(x)$ if $x \in \Omega$.

Now suppose $x, y \in \mathbb{R}^n$ and consider $|\overline{h}(x) - \overline{h}(y)|$. Without loss of generality assume $\overline{h}(x) \geq \overline{h}(y)$. (If not, repeat the following argument with $x$ and $y$ interchanged.) Pick $w \in \Omega$ such that

$$h(w) + K|y - w| - \varepsilon < \overline{h}(y).$$

Then

$$|\overline{h}(x) - \overline{h}(y)| = \overline{h}(x) - \overline{h}(y) \leq h(w) + K|x - w| -
[h(w) + K|y - w| - \varepsilon] \leq K|x - y| + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary,

$$|\overline{h}(x) - \overline{h}(y)| \leq K|x - y|$$

and this proves the theorem.

This yields a simple corollary to Theorem 33.6.12.

**Corollary 33.6.14** Let $h : \Omega \to \mathbb{R}^n$ be Lipschitz continuous and one to one where $\Omega$ is a Lebesgue measurable set. Then if $f \geq 0$ is Lebesgue measurable,

$$\int_{h(\Omega)} f(y) \, dm_n = \int_{\Omega} f(h(x)) |\det D\overline{h}(x)| \, dm_n. \quad (33.6.28)$$

where $\overline{h}$ denotes a Lipschitz extension of $h$. 

Chapter 34

The Area And Coarea Formulas

34.1 The Area Formula Again

Recall the area formula presented earlier. For convenience, here it is.

**Theorem 34.1.1** Let \( g : h(A) \to [0, \infty] \) be \( \mathcal{H}^n \) measurable where \( h \) is a continuous function and \( A \) is a Lebesgue measurable set satisfying certain conditions. That is, \( U \) is an open set in \( \mathbb{R}^n \) on which \( h \) is defined and \( A \subseteq U \) is a Lebesgue measurable set, \( m \geq n \), and

\[
\text{Det}(h(x)) \text{ exists for all } x \in A, \tag{34.1.1}
\]

Also assume that for every \( x \in A \), there exists \( r_x \) and \( L_x \) such that for all \( y, z \in B(x, r_x) \),

\[
|h(z) - h(y)| \leq L_x |x - y| \tag{34.1.3}
\]

Then \( x \to (g \circ h)(x) J(x) \) is Lebesgue measurable and

\[
\int_{h(A)} g(y) \, d\mathcal{H}^n = \int_A g(h(x)) J(x) \, dm
\]

where \( J(x) = \det(U(x)) = \det(Dh(x)^* \, Dh(x))^{1/2} \).

Obviously, one can obtain improved versions of this important theorem by using Rademacher’s theorem and condition 34.1.3. As mentioned earlier, a function which satisfies 34.1.3 is called locally Lipschitz at \( x \). Here is a simple lemma which is in the spirit of similar lemmas presented in the chapter on Hausdorff measures.
Lemma 34.1.2 Let $U$ be an open set in $\mathbb{R}^n$ and let $h : U \to \mathbb{R}^m$ where $m \geq n$. Let $A \subseteq U$ and let $h$ be locally Lipschitz at every point of $A$. Then if $N \subseteq A$ has Lebesgue measure zero, it follows that $H^n(h(N)) = 0$.

Proof: Let $N_k$ be defined as

$$N_k \equiv \{ x \in N : \text{for some } R_x > 0, |h(z) - h(y)| \leq k |z - y| \text{ for all } y, z \in B(x, R_x) \}$$

Thus $N_k \uparrow N$. Let $\varepsilon > 0$ be given and let $U \supseteq V_k \supseteq N$ be open and $m_n(V_k) < \frac{\varepsilon}{5^m k^n}$. Now fix $\delta > 0$. For $x \in N_k$ let $B(x, 5r_x) \subseteq V_k$ such that $r_x < \min\left(\frac{\delta}{5}, R_x\right)$. By the Vitali covering theorem, there exists a disjoint sequence of these balls, $\{B_i\}_{i=1}^\infty$ such that $\{\hat{B}_i\}_{i=1}^\infty$, the corresponding sequence of balls having the same centers but five times the radius covers $N_k$. Then $\text{diam}(\hat{B}_i) < 2\delta/k$. Hence $\{h(\hat{B}_i)\}_{i=1}^\infty$ covers $h(N_k)$ and $\text{diam}\left(h(\hat{B}_i)\right) < 2\delta$. It follows

$$H^n_{2\delta}(h(N_k)) \leq \sum_{i=1}^\infty \alpha(n) r(h(\hat{B}_i))^n$$

$$\leq \sum_{i=1}^\infty \alpha(n) k^n 5^n r(B_i)^n$$

$$= 5^n k^n \sum_{i=1}^\infty m_n(B_i) \leq 5^n k^n m_n(V_k) < \varepsilon$$

Since $\delta$ was arbitrary, this shows $\mathcal{H}^n(h(N_k)) \leq \varepsilon$. Since $k$ was arbitrary, this shows $\mathcal{H}^n(h(N)) = \lim_{k \to \infty} \mathcal{H}^n(h(N_k)) \leq \varepsilon$. Since $\varepsilon$ is arbitrary, this shows $\mathcal{H}^n(h(N)) = 0$. This proves the lemma.

Now with this lemma, here is one of many possible generalizations of the area formula.

Theorem 34.1.3 Let $U$ be an open set in $\mathbb{R}^n$ and $h : U \to \mathbb{R}^m$. Let $h$ be locally Lipschitz and one to one on $A$, a Lebesgue measurable subset of $U$ and let $g : h(A) \to \mathbb{R}$ be a nonnegative $\mathcal{H}^n$ measurable function. Then

$$x \to (g \circ h)(x) J(x)$$

is Lebesgue measurable and

$$\int_{h(A)} g(y) d\mathcal{H}^n = \int_A g(h(x)) J(x) \, dm_n$$

where $J(x) = \det(U(x)) = \det(Dh(x)^*) Dh(x)^{1/2}$.

Proof: For $x \in A$, there exists a ball, $B_x$ on which $h$ is Lipschitz. By Rademacher’s theorem, $h$ is differentiable a.e. on $B_x$. There is a countable cover...
of $A$ consisting of such balls on which $h$ is Lipschitz. Therefore, $h$ is differentiable on $A_0 \subseteq A$ where $m_n (A \setminus A_0) = 0$. Then by the earlier area formula,

$$\int_{h(A)} g(y) \, d\mathcal{H}^n = \int_{A_0} g(h(x)) \, J(x) \, dm_n$$

By Lemma

$$\int_{h(A)} g(y) \, d\mathcal{H}^n = \int_{h(A_0)} g(y) \, d\mathcal{H}^n = \int_{A_0} g(h(x)) \, J(x) \, dm_n = \int_A g(h(x)) \, J(x) \, dm_n$$

This proves the theorem.

Note how a special case of this occurs when $h$ is one to one and $C^1$. Of course this yields the earlier change of variables formula as a still more special case.

In addition to this, recall the divergence theorem, Theorem 36.6.14 on Page 1028. This theorem was stated for bounded open sets which have a Lipschitz boundary. This definition of Lipschitz boundary involved an assumption that certain Lipschitz mappings had a derivative a.e. Rademacher’s theorem makes this assumption redundant. Therefore, the statement of Theorem 36.6.14 remains valid with the following definition of a Lipschitz boundary.

**Definition 34.1.4** A bounded open set, $U \subseteq \mathbb{R}^n$ is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, $Q_1, \cdots, Q_N$,

$$Q_i = \prod_{j=1}^{n} (a^i_j, b^i_j)$$

such that $\partial U \equiv U \setminus U$ is contained in their union. Also, for each $Q_i$, there exists $k$ and a Lipschitz function, $g_i$ such that $U \cap Q_i$ is of the form

$$\left\{ \mathbf{x} : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} (a^i_j, b^i_j) \times \prod_{j=k+1}^{n} (a^i_j, b^i_j) \text{ and } a^i_k < x_k < b^i_k \right\} \quad (34.1.4)$$

or else of the form

$$\left\{ \mathbf{x} : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} (a^i_j, b^i_j) \times \prod_{j=k+1}^{n} (a^i_j, b^i_j) \text{ and } g_i (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) < x_k < b^i_j \right\} \quad (34.1.5)$$

Also, there exists an open set, $Q_0$ such that $Q_0 \subseteq \overline{U} \subseteq U$ and $\overline{U} \subseteq Q_0 \cup Q_1 \cup \cdots \cup Q_N$. 

CHAPTER 34. THE AREA AND COAREA FORMULAS

34.2 Mappings That Are Not One To One

Next I will consider the case where \( h \) is not necessarily one to one. Recall the major theorem presented earlier on which the proof of the area formula depended, Theorem 26.5.10 on Page 1002. Here it is.

**Theorem 34.2.1** Let \( h : U \to \mathbb{R}^m \) where \( U \) is an open set in \( \mathbb{R}^n \) for \( n \leq m \) and suppose \( h \) is locally Lipschitz at every point of a Lebesgue measurable subset, \( A \) of \( U \). Also suppose that for every \( x \in A \), \( D h(x) \) exists. Then for \( x \in A \),

\[
J(x) = \lim_{r \to 0} \frac{\mathcal{H}^n(h(B(x, r)))}{m_n(B(x, r))},
\]

where \( J(x) \equiv \det(U(x)) = \det(Dh(x)^*Dh(x))^{1/2} \).

The next lemma is a version of Sard’s lemma.

**Lemma 34.2.2** Let \( h : U \to \mathbb{R}^m \) where \( U \) is an open set in \( \mathbb{R}^n \) for \( n \leq m \) and suppose \( h \) is locally Lipschitz at every point of a Lebesgue measurable subset, \( A \) of \( U \). Let

\[
N \equiv \{ x \in A : D h(x) \text{ does not exist} \}
\]

and let

\[
S \equiv \{ x \in A_0 \equiv A \setminus N : J(x) = 0 \}
\]

Then \( \mathcal{H}^n(h(S \cup N)) = 0 \).

**Proof:** By Rademacher’s theorem, \( N \) has measure 0. Therefore, \( \mathcal{H}^n(h(N)) = 0 \) by Lemma 34.1.2.

It remains to show \( \mathcal{H}^n(h(S)) = 0 \). Let \( S_k = B(0, k) \cap S \) for \( k \) a positive integer large enough that \( 5r_x < \min(R_x, 1) \) and if \( r \leq 5r_x \),

\[
\frac{\mathcal{H}^n(h(B(x, r)))}{m_n(B(x, r))} < \varepsilon \frac{5^n}{5^n k^n}, \quad B(x, r) \subseteq B(0, k) \cap U
\]

Then by the Vitali covering theorem, there exists a sequence of disjoint balls of this sort, \( \{B_i\}_{i=1}^\infty \) such that the balls having 5 times the radius but the same center, \( \{\hat{B}_i\}_{i=1}^\infty \) cover \( S_k \). Then \( \{h(\hat{B}_i)\}_{i=1}^\infty \) covers \( h(S_k) \). Then from Lemma 34.2.2

\[
\mathcal{H}^n(h(S_k)) \leq \sum_{i=1}^\infty \mathcal{H}^n(h(\hat{B}_i)) \leq \sum_{i=1}^\infty 5^n \mathcal{H}^n(h(B_i))
\]

\[
\leq \sum_{i=1}^\infty 5^n \frac{\varepsilon}{5^n k^n} m_n(B_i) \leq \frac{\varepsilon}{k^n} m_n(B(0, k)) = \varepsilon \alpha(n)
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows \( \mathcal{H}^n(h(S_k)) = 0 \) and now letting \( k \to 0 \), it follows \( \mathcal{H}^n(h(S)) = 0 \). This proves the lemma.

The following very technical lemma provides the necessary theory to generalize to functions which are not one to one.
Lemma 34.2.3 Let $h : U \rightarrow \mathbb{R}^m$ where $U$ is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $h$ is locally Lipschitz at every point of a Lebesgue measurable subset, $A$ of $U$. Let

$$ N \equiv \{ x \in A : Dh(x) \text{ does not exist} \} $$

and let

$$ S \equiv \{ x \in A_0 = A \setminus N : J(x) = 0 \} $$

Let $B = A \setminus (S \cup N)$. Then there exist measurable disjoint sets, $\{E_i\}_{i=1}^{\infty}$ such that $A = \bigcup_{i=1}^{\infty} E_i$ and $h$ is one to one on $E_i$. Furthermore, $h^{-1}$ is Lipschitz on $h(E_i)$.

**Proof:** Let $C$ be a dense countable subset of $B$ and let $F$ be a countable dense subset of the invertible elements of $\mathcal{L} (\mathbb{R}^n, \mathbb{R}^m)$. For $i$ a positive integer and $T \in F, c \in C$

$$ E(c,T,i) \equiv \left\{ b \in B \left( c, \frac{1}{i} \right) \cap B : (a), (b) \text{ both hold} \right\} $$

where $(a), (b)$ are given by

$$ \frac{2}{3} |Tv| \leq |U(b)v| \text{ for all } v \quad (a) $$

$$ |h(a) - h(b) - Dh(b)(a-b)| \leq \frac{1}{2} |T(a-b)| \quad (b) $$

for all $a \in B(b, \frac{2}{i})$.

First I will show these sets, $E(c,T,i)$ cover $B$ and that they are measurable sets. To begin with consider the measurability question. Inequality $(a)$ is the same as saying

$$ \frac{2}{3} |Tv| \leq |Dh(b)v| \text{ for all } v $$

which is the same as saying

$$ \frac{2}{3} |v| \leq |Dh(b)T^{-1}v| \text{ for all } v.$$
Let \( \{v_i\} \) denote a dense countable subset of \( \mathbb{R}^n \). Letting

\[
S_i \equiv \left\{ b : \frac{2}{3} |v_i| \leq |Dh(b) T^{-1} v_i| \right\}
\]

it follows easily that \( S_i \) is measurable because the component functions of the matrix of \( Dh(b) \) are limits of difference quotients of continuous functions so they are Borel measurable. (Note that if \( B \) were Borel, then \( S_i \) would also be Borel.) Now by continuity,

\[
\bigcup_{i=1}^{\infty} S_i = \left\{ b : \frac{2}{3} |v| \leq |Dh(b) T^{-1} v| \text{ for all } v \right\}
\]

and so this set is measurable also. Inequality \((b)\) also determines a measurable set by similar reasoning. It is the same as saying that for all \(|v| < 2/i\),

\[
|h(b + v) - h(b) - Dh(b)(v)| \leq \frac{1}{2} |T(v)|
\]

Use \( \{v_i\} \) a countable dense subset of \( B(0, 2/i) \) in a similar fashion to \((a)\).

Next I need to show these sets cover \( B \). Let \( x \in B \). Then pick \( c_i \in B(\bar{x}, \frac{1}{i}) \) and \( T_i \in B(U(x), \frac{1}{i}) \). I need to show that \( x \in E(c_i, T_i, i) \) for \( i \) large enough. For \( i \) large enough, \( ||T_i U(x)^{-1}|| < \frac{3}{2} \). Therefore, for such \( i \)

\[
|T_i U(x)^{-1}(v)| < \frac{3}{2} |v|
\]

for all \( v \) and so

\[
|T_i w| < \frac{3}{2} |U(x) w|
\]

for all \( w \). Next consider \((b)\). An equivalent norm is \( v \rightarrow |U(x) v| \) and so, for \( i \) large enough,

\[
|h(a) - h(x) - Dh(x)(a - x)| \leq \frac{1}{8} |U(x)(a - x)| \quad (34.2.10)
\]

whenever \( |a - x| < 2/i \). Now also, for \( i \) large enough, \( ||U(x) T_i^{-1}|| < 4 \) and so for all \( w \),

\[
|U(x) T_i^{-1} w| < 4 |w|
\]

which implies

\[
|U(x) v| < 4 |T_i v|
\]

Applying this in \((34.2.10)\) yields

\[
|h(a) - h(x) - Dh(x)(a - x)| \leq \frac{1}{2} |T_i (a - x)|
\]

with implies \( x \in E(c_i, T_i, i) \).
Next I need to show \( h \) is one to one on \( E(c,T,i) \). Suppose \( b_1, b_2 \in E(c,T,i) \).

From (b) and (a),

\[
|T(b_2 - b_1)| \leq \frac{3}{2} |U(b_1)(b_2 - b_1)| = \frac{3}{2} |Dh(b_1)(b_2 - b_1)|
\]

which is a contradiction unless \( b_2 = b_1 \).

There are clearly countably many \( E(c,T,i) \). Denote them as \( \{F_i\}_{i=1}^{\infty} \).

Then let \( E_1 = F_1 \) and if \( E_1, \cdots, E_m \) have been chosen, let

\[
E_{m+1} = F_{m+1} \setminus \bigcup_{i=1}^{m} E_i.
\]

Thus the \( E_i \) are disjoint measurable sets whose union is \( B \) and \( h \) is one to one on each \( E_i \).

Now consider one of the \( E_i \). This is a subset of some \( E(c,T,i) \). Let \( a, b \in E_i \).

Then using (a) and (b),

\[
|T(a - b)| \leq \frac{3}{2} |U(b)(a - b)|
\]

\[
= \frac{3}{2} |Dh(b)(a - b)|
\]

\[
\leq \frac{3}{2} |h(a) - h(b)| + \frac{3}{4} |T(a - b)|.
\]

Hence

\[
\frac{1}{4} |T(a - b)| \leq \frac{3}{2} |h(a) - h(b)|
\]

Since \( v \to |Tv| \) is an equivalent norm, there exists some \( r > 0 \) such that \( |Tv| \geq r |v| \) for all \( v \). Therefore,

\[
|a - b| \leq \frac{6}{r} |h(a) - h(b)|.
\]

In other words,

\[
|h^{-1}(h(a)) - h^{-1}(h(b))| = |a - b| \leq \frac{6}{r} |h(a) - h(b)|.
\]

which completes the proof.

With these lemmas, here is the main theorem which is a generalization of Theorem 34.1.4. First remember that from Lemma 36.5.4 on Page 1276 a locally Lipschitz function maps Lebesgue measurable sets to Hausdorff measurable sets.

**Theorem 34.2.4** Let \( U \) be an open set in \( \mathbb{R}^n \) and \( h : U \to \mathbb{R}^m \). Let \( h \) be locally Lipschitz on \( A \), a Lebesgue measurable subset of \( U \) and let \( g : h(A) \to \mathbb{R} \) be a nonnegative \( \mathcal{H}^n \) measurable function. Also let

\[
\#(y) \equiv \text{Number of elements of } h^{-1}(y)
\]
Then $\#$ is $\mathcal{H}^n$ measurable,

$$x \to (g \circ h)(x) J(x)$$

is Lebesgue measurable, and

$$\int_{h(A)} \#(y) g(y) d\mathcal{H}^n = \int_A g(h(x)) J(x) dm_n$$

where $J(x) = \det(U(x)) = \det(Dh(x)^* Dh(x))^{1/2}$.

**Proof:** Let $B = A \setminus (S \cup N)$ where $S$ is the set of points where $J(x) = 0$ and $N$ is the set of points, $x$ of $A$ where $Dh(x)$ does not exist. Also from Lemma 72.2.3 there exists $\{E_i\}_{i=1}^\infty$, a sequence of disjoint measurable sets whose union equals $B$ such that $h$ is one to one on each $E_i$. Then from Theorem 72.1.3

$$\int_A g(h(x)) J(x) dm_n$$

$$= \int_B g(h(x)) J(x) dm_n = \sum_{i=1}^\infty \int_{h(E_i)} g(h(x)) J(x) dm_n$$

$$= \sum_{i=1}^\infty \int_{h(E_i)} g(y) d\mathcal{H}^n = \int_{h(B)} \left( \sum_{i=1}^\infty \mathcal{X}_{h(E_i)}(y) \right) g(y) d\mathcal{H}^n. \quad (34.2.11)$$

Now $\#(y) = (\sum_{i=1}^\infty \mathcal{X}_{h(E_i)}(y))$ on $h(B)$ and $\#$ differs from this $\mathcal{H}^n$ measurable function only on $h(S \cup N)$, which by Lemma 72.2.3 is a set of $\mathcal{H}^n$ measure zero. Therefore, $\#$ is $\mathcal{H}^n$ measurable and the last term of (34.2.11) equals

$$\int_{h(A)} \left( \sum_{i=1}^\infty \mathcal{X}_{h(E_i)}(y) \right) g(y) d\mathcal{H}^n = \int_{h(A)} \#(y) g(y) d\mathcal{H}^n.$$

This proves the theorem.

### 34.3 The Coarea Formula

The coarea formula involves a function, $h$ which maps a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ where $m \leq n$ instead of $m \geq n$ as in the area formula. The symbol, $\text{Lip}(h)$ will denote the Lipschitz constant for $h$.

It is possible to obtain the coarea formula as a computation involving the area formula and some simple linear algebra and this is the approach taken here. To begin with, here is the necessary linear algebra.

**Theorem 34.3.1** Let $A$ be an $m \times n$ matrix and let $B$ be an $n \times m$ matrix for $m \leq n$. Then

$$p_{BA}(t) = t^{n-m}p_{AB}(t),$$

so the eigenvalues of $BA$ and $AB$ are the same including multiplicities except that $BA$ has $n-m$ extra zero eigenvalues.
34.3. THE COAREA FORMULA

**Proof:** Use block multiplication to write
\[
\begin{pmatrix}
AB & 0 \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix}
= 
\begin{pmatrix}
AB & ABA \\
B & BA
\end{pmatrix}
\]
\[
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
B & BA
\end{pmatrix}
= 
\begin{pmatrix}
AB & ABA \\
B & BA
\end{pmatrix}.
\]

Therefore,
\[
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix}^{-1}
\begin{pmatrix}
AB & 0 \\
B & 0
\end{pmatrix}
\begin{pmatrix}
I & A \\
0 & I
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 \\
B & BA
\end{pmatrix}
\]

It follows that \( \begin{pmatrix}
0 & 0 \\
B & BA
\end{pmatrix} \) and \( \begin{pmatrix}
AB & 0 \\
B & 0
\end{pmatrix} \) have the same characteristic polynomials because the two matrices are similar. Thus
\[
\det \begin{pmatrix}
tI - AB & 0 \\
-B & tI
\end{pmatrix} = \det \begin{pmatrix}
tI & 0 \\
-B & tI - BA
\end{pmatrix}
\]

and so noting that \( BA \) is an \( n \times n \) matrix and \( AB \) is an \( m \times m \) matrix,
\[
t^m \det (tI - BA) = t^n \det (tI - AB)
\]

and so \( \det (tI - BA) = p_{BA}(t) = t^{n-m} \det (tI - AB) = t^{n-m} p_{AB}(t) \). This proves the theorem.

The following corollary is what will be used to prove the coarea formula.

**Corollary 34.3.2** Let \( A \) be an \( m \times n \) matrix. Then
\[
\det (I + AA^*) = \det (I + A^*A) .
\]

**Proof:** Assume \( m \leq n \). From Theorem [34.3.1] \( AA^* \) and \( A^*A \) have the eigenvalues, \( \lambda_1, \ldots, \lambda_m \), necessarily nonnegative, with the same multiplicities and some zero eigenvalues which have differing multiplicities. The eigenvalues, \( \lambda_1, \ldots, \lambda_m \) are the zeros of \( p_{AA^*}(t) \). Thus there is an orthogonal transformation, \( P \) such that
\[
A^*A = P
\begin{pmatrix}
\lambda_1 & & \\
& \ddots & \\
& & \lambda_m
\end{pmatrix}
P^*.
\]

Therefore,
\[
I + A^*A = P
\begin{pmatrix}
\lambda_1 + 1 & & \\
& \ddots & \\
& & \lambda_m + 1
\end{pmatrix}
P^*.
\]
and so
\[ \det (I + A^*A) = \det \begin{pmatrix} \lambda_1 + 1 & 0 \\ \vdots & \ddots \\ 0 & \lambda_m + 1 \end{pmatrix} = \det (I + AA^*). \]

This proves the corollary.

The other main ingredient is the following version of the chain rule.

**Theorem 34.3.3** Let $h$ and $g$ be locally Lipschitz mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $h(g(x)) = x$ on $A$, a Lebesgue measurable set. Then for a.e. $x \in A$, $Dg(h(x))$, $Dh(x)$, and $D(g \circ h)(x)$ all exist and
\[ I = D(g \circ h)(x) = Dg(h(x))Dh(x). \]

The proof of this theorem is based on the following lemma.

**Lemma 34.3.4** If $h : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, then if $h(x) = 0$ for all $x \in A$, then $\det(Dh(x)) = 0$ a.e.

**Proof:** By the case of the Area formula which involves mappings which are not one to one, $0 = \int_{\emptyset} \#(y) \, dy = \int_A |\det(Dh(x))| \, dx$ and so $\det(Dh(x)) = 0$ a.e.

**Proof of the theorem:** On $A$, $g(h(x)) - x = 0$ and so by the lemma, there exists a set of measure zero, $N_1$ such that if $x \notin N_1$, $D(g \circ h)(x) - I = 0$. Let $M$ be the set of measure zero of points in $h(\mathbb{R}^n)$ where $g$ fails to be differentiable and let $N_2 \equiv g(M) \cap A$, also a set of measure zero because locally Lipschitz maps take sets of measure zero to sets of measure zero. Finally let $N_3$ be the set of points where $h$ fails to be differentiable. Then if $x \notin N_1 \cup N_2 \cup N_3$, the chain rule implies $I = D(g \circ h)(x) = Dg(h(x))Dh(x)$. This proves the theorem.

**Lemma 34.3.5** Let $h : \mathbb{R}^p \to \mathbb{R}^m$ be Lipschitz continuous and $\delta > 0$. Then if $A \subseteq \mathbb{R}^p$ is either open or compact,
\[ y \to H_\delta^p(A \cap h^{-1}(y)) \]
is Borel measurable.

**Proof:** Suppose first that $A$ is compact and suppose for $\delta > 0$,
\[ H_\delta^p(A \cap h^{-1}(y)) < t \]
Then there exist sets $S_i$, satisfying
\[ \operatorname{diam}(S_i) < \delta, A \cap h^{-1}(y) \subseteq \bigcup_{i=1}^{\infty} S_i, \]
and
\[ \sum_{i=1}^{\infty} \alpha(s)(r(S_i))^s < t. \]
I claim these sets can be taken to be open sets. Choose \( \lambda > 1 \) but close enough to 1 that
\[
\sum_{i=1}^{\infty} \alpha(s) (\lambda r(S_i))^s < t
\]
Replace \( S_i \) with \( S_i + B(0, \eta_i) \) where \( \eta_i \) is small enough that
\[
\text{diam}(S_i) + 2\eta_i < \lambda \text{diam}(S_i)
\]
and so \( r(S_i + B(0, \eta_i)) \leq \lambda r(S_i) \). Thus
\[
\sum_{i=1}^{\infty} \alpha(s) r(S_i + B(0, \eta_i))^s < t.
\]

Hence you could replace \( S_i \) with \( S_i + B(0, \eta_i) \) and so one can assume the sets \( S_i \) are open.

**Claim:** If \( z \) is close enough to \( y \), then \( A \cap h^{-1}(z) \subseteq \bigcup_{i=1}^{\infty} S_i \).

**Proof:** If not, then there exists a sequence \( \{z_k\} \) such that
\[
z_k \to y, \quad x_k \in (A \cap h^{-1}(z_k)) \setminus \bigcup_{i=1}^{\infty} S_i.
\]
By compactness of \( A \), there exists a subsequence still denoted by \( k \) such that
\[
z_k \to y, \quad x_k \to x \in A \setminus \bigcup_{i=1}^{\infty} S_i.
\]
Hence
\[
h(x) = \lim_{k \to \infty} h(x_k) = \lim_{k \to \infty} z_k = y.
\]
But \( x \notin \bigcup_{i=1}^{\infty} S_i \) contrary to the assumption that \( A \cap h^{-1}(y) \subseteq \bigcup_{i=1}^{\infty} S_i \).
It follows from this claim that whenever \( z \) is close enough to \( y \),
\[
\mathcal{H}_\delta^s \left( A \cap h^{-1}(z) \right) < t.
\]
This shows
\[
\{z \in \mathbb{R}^p : \mathcal{H}_\delta^s \left( A \cap h^{-1}(z) \right) < t\}
\]
is an open set and so \( y \to \mathcal{H}_\delta^s \left( A \cap h^{-1}(y) \right) \) is Borel measurable whenever \( A \) is compact. Now let \( V \) be an open set and let
\[
A_k \uparrow V, \ A_k \text{ compact.}
\]
Then
\[
\mathcal{H}_\delta^s (V \cap h^{-1}(y)) = \lim_{k \to \infty} \mathcal{H}_\delta^s (A_k \cap h^{-1}(y))
\]
so \( y \to \mathcal{H}_\delta^s (V \cap h^{-1}(y)) \) is Borel measurable for all \( V \) open. This proves the lemma.
Lemma 34.3.6 Let $h : \mathbb{R}^p \to \mathbb{R}^m$ be Lipschitz continuous. Suppose $A$ is either open or compact in $\mathbb{R}^p$. Then $y \to H^s (A \cap h^{-1}(y))$ is also Borel measurable and

$$\int_{\mathbb{R}^m} H^s (A \cap h^{-1}(y)) \, dy \leq 2^m (\text{Lip} (h))^m \frac{\alpha(s) \alpha(m)}{\alpha(s + m)} H^{s + m} (A)$$

In particular, if $s = n - m$ and $p = n$

$$\int_{\mathbb{R}^m} H^{n - m} (A \cap h^{-1}(y)) \, dy \leq 2^m (\text{Lip} (h))^m \frac{\alpha(n - m) \alpha(m)}{\alpha(n)} m_n (A)$$

Proof: From Lemma 34.3.5 $y \to H^s (A \cap h^{-1}(y))$ is Borel measurable for each $\delta > 0$. Without loss of generality, $H^{s + m} (A) < \infty$. Now let $B_i$ be closed sets with $\text{diam} (B_i) < \delta, A \subseteq \bigcup_{i=1}^{\infty} B_i$, and

$$H^{s + m} (A) + \varepsilon > \sum_{i=1}^{\infty} \alpha(s + m) r(B_i)^{s + m}.$$ 

Note each $B_i$ is compact so $y \to H^s (B_i \cap h^{-1}(y))$ is Borel measurable. Thus

$$\int_{\mathbb{R}^m} H^s (A \cap h^{-1}(y)) \, dy$$

$$\leq \int_{\mathbb{R}^m} \sum_i H^s (B_i \cap h^{-1}(y)) \, dy$$

$$= \sum_i \int_{\mathbb{R}^m} H^s (B_i \cap h^{-1}(y)) \, dy$$

$$\leq \sum_i \int_{h(B_i)} H^s (B_i) \, dy$$

$$= \sum_i m_m (h(B_i)) H^s (B_i)$$

$$\leq \sum_i (\text{Lip} (h))^m 2^m \alpha(m) r(B_i)^m \alpha(s) r(B_i)^s$$

$$= (\text{Lip} (h))^m \frac{\alpha(m) \alpha(s)}{\alpha(m + s)} 2^m \sum_i \alpha(s + m) r(B_i)^{m + s}$$

$$\leq (\text{Lip} (h))^m \frac{\alpha(m) \alpha(s)}{\alpha(m + s)} 2^m (H^{s + m}_\delta (A) + \varepsilon)$$

Since $\varepsilon$ is arbitrary,

$$\int_{\mathbb{R}^m} H^s (A \cap h^{-1}(y)) \, dy \leq (\text{Lip} (h))^m \frac{\alpha(m) \alpha(s)}{\alpha(m + s)} 2^m H^{s + m}_\delta (A)$$

Taking a limit as $\delta \to 0$ this proves the lemma.

Next I will show that whenever $A$ is Lebesgue measurable,

$$y \to H^{n - m} (A \cap h^{-1}(y))$$

is $m_m$ measurable and the above estimate holds.
34.3. THE COAREA FORMULA

Lemma 34.3.7 Let $A$ be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $h : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then

$$ y \to \mathcal{H}^{n-m}(A \cap h^{-1}(y)) $$

is Lebesgue measurable. Furthermore, for all $A$ Lebesgue measurable,

$$ \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap h^{-1}(y)) \, dy \leq 2^m (\text{Lip}(h))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(A) $$

Proof: Let $A$ be a bounded Lebesgue measurable set in $\mathbb{R}^n$. Then by inner and outer regularity of Lebesgue measure there exists an increasing sequence of compact sets, $\{K_k\}$ contained in $A$ and a decreasing sequence of open sets, $\{V_k\}$ containing $A$ such that $m_n(V_k \setminus K_k) < 2^{-k}$. Thus $m_n(V_k) \leq m_n(A) + 1$. By Lemma 34.3.7,

$$ \int_{\mathbb{R}^m} \mathcal{H}^{n-m}_\delta(V_k \cap h^{-1}(y)) \, dy \leq 2^m (\text{Lip}(h))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (m_n(A) + 1). $$

Then

$$ \mathcal{H}^{n-m}_\delta(K_k \cap h^{-1}(y)) \leq \mathcal{H}^{n-m}_\delta(A \cap h^{-1}(y)) \leq \mathcal{H}^{n-m}_\delta(V_k \cap h^{-1}(y)) $$

(34.3.12)

By Lemma 34.3.7,

$$ = \int_{\mathbb{R}^m} (\mathcal{H}^{n-m}_\delta(V_k \cap h^{-1}(y)) - \mathcal{H}^{n-m}_\delta(K_k \cap h^{-1}(y))) \, dy $$

$$ = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}_\delta((V_k \setminus V_k) \cap h^{-1}(y)) \, dy $$

$$ \leq 2^m (\text{Lip}(h))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(V_k \setminus K_k) $$

$$ < 2^m (\text{Lip}(h))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} 2^{-k} $$

Let the Borel measurable functions, $g$ and $f$ be defined by

$$ g(y) \equiv \lim_{k \to \infty} \mathcal{H}^{n-m}_\delta(V_k \cap h^{-1}(y)), \quad f(y) \equiv \lim_{k \to \infty} \mathcal{H}^{n-m}_\delta(K_k \cap h^{-1}(y)) $$

It follows from the dominated convergence theorem and (34.3.12) that

$$ f(y) \leq \mathcal{H}^{n-m}_\delta(A \cap h^{-1}(y)) \leq g(y) $$

and

$$ \int_{\mathbb{R}^m} (g(y) - f(y)) \, dy = 0. $$

By completeness of $m_m$, this establishes $\mathcal{Y} \to \mathcal{H}^{n-m}_\delta(A \cap h^{-1}(y))$ is Lebesgue measurable. Then by Lemma 34.3.7 again,

$$ \int_{\mathbb{R}^m} \mathcal{H}^{n-m}_\delta(A \cap h^{-1}(y)) \, dy \leq 2^m (\text{Lip}(h))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(A). $$
Letting $\delta \to 0$ and using the monotone convergence theorem yields the desired inequality for $H^{n-m}(A \cap h^{-1}(y))$.

The case where $A$ is not bounded can be handled by considering $A_r = A \cap B(0, r)$ and letting $r \to \infty$. This proves the lemma.

By fussing with the isodiametric inequality one can remove the factor of $2^n$ in the above inequalities obtaining much more attractive formulas. This is done in [13]. See also [17] which follows [13] and [17]. This last reference probably has the most complete treatment of these topics.

With these lemmas, it is now possible to give a proof of the coarea formula.

Define $A(n, m)$ as all possible ordered lists of $m$ numbers taken from $\{1, 2, \cdots, n\}$.

**Lemma 34.3.8** Let $A$ be a measurable set in $\mathbb{R}^n$ and let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map where $m \leq n$ which is differentiable at every point of $A$ and for which

$$Jh(x) \equiv \det(Dh(x)Dh(x)^*)^{1/2} \neq 0.$$ 

Then the following formula holds along with all measurability assertions needed for it to make sense.

$$\int_{\mathbb{R}^m} H^{n-m}(A \cap h^{-1}(y)) \, dy = \int_A Jh(x) \, dx \quad (34.3.13)$$

**Proof:** For $x \in \mathbb{R}^n$, and $i \in \Lambda(n, m)$, with $i = (i_1, \cdots, i_m)$, define $x_i \equiv (x_{i_1}, \cdots, x_{i_m})$, and $\pi_i x \equiv x_i$. Also for $i \in \Lambda(n, m)$, let $i_{\text{c}} \in \Lambda(n, n-m)$ consist of the remaining indices taken in order. For $h : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$, define $Jh(x) \equiv \det(Dh(x)Dh(x)^*)^{1/2}$. For each $i \in \Lambda(n, m)$, define

$$h_i(x) \equiv \begin{pmatrix} h(x) \\ x_{i_{\text{c}}} \end{pmatrix}.$$ 

By Lemma 34.3.8, there exist disjoint measurable sets $\{F^i_j\}_{j=1}^\infty$ such that $h_i$ is one to one on $F^i_j$, $(h_i)^{-1}$ is Lipschitz on $h_i(F^i_j)$, and

$$\bigcup_{j=1}^\infty F^i_1 = \{x \in A : \det(Dh_i(x)) \neq 0\}.$$ 

For $x \in A$, $\det(D_{x_i}h(x)) \neq 0$ for some $i \in \Lambda(n, m)$. But $\det(D_{x_i}h(x)) = \det(Dh_i(x))$ and so $x \in F^i_j$ for some $i$ and $j$. Hence

$$\cup_{i,j} F^i_j = A.$$ 

Now let $\{E^i_k\}$ be measurable sets such that $E_k^i \subseteq F^i_k$ for some $k$, the sets are disjoint, and their union coincides with $\cup_{i,j} F^i_j$. Then

$$\int_A Jh(x) \, dx = \sum_{i \in \Lambda(n, m)} \sum_{j=1}^\infty \int_{E^i_j \cap A} \det(Dh(x)Dh(x)^*)^{1/2} \, dx. \quad (34.3.14)$$
Let \( g : \mathbb{R}^n \to \mathbb{R}^n \) be a Lipschitz extension of \((h^i)^{-1}\) so \( g \circ h^i(x) = x \) for all \( x \in E_j^i \). First, using Theorem 34.3.15, and the fact that Lipschitz mappings take sets of measure zero to sets of measure zero, replace \( E_j^i \) with a measurable set, \( E_j^i \subseteq E_j^i \) such that \( E_j^i \setminus E_j^i \) has measure zero and

\[
Dh^i(g(y)) Dg(y) = I
\]
on \( h^i(E_j^i) \). Changing the variables using the area formula, the expression in \((34.3.15)\) equals

\[
\int_A Jh(x) \, dx = 
\sum_{i \in \Lambda(n,n)} \sum_{j=1}^{\infty} \int_{h(E_j^i \cap A)} \det \left(Dh(g(y)) Dg(y) \right)^{1/2} |\det Dh^i(g(y))|^{-1} \, dy.
\]

Note the integrands are all Borel measurable functions because they are continuous functions of the entries of matrices which entries come from taking limits of difference quotients of continuous functions. Thus,

\[
\int_{E_j^i \cap A} \det \left(Dh(x) Dh(x)^* \right)^{1/2} \, dx = 
\int_{\mathbb{R}^n} X_h'(E_j^i \cap A) (y) \det \left(Dh(g(y)) Dg(y) \right)^{1/2} |\det Dh^i(g(y))|^{-1} \, dy
\]

\[
= \int_{\mathbb{R}^n} \int_{\pi_{i\cdot}^{-1}(y) \cap E_j^i \cap A} \det \left(Dh(g(y)) Dg(y) \right)^{1/2} |\det Dh_{x_{i\cdot}}(g(y))|^{-1} \, dy_2 dy_1
\]

where \( y_1 = h(x) \) and \( y_2 = x_{i\cdot} \). Thus

\[
y_2 = \pi_{i\cdot} g(y) = \pi_{i\cdot} h^i(x) = x_{i\cdot}.
\]

Now consider the inner integral in \((34.3.16)\) in which \( y_1 \) is fixed. The integrand equals

\[
\det \left[ \left( \begin{array}{cc} D_{x_{i\cdot}} h(g(y)) & D_{x_{i\cdot}} h^*(g(y)) \\ D_{x_{i\cdot}} h^*(g(y)) & D_{x_{i\cdot}} h^*(g(y)) \end{array} \right) \right]^{1/2} |\det Dh_{x_{i\cdot}}(g(y))|^{-1}.
\]

I want to massage the above expression slightly. Since \( y_1 \) is fixed, and \( y_1 = h(\pi_{i\cdot} g(y), \pi_{i\cdot} g(y)) = h(g(y)) \), it follows from \((34.3.17)\) that

\[
0 = D_{x_{i\cdot}} h(g(y)) D_{y_{i\cdot}} \pi_{i\cdot} g(y) + D_{x_{i\cdot}} h^*(g(y)) D_{y_{i\cdot}} \pi_{i\cdot} g(y)
= D_{x_{i\cdot}} h(g(y)) D_{y_{i\cdot}} \pi_{i\cdot} g(y) + D_{x_{i\cdot}} h^*(g(y))
\]
Letting $A \equiv D_{x_i} h(g(y))$ and $B \equiv D_{y_i} \pi_i g(y)$ and using the above formula, is of the form

$$\det \left[ \left( \begin{array}{cc} A & -AB \\ -B^* A^* \\ \end{array} \right) \right]^{1/2} |\det A|^{-1}$$

$$= \det [AA^* + ABB^* A^*]^{1/2} |\det A|^{-1}$$

$$= \det [A(I + BB^*) A^*]^{1/2} |\det A|^{-1}$$

$$= (\det(A) \det(A^*))^{1/2} \det (I + BB^*)^{1/2} |\det A|^{-1}$$

$$= \det (I + BB^*)^{1/2},$$

which, by Corollary 34.3.2, equals $\det (I + B^* B)^{1/2}$. (Note the size of the identity changes in these two expressions, the first being an $m \times m$ matrix and the second being a $n - m \times n - m$ matrix.)

By 34.3.17 $\pi_1 g(y) = y_2$ and so,

$$\det (I + B^* B)^{1/2} = \det \left[ \left( \begin{array}{cc} B^* & I \\ \end{array} \right) \right]^{1/2}$$

$$= \det \left[ \left( \begin{array}{cc} D_{y_2} \pi_1 g(y)^* & D_{x_2} \pi_1 g(y) \\ D_{y_2} \pi_1 g(y)^* & D_{x_2} \pi_1 g(y) \\ \end{array} \right) \right]^{1/2}$$

$$= \det \left( D_{y_2} g(y)^* D_{y_2} g(y) \right)^{1/2}.$$

Therefore, 34.3.16 reduces to

$$\int_{E_j \cap A} \det \left( D h(x) D h(x)^* \right)^{1/2} dx =$$

$$\int_{\mathbb{R}^m} \int_{\pi_i (h^{-1}(y)) \cap E_j \cap A} \det \left( D_{y_2} g(y)^* D_{y_2} g(y) \right)^{1/2} dy dy_1. \quad (34.3.19)$$

By the area formula applied to the inside integral, this integral equals

$$H^{n-m} \left( h^{-1}(y_1) \cap \bar{E}_j \cap A \right)$$

and so

$$\int_{E_j \cap A} \det \left( D h(x) D h(x)^* \right)^{1/2} dx =$$

$$\int_{\mathbb{R}^m} H^{n-m} \left( h^{-1}(y_1) \cap \bar{E}_j \cap A \right) dy_1.$$

Using Lemma 34.3.1, along with the inner regularity of Lebesgue measure, $\bar{E}_j^i$ can be replaced with $E_j^i$. Therefore, summing the terms over all $i$ and $j$,

$$\int_A \det \left( D h(x) D h(x)^* \right)^{1/2} dx = \int_{\mathbb{R}^m} H^{n-m} \left( h^{-1}(y) \cap A \right) dy.$$
This proves the lemma.

Now the following is the coarea formula.

**Corollary 34.3.9** Let $A$ be a measurable set in $\mathbb{R}^n$ and let $h : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map where $m \leq n$. Then the following formula holds along with all measurability assertions needed for it to make sense.

\[
\int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left( A \cap h^{-1}(y) \right) dy = \int_A Jh(x) \, dx \tag{34.3.20}
\]

where $Jh(x) \equiv \det \left(Dh(x) \cdot Dh(x)^*\right)^{1/2}$.

**Proof:** By Lemma 34.3.7 again, this formula is true for all measurable $A \subseteq \mathbb{R}^n \setminus S$. It remains to verify the formula for all measurable sets, $A$, whether or not they intersect $S$.

Consider the case where

\[
A \subseteq S \equiv \{ x : J(Dh(x)) = 0 \}.
\]

Let $A$ be compact so that by Lemma 34.3.6, $y \mapsto \mathcal{H}^{n-m} \left( A \cap h^{-1}(y) \right)$ is Borel. For $\varepsilon > 0$, define $k, p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ by

\[
k(x, y) = h(x) + \varepsilon y, \quad p(x, y) = y.
\]

Then

\[
Dk(x, y) = (Dh(x), \varepsilon I) = (UR, \varepsilon I)
\]

where the dependence of $U$ and $R$ on $x$ has been suppressed. Thus

\[
Jk^2 = \det (UR, \varepsilon I) \left( \frac{R^*U}{\varepsilon I} \right) = \det (U^2 + \varepsilon^2 I)
\]

\[
= \det (Q^*DQQ^*DQ + \varepsilon^2 I) = \det (D^2 + \varepsilon^2 I)
\]

\[
= \prod_{i=1}^{m} \left( \lambda_i^2 + \varepsilon^2 \right) \in [\varepsilon^{2m}, C^2 \varepsilon^2] \tag{34.3.21}
\]

since one of the $\lambda_i$ equals 0. All the eigenvalues must be bounded independent of $x$, since $\|Dh(x)\|$ is bounded independent of $x$ due to the assumption that $h$ is Lipschitz. Since $Jk \neq 0$, the first part of the argument implies

\[
\varepsilon Cm_{n+m} \left( A \times B(0,1) \right) \geq \int_{A \times B(0,1)} |Jk| \, dm_{n+m}
\]

\[
= \int_{\mathbb{R}^m} \mathcal{H}^n \left( k^{-1}(y) \cap A \times B(0,1) \right) dy
\]

Which by Lemma 34.3.8

\[
\geq C_{nm} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left( k^{-1}(y) \cap p^{-1}(w) \cap A \times B(0,1) \right) \, dw \, dy \tag{34.3.22}
\]
where $C_{nm} = \frac{\alpha(n)}{\alpha(n-m)\alpha(m)}$.

Claim:

\[
\mathcal{H}^{n-m}(k^{-1}(y) \cap p^{-1}(w) \cap A \times B(0,1)) \\
\geq \lambda_{B(0,1)}(w) \mathcal{H}^{n-m}(h^{-1}(y - \varepsilon w) \cap A).
\]

Proof of the claim: If $w \notin \overline{B(0,1)}$, there is nothing to prove so assume $w \in \overline{B(0,1)}$. For such $w$,

\[
(x, w_1) \in k^{-1}(y) \cap p^{-1}(w) \cap A \times B(0,1)
\]

if and only if $h(x) + \varepsilon w_1 = y$, $w_1 = w$, and $x \in A$, if and only if

\[
(x, w_1) \in h^{-1}(y - \varepsilon w) \cap A \times \{w\}.
\]

Therefore for $w \in \overline{B(0,1)}$,

\[
\mathcal{H}^{n-m}(k^{-1}(y) \cap p^{-1}(w) \cap A \times B(0,1)) \\
\geq \mathcal{H}^{n-m}(h^{-1}(y - \varepsilon w) \cap A \times \{w\}) = \mathcal{H}^{n-m}(h^{-1}(y - \varepsilon w) \cap A).
\]

(Actually equality holds in the claim.) From the claim, (34.3.24) is at least as large as

\[
C_{nm} \int_{R^m} \int_{B(0,1)} \mathcal{H}^{n-m}(h^{-1}(y - \varepsilon w) \cap A) \, dw \, dy \quad (34.3.23)
\]

\[
= C_{nm} \int_{B(0,1)} \int_{R^m} \mathcal{H}^{n-m}(h^{-1}(y - \varepsilon w) \cap A) \, dy \, dw
\]

\[
= \frac{\alpha(n)}{\alpha(n-m)} \int_{R^m} \mathcal{H}^{n-m}(h^{-1}(y) \cap A) \, dy. \quad (34.3.24)
\]

The use of Fubini’s theorem is justified because the integrand is Borel measurable.

Now by (34.3.24), it follows since $\varepsilon > 0$ is arbitrary,

\[
\int_{R^m} \mathcal{H}^{n-m}(A \cap h^{-1}(y)) \, dy = 0 = \int_{A} Jh(x) \, dx.
\]

Since this holds for arbitrary compact sets in $S$, it follows from Lemma and inner regularity of Lebesgue measure that the equation holds for all measurable subsets of $S$. This completes the proof of the coarea formula. There is a simple corollary to this theorem in the case of locally Lipschitz maps.

Corollary 34.3.10 Let $h : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$ and $h$ is locally Lipschitz. Then the Coarea formula, (34.3.13), holds for $h$. 
Proof: The assumption that \( h \) is locally Lipschitz implies that for each \( r > 0 \) it follows \( h \) is Lipschitz on \( B(0, r) \). To see this, cover the compact set, \( B(0, r) \) with finitely many balls on which \( h \) is Lipschitz.

Let \( A \subseteq B(0, r) \) and let \( h_r \) be Lipschitz with 
\[
h(x) = h_r(x)
\]
for \( x \in B(0, r + 1) \). Then
\[
\int_A J(Dh(x)) \, dx = \int_A J(Dh_r(x)) \, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap h_r^{-1}(y)) \, dy
\]
\[
= \int_{h_r(A)} \mathcal{H}^{n-m}(A \cap h_r^{-1}(y)) \, dy = \int_{h(A)} \mathcal{H}^{n-m}(A \cap h^{-1}(y)) \, dy
\]
\[
= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap h^{-1}(y)) \, dy
\]
Now for arbitrary measurable \( A \) the above shows for \( k = 1, 2, \ldots \)
\[
\int_{A \cap B(0, k)} J(Dh(x)) \, dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap B(0, k) \cap h^{-1}(y)) \, dy.
\]
Use the monotone convergence theorem to obtain (34.3.13).

From the definition of Hausdorff measure it follows \( \mathcal{H}^0(E) \) equals the number of elements in \( E \). Thus, if \( n = m \), the Coarea formula implies
\[
\int_A J(Dh(x)) \, dx = \int_{h(A)} \mathcal{H}^0(A \cap h^{-1}(y)) \, dy = \int_{h(A)} \#(y) \, dy
\]
This gives a version of Sard’s theorem by letting \( S = A \).

### 34.4 A Nonlinear Fubini’s Theorem

Coarea formula holds for \( h : \mathbb{R}^n \rightarrow \mathbb{R}^m, n \geq m \) if whenever \( A \) is a Lebesgue measurable subset of \( \mathbb{R}^n \), the following formula is valid.
\[
\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap h^{-1}(y)) \, dy = \int_A Jh(x) \, dx \quad (34.4.25)
\]
Note this is the same as
\[
\int_A J(Dh(x)) \, dx = \int_{h(A)} \mathcal{H}^{n-m}(A \cap h^{-1}(y)) \, dy
\]
because if \( y \notin h(A) \), then \( h^{-1}(y) = \emptyset \). Now let
\[
s(x) = \sum_{i=1}^{p} c_i X_{E_i}(x)
\]
where \( E_i \) is measurable and \( c_i \geq 0 \). Then

\[
\int_{\mathbb{R}^n} s(x) J ((Dh(x))) \, dx = \sum_{i=1}^{p} c_i \int_{E_i} J (Dh(x)) \, dx
\]

\[
= \sum_{i=1}^{p} c_i \int_{h(E_i)} H^{n-m} (E_i \cap h^{-1}(y)) \, dy
\]

\[
= \int_{h(\mathbb{R}^n)} \sum_{i=1}^{p} c_i H^{n-m} (E_i \cap h^{-1}(y)) \, dy
\]

\[
= \int_{h(\mathbb{R}^n)} \left[ \int_{h^{-1}(y)} s \, dH^{n-m} \right] \, dy
\]

\[
= \int_{h(\mathbb{R}^n)} \left[ \int_{h^{-1}(y)} s \, dH^{n-m} \right] \, dy.
\] (34.4.26)

**Theorem 34.4.1** Let \( g \geq 0 \) be Lebesgue measurable and let

\[ h : \mathbb{R}^n \to \mathbb{R}^m, \ n \geq m \]

satisfy the Coarea formula. For example, it could be locally Lipschitz. Then

\[
\int_{\mathbb{R}^n} g(x) J ((Dh(x))) \, dx = \int_{h(\mathbb{R}^n)} \left[ \int_{h^{-1}(y)} g \, dH^{n-m} \right] \, dy.
\]

**Proof:** Let \( s_i \uparrow g \) where \( s_i \) is a simple function satisfying 34.4.26. Then let \( i \to \infty \) and use the monotone convergence theorem to replace \( s_i \) with \( g \). This proves the change of variables formula.

Note that this formula is a nonlinear version of Fubini’s theorem. The “\( n-m \) dimensional surface”, \( h^{-1}(y) \), plays the role of \( \mathbb{R}^{n-m} \) and \( H^{n-m} \) is like \( n-m \) dimensional Lebesgue measure. The term, \( J ((Dh(x))) \), corrects for the error occurring because of the lack of flatness of \( h^{-1}(y) \).
Chapter 35

Integration On Manifolds

You can do integration on various manifolds by using the Hausdorff measure of an appropriate dimension. However, it is possible to discuss this through the use of the Riesz representation theorem and some of the machinery for accomplishing this is interesting for its own sake so I will present this alternate point of view.

35.1 Partitions Of Unity

This material has already been mostly discussed starting on Page 1013. However, that was a long time ago and it seems like it might be good to go over it again and so, for the sake of convenience, here it is again.

Definition 35.1.1 Let $C$ be a set whose elements are subsets of $\mathbb{R}^n$. Then $C$ is said to be locally finite if for every $x \in \mathbb{R}^n$, there exists an open set, $U_x$ containing $x$ such that $U_x$ has nonempty intersection with only finitely many sets of $C$.

Lemma 35.1.2 Let $C$ be a set whose elements are open subsets of $\mathbb{R}^n$ and suppose $\bigcup C \supseteq H$, a closed set. Then there exists a countable list of open sets, $\{U_i\}_{i=1}^{\infty}$ such that each $U_i$ is bounded, each $U_i$ is a subset of some set of $C$, and $\bigcup_{i=1}^{\infty} U_i \supseteq H$.

Proof: Let $W_k \equiv B(0, k), W_0 = W_{-1} = \emptyset$. For each $x \in H \cap \overline{W_k}$ there exists an open set, $U_x$ such that $U_x$ is a subset of some set of $C$ and $U_x \subseteq W_{k+1} \setminus \overline{W_k}$. Then since $H \cap \overline{W_k}$ is compact, there exist finitely many of these sets, $\{U_i^k\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_k}$. If $H \cap \overline{W_k} = \emptyset$, let $m(k) = 0$ and there are no such sets obtained. The desired countable list of open sets is $\bigcup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $x \in \mathbb{R}^n$, then $x \in W_k$ where $k$ is the first positive integer with $x \in W_k$. Then $W_k \setminus \overline{W_{k-1}}$ is an open set containing $x$ and this open set can have nonempty intersection only with a set of $\{U_i^k\}_{i=1}^{m(k)} \cup \{U_i^{k-1}\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\bigcup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$ is locally finite.

1The definition applies with no change to a general topological space in place of $\mathbb{R}^n$. 

1315
The set, $\{U_i\}_{i=1}^\infty$ is said to be a locally finite cover of $H$. The following lemma gives some important reasons why a locally finite list of sets is so significant. First of all consider the rational numbers, $\{r_i\}_{i=1}^\infty$ each rational number is a closed set.

$$Q = \{r_i\}_{i=1}^\infty = \bigcup_{i=1}^\infty \overline{\{r_i\}} \neq \bigcup_{i=1}^\infty \{r_i\} = \mathbb{R}$$

The set of rational numbers is definitely not locally finite.

**Lemma 35.1.3** Let $\mathcal{C}$ be locally finite. Then

$$\overline{\mathcal{C}} = \bigcup \{\overline{H} : H \in \mathcal{C}\}.$$ 

Next suppose the elements of $\mathcal{C}$ are open sets and that for each $U \in \mathcal{C}$, there exists a differentiable function, $\psi_U$, having $\text{spt} (\psi_U) \subseteq U$. Then you can define the following finite sum for each $x \in \mathbb{R}^n$

$$f(x) \equiv \sum \{\psi_U(x) : x \in U \in \mathcal{C}\}.$$ 

Furthermore, $f$ is also a differentiable function and

$$Df(x) = \sum \{D\psi_U(x) : x \in U \in \mathcal{C}\}.$$ 

**Proof:** Let $p$ be a limit point of $\cup \mathcal{C}$ and let $W$ be an open set which intersects only finitely many sets of $\mathcal{C}$. Then $p$ must be a limit point of one of these sets. It follows $p \in \bigcup \{\overline{H} : H \in \mathcal{C}\}$ and so $\overline{\mathcal{C}} \subseteq \bigcup \{\overline{H} : H \in \mathcal{C}\}$. The inclusion in the other direction is obvious.

Now consider the second assertion. Letting $x \in \mathbb{R}^n$, there exists an open set, $W$ intersecting only finitely many open sets of $\mathcal{C}$, $U_1, U_2, \cdots , U_m$. Then for all $y \in W$,

$$f(y) = \sum_{i=1}^m \psi_{U_i}(y)$$

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

**Definition 35.1.4** Let $K$ be a closed subset of an open set, $U$. $K \prec f \prec U$ if $f$ is continuous, has values in $[0, 1]$, equals 1 on $K$, and has compact support contained in $U$.

**Lemma 35.1.5** Let $U$ be a bounded open set and let $K$ be a closed subset of $U$. Then there exist an open set, $W$, such that $W \subseteq \overline{W} \subseteq U$ and a function, $f \in C_\infty^\infty (U)$ such that $K \prec f \prec U$.

\[\text{If each } \psi_U \text{ were only continuous, one could conclude } f \text{ is continuous. Here the main interest is differentiable.}\]
35.1. PARTITIONS OF UNITY

Proof: The set, $K$ is compact so is at a positive distance from $U^C$. Let

$$W \equiv \{ x : \text{dist} (x, K) < 3^{-1} \text{dist} (K, U^C) \}.$$ 

Also let

$$W_1 \equiv \{ x : \text{dist} (x, K) < 2^{-1} \text{dist} (K, U^C) \}$$

Then it is clear

$$K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$$

Now consider the function,

$$h(x) = \frac{\text{dist} (x, W^C)}{\text{dist} (x, W^C) + \text{dist} (x, \overline{W})}$$

Since $\overline{W}$ is compact it is at a positive distance from $W^C$ and so $h$ is a well defined continuous function which has compact support contained in $W_1$, equals 1 on $W$, and has values in $[0, 1]$. Now let $\phi_k$ be a mollifier. Letting

$$k^{-1} < \min \left( \text{dist} (K, W^C), 2^{-1} \text{dist} (\overline{W_1}, U^C) \right),$$

it follows that for such $k$, the function, $h \ast \phi_k \in C_c^\infty (U)$, has values in $[0, 1]$, and equals 1 on $K$. Let $f = h \ast \phi_k$.

The above lemma is used repeatedly in the following.

Lemma 35.1.6 Let $K$ be a closed set and let $\{V_i\}_{i=1}^\infty$ be a locally finite list of bounded open sets whose union contains $K$. Then there exist functions, $\psi_i \in C_c^\infty (V_i)$ such that for all $x \in K$,

$$1 = \sum_{i=1}^\infty \psi_i (x)$$

and the function $f(x)$ given by

$$f(x) = \sum_{i=1}^\infty \psi_i (x)$$

is in $C^\infty (\mathbb{R}^n)$.

Proof: Let $K_1 = K \setminus \bigcup_{i=2}^\infty V_i$. Thus $K_1$ is compact because $K_1 \subseteq V_1$. Let

$$K_1 \subseteq W_1 \subseteq \overline{W_1} \subseteq V_1$$

Thus $W_1, V_2, \cdots, V_n$ covers $K$ and $\overline{W_1} \subseteq V_1$. Suppose $W_1, \cdots, W_r$ have been defined such that $\overline{W_i} \subseteq V_i$ for each $i$, and $W_1, \cdots, W_r, V_{r+1}, \cdots, V_n$ covers $K$. Then let

$$K_{r+1} = K \setminus \left( \left( \bigcup_{i=r+2}^\infty V_i \right) \cup \left( \bigcup_{j=1}^r W_j \right) \right).$$
It follows $K_{r+1}$ is compact because $K_{r+1} \subseteq V_{r+1}$. Let $W_{r+1}$ satisfy

$$K_{r+1} \subseteq W_{r+1} \subseteq \overline{W}_{r+1} \subseteq V_{r+1}$$

Continuing this way defines a sequence of open sets, $\{W_i\}_{i=1}^\infty$ with the property

$$\overline{W}_i \subseteq V_i, K \subseteq \bigcup_{i=1}^\infty W_i.$$ 

Note $\{W_i\}_{i=1}^\infty$ is locally finite because the original list, $\{V_i\}_{i=1}^\infty$ was locally finite.

Now let $U_i$ be open sets which satisfy

$$W_i \subseteq U_i \subseteq U_i \subseteq V_i.$$ 

Similarly, $\{U_i\}_{i=1}^\infty$ is locally finite.

Since the set, $\{W_i\}_{i=1}^\infty$ is locally finite, it follows $\bigcup_{i=1}^\infty W_i = \bigcup_{i=1}^\infty \overline{W}_i$ and so it is possible to define $\phi_i$ and $\gamma$, infinitely differentiable functions having compact support such that

$$\overline{U}_i \prec \phi_i \prec V_i, \bigcup_{i=1}^\infty W_i \prec \gamma \prec \bigcup_{i=1}^\infty U_i.$$ 

Now define

$$\psi_i(x) = \begin{cases} \gamma(x)\phi_i(x)/\sum_{j=1}^\infty \phi_j(x) & \text{if } \sum_{j=1}^\infty \phi_j(x) \neq 0, \\ 0 & \text{if } \sum_{j=1}^\infty \phi_j(x) = 0. \end{cases}$$

If $x$ is such that $\sum_{j=1}^\infty \phi_j(x) = 0$, then $x \notin \bigcup_{i=1}^\infty U_i$ because $\phi_i$ equals one on $U_i$. Consequently $\gamma(y) = 0$ for all $y$ near $x$ thanks to the fact that $\bigcup_{i=1}^\infty U_i$ is closed and so $\psi_i(y) = 0$ for all $y$ near $x$. Hence $\psi_i$ is infinitely differentiable at such $x$. If $\sum_{j=1}^\infty \phi_j(x) \neq 0$, this situation persists near $x$ because each $\phi_j$ is continuous and so $\psi_i$ is infinitely differentiable at such points also thanks to Lemma 35.1.3. Therefore $\psi_i$ is infinitely differentiable. If $x \in K$, then $\gamma(x) = 1$ and so $\sum_{j=1}^\infty \psi_j(x) = 1$. Clearly $0 \leq \psi_i(x) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The method of proof of this lemma easily implies the following useful corollary.

**Corollary 35.1.7** If $H$ is a compact subset of $V_i$ for some $V_i$ there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Lemma 35.1.6.

**Proof:** Keep $V_i$ the same but replace $V_j$ with $\widetilde{V}_j \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i$, $\phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on $H$.
35.2. **Integration On Manifolds**

Manifolds are things which locally appear to be \( \mathbb{R}^n \) for some \( n \). The extent to which they have such a local appearance varies according to various analytical characteristics which the manifold possesses.

**Definition 35.2.1** Let \( U \subseteq \mathbb{R}^n \) be an open set and let \( h : U \to \mathbb{R}^m \). Then for \( r \in [0, 1] \), \( h \in C^{k,r}(U) \) for \( k \) a nonnegative integer means that \( D^\alpha h \) exists for all \( |\alpha| \leq k \) and each \( D^\alpha h \) is Hölder continuous with exponent \( r \). That is

\[
|D^\alpha h(x) - D^\alpha h(y)| \leq K |x - y|^r.
\]

Also \( h \in C^{k,r}(\overline{U}) \) if it is the restriction of a function of \( C^{k,r}(\mathbb{R}^n) \) to \( U \).

**Definition 35.2.2** Let \( \Gamma \) be a closed subset of \( \mathbb{R}^p \) where \( p \geq n \). Suppose \( \Gamma = \bigcup_{i=1}^{\infty} \Gamma_i \) where \( \Gamma_i = \Gamma \cap W_i \) for \( W_i \) a bounded open set. Suppose also \( \{W_i\}_{i=1}^{\infty} \) is locally finite. This means every bounded open set intersects only finitely many. Also suppose there are open bounded sets, \( U_i \) having Lipschitz boundaries and functions \( h_i : U_i \to \Gamma_i \) which are one to one, onto, and in \( C^{m,1}(U_i) \). Suppose also there exist functions, \( g_i : W_i \to U_i \) such that \( g_i \) is \( C^{m,1}(W_i) \), and \( g_i \circ h_i = id \) on \( U_i \) while \( h_i \circ g_i = id \) on \( \Gamma_i \). The collection of sets, \( \Gamma_j \) and mappings, \( g_j, \{\Gamma_j, g_j\} \) is called an atlas and an individual entry in the atlas is called a chart. Thus \( (\Gamma_j, g_j) \) is a chart. Then \( \Gamma \)}
as just described is called a $C^{m,1}$ manifold. The number, $m$ is just a nonnegative integer. When $m = 0$ this would be called a Lipschitz manifold, the least smooth of the manifolds discussed here.

For example, take $p = n + 1$ and let

$$h_i (u) = (u_1, \ldots, u_i, \phi_i (u), u_{i+1}, \ldots, u_n)^T$$

for $u = (u_1, \ldots, u_i, u_{i+1}, \ldots, u_n)^T \in U_i$ for $\phi_i \in C^{m,1} (U_i)$ and $g_i : U_i \times R \rightarrow U_i$ given by

$$g_i (u_1, \ldots, u_i, y, u_{i+1}, \ldots, u_n) \equiv u$$

for $i = 1, 2, \ldots, p$. Then for $u \in U_i$, the definition gives

$$g_i \circ h_i (u) = g_i (u_1, \ldots, u_i, \phi_i (u), u_{i+1}, \ldots, u_n) = u$$

and for $\Gamma_i \equiv h_i (U_i)$ and $(u_1, \ldots, u_i, \phi_i (u), u_{i+1}, \ldots, u_n)^T \in \Gamma_i$,

$$h_i \circ g_i (u_1, \ldots, u_i, \phi_i (u), u_{i+1}, \ldots, u_n) = h_i (u) = (u_1, \ldots, u_i, \phi_i (u), u_{i+1}, \ldots, u_n)^T.$$

This example can be used to describe the boundary of a bounded open set and since $\phi_i \in C^{m,1} (U_i)$, such an open set is said to have a $C^{m,1}$ boundary. Note also that in this example, $U_i$ could be taken to be $R^n$ or if $U_i$ is given, both $h_i$ and and $g_i$ can be taken as restrictions of functions defined on all of $R^n$ and $R^p$ respectively.

The symbol, $I$ will refer to an increasing list of $n$ indices taken from $\{1, \ldots, p\}$. Denote by $\Lambda (p, n)$ the set of all such increasing lists of $n$ indices.

Let

$$J_i (u) \equiv \left[ \sum_{I \in \Lambda (p, n)} \left( \frac{\partial (x^1 \ldots x^n)}{\partial (u^1 \ldots u^n)} \right)^2 \right]^{1/2}$$

where here the sum is taken over all possible increasing lists of $n$ indices, $I$, from $\{1, \ldots, p\}$ and $x = h_i u$. Thus there are $p!$ terms in the sum. In this formula, $\frac{\partial (x^1 \ldots x^n)}{\partial (u^1 \ldots u^n)}$ is defined to be the determinant of the following matrix.

$$\begin{pmatrix} \frac{\partial x^1}{\partial u_1} & \ldots & \frac{\partial x^1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial u_1} & \ldots & \frac{\partial x^n}{\partial u_n} \end{pmatrix}.$$ 

Note that if $p = n$ there is only one term in the sum, the absolute value of the determinant of $Dx (u)$. Define a positive linear functional, $\Lambda$ on $C_c (\Gamma)$ as follows: First let $\{\psi_i\}$ be a $C^\infty$ partition of unity subordinate to the open sets, $\{W_i\}$. Thus $\psi_i \in C^\infty_c (W_i)$ and $\sum_i \psi_i (x) = 1$ for all $x \in \Gamma$. Then

$$\Lambda f \equiv \sum_{i=1}^\infty \int_{g_i \Gamma_i} f \psi_i (h_i (u)) J_i (u) \, du.$$ (35.2.1)

Is this well defined?
Lemma 35.2.3  The functional defined in 35.2.1 does not depend on the choice of atlas or the partition of unity.

Proof: In 35.2.1, let \{\psi_i\} be a \(C^\infty\) partition of unity which is associated with the atlas \((\Gamma_i, g_i)\) and let \{\eta_i\} be a \(C^\infty\) partition of unity associated in the same manner with the atlas \((\Gamma'_i, g'_i)\). In the following argument, the local finiteness of the \(\Gamma_i\) implies that all sums are finite. Using the change of variables formula with \(u = (g_i \circ h'_j) v\)

\[
\sum_{i=1}^\infty \int_{g_i \Gamma_i} \psi_i f(h_i(u)) J_i(u) \, du = \tag{35.2.2}
\]

\[
\sum_{i=1}^\infty \sum_{j=1}^\infty \int_{g_i \Gamma_i} \eta_j \psi_i f(h_i(u)) J_i(u) \, du = \sum_{i=1}^\infty \sum_{j=1}^\infty \int_{g'_j (\Gamma_i \cap \Gamma'_j)} \eta_j (h'_j(v)) \psi_i (h'_j(v)) f(h'_j(v)) J_i(u) \left| \frac{\partial (u^1 \cdots u^n)}{\partial (v^1 \cdots v^n)} \right| \, dv
\]

\[
= \sum_{i=1}^\infty \sum_{j=1}^\infty \int_{g'_j (\Gamma_i \cap \Gamma'_j)} \eta_j (h'_j(v)) \psi_i (h'_j(v)) f(h'_j(v)) J_j(v) \, dv. \tag{35.2.3}
\]

Thus the definition of \(\Lambda f\) using \((\Gamma_i, g_i)\) is defined as

\[
\sum_{i=1}^\infty \int_{g_i \Gamma_i} \psi_i f(h_i(u)) J_i(u) \, du = \sum_{i=1}^\infty \sum_{j=1}^\infty \int_{g'_j (\Gamma_i \cap \Gamma'_j)} \eta_j (h'_j(v)) \psi_i (h'_j(v)) f(h'_j(v)) J_j(v) \, dv
\]

the definition of \(\Lambda f\) using \((V_i, g'_i)\).

This proves the lemma.

This lemma and the Riesz representation theorem for positive linear functionals implies the part of the following theorem which says the functional is well defined.

Theorem 35.2.4  Let \(\Gamma\) be a \(C^{m,1}\) manifold. Then there exists a unique Radon measure, \(\mu\), defined on \(\Gamma\) such that whenever \(f\) is a continuous function having compact support which is defined on \(\Gamma\) and \((\Gamma_i, g_i)\) denotes an atlas and \{\psi_i\} a partition of unity subordinate to this atlas,

\[
\Lambda f = \int_{\Gamma} f \, d\mu = \sum_{i=1}^\infty \int_{g_i \Gamma_i} \psi_i f(h_i(u)) J_i(u) \, du. \tag{35.2.4}
\]
Also, a subset, \( A \), of \( \Gamma \) is \( \mu \) measurable if and only if for all \( r, g_r(\Gamma_r \cap A) \) is \( \nu_r \) measurable where \( \nu_r \) is the measure defined by

\[
\nu_r(\Gamma_r \cap A) = \int_{g_r(\Gamma_r \cap A)} J_r(u) \, du
\]

**Proof:** To begin, here is a claim.

**Claim:** A set, \( S \subseteq \Gamma_i \), has \( \mu \) measure zero if and only if \( g_i S \) has measure zero in \( g_i \Gamma_i \) with respect to the measure, \( \nu_i \).

**Proof of the claim:** Let \( \varepsilon > 0 \) be given. By outer regularity, there exists a set, \( V \subseteq \Gamma_i \), open\(^3\) in \( \Gamma \) such that \( \mu(V) < \varepsilon \) and \( S \subseteq V \subseteq \Gamma_i \). Then \( g_i V \) is open in \( \mathbb{R}^n \) and contains \( g_i S \). Letting \( h \prec g_i V \) and \( h_1(x) \equiv h(g_i(x)) \) for \( x \in \Gamma_i \), it follows \( h_1 \prec V \). By Corollary 35.1.7 on Page 1318 there exists a partition of unity such that \( spt(h_j) \subseteq \{ x \in \mathbb{R}^p : \psi_i(x) = 1 \} \). Thus \( \psi_j h_1(h_j(u)) = 0 \) unless \( j = i \) when this reduces to \( h_1(h_i(u)) \). It follows

\[
\varepsilon \geq \mu(V) \geq \sum_{j=1}^{\infty} \int_{g_i \Gamma_j} \psi_j h_1(h_j(u)) J_j(u) \, du
\]

\[
= \int_{g_i \Gamma_i} h_1(h_i(u)) J_i(u) \, du = \int_{g_i \Gamma_i} h(u) J_i(u) \, du
\]

Now this holds for all \( h \prec g_i V \) and so

\[
\int_{g_i V} J_i(u) \, du \leq \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this shows \( g_i V \) has measure no more than \( \varepsilon \) with respect to the measure, \( \nu_i \). Since \( \varepsilon \) is arbitrary, \( g_i S \) has measure zero.

Consider the converse. Suppose \( g_i S \) has \( \nu_i \) measure zero. Then there exists an open set, \( O \subseteq g_i \Gamma_i \) such that \( O \supseteq g_i S \) and

\[
\int_O J_i(u) \, du < \varepsilon.
\]

Thus \( h_i(O) \) is open in \( \Gamma \) and contains \( S \). Let \( h \prec h_i(O) \) be such that

\[
\int_{\Gamma} h d\mu + \varepsilon > \mu(h_i(O)) \geq \mu(S) \quad (35.2.5)
\]
As in the first part, Corollary 35.1.7 on Page 1318 implies there exists a partition of unity such that \( h(x) = 0 \) off the set, \( \{ x \in \mathbb{R}^p : \psi_i(x) = 1 \} \)
and so as in this part of the argument,
\[
\int_{\Gamma} h d\mu = \sum_{j=1}^\infty \int_{g_j U_j} \psi_j h(h_j(u)) J_j(u) du
= \int_{g_r \Gamma_r} h(h_i(u)) J_i(u) du
= \int_{g_r \Gamma_r} h(h_i(u)) J_i(u) du
\leq \int_{O} J_i(u) du < \varepsilon
\]
(35.2.6)
and so from 35.2.5 and 35.2.6 \( \mu(S) \leq 2\varepsilon \). Since \( \varepsilon \) is arbitrary, this proves the claim.

For the last part of the theorem, it suffices to let \( A \subseteq \Gamma_r \) because otherwise, the above argument would apply to \( A \cap \Gamma_r \).
Thus let \( A \subseteq \Gamma_r \) be \( \mu \) measurable. By the regularity of the measure, there exists an \( F_\alpha \) set, \( F \) and a \( G_\delta \) set, \( G \) such that \( \Gamma_r \supseteq G \supseteq A \supseteq F \) and \( \mu(G \setminus F) = 0 \).
(Recall a \( G_\delta \) set is a countable intersection of open sets and an \( F_\sigma \) set is a countable union of closed sets.) Then since \( \Gamma_r \) is compact, it follows each of the closed sets whose union equals \( F \) is a compact set. Thus if \( F = \cup_{k=1}^\infty F_k \), \( g_r(F_k) \) is also a compact set and so \( g_r(F) = \cup_{k=1}^\infty g_r(F_k) \) is a Borel set. Similarly, \( g_r(G) \) is also a Borel set. Now by the claim,
\[
\int_{g_r(G \setminus F)} J_r(u) du = 0.
\]
Since \( g_r \) is one to one,
\[
g_rG \setminus g_rF = g_r(G \setminus F)
\]
and so
\[
g_r(F) \subseteq g_r(A) \subseteq g_r(G)
\]
where \( g_r(G) \setminus g_r(F) \) has measure zero. By completeness of the measure, \( \nu_r, g_r(A) \) is measurable. It follows that if \( A \subseteq \Gamma \) is \( \mu \) measurable, then \( g_r(\Gamma_r \cap A) \) is \( \nu_r \) measurable for all \( r \). The converse is entirely similar. This proves the theorem.

**Corollary 35.2.5** Let \( f \in L^1(\Gamma; \mu) \) and suppose \( f(x) = 0 \) for all \( x \notin \Gamma_r \) where \( (\Gamma_r, g_r) \) is a chart. Then
\[
\int_{\Gamma} f d\mu = \int_{\Gamma_r} f d\mu = \int_{g_r \Gamma_r} f(h_r(u)) J_r(u) du.
\]
(35.2.7)
Furthermore, if \( \{ (\Gamma_i, g_i) \} \) is an atlas and \( \{ \psi_i \} \) is a partition of unity as described earlier, then for any \( f \in L^1(\Gamma, \mu) \),
\[
\int_{\Gamma} f d\mu = \sum_{r=1}^\infty \int_{g_r \Gamma_r} \psi_r f(h_r(u)) J_r(u) du.
\]
(35.2.8)
\textbf{Proof:} Let \( f \in L^1(\Gamma, \mu) \) with \( f = 0 \) off \( \Gamma_r \). Without loss of generality assume \( f \geq 0 \) because if the formulas can be established for this case, the same formulas are obtained for an arbitrary complex valued function by splitting it up into positive and negative parts of the real and imaginary parts in the usual way. Also, let \( K \subseteq \Gamma_r \) a compact set. Since \( \mu \) is a Radon measure there exists a sequence of continuous functions, \( \{f_k\} \), \( f_k \in C_c(\Gamma_r) \), which converges to \( f \) in \( L^1(\Gamma, \mu) \) and for \( \mu \) a.e. \( x \). Take the partition of unity, \( \{\psi_i\} \) to be such that

\[
K \subseteq \{ x : \psi_r(x) = 1 \}.
\]

Therefore, the sequence \( \{f_k(h_r(\cdot))\} \) is a Cauchy sequence in the sense that

\[
\lim_{k,l \to \infty} \int_{g_r(K)} |f_k(h_r(u)) - f_l(h_r(u))| J_r(u) \, du = 0
\]

It follows there exists \( g \) such that

\[
\int_{g_r(K)} |f_k(h_r(u)) - g(u)| J_r(u) \, du \to 0,
\]

and

\[
g \in L^1(g_rK; \nu_r).
\]

By the pointwise convergence and the claim used in the proof of Theorem 35.2.4, \( g(u) = f(h_r(u)) \)

for \( \mu \) a.e. \( h_r(u) \in K \). Therefore,

\[
\int_K f \, d\mu = \lim_{k \to \infty} \int_K f_k \, d\mu = \lim_{k \to \infty} \int_{g_r(K)} f_k(h_r(u)) J_r(u) \, du
\]

\[
= \int_{g_r(K)} g(u) J_r(u) \, du = \int_{g_r(K)} f(h_r(u)) J_r(u) \, du. \quad (35.2.9)
\]

Now let \( \cdots K_j \subseteq K_{j+1} \cdots \) and \( \bigcup_{j=1}^{\infty} K_j = \Gamma_r \) where \( K_j \) is compact for all \( j \). Replace \( K \) in (35.2.4) with \( K_j \) and take a limit as \( j \to \infty \). By the monotone convergence theorem,

\[
\int_{\Gamma_r} f \, d\mu = \int_{g_r(\Gamma_r)} f(h_r(u)) J_r(u) \, du.
\]

This establishes (35.2.4).

To establish (35.2.5), let \( f \in L^1(\Gamma, \mu) \) and let \( \{(\Gamma_i, g_i)\} \) be an atlas and \( \{\psi_i\} \) be a partition of unity. Then \( f\psi_i \in L^1(\Gamma, \mu) \) and is zero off \( \Gamma_i \). Therefore, from what was just shown,

\[
\int_{\Gamma} f \, d\mu = \sum_{i=1}^{\infty} \int_{\Gamma_i} f \psi_i \, d\mu
\]

\[
= \sum_{r=1}^{\infty} \int_{g_r(\Gamma_r)} \psi_r f(h_r(u)) J_r(u) \, du
\]
35.3 Comparison With $\mathcal{H}^n$

The above gives a measure on a manifold, $\Gamma$. I will now show that the measure obtained is nothing more than $\mathcal{H}^n$, the $n$ dimensional Hausdorff measure. Recall $\Lambda(p,n)$ was the set of all increasing lists of $n$ indices taken from $\{1,2,\cdots,p\}$.

Recall $J_i(u) \equiv \left[ \sum_{I \in \Lambda(p,n)} \left( \frac{\partial (x^i_1 \cdots x^i_n)}{\partial (u^1 \cdots u^n)} \right)^2 \right]^{1/2}$

where here the sum is taken over all possible increasing lists of $n$ indices, $I$, from $\{1,\cdots,p\}$ and $x = h_i u$ and the functional was given as

$$\Lambda f \equiv \sum_{i=1}^{\infty} \int_{\mathcal{G}_i \Gamma_i} f\psi_i(h_i(u)) J_i(u) \, du$$

(35.3.10)

where the $\{\psi_i\}_{i=1}^{\infty}$ was a partition of unity subordinate to the open sets, $\{W_i\}_{i=1}^{\infty}$ as described above. I will show

$$J_i(u) = \det \left( Dh(u)^* Dh(u) \right)^{1/2}$$

and then use the area formula. The key result is really a special case of the Binet Cauchy theorem and this special case is presented in the next lemma.

**Lemma 35.3.1** Let $A = (a_{ij})$ be a real $p \times n$ matrix in which $p \geq n$. For $I \in \Lambda(p,n)$ denote by $A_I$ the $n \times n$ matrix obtained by deleting from $A$ all rows except for those corresponding to an element of $I$. Then

$$\sum_{I \in \Lambda(p,n)} \det(A_I)^2 = \det(A^* A)$$

**Proof:** For $(j_1,\cdots,j_n) \in \Lambda(p,n)$, define $\theta(j_k) \equiv k$. Then let for $\{k_1,\cdots,k_n\} = \{j_1,\cdots,j_n\}$ define

$$\text{sgn}(k_1,\cdots,k_n) \equiv \text{sgn}(\theta(k_1),\cdots,\theta(k_n)).$$

Then from the definition of the determinant and matrix multiplication,

$$\det(A^* A) = \sum_{i_1,\cdots,i_n} \text{sgn}(i_1,\cdots,i_n) \sum_{k_1=1}^{p} a_{k_1 i_1} a_{k_1 1} \sum_{k_2=1}^{p} a_{k_2 i_2} a_{k_2 2} \cdots \sum_{k_n=1}^{p} a_{k_n i_n} a_{k_n n}$$

$$= \sum_{J \in \Lambda(p,n) \{k_1,\cdots,k_n\} = \{j_1,\cdots,j_n\}} \sum_{i_1,\cdots,i_n} \text{sgn}(i_1,\cdots,i_n) a_{k_1 i_1} a_{k_1 1} a_{k_2 i_2} a_{k_2 2} \cdots a_{k_n i_n} a_{k_n n}$$
\[ \sum_{J \in \Lambda(p, n)} \sum_{\{k_1, \ldots, k_n\} = J} \text{sgn}(i_1, \ldots, i_n) a_{k_1 i_1} a_{k_2 i_2} \cdots a_{k_n i_n} \cdot a_{k_1} a_{k_2} \cdots a_{k_n} \]

\[ = \sum_{J \in \Lambda(p, n)} \sum_{\{k_1, \ldots, k_n\} = J} \text{sgn}(i_1, \ldots, i_n) \det(A_J) a_{k_1} a_{k_2} \cdots a_{k_n} \]

\[ = \sum_{J \in \Lambda(p, n)} \det(A_J) \det(A_J) \]

and this proves the lemma.

It follows from this lemma that
\[ J_i(u) = \det(Dh(u)^* Dh(u))^{1/2}. \]

From \[\text{35.3.10}\] and the area formula, the functional equals
\[ \Lambda f = \sum_{i=1}^{\infty} \int_{E_i \cap \Gamma} f \psi_i(h_i(u)) J_i(u) \, du \]
\[ = \sum_{i=1}^{\infty} \int_{E_i} f \psi_i(y) \, d\mathcal{H}^n = \int_{\Gamma} f(y) \, d\mathcal{H}^n. \]

Now \(\mathcal{H}^n\) is a Borel measure defined on \(\Gamma\) which is finite on all compact subsets of \(\Gamma\). This finiteness follows from the above formula. If \(K\) is a compact subset of \(\Gamma\), then there exists an open set, \(W\) whose closure is compact and a continuous function with compact support, \(f\) such that \(K < f < W\). Then \(\mathcal{H}^n(K) \leq \int_{\Gamma} f(y) \, d\mathcal{H}^n < \infty\) because of the above formula.

**Lemma 35.3.2** \(\mu = \mathcal{H}^n\) on every \(\mu\) measurable set.

**Proof:** The Riesz representation theorem shows that
\[ \int_{\Gamma} f \, d\mu = \int_{\Gamma} f \, d\mathcal{H}^n \]
for every continuous function having compact support. Therefore, since every open set is the countable union of compact sets, it follows \(\mu = \mathcal{H}^n\) on all open sets. Since compact sets can be obtained as the countable intersection of open sets, these two measures are also equal on all compact sets. It follows they are also equal on all countable unions of compact sets. Suppose now that \(E\) is a \(\mu\) measurable set of finite measure. Then there exist sets, \(F, G\) such that \(G\) is the countable intersection of open sets each of which has finite measure and \(F\) is the countable union of compact sets such that \(\mu(G \setminus F) = 0\) and \(F \subseteq E \subseteq G\). Thus \(\mathcal{H}^n(G \setminus F) = 0\),
\[ \mathcal{H}^n(G) = \mu(G) = \mu(F) = \mathcal{H}^n(F) \]
By completeness of \(\mathcal{H}^n\) it follows \(E\) is \(\mathcal{H}^n\) measurable and \(\mathcal{H}^n(E) = \mu(E)\). If \(E\) is not of finite measure, consider \(E_r \equiv E \cap B(0, r)\). This is contained in the compact set \(\Gamma \cap \overline{B(0, r)}\) and so \(\mu(E_r)\) if finite. Thus from what was just shown, \(\mathcal{H}^n(E_r) = \mu(E_r)\) and so, taking \(r \to \infty\) \(\mathcal{H}^n(E) = \mu(E)\).

This shows you can simply use \(\mathcal{H}^n\) for the measure on \(\Gamma\).
Chapter 36

Basic Theory Of Sobolev Spaces

**Definition 36.0.3** Let $U$ be an open set of $\mathbb{R}^n$. Define $X^{m,p}(U)$ as the set of all functions in $L^p(U)$ whose weak partial derivatives up to order $m$ are also in $L^p(U)$ where $1 \leq p$. The norm in this space is given by

$$||u||_{m,p} = \left( \int_U \sum_{|\alpha| \leq m} |D^\alpha u|^p \, dx \right)^{1/p},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \sum \alpha_i$. Here $D^\alpha u \equiv u \in C^\infty_c(\mathbb{R}^n)$. Thus $C^\infty_c(\overline{U}) \subseteq W^{m,p}(U)$. The Sobolev space, $W^{m,p}(U)$ is defined to be the closure of $C^\infty_c(\overline{U})$ in $X^{m,p}(U)$ with respect to the above norm. Denote this norm by $||u||_{W^{m,p}(U)}, ||u||_{X^{m,p}(U)}$, or $||u||_{m,p,U}$ when it is important to identify the open set, $U$.

Also the following notation will be used pretty consistently.

**Definition 36.0.4** Let $u$ be a function defined on $U$. Define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

**Theorem 36.0.5** Both $X^{m,p}(U)$ and $W^{m,p}(U)$ are separable reflexive Banach spaces provided $p > 1$.

---

1You could also let the norm be given by $||u||_{m,p} = \sum_{|\alpha| \leq m} ||D^\alpha u||_p$ or $||u||_{m,p} = \max \{ ||D^\alpha u||_p : |\alpha| \leq m \}$ because all norms are equivalent on $\mathbb{R}^p$ where $p$ is the number of multi indices no larger than $m$. This is used whenever convenient.
Suppose \( X \) is a completely separable metric space, note that \( X \) is a Banach space, and therefore \( \Lambda (X) \) is an isometry of \( X \). Finally, \( \Lambda \) is reflexive and any closed subspace of a reflexive Banach space is reflexive. Now \( \Lambda \) is one to one because one of the multi indices is \( \alpha = 0 \). Also

\[
\Lambda (X^{m,p}(U))
\]

is a closed subspace of \( L^p(U)^w \). To see this, suppose

\[
(u_k, D^{a_1} u_k, \ldots, D^{a_w} u_k) \to (f_1, f_2, \ldots, f_w)
\]

in \( L^p(U)^w \). Then \( u_k \to f_1 \) in \( L^p(U) \) and \( D^{a_j} u_k \to f_j \) in \( L^p(U) \). Therefore, letting \( \phi \in C^\infty_c(U) \) and letting \( k \to \infty \),

\[
\int_U (D^{a_j} u_k) \phi dx = (-1)^{|\alpha|} \int_U u_k D^{a_j} \phi dx
\]

\[
\int_U f_j \phi dx = (-1)^{|\alpha|} \int_U f_1 D^{a_j} \phi dx
\]

It follows \( D^{a_j} (f_1) = f_j \) and so \( \Lambda (X^{m,p}(U)) \) is closed as claimed. This is clearly also a subspace of \( L^p(U)^w \) and so it follows that \( \Lambda (X^{m,p}(U)) \) is a reflexive Banach space. This is because \( L^p(U)^w \), being the product of reflexive Banach spaces, is reflexive and any closed subspace of a reflexive Banach space is reflexive. Now \( \Lambda \) is an isometry of \( X^{m,p}(U) \) and \( \Lambda (X^{m,p}(U)) \) which shows that \( X^{m,p}(U) \) is a reflexive Banach space. Finally, \( W^{m,p}(U) \) is a closed subspace of the reflexive Banach space, \( X^{m,p}(U) \) and so it is also reflexive. To see \( X^{m,p}(U) \) is separable, note that \( L^p(U)^w \) is separable because it is the finite product of the separable hence completely separable metric space, \( L^p(U) \) and \( \Lambda (X^{m,p}(U)) \) is a subset of \( L^p(U)^w \). Therefore, \( \Lambda (X^{m,p}(U)) \) is separable and since \( \Lambda \) is an isometry, it follows \( X^{m,p}(U) \) is separable also. Now \( W^{m,p}(U) \) must also be separable because it is a subset of \( X^{m,p}(U) \).

The following theorem is obvious but is worth noting because it says that if a function has a weak derivative in \( L^p(U) \) on a large open set, \( U \) then the restriction of this weak derivative is also the weak derivative for any smaller open set.

**Theorem 36.0.6** Suppose \( U \) is an open set and \( U_0 \subset U \) is another open set. Suppose also \( D^a u \in L^p(U) \). Then for all \( \psi \in C^\infty_c(U_0) \),

\[
\int_{U_0} (D^a u) \psi dx = (-1)^{|\alpha|} \int_{U_0} u (D^a \psi).
\]

The following theorem is a fundamental approximation result for functions in \( X^{m,p}(U) \).

**Theorem 36.0.7** Let \( U \) be an open set and let \( U_0 \) be an open subset of \( U \) with the property that \( \text{dist}(\overline{U_0}, U^c) > 0 \). Then if \( u \in X^{m,p}(U) \) and \( \tilde{u} \) denotes the zero extension of \( u \) off \( U \),

\[
\lim_{t \to \infty} \| \tilde{u} * \phi_t - u \|_{X^{m,p}(U_0)} = 0.
\]
Proof: Always assume $l$ is large enough that $1/l < \text{dist}(U_0, U^C)$. Thus for $x \in U_0$,
\[
\tilde{u} \ast \phi_l (x) = \int_{B(0, \frac{1}{l})} u(x - y) \phi_l (y) \, dy. \quad (36.0.1)
\]
The theorem is proved if it can be shown that $D^\alpha (\tilde{u} \ast \phi_l) \rightarrow D^\alpha u$ in $L^p(U_0)$. Let $\psi \in C_c^\infty (U_0)$
\[
D^\alpha (\tilde{u} \ast \phi_l) \ast \psi = (-1)^{|\alpha|} \int_{U_0} \int_{U_0} (\tilde{u} \ast \phi_l) (x) \, D^\alpha \psi (x) \, dx \, dy = (-1)^{|\alpha|} \int_{U} u(y) \int_{U_0} \phi_l (x - y) \, D^\alpha \psi (x) \, dxdy.
\]
Also,
\[
(D^\alpha u \ast \psi)(x) = \int_{U_0} \left( \int_{U} D^\alpha u (y) \phi_l (x - y) \, dy \right) \psi (x) \, dx \equiv (-1)^{|\alpha|} \int_{U} u(y) \int_{U_0} \phi_l (x - y) \, D^\alpha \psi (x) \, dxdy.
\]
It follows that $D^\alpha (\tilde{u} \ast \phi_l) = (D^\alpha u \ast \phi_l)$ as weak derivatives defined on $C_c^\infty (U_0)$.
Therefore,
\[
\|D^\alpha (\tilde{u} \ast \phi_l) - D^\alpha u\|_{L^p(U_0)} = \|D^\alpha u \ast \phi_l - D^\alpha u\|_{L^p(U_0)} \leq \|D^\alpha u \ast \phi_l - D^\alpha u\|_{L^p(\mathbb{R}^n)} \to 0.
\]
This proves the theorem.

As part of the proof of the theorem, the following corollary was established.

**Corollary 36.0.8** Let $U_0$ and $U$ be as in the above theorem. Then for all $l$ large enough and $\phi_l$ a mollifier,
\[
D^\alpha (\tilde{u} \ast \phi_l) = (D^\alpha u \ast \phi_l) \quad (36.0.2)
\]
as distributions on $C^\infty_c (U_0)$.
**Definition 36.0.9** Let $U$ be an open set. $C^\infty(U)$ denotes the set of functions which are defined and infinitely differentiable on $U$.

Note that $f(x) = \frac{1}{x}$ is a function in $C^\infty(0, 1)$. However, it is not equal to the restriction to $(0, 1)$ of some function which is in $C^\infty(\mathbb{R})$. This illustrates the distinction between $C^\infty(U)$ and $C^\infty(\overline{U})$. The set, $C^\infty(\overline{U})$ is a subset of $C^\infty(U)$. The following theorem is known as the Meyer Serrin theorem.

**Theorem 36.0.10** (Meyer Serrin) Let $U$ be an open subset of $\mathbb{R}^n$. Then if $\delta > 0$ and $u \in X^{m,p}(U)$, there exists $J \in C^\infty(U)$ such that $||J - u||_{m,p,U} < \delta$.

**Proof:** Let $\cdots U_k \subseteq \overline{U_k} \subseteq U_{k+1} \cdots$ be a sequence of open subsets of $U$ whose union equals $U$ such that $\overline{U_k}$ is compact for all $k$. Also let $U_{-3} = U_{-2} = U_{-1} = U_0 = \emptyset$. Now define $V_k \equiv U_{k+1} \setminus U_{k-1}$. Thus $\{V_k\}_{k=1}^\infty$ is an open cover of $U$. Note the open cover is locally finite and therefore, there exists a partition of unity subordinate to this open cover, $\{\eta_k\}_{k=1}^\infty$ such that each $\text{spt } (\eta_k) \in C^\infty_c(V_k)$. Let $\psi_m$ denote the sum of all the $\eta_k$ which are non zero at some point of $V_m$. Thus

$$\text{spt } (\psi_m) \subseteq U_{m+2} \setminus \overline{U_{m-2}}, \psi_m \in C^\infty_c(U), \sum_{m=1}^\infty \psi_m(x) = 1 \quad (36.0.3)$$

for all $x \in U$, and $\psi_m u \in W^{m,p}(U_{m+2})$.

Now let $\phi_l$ be a mollifier and consider

$$J \equiv \sum_{m=0}^\infty u\psi_m \ast \phi_{l_m} \quad (36.0.4)$$

where $l_m$ is chosen large enough that the following two conditions hold:

$$\text{spt } (u\psi_m \ast \phi_{l_m}) \subseteq U_{m+3} \setminus \overline{U_{m-3}}, \quad (36.0.5)$$

$$|| (u\psi_m \ast \phi_{l_m} - u\psi_m) ||_{m,p,U_{m+3}} = || (u\psi_m \ast \phi_{l_m} - u\psi_m) ||_{m,p,U} < \frac{\delta}{2m+5}, \quad (36.0.6)$$

where $\delta$ is obtained from Theorem 36.0.7. Because of \textbf{36.0.6}, only finitely many terms of the series in \textbf{36.0.4} are nonzero and therefore, $J \in C^\infty(U)$. Now let $N > 10$, some large value.

$$|| J - u ||_{m,p,U_{N-3}} = \left\| \sum_{k=0}^N \left( u\psi_k \ast \phi_{l_k} - u\psi_k \right) \right\|_{m,p,U_{N-3}}$$

$$\leq \sum_{k=0}^N \left\| u\psi_k \ast \phi_{l_k} - u\psi_k \right\|_{m,p,U_{N-3}}$$

$$\leq \sum_{k=0}^N \frac{\delta_k}{2m+5} < \delta.$$
Now apply the monotone convergence theorem to conclude that \(|J - u|_{m,p,U} \leq \delta\). This proves the theorem.

Note that \(J = 0\) on \(\partial U\). Later on, you will see that this is pathological.

In the study of partial differential equations it is the space \(W^{m,p}(U)\) which is of the most use, not the space \(X^{m,p}(U)\). This is because of the density of \(C^\infty(U)\).

Nevertheless, for reasonable open sets, \(U\), the two spaces coincide.

**Definition 36.0.11** An open set, \(U \subseteq \mathbb{R}^n\) is said to satisfy the segment condition if for all \(z \in \overline{U}\), there exists an open set \(U_z\) containing \(z\) and a vector \(a\) such that

\[
\overline{U} \cap U_z + ta \subseteq U
\]

for all \(t \in (0, 1)\).

You can imagine open sets which do not satisfy the segment condition. For example, a pair of circles which are tangent at their boundaries. The condition in the above definition breaks down at their point of tangency.

Here is a simple lemma which will be used in the proof of the following theorem.

**Lemma 36.0.12** If \(u \in W^{m,p}(U)\) and \(\psi \in C^\infty_c(\mathbb{R}^n)\), then \(u\psi \in W^{m,p}(U)\).

**Proof:** Let \(|\alpha| \leq m\) and let \(\phi \in C^\infty_c(U)\). Then

\[
(D_{x_i} (u\psi))(\phi) = - \int_U u\psi \phi_{x_i} \, dx
\]

\[
= - \int_U u ((\psi \phi)_{x_i} - \phi \psi_{x_i}) \, dx
\]

\[
= (D_{x_i} u)(\psi \phi) + \int_U u \psi_{x_i} \phi \, dx
\]

\[
= \int_U (\psi D_{x_i} u + u \psi_{x_i}) \phi \, dx
\]

Therefore, \(D_{x_i} (u\psi) = \psi D_{x_i} u + u \psi_{x_i} \in L^p(U)\). In other words, the product rule holds. Now considering the terms in the last expression, you can do the same
argument with each of these as long as they all have derivatives in $L^p(U)$. Therefore, continuing this process the lemma is proved.

**Theorem 36.0.13** Let $U$ be an open set and suppose there exists a locally finite covering of $U$ which is of the form $\{U_i\}_{i=1}^\infty$ such that each $U_i$ is a bounded open set which satisfies the conditions of Definition 36.0.11. Thus there exist vectors, $a_i$ such that for all $t \in (0, 1)$,
\[ U_i \cap U + ta_i \subseteq U. \]
Then $C^\infty(U)$ is dense in $X^{m,p}(U)$ and so $W^{m,p}(U) = X^{m,p}(U)$.

**Proof:** Let $\{\psi_i\}_{i=1}^\infty$ be a partition of unity subordinate to the given open cover with $\psi_i \in C^\infty_c(U_i)$ and let $u \in X^{m,p}(U)$. Thus
\[ u = \sum_{k=1}^\infty \psi_k u. \]
Consider $U_k$ for some $k$. Let $a_k$ be the special vector associated with $U_k$ such that
\[ ta_k + U \cap \overline{U_k} \subseteq U \tag{36.0.7} \]
for all $t \in (0, 1)$ and consider only $t$ small enough that
\[ \text{spt} (\psi_k) - ta_k \subseteq U_k \tag{36.0.8} \]
Pick $l(t) > 1/t$ which is also large enough that
\[ ta_k + U \cap \overline{U_k} + B \left(0, \frac{1}{l(t)}\right) \subseteq U, \ \text{spt} (\psi_k) + B \left(0, \frac{1}{l(t_k)}\right) - ta_k \subseteq U_k. \tag{36.0.9} \]
This can be done because $ta_k + U \cap \overline{U_k}$ is a compact subset of $U$ and so has positive distance to $U^C$ and $\text{spt} (\psi_k) - ta_k$ is a compact subset of $U_k$ having positive distance to $U_k^C$. Let $t_k$ be such a value for $t$ and for $\phi_i$ a mollifier, define
\[ v_{t_k}(x) = \int_{\mathbb{R}^n} \tilde{u}(x + t_k a_k - y) \psi_k (x + t_k a_k - y) \phi_{l(t_k)} (y) \, dy \tag{36.0.10} \]
where as usual, $\tilde{u}$ is the zero extension of $u$ off $U$. For $v_{t_k}(x) \neq 0$, it is necessary that $x + t_k a_k - y \in \text{spt} (\psi_k)$ for some $y \in B \left(0, \frac{1}{l(t_k)}\right)$. Therefore, using \[ \text{(Wكر)} \] for $v_{t_k}(x) \neq 0$, it is necessary that
\[ x \in y - t_k a_k + U \cap \text{spt} (\psi_k) \subseteq B \left(0, \frac{1}{l(t_k)}\right) + \text{spt} (\psi_k) - t_k a_k \]
\[ \text{This is never a problem in } \mathbb{R}^n. \text{ In fact, every open covering has a locally finite subcovering in } \mathbb{R}^n \text{ or more generally in any metric space due to Stone’s theorem. These are issues best left to you in case you are interested. I am usually interested in bounded sets, } U, \text{ and for these, there is a finite covering.} \]
showing that $v_{t_k}$ has compact support in $U_k$. Now change variables in (36.0.10) to obtain

$$v_{t_k}(x) = \int_{\mathbb{R}^n} \tilde{u}(y) \psi_k(y) \phi_{l(t_k)}(x + t_k a_k - y) \, dy.$$  \hspace{1cm} (36.0.11)

For $x \in U \cap U_k$, the above equals zero unless $y - t_k a_k - x \in B(0, \frac{1}{l(t_k)})$ which implies by (36.0.9) that

$$y \in t_k a_k + U \cap U_k + B(0, \frac{1}{l(t_k)}) \subseteq U.$$

Therefore, for such $x \in U \cap U_k$, (36.0.11) reduces to

$$v_{t_k}(x) = \int_{\mathbb{R}^n} u(y) \psi_k(y) \phi_{l(t_k)}(x + t_k a_k - y) \, dy = \int_{U} u(y) \psi_k(y) \phi_{l(t_k)}(x + t_k a_k - y) \, dy.$$

It follows that for $|\alpha| \leq m$, and $x \in U \cap U_k$

$$D^\alpha v_{t_k}(x) = \int_{U} u(y) \psi_k(y) D^\alpha \phi_{l(t_k)}(x + t_k a_k - y) \, dy = \int_{U} D^\alpha (u \psi_k)(y) \phi_{l(t_k)}(x + t_k a_k - y) \, dy = \int_{\mathbb{R}^n} D^\alpha (u \psi_k)(x + t_k a_k - y) \phi_{l(t_k)}(y) \, dy.$$  \hspace{1cm} (36.0.12)

Actually, this formula holds for all $x \in U$. If $x \in U$ but $x \notin U_k$, then the left side of the above formula equals zero because, as noted above, spt($v_{t_k}$) $\subseteq U_k$. The integrand of the right side equals zero unless

$$x \in B \left( 0, \frac{1}{l(t_k)} \right) + \text{spt} (\psi_k) - t_k a_k \subseteq U_k$$

by (36.0.10) and here $x \notin U_k$.

Next an estimate is obtained for $\|D^\alpha v_{t_k} - D^\alpha (u \psi_k)\|_{L^p(U)}$. By (36.0.12),

$$\|D^\alpha v_{t_k} - D^\alpha (u \psi_k)\|_{L^p(U)} \leq$$
\begin{align*}
\left( \int_U \left( \int_{\mathbb{R}^n} \left| \widetilde{D^\alpha (u \psi_k)} (x + t_k a_k - y) - \widetilde{D^\alpha (u \psi_k)} (x) \right| \phi_l(t_k) (y) \, dy \right) \, dx \right)^{1/p} & \\
\leq & \\
\int_{\mathbb{R}^n} \phi_l(t_k) (y) \left( \int_U \left| \widetilde{D^\alpha (u \psi_k)} (x + t_k a_k - y) - \widetilde{D^\alpha (u \psi_k)} (x) \right|^p \, dx \right)^{1/p} \, dy \\
\leq & \varepsilon \frac{2^{-k}}{
}
\end{align*}
whenever \( t_k \) is taken small enough. Pick \( t_k \) this small and let \( w_k \equiv v_{t_k} \). Thus

\[ ||D^\alpha w_k - D^\alpha (u \psi_k)||_{L^p (U)} \leq \frac{\varepsilon}{2^k} \]  
and \( w_k \in C_c^\infty (\mathbb{R}^n) \). Now let

\[ J (x) \equiv \sum_{k=1}^{\infty} w_k \]

Since the \( U_k \) are locally finite and \( \text{spt} (w_k) \subseteq U_k \) for each \( k \), it follows

\[ D^\alpha J = \sum_{k=0}^{\infty} D^\alpha w_k \]
and the sum is always finite. Similarly,

\[ D^\alpha \sum_{k=1}^{\infty} (\psi_k u) = \sum_{k=1}^{\infty} D^\alpha (\psi_k u) \]
and the sum is always finite. Therefore,

\[ ||D^\alpha J - D^\alpha u||_{L^p (U)} = \left( \sum_{k=1}^{\infty} ||D^\alpha w_k - D^\alpha (\psi_k u)||_{L^p (U)} \right) \]
\[ \leq \sum_{k=1}^{\infty} ||D^\alpha w_k - D^\alpha (\psi_k u)||_{L^p (U)} \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon. \]

By choosing \( t_k \) small enough, such an inequality can be obtained for

\[ ||D^\beta J - D^\beta u||_{L^p (U)} \]
for each multi index, \( \beta \) such that \( |\beta| \leq m \). Therefore, there exists

\[ J \in C_c^\infty (\mathbb{R}^n) \]
such that\n
\[ ||J - u||_{W^{m,p} (U)} \leq \varepsilon K \]
where \( K \) equals the number of multi indices no larger than \( m \). Since \( \varepsilon \) is arbitrary, this proves the theorem.
Corollary 36.0.14 Let \( U \) be an open set which has the segment property. Then \( W^{m,p}(U) = X^{m,p}(U) \).

**Proof:** Start with an open covering of \( U \) whose sets satisfy the segment condition and obtain a locally finite refinement consisting of bounded sets which are of the sort in the above theorem.

Now consider a situation where \( h : U \rightarrow V \) where \( U \) and \( V \) are two open sets in \( \mathbb{R}^n \) and \( D^\alpha h \) exists and is continuous and bounded if \( |\alpha| < m - 1 \) and \( D^\alpha h \) is Lipschitz if \( |\alpha| = m - 1 \).

**Definition 36.0.15** Whenever \( h : U \rightarrow V \), define \( h^* \) mapping the functions which are defined on \( V \) to the functions which are defined on \( U \) as follows.

\[ h^* f(x) \equiv f(h(x)). \]

\( h : U \rightarrow V \) is bilipschitz if \( h \) is one to one, onto and Lipschitz and \( h^{-1} \) is also one to one, onto and Lipschitz.

**Theorem 36.0.16** Let \( h : U \rightarrow V \) be one to one and onto where \( U \) and \( V \) are two open sets. Also suppose that \( D^\alpha h \) and \( D^\alpha (h^{-1}) \) exist and are Lipschitz continuous if \( |\alpha| \leq m - 1 \) for \( m \) a positive integer. Then \( h^* : W^{m,p}(V) \rightarrow W^{m,p}(U) \) is continuous, linear, one to one, and has an inverse with the same properties, the inverse being \( (h^{-1})^* \).

**Proof:** It is clear that \( h^* \) is linear. It is required to show it is one to one and continuous. First suppose \( h^* f = 0 \). Then

\[ 0 = \int_V |f(h(x))|^p \, dx \]

and so \( f(h(x)) = 0 \) for a.e. \( x \in U \). Since \( h \) is Lipschitz, it takes sets of measure zero to sets of measure zero. Therefore, \( f(y) = 0 \) a.e. This shows \( h^* \) is one to one.

By the Meyer Serrin theorem, Theorem 36.0.10, it suffices to verify that \( h^* \) is continuous on functions in \( C^\infty(V) \). Let \( f \) be such a function. Then using the chain rule and product rule, \( (h^* f)_i(x) = f_k(h(x)) h_{k,i}(x) \),

\[ (h^* f)_{ij}(x) = (f_{k,h(x)} h_{k,i}(x))_{ij} = f_{kl}(h(x)) h_{l,j}(x) h_{k,i}(x) + f_{k,h(x)} h_{k,ij}(x) \]

etc. In general, for \( |\alpha| \leq m - 1 \), successive applications of the product rule and chain rule yield that \( D^\alpha (h^* f)(x) \) has the form

\[ D^\alpha (h^* f)(x) = \sum_{|\beta| \leq |\alpha|} h^* (D^\beta f)(x) g_\beta(x) \]
where \( g_\beta \) is a bounded Lipschitz function with Lipschitz constant dependent on \( h \) and its derivatives. It only remains to take one more derivative of the functions, \( D^\alpha f \) for \( |\alpha| = m - 1 \). This can be done again but this time you have to use Rademacher’s theorem which assures you that the derivative of a Lipschitz function exists a.e. in order to take the partial derivative of the \( g_\beta (x) \). When this is done, the above formula remains valid for all \( |\alpha| \leq m \). Therefore, using the change of variables formula for multiple integrals, Corollary 33.6.14 on Page 1293,

\[
\int_U |D^\alpha (h^* f) (x)|^p \, dx \leq C_{m,p,h} \sum_{|\beta| \leq m} \int_U |h^* (D^\beta f) (x)|^p \, dx
\]

\[
= C_{m,p,h} \sum_{|\beta| \leq m} \int_U |(D^\beta f) (h(x))|^p \, dx
\]

\[
= C_{m,p,h} \sum_{|\beta| \leq m} \int_V |(D^\beta f) (y)|^p |\det Dh^{-1} (y)| \, dy
\]

\[
\leq C_{m,p,h,h^{-1}} \|f\|_{m,p,V}
\]

This shows \( h^* \) is continuous on \( C^\infty (V) \cap W^{m,p} (U) \) and since this set is dense, this proves \( h^* \) is continuous. The same argument applies to \((h^{-1})^* \) and now the definitions of \( h^* \) and \((h^{-1})^* \) show these are inverses.

### 36.1 Embedding Theorems For \( W^{m,p} (\mathbb{R}^n) \)

Recall Theorem 33.5.1 which is listed here for convenience.

**Theorem 36.1.1** Suppose \( u, u_i \in L^p_{loc} (\mathbb{R}^n) \) for \( i = 1, \ldots, n \) and \( p > n \). Then \( u \) has a representative, still denoted by \( u \), such that for all \( x, y \in \mathbb{R}^n \),

\[
|u (x) - u (y)| \leq C \left( \int_{B(x,2|y-x|)} |\nabla u|^p \, dz \right)^{1/p} |x - y|^{(1-n/p)}. \quad (36.1.13)
\]

This amazing result shows that every \( u \in W^{m,p} (\mathbb{R}^n) \) has a representative which is continuous provided \( p > n \).

Using the above inequality, one can give an important embedding theorem.

**Definition 36.1.2** Let \( X, Y \) be two Banach spaces and let \( f : X \to Y \) be a function. Then \( f \) is a compact map if whenever \( S \) is a bounded set in \( X \), it follows that \( f (S) \) is precompact in \( Y \).

**Theorem 36.1.3** Let \( U \) be a bounded open set and for \( u \) a function defined on \( \mathbb{R}^n \), let \( r_U u (x) \equiv u (x) \) for \( x \in \overline{U} \). Then if \( p > n, r_U : W^{1,p} (\mathbb{R}^n) \to C (\overline{U}) \) is continuous and compact.
36.1. **EMBEDDING THEOREMS FOR** $W^{1,p}(\mathbb{R}^n)$

**Proof:** First suppose $u_k \to 0$ in $W^{1,p}(\mathbb{R}^n)$. Then if $r_U u_k$ does not converge to 0, it follows there exists a sequence, still denoted by $k$ and $\varepsilon > 0$ such that $u_k \to 0$ in $W^{1,p}(\mathbb{R}^n)$ but $\|r_U u_k\|_\infty \geq \varepsilon$. Selecting a further subsequence which is still denoted by $k$, you can also assume $u_k(x) \to 0$ a.e. Pick such an $x_0 \in U$ where this convergence takes place. Then from (1), for all $x \in U$,

$$|u_k(x)| \leq |u_k(x_0)| + C \|u_k\|_{1,p,\mathbb{R}^n} \text{diam}(U)$$

showing that $u_k$ converges uniformly to 0 on $U$ contrary to $\|r_U u_k\|_\infty \geq \varepsilon$. Therefore, $r_U$ is continuous as claimed.

Next let $S$ be a bounded subset of $W^{1,p}(\mathbb{R}^n)$ with $\|u\|_{1,p} < M$ for all $u \in S$. Then for $u \in S$

$$r^p m_n ([|u| > r] \cap U) \leq \int_{[|u| > r] \cap U} |u|^p \, dm_n \leq M^p$$

and so

$$m_n ([|u| > r] \cap U) \leq \frac{M^p}{r^p}.$$ 

Now choosing $r$ large enough, $M^p/r^p < m_n(U)$ and so, for such $r$, there exists $x_u \in U$ such that $|u(x_u)| \leq r$. Therefore from (1), whenever $x, y \in U$ implies

$$|u(x) - u(y)| \leq CM |x - y|^{1 - \frac{n}{p}}$$

showing that $\{r_U u : u \in S\}$ is uniformly bounded. But also, for $x, y \in U$ implies

$$|u(x) - u(y)| \leq CM |x - y|^{1 - \frac{n}{p}}$$

showing that $\{r_U u : u \in S\}$ is equicontinuous. By the Ascoli Arzelà theorem, it follows $r_U(S)$ is precompact and so $r_U$ is compact.

**Definition 36.1.4** Let $\alpha \in (0, 1]$ and $K$ a compact subset of $\mathbb{R}^n$

$$C^\alpha(K) \equiv \{f \in C(K) : \rho_\alpha(f) + \|f\| \equiv \|f\|_\alpha < \infty\}$$

where

$$\|f\| \equiv \|f\|_\infty \equiv \sup\{|f(x)| : x \in K\}$$

and

$$\rho_\alpha(f) \equiv \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} : x, y \in K, x \neq y \right\}.$$ 

Then $(C^\alpha(K), \|\|_\alpha)$ is a complete normed linear space called a Holder space.

The verification that this is a complete normed linear space is routine and is left for you. More generally, one considers the following class of Holder spaces.
Definition 36.1.5 Let $K$ be a compact subset of $\mathbb{R}^n$ and let $\lambda \in (0, 1]$. $C^{m,\lambda}(K)$ denotes the set of functions, $u$ which are restrictions of functions defined on $\mathbb{R}^n$ to $K$ such that for $|\alpha| \leq m$,

$$D^\alpha u \in C(K)$$

and if $|\alpha| = m$,

$$D^\alpha u \in C^\lambda(K).$$

Thus $C^{0,\lambda}(K) = C^\lambda(K)$. The norm of a function in $C^{m,\lambda}(K)$ is given by

$$||u||_{m,\lambda} \equiv \sup_{|\alpha|=m} \rho_\lambda(D^\alpha u) + \sum_{|\alpha|\leq m} ||D^\alpha u||_\infty.$$ 

Lemma 36.1.6 Let $m$ be a positive integer, $K$ a compact subset of $\mathbb{R}^n$, and let $0 < \beta < \lambda \leq 1$. Then the identity map from $C^{m,\lambda}(K)$ into $C^{m,\beta}(K)$ is compact.

Proof: First note that the containment is obvious because for any function, $f$,

$$\rho_\lambda(f) \equiv \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\lambda} : x, y \in K, x \neq y \right\} < \infty,$$

Then

$$\rho_\beta(f) \equiv \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^{\beta}} : x, y \in K, x \neq y \right\}$$

$$= \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\lambda} : x, y \in K, x \neq y \right\}$$

$$\leq \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^{\lambda-\beta}} \cdot \text{diam}(K)^{\lambda-\beta} : x, y \in K, x \neq y \right\} < \infty.$$ 

Suppose the identity map, id, is not compact. Then there exists $\varepsilon > 0$ and a sequence, $\{f_k\}_{k=1}^\infty \subseteq C^{m,\lambda}(K)$ such that $||f_k||_{m,\lambda} < M$ for all $k$ but $||f_k - f_l||_\beta \geq \varepsilon$ whenever $k \neq l$. By the Ascoli Arzela theorem, there exists a subsequence of this, still denoted by $f_k$ such that $\sum_{|\alpha|\leq m} ||D^\alpha (f_l - f_k)||_\infty < \delta$ where $\delta$ satisfies

$$0 < \delta < \min \left( \frac{\varepsilon}{2}, \frac{\varepsilon}{8} \left( \frac{\varepsilon}{8M} \right)^{\beta/(\lambda-\beta)} \right).$$

Therefore, $\sup_{|\alpha|=m} \rho_\beta(D^\alpha (f_k - f_l)) \geq \varepsilon - \delta$ for all $k \neq l$. It follows that there exist pairs of points and a multi index, $\alpha$ with $|\alpha| = m$, $\{x_{kl}, y_{kl}, \alpha\}$ such that

$$\frac{\varepsilon - \delta}{2} < \frac{|(D^\alpha f_k - D^\alpha f_l)(x_{kl}) - ((D^\alpha f_k - D^\alpha f_l)(y_{kl}))|}{|x_{kl} - y_{kl}|^\beta} \leq 2M |x_{kl} - y_{kl}|^{\lambda-\beta}$$

and so considering the ends of the above inequality,

$$\left( \frac{\varepsilon - \delta}{4M} \right)^{1/(\lambda-\beta)} < |x_{kl} - y_{kl}|.$$

Now also, since $\sum_{|\alpha| \leq m} ||D^\alpha (f_l - f_k)||_\infty < \delta$, it follows from the first inequality in 36.1.15 that
\[
\frac{\varepsilon - \delta}{2} < \frac{2\delta}{(\varepsilon/2\delta)^{\beta/(\lambda - \beta)}}.
\]
Since $\delta < \varepsilon/2$, this implies
\[
\frac{\varepsilon}{4} < \frac{2\delta}{(\varepsilon/8\delta)^{\beta/(\lambda - \beta)}}
\]
and so
\[
\left(\frac{\varepsilon}{8}\right) \left(\frac{\varepsilon}{8\delta}\right)^{\beta/(\lambda - \beta)} < \delta
\]
contrary to 36.1.14. This proves the lemma.

**Corollary 36.1.7** Let $p > n, U$ and $r_U$ be as in Theorem 36.1.3 and let $m$ be a nonnegative integer. Then $r_U : W^{m+1,p} (\mathbb{R}^n) \to C^{m,\lambda} (\overline{U})$ is continuous as a map into $C^{m,\lambda} (\overline{U})$ for all $\lambda \in [0, 1 - \frac{n}{p}]$ and $r_U$ is compact if $\lambda < 1 - \frac{n}{p}$.

**Proof:** Suppose $u_k \to 0$ in $W^{m+1,p} (\mathbb{R}^n)$. Then from 36.1.3, if $\lambda \leq 1 - \frac{n}{p}$ and $|\alpha| = m$
\[
\rho_\lambda (D^\alpha u_k) \leq C ||D^\alpha u_k||_{1,p} \text{diam} (U)^{1 - \frac{n}{p} - \lambda}.
\]
Therefore, $\rho_\lambda (D^\alpha u_k) \to 0$. From Theorem 36.1.3 it follows that for $|\alpha| \leq m$,
\[
||D^\alpha u_k|| \to 0
\]
and so $||u_k||_{m,\lambda} \to 0$. This proves the claim about continuity. The claim about compactness for $\lambda < 1 - \frac{n}{p}$ follows from Lemma 36.1.4 and this.

(Bounded in $W^{m,p} (\mathbb{R}^n) \xrightarrow{r_U} Bounded in C^{m,1 - \frac{n}{p}} (\overline{U}) \xrightarrow{id} Compact in C^{m,\lambda} (\overline{U})$.)

It is just as important to consider the case where $p < n$. To do this case the following lemma due to Gagliardo [11] will be of interest. See also [11].

**Lemma 36.1.8** Suppose $n \geq 2$ and $w_j$ does not depend on the $j^{th}$ component of $x, x_j$. Then
\[
\int_{\mathbb{R}^n} \prod_{j=1}^n |w_j (x)| dm_n \leq \prod_{i=1}^n \left(\int_{\mathbb{R}^{n-1}} |w_j (x)|^{n-1} dm_{n-1}\right)^{1/(n-1)}.
\]
In this inequality, assume all the functions are continuous so there can be no measurability questions.

**Proof:** First note that for $n = 2$ the inequality reduces to the statement
\[
\int \int |w_1 (x_2)| |w_2 (x_1)| dx_1 dx_2 \leq \int |w_1 (x_2)| dx_2 \int |w_2 (x_1)| dx_1
\]
which is obviously true. Suppose then that the inequality is valid for some \( n \). Using Fubini’s theorem, Holder’s inequality, and the induction hypothesis,

\[
\int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1} |w_j(x)| \, dm_{n+1}
= \int_{\mathbb{R}^n} \int_{\mathbb{R}} |w_{n+1}(x)| \prod_{j=1}^{n} |w_j(x)| \, dm_n \, dx_{n+1}
= \int_{\mathbb{R}^n} |w_{n+1}(x)| \int_{\mathbb{R}} \prod_{j=1}^{n} |w_j(x)| \, dx_{n+1} \, dm_n
= \int_{\mathbb{R}^n} |w_{n+1}(x)| \left( \int_{\mathbb{R}} \prod_{j=1}^{n} |w_j(x)|^{1/n} \, dx_{n+1} \right)^{1/n} \, dm_n
= \int_{\mathbb{R}^n} |w_{n+1}(x)| \prod_{j=1}^{n} \left( \int_{\mathbb{R}} |w_j(x)|^{n} \, dx_{n+1} \right)^{1/n} \, dm_n
\leq \left( \int_{\mathbb{R}^n} |w_{n+1}(x)|^{n} \, dm_n \right)^{1/n}.
\]

This proves the lemma.
Lemma 36.1.9 If $\phi \in C^\infty_c(\mathbb{R}^n)$ and $n \geq 1$, then
\[
\|\phi\|_{n/(n-1)} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} \left\| \frac{\partial \phi}{\partial x_j} \right\|_1.
\]

Proof: The case where $n = 1$ is obvious if $n/(n-1)$ is interpreted as $\infty$. Assume then that $n > 1$ and note that for $a_i \geq 0$,
\[
\prod_{i=1}^{n} a_i \leq \left( \sum_{j=1}^{n} a_i \right)^n.
\]
In fact, the term on the left is one of many terms of the expression on the right. Therefore, taking $n^{th}$ roots
\[
\prod_{i=1}^{n} a_i^{1/n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} a_i.
\]
Then observe that for each $j = 1, 2, \ldots, n$,
\[
|\phi(x)| \leq \int_{-\infty}^{\infty} |\phi_j(x)| \, dx_j
\]
so
\[
\|\phi\|_{n/(n-1)} \leq \int_{\mathbb{R}^n} |\phi_j(x)|^{n/(n-1)} \, dm_n
\]
\[
\leq \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left( \int_{-\infty}^{\infty} |\phi_j(x)| \, dx_j \right)^{1/(n-1)} \, dm_n
\]
and from Lemma 36.1.8 this is dominated by
\[
\leq \prod_{j=1}^{n} \left( \int_{\mathbb{R}^n} |\phi_j(x)| \, dm_n \right)^{1/(n-1)}.
\]
Hence
\[
\prod_{i=1}^{n} a_i^{1/n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} a_i
\]
\[
\|\phi\|_{n/(n-1)} \leq \prod_{j=1}^{n} \left( \int_{\mathbb{R}^n} |\phi_j(x)| \, dm_n \right)^{1/n}
\]
\[
\leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} \int_{\mathbb{R}^n} |\phi_j(x)| \, dm_n
\]
\[
= \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} \|\phi_j\|_1
\]
and this proves the lemma.

The above lemma is due to Gagliardo and Nirenberg.

With this lemma, it is possible to prove a major embedding theorem which follows.
Theorem 36.1.10 Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then if $f \in W^{1,p}(\mathbb{R}^n)$,

$$||f||_q \leq \frac{1}{\sqrt{n}} \frac{(n-1)p}{n-p} ||f||_{1,p,\mathbb{R}^n}.$$ 

Proof: From the definition of $W^{1,p}(\mathbb{R}^n)$, $C^1_c(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. Here $C^1_c(\mathbb{R}^n)$ is the space of continuous functions having continuous derivatives which have compact support. The desired inequality will be established for such $\phi$ and then the density of this set in $W^{1,p}(\mathbb{R}^n)$ will be exploited to obtain the inequality for all $f \in W^{1,p}(\mathbb{R}^n)$. First note that the case where $p = 1$ follows immediately from the above lemma and so it is only necessary to consider the case where $p > 1$.

Let $\phi \in C^1_c(\mathbb{R}^n)$ and consider $|\phi|^r$ where $r > 1$. Then a short computation shows $|\phi|^r \in C^1_c(\mathbb{R}^n)$ and $||\phi|^r||_{1,p} = r |\phi|^{r-1} |\phi|$. Therefore, from Lemma 36.1.9,

$$\left(\int |\phi|^\frac{rn}{n-p} dm_n\right)^{(n-1)/n} \leq \frac{r}{\sqrt{n}} \sum_{i=1}^{n} \int |\phi|^{r-1} |\phi, i| dm_n$$

$$\leq \frac{r}{\sqrt{n}} \sum_{i=1}^{n} \left(\int |\phi, i|^p\right)^{1/p} \left(\int (|\phi|^{r-1})^{p/(p-1)} dm_n\right)^{(p-1)/p}.$$ 

Now choose $r$ such that

$$\frac{(r-1)p}{p-1} = \frac{rn}{n-1}.$$ 

That is, let $r = \frac{p(n-1)}{n-p} > 1$ and so $\frac{rn}{n-1} = \frac{np}{n-p}$. Then this reduces to

$$\left(\int |\phi|^\frac{np}{n-p} dm_n\right)^{(n-1)/n} \leq \frac{r}{\sqrt{n}} \sum_{i=1}^{n} \left(\int |\phi, i|^p\right)^{1/p} \left(\int |\phi|^\frac{np}{n-p} dm_n\right)^{(p-1)/p}.$$ 

Also, $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$ and so, dividing both sides by the last term yields

$$\left(\int |\phi|^\frac{np}{n-p} dm_n\right)^{\frac{n-p}{np}} \leq \frac{r}{\sqrt{n}} \sum_{i=1}^{n} \left(\int |\phi, i|^p\right)^{1/p} \leq \frac{r}{\sqrt{n}} ||\phi||_{1,p,\mathbb{R}^n}.$$ 

Letting $q = \frac{np}{n-p}$, it follows $\frac{1}{q} = \frac{n-p}{np} = \frac{1}{p} - \frac{1}{n}$ and

$$||\phi||_q \leq \frac{r}{\sqrt{n}} ||\phi||_{1,p,\mathbb{R}^n}.$$
36.1. EMBEDDING THEOREMS FOR $W^{m,p} (\mathbb{R}^N)$

Now let $f \in W^{m,p} (\mathbb{R}^n)$ and let $\|\phi_k - f\|_{1,p,\mathbb{R}^n} \to 0$ as $k \to \infty$. Taking another subsequence, if necessary, you can also assume $\phi_k (x) \to f (x)$ a.e. Therefore, by Fatou’s lemma,

$$
\|f\|_q \leq \liminf_{k \to \infty} \left( \int_{\mathbb{R}^n} |\phi_k (x)|^q \, dm_n \right)^{1/q} \\
\leq \liminf_{k \to \infty} \frac{r}{\sqrt{n}} \|\phi_k\|_{1,p,\mathbb{R}^n} = \|f\|_{1,p,\mathbb{R}^n}.
$$

This proves the theorem.

**Corollary 36.1.11** Suppose $mp < n$. Then $W^{m,p} (\mathbb{R}^n) \subseteq L^q (\mathbb{R}^n)$ where $q = \frac{np}{n - mp}$ and the identity map, $\text{id} : W^{m,p} (\mathbb{R}^n) \to L^q (\mathbb{R}^n)$ is continuous.

**Proof:** This is true if $m = 1$ according to Theorem 36.1.10. Suppose it is true for $m - 1$ where $m > 1$. If $u \in W^{m,p} (\mathbb{R}^n)$ and $|\alpha| \leq 1$, then $D^\alpha u \in W^{m-1,p} (\mathbb{R}^n)$ so by induction, for all such $\alpha$,

$$
D^\alpha u \in L^{\frac{np}{n - (m-1)p}} (\mathbb{R}^n).
$$

Thus $u \in W^{1,q_1} (\mathbb{R}^n)$ where

$$
q_1 = \frac{np}{n - (m-1)p}.
$$

By Theorem 36.1.10, it follows that $u \in L^q (\mathbb{R}^n)$ where

$$
\frac{1}{q} = \frac{n - (m-1)p}{np} = \frac{1}{n} - \frac{m-p}{np}.
$$

This proves the corollary.

There is another similar corollary of the same sort which is interesting and useful.

**Corollary 36.1.12** Suppose $m \geq 1$ and $j$ is a nonnegative integer satisfying $jp < n$. Then $W^{m+j,p} (\mathbb{R}^n) \subseteq W^{m,q} (\mathbb{R}^n)$ for

$$
q = \frac{np}{n - jp}
$$

and the identity map is continuous.

**Proof:** If $|\alpha| \leq m$, then $D^\alpha u \in W^{j,p} (\mathbb{R}^n)$ and so by Corollary 36.1.11, $D^\alpha u \in L^q (\mathbb{R}^n)$ where $q$ is given above. This means $u \in W^{m,q} (\mathbb{R}^n)$.

The above corollaries imply yet another interesting corollary which involves embeddings in the Holder spaces.
Corollary 36.1.13 Suppose \( jp < n < (j + 1)p \) and let \( m \) be a positive integer. Let \( U \) be any bounded open set in \( \mathbb{R}^n \). Then letting \( r_U \) denote the restriction to \( U \), \( r_U : W^{m+j,p}(\mathbb{R}^n) \to C^{m-1,\lambda}(\overline{U}) \) is continuous for every \( \lambda \leq \lambda_0 \equiv (j + 1) - \frac{n}{p} \) and if \( \lambda < (j + 1) - \frac{n}{p} \), then \( r_U \) is compact.

Proof: From Corollary 36.1.12 \( W^{m+j,p}(\mathbb{R}^n) \subseteq W^{m,q}(\mathbb{R}^n) \) where \( q \) is given by 36.1.16. Therefore, \( \frac{np}{n-jp} > n \) and so by Corollary 36.1.7, \( W^{m,q}(\mathbb{R}^n) \subseteq C^{m-1,\lambda}(\overline{U}) \) for all \( \lambda \) satisfying

\[
0 < \lambda < 1 - \frac{(n-jp)n}{np} = \frac{p(j+1)-n}{p} = (j+1) - \frac{n}{p}.
\]

The assertion about compactness follows from the compactness of the embedding of \( C^{m-1,\lambda_0}(\overline{U}) \) into \( C^{m-1,\lambda}(\overline{U}) \) for \( \lambda < \lambda_0 \). See Lemma 36.1.6.

There are other embeddings of this sort available. You should see Adams [1] for a more complete listing of these. Next are some theorems about compact embeddings. This requires some consideration of which subsets of \( L^p(U) \) are compact. The main theorem is the following. See [1].

Theorem 36.1.14 Let \( K \) be a bounded subset of \( L^p(U) \) and suppose that for all \( \varepsilon > 0 \), there exist a \( \delta > 0 \) such that if \( ||h|| > \delta \), then

\[
\int_{\mathbb{R}^n} |\overline{u}(x+h) - \overline{u}(x)|^p \, dx < \varepsilon^p
\]

(36.1.17)

Suppose also that for each \( \varepsilon > 0 \) there exists an open set, \( G \subseteq U \) such that \( \overline{G} \) is compact and for all \( u \in K \),

\[
\int_{U \setminus \overline{G}} |u(x)|^p \, dx < \varepsilon^p
\]

(36.1.18)

Then \( K \) is precompact in \( L^p(\mathbb{R}^n) \).

Proof: To save fussing first consider the case where \( U = \mathbb{R}^n \) so that \( \overline{u} = u \). Suppose the two conditions hold and let \( \phi_k \) be a mollifier of the form \( \phi_k(x) = k^n \phi(kx) \) where \( \text{spt}(\phi) \subseteq B(0,1) \). Consider

\[
K_k \equiv \{ u * \phi_k : u \in K \}.
\]

and verify the conditions for the Ascoli Arzela theorem for these functions defined on \( G \). Say \( ||u||_p \leq M \) for all \( u \in K \).
First of all, for \( u \in K \) and \( x \in \mathbb{R}^n \),
\[
|u \ast \phi_k(x)|^p \leq \left( \int |u(x - y) \phi_k(y)| dy \right)^p
\]
\[
= \left( \int |u(y) \phi_k(x - y)| dy \right)^p
\]
\[
\leq \int |u(y)|^p \phi_k(x - y) dy
\]
\[
\leq \left( \sup_{z \in \mathbb{R}^n} \phi_k(z) \right) \int |u(y)| dy \leq M \left( \sup_{z \in \mathbb{R}^n} \phi_k(z) \right)
\]
showing the functions in \( K_k \) are uniformly bounded.

Next suppose \( x, x_1 \in K_k \) and consider
\[
|u \ast \phi_k(x) - u \ast \phi_k(x_1)|
\]
\[
\leq \int |u(x - y) - u(x_1 - y)| \phi_k(y) dy
\]
\[
\leq \left( \int |u(x - y) - u(x_1 - y)|^p dy \right)^{1/p} \left( \int \phi_k(y)^q dy \right)^{q/p}
\]
which by assumption is small independent of the choice of \( u \) whenever \( |x - x_1| \) is small enough. Note that \( k \) is fixed in the above. Therefore, the set, \( K_k \) is precompact in \( C(G) \) thanks to the Ascoli Arzela theorem. Next consider how well \( u \in K \) is approximated by \( u \ast \phi_k \) in \( L^p(\mathbb{R}^n) \). By Minkowski’s inequality,
\[
\left( \int |u(x) - u \ast \phi_k(x)|^p dx \right)^{1/p}
\]
\[
\leq \left( \int \left( \int |u(x) - u(x - y)| \phi_k(y) dy \right)^p dx \right)^{1/p}
\]
\[
\leq \int_{B(0, \frac{1}{k})} \phi_k(y) \left( \int |u(x) - u(x - y)|^p dx \right)^{1/p} dy.
\]
Now let \( \eta > 0 \) be given. From there exists \( k \) large enough that for all \( u \in K \),
\[
\int_{B(0, \frac{1}{k})} \phi_k(y) \left( \int |u(x) - u(x - y)|^p dx \right)^{1/p} dy \leq \int_{B(0, \frac{1}{k})} \phi_k(y) \eta dy = \eta.
\]
Now let \( \varepsilon > 0 \) be given and let \( \delta \) and \( G \) correspond to \( \varepsilon \) as given in the hypotheses and let \( 1/k < \delta \) and also \( k \) is large enough that for all \( u \in K \),
\[
||u - u \ast \phi_k||_p < \varepsilon
\]
as in the above inequality. By the Ascoli Arzela theorem there exists an
\[
\left( \frac{\varepsilon}{m(G + B(0, 1))} \right)^{1/p}
\]
net for $K_k$ in $C(G)$. That is, there exist $\{u_i\}_{i=1}^m \subseteq K$ such that for any $u \in K$,

$$\|u * \phi_k - u_j * \phi_k\|_\infty < \left(\frac{\varepsilon}{m(G + B(0,1))}\right)^{1/p}$$

for some $j$. Letting $u \in K$ be given, let $u \in \{u_i\}_{i=1}^m \subseteq K$ be such that the above inequality holds. Then

$$\|u - u_j\|_p \leq \|u - u * \phi_k\|_p + \|u * \phi_k - u_j * \phi_k\|_p + \|u_j * \phi_k - u_j\|_p$$

$$\leq 2\varepsilon + \|u * \phi_k - u_j * \phi_k\|_p$$

$$\leq 2\varepsilon + \frac{1}{\varepsilon}$$

$$\leq \int_{G+B(0,1)} |u * \phi_k - u_j * \phi_k| \, dx$$

$$\leq 2\varepsilon + \frac{1}{\varepsilon}$$

and since $\varepsilon > 0$ is arbitrary, this shows that $K$ is totally bounded and is therefore precompact.

Now for an arbitrary open set, $U$ and $K$ given in the hypotheses of the theorem, let $\overline{K} \equiv \{u : u \in K\}$ and observe that $\overline{K}$ is precompact in $L^p(\mathbb{R}^n)$. But this is the same as saying that $K$ is precompact in $L^p(U)$. This proves the theorem.

Actually the converse of the above theorem is also true [1] but this will not be needed so I have left it as an exercise for anyone interested.

**Lemma 36.1.15** Let $u \in W^{1,1}(U)$ for $U$ an open set and let $\phi \in C_c(\mathbb{R}^n)$. Then there exists a constant,

$$C \left(\phi, \|u\|_{1,1,U}\right),$$
36.1. EMBEDDING THEOREMS FOR $W^{1,p}(\mathbb{R}^N)$

Depending only on the indicated quantities such that whenever $v \in \mathbb{R}^n$ with $|v| < \text{dist } (\text{spt } \phi, U^C)$, it follows that

$$\int_{\mathbb{R}^n} \left| \tilde{\phi}u(x + v) - \tilde{\phi}u(x) \right| \, dx \leq C \left( \phi, ||u||_{1,1,U} \right) |v|.$$ 

**Proof:** First suppose $u \in C^\infty (\overline{U})$. Then for any $x \in \text{spt } \phi \cup (\text{spt } \phi - v) \equiv G_v$, the chain rule implies

$$|\phi u (x + v) - \phi u (x)| \leq \int_0^1 \sum_{i=1}^n (\phi u)_{,i} (x + tv) v_i \, dt$$

$$\leq \int_0^1 \sum_{i=1}^n \left| (\phi, u + u_i \phi) (x + tv) \right| dt |v|.$$ 

Therefore, for such $u$,

$$\int_{\mathbb{R}^n} \left| \tilde{\phi}u(x + v) - \tilde{\phi}u(x) \right| \, dx$$

$$= \int_{G_v} |\phi u (x + v) - \phi u (x)| \, dx$$

$$\leq \int_{G_v} \int_0^1 \sum_{i=1}^n \left| (\phi, u + u_i \phi) (x + tv) \right| dt dx |v|$$

$$\leq \int_0^1 \int_{G_v} \sum_{i=1}^n \left| (\phi, u + u_i \phi) (x + tv) \right| dx dt |v|$$

$$\leq C \left( \phi, ||u||_{1,1,U} \right) |v|$$

where $C$ is a continuous function of $||u||_{1,1,U}$. Now for general $u \in W^{1,1} (U)$, let $u_k \to u$ in $W^{1,1} (U)$ where $u_k \in C^\infty (\overline{U})$. Then for $|v| < \text{dist } (\text{spt } \phi, U^C)$,

$$\int_{\mathbb{R}^n} \left| \tilde{\phi}u(x + v) - \tilde{\phi}u(x) \right| \, dx$$

$$= \int_{G_v} |\phi u (x + v) - \phi u (x)| \, dx$$

$$= \lim_{k \to \infty} \int_{G_v} |\phi u_k (x + v) - \phi u_k (x)| \, dx$$

$$\leq \lim_{k \to \infty} C \left( \phi, ||u_k||_{1,1,U} \right) |v|$$

$$= C \left( \phi, ||u||_{1,1,U} \right) |v|.$$ 

This proves the lemma.
Lemma 36.1.16 Let $U$ be a bounded open set and define for $p > 1$

$$S \equiv \left\{ u \in W^{1,1}(U) \cap L^p(U) : ||u||_{1,1,U} + ||u||_{L^p(U)} \leq M \right\}$$

(36.1.19)

and let $\phi \in C_c^\infty(U)$ and  

$$S_1 \equiv \{ u \phi : u \in S \}.$$  

(36.1.20)

Then $S_1$ is precompact in $L^q(U)$ where $1 \leq q < p$.

**Proof:** This depends on Theorem [36.1.14]. The second condition is satisfied by taking $G \equiv \text{spt}(\phi)$. Thus, for $w \in S_1$,

$$\int_{U \setminus G} |w(x)|^q \, dx = 0 < \varepsilon^p.$$  

It remains to satisfy the first condition. It is necessary to verify there exists $\delta > 0$ such that if $|v| < \delta$, then

$$\int_{\mathbb{R}^n} |\widetilde{\phi} u(x + v) - \tilde{\phi} u(x)|^q \, dx < \varepsilon^p.$$  

(36.1.21)

Let $\text{spt}(\phi) \cup (\text{spt}(\phi) - v) \equiv G_v$. Now if $h$ is any measurable function, and if $\theta \in (0,1)$ is chosen small enough that $\theta q < 1$,

$$\int_{G_v} |h|^q \, dx = \int_{G_v} |h|^\theta q |h|^{(1-\theta)q} \, dx \leq \left( \int_{G_v} |h| \, dx \right)^\theta \left( \int_{G_v} |h|^{(1-\theta)q} \right)^{1-\theta q} = \left( \int_{G_v} |h| \, dx \right)^\theta \left( \int_{G_v} |h|^\left(1-\frac{\theta q}{1-\theta q}\right) \right)^{1-\theta q}.$$  

(36.1.22)

Now let $\theta$ also be small enough that there exists $r > 1$ such that $r \left(\frac{1-\theta q}{1-\theta q}\right) = p$

and use Holder’s inequality in the last factor of the right side of (36.1.22). Then

$$\int_{G_v} |h|^{\left(1-\frac{\theta q}{1-\theta q}\right)} \, dx = C \left(||h||_{L^p(G_v)}, m_n(G_v)\right) \left( \int_{G_v} |h| \, dx \right)^\theta.$$  

Therefore, for $u \in S$,

$$\int_{\mathbb{R}^n} |\tilde{\phi} u(x + v) - \tilde{\phi} u(x)|^q \, dx = \int_{G_v} |\tilde{\phi} u(x + v) - \tilde{\phi} u(x)|^q \, dx \leq$$
36.1. EMBEDDING THEOREMS FOR $W^{M,p}(\mathbb{R}^N)$

\[
C \left( \|\phi u (+v) - \phi u (\cdot)\|_{L^p(G_\nu)}, m_n(G_\nu) \right) \left( \int_{G_\nu} |\phi u (x + v) - \phi u (x)| \, dx \right)^{\theta_q} \\
\leq C \left( 2 \|\phi u (\cdot)\|_{L^p(U)}, m_n(U) \right) \left( \int_{G_\nu} |\phi u (x + v) - \phi u (x)| \, dx \right)^{\theta_q} \\
\leq C (\phi, M, m_n(U)) \left( \int_{G_\nu} \left| \widetilde{\phi u} (x + v) - \widetilde{\phi u} (x) \right| \, dx \right)^{\theta_q} \\
= C (\phi, M, m_n(U)) \left( \int_{\mathbb{R}^n} \left| \widetilde{\phi u} (x + v) - \widetilde{\phi u} (x) \right| \, dx \right)^{\theta_q}. \quad (36.1.23)
\]

Now by Lemma 36.1.16,

\[
\int_{\mathbb{R}^n} \left| \widetilde{\phi u} (x + v) - \widetilde{\phi u} (x) \right| \, dx \leq C \left( \phi, \|u\|_{1,1,V} \right) |v| \quad (36.1.24)
\]

and so from 36.1.23 and 36.1.24, and adjusting the constants

\[
\int_{\mathbb{R}^n} \left| \widetilde{\phi u} (x + v) - \widetilde{\phi u} (x) \right|^q \, dx \leq C (\phi, M, m_n(U)) \left( C \left( \phi, \|u\|_{1,1,U} \right) |v| \right)^{\theta_q} \\
= C (\phi, M, m_n(U)) |v|^{\theta_q}
\]

which verifies 36.1.23 whenever $|v|$ is sufficiently small. This proves the lemma because the conditions of Theorem 36.1.17 are satisfied.

**Theorem 36.1.17** Let $U$ be a bounded open set and define for $p > 1$

\[
S \equiv \left\{ u \in W^{1,1}(U) \cap L^p(U) : \|u\|_{1,1,U} + \|u\|_{L^p(U)} \leq M \right\} \quad (36.1.25)
\]

Then $S$ is precompact in $L^q(U)$ where $1 \leq q < p$.

**Proof:** If suffices to show that for every sequence, $\{u_k\}_{k=1}^\infty \subseteq S$ has a subsequence which converges in $L^q(U)$. Let $\{K_m\}_{m=1}^\infty$ denote a sequence of compact subsets of $U$ with the property that $K_m \subseteq K_{m+1}$ for all $m$ and $\cup_{m=1}^\infty K_m = U$. Now let $\phi_m \in C_c^\infty(U)$ such that $\phi_m (x) \in [0,1]$ and $\phi_m (x) = 1$ for all $x \in K_m$. Let $S_m \equiv \{ \phi_m u : u \in S \}$. By Lemma 36.1.16 there exists a subsequence of $\{u_k\}_{k=1}^\infty$, denoted here by $\{u_{1,k}\}_{k=1}^\infty$, such that $\{\phi_1 u_{1,k}\}_{k=1}^\infty$ converges in $L^q(U)$. Now $S_2$ is also precompact in $L^q(U)$ and so there exists a subsequence of $\{u_{1,k}\}_{k=1}^\infty$, denoted by $\{u_{2,k}\}_{k=1}^\infty$ such that $\{\phi_2 u_{2,k}\}_{k=1}^\infty$ converges in $L^2(U)$. Thus it is also the case that $\{\phi_1 u_{1,k}\}_{k=1}^\infty$ converges in $L^q(U)$, and so forth. Continue taking subsequences in this manner such that for all $m$, $\{\phi_m u_{m,k}\}_{k=1}^\infty$ converges in $L^q(U)$, and so forth. Let $w_{m,k} \equiv \{u_{m,k}\}_{k=1}^\infty$ so that $\{w_k\}_{k=m}^\infty$ is a subsequence of $\{u_m\}_{m=1}^\infty$. Then it follows for all $k$, $\{\phi_k w_k\}_{m=1}^\infty$
must converge in \( L^q(U) \). For \( u \in S \),
\[
\|u - \phi_k u\|_{L^q(U)}^q = \int_U |u|^q (1 - \phi_k)^q \, dx \\
\leq \left( \int_U |u|^p \, dx \right)^{q/p} \left( \int_U (1 - \phi_k)^{qr} \, dx \right)^{1/r} \\
\leq M \left( \int_U (1 - \phi_k)^{qr} \, dx \right)^{1/r}
\]
where \( q/p + 1/r = 1 \). Now \( \phi_l(x) \to X \) and so the integrand in the last integral converges to 0 by the dominated convergence theorem. Therefore, \( k \) may be chosen large enough that for all \( u \in S \),
\[
\|u - \phi_k u\|_{L^q(U)}^q \leq \left( \frac{\epsilon}{3} \right)^q .
\]
Fix such a value of \( k \). Then
\[
\|w_q - w_p\|_{L^q(U)} \leq 2\|w_q - \phi_k w_q\|_{L^q(U)} + ||\phi_k w_q - \phi_k w_p\|_{L^q(U)} + ||w_p - \phi_k w_p\|_{L^q(U)} \\
\leq \frac{2\epsilon}{3} + ||\phi_k w_q - \phi_k w_p\|_{L^q(U)} .
\]
But \( \{\phi_k w_m\}_{m=1}^\infty \) converges in \( L^q(U) \) and so the last term in the above is less than \( \epsilon/3 \) whenever \( p, q \) are large enough. Thus \( \{w_m\}_{m=1}^\infty \) is a Cauchy sequence and must therefore converge in \( L^q(U) \). This proves the theorem.

### 36.2 An Extension Theorem

**Definition 36.2.1** An open subset, \( U \), of \( \mathbb{R}^n \) has a Lipschitz boundary if it satisfies the following conditions. For each \( p \in \partial U = \overline{U} \setminus U \), there exists an open set, \( Q \), containing \( p \), an open interval \( (a,b) \), a bounded open box \( B \subseteq \mathbb{R}^{n-1} \), and an orthogonal transformation \( R \) such that
\[
RQ = B \times (a,b) , \tag{36.2.26}
\]
\[
R (Q \cap U) = \{ y \in \mathbb{R}^n : \tilde{y} \in B, \ a < y_n < g(\tilde{y}) \} \tag{36.2.27}
\]
where \( g \) is Lipschitz continuous on \( \overline{B} \), \( a < \min \{ g(x) : x \in \overline{B} \} \), and
\[
\tilde{y} \equiv (y_1, \cdots , y_{n-1}).
\]
Letting \( W = Q \cap U \) the following picture describes the situation.
36.2. AN EXTENSION THEOREM

The following lemma is important.

**Lemma 36.2.2** If $U$ is an open subset of $\mathbb{R}^n$ which has a Lipschitz boundary, then it satisfies the segment condition and so $X^{m,p}(U) = W^{m,p}(U)$.

**Proof:** For $x \in \partial U$, simply look at a single open set, $Q_x$ described in the above which contains $x$. Then consider an open set whose intersection with $U$ is of the form $R^T(\{y : \hat{y} \in B, g(\hat{y}) - \varepsilon < y_n < g(\hat{y})\})$ and a vector of the form $\varepsilon R^T(-e_n)$ where $\varepsilon$ is chosen smaller than $\min \{g(x) : x \in B\} - a$. There is nothing to prove for points of $U$.

One way to extend many of the above theorems to more general open sets than $\mathbb{R}^n$ is through the use of an appropriate extension theorem. In this section, a fairly general one will be presented.

**Lemma 36.2.3** Let $B \times (a, b)$ be as described in Definition [36.2.1] and let

$$V^- \equiv \{(\hat{y}, y_n) : y_n < g(\hat{y})\}, \quad V^+ \equiv \{(\hat{y}, y_n) : y_n > g(\hat{y})\},$$

for $g$ a Lipschitz function of the sort described in this definition. Suppose $u^+$ and $u^-$ are Lipschitz functions defined on $V^+$ and $V^-$ respectively and suppose that $u^+(\hat{y}, g(\hat{y})) = u^-(\hat{y}, g(\hat{y}))$ for all $\hat{y} \in B$. Let

$$u(\hat{y}, y_n) \equiv \begin{cases} u^+(\hat{y}, y_n) & \text{if } (\hat{y}, y_n) \in V^+ \\ u^-(\hat{y}, y_n) & \text{if } (\hat{y}, y_n) \in V^- \end{cases}$$

and suppose $\text{spt}(u) \subseteq B \times (a, b)$. Then extending $u$ to be 0 off of $B \times (a, b)$, $u$ is continuous and the weak partial derivatives, $u_{i,i}$, are all in $L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $p > 1$ and $u_{i,i} = (u^+),_i$ on $V^+$ and $u_{i,i} = (u^-),_i$ on $V^-$.  

**Proof:** Consider the following picture which is descriptive of the situation.
Note first that $u$ is Lipschitz continuous. To see this, consider $|u(y_1) - u(y_2)|$ where $(\hat{y}_1, y_{n_1}^1) = y_i$. There are various cases to consider depending on whether $y_{n_1}^1$ is above $g(\hat{y}_1)$. Suppose $y_{n_1}^1 < g(\hat{y}_1)$ and $y_{n_2}^2 > g(\hat{y}_2)$. Then letting $K \geq \max(\text{Lip}(u^+), \text{Lip}(u^-), \text{Lip}(g))$,

$$|u(\hat{y}_1, y_{n_1}^1) - u(\hat{y}_2, y_{n_2}^2)| \leq K|y_{n_1}^1 - y_{n_2}^2| + K[g(\hat{y}_2) - g(\hat{y}_1)] + g(\hat{y}_1) - y_{n_1}^1 + y_{n_2}^2 - g(\hat{y}_2)$$

The other cases are similar. Thus $u$ is a Lipschitz continuous function which has compact support. By Corollary 36.2.4 on Page 1284 it follows that $u_{i,j} \in L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $p > 1$. It remains to verify $u_{i,j} = (u^+)_{i,j}$ on $V^+$ and $u_{i,j} = (u^-)_{i,j}$ on $V^-$. The last claim is obvious from the definition of weak derivatives.

**Lemma 36.2.4** In the situation of Lemma 36.2.3 let $u \in C^1(\mathbb{R}^n) \cap C^1_c(\mathbb{R}^n)$ and define

$$w(\tilde{y}, y_n) \equiv \begin{cases} u(\tilde{y}, y_n) & \text{if } \tilde{y} \in B \text{ and } y_n \leq g(\tilde{y}) \ , \\ u(\tilde{y}, 2g(\tilde{y}) - y_n) & \text{if } \tilde{y} \in B \text{ and } y_n > g(\tilde{y}) \\ 0 & \text{if } \tilde{y} \notin B. \end{cases}$$

Then $w \in W^{1,p}(\mathbb{R}^n)$ and there exists a constant, $C$ depending only on Lip $(g)$ and dimension such that

$$\|w\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$
36.2. AN EXTENSION THEOREM

\textbf{Proof:} As in the previous lemma, \( w \) is Lipschitz continuous and has compact support so it is clear \( w \in W^{1,p}(\mathbb{R}^n) \). The main task is to find \( w_{,i} \) for \( \tilde{y} \in B \) and \( y_n > g(\tilde{y}) \) and then to extract an estimate of the right sort. Denote by \( U \) the set of points of \( \mathbb{R}^n \) with the property that \( (\tilde{y}, y_n) \in U \) if and only if \( \tilde{y} \notin B \) or \( \tilde{y} \in B \) and \( y_n > g(\tilde{y}) \). Then letting \( \phi \in C_c^\infty(U) \), suppose first that \( i < n \). Then

\[
\int_U w(\tilde{y}, y_n) \phi_{,i}(y) \, dy
\]

\[
= \lim_{h \to 0} \int_U \phi(y) \frac{u(\tilde{y} - he_i^{n-1}, 2g(\tilde{y} - he_i^{n-1}) - y_n) - u(\tilde{y}, 2g(\tilde{y}) - y_n)}{h} \, dy
\]

\[
= \lim_{h \to 0} \left\{ \int_U \phi(y) \left[ D_1 u(\tilde{y}, 2g(\tilde{y}) - y_n) \left( he_i^{n-1} \right) + 2D_2 u(\tilde{y}, 2g(\tilde{y}) - y_n) \left( g(\tilde{y} - he_i^{n-1}) - g(\tilde{y}) \right) \right] \, dy + \frac{1}{h} \int_U \phi(y) \left[ o \left( g(\tilde{y} - he_i^{n-1}) - g(\tilde{y}) \right) + o(h) \right] \, dy \right\}
\]

where \( e_i^{n-1} \) is the unit vector in \( \mathbb{R}^{n-1} \) having all zeros except for a 1 in the \( i^{th} \) position. Now by Rademacher’s theorem, \( Dg(\tilde{y}) \) exists for a.e. \( \tilde{y} \) and so except for a set of measure zero, the expression, \( o \left( g(\tilde{y} - he_i^{n-1}) - g(\tilde{y}) \right) \) is \( o(h) \) and also for \( \tilde{y} \) not in the exceptional set,

\[
g(\tilde{y} - he_i^{n-1}) - g(\tilde{y}) = -h Dg(\tilde{y}) e_i^{n-1} + o(h).
\]

Therefore, since the integrand in (36.2.28) has compact support and because of the Lipschitz continuity of all the functions, the dominated convergence theorem may be applied to obtain

\[
\int_U w(\tilde{y}, y_n) \phi_{,i}(y) \, dy = \int_U \phi(y) \left[ -D_1 u(\tilde{y}, 2g(\tilde{y}) - y_n) \left( e_i^{n-1} \right) + 2D_2 u(\tilde{y}, 2g(\tilde{y}) - y_n) \left( Dg(\tilde{y}) e_i^{n-1} \right) \right] \, dy
\]

\[
= \int_U \phi(y) \left[ -\frac{\partial u}{\partial y_i}(\tilde{y}, 2g(\tilde{y}) - y_n) + 2 \frac{\partial u}{\partial y_n}(\tilde{y}, 2g(\tilde{y}) - y_n) \frac{\partial g(\tilde{y})}{\partial y_i} \right] \, dy
\]

and so

\[
w_{,i}(y) = \frac{\partial u}{\partial y_i}(\tilde{y}, 2g(\tilde{y}) - y_n) - 2 \frac{\partial u}{\partial y_n}(\tilde{y}, 2g(\tilde{y}) - y_n) \frac{\partial g(\tilde{y})}{\partial y_i} \quad (36.2.29)
\]

whenever \( i < n \) which is what you would expect from a formal application of the chain rule. Next suppose \( i = n \).

\[
\int_U w(\tilde{y}, y_n) \phi_{,n}(y) \, dy
\]
\[ \lim_{h \to 0} \int_U \frac{u(y, 2g(y) - (y_n + h)) - u(y, 2g(y) - y_n)}{h} \phi(y) \, dy \]
\[ = \lim_{h \to 0} \int_U \frac{D_2u(y, 2g(y) - y_n) h + o(h)}{h} \phi(y) \, dy \]
\[ = \int_U \frac{\partial u}{\partial y_n} (y, 2g(y) - y_n) \phi(y) \, dy \]
showing that
\[ w_n(y) = -\frac{\partial u}{\partial y_n} (y, 2g(y) - y_n) \] (36.2.30)
which is also expected.

From the definition, for \( y \in \mathbb{R}^n \setminus U \equiv \{(y_n) : y_n \leq g(y)\} \) it follows \( w_i = u_i \) and on \( U, w_i \) is given by (36.2.29) and (36.2.30). Consider \( \|w_i\|_{L^p(U)}^p \) for \( i < n \). From (36.2.30)
\[ \|w_i\|_{L^p(U)}^p = \int_U \left| \frac{\partial u}{\partial y_i} (y, 2g(y) - y_n) - 2 \frac{\partial u}{\partial y_n} (y, 2g(y) - y_n) \frac{\partial g(y)}{\partial y_i} \right|^p \, dy \]
\[ \leq 2^{p-1} \int_U \left| \frac{\partial u}{\partial y_i} (y, 2g(y) - y_n) \right|^p \, dy + 2^p \left| \frac{\partial u}{\partial y_n} (y, 2g(y) - y_n) \right|^p \text{Lip}(g)^p \, dy \]
\[ \leq 4^p (1 + \text{Lip}(g))^p \int_U \left| \frac{\partial u}{\partial y_i} (y, 2g(y) - y_n) \right|^p \, dy \]
\[ + \left| \frac{\partial u}{\partial y_n} (y, 2g(y) - y_n) \right|^p \, dy \]
\[ = 4^p (1 + \text{Lip}(g))^p \int_B \int_{g(y)}^{g(y')} \left| \frac{\partial u}{\partial y_i} (y, z_n) \right|^p \, dz_n \, d\tilde{y} \]
\[ + \left| \frac{\partial u}{\partial y_n} (y, z_n) \right|^p \, dz_n \, d\tilde{y} \]
\[ = 4^p (1 + \text{Lip}(g))^p \int_B \int_{g(y)}^{g(y')} \left| \frac{\partial u}{\partial y_i} (y, z_n) \right|^p \, dz_n \, d\tilde{y} + 4^p (1 + \text{Lip}(g))^p \int_B \int_{g(y)}^{g(y')} \left| \frac{\partial u}{\partial y_n} (y, z_n) \right|^p \, dz_n \, d\tilde{y} \]
\[ \leq 4^p (1 + \text{Lip}(g))^p \|u\|_{1,p,V}^p \]
Now by similar reasoning,

\[
\|w_n\|_{L^p(U)}^p = \int_U \left| -\frac{\partial u}{\partial y_n}(\tilde{y}, 2g(\tilde{y}) - y_n) \right|^p \, dy
\]

\[
= \int_B \int_{g(\tilde{y})}^\infty \left| -\frac{\partial u}{\partial y_n}(\tilde{y}, 2g(\tilde{y}) - y_n) \right|^p \, dy_n d\tilde{y}
\]

\[
= \int_B \int_a^{g(\tilde{y})} \left| -\frac{\partial u}{\partial y_n}(\tilde{y}, z_n) \right|^p \, dz_n d\tilde{y} = \|u_n\|_{1,p,V}^p.
\]

It follows

\[
\|w\|_{1,p,\mathbb{R}^n}^p = \|w\|_{1,p,U}^p + \|u\|_{1,p,V}^p
\]

\[
\leq 4^p n (1 + \text{Lip}(g)^p) \|u\|_{1,p,V}^p + \|u\|_{1,p,V}^p
\]

and so

\[
\|w\|_{1,p,\mathbb{R}^n}^p \leq 4^p n (2 + \text{Lip}(g)^p) \|u\|_{1,p,V}^p
\]

which implies

\[
\|w\|_{1,p,\mathbb{R}^n} \leq 4n^{1/p} (2 + \text{Lip}(g)^p)^{1/p} \|u\|_{1,p,V}.
\]

It is obvious that \(E_0\) is a continuous linear mapping. This proves the lemma.

Now recall Definition 36.2.1, listed here for convenience.

**Definition 36.2.5** An open subset, \(U\), of \(\mathbb{R}^n\) has a Lipschitz boundary if it satisfies the following conditions. For each \(p \in \partial U \equiv \overline{U} \setminus U\), there exists an open set, \(Q\), containing \(p\), an open interval \((a, b)\), a bounded open box \(B \subseteq \mathbb{R}^{n-1}\), and an orthogonal transformation \(R\) such that

\[
RQ = B \times (a, b),
\]

\[
R(Q \cap U) = \{ y \in \mathbb{R}^n : \tilde{y} \in B, \, a < y_n < g(\tilde{y}) \}\]

where \(g\) is Lipschitz continuous on \(\overline{B}, a < \min \{ g(x) : x \in \overline{B} \}\), and

\[
\tilde{y} = (y_1, \cdots, y_{n-1}).
\]

Letting \(W = Q \cap U\) the following picture describes the situation.
Chapter 36. Basic Theory of Sobolev Spaces

Lemma 36.2.6 In the situation of Definition 36.2.1, let \( u \in C^1(U) \cap C^1_c(Q) \) and define

\[
E u := R^* (R^T)^* u.
\]

where \( (R^T)^* \) maps \( W^{1,p}(U \cap Q) \) to \( W^{1,p}(R^n) \). Then \( E \) is linear and satisfies

\[
||E u||_{W^{1,p}(R^n)} \leq C ||u||_{W^{1,p}(Q \cap U)}, \quad E u(x) = u(x) \text{ for } x \in Q \cap U.
\]

where \( C \) depends only on the dimension and \( \text{Lip}(g) \).

Proof: This follows from Theorem 36.0.16 and Lemma 36.2.4.

The following theorem is a general extension theorem for Sobolev spaces.

Theorem 36.2.7 Let \( U \) be a bounded open set which has Lipschitz boundary. Then for each \( p \geq 1 \), there exists \( E \in L(W^{1,p}(U), W^{1,p}(R^n)) \) such that \( E u(x) = u(x) \) a.e. \( x \in U \).

Proof: Let \( \partial U \subseteq \bigcup_{i=1}^p Q_i \) Where the \( Q_i \) are as described in Definition 36.2.8. Also let \( R_i \) be the orthogonal transformation and \( g_i \) the Lipschitz functions associated with \( Q_i \) as in this definition. Now let \( Q_0 \subseteq Q_0 \subseteq U \) be such that \( U \subseteq \bigcup_{i=0}^p Q_i \), and let \( \psi_i \in C^\infty_c(Q_i) \) with \( \psi_i(x) \in [0,1] \) and \( \sum_{i=0}^p \psi_i(x) = 1 \) on \( U \). For \( u \in C^\infty_c(U) \), let \( E^0(\psi_0 u) \equiv \psi_0 u \) on \( Q_0 \) and 0 off \( Q_0 \). Thus

\[
||E^0(\psi_0 u)||_{1,p,R^n} = ||\psi_0 u||_{1,p,U}.
\]

For \( i \geq 1 \), let

\[
E^i(\psi_i u) \equiv R^*_i E_0(R^T)^*(\psi_i u).
\]

Thus, by Lemma 36.2.4,

\[
||E^i(\psi_i u)||_{1,p,R^n} \leq C ||\psi_i u||_{1,p,c \cap U}
\]

where the constant depends on \( \text{Lip}(g_i) \) but is independent of \( u \in C^\infty_c(U) \). Now define \( E \) as follows.

\[
E u : = \sum_{i=0}^p E^i(\psi_i u).
\]

Thus for \( u \in C^\infty_c(U) \), it follows \( E u(x) = u(x) \) for all \( x \in U \). Also,

\[
||E u||_{1,p,R^n} \leq \sum_{i=0}^p ||E^i(\psi_i u)|| \leq \sum_{i=0}^p C_i ||\psi_i u||_{1,p,c \cap U}
\]

\[
= \sum_{i=0}^p C_i ||\psi_i u||_{1,p,U} \leq \sum_{i=0}^p C_i ||u||_{1,p,U}
\]

\[
\leq (p+1) \sum_{i=0}^p C_i ||u||_{1,p,U} \equiv C ||u||_{1,p,U}.
\]
36.3. **GENERAL EMBEDDING THEOREMS**

where $C$ depends on the $\psi_i$ and the $g_i$ but is independent of $u \in C^\infty (\overline{U})$. Therefore, by density of $C^\infty (\overline{U})$ in $W^{1,p} (U)$, $E$ has a unique continuous extension to $W^{1,p} (U)$ still denoted by $E$ satisfying the inequality determined by the ends of $36.2.3$. It remains to verify that $Eu (x) = u (x)$ a.e. for $x \in U$.

Let $u_k \to u$ in $W^{1,p} (U)$ where $u_k \in C^\infty (\overline{U})$. Therefore, by 36.2.3, $Eu_k \to Eu$ in $W^{1,p} (\mathbb{R}^n)$. Since $Eu_k (x) = u_k (x)$ for each $k$,

$$\|u - Eu\|_{L^p(U)} = \lim_{k \to \infty} \|u_k - Eu_k\|_{L^p(U)} = \lim_{k \to \infty} \|Eu_k - Eu_k\|_{L^p(U)} = 0$$

which shows $u (x) = Eu (x)$ for a.e. $x \in U$ as claimed. This proves the theorem.

**Definition 36.2.8** Let $U$ be an open set. Then $W_0^{m,p} (U)$ is the closure of the set, $C_0^\infty (U)$ in $W^{m,p} (U)$.

**Corollary 36.2.9** Let $U$ be a bounded open set which has Lipschitz boundary and let $W$ be an open set containing $\overline{U}$. Then for each $p \geq 1$, there exists $E_W \in \mathcal{L} \left(W^{1,p} (U) , W_0^{1,p} (W) \right)$ such that $E_W u (x) = u (x)$ a.e. $x \in U$.

**Proof:** Let $\psi \in C_0^\infty (W)$ and $\psi = 1$ on $U$. Then let $E_W u \equiv \psi Eu$ where $E$ is the extension operator of Theorem 36.2.7.

Extension operators of the above sort exist for many open sets, $U$, not just for bounded ones. In particular, the above discussion would apply to an open set, $U$, not necessarily bounded, if you relax the condition that the $Q_i$ must be bounded but require the existence of a finite partition of unity $\{\psi_i\}_{i=1}$ having the property that $\psi_i$ and $\psi_i j$ are uniformly bounded for all $i, j$. The proof would be identical to the above. My main interest is in bounded open sets so the above theorem will suffice. Such an extension operator will be referred to as a $(1, p)$ extension operator.

### 36.3 General Embedding Theorems

With the extension theorem it is possible to give a useful theory of embeddings.

**Theorem 36.3.1** Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and let $U$ be any open set for which there exists a $(1, p)$ extension operator. Then if $u \in W^{1,p} (U)$, there exists a constant independent of $u$ such that

$$\|u\|_{L^q(U)} \leq C \|u\|_{1,p,U}.$$  

If $U$ is bounded and $r < q$, then $\text{id} : W^{1,p} (U) \to L^r (U)$ is also compact.

**Proof:** Let $E$ be the $(1, p)$ extension operator. Then by Theorem 36.2.7 on Page 1356,

$$\|u\|_{L^q(U)} \leq \|Eu\|_{L^q(\mathbb{R}^n)} \leq \frac{1}{\sqrt{n}} \left(\frac{n-1}{n} \right)^p \|Eu\|_{1,p,\mathbb{R}^n} \leq C \|u\|_{1,p,U}.$$
It remains to prove the assertion about compactness. If \( S \subseteq W^{1,p}(U) \) is bounded then
\[
\sup_{u \in S} \left\{ ||u||_{1,1,U} + ||u||_{L^q(U)} \right\} < \infty
\]
and so by Theorem 36.1.17 on Page 1349, it follows \( S \) is precompact in \( L^r(U) \). This proves the theorem.

**Corollary 36.3.2** Suppose \( mp < n \) and \( U \) is an open set satisfying the segment condition which has a \((1,p)\) extension operator for all \( p \). Then \( \text{id} \in L(W^{m,p}(U), L^q(U)) \) where \( q = \frac{np}{n-mp} \).

**Proof:** This is true if \( m = 1 \) according to Theorem 36.3.1. Suppose it is true for \( m-1 \) where \( m > 1 \). If \( u \in W^{m,p}(U) \) and \( |\alpha| \leq 1 \), then \( D^\alpha u \in W^{m-1,p}(U) \) so by induction, for all such \( \alpha \),
\[
D^\alpha u \in L^{\frac{np}{n-(m-1)p}}(U).
\]
Thus, since \( U \) has the segment condition, \( u \in W^{1,q}(U) \) where
\[
q_1 = \frac{np}{n-(m-1)p}
\]
By Theorem 36.3.1 it follows \( u \in L^q(\mathbb{R}^n) \) where
\[
1 = \frac{n-(m-1)p}{np} - \frac{1}{q_1} = \frac{n-mp}{np}.
\]
This proves the corollary.

There is another similar corollary of the same sort which is interesting and useful.

**Corollary 36.3.3** Suppose \( m \geq 1 \) and \( j \) is a nonnegative integer satisfying \( jp < n \). Also suppose \( U \) has a \((1,p)\) extension operator for all \( p \geq 1 \) and satisfies the segment condition. Then
\[
\text{id} \in L(W^{m+j,p}(U), W^{m,q}(U))
\]
where
\[
q \equiv \frac{np}{n-jp}.
\]
If, in addition to the above, \( U \) is bounded and \( 1 \leq r < q \), then
\[
\text{id} \in L(W^{m+j,p}(U), W^{m,r}(U))
\]
and is compact.

**Proof:** If \( |\alpha| \leq m \), then \( D^\alpha u \in W^{j,p}(U) \) and so by Corollary 36.3.2, \( D^\alpha u \in L^q(U) \) where \( q \) is given above. Since \( U \) has the segment property, this means \( u \in W^{m,q}(U) \). It remains to verify the assertion about compactness of id.

Let \( S \) be bounded in \( W^{m+j,p}(U) \). Then \( S \) is bounded in \( W^{m,q}(U) \) by the first part. Now let \( \{u_k\}_{k=1}^\infty \) be any sequence in \( S \). The corollary will be proved
36.3. GENERAL EMBEDDING THEOREMS

If it is shown that any such sequence has a convergent subsequence in $W^{m,r}(U)$. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ denote the indices satisfying $|\alpha| \leq m$. Then for each of these indices, $\alpha$,

$$\sup_{u \in S} \left\{ \|D^\alpha u\|_{1,1,U} + \|D^\alpha u\|_{L^s(U)} \right\} < \infty$$

and so for each such $\alpha$, satisfying $|\alpha| \leq m$, it follows from Lemma 36.1.10 on Page 1349 that $\{D^\alpha u : u \in S\}$ is precompact in $L^r(U)$. Therefore, there exists a subsequence, still denoted by $u_k$ such that $D^{\alpha_1} u_k$ converges in $L^r(U)$. Applying the same lemma, there exists a subsequence of this subsequence such that both $D^{\alpha_1} u_k$ and $D^{\alpha_2} u_k$ converge in $L^r(U)$. Continue taking subsequences until you obtain a subsequence, $\{u_k\}_{k=1}^\infty$ for which $\{D^\alpha u_k\}_{k=1}^\infty$ converges in $L^r(U)$ for all $|\alpha| \leq m$. But this must be a convergent subsequence in $W^{m,r}(U)$ and this proves the corollary.

**Theorem 36.3.4** Let $U$ be a bounded open set having a $(1,p)$ extension operator and let $p > n$. Then $\text{id} : W^{1,p}(U) \to C(\overline{U})$ is continuous and compact.

**Proof:** Theorem 36.3.2 on Page 1348 implies $r_U : W^{1,p}(\mathbb{R}^n) \to C(\overline{U})$ is continuous and compact. Thus

$$\|u\|_{\infty,U} = \|Eu\|_{\infty,U} \leq C \|Eu\|_{1,p,\mathbb{R}^n} \leq C \|u\|_{1,p,U}.$$

This proves continuity. If $S$ is a bounded set in $W^{1,p}(U)$, then define $S_1 = \{ Eu : u \in S \}$. Then $S_1$ is a bounded set in $W^{1,p}(\mathbb{R}^n)$ and so by Theorem 36.3.2 the set of restrictions to $U$, is precompact. However, the restrictions to $U$ are just the functions of $S$. Therefore, id is compact as well as continuous.

**Corollary 36.3.5** Let $p > n$, let $U$ be a bounded open set having a $(1,p)$ extension operator which also satisfies the segment condition, and let $m$ be a nonnegative integer. Then $\text{id} : W^{m+1,p}(U) \to C^{m,\lambda}(\overline{U})$ is continuous for all $\lambda \in [0,1 - \frac{n}{p}]$ and $\text{id}$ is compact if $\lambda < 1 - \frac{n}{p}$.

**Proof:** Let $u_k \to 0$ in $W^{m+1,p}(U)$. Then it follows that for each $|\alpha| \leq m$, $D^\alpha u_k \to 0$ in $W^{1,p}(U)$. Therefore,

$$E(D^\alpha u_k) \to 0 \text{ in } W^{1,p}(\mathbb{R}^n).$$

Then from Morrey’s inequality, 36.1.13 on Page 1348, if $\lambda \leq 1 - \frac{n}{p}$ and $|\alpha| = m$

$$\rho_\lambda(E(D^\alpha u_k)) \leq C \|E(D^\alpha u_k)\|_{1,p,\mathbb{R}^n} \text{ diam}(U)^{1 - \frac{n}{p} - \lambda}.$$

Therefore, $\rho_\lambda(E(D^\alpha u_k)) = \rho_\lambda(D^\alpha u_k) \to 0$. From Theorem 36.3.2 it follows that for $|\alpha| \leq m$, $\|D^\alpha u_k\|_{\infty} \to 0$ and so $\|u_k\|_{m,\lambda} \to 0$. This proves the claim about continuity. The claim about compactness for $\lambda < 1 - \frac{n}{p}$ follows from Lemma 36.3.2 on Page 1348 and this. (Bounded in $W^{m,p}(U)$ $\text{id}$ $\to$ Bounded in $C^{m,1 - \frac{n}{p},\lambda}(\overline{U})$ $\text{id}$ Compact in $C^{m,\lambda}(\overline{U}).$)
**Theorem 36.3.6** Suppose \( j p < n < (j + 1)p \) and let \( m \) be a positive integer. Let \( U \) be any bounded open set in \( \mathbb{R}^n \) which has a \((1, p)\) extension operator for each \( p \geq 1 \) and the segment property. Then \( \text{id} \in \mathcal{L} \left( W^{m+j,p}(U), C^{m-1, \lambda}(U) \right) \) for every \( \lambda \leq \lambda_0 \equiv (j + 1) - \frac{n}{p} \) and if \( \lambda < (j + 1) - \frac{n}{p} \), \( \text{id} \) is compact.

**Proof:** From Corollary 36.3.3 \( W^{m+j,p}(U) \subseteq W^{m,q}(U) \) where \( q \) is given by 36.3.35. Therefore,

\[
\frac{np}{n - j p} > n
\]

and so by Corollary 36.3.4, \( W^{m,q}(U) \subseteq C^{m-1, \lambda}(U) \) for all \( \lambda \) satisfying

\[
0 < \lambda < 1 - \frac{(n - j p)n}{np} = \frac{p(j + 1) - n}{p} = (j + 1) - \frac{n}{p}.
\]

The assertion about compactness follows from the compactness of the embedding of \( C^{m-1, \lambda_0}(U) \) into \( C^{m-1, \lambda}(U) \) for \( \lambda < \lambda_0 \), Lemma 36.1.6 on Page 1338.

### 36.4 More Extension Theorems

The theorem about the existence of a \((1, p)\) extension is all that is needed to obtain general embedding theorems for Sobolev spaces. However, a more general theory is needed in order to tie the theory of Sobolev spaces presented thus far to a very appealing description using Fourier transforms. First the problem of extending \( W^k,p(H) \) to \( W^k,p(\mathbb{R}^n) \) is considered for \( H \) a half space

\[
H^- = \{ y \in \mathbb{R}^n : y_n < 0 \}.
\]

I am following Adams [1].

**Lemma 36.4.1** Let \( H^- \) be a half space as in 36.4.35. Let \( H^+ \) be the half space in which \( y_n < 0 \) is replaced with \( y_n > 0 \). Also let \( (y', y_n) = y \)

\[
u(y', y_n) \equiv \begin{cases} u^+(y', y_n) & \text{if } y \in H^+ \\ u^-(y', y_n) & \text{if } y \in H^- \end{cases},
\]

suppose \( u^+ \in C^\infty \left( \overline{H^+} \right) \) and \( u^- \in C^\infty \left( \overline{H^-} \right) \), and that for \( l \leq k - 1 \),

\[
D^l u^+(y', 0) = D^l u^-(y', 0).
\]

Then \( u \in W^{k, p}(\mathbb{R}^n) \). Furthermore,

\[
D^\alpha u(y', y_n) \equiv \begin{cases} D^\alpha u^+(y', y_n) & \text{if } y \in H^+ \\ D^\alpha u^-(y', y_n) & \text{if } y \in H^- \end{cases}
\]
Proof: Consider the following for $\phi \in C^\infty_c(\mathbb{R}^n)$ and $|\alpha| \leq k$.

\[
(-1)\lvert \alpha \rvert \left( \int_{\mathbb{R}^{n-1}} \int_0^\infty u^+ D^\alpha \phi dy_1 dy' + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 u^- D^\alpha \phi dy_1 dy' \right).
\]

Integrating by parts, this yields

\[
(-1)^{|\alpha|} (-1)^{|\beta|} \left( \int_{\mathbb{R}^{n-1}} \int_0^\infty D^{\beta} u^+ D^{\alpha_\omega} \phi dy_1 dy' + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 D^{\beta} u^- D^{\alpha_\omega} \phi dy_1 dy' \right)
\]

where $\beta \equiv (\alpha_1, \alpha_2, \cdots \alpha_{n-1}, 0)$. Do integration by parts on the inside integral and by assumption, the boundary terms will cancel and the whole thing reduces to

\[
(-1)^{|\alpha|} (-1)^{|\beta|} (-1)^{|\alpha_n|} \left( \int_{\mathbb{R}^{n-1}} \int_0^\infty D^\alpha u^+ \phi dy_1 dy' + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 D^\alpha u^- \phi dy_1 dy' \right)
\]

which proves the lemma.

Lemma 36.4.2 Let $H^-$ be the half space in 36.4.35 and let $u \in C^\infty(H^-)$. Then there exists a mapping,

\[
E : C^\infty(H^-) \to W^{k,p}(\mathbb{R}^n)
\]

and a constant, $C$ which is independent of $u \in C^\infty(H^-)$ such that $E$ is linear and for all $l \leq k$,

\[
\lVert Eu \rVert_{l,p,\mathbb{R}^n} \leq C \lVert u \rVert_{l,p,H^-}.
\]

Proof: Define

\[
Eu(x',x_n) = \begin{cases} u(x',x_n) & \text{if } x_n < 0 \\ \sum_{j=1}^k \lambda_j u(x',-jx_n) & \text{if } x_n \geq 0 \end{cases}
\]

where the $\lambda_j$ are chosen in such a way that for $l \leq k - 1$,

\[
D^{\alpha_\omega} u(x',0) - D^{\alpha_\omega} \left( \sum_{j=1}^k \lambda_j u \right)(x',0) = 0
\]
so that Lemma 36.4.1 may be applied. Do there exist such \( \lambda_j \)? It is necessary to have the following hold for each \( r = 0, 1, \ldots, k - 1 \).

\[
\sum_{j=1}^{k} (-j)^r \lambda_j D^{r\alpha} u(x',0) = D^{r\alpha} u(x',0).
\]

This is satisfied if

\[
\sum_{j=1}^{k} (-j)^r \lambda_j = 1
\]

for \( r = 0, 1, \ldots, k - 1 \). This is a system of \( k \) equations for the \( k \) variables, the \( \lambda_j \).

The matrix of coefficients is of the form

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
-1 & -2 & -3 & \cdots & -k \\
1 & 4 & 9 & \cdots & k^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(-1)^k & (-2)^k & (-3)^k & \cdots & (-k)^k
\end{pmatrix}
\]

This matrix has an inverse because its determinant is nonzero.

Now from Lemma 36.4.1, it follows from the above description of \( E \) that for \( |\alpha| \leq k \),

\[
D^\alpha (Eu)(x',x_n) \equiv \begin{cases} 
D^\alpha u(x',x_n) & \text{if } x_n < 0 \\
\sum_{j=1}^{k} \lambda_j (-j)^{\alpha_n} (D^\alpha u)(x',-jx_n) & \text{if } x_n \geq 0 
\end{cases}
\]

It follows that \( E \) is linear and there exists a constant, \( C \) independent of \( u \) such that 36.4.36 holds. This proves the lemma.

**Corollary 36.4.3** Let \( H^- \) be the half space of 36.4.3. There exists \( E \) with the property that \( E : W^{l,p}(H^-) \to W^{l,p}(\mathbb{R}^n) \) and is linear and continuous for each \( l \leq k \).

**Proof:** This immediate from the density of \( C^\infty_c(\overline{H^-}) \) in \( W^{k,p}(\overline{H^-}) \) and Lemma 36.3.3.

There is nothing sacred about a half space or this particular half space. It is clear that everything works as well for a half space of the form

\[
H^-_k = \{ x : x_k < 0 \}.
\]

Thus the half space featured in the above discussion is \( H^-_n \).

**Corollary 36.4.4** Let \( \{k_1, \ldots, k_r\} \subseteq \{1, \ldots, n\} \) where the \( k_i \) are distinct and let

\[
H^-_{k_1 \cdots k_r} = H^-_{k_1} \cap H^-_{k_2} \cap \cdots \cap H^-_{k_r}.
\]

(36.4.37)

Then there exists \( E : W^{k,p}(H^-_{k_1 \cdots k_r}) \to W^{k,p}(\mathbb{R}^n) \) such that \( E \) is linear and continuous.
Proof: Follow the above argument with minor modifications to first extend from $H_{k_1\cdots k_r}$ to $H_{k_1\cdots k_r-1}$ and then from $H_{k_1\cdots k_r-1}$ to $H_{k_1\cdots k_r-2}$ etc.

This easily implies the ability to extend off bounded open sets which near their boundaries look locally like an intersection of half spaces.

Theorem 36.4.5 Let $U$ be a bounded open set and suppose $U_0, U_1, \ldots, U_m$ are open sets with the property that $U \subseteq \bigcup_{k=0}^m U_k$, $U_0 \subseteq U$, and $\partial U \subseteq \bigcup_{k=1}^m U_k$. Suppose also there exist one to one and onto functions, $h_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $h_k(U_k \cap U) = W_k$ where $W_k$ equals the intersection of a bounded open set with a finite intersection of half spaces, $H_{k_1\cdots k_r}^-$, as in 36.4.37 such that $h_k(\partial U \cap U_k) \subseteq \partial H_{k_1\cdots k_r}^-$. Suppose also that for all $|\alpha| \leq k - 1$, $D^\alpha h_k$ and $D^\alpha h_k^{-1}$ exist and are Lipschitz continuous. Then there letting $W$ be an open set which contains $\overline{U}$, there exists $E : W^{k,p}(U) \rightarrow W^{k,p}(W)$ such that $E$ is a linear continuous map from $W^{l,p}(U)$ to $W^{l,p}(W)$ for each $l \leq k$.

Proof: Let $\psi_j \in C_\infty(U_j)$, $\psi_j(x) \in [0, 1]$ for all $x \in \mathbb{R}^n$, and $\sum_{j=0}^m \psi_j(x) = 1$ on $U$. This is a $C_\infty$ partition of unity on $U$. By Theorem 36.0.10, $(h_j^{-1})^* u \psi_j \in W^{k,p}(W_j)$. By the assumption that $h_j(\partial U \cap U_j) \subseteq \partial H_{k_1\cdots k_r}^-$, the zero extension of $(h_j^{-1})^* u \psi_j$ to the rest of $H_{k_1\cdots k_r}^-$ results in an element of $W^{k,p}(H_{k_1\cdots k_r}^-)$. Apply Corollary 36.4.2 to conclude there exists $E_j : W^{k,p}(H_{k_1\cdots k_r}^-) \rightarrow W^{k,p}(\mathbb{R}^n)$ which is continuous and linear. Abusing notation slightly, by using $(h_j^{-1})^* u \psi_j$ as the above zero extension, it follows $E_j \left( (h_j^{-1})^* u \psi_j \right) \in W^{k,p}(\mathbb{R}^n)$. Now let $\eta$ be a function in $C_\infty(h(W))$ such that $\eta(y) = 1$ on $h(U)$. Then Define

$$Eu \equiv \sum_{j=0}^m h_j^\ast \eta E_j \left( (h_j^{-1})^* u \psi_j \right).$$

Clearly $Eu(x) = u(x)$ if $x \in U$. It is also clear that $E$ is linear. It only remains to verify $E$ is continuous. In what follows, $C_j$ will denote a constant which is
independent of $u$ which may change from line to line. By Theorem 36.0.16,

$$
\|Eu\|_{k,p,W} \leq \sum_{j=0}^{m} \left( h_j^* \eta E_j \left( (h_j^{-1})^* (u\psi_j) \right) \right)_{k,p,W} \\
\leq \sum_{j=0}^{m} C_j \left( \eta E_j \left( (h_j^{-1})^* (u\psi_j) \right) \right)_{k,p,h(W)} \\
= \sum_{j=0}^{m} C_j \left( \eta E_j \left( (h_j^{-1})^* (u\psi_j) \right) \right)_{k,p,\mathbb{R}^n} \\
\leq \sum_{j=0}^{m} C_j \left( E_j \left( (h_j^{-1})^* (u\psi_j) \right) \right)_{k,p,\mathbb{R}^n} \\
\leq \sum_{j=0}^{m} C_j \left( (h_j^{-1})^* (u\psi_j) \right)_{k,p,h(U\cap U_j)} \\
\leq \sum_{j=0}^{m} C_j \left( u\psi_j \right)_{k,p,U\cap U_k} \\
\leq \sum_{j=0}^{m} C_j \left( u \right)_{k,p,U\cap U_k} \leq \sum_{j=0}^{m} C_j \left( u \right)_{k,p,U}.
$$

Similarly $E : W^{l,p} (U) \rightarrow W^{l,p} (U)$ for $l \leq k$. This proves the theorem.

**Definition 36.4.6** When $E$ is a linear continuous map from $W^{l,p} (U)$ to $W^{l,p} (\mathbb{R}^n)$ for each $l \leq k$, it is called a strong $(k,p)$ extension map.

There is also a very easy sort of extension theorem for the space, $W^{m,p}_0 (U)$ which does not require any assumptions on the boundary of $U$ other than $m_n (\partial U) = 0$. First here is the definition of $W^{m,p}_0 (U)$.

**Definition 36.4.7** Denote by $W^{m,p}_0 (U)$ the closure of $C_\infty^\infty (U)$ in $W^{m,p} (U)$.

**Theorem 36.4.8** For $u \in W^{m,p}_0 (U)$, define

$$
Eu (x) \equiv \begin{cases} 
    u (x) & \text{if } x \in U \\
    0 & \text{if } x \notin U
\end{cases}
$$

Then $E$ is a strong $(k,p)$ extension map.

**Proof:** Letting $l \leq m$, it is clear that for $|\alpha| \leq l$,

$$
D^\alpha Eu = \begin{cases} 
    D^\alpha u & \text{for } x \in U \\
    0 & \text{for } x \notin U
\end{cases}.
$$

This follows because, since $m_n (\partial U) = 0$ it suffices to consider $\phi \in C_\infty^\infty (U)$ and $\phi \in C_\infty^\infty \left( \overline{U}^C \right)$. Therefore, $\|Eu\|_{l,p,\mathbb{R}^n} = \|u\|_{l,p,U}$.  


36.4. MORE EXTENSION THEOREMS

There are many other extension theorems and if you are interested in pursuing this further, consult Adams [1]. One of the most famous which is discussed in this reference is due to Calderon and depends on the theory of singular integrals.
Chapter 37

Sobolev Spaces Based On $L^2$

37.1 Fourier Transform Techniques

Much insight can be obtained easily through the use of Fourier transform methods. This technique will be developed in this chapter. When this is done, it is necessary to use Sobolev spaces of the form $W^{k,2}(U)$, those Sobolev spaces which are based on $L^2(U)$. It is true there are generalizations which use Fourier transform methods in the context of $L^p$ but the spaces so considered are called Bessel potential spaces. They are not really Sobolev spaces. Furthermore, it is Mihlin’s theorem rather than the Plancherel theorem which is the main tool of the analysis. This is a hard theorem.

It is convenient to consider the Schwartz class of functions, $\mathcal{S}$. These are functions which have infinitely many derivatives and vanish quickly together with their derivatives as $|x| \to \infty$. In particular, $C_c^\infty(\mathbb{R}^n)$ is contained in $\mathcal{S}$ which is not true of the functions, $\mathcal{G}$ used earlier in defining the Fourier transforms which are a suspace of $\mathcal{S}$. Recall the following definition.

**Definition 37.1.1** $f \in \mathcal{S}$, the Schwartz class, if $f \in C^\infty(\mathbb{R}^n)$ and for all positive integers $N$,

$$\rho_N(f) < \infty$$

where

$$\rho_N(f) = \sup\{(1 + |x|^2)^N|D^\alpha f(x)| : x \in \mathbb{R}^n, |\alpha| \leq N\}.$$  

Thus $f \in \mathcal{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and

$$\sup\{|x^\beta D^\alpha f(x)| : x \in \mathbb{R}^n\} < \infty \quad (37.1.1)$$

for all multi indices $\alpha$ and $\beta$.

Thus all partial derivatives of a function in $\mathcal{S}$ are in $L^p(\mathbb{R}^n)$ for all $p \geq 1$. Therefore, for $f \in \mathcal{S}$, the Fourier and inverse Fourier transforms are given in the
usual way,

\[ Ff(t) = \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(x) e^{-it \cdot x} \, dx, \quad F^{-1} f(t) = \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(x) e^{it \cdot x} \, dx. \]

Also recall that the Fourier transform and its inverse are one to one and onto maps from \( \mathcal{S} \) to \( \mathcal{S} \).

To tie the Fourier transform technique in with what has been done so far, it is necessary to make the following assumption on the set, \( U \). This assumption is made so that it is possible to consider elements of \( W^{k,2}(U) \) as restrictions of elements of \( W^{k,2}(\mathbb{R}^n) \).

**Assumption 37.1.2** Assume \( U \) satisfies the segment condition and that for any \( m \) of interest, there exists \( E \in \mathcal{L}(W^{m,p}(U),W^{m,p}(\mathbb{R}^n)) \) such that for each \( k \leq m, \ E \in \mathcal{L}(W^{k,p}(U),W^{k,p}(\mathbb{R}^n)) \). That is, there exists a strong \((m,p)\) extension operator.

**Lemma 37.1.3** The Schwartz class, \( \mathcal{S} \), is dense in \( W^{m,p}(\mathbb{R}^n) \).

**Proof:** The set, \( \mathbb{R}^n \) satisfies the segment condition and so \( C_c^\infty(\mathbb{R}^n) \) is dense in \( W^{m,p}(\mathbb{R}^n) \). However, \( C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S} \). This proves the lemma.

Recall now Plancherel’s theorem which states that \( ||f||_{0,2,\mathbb{R}^n} = ||Ff||_{0,2,\mathbb{R}^n} \) whenever \( f \in L^2(\mathbb{R}^n) \). Also it is routine to verify from the definition of the Fourier transform that for \( u \in \mathcal{S} \),

\[ F\partial_k u = ix_k Fu. \]

From this it follows that

\[ ||D^\alpha u||_{0,2,\mathbb{R}^n} = ||x^\alpha Fu||_{0,2,\mathbb{R}^n}. \]

Here \( x^\alpha \) denotes the function \( x \rightarrow x^\alpha \). Therefore,

\[ ||u||_{m,2,\mathbb{R}^n} = \left( \int_{\mathbb{R}^n} \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} |Fu(x)|^2 \, dx \right)^{1/2}. \]

Also, it is not hard to verify that

\[ \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \leq \left( 1 + \sum_{j=1}^n x_j^2 \right)^m \leq C(n,m) \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \]

where \( C(n,m) \) is the largest of the multinomial coefficients obtained in the expansion,

\[ \left( 1 + \sum_{j=1}^n x_j^2 \right)^m. \]
Therefore, for all $u \in \mathcal{S}$,

$$||u||_{m,2,\mathbb{R}^n} \leq \left(\int_{\mathbb{R}^n} \left(1 + |x|^2\right)^m |Fu(x)|^2 \, dx \right)^{1/2} \leq C(m,n) ||u||_{m,2,\mathbb{R}^n}. \quad (37.1.2)$$

This motivates the following definition.

**Definition 37.1.4** Let $H^m(\mathbb{R}^n) \equiv$

$$\left\{ u \in L^2(\mathbb{R}^n) : ||u||_{H^m(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} \left(1 + |x|^2\right)^m |Fu(x)|^2 \, dx \right)^{1/2} < \infty \right\}. \quad (37.1.3)$$

**Lemma 37.1.5** $\mathcal{S}$ is dense in $H^m(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n) = W^{2,m}(\mathbb{R}^n)$. Furthermore, the norms are equivalent.

**Proof:** First it is shown that $\mathcal{S}$ is dense in $H^m(\mathbb{R}^n)$. Let $u \in H^m(\mathbb{R}^n)$. Let $\mu(E) \equiv \int_E \left(1 + |x|^2\right)^m \, dx$. Thus $\mu$ is a regular measure and $u \in H^m(\mathbb{R}^n)$ just means that $Fu \in L^2(\mu)$, the space of functions which are in $L^2(\mathbb{R}^n)$ with respect to this measure, $\mu$. Therefore, from the regularity of the measure, $\mu$, there exists $u_k \in C_c(\mathbb{R}^n)$ such that

$$||u_k - Fu||_{L^2(\mu)} \to 0.$$

Now let $\psi_e$ be a mollifier and pick $\epsilon_k$ small enough that

$$||u_k \star \psi_{\epsilon_k} - u_k||_{L^2(\mu)} < \frac{1}{2^k}.$$  

Then $u_k \star \psi_{\epsilon_k} \in C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}$. Therefore, there exists $w_k \in \mathcal{S}$ such that $FW_k = u_k \star \psi_{\epsilon_k}$. It follows

$$||FW_k - Fu||_{L^2(\mu)} \leq ||FW_k - u_k||_{L^2(\mu)} + ||u_k - Fu||_{L^2(\mu)}$$

and these last two terms converge to 0 as $k \to \infty$. Therefore, $w_k \to u$ in $H^m(\mathbb{R}^n)$ and this proves the first part of this lemma.

Now let $u \in H^m(\mathbb{R}^n)$. By what was just shown, there exists a sequence, $u_k \to u$ in $H^m(\mathbb{R}^n)$ where $u_k \in \mathcal{S}$. It follows from (37.1.2) that

$$||u_k - u||_{H^m} \geq ||u_k - u||_{m,2,\mathbb{R}^n}$$

and so $\{u_k\}$ is a Cauchy sequence in $W^{m,2}(\mathbb{R}^n)$. Therefore, there exists $w \in W^{m,2}(\mathbb{R}^n)$ such that

$$||u_k - w||_{m,2,\mathbb{R}^n} \to 0.$$

But this implies

$$0 = \lim_{k \to \infty} ||u_k - w||_{0,2,\mathbb{R}^n} = \lim_{k \to \infty} ||u_k - u||_{0,2,\mathbb{R}^n}$$

showing $u = w$ which verifies $H^m(\mathbb{R}^n) \subseteq W^{2,m}(\mathbb{R}^n)$. The opposite inclusion is proved the same way, using density of $\mathcal{S}$ and the fact that the norms in both spaces
are larger than the norms in $L^2 (\mathbb{R}^n)$. The equivalence of the norms follows from the density of $\mathcal{S}$ and the equivalence of the norms on $\mathcal{S}$. This proves the lemma.

The conclusion of this lemma with the density of $\mathcal{S}$ and \eqref{37.1.2} implies you can use either norm, $\|u\|_{H^m (\mathbb{R}^n)}$ or $\|u\|_{m,2,\mathbb{R}^n}$ when working with these Sobolev spaces.

What of open sets satisfying Assumption \eqref{37.1.2}? How does $W^{m,2} (U)$ relate to the Fourier transform?

\begin{definition}
Let $U$ be an open set in $\mathbb{R}^n$. Then 
\begin{equation}
H^m (U) \equiv \{ u : u = v|_U \text{ for some } v \in H^m (\mathbb{R}^n) \}
\end{equation}
Here the notation, $v|_U$ means $v$ restricted to $U$. Define the norm in this space by 
\begin{equation}
\|u\|_{H^m (U)} \equiv \inf \left\{ \|v\|_{H^m (\mathbb{R}^n)} : v|_U = u \right\}.
\end{equation}
\end{definition}

\begin{lemma}
$H^m (U)$ is a Banach space.
\end{lemma}

**Proof:** First it is necessary to verify that the given norm really is a norm. Suppose then that $u = 0$. Is $\|u\|_{H^m (U)} = 0$? Of course it is. Just take $v \equiv 0$. Then $v|_U = u$ and $\|v\|_{H^m} = 0$. Next suppose $\|u\|_{H^m (U)} = 0$. Does it follow that $u = 0$? Letting $\varepsilon > 0$ be given, there exists $v \in H^m (\mathbb{R}^n)$ such that $v|_U = u$ and $\|v\|_{H^m (\mathbb{R}^n)} < \varepsilon$. Therefore, 
\begin{equation}
\|u\|_{0,U} \leq \|v\|_{0,\mathbb{R}^n} \leq \|v\|_{H^m (U)} < \varepsilon.
\end{equation}
Since $\varepsilon > 0$ is arbitrary, it follows $u = 0$ a.e. Next suppose $u_i \in H^m (U)$ for $i = 1, 2$. There exists $v_i \in H^m (\mathbb{R}^n)$ such that 
\begin{equation}
\|v_i\|_{H^m (\mathbb{R}^n)} < \|u_i\|_{H^m (U)} + \varepsilon.
\end{equation}
Therefore, 
\begin{align*}
\|u_1 + u_2\|_{H^m (U)} &\leq \|v_1 + v_2\|_{H^m (\mathbb{R}^n)} \leq \|v_1\|_{H^m (\mathbb{R}^n)} + \|v_2\|_{H^m (\mathbb{R}^n)} \\
&\leq \|u_1\|_{H^m (U)} + \|u_2\|_{H^m (U)} + 2\varepsilon
\end{align*}
and since $\varepsilon > 0$ is arbitrary, this shows the triangle inequality.

The interesting question is the one about completeness. Suppose then $\{u_k\}$ is a Cauchy sequence in $H^m (U)$. There exists $N_k$ such that if $k, l \geq N_k$, it follows $\|u_k - u_l\|_{H^m (U)} < \frac{1}{2^l}$ and the numbers, $N_k$ can be taken to be strictly increasing in $k$. Thus for $l \geq N_k$, $\|u_l - u_{N_k}\|_{H^m (U)} < 1/2^l$. Therefore, there exists $w_l \in H^m (\mathbb{R}^n)$ such that 
\begin{equation}
w_l|_U = u_l - u_{N_k}, \quad \|w_l\|_{H^m (\mathbb{R}^n)} < \frac{1}{2^l}.
\end{equation}
Also let $v_{N_k}|_U = u_{N_k}$ with $v_{N_k} \in H^m (\mathbb{R}^n)$ and 
\begin{equation}
\|v_{N_k}\|_{H^m (\mathbb{R}^n)} < \|u_{N_k}\|_{H^m (U)} + \frac{1}{2^k}.
\end{equation}
37.1. FOURIER TRANSFORM TECHNIQUES

1371

Now for \( l > N_k \), define \( v_l \) by \( v_l - v_{N_k} = w_{N_k} \) so that \( \| v_l - v_{N_k} \|_{H^m(\mathbb{R}^n)} < 1/2^k \). In particular,

\[
\left\| v_{N_k+1} - v_{N_k} \right\|_{H^m(\mathbb{R}^n)} < 1/2^k
\]

which shows that \( \{ v_{N_k} \}_{k=1}^\infty \) is a Cauchy sequence. Consequently it must converge to \( v \in H^m(\mathbb{R}^n) \). Let \( u = v|_U \). Then

\[
\| u - v_{N_k} \|_{H^m(U)} \leq \| v - v_{N_k} \|_{H^m(\mathbb{R}^n)}
\]

which shows the subsequence, \( \{ u_{N_k} \} \) converges to \( u \). Since \( \{ u_k \} \) is a Cauchy sequence, it follows it too must converge to \( u \). This proves the lemma.

The main result is next.

**Theorem 37.1.8** Suppose \( U \) satisfies Assumption [37.1.3]. Then for \( m \) a nonnegative integer, \( H^m(U) = W^{m,2}(U) \) and the norms are equivalent.

**Proof:** Let \( u \in H^m(U) \). Then there exists \( v \in H^m(\mathbb{R}^n) \) such that \( v|_U = u \). Hence \( v \in W^{k,2}(\mathbb{R}^n) \) and so all its weak derivatives up to order \( m \) are in \( L^2(\mathbb{R}^n) \). Therefore, the restrictions of these weak derivatives are in \( L^2(U) \). Since \( U \) satisfies the segment condition, it follows \( u \in W^{m,2}(U) \) which shows \( H^m(U) \subseteq W^{m,2}(U) \).

Next take \( u \in W^{m,2}(U) \). Then \( Eu \in W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n) \) and this shows \( u \in H^m(U) \). This has shown the two spaces are the same. It remains to verify their norms are equivalent. Let \( u \in H^m(U) \) and let \( v|_U = u \) where \( v \in H^m(\mathbb{R}^n) \) and

\[
||u||_{H^m(U)} + \varepsilon > ||v||_{H^m(\mathbb{R}^n)}.
\]

Then recalling that \( ||.||_{H^m(\mathbb{R}^n)} \) and \( ||.||_{m,2,\mathbb{R}^n} \) are equivalent norms for \( H^m(\mathbb{R}^n) \), there exists a constant, \( C \) such that

\[
||u||_{H^m(U)} + \varepsilon > ||v||_{H^m(\mathbb{R}^n)} \geq C ||v||_{m,2,\mathbb{R}^n} \geq C ||u||_{m,2,U}
\]

Now consider the two Banach spaces,

\[
\left( H^m(U), ||.||_{H^m(U)} \right), \left( W^{m,2}(U), ||.||_{m,2,U} \right)
\]

The above inequality shows since \( \varepsilon > 0 \) is arbitrary that \( \text{id} : \left( H^m(U), ||.||_{H^m(U)} \right) \to \left( W^{m,2}(U), ||.||_{m,2,U} \right) \) is continuous. By the open mapping theorem, it follows id is continuous in the other direction. Thus there exists a constant, \( K \) such that

\[
||u||_{H^m(U)} \leq K ||u||_{k,2,U}.
\]

Hence the two norms are equivalent as claimed.

Specializing Corollary [36.4.7] and Theorem [36.4.8] starting on Page [448] to the case of \( p = 2 \) while also assuming more on \( U \) yields the following embedding theorems.

**Theorem 37.1.9** Suppose \( m \geq 0 \) and \( j \) is a nonnegative integer satisfying \( 2j < n \). Also suppose \( U \) is an open set which satisfies Assumption [37.1.3]. Then \( \text{id} \in \mathcal{L}(H^{m+j}(U), W^{m,q}(U)) \) where

\[
q = \frac{2n}{n - 2j}.
\]
If, in addition to the above, $U$ is bounded and $1 \leq r < q$, then
$$\text{id} \in \mathcal{L}(H^{m+j}(U), W^{m,r}(U))$$

and is compact.

**Theorem 37.1.10** Suppose for $j$ a nonnegative integer, $2j < n < 2(j+1)$ and let $m$ be a positive integer. Let $U$ be any bounded open set in $\mathbb{R}^n$ which satisfies Assumption $37.1.2$. Then $\text{id} \in \mathcal{L}(H^{m+j}(U), C^{m-1,\lambda}(U))$ for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{2}$ and if $\lambda < (j+1) - \frac{n}{2}$, $\text{id}$ is compact.

### 37.2 Fractional Order Spaces

What has been gained by all this? The main thing is that $H^{m+s}(U)$ makes sense for any $s \in (0,1)$ and $m$ an integer. You simply replace $m$ with $m+s$ in the above for $s \in (0,1)$. This gives what is meant by $H^{m+s}(\mathbb{R}^n)$

**Definition 37.2.1** For $m$ an integer and $s \in (0,1)$, let $H^{m+s}(\mathbb{R}^n) \equiv \{ u \in L^2(\mathbb{R}^n) : ||u||_{H^{m+s}(\mathbb{R}^n)} \equiv \left( \int_{\mathbb{R}^n} \left( 1 + |x|^2 \right)^{m+s} |Fu(x)|^2 \, dx \right)^{1/2} < \infty \}$. (37.2.7)

You could also simply refer to $H^t(\mathbb{R}^n)$ where $t$ is a real number replacing the $m+s$ in the above formula with $t$ but I want to emphasize the notion that $t = m+s$ where $m$ is a nonnegative integer. Therefore, I will often write $m+s$. Let $U$ be an open set in $\mathbb{R}^n$. Then

$$H^{m+s}(U) \equiv \{ u : u = v|_U \text{ for some } v \in H^{m+s}(\mathbb{R}^n) \}.$$ (37.2.8)

Define the norm in this space by

$$||u||_{H^{m+s}(U)} \equiv \inf \left\{ ||v||_{H^{m+s}(\mathbb{R}^n)} : v|_U = u \right\}.$$ (37.2.9)

**Lemma 37.2.2** $H^{m+s}(U)$ is a Banach space.

**Proof:** Just repeat the proof of Lemma 37.1.7.

The theorem about density of $\mathcal{S}$ also remains true in $H^{m+s}(\mathbb{R}^n)$. Just repeat the proof of that part of Lemma 37.1.5 replacing the integer, $m$, with the symbol, $m+s$.

**Lemma 37.2.3** $\mathcal{S}$ is dense in $H^{m+s}(\mathbb{R}^n)$.

In fact, more can be said.

**Corollary 37.2.4** Let $U$ be an open set and let $\mathcal{S}|_U$ denote the restrictions of functions of $\mathcal{S}$ to $U$. Then $\mathcal{S}|_U$ is dense in $H^t(U)$. 

37.2. FRACTIONAL ORDER SPACES

Proof: Let \( u \in H^t(U) \) and let \( v \in H^t(\mathbb{R}^n) \) such that \( v|_U = u \) a.e. Then since \( \mathcal{S} \) is dense in \( H^t(\mathbb{R}^n) \), there exists \( w \in \mathcal{S} \) such that
\[
||w - v||_{H^t(\mathbb{R}^n)} < \varepsilon.
\]
It follows that
\[
||u - w||_{H^t(U)} \leq ||u - v||_{H^t(U)} + ||v - w||_{H^t(U)} \leq 0 + ||v - w||_{H^t(\mathbb{R}^n)} < \varepsilon.
\]

These fractional order spaces are important when trying to understand the trace on the boundary. The Fourier transform description also makes it very easy to establish interesting inequalities such as interpolation inequalities.

Lemma 37.2.5 Let \( 0 \leq r < s < t \). Then if \( u \in H^t(\mathbb{R}^n) \),
\[
||u||_{H^r(\mathbb{R}^n)} \leq ||u||_{H^r(\mathbb{R}^n)}^{\theta} ||u||_{H^r(\mathbb{R}^n)}^{1-\theta}
\]
where \( \theta \) is a positive number such that \( \theta r + (1 - \theta) t = s \).

Proof: This follows from Holder’s inequality applied to the measure \( \mu \) given by
\[
\mu(E) = \int_E |Fu|^2 \, dx
\]
Thus
\[
\int \left( 1 + |x|^2 \right)^{s} |Fu|^2 \, dx = \int \left( 1 + |x|^2 \right)^{\theta r} \left( 1 + |x|^2 \right)^{(1-\theta)t} |Fu|^2 \, dx \leq \left( \int \left( 1 + |x|^2 \right)^{r} |Fu|^2 \, dx \right)^{\theta} \left( \int \left( 1 + |x|^2 \right)^{(1-\theta)t} |Fu|^2 \, dx \right)^{1-\theta} = ||u||_{H^{r}(\mathbb{R}^n)}^{2\theta} ||u||_{H^{r}(\mathbb{R}^n)}^{2(1-\theta)}.
\]
Taking square roots yields the desired inequality.

Corollary 37.2.6 Let \( U \) be an open set satisfying Assumption 37.1.2 and let \( p < q \) where \( p, q \) are two nonnegative integers. Also let \( t \in (p, q) \). Then exists a constant, \( C \) independent of \( u \in H^q(U) \) such that for all \( u \in H^q(U) \),
\[
||u||_{H^t(U)} \leq C ||u||_{H^r(U)}^{\theta} ||u||_{H^s(U)}^{1-\theta}
\]
where \( \theta \) is such that \( t = \theta p + (1-\theta) q \).
CHAPTER 37. SOBOLEV SPACES BASED ON $L^2$

Proof: Let $E \in \mathcal{L}(H^q(U), H^q(\mathbb{R}^n))$ such that for all positive integers, $l$ less than or equal to $q$, $E \in \mathcal{L}(H^l(U), H^l(\mathbb{R}^n))$. Then $Eu|_U = u$ and $Eu \in H^l(\mathbb{R}^n)$. Therefore, by Lemma 37.2.5,

$$
||u||_{H^l(U)} \leq ||Eu||_{H^l(\mathbb{R}^n)} \leq ||Eu||_{H^q(\mathbb{R}^n)} ||u||_{H^q(U)}^{1-\theta} \leq C ||u||_{H^q(U)} \theta ||u||_{H^q(U)}^{1-\theta}.
$$

Now recall the very important Theorem 36.0.16 on Page 1335 which is listed here for convenience.

Theorem 37.2.7 Let $h : U \to V$ be one to one and onto where $U$ and $V$ are two open sets. Also suppose that $D^\alpha h$ and $D^\alpha (h^{-1})$ exist and are Lipschitz continuous if $|\alpha| \leq m - 1$ for $m$ a positive integer. Then

$$
h^*: W^{m,p}(V) \to W^{m,p}(U)
$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $(h^{-1})^*$.

37.3 An Intrinsic Norm

Is there something like this for the fractional order spaces? Yes there is. However, in order to prove it, it is convenient to use an equivalent norm for $H^{m+s}(\mathbb{R}^n)$ which does not depend explicitly on the Fourier transform. The following theorem is similar to one in [61]. It describes the norm in $H^{m+s}(\mathbb{R}^n)$ in terms which are free of the Fourier transform. This is also called an intrinsic norm [1].

Theorem 37.3.1 Let $s \in (0, 1)$ and let $m$ be a nonnegative integer. Then an equivalent norm for $H^{m+s}(\mathbb{R}^n)$ is

$$
||u||_{H^{m+s}}^2 = ||u||_{m,2,\mathbb{R}^n}^2 + \sum_{|\alpha|=m} \int \int |D^\alpha u(x) - D^\alpha u(y)|^2 |x-y|^{-n-2s} dx dy.
$$

Also if $|\beta| \leq m$, there are constants, $m(s)$ and $M(s)$ such that

$$
m(s) \int |F u(z)|^2 |z|^{2s} dz \leq \int \int |D^\beta u(x) - D^\beta u(y)|^2 |x-y|^{-n-2s} dx dy
$$

$$
\leq M(s) \int |F u(z)|^2 |z|^{2s} dz
$$

(37.3.10)

Proof: Let $u \in \mathcal{S}$ which is dense in $H^{m+s}(\mathbb{R}^n)$. The Fourier transform of the function, $y \to D^\alpha u(x+y) - D^\alpha u(y)$ equals

$$
(e^{ix} - 1) F D^\alpha u(z).
$$
Now by Fubini’s theorem and Plancherel’s theorem along with the above, taking \(|\alpha| = m|\),
\[
\int \int |D^\alpha u(x) - D^\alpha u(y)|^2 |x - y|^{-n-2s} \, dx \, dy
\]
\[
= \int \int |D^\alpha u(y + t) - D^\alpha u(y)|^2 |t|^{-n-2s} \, dt \, dy
\]
\[
= \int |t|^{-n-2s} \int |D^\alpha u(y + t) - D^\alpha u(y)|^2 \, dy \, dt
\]
\[
= \int |t|^{-n-2s} \left( \int |e^{i\alpha t} - 1| F D^\alpha u(z) |^2 \, dz \right) dt
\]
\[
= \int |F D^\alpha u(z)|^2 \left( \int |t|^{-n-2s} \left| (e^{i\alpha t} - 1) \right|^2 \, dt \right) dz. \quad (37.3.11)
\]

Consider the inside integral, the one taken with respect to \(t\).
\[
G(z) = \left( \int |t|^{-n-2s} \left| (e^{i\alpha t} - 1) \right|^2 \, dt \right).
\]

The essential thing to notice about this function of \(z\) is that it is a positive real number whenever \(z \neq 0\). This is because for small \(|t|\), the integrand is dominated by \(C|t|^{-n+2(1-s)}\). Changing to polar coordinates, you see that
\[
\int_{|t| \leq 1} |t|^{-n-2s} \left| (e^{i\alpha t} - 1) \right|^2 \, dt < \infty
\]

Next, for \(|t| > 1\), the integrand is bounded by \(4|t|^{-n-2s}\), and changing to polar coordinates shows
\[
\int_{|t| > 1} |t|^{-n-2s} \left| (e^{i\alpha t} - 1) \right|^2 \, dt \leq 4 \int_{|t| > 1} |t|^{-n-2s} \, dt < \infty.
\]

Now for \(\alpha > 0\),
\[
G(\alpha z) = \int |t|^{-n-2s} \left| (e^{i\alpha t} - 1) \right|^2 \, dt
\]
\[
= \int |t|^{-n-2s} \left| (e^{i\alpha t} - 1) \right|^2 \, dt
\]
\[
= \int \frac{r}{\alpha}^{-n-2s} \left| (e^{ir} - 1) \right|^2 \frac{1}{\alpha^s} \, dr
\]
\[
= \alpha^{2s} \int |r|^{-n-2s} \left| (e^{ir} - 1) \right|^2 \, dr = \alpha^{2s} G(z).
\]

Also \(G\) is continuous and strictly positive. Letting
\[
0 < m(s) = \min \{ G(w) : |w| = 1 \} \]
and 

\[ M(s) = \max \{ G(w) : |w| = 1 \}, \]

it follows from this, and letting \( \alpha = |z|, w = z/|z| \), that

\[ G(z) \in \left( m(s) |z|^{2s}, M(s) |z|^{2s} \right). \]

More can be said but this will suffice. Also observe that for \( s \in (0, 1) \) and \( b > 0, \)

\((1 + b)^s \leq 1 + b^s, 2^{1-s}(1 + b)^s \geq 1 + b^s.\)

In what follows, \( C(s) \) will denote a constant which depends on the indicated quantities which may be different on different lines of the argument. Then from (7.3.11),

\[
\int \int |D^{\alpha}u(x) - D^{\alpha}u(y)|^2 |x - y|^{-n-2s} \, dx \, dy \\
\leq M(s) \int \left| FD^{\alpha}u(z) \right|^2 |z|^{2s} \, dz \\
= M(s) \int \left| Fu(z) \right|^2 |z|^2 |z|^{2s} \, dz.
\]

No reference was made to \( |\alpha| = m \) and so this establishes the top half of (7.3.11). Therefore,

\[
|||u|||_{m+s}^2 \equiv ||u||^2_{m,2,\mathbb{R}^n} + \sum_{|\alpha|=m} \int \int |D^{\alpha}u(x) - D^{\alpha}u(y)|^2 |x - y|^{-n-2s} \, dx \, dy \\
\leq C \int \left( 1 + |z|^2 \right)^m |Fu(z)|^2 \, dz + M(s) \int |Fu(z)|^2 \sum_{|\alpha|=m} |z|^{2s} |z|^{2s} \, dz
\]

Recall that

\[
\sum_{|\alpha| \leq m} z_1^{2\alpha_1} \cdots z_n^{2\alpha_n} \leq \left( 1 + \sum_{j=1}^n z_j^2 \right)^m \leq C(n,m) \sum_{|\alpha| \leq m} z_1^{2\alpha_1} \cdots z_n^{2\alpha_n}. \quad (37.3.12)
\]

Therefore, where \( C(n, m) \) is the largest of the multinomial coefficients obtained in the expansion,

\[
\left( 1 + \sum_{j=1}^n z_j^2 \right)^m.
\]

Therefore,

\[
|||u|||_{m+s}^2 \leq C \int \left( 1 + |z|^2 \right)^m |Fu(z)|^2 \, dz + M(s) \int |Fu(z)|^2 \sum_{|\alpha|=m} |z|^{2s} |z|^{2s} \, dz \\
\leq C \int \left( 1 + |z|^2 \right)^{m+s} |Fu(z)|^2 \, dz + M(s) \int |Fu(z)|^2 \left( 1 + |z|^2 \right)^m |z|^{2s} \, dz \\
\leq C \int \left( 1 + |z|^2 \right)^{m+s} |Fu(z)|^2 \, dz = C ||u||_{H^{m+s}(\mathbb{R}^n)}.
\]
37.3. AN INTRINSIC NORM

It remains to show the other inequality. From \(37.3.11\),

\[
\int \int |D^\alpha u(x) - D^\alpha u(y)|^2 |x - y|^{-n-2s} \, dx \, dy
\]

\[
\geq m(s) \int |FD^\alpha u(z)|^2 |z|^{2s} \, dz
\]

\[
= m(s) \int |Fu(z)|^2 |z|^{2s} \, dz.
\]

No reference was made to \(|\alpha| = m\) and so this establishes the bottom half of \(37.3.10\). Therefore, from \(37.3.12\),

\[
|||u|||_{H^{m+s}(\mathbb{R}^n)}^2 \geq C \int (1 + |z|^2)^m |Fu(z)|^2 |z|^{2s} \, dz
\]

\[
\geq C \int (1 + |z|^2)^m |Fu(z)|^2 \, dz + m(s) \int |Fu(z)|^2 \sum_{|\alpha| = m} |z|^2 |z|^{2s} \, dz
\]

\[
= C \int (1 + |z|^2)^m \left(1 + |z|^{2s}\right) |Fu(z)|^2 \, dz
\]

\[
\geq C \int \left(1 + |z|^2\right)^m \left(1 + |z|^2\right)^s |Fu(z)|^2 \, dz
\]

\[
= C \int \left(1 + |z|^2\right)^{m+s} |Fu(z)|^2 \, dz = ||u||_{H^{m+s}(\mathbb{R}^n)}.
\]

This proves the theorem.

With the above intrinsic norm, it becomes possible to prove the following version of Theorem \(37.2.7\).

**Lemma 37.3.2** Let \(h : \mathbb{R}^n \rightarrow \mathbb{R}^n\) be one to one and onto. Also suppose that \(D^\alpha h\) and \(D^\alpha \left(h^{-1}\right)\) exist and are Lipschitz continuous if \(|\alpha| \leq m\) for \(m\) a positive integer. Then

\(h^* : H^{m+s}(\mathbb{R}^n) \rightarrow H^{m+s}(\mathbb{R}^n)\)

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being \(\left(h^{-1}\right)^*\).
Proof: Let $u \in \mathcal{S}$. From Theorem 37.3.10 and the equivalence of the norms in $W^{m,2}(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$,

\[
\left\| \mathbf{h}^* u \right\|_{H^m(\mathbb{R}^n)}^2 + \int \sum_{|\alpha|=m} |D^\alpha h^* u(x) - D^\alpha h^* u(y)|^2 |x-y|^{-n-2s} dxdy \leq C \left\| u \right\|_{H^m(\mathbb{R}^n)}^2 + \int \sum_{|\alpha|=m} |D^\alpha h^* u(x) - D^\alpha h^* u(y)|^2 |x-y|^{-n-2s} dxdy
\]

\[
= C \left\| u \right\|_{H^m(\mathbb{R}^n)}^2 + \int \sum_{|\alpha|=m} \left| \sum_{|\beta(\alpha)| \leq m} h^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(x) \right| \left| h^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(y) \right| |x-y|^{-n-2s} dxdy
\]

\[
\leq C \left\| u \right\|_{H^m(\mathbb{R}^n)}^2 + C \int \sum_{|\alpha|=m} \sum_{|\beta(\alpha)| \leq m} \left| h^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(x) \right| \left| h^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(y) \right| |x-y|^{-n-2s} dxdy
\]

A single term in the last sum corresponding to a given $\alpha$ is then of the form,

\[
\int \int \left| h^* (D^\beta u) g_{\beta}(x) - h^* (D^\beta u) g_{\beta}(y) \right|^2 |x-y|^{-n-2s} dxdy
\]

(37.3.14)

\[
\leq \int \int \left| h^* (D^\beta u)(x) g_{\beta}(x) - h^* (D^\beta u)(y) g_{\beta}(x) \right|^2 |x-y|^{-n-2s} dxdy + \int \int \left| h^* (D^\beta u)(y) g_{\beta}(x) - h^* (D^\beta u)(y) g_{\beta}(y) \right|^2 |x-y|^{-n-2s} dxdy
\]

\[
\leq C(h) \int \int \left| h^* (D^\beta u)(x) - h^* (D^\beta u)(y) \right|^2 |x-y|^{-n-2s} dxdy + \int \int \left| h^* (D^\beta u)(y) \right|^2 |g_{\beta}(x) - g_{\beta}(y)|^2 |x-y|^{-n-2s} dxdy
\]

Changing variables, and then using the names of the old variables to simplify the notation,

\[
\leq C(h, h^{-1}) \int \int \left| (D^\beta u)(x) - (D^\beta u)(y) \right|^2 |x-y|^{-n-2s} dxdy + \int \int \left| h^* (D^\beta u)(y) \right|^2 |g_{\beta}(x) - g_{\beta}(y)|^2 |x-y|^{-n-2s} dxdy
\]

By 37.3.10,

\[
\leq C(h) \int \int |F(u)(z)|^2 |z\beta|^2 |z|^{2s} dz + \int \int \left| h^* (D^\beta u)(y) \right|^2 |g_{\beta}(x) - g_{\beta}(y)|^2 |x-y|^{-n-2s} dxdy.
\]
In the second term, let \( t = x - y \). Then this term is of the form

\[
\int |h^* (D^\beta u)(y)|^2 \int |g_\beta(y + t) - g_\beta(y)|^2 |t|^{-n-2s} dtdy \quad (37.3.15)
\]

\[
\leq C \int |h^* (D^\beta u)(y)|^2 dy \leq C \|u\|^2_{H^m(\mathbb{R}^n)}. \quad (37.3.16)
\]

because the inside integral equals a constant which depends on the Lipschitz constants and bounds of the function, \( g_\beta \) and these things depend only on \( h \). The reason this integral is finite is that for \(|t| \leq 1\),

\[
|g_\beta(y + t) - g_\beta(y)|^2 |t|^{-n-2s} \leq K |t|^{-n-2s}
\]

and using polar coordinates, you see

\[
\int_{|t| \leq 1} |g_\beta(y + t) - g_\beta(y)|^2 |t|^{-n-2s} dt < \infty.
\]

Now for \(|t| > 1\), the integrand in (37.3.15) is dominated by \( 4 |t|^{-n-2s} \) and using polar coordinates, this yields

\[
\int_{|t| > 1} |g_\beta(y + t) - g_\beta(y)|^2 |t|^{-n-2s} dt \leq 4 \int_{|t| > 1} |t|^{-n-2s} dt < \infty.
\]

It follows (37.3.16) is dominated by an expression of the form

\[
C(h) \int |F(u)(z)|^2 |z|^2s dz + C \|u\|^2_{H^m(\mathbb{R}^n)}
\]

and so the sum in (37.3.13) is dominated by

\[
C(m, h) \int |F(u)(z)|^2 |z|^2s \sum_{|\beta| \leq m} |z| |z|^m dz + C \|u\|^2_{H^m(\mathbb{R}^n)}
\]

\[
\leq C(m, h) \int |F(u)(z)|^2 \left(1 + |z|^2\right)^s \left(1 + |z|^2\right)^m dz + C \|u\|^2_{H^m(\mathbb{R}^n)}
\]

\[
\leq C \|u\|^2_{H^{m+s}(\mathbb{R}^n)}. \]

This proves the theorem because the assertion about \( h^{-1} \) is obvious. Just replace \( h \) with \( h^{-1} \) in the above argument.

Next consider the case where \( U \) is an open set.

**Lemma 37.3.3** Let \( h(U) \subseteq V \) where \( U \) and \( V \) are open subsets of \( \mathbb{R}^n \) and suppose that \( h, h^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) are both functions in \( C^{m,1}(\mathbb{R}^n) \). Recall this means \( D^\alpha h \) and \( D^\alpha h^{-1} \) exist and are Lipschitz continuous for all \(|\alpha| \leq m\). Then \( h^* \in L(H^{m+s}(V), H^{m+s}(U)) \).
37.3.2

Let \( v \in H^{m+s}(V) \) and \( \phi \in C^{m,1}(\mathbb{R}^n) \) such that \( v|_V = u \). Then from the above, \( h^*v \in H^{m+s}(\mathbb{R}^n) \) and so \( h^*u \in H^{m+s}(U) \) because \( h^*v = h^*u \).

Then by Lemma 37.3.4,

\[
||h^*u||_{H^{m+s}(U)} \leq ||h^*v||_{H^{m+s}(\mathbb{R}^n)} \leq C ||v||_{H^{m+s}(\mathbb{R}^n)}
\]

Since this is true for all \( v \in H^{m+s}(\mathbb{R}^n) \), it follows that

\[
||h^*u||_{H^{m+s}(U)} \leq C ||u||_{H^{m+s}(V)}.
\]

With harder work, you don’t need to have \( h, h^{-1} \) defined on all of \( \mathbb{R}^n \) but I don’t feel like including the details so this lemma will suffice.

Another interesting application of the intrinsic norm is the following.

Lemma 37.3.4 Let \( \phi \in C^{m,1}(\mathbb{R}^n) \) and suppose \( \text{spt}(\phi) \) is compact. Then there exists a constant, \( C_\phi \) such that whenever \( u \in H^{m+s}(\mathbb{R}^n) \),

\[
||\phi u||_{H^{m+s}(\mathbb{R}^n)} \leq C_\phi ||u||_{H^{m+s}(\mathbb{R}^n)}.
\]

Proof: It is a routine exercise in the product rule to verify that \( ||\phi u||_{H^{m+s}(\mathbb{R}^n)} \leq C_\phi ||u||_{H^{m+s}(\mathbb{R}^n)} \). It only remains to consider the term involving the integral. A typical term is

\[
\int \int |D^\alpha \phi u(x) - D^\alpha \phi u(y)|^2 |x - y|^{-n-2s} \, dx \, dy.
\]

This is a finite sum of terms of the form

\[
\int \int |D^\gamma \phi(x) D^\beta u(x) - D^\gamma \phi(y) D^\beta u(y)|^2 |x - y|^{-n-2s} \, dx \, dy
\]

where \(|\gamma| \leq m\).

\[
\leq 2 \int \int |D^\gamma \phi(x)|^2 |D^\beta u(x) - D^\beta u(y)|^2 |x - y|^{-n-2s} \, dx \, dy
+ 2 \int \int |D^\beta u(y)|^2 |D^\gamma \phi(x) - D^\gamma \phi(y)|^2 |x - y|^{-n-2s} \, dx \, dy
\]

By Lemma and the Lipschitz continuity of all the derivatives of \( \phi \), this is dominated by

\[
CM(s) \int |F u(z)|^2 |z \beta|^2 |z|^{2s} \, dz
\]

\[
+ K \int \int |D^\beta u(y)|^2 |x - y|^2 |x - y|^{-n-2s} \, dx \, dy
= CM(s) \int |F u(z)|^2 |z \beta|^2 |z|^{2s} \, dz
\]

\[
+ K \int |D^\beta u(y)|^2 \int |t|^{-n+2(1-s)} \, dt \, dy
\]

\[
\leq C(s) \left( \int |F u(z)|^2 |z \beta|^2 |z|^{2s} \, dz + K \int |D^\beta u(y)|^2 \, dy \right)
\]

\[
\leq C(s) \left( 1 + |y|^2 \right)^{m+s} |F u(y)|^2 \, dy.
\]
Since there are only finitely many such terms, this proves the lemma.

**Corollary 37.3.5** Let \( t = m + s \) for \( s \in [0,1) \) and let \( U, V \) be open sets. Let \( \phi \in C_c^{m,1}(V) \). This means \( \text{spt}(\phi) \subseteq V \) and \( \phi \in C^{m,1}(\mathbb{R}^n) \). Then if \( u \in H^t(U) \) it follows that \( u\phi \in H^t(U \cap V) \) and \( \|u\phi\|_{H^t(U \cap V)} \leq C_\phi \|u\|_{H^t(U)} \).

**Proof:** Let \( v|_U = u \) a.e. where \( v \in H^t(\mathbb{R}^n) \). Then by Lemma 37.3.4, \( \phi v \in H^t(\mathbb{R}^n) \) and \( \phi v|_{U \cap V} = \phi u \) a.e. Therefore, \( \phi u \in H^t(U \cap V) \) and

\[
\|\phi u\|_{H^t(U \cap V)} \leq \|\phi v\|_{H^t(\mathbb{R}^n)} \leq C_\phi \|v\|_{H^t(\mathbb{R}^n)}.
\]

Taking the infimum for all such \( v \) whose restrictions equal \( u \), this yields

\[
\|\phi u\|_{H^t(U \cap V)} \leq C_\phi \|u\|_{H^t(U)}.
\]

This proves the corollary.

### 37.4 Embedding Theorems

The Fourier transform description of Sobolev spaces makes possible fairly easy proofs of various embedding theorems.

**Definition 37.4.1** Let \( C^m_0(\mathbb{R}^n) \) denote the functions which are \( m \) times continuously differentiable and for which

\[
\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |D^\alpha u(x)| \equiv \|u\|_{C^m_0(\mathbb{R}^n)} < \infty.
\]

For \( U \) an open set, \( C^m(U) \) denotes the functions which are restrictions of \( C^m_0(\mathbb{R}^n) \) to \( U \).

It is clear this is a Banach space, the proof being a simple exercise in the use of the fundamental theorem of calculus along with standard results about uniform convergence.

**Lemma 37.4.2** Let \( u \in \mathcal{S} \) and let \( \frac{n}{2} + m < t \). Then there exists \( C \) independent of \( u \) such that

\[
\|u\|_{C^m_0(\mathbb{R}^n)} \leq C \|u\|_{H^t(\mathbb{R}^n)}.
\]
CHAPTER 37. SOBOLEV SPACES BASED ON $L^2$

Proof: Using the fact that the Fourier transform maps $\mathcal{S}$ to $\mathcal{S}$ and the definition of the Fourier transform,

$$|D^\alpha u(x)| \leq C \|FD^\alpha u\|_{L^1(\mathbb{R}^n)}$$

$$= C \int |\xi^\alpha| |Fu(x)| \, dx$$

$$\leq C \int (1 + |x|^2)^{\alpha/2} |Fu(x)| \, dx$$

$$\leq C \int (1 + |x|^2)^{m/2} (1 + |x|^2)^{t/2} |Fu(x)| \, dx$$

$$\leq C \left( \int (1 + |x|^2)^{m-t} \, dx \right)^{1/2} \left( \int (1 + |x|^2)^t |Fu(x)|^t \right)^{1/2}$$

$$\leq C \|u\|_{H^t(\mathbb{R}^n)}$$

because for the given values of $t$ and $m$ the first integral is finite. This follows from a use of polar coordinates. Taking sup over all $x \in \mathbb{R}^n$ and $|\alpha| \leq m$, this proves the lemma.

**Corollary 37.4.3** Let $u \in H^t(\mathbb{R}^n)$ where $t > m + \frac{n}{2}$. Then $u$ is a.e. equal to a function of $C^m_b(\mathbb{R}^n)$ still denoted by $u$. Furthermore, there exists a constant, $C$ independent of $u$ such that

$$\|u\|_{C^m_b(\mathbb{R}^n)} \leq C \|u\|_{H^t(\mathbb{R}^n)}.$$

**Proof:** This follows from the above lemma. Let $\{u_k\}$ be a sequence of functions of $\mathcal{S}$ which converges to $u$ in $H^t$ and a.e. Then by the inequality of the above lemma, this sequence is also Cauchy in $C^m_b(\mathbb{R}^n)$ and taking the limit,

$$\|u\|_{C^m_b(\mathbb{R}^n)} = \lim_{k \to \infty} \|u_k\|_{C^m_b(\mathbb{R}^n)} \leq C \lim_{k \to \infty} \|u_k\|_{H^t(\mathbb{R}^n)} = C \|u\|_{H^t(\mathbb{R}^n)}.$$

What about open sets, $U$?

**Corollary 37.4.4** Let $t > m + \frac{n}{2}$ and let $U$ be an open set with $u \in H^t(U)$. Then $u$ is a.e. equal to a function of $C^m(\overline{U})$ still denoted by $u$. Furthermore, there exists a constant, $C$ independent of $u$ such that

$$\|u\|_{C^m(\overline{U})} \leq C \|u\|_{H^t(U)}.$$

**Proof:** Let $u \in H^t(U)$ and let $v \in H^t(\mathbb{R}^n)$ such that $v|_U = u$. Then

$$\|u\|_{C^m(\overline{U})} \leq \|v\|_{C^m_b(\mathbb{R}^n)} \leq C \|v\|_{H^t(\mathbb{R}^n)}.$$

Now taking the inf for all such $v$ yields

$$\|u\|_{C^m(\overline{U})} \leq C \|u\|_{H^t(U)}.$$
37.5 The Trace On The Boundary Of A Half Space

It is important to consider the restriction of functions in a Sobolev space onto a smaller dimensional set such as the boundary of an open set.

**Definition 37.5.1** For \( u \in \mathcal{S} \), define \( \gamma u \) a function defined on \( \mathbb{R}^{n-1} \) by \( \gamma u(x') \equiv u(x', 0) \) where \( x' \in \mathbb{R}^{n-1} \) is defined by \( x = (x', x_n) \).

The following elementary lemma featuring trig. substitutions is the basis for the proof of some of the arguments which follow.

**Lemma 37.5.2** Consider the integral,

\[
\int_{\mathbb{R}} (a^2 + x^2)^{-t} dx.
\]

for \( a > 0 \) and \( t > 1/2 \). Then this integral is of the form \( C_t a^{-2t+1} \) where \( C_t \) is some constant which depends on \( t \).

**Proof:** Letting \( x = a \tan \theta \),

\[
\int_{\mathbb{R}} (a^2 + x^2)^{-t} dx = a^{-2t+1} \int_{-\pi/2}^{\pi/2} \cos^{2t-2} (\theta) d\theta
\]

and since \( t > 1/2 \) the last integral is finite. This yields the desired conclusion and proves the lemma.

**Lemma 37.5.3** Let \( u \in \mathcal{S} \). Then there exists a constant, \( C_n \), depending on \( n \) but independent of \( u \in \mathcal{S} \) such that

\[
F\gamma u(x') = C_n \int_{\mathbb{R}} F(u(x', x_n)) \, dx_n.
\]

**Proof:** Using the dominated convergence theorem,

\[
\int_{\mathbb{R}} F(u(x', x_n)) \, dx_n \equiv \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} F(u(x', x_n)) \, dx_n
\]

\[
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} e^{-i(x' \cdot y + x_n y_n)} u(y', y_n) \, dy' \, dy_n \, dx_n
\]

\[
= \lim_{\varepsilon \to 0} \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} u(y', y_n) e^{-i x' \cdot y'} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} e^{-i x_n y_n} \, dx_n \, dy' \, dy_n
\]

Now \( -(\varepsilon x_n)^2 - ix_n y_n = -\varepsilon^2 \left( x_n + \frac{iy}{2} \right)^2 - \varepsilon^2 \frac{y^2}{4} \) and so the above reduces to an expression of the form

\[
\lim_{\varepsilon \to 0} K_n \int_{\mathbb{R}^n} u(y', y_n) e^{-i x' \cdot y'} \, dy' \, dy_n = K_n \int_{\mathbb{R}^n} u(y', 0) e^{-i x' \cdot y'} \, dy' = K_n F\gamma u(x')
\]
and this proves the lemma with \( C_n \equiv K_n^{-1} \).

Earlier \( H^1(\mathbb{R}^n) \) was defined and then for \( U \) an open subset of \( \mathbb{R}^n \), \( H^1(U) \) was defined to be the space of restrictions of functions of \( H^1(\mathbb{R}^n) \) to \( U \) and a norm was given which made \( H^1(U) \) into a Banach space. The next task is to consider \( \mathbb{R}^{n-1} \times \{0\} \), a smaller dimensional subspace of \( \mathbb{R}^n \) and examine the functions defined on this set, denoted by \( \mathbb{R}^{n-1} \) for short which are restrictions of functions in \( H^1(\mathbb{R}^n) \). You note this is somewhat different because heuristically, the dimension of the domain of the function is changing. An open set in \( \mathbb{R}^n \) is considered an \( n \) dimensional thing but \( \mathbb{R}^{n-1} \) is only \( n-1 \) dimensional. I realize this is vague because the standard definition of dimension requires a vector space and an open set is not a vector space. However, think in terms of fatness. An open set is fat in \( n \) directions whereas \( \mathbb{R}^{n-1} \) is only fat in \( n-1 \) directions. Therefore, something interesting is likely to happen.

Let \( \mathcal{S} \) denote the Schwartz class of functions on \( \mathbb{R}^n \) and \( \mathcal{S}' \) the Schwartz class of functions on \( \mathbb{R}^{n-1} \). Also, \( y' \in \mathbb{R}^{n-1} \) while \( y \in \mathbb{R}^n \). Let \( u \in \mathcal{S} \). Then from Lemma \( \text{[37.5.17]} \) and \( s > 0 \),

\[
\int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s |F\gamma u(y')|^2 \, dy' = C_n \int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s \left| \int_{\mathbb{R}} F(u(y',y_n)) \, dy_n \right|^2 \, dy' \\
= C_n \int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s \left| \int_{\mathbb{R}} F(u(y',y_n)) \left(1 + |y|^2\right)^{t/2} \left(1 + |y|^2\right)^{-t/2} \, dy_n \right|^2 \, dy' \\
\leq C_n \int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s \int_{\mathbb{R}} |F(u(y',y_n))|^2 \left(1 + |y|^2\right)^{t} \, dy_n \int_{\mathbb{R}} \left(1 + |y|^2\right)^{-t} \, dy_n \, dy'.
\]

(37.5.17)

Then by the Cauchy Schwarz inequality,

\[
\leq C_n \int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s \int_{\mathbb{R}} |F(u(y',y_n))|^2 \left(1 + |y|^2\right)^{t} \, dy_n \int_{\mathbb{R}} \left(1 + |y|^2\right)^{-t} \, dy_n \, dy'.
\]

Consider

\[
\int_{\mathbb{R}} \left(1 + |y|^2\right)^{-t} \, dy_n = \int_{\mathbb{R}} \left(1 + |y'|^2 + y_n^2\right)^{-t} \, dy_n
\]

by Lemma \( \text{[37.5.2]} \) and taking \( a = \left(1 + |y'|^2\right)^{1/2} \), this equals

\[
C_t \left(1 + |y'|^2\right)^{1/2} = C_t \left(1 + |y'|^2\right)^{-2t+1/2}.
\]

Now using this in \( \text{[37.5.17]} \),

\[
\int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s |F\gamma u(y')|^2 \, dy' \\
\leq C_{n,t} \int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^s \int_{\mathbb{R}} |F(u(y',y_n))|^2 \left(1 + |y|^2\right)^{t} \, dy_n \cdot \left(1 + |y'|^2\right)^{-2t+1/2} \, dy' \\
= C_{n,t} \int_{\mathbb{R}^{n-1}} \left(1 + |y'|^2\right)^{s+(2t-1)/2} \int_{\mathbb{R}} |F(u(y',y_n))|^2 \left(1 + |y|^2\right)^{t} \, dy_n \, dy'.
\]
What is the correct choice of \( t \) so that the above reduces to \( \|u\|_{H^t(\mathbb{R}^n)}^2 \)? It is clearly the one for which

\[
s + (-2t + 1)/2 = 0
\]

which occurs when \( t = s + 1/2 \). Then for this choice of \( t \), the following inequality is obtained for any \( u \in \mathcal{S} \).

\[
\|\gamma u\|_{H^{t-1/2}((\mathbb{R}^{n-1})^t)} \leq C_{n,t} \|u\|_{H^t(\mathbb{R}^n)}.
\]

(37.5.18)

This has proved part of the following theorem.

**Theorem 37.5.4** For each \( t > 1/2 \) there exists a unique mapping

\[
\gamma \in \mathcal{L}\left(H^t(\mathbb{R}^n), H^{t-1/2}((\mathbb{R}^{n-1})^t)\right)
\]

which has the property that for \( u \in \mathcal{S} \), \( \gamma u(x') = u(x', 0) \). In addition to this, \( \gamma \) is onto. In fact, there exists a continuous map, \( \zeta \in \mathcal{L}\left(H^{t-1/2}((\mathbb{R}^{n-1})^t), H^t(\mathbb{R}^n)\right) \) such that \( \gamma \circ \zeta = \text{id} \).

**Proof:** It only remains to verify that \( \gamma \) is onto and that the continuous map, \( \zeta \) exists. Now define

\[
\phi(y) \equiv \phi(y', y_n) \equiv \frac{(1 + |y'|^2)^{t-1/2}}{(1 + |y|^2)^t}.
\]

Then for \( u \in \mathcal{S}' \), let

\[
\zeta u(x) \equiv CF^{-1}(\phi Fu)(x) =
C \int_{\mathbb{R}^n} e^{iy \cdot x} \frac{(1 + |y'|^2)^{t-1/2}}{(1 + |y|^2)^t} Fu(y') dy
\]

(37.5.19)

Here the inside Fourier transform is taken with respect to \( \mathbb{R}^{n-1} \) because \( u \) is only defined on \( \mathbb{R}^{n-1} \) and \( C \) will be chosen in such a way that \( \gamma \circ \zeta = \text{id} \). First the existence of \( C \) such that \( \gamma \circ \zeta = \text{id} \) will be shown. Since \( u \in \mathcal{S}' \) it follows

\[
y \rightarrow \frac{(1 + |y'|^2)^{t-1/2}}{(1 + |y|^2)^t} Fu(y')
\]

is in \( \mathcal{S} \). Hence the inverse Fourier transform of this function is also in \( \mathcal{S} \) and so for \( u \in \mathcal{S}' \), it follows \( \zeta u \in \mathcal{S} \). Therefore, to check \( \gamma \circ \zeta = \text{id} \) it suffices to plug in \( x_n = 0 \). From Lemma 37.5.2 this yields

\[
\gamma(\zeta u)(x', 0)
= C \int_{\mathbb{R}^n} e^{iy' \cdot x'} \frac{(1 + |y'|^2)^{t-1/2}}{(1 + |y|^2)^t} Fu(y') dy
\]
Suppose Lemma 37.5.5 on the boundary of an open set is considered.

This proves the theorem because \( \mathfrak{S} \) is dense in \( \mathbb{R}^n \).

Actually, the assertion that \( \gamma u (x') = u (x', 0) \) holds for more functions, \( u \) than just \( u \in \mathfrak{S} \). I will make no effort to obtain the most general description of such functions but the following is a useful lemma which will be needed when the trace on the boundary of an open set is considered.

**Lemma 37.5.5** Suppose \( u \) is continuous and \( u \in H^1 (\mathbb{R}^n) \). Then there exists a set of \( m_1 \) measure zero, \( N \) such that if \( x_n \notin N \), then for every \( \phi \in L^2 (\mathbb{R}^{n-1}) \)

\[
(\gamma u, \phi)_H + \int_0^{x_n} (u_n (\cdot, t), \phi)_H \, dt = (u (\cdot, x_n), \phi)_H
\]
where here
\[ (f, g)_H = \int_{\mathbb{R}^{n-1}} f \bar{g} dx', \]
just the inner product in \( L^2 (\mathbb{R}^{n-1}) \). Furthermore,
\[ u (\cdot, 0) = \gamma u \text{ a.e. } x'. \]

**Proof:** Let \( \{ u_k \} \) be a sequence of functions from \( \mathcal{S} \) which converges to \( u \) in \( H^1 (\mathbb{R}^n) \) and let \( \{ \phi_k \} \) denote a countable dense subset of \( L^2 (\mathbb{R}^{n-1}) \). Then
\[
(\gamma u_k, \phi_j)_H + \int_0^{x_n} (u_{k,n} (\cdot,t), \phi_j)_H dt = (u_k (\cdot, x_n), \phi_j)_H. \tag{37.5.20}
\]
Now
\[
\begin{align*}
&\left( \int_0^\infty \left| (u_k (\cdot, x_n), \phi_j)_H - (u (\cdot, x_n), \phi_j)_H \right|^2 dx_n \right)^{1/2} \\
&= \left( \int_0^\infty \left| (u_k (\cdot, x_n) - u (\cdot, x_n), \phi_j)_H \right|^2 dx_n \right)^{1/2} \\
&\leq \left( \int_0^\infty |u_k (\cdot, x_n) - u (\cdot, x_n)|^2_H |\phi_j|^2 dx_n \right)^{1/2} \\
&= |\phi_j|_H^2 \left( \int_0^\infty \int_{\mathbb{R}^{n-1}} |u_k (x', x_n) - u (x', x_n)|^2 dx' dx_n \right)^{1/2}
\end{align*}
\]
which converges to zero. Therefore, there exists a set of measure zero, \( N_j \) and a subsequence, still denoted by \( k \) such that if \( x_n \notin N_j \), then
\[ (u_k (\cdot, x_n), \phi_j)_H \to (u (\cdot, x_n), \phi_j)_H. \]
Now by Theorem [37.5.8], \( \gamma u_k \to \gamma u \) in \( H = L^2 (\mathbb{R}^{n-1}) \). It only remains to consider the term of interest which involves an integral.
\[
\begin{align*}
&\left| \int_0^{x_n} (u_{k,n} (\cdot,t), \phi_j)_H dt - \int_0^{x_n} (u_{n,n} (\cdot,t), \phi_j)_H dt \right| \\
&\leq \int_0^{x_n} \left| (u_{k,n} (\cdot, t) - u_{n,n} (\cdot, t), \phi_j)_H \right| dt \\
&\leq \int_0^{x_n} |u_{k,n} (\cdot, t) - u_{n,n} (\cdot, t)|_H |\phi_j|_H dt \\
&\leq \left( \int_0^{x_n} |u_{k,n} (\cdot, t) - u_{n,n} (\cdot, t)|^2_H dt \right)^{1/2} \left( \int_0^{x_n} |\phi_j|^2_H dt \right)^{1/2} \\
&= x_n^{1/2} |\phi_j|_H \left( \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |u_{k,n} (x', t) - u_{n,n} (x', t)|^2 dx' dt \right)^{1/2}
\end{align*}
\]
and this converges to zero as $k \to \infty$. Therefore, using a diagonal sequence argument, there exists a subsequence, still denoted by $k$ and a set of measure zero, $N \equiv \bigcup_{j=1}^{\infty} N_j$ such that for $x' \notin N$, you can pass to the limit in and obtain that for all $\phi_j$,

$$
(\gamma u, \phi_j)_H + \int_0^{x_n} (u, \phi_j) dt = (u, \phi_j)_H.
$$

By density of $\{\phi_j\}$, this equality holds for all $\phi \in L^2(\mathbb{R}^{n-1})$. In particular, the equality holds for every $\phi \in C_c(\mathbb{R}^{n-1})$. Since $u$ is uniformly continuous on the compact set, $\text{spt}(\phi) \times [0,1]$, there exists a sequence, $(x_n)_k \to 0$ such that the above equality holds for $x_n$ replaced with $(x_n)_k$ and $\phi$ in place of $\phi_j$. Now taking $k \to \infty$, this uniform continuity implies

$$(\gamma u, \phi)_H = (u(\cdot,0), \phi)_H.$$

This implies since $C_c(\mathbb{R}^{n-1})$ is dense in $L^2(\mathbb{R}^{n-1})$ that $\gamma u = u(\cdot,0)$ a.e. and this proves the lemma.

**Lemma 37.5.6** Suppose $U$ is an open subset of $\mathbb{R}^n$ of the form

$$U \equiv \{u \in \mathbb{R}^n : u' \in U' \text{ and } 0 < u_n < \phi(u')\}$$

where $U'$ is an open subset of $\mathbb{R}^{n-1}$ and $\phi(u')$ is a positive function such that $\phi(u') \leq \infty$ and

$$\inf \{\phi(u') : u' \in U'\} = \delta > 0$$

Suppose $v \in H^1(\mathbb{R}^n)$ such that $v = 0$ a.e. on $U$. Then $\gamma v = 0$ a.e. point of $U'$. Also, if $v \in H^1(\mathbb{R}^n)$ and $\phi \in C_c(\mathbb{R}^n)$, then $\gamma v \gamma \phi = \gamma (\phi v)$.

**Proof:** First consider the second claim. Let $v \in H^1(\mathbb{R}^n)$ and let $v_k \to v$ in $H^1(\mathbb{R}^n)$ where $v_k \in \mathcal{S}$. Then from Lemma 37.4.3 and Theorem 37.5.3:

$$||\gamma (\phi v) - \gamma \phi \gamma v||_{H^{1/2}(\mathbb{R}^n)} = \lim_{k \to \infty} ||\gamma (\phi v_k) - \gamma \phi \gamma v_k||_{H^{1/2}(\mathbb{R}^n)} = 0$$

because each term in the sequence equals zero due to the observation that for $v_k \in \mathcal{S}$ and $\phi \in C_c(\mathcal{U})$, $\gamma (\phi v_k) = \gamma v_k \gamma \phi$.

Now suppose $v = 0$ a.e. on $U$. Define for $0 < r < \delta$, $v_r(x) \equiv v(x', x_n + r)$.

**Claim:** If $u \in H^1(\mathbb{R}^n)$, then

$$\lim_{r \to 0} ||v_r - v||_{H^1(\mathbb{R}^n)} = 0.$$

**Proof of claim:** First of all, let $v \in \mathcal{S}$. Then $v \in H^m(\mathbb{R}^n)$ for all $m$ and so by Lemma 37.3.4,

$$||v_r - v||_{H^1(\mathbb{R}^n)} \leq ||v_r - v||^\theta_{H^m(\mathbb{R}^n)} ||v_r - v||^{1-\theta}_{H^{m+1}(\mathbb{R}^n)}$$
where \( t \in [m, m + 1) \). It follows from continuity of translation in \( L^p (\mathbb{R}^n) \) that
\[
\lim_{r \to 0} \| v_r - v \|_{H^m(\mathbb{R}^n)}^{1-\theta} \| v_r - v \|_{H^{m+1}(\mathbb{R}^n)}^{\theta} = 0
\]
and so the claim is proved if \( v \in \mathcal{S} \). Now suppose \( u \in H^t (\mathbb{R}^n) \) is arbitrary. By density of \( \mathcal{S} \) in \( H^t (\mathbb{R}^n) \), there exists \( v \in \mathcal{S} \) such that
\[
\| u - v \|_{H^t(\mathbb{R}^n)} < \varepsilon/3.
\]
Therefore,
\[
\| u_r - u \|_{H^t(\mathbb{R}^n)} \leq \| u_r - v \|_{H^t(\mathbb{R}^n)} + \| v_r - v \|_{H^t(\mathbb{R}^n)} + \| v - u \|_{H^t(\mathbb{R}^n)}
\]
\[
= 2\varepsilon/3 + \| v_r - v \|_{H^t(\mathbb{R}^n)}.
\]
Now using what was just shown, it follows that for \( r \) small enough, \( \| u_r - u \|_{H^t(\mathbb{R}^n)} < \varepsilon \) and this proves the claim.

Now suppose \( v \in H^t (\mathbb{R}^n) \). By the claim,
\[
\| v_r - v \|_{H^t(\mathbb{R}^n)} \to 0
\]
and so by continuity of \( \gamma \),
\[
\gamma v_r \to \gamma v \text{ in } H^{t-1/2} (\mathbb{R}^{n-1}). \tag{37.5.21}
\]
Note \( v_r = 0 \) a.e. on \( U_r \equiv \{ u \in \mathbb{R}^n : u' \in U' \text{ and } -r < u_n < \phi (u') - r \} \).

Let \( \phi \in C_0^\infty (U_r) \) and consider \( \phi v_r \). Then it follows \( \phi v_r = 0 \) a.e. on \( \mathbb{R}^n \). Let \( w \equiv 0 \). Then \( w \in \mathcal{S} \) and so \( \gamma w = 0 = \gamma (\phi v_r) = \gamma \phi \gamma v_r \) in \( H^{t-1/2} (\mathbb{R}^{n-1}) \). It follows that for \( m_{n-1} \) a.e. \( x' \in [\phi \neq 0] \cap \mathbb{R}^{n-1} \), \( \gamma v_r (x') = 0 \). Now let \( U' = \cup_{k=1}^\infty K_k \) where the \( K_k \) are compact sets such that \( K_k \subseteq K_{k+1} \) and let \( \phi_k \in C_0^\infty (U) \) such that \( \phi_k \) has values in \([0, 1]\) and \( \phi_k (x') = 1 \) if \( x' \in K_k \). Then from what was just shown, \( \gamma v_r = 0 \) for a.e. point in \( U' \). Therefore, since each \( \gamma v_r = 0 \), it follows from [Notes] that \( \gamma v = 0 \) also. This proves the lemma.

**Theorem 37.5.7** Let \( t > 1/2 \) and let \( U \) be of the form
\[
\{ u \in \mathbb{R}^n : u' \in U' \text{ and } 0 < u_n < \phi (u') \}
\]
where \( U' \) is an open subset of \( \mathbb{R}^{n-1} \) and \( \phi (u') \) is a positive function such that \( \phi (u') \leq \infty \) and
\[
\inf \{ \phi (u') : u' \in U' \} = \delta > 0.
\]
Then there exists a unique
\[
\gamma \in \mathcal{L} \left( H^t (U), H^{t-1/2} (U') \right)
\]
which has the property that if \( u = v|_U \) where \( v \) is continuous and also a function of \( H^1 (\mathbb{R}^n) \), then \( \gamma u (x') = u (x', 0) \) for a.e. \( x' \in U' \).
Proof: Let \( u \in H^t(U) \). Then \( u = v|_U \) for some \( v \in H^t(\mathbb{R}^n) \). Define

\[
\gamma u \equiv \gamma v|_{U'}
\]

Is this well defined? The answer is yes because if \( v_i|_U = u \) a.e., then \( \gamma(v_1 - v_2) = 0 \) a.e. on \( U' \) which implies \( \gamma v_1 = \gamma v_2 \) a.e. and so the two different versions of \( \gamma u \) differ only on a set of measure zero.

If \( u = v|_U \) where \( v \) is continuous and also a function of \( H^1(\mathbb{R}^n) \), then for a.e. \( x' \in \mathbb{R}^n - 1 \), it follows from Lemma 37.5.5 on Page 1386 that \( \gamma v(x') = u(x',0) \).

Hence, it follows that for a.e. \( x' \in U' \), \( \gamma u(x') \equiv u(x',0) \).

In particular, \( \gamma \) is determined by \( \gamma u(x') = u(x',0) \) on \( \mathcal{G}|_U \) and the density of \( \mathcal{G}|_U \) and continuity of \( \gamma \) shows \( \gamma \) is unique.

It only remains to show \( \gamma \) is continuous. Let \( u \in H^t(U) \). Thus there exists \( v \in H^t(\mathbb{R}^n) \) such that \( u = v|_U \).

Then

\[
||\gamma u||_{H^{t-1/2(U')}} \leq ||\gamma v||_{H^{t-1/2(\mathbb{R}^n - 1)}} \leq C ||v||_{H^t(\mathbb{R}^n)}
\]

for \( C \) independent of \( v \). Then taking the inf for all such \( v \in H^t(\mathbb{R}^n) \) which are equal to \( u \) a.e. on \( U \), it follows

\[
||\gamma u||_{H^{t-1/2(U')}} \leq C ||v||_{H^t(\mathbb{R}^n)}
\]

and this proves \( \gamma \) is continuous.

### 37.6 Sobolev Spaces On Manifolds

#### 37.6.1 General Theory

The type of manifold, \( \Gamma \) for which Sobolev spaces will be defined on is:

**Definition 37.6.1**

1. \( \Gamma \) is a closed subset of \( \mathbb{R}^p \) where \( p \geq n \).

2. \( \Gamma = \bigcup_{i=1}^{\infty} \Gamma_i \) where \( \Gamma_i = \Gamma \cap W_i \) for \( W_i \) a bounded open set.

3. \( \{W_i\}_{i=1}^{\infty} \) is locally finite.

4. There are open bounded sets, \( U_i \) and functions \( h_i : U_i \to \Gamma_i \) which are one to one, onto, and in \( C^{m,1}(U_i) \). There exists a constant, \( C \), such that \( C \geq \text{Lip} h_r \) for all \( r \).

5. There exist functions, \( g_i : W_i \to U_i \) such that \( g_i \) is \( C^{m,1}(W_i) \), and \( g_i \circ h_i = \text{id} \) on \( U_i \) while \( h_i \circ g_i = \text{id} \) on \( \Gamma_i \). This will be referred to as a \( C^{m,1} \) manifold.

**Lemma 37.6.2** Let \( g_i, h_i, U_i, W_i, \) and \( \Gamma_i \) be as defined above. Then

\[
g_i \circ h_i : U_i \cap h^{-1}_i(\Gamma_i) \to U_i \cap h^{-1}_i(\Gamma_i)
\]

is \( C^{m,1} \). Furthermore, the inverse of this map is \( g_k \circ h_i \).
Let $H$ be a Cauchy sequence in $H$. It follows easily by approximating with simple functions that for ever nonnegative $\mu$ measurable function, $u$ is in $H$ with restriction to $\Gamma$ of any function.

Now since $h_k(x) \in \Gamma$, $g_i(h_k(x)) \in h_k^{-1}(\Gamma)$ also and this proves the mappings do what they should in terms of mapping the two open sets. That $g_i \circ h_k$ is $C^{m,1}$ follows immediately from the chain rule and the assumptions that the functions $g_i$ and $h_k$ are $C^{m,1}$. The claim about the inverse follows immediately from the definitions of the functions.

Let $\{\psi_i\}_{i=1}^\infty$ be a partition of unity subordinate to the open cover $\{W_i\}$ satisfying $\psi_i \in C_c^\infty (W_i)$. Then the following definition provides a norm for $H^{m+s}(\Gamma)$.

**Definition 37.6.3** Let $s \in (0,1)$ and $m$ is a nonnegative integer. Also let $\mu$ denote the surface measure for $\Gamma$ defined in the last section. A measurable function, $u$ is in $H^{m+s}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma, U_i, h_i, g_i\}_{i=1}^\infty$ is described above, $h_i^* (u\psi_i) \in H^{m+s}(U_i)$ and

$$||u||_{H^{m+s}(\Gamma)} \equiv \left( \sum_{i=1}^\infty ||h_i^* (u\psi_i)||_{H^{m+s}(U_i)}^2 \right)^{1/2} < \infty.$$  

Are there functions which are in $H^{m+s}(\Gamma)$? The answer is yes. Just take the restriction to $\Gamma$ of any function, $u \in C_\infty (\mathbb{R}^m)$. Then each $h_i^* (u\psi_i) \in H^{m+s}(U_i)$ and the sum is finite because spt $u$ has nonempty intersection with only finitely many $W_i$.

It is not at all obvious this norm is well defined. What if $\{W'_i, \psi'_i, \Gamma'_i, U_i, h'_i, g'_i\}_{i=1}^\infty$ is as described above? Would the two norms be equivalent? If they aren’t, then this is not a good way to define $H^{m+s}(\Gamma)$ because it would depend on the choice of partition of unity and functions, $h_i$ and choice of the open sets, $U_i$. To begin with pick a particular choice for $\{W_i, \psi_i, \Gamma, U_i, h_i, g_i\}_{i=1}^\infty$.

**Lemma 37.6.4** $H^{m+s}(\Gamma)$ as just described, is a Banach space.

**Proof:** Let $\{u_j\}_{j=1}^\infty$ be a Cauchy sequence in $H^{m+s}(\Gamma)$. Then $\{h_i^* (u_j\psi_i)\}_{j=1}^\infty$ is a Cauchy sequence in $H^{m+s}(U_i)$ for each $i$. Therefore, for each $i$, there exists $w_i \in H^{m+s}(U_i)$ such that

$$\lim_{j \to \infty} h_i^* (u_j\psi_i) = w_i \text{ in } H^{m+s}(U_i). \tag{37.6.22}$$

It is required to show there exists $u \in H^{m+s}(\Gamma)$ such that $w_i = h_i^* (u\psi_i)$ for each $i$.

Now from Corollary 37.6.2 it follows easily by approximating with simple functions that for ever nonnegative $\mu$ measurable function, $f$,

$$\int_{\Gamma} f \, d\mu = \sum_{r=1}^\infty \int_{\mathbb{R}^m} \psi_r f (h_r(u)) J_r(u) \, du.$$
Therefore, \[
\int_{\Gamma} |u_j - u_k|^2 \, d\mu = \sum_{r=1}^{\infty} \int_{\mathcal{G}, \Gamma_r} \psi_r |u_j - u_k|^2 (h_r(u)) \, J_r(u) \, du 
\leq C \sum_{r=1}^{\infty} \int_{\mathcal{G}, \Gamma_r} \psi_r |u_j - u_k|^2 (h_r(u)) \, du 
= C \sum_{r=1}^{\infty} ||h^*_r(\psi_r |u_j - u_k|)||^2_{0,2,U_r} 
\leq C ||u_j - u_k||_{H^{m+s}(\Gamma)}
\]
and it follows there exists \( u \in L^2(\Gamma) \) such that
\[
||u_j - u||_{0,2,\Gamma} \to 0.
\]
and a subsequence, still denoted by \( u_j \) such that \( u_j(x) \to u(x) \) for \( \mu \) a.e. \( x \in \Gamma \). It is required to show that \( u \in H^{m+s}(\Gamma) \) such that \( w_i = h^*_i(u \psi_i) \) for each \( i \). First of all, \( u \) is measurable because it is the limit of measurable functions. The pointwise convergence just established and the fact that sets of measure zero on \( \Gamma \) correspond to sets of measure zero on \( U_i \) which was discussed in the claim found in the proof of Theorem 35.2.4 on Page 1321 shows that \( h^*_i(u \psi_i) \to w_i \) a.e. \( x \).

Therefore, \( h^*_i(u \psi_i)(x) \to h^*_i(u \psi_i)(x) \)

and this shows that \( h^*_i(u \psi_i) \in H^{m+s}(U_i) \) . It remains to verify that \( u \in H^{m+s}(\Gamma) \).

This follows from Fatou’s lemma. From Lemma 37.6.22
\[
||h^*_i(u \psi_i)||_{H^{m+s}(U_i)} \to ||h^*_i(u \psi_i)||_{H^{m+s}(U_i)}
\]
and so
\[
\sum_{i=1}^{\infty} ||h^*_i(u \psi_i)||_{H^{m+s}(U_i)} \leq \liminf_{j \to \infty} \sum_{i=1}^{\infty} ||h^*_i(u_j \psi_i)||_{H^{m+s}(U_i)} 
= \liminf_{j \to \infty} ||u_j||_{H^{m+s}(\Gamma)} < \infty.
\]

This proves the lemma.

In fact any two such norms are equivalent. This follows from the open mapping theorem. Suppose \( ||\cdot||_1 \) and \( ||\cdot||_2 \) are two such norms and consider the norm \( ||\cdot||_3 = \max(||\cdot||_1, ||\cdot||_2) \). Then \( (H^{m+s}(\Gamma), ||\cdot||_3) \) is also a Banach space and the identity map from this Banach space to \( (H^{m+s}(\Gamma), ||\cdot||_i) \) for \( i = 1, 2 \) is continuous. Therefore, by the open mapping theorem, there exist constants, \( C, C' \) such that for all \( u \in H^{m+s}(\Gamma) \),
\[
||u||_1 \leq ||u||_3 \leq C ||u||_2 \leq C ||u||_3 \leq CC' ||u||_1
\]
Therefore,
\[ ||u||_1 \leq C ||u||_2, \quad ||u||_2 \leq C'||u||_1. \]

This proves the following theorem.

**Theorem 37.6.5** Let \( \Gamma \) be described above. Defining \( H^t(\Gamma) \) as in Definition 37.6.3, any two norms like those given in this definition are equivalent.

Suppose \((\Gamma, W_i, U_i, \Gamma_i, h_i, g_i)\) are as defined above where \( h_i, g_i \) are \( C^{m,1} \) functions. Take \( W, \) an open set in \( \mathbb{R}^p \) and define \( \Gamma' \equiv W \cap \Gamma \). Then letting
\[ W'_i \equiv W \cap W_i, \quad \Gamma'_i \equiv W'_i \cap \Gamma, \]
and
\[ U'_i \equiv g_i(\Gamma'_i) = h_i^{-1}(W'_i \cap \Gamma), \]
it follows that \( U'_i \) is an open set because \( h_i \) is continuous and \((\Gamma'_i, W'_i, U'_i, \Gamma'_i, h'_i, g'_i)\) is also a \( C^{m,1} \) manifold if you define \( h'_i \) to be the restriction of \( h_i \) to \( U'_i \) and \( g'_i \) to be the restriction of \( g_i \) to \( W'_i \).

As a case of this, consider a \( C^{m,1} \) manifold, \( \Gamma \) where \((\Gamma, W_i, U_i, \Gamma_i, h_i, g_i)\) are as described in Definition 37.6.1 and the submanifold consisting of \( \Gamma_i \). The next lemma shows there is a simple way to define a norm on \( H^t(\Gamma_i) \) which does not depend on dragging in a partition of unity.

**Lemma 37.6.6** Suppose \( \Gamma \) is a \( C^{m,1} \) manifold and \((\Gamma, W_i, U_i, \Gamma_i, h_i, g_i)\) are as described in Definition 37.6.1. Then for \( t \in [m, m+s) \), it follows that if \( u \in H^t(\Gamma) \), then \( u \in H^t(\Gamma_k) \) and the restriction map is continuous. Also an equivalent norm for \( H^t(\Gamma_k) \) is given by
\[ |||u|||_t \equiv ||h^*_k u||_{H^t(U_k)}. \]

**Proof:** Let \( u \in H^t(\Gamma) \) and let \((\Gamma_k, W'_i, U'_i, \Gamma'_i, h'_i, g'_i)\) be the sets and functions which define what is meant by \( \Gamma_k \) being a \( C^{m,1} \) manifold as described in Definition 37.6.1. Also let \((\Gamma, W_i, U_i, \Gamma_i, h_i, g_i)\) be pertinent to \( \Gamma \) in the same way and let \( \{\phi_j\} \) be a \( C^\infty \) partition of unity for the \( \{W_j\} \). Since the \( \{W'_i\} \) are locally finite, only finitely many can intersect \( \Gamma_k \), say \( \{W'_1, \cdots, W'_{s}\} \). Also only finitely many of the \( W_i \) can intersect \( \Gamma_k \), say \( \{W_1, \cdots, W_q\} \). Then letting \( \{\psi'_i\} \) be a \( C^\infty \) partition of
unity subordinate to the $\{W_i\}$.

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{q} \phi_j u \psi_i \right|_{H^r(U_i')} = \sum_{i=1}^{s} \left| \sum_{j=1}^{q} \phi_j u \psi_i \right|_{H^r(U_i')} \leq \sum_{j=1}^{q} \sum_{i=1}^{s} \left| \sum_{j=1}^{q} \phi_j u \psi_i \right|_{H^r(U_i')} = \sum_{j=1}^{q} \sum_{i=1}^{s} \left| (g_j \circ h'_i) \ast h'_i \phi_j u \psi_i \right|_{H^r(U_i')} \leq C \sum_{j=1}^{q} \sum_{i=1}^{s} \left| (g_j \circ h'_i) \ast h'_i \phi_j u \psi_i \right|_{H^r(U_j)}.$$  

By Lemma 37.3.3 on page 1379, there exists a single constant, $C$ such that the above is dominated by

$$C \sum_{j=1}^{q} \sum_{i=1}^{s} \left| (g_j \circ h'_i) \ast h'_i \phi_j u \psi_i \right|_{H^r(U_j)}.$$  

Now by Corollary 37.3.5 on Page 1381, this is no larger than

$$C \sum_{j=1}^{q} \sum_{i=1}^{s} \left| (g_j \circ h'_i) \ast h'_i \phi_j u \psi_i \right|_{H^r(U_j)} \leq C \sum_{j=1}^{q} \sum_{i=1}^{s} \left| \sum_{j=1}^{q} \phi_j u \psi_i \right|_{H^r(U_j)} = C \sum_{j=1}^{q} \left| \sum_{j=1}^{q} \phi_j u \psi_i \right|_{H^r(U_j)} < \infty.$$  

This shows that $u$ restricted to $\Gamma_k$ is in $H^r(\Gamma_k)$. It also shows that the restriction map of $H^r(\Gamma)$ to $H^r(\Gamma_k)$ is continuous.

Now consider the norm $|||\cdot|||$ on $\Gamma_k$. For $u \in H^r(\Gamma_k)$, let $(\Gamma_k, W_i', U_i', \Gamma_i', h'_i, g'_i)$ be sets and functions which define an atlas for $\Gamma_k$. Since the $\{W_i\}$ are locally finite, only finitely many can have nonempty intersection with $\Gamma_k$, say $\{W_1, \cdots, W_s\}$. Thus $i \leq s$ for some finite $s$. The problem is to compare $|||\cdot|||$ with $||\cdot|||_{H^r(\Gamma_k)}$. As above,
let \( \{ \psi'_j \} \) denote a \( C^\infty \) partition of unity subordinate to the \( \{ W'_j \} \). Then

\[
\| u \|_t = \| h_k^* u \|_{H'(U_k)} = \left\| h_k^* \sum_{j=1}^s \psi'_j u \right\|_{H'(U_k)}
\]

\[
\leq \sum_{j=1}^s \left\| h_k^* (\psi'_j u) \right\|_{H'(U_k)}
\]

\[
= \sum_{j=1}^s \left\| (g'_j \circ h_k)^* h_j^* (\psi'_j u) \right\|_{H'(U'_j)}
\]

\[
\leq C \sum_{j=1}^s \left\| h_j^*(\psi'_j u) \right\|_{H'(U'_j)}.
\]

\[
\leq C \left( \sum_{j=1}^s \left\| h_j^*(\psi'_j u) \right\|^2_{H'(U'_j)} \right)^{1/2}
\]

where Lemma 37.3.3 on page 1379 was used in the last step. Now also, from Lemma 37.3.3 on page 1379

\[
\| u \|_{H'(\Gamma_k)} = \left( \sum_{j=1}^s \left\| h_j^*(\psi'_j u) \right\|^2_{H'(U'_j)} \right)^{1/2}
\]

\[
= \left( \sum_{j=1}^s \left\| (g_k \circ h'_j)^* h_k^*(\psi'_j u) \right\|^2_{H'(U'_j)} \right)^{1/2}
\]

\[
\leq C \left( \sum_{j=1}^s \left\| h_k^*(\psi'_j u) \right\|^2_{H'(U_k)} \right)^{1/2}
\]

\[
\leq C \left( \sum_{j=1}^s \left\| h_k^*(\psi'_j u) \right\|^2_{H'(U_k)} \right)^{1/2} = C_s \| h_k^* u \|_{H'(U_k)} = \| u \|_t.
\]

This proves the lemma.

### 37.6.2 The Trace On The Boundary

**Definition 37.6.7** A bounded open subset, \( \Omega \), of \( \mathbb{R}^n \) has a \( C^{m,1} \) boundary if it satisfies the following conditions. For each \( p \in \Gamma \equiv \Omega \setminus \Omega \), there exists an open set, \( W \), containing \( p \), an open interval \((0,b)\), a bounded open box \( U' \subseteq \mathbb{R}^{n-1} \), and an affine orthogonal transformation, \( R_W \), consisting of a distance preserving linear transformation followed by a translation such that

\[
R_W W = U' \times (0,b),
\]  

\[(37.6.23)\]
CHAPTER 37. SOBOLEV SPACES BASED ON $L^2$

\[ R_W(W \cap \Omega) = \{ u \in \mathbb{R}^n : u' \in U', 0 < u_n < \phi_W(u') \} \equiv U_W \quad (37.6.24) \]

where $\phi_W \in C^{m,1}(\mathbb{R}^n)$ meaning $\phi_W$ is the restriction to $U'$ of a function, still denoted by $\phi_W$ which is in $C^{m,1}(\mathbb{R}^n)$ and

\[ \inf \{ \phi_W(u') : u' \in U' \} > 0 \]

The following picture depicts the situation.

For the situation described in the above definition, let $h_W : U' \to \Gamma \cap W$ be defined by

\[ h_W(u') \equiv R_W^{-1}(u', \phi_W(u')) \, , \, g_W(x) \equiv (R_Wx)', \, H_W(u) \equiv R_W^{-1}(u', \phi_W(u') - u_n). \]

where $x' \equiv (x_1, \cdots, x_{n-1})$ for $x = (x_1, \cdots, x_n)$. Thus $g_W \circ h_W = \text{id}$ on $U'$ and $h_W \circ g_W = \text{id}$ on $\Gamma \cap W$. Also note that $H_W$ is defined on all of $\mathbb{R}^n$ is $C^{m,1}$, and has an inverse with the same properties. To see this, let $G_W(u) = (u', \phi_W(u') - u_n)$.

Then $H_W = R_W^{-1} \circ G_W$ and $G_W^{-1} = (u', \phi_W(u') - u_n)$ and so $H_W^{-1} = G_W^{-1} \circ R_W$.

Note also that as indicated in the picture,

\[ R_W(W \cap \Omega) = \{ u \in \mathbb{R}^n : u' \in U' \text{ and } 0 < u_n < \phi_W(u') \}. \]

Since $\Gamma = \partial \Omega$ is compact, there exist finitely many of these open sets, $W$, denoted by $\{W_i\}_{i=1}^q$ such that $\Gamma \subseteq \bigcup_{i=1}^q W_i$. Let the corresponding sets, $U'$ be denoted by $U'_i$ and let the functions, $\phi$ be denoted by $\phi_i$. Also let $h_i = h_{W_i}$ etc. Now let $\{\psi_i\}_{i=1}^q$ be a $C^\infty$ partition of unity subordinate to the $\{W_i\}_{i=1}^q$. If $u \in H^t(\Omega)$, then by Corollary \textsf{LMS} on Page \textsf{FLK} it follows that $u\psi_i \in H^t(W_i \cap \Omega)$. Now

\[ H_i : U_i \equiv \{ u \in \mathbb{R}^n : u' \in U'_i, 0 < u_n < \phi_i(u') \} \to W_i \cap \Omega \]

and $H_i$ and its inverse are defined on $\mathbb{R}^n$ and are in $C^{m,1}(\mathbb{R}^n)$. Therefore, by Lemma \textsf{LXVIII} on Page \textsf{LML}

\[ H_i^* \in \mathcal{L}(H^t(W_i \cap \Omega), H^t(U_i)). \]

Provide $t = m + s$ where $s > 0$.

Now it is possible to define the trace on $\Gamma \equiv \partial \Omega$. For $u \in H^t(\Omega)$,

\[ \gamma u \equiv \sum_{i=1}^q g_i^* (\gamma H_i^*(u\psi_i)). \quad (37.6.25) \]
I must show it satisfies what it should. Recall the definition of what it means for a function to be in $H^{t-1/2}(\Gamma)$ where $t = m + s$.

**Definition 37.6.8** Let $s \in (0, 1)$ and $m$ is a nonnegative integer. Also let $\mu$ denote the surface measure for $\Gamma$. A $\mu$ measurable function, $u$ is in $H^{m+s}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, h_i, g_i\}_{i=1}^{\infty}$ is described above, $h_i^*(u\psi_i) \in H^{m+s}(U_i)$ and

$$||u||_{H^{m+s}(\Gamma)} \equiv \left( \sum_{i=1}^{\infty} ||h_i^*(u\psi_i)||_{H^{m+s}(U_i)}^2 \right)^{1/2} < \infty.$$

Recall that all these norms which are obtained from various partitions of unity and functions, $h_i$ and $g_i$, are equivalent. Here there are only finitely many $W_i$ so the sum is a finite sum. The theorem is the following.

**Theorem 37.6.9** Let $\Omega$ be a bounded open set having $C^{m,1}$ boundary as discussed above in Definition 37.6.7. Then for $t \leq m + 1$, there exists a unique $\gamma \in \mathcal{L} \left( H^t(\Omega), H^{t-1/2}(\Gamma) \right)$ which has the property that for $\mu$ the measure on the boundary,

$$\gamma u(x) = u(x) \text{ for } \mu \text{ a.e. } x \in \Gamma \text{ whenever } u \in \mathcal{S}|\Omega. \quad (37.6.26)$$

**Proof:** First consider the claim that $\gamma \in \mathcal{L} \left( H^t(\Omega), H^{t-1/2}(\Gamma) \right)$. This involves first showing that for $u \in H^t(\Omega)$, $\gamma u \in H^{t-1/2}(\Gamma)$. To do this, use the above definition.

$$h_j^*(\psi_j(\gamma u)) = \sum_{i=1}^{q} h_j^* \left( \psi_j g_i^* \left( \gamma H_i^* \left( u\psi_i \right) \right) \right)$$

$$= \sum_{i=1}^{q} \left( h_j^* \psi_j \right) \left( h_j^* \left( g_i^* \left( \gamma H_i^* \left( u\psi_i \right) \right) \right) \right)$$

$$= \sum_{i=1}^{q} \left( h_j^* \psi_j \right) \left( g_i \circ h_j \right)^* \left( \gamma H_i^* \left( u\psi_i \right) \right) \quad (37.6.27)$$

First note that

$$\gamma H_i^*(u\psi_i) \in H^{t-1/2}(U_i')$$

Now $g_i \circ h_j$ and its inverse, $g_j \circ h_i$, are both functions in $C^{m,1} \left( \mathbb{R}^{n-1} \right)$ and

$$g_i \circ h_j : U_i' \rightarrow U_j'.$$

Therefore, by Lemma 37.3. on Page 1374,

$$(g_i \circ h_j)^* \left( \gamma H_i^* \left( u\psi_i \right) \right) \in H^{t-1/2}(U_j')$$
and
\[ \| (g_i \circ h_j)^* (\gamma H_i^\ast (u \psi_i)) \|_{H^{t-1/2} (U')} \leq C_{ij} \| \gamma H_i^\ast (u \psi_i) \|_{H^{t-1/2} (U')} \cdot \]

Also \( h_j^\ast \psi_j \in C^{m,1} (U'_j) \) and has compact support in \( U'_j \) and so by Corollary [37.3.5](#) on Page 1381 and \( 37.6.26 \)
\[ (h_j^\ast \psi_j) (g_i \circ h_j)^* (\gamma H_i^\ast (u \psi_i)) \in H^{t-1/2} (U'_j) \]
and
\[ \| (h_j^\ast \psi_j) (g_i \circ h_j)^* (\gamma H_i^\ast (u \psi_i)) \|_{H^{t-1/2} (U')} \leq C_{ij} \| (g_i \circ h_j)^* (\gamma H_i^\ast (u \psi_i)) \|_{H^{t-1/2} (U')} \leq C_{ij} \| \gamma H_i^\ast (u \psi_i) \|_{H^{t-1/2} (U')} \cdot \] (37.6.28)
(37.6.29)

This shows \( \gamma u \in H^{t-1/2} (\Gamma) \) because each \( h_j^\ast (\psi_j (\gamma u)) \in H^{t-1/2} (U'_j) \). Also from \( 37.3.5 \) and \( 37.6.26 \)
\[ \| \gamma u \|_{H^{t-1/2} (\Gamma)}^2 \leq \sum_{j=1}^q \| h_j^\ast (\psi_j (\gamma u)) \|_{H^{t-1/2} (U')}^2 \]

\[ = \sum_{j=1}^q \| h_j^\ast (\psi_j (\gamma u)) \|_{H^{t-1/2} (U')}^2 \]
\[ = \sum_{j=1}^q \left\| \sum_{i=1}^q (h_j^\ast \psi_j) (g_i \circ h_j)^* (\gamma H_i^\ast (u \psi_i)) \right\|_{H^{t-1/2} (U')}^2 \]
\[ \leq C_q \sum_{j=1}^q \sum_{i=1}^q \left\| (h_j^\ast \psi_j) (g_i \circ h_j)^* (\gamma H_i^\ast (u \psi_i)) \right\|_{H^{t-1/2} (U')}^2 \]
\[ \leq C_q \sum_{j=1}^q \sum_{i=1}^q C_{ij} \left\| (\gamma H_i^\ast (u \psi_i)) \right\|_{H^{t-1/2} (U')}^2 \]
\[ \leq C_q \sum_{i=1}^q \left\| (\gamma H_i^\ast (u \psi_i)) \right\|_{H^{t-1/2} (U')}^2 \]
\[ \leq C_q \sum_{i=1}^q \left\| (\gamma H_i^\ast (u \psi_i)) \right\|_{H^{t-1/2} (U')}^2 \]
\[ \leq C_q \sum_{i=1}^q \| H_i^\ast (u \psi_i) \|_{H^{t-1/2} (\Omega)^{\infty}}^2 \]
\[ \leq C_q \sum_{i=1}^q \| u \psi_i \|_{H^{t/(\omega, \infty)}}^2 \leq C_q \| u \|_{H^{t/(\omega, \infty)}}^2 \cdot \]

Does \( \gamma \) satisfy \( 37.3.5 \)? Let \( x \in \Gamma \) and \( u \in \mathcal{G}|_{\Omega} \). Let
\[ I_x = \{ i \in \{ 1, 2, \ldots, q \} : x = h_i (u'_i) \text{ for some } u'_i \in U'_i \} . \]
Then
\[ \gamma u(x) = \sum_{i \in I_a} (\gamma H_i^*(u \psi_i))(g_i(x)) \]
\[ = \sum_{i \in I_a} (\gamma H_i^*(u \psi_i))(g_i(h_i(u'_i))) \]
\[ = \sum_{i \in I_a} (\gamma H_i^*(u \psi_i))(u'_i). \]

Now because \( H_i \) is Lipschitz continuous and \( u \psi \in \mathcal{S} \), it follows that \( H_i^*(u \psi_i) \in H^1(\mathbb{R}^n) \) and is continuous and so by Theorem 37.5.7 on Page 1389 for a.e. \( u'_i \),
\[ = \sum_{i \in I_a} H_i^*(u \psi_i)(u'_i, 0) \]
\[ = \sum_{i \in I_a} h_i^*(u \psi_i)(u'_i) \]
\[ = \sum_{i \in I_a} (u \psi_i)(h_i(u'_i)) = u(x) \text{ for } \mu \text{ a.e. } x. \] (37.6.30)

This verifies 37.6.26 and completes the proof of the theorem.
CHAPTER 37. SOBOLEV SPACES BASED ON $L^2$
Chapter 38

Weak Solutions

38.1 The Lax Milgram Theorem

The Lax Milgram theorem is a fundamental result which is useful for obtaining weak solutions to many types of partial differential equations. It is really a general theorem in functional analysis.

Definition 38.1.1 Let $A \in \mathcal{L}(V, V')$ where $V$ is a Hilbert space. Then $A$ is said to be coercive if

$$A(v)(v) \geq \delta \|v\|^2$$

for some $\delta > 0$.

Theorem 38.1.2 (Lax Milgram) Let $A \in \mathcal{L}(V, V')$ be coercive. Then $A$ maps one to one and onto.

Proof: The proof that $A$ is onto involves showing $A(V)$ is both dense and closed.

Consider first the claim that $A(V)$ is closed. Let $Ax_n \to y^* \in V'$. Then

$$\delta \|x_n - x_m\|^2 \leq \|Ax_n - Ax_m\|_{V'} \|x_n - x_m\|_V.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $V$. It follows $x_n \to x \in V$ and since $A$ is continuous, $Ax_n \to Ax$. This shows $A(V)$ is closed.

Now let $R : V \to V'$ denote the Riesz map defined by $Rx(y) = (y, x)$. Recall that the Riesz map is one to one, onto, and preserves norms. Therefore, $R^{-1}(A(V))$ is a closed subspace of $V$. If there $R^{-1}(A(V)) \neq V$, then $(R^{-1}(A(V)))^\perp \neq \{0\}$. Let $x \in (R^{-1}(A(V)))^\perp$ and $x \neq 0$. Then in particular,

$$0 = (x, R^{-1}Ax) = R\left(R^{-1}(A(x))\right)(x) = A(x)(x) \geq \delta \|x\|^2,$$

a contradiction to $x \neq 0$. Therefore, $R^{-1}(A(V)) = V$ and so $A(V) = R(V) = V'$. 

1401
Since \( A(V) \) is both closed and dense, \( A(V) = V' \). This shows \( A \) is onto.

If \( Ax = Ay \), then
\[
0 = A(x - y)(x - y) \geq \delta \|x - y\|^2_V, \n\]
and this shows \( A \) is one to one. This proves the theorem.

Here is a simple example which illustrates the use of the above theorem. In the example the repeated index summation convention is being used. That is, you sum over the repeated indices.

**Example 38.1.3** Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( V \) be a closed subspace of \( H^1(U) \). Let \( \alpha^{ij} \in L^\infty(U) \) for \( i, j = 1, 2, \cdots, n \).

Now define \( A : V \rightarrow V' \) by
\[
A(v)(u) \equiv \int_U (\alpha^{ij}(x) \, u_{,i}(x) \, v_{,j}(x) + u(x) \, v(x)) \, dx.
\]
Suppose also that
\[
\alpha^{ij} v_{i} v_{j} \geq \delta |v|^2
\]
whenever \( v \in \mathbb{R}^n \). Then \( A \) maps \( V \) to \( V' \) one to one and onto.

Here is why. It is obvious that \( A \) is in \( \mathcal{L}(V, V') \). It only remains to verify that it is coercive.

\[
A(u)(u) \equiv \int_U (\alpha^{ij}(x) \, u_{,i}(x) \, u_{,j}(x) + u(x) \, u(x)) \, dx
\]
\[
\geq \int_U \delta |\nabla u(x)|^2 + |u(x)|^2 \, dx
\]
\[
\geq \delta \|u\|^2_{H^1(U)}
\]

This proves coercivity and verifies the claim.

What has been obtained in the above example? This depends on how you choose \( V \). In Example 38.1.3 suppose \( U \) is a bounded open set with \( C^{0,1} \) boundary and \( V = H^1_0(U) \) where
\[
H^1_0(U) \equiv \{ u \in H^1(U) : \gamma u = 0 \}
\]
Also suppose \( f \in L^2(U) \). Then you can consider \( F \in V' \) by defining
\[
F(v) \equiv \int_U f(x) \, v(x) \, dx.
\]

According to the Lax Milgram theorem and the verification of its conditions in Example 38.1.3, there exists a unique solution to the problem of finding \( u \in H^1_0(U) \) such that for all \( v \in H^1_0(U) \),
\[
\int_U (\alpha^{ij}(x) \, u_{,i}(x) \, v_{,j}(x) + u(x) \, v(x)) \, dx = \int_U f(x) \, v(x) \, dx \quad (38.1.1)
\]
In particular, this holds for all \( v \in C^\infty_c(U) \). Thus for all such \( v \),
\[
\int_U \left( - (\alpha^{ij}(x) \, u_{,i}(x))_{,j} + u(x) - f(x) \right) \, v(x) \, dx = 0.
\]
Therefore, in terms of weak derivatives,

\[-(\alpha^{ij} u)_{,i,j} + u = f\]

and since \(u \in H_0^1(U)\), it must be the case that \(\gamma u = 0\) on \(\partial U\). This is why the solution to (38.1.1) is referred to as a weak solution to the boundary value problem

\[-(\alpha^{ij}(x) u_{,i}(x))_{,j} + u(x) = f(x), u = 0 \text{ on } \partial U.\]

Of course you then begin to ask the important question whether \(u\) really has two derivatives. It is not immediately clear that just because \[-(\alpha^{ij}(x) u_{,i}(x))_{,j} \in L^2(U)\] it follows that the second derivatives of \(u\) exist. Actually this will often be true and is discussed somewhat in the next section.

Next suppose you choose \(V = H^1(U)\) and let \(g \in H^{1/2}(\partial U)\). Define \(F \in V'\) by

\[F(v) = \int_U f(x)v(x)dx + \int_{\partial U} g(x)\gamma v(x)d\mu.\]

Everything works the same way and you get the existence of a unique \(u \in H^1(U)\) such that for all \(v \in H^1(U)\),

\[\int_U (\alpha^{ij}(x) u_{,i}(x)v_{,j}(x) + u(x)v(x))dx = \int_U f(x)v(x)dx + \int_{\partial U} g(x)\gamma v(x)d\mu,\]

is satisfied. If you pretend \(u\) has all second order derivatives in \(L^2(U)\) and apply the divergence theorem, you find that you have obtained a weak solution to

\[-(\alpha^{ij} u_{,i})_{,j} + u = f, \alpha^{ij} u_{,i} n_j = g \text{ on } \partial U\]

where \(n_j\) is the \(j^{th}\) component of \(n\), the unit outer normal. Therefore, \(u\) is a weak solution to the above boundary value problem.

The conclusion is that the Lax Milgram theorem gives a way to obtain existence and uniqueness of weak solutions to various boundary value problems. The following theorem is often very useful in establishing coercivity. To prove this theorem, here is a definition.

**Definition 38.1.4** Let \(U\) be an open set and \(\delta > 0\). Then

\[U_\delta = \{x \in U : \text{dist}(x, U^c) > \delta\}.\]

**Theorem 38.1.5** Let \(U\) be a connected bounded open set having \(C^{0,1}\) boundary such that for some sequence, \(\eta_k \downarrow 0\),

\[U = \bigcup_{k=1}^\infty U_{\eta_k}\]

and \(U_{\eta_k}\) is a connected open set. Suppose \(\Gamma \subseteq \partial U\) has positive surface measure and that

\[V = \{u \in H^1(U) : \gamma u = 0 \text{ a.e. on } \Gamma\}.\]
Then the norm $|||\cdot|||$ given by
\[
|||u||| = \left( \int_U |\nabla u|^2 \, dx \right)^{1/2}
\]
is equivalent to the usual norm on $V$.

**Proof:** First it is necessary to verify this is actually a norm. It clearly satisfies all the usual axioms of a norm except for the condition that $|||u||| = 0$ if and only if $u = 0$. Suppose then that $|||u||| = 0$. Let $\delta_0 = \eta_k$ for one of those $\eta_k$ mentioned above and define
\[
u_\delta (x) = \int_{B(0,\delta)} u(x - y) \phi_\delta (y) \, dy
\]
where $\phi_\delta$ is a mollifier having support in $B(0,\delta)$. Then changing the variables, it follows that for $x \in U_{\delta_0}$
\[
u_\delta (x) = \int_{B(x,\delta)} u(t) \phi_\delta (x - t) \, dt = \int_{U} u(t) \phi_\delta (x - t) \, dt
\]
and so $\nu_\delta \in C^\infty (U_{\delta_0})$ and
\[
\nabla \nu_\delta (x) = \int_{U} u(t) \nabla \phi_\delta (x - t) \, dt = \int_{B(0,\delta)} \nabla u(x - y) \phi_\delta (y) \, dy = 0.
\]
Therefore, $\nu_\delta$ equals a constant on $U_{\delta_0}$ because $U_{\delta_0}$ is a connected open set and $\nu_\delta$ is a smooth function defined on this set which has its gradient equal to 0. By Minkowski’s inequality,
\[
\left( \int_{U_{\delta_0}} |u(x) - \nu_\delta (x)|^2 \, dx \right)^{1/2} \leq \int_{B(0,\delta)} \phi_\delta (y) \left( \int_{U_{\delta_0}} |u(x) - u(x - y)|^2 \, dx \right)^{1/2} \, dy
\]
and this converges to 0 as $\delta \to 0$ by continuity of translation in $L^2$. It follows there exists a sequence of constants, $c_\delta \equiv u_\delta (x)$ such that $\{c_\delta\}$ converges to $u$ in $L^2(U_{\delta_0})$. Consequently, a subsequence, still denoted by $u_\delta$, converges to $u$ a.e. By Eggoroff’s theorem there exists a set, $N_k$ having measure no more than $3^{-k}m_n (U_{\delta_0})$ such that $\nu_\delta$ converges to $u$ uniformly on $N_k \setminus K$. Now $\sum_k m_n (N_k) \leq \frac{1}{2} m_n (U_{\delta_0})$ and so there exists $x_0 \in U_{\delta_0} \setminus \cup_{k=1}^\infty K$. Therefore, if $x \notin N_k$ it follows $u(x) = u(x_0)$ and so, if $u(x) \neq u(x_0)$ it must be the case that $x \in \cap_{k=1}^\infty K$, a set of measure zero. This shows that $u$ equals a constant a.e. on $U_{\delta_0} = U_n$. Since $k$ is arbitrary, $u$ shows $u$ is a.e. equal to a constant on $U$. Therefore, $u$ equals the restriction of a function of $\mathcal{S}$ to $U$ and so $\gamma u$ equals this constant in $L^2(\partial \Omega)$. Since the surface measure of $\Gamma$ is positive, the constant must equal zero. Therefore, $|||\cdot|||$ is a norm.
It remains to verify that it is equivalent to the usual norm. It is clear that \( \| u \| \leq \| u \|_{1,2} \). What about the other direction? Suppose it is not true that for some constant, \( K \), \( \| u \|_{1,2} \leq K \| u \| \). Then for every \( k \in \mathbb{N} \), there exists \( u_k \in V \) such that
\[
\| u_k \|_{1,2} > k \| u_k \|.
\]
Replacing \( u_k \) with \( u_k / \| u_k \|_{1,2} \), it can be assumed that \( \| u_k \|_{1,2} = 1 \) for all \( k \).

Therefore, using the compactness of the embedding of \( H^1(U) \) into \( L^2(U) \), there exists a subsequence, still denoted by \( u_k \) such that
\[
\begin{align*}
u_k &\to u \text{ weakly in } V, \\
u_k &\to u \text{ strongly in } L^2(U), \\
\| u_k \| &\to 0, \\
u_k &\to u \text{ weakly in } (V, \| \cdot \|).
\end{align*}
\]
From 38.1.5 and 38.1.6, it follows \( u = 0 \). Therefore, \( \| u_k \|_{L^2(U)} \to 0 \). This with 38.1.5 contradicts the fact that \( \| u_k \|_{1,2} = 1 \) and this proves the equivalence of the two norms.

The proof of the above theorem yields the following interesting corollary.

**Corollary 38.1.6** Let \( U \) be a connected open set with the property that for some sequence, \( \eta_k \downarrow 0 \),
\[
U = \bigcup_{k=1}^{\infty} U_{\eta_k}
\]
for \( U_{\eta_k} \) a connected open set and suppose \( u \in W^{1, p}(U) \) and \( \nabla u = 0 \) a.e. Then \( u \) equals a constant a.e.

**Example 38.1.7** Let \( U \) be a bounded open connected subset of \( \mathbb{R}^n \) and let \( V \) be a closed subspace of \( H^1(U) \) defined by
\[
V \equiv \{ u \in H^1(U) : \gamma u = 0 \text{ on } \Gamma \}
\]
where the surface measure of \( \Gamma \) is positive.

Let \( \alpha^{ij} \in L^\infty(U) \) for \( i, j = 1, 2, \cdots, n \) and define \( A : V \to V' \) by
\[
A(u)(v) \equiv \int_U \alpha^{ij}(x) u_i(x) v_j(x) dx.
\]
for
\[
\alpha^{ij} v_i v_j \geq \delta |v|^2
\]
whenever \( v \in \mathbb{R}^n \). Then \( A \) maps \( V \) to \( V' \) one to one and onto.

This follows from Theorem 38.1.5 using the equivalent norm defined there. Define \( F \in V' \) by
\[
\int_U f(x) v(x) dx + \int_{\partial U \setminus \Gamma} g(x) \gamma v(x) dx
\]
for \( f \in L^2(U) \) and \( g \in H^{1/2}(\partial U) \). Then the equation,
\[
Au = F \quad \text{in } V'
\]
which is equivalent to \( u \in V \) and for all \( v \in V \),
\[
\int_U \alpha_{ij}^i (x) u_{,i} (x) v_{,j} (x) \, dx = \int_U f(x) v(x) \, dx + \int_{\partial U \setminus \Gamma} g(x) \gamma v(x) \, d\mu
\]
is a weak solution for the boundary value problem,
\[
- (\alpha_{ij}^i u_{,i})_{,j} = f \text{ in } U, \quad \alpha_{ij}^i u_{,i} n_j = g \text{ on } \partial U \setminus \Gamma, \quad u = 0 \text{ on } \Gamma
\]
as you can verify by using the divergence theorem formally.

### 38.2 An Application Of The Mountain Pass Theorem

Recall the mountain pass theorem 22.1.3.

**Theorem 38.2.1** Let \( H \) be a Hilbert space and let \( I : H \to \mathbb{R} \) be a \( C^1 \) functional having \( I' \) Lipschitz continuous and such that \( I \) satisfies the Palais Smale condition. Suppose \( I(0) = 0 \) and \( I(u) \geq a > 0 \) for all \( \|u\| = r \). Suppose also that there exists \( v, \|v\| > r \) such that \( I(v) \leq 0 \). Then define
\[
\Gamma \equiv \{ g \in C([0, 1]; H) : g(0) = 0, g(1) = v \}
\]
Let
\[
c \equiv \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} I(g(t))
\]
Then \( c \) is a critical value of \( I \) meaning that there exists \( u \) such that \( I(u) = c \) and \( I'(u) = 0 \). In particular, there is \( u \neq 0 \) such that \( I'(u) = 0 \).

This nice example is in Evans [72]. Let the Hilbert space be \( H^1_0(U) \) where \( U \) is a bounded open set. To avoid cases, assume \( U \) is in \( \mathbb{R}^3 \) or higher. The main results will work in general but it would involve cases. Consider the functional
\[
\frac{1}{2} \|u\|_{H^1_0}^2 - \int_U F(u) \, dx \equiv I_1(u) - I_2(u)
\]
where \( F'(u) = f(u), f(0) = 0 \). Here it is assumed that
\[
|f(u)| \leq C(1 + |u|^p), \quad |f'(u)| \leq C \left(1 + |u|^{p-1}\right), \quad 1 < p < \frac{n+2}{n-2} \quad (38.2.8)
\]
Also suppose that
\[
0 \leq F(u) \leq \gamma f(u) u \quad \text{where } 0 < \gamma < 1/2 \quad (38.2.9)
\]
and finally that
\[
\alpha |u|^{p+1} \leq F(u) \leq A |u|^{p+1}, \quad \alpha, A > 0 \quad (38.2.10)
\]
Let \( R : H^1_0(U) \to H^{-1}(U) \) be the Riesz map.
38.2. AN APPLICATION OF THE MOUNTAIN PASS THEOREM

Showing Functional is $C^{1,1}$

Then it is not hard to verify that $(I_1(u), v) = (u, v)$ and so it is clearly the case that $I'(u)$ exists and is a continuous function of $u$. In addition to this, it is Lipschitz.

Next consider $I_2$.

$$I_2(u + v) - I_2(u) = \int_U F(u + v) - F(u) \, dx$$

$$= \int_U f(u) v + \frac{1}{2} f'(\hat{u}) v^2 \, dx, \quad \hat{u} \in [u, u + v]$$

Now $H_0^1(U)$ embeds continuously into $L^{2n/(n-2)}(U)$. Because of the estimate for $f(u)$, we can regard $f(u)$ as being in $H^{-1}(U)$ as follows.

$$\left| \int_U f(u) v \, dx \right| \leq \left( \int_U |f(u)|^{2n/(n+2)} \, dx \right)^{(n+2)/2n} \left( \int_U |v|^{2n/(n-2)} \, dx \right)^{(n-2)/2n}$$

$$\leq \left( \int_U C \left( 1 + |u|^{2n/(n-2)} \right) dx \right)^{(n+2)/2n} \|v\|_{H_0^1}$$

where $C$ will be adjusted as needed here and elsewhere. Thus, writing in terms of the inner product on $H_0^1$,

$$(I_2'(u), v)_{H_0^1} = (R^{-1} f(u), v)_{H_0^1}$$

This is so if the $\frac{1}{2} f'(\hat{u}) v^2$ term is as it should be. We need to verify that

$$\frac{\int_U \left| \frac{1}{2} f' \left( \hat{u} \right) v^2 \right| \, dx}{\|v\|_{H_0^1}} \to 0$$

However, we can use the estimate and write that this is no larger than

$$\frac{\int_U C \left( 1 + |v|^{p-1} + |u|^{p-1} \right) |v|^2 \, dx}{\left( \int_U |v|^{2n/(n-2)} \, dx \right)^{(n-2)/2n}}$$

(38.2.11)

Then consider the term involving $|u|$.

$$\int_U |u|^{p-1} |v|^2 \, dx \leq \left( \int_U |u|^{p+1} \right)^{\frac{p-1}{p+1}} \left( \int_U |v|^{p+1} \right)^{2/(p+1)}$$

Now $p + 1 \leq \frac{2n}{n-2}$ and so the first factor is finite. As to the second, it equals

$$\left( \left( \int_U |v|^{2n/(n-2)} \right)^{\frac{1}{n}(n-2)} \right)^2$$
and so this term from \(38.2.11\) is \(o(v)\) on \(H^1_0(U)\). The term involving \(|v|^{p-1}\) is obviously \(o(v)\). Consider the constant term.

\[
\int_U |v|^2 \, dx \leq \left( \int_U |v|^{2n/(n-2)} \, dx \right)^{(n-2)/2n} \]

so it is also all right. Thus the derivative is as claimed. Is this derivative Lipschitz on bounded sets?

\[
f(\hat{u}) - f(u) = \int_0^1 f'(u + t(\hat{u} - u))(\hat{u} - u) \, dt
\]

Thus in \(H^{-1}\) and using the estimates,

\[
\left| \int_U (f(\hat{u}) - f(u)) \, v \, dx \right| \leq \int_U \int_0^1 C \left( 1 + |u + t(\hat{u} - u)|^{p-1} \right) |\hat{u} - u| \, dt \, dx
\]

\[
= \int_0^1 \int_U C \left( 1 + |u + t(\hat{u} - u)|^{p-1} \right) |\hat{u} - u| \, dx \, dt
\]

\[
\leq C \left( \int_U \left( 1 + |\hat{u}|^{p-1} + |u|^{p-1} \right)^{2n/(n+2)} \right)^{\frac{n+2}{2n}} \left( \int_U |\hat{u} - u|^{2n/(n-2)} \right)^{(n-2)/2n}
\]

\[
\leq C \left( \int_U \left( 1 + |\hat{u}|^{p-1} + |u|^{p-1} \right)^{2n/(n+2)} \right)^{\frac{n+2}{2n}} \|\hat{u} - u\|_{H^1_0(U)}
\]

Now \((p-1) \frac{2n}{n+2} \leq \left( \frac{n+2}{n+2} - 1 \right) \frac{2n}{n+2} = 8 \frac{n}{n^2-4} \leq 2 \frac{n}{n-2} \) and so the derivative is Lipschitz on bounded sets of \(H^1_0(U)\).

**Palais Smale Conditions**

Here we verify the Palais Smale conditions. Suppose then that \(I(u_k)\) is bounded and \(I'(u_k) \to 0\) in \(H^1_0(U)\). Then

\[
\left| \frac{1}{2} \|u_k\|_{H^1_0}^2 - \int_U F'(u_k) \, dx \right| \leq C
\]

(38.2.12)

Since \(I'(u_k) \to 0\),

\[
u_k - R^{-1} f(u_k) \to 0 \text{ in } H^1_0(U)
\]

(38.2.13)

Take inner product of the second term with \(u_k\).

\[
(R^{-1} f(u_k) , u_k) \equiv \langle f(u_k) , u_k \rangle_{H^{-1}, H^1_0} = \int_U f(u_k) u_k \, dx
\]

Then by assumption, for \(\varepsilon > 0\), and all \(k\) large enough,

\[
|(I'(u_k) , u_k)| \leq \left| \|u_k\|^2_{H^1_0(U)} - \int_U f(u_k) u_k \, dx \right| \leq \varepsilon \|u_k\|_{H^1_0(U)}
\]
38.2. AN APPLICATION OF THE MOUNTAIN PASS THEOREM

Then also for large $k$, letting $\varepsilon = 1$,

$$\left| \int_U f(u_k) u_k dx \right| \leq \|u_k\|_{H_0^1(U)}^2 + \|u_k\|_{H_0^1(U)}$$

Now from the estimates assumed and (38.2.12),

$$\frac{1}{2} \|u_k\|_{H_0^1}^2 \leq C + \int_U F(u_k) dx \leq C + \gamma \int_U f(u_k) u_k dx \leq C + \gamma \left( \|u_k\|_{H_0^1(U)}^2 + \|u_k\|_{H_0^1(U)} \right)$$

and since $\gamma < 1/2$,

$$\left( \frac{1}{2} - \gamma \right) \|u_k\|_{H_0^1}^2 \leq C + \|u_k\|_{H_0^1(U)}$$

and so $\|u_k\|_{H_0^1(U)}$ is bounded. Hence it has a subsequence still denoted as $u_k$ which converges weakly in $H_0^1(U)$ to $u \in H_0^1(U)$. Since $p < \frac{n+2}{n-2}$, it follows that

$$p + 1 < \frac{n + 2}{n - 2} + 1 = \frac{2n}{n - 2}$$

and so by compactness of the embedding, it follows that $u_k \to u$ strongly in $L^{p+1}(U)$. We can assume convergence also takes place pointwise by taking a suitable subsequence.

Now $|f(u)v| \leq C (1 + |u|^p) |v|$. Therefore, adjusting the constants,

$$|f(u)v| \leq C (1 + |u|^p) |v| \leq C \left( 1 + |u|^{p+1} \right)^{(p+1)/p} |v|$$

$$\left| \int_U f(u) v dx \right| \leq C \left( \int_U (1 + |u|^{p+1}) \right)^{p/(p+1)} \left( \int_U |v|^{p+1} \right)^{1/(p+1)} \leq C \left( \int_U (1 + |u|^{p+1}) \right)^{p/(p+1)} \|v\|_{H_0^1(U)}$$

and so

$$\|f(u)\|_{H^{-1}(U)} \leq \|f(u)\|_{L^{p+1}(U)} \leq C \left( \int_U (1 + |u|^{p+1}) \right)^{p/(p+1)}$$

It follows that

$$f(u_k) \to f(u) \text{ pointwise}$$

and also

$$|f(u_k) - f(u)|^{p+1} \leq C_p \left( |f(u_k)|^{p+1} + |f(u)|^{p+1} \right)$$

where

$$\lim_{k \to \infty} \int_U \left( |f(u_k)|^{p+1} + |f(u)|^{p+1} \right) dx = \int_U 2 |f(u)|^{p+1} dx$$
then by the dominated convergence theorem or more precisely Corollary 9.4.10,

\[
\lim_{k \to \infty} \left( \int_{U} |f(u_k) - f(u)|^{p+1} \right)^{1/(p+1)} = 0
\]

It follows that \( f(u_k) \to f(u) \) in \( H^{-1}(U) \). Hence \( R^{-1}f(u_k) \to R^{-1}f(u) \) in \( H^{1}_0(U) \) and so from 38.2.13, \( u_k \to u \) strongly in \( H^{1}_0(U) \). Thus \( \{u_k\} \) is precompact. This verifies the Palais Smale conditions.

mountain pass conditions

It is clear that \( I(0) = 0 \). It remains to verify that for some \( r > 0 \), \( I(u) \geq a > 0 \) whenever \( \|u\|_{H^{1}_0(U)} = r \) and for some \( v \) with \( \|v\| > r \), \( I(v) = 0 \). Now consider \( ru \) where \( \|u\| = 1 \).

\[
I(ru) = \frac{r^2}{2} - \int_{U} F(ru) \, dx
\]

From the assumed estimates and Sobolev embedding,

\[
I(ru) \geq \frac{1}{2} r^2 - \int_{U} A|u|^{p+1} \alpha^{p+1} dx \geq \frac{1}{2} r^2 - CAR^{p+1} \|u\|^{p+1}_{H^{1}_0(U)}
\]

Now this is independent of \( u \) such that \( \|u\| = 1 \). Then the derivative of the right side is

\[
r - \frac{p + 1}{2} CAR^p
\]

where \( p > 1 \). Thus this is positive for a while and then when \( r \) is larger, it becomes negative. Thus there is \( r_0 > 0 \) where \( \frac{r^2}{2} - CAR^{p+1} \equiv a > 0 \). Hence when \( \|u\| = r_0 \), you have \( I(u) \geq a > 0 \). This is part of the mountain pass conditions. Now consider the other part. Letting \( \|u\| = 1 \) be fixed, the estimates imply

\[
I(ru) \leq \frac{1}{2} r^2 - \int_{U} A|u|^{p+1} \alpha^{p+1} dx \leq \frac{1}{2} r^2 - r^{p+1} C
\]

Hence, for \( r \) large enough, the right side becomes negative because \( p + 1 > 2 \). Therefore, \( r \to I(ru) \) is positive for small \( r \) and is eventually negative as \( r \) gets larger. hence there is some value of \( r \) where this equals 0. Then \( v = ru \). This verifies the conditions for the mountain pass theorem.

conclusions

It follows from the mountain pass theorem that there is some \( u \neq 0 \) such that \( I'(u) = 0 \). From the above computations,

\[
u - R^{-1}f(u) = 0
\]
Now $R = -\Delta$ the Laplacian. In terms of weak derivatives,
\[
\langle -\Delta u, v \rangle_{H^{-1}, H^1_0} = \int_U \nabla u \cdot \nabla v \, dx = (u, v)_{H^1_0(U)}
\]
and so in terms of weak derivatives,
\[
-\Delta u = f(u) \text{ in } H^{-1}(U), \quad u \in H^1_0(U) \text{ so } u = 0 \text{ on } \partial U.
\]

This proves the following theorem.

**Theorem 38.2.2** Suppose the conditions 38.2.8 - 38.2.10 hold. Then there exists a nonzero $u \in H^1_0(U)$ such that
\[
-\Delta u = f(u)
\]

One can verify that an example of such a function $f(u)$ is
\[
f(u) = |u|^{p-2} u
\]

This is very exciting to a large number of people because it gives an interesting example of non uniqueness of a boundary value problem. It is clear that $u = 0$ works.
CHAPTER 38. WEAK SOLUTIONS
Chapter 39

Korn’s Inequality

A fundamental inequality used in elasticity to obtain coercivity and then apply the Lax Milgram theorem or some other theorem is Korn’s inequality. The proof given here of this fundamental result follows \cite{92} and \cite{43}.

39.1 A Fundamental Inequality

The proof of Korn’s inequality depends on a fundamental inequality involving negative Sobolev space norms. The theorem to be proved is the following.

**Theorem 39.1.1** Let \( f \in L^2(\Omega) \) where \( \Omega \) is a bounded Lipschitz domain. Then there exist constants, \( C_1 \) and \( C_2 \) such that

\[
C_1 \| f \|_{0,2,\Omega} \leq \left( \| f \|_{-1,2,\Omega} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right) \leq C_2 \| f \|_{0,2,\Omega},
\]

where here \( \| \cdot \|_{0,2,\Omega} \) represents the \( L^2 \) norm and \( \| \cdot \|_{-1,2,\Omega} \) represents the norm in the dual space of \( H^1_0(\Omega) \), denoted by \( H^{-1}(\Omega) \).

Similar conventions will apply for any domain in place of \( \Omega \). The proof of this theorem will proceed through the use of several lemmas.

**Lemma 39.1.2** Let \( U^- \) denote the set,

\[
\{ (x,x_n) \in \mathbb{R}^n : x_n < g(x) \}
\]

where \( g : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) is Lipschitz and denote by \( U^+ \) the set

\[
\{ (x,x_n) \in \mathbb{R}^n : x_n > g(x) \}.
\]

Let \( f \in L^2(U^-) \) and extend \( f \) to all of \( \mathbb{R}^n \) in the following way.

\[
f(x,x_n) \equiv -3f(x,2g(x) - x_n) + 4f(x,3g(x) - 2x_n).
\]
Then there is a constant, \( C_g \), depending on \( g \) such that

\[
\|f\|_{-1,2,\mathbb{R}^n} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} \leq C_g \left( \|f\|_{-1,2,U^-} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,U^-} \right).
\]

**Proof:** Let \( \phi \in C_\infty_c (\mathbb{R}^n) \). Then,

\[
\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_n} \, dx = \int_{U^+} \frac{\partial \phi}{\partial x_n} \left( -3f (x,2g(x) - x_n) + 4f (x,3g(x) - 2x_n) \right) \, dx
\]

\[
+ \int_{U^-} f \frac{\partial \phi}{\partial x_n} \, dx. \tag{39.1.1}
\]

Consider the first integral on the right in (39.1.1) Changing the variables, letting \( y_n = 2g(x) - x_n \) in the first term of the integrand and \( 3g(x) - 2x_n \) in the next, it equals

\[
-3 \int_{U^-} \frac{\partial \phi}{\partial x_n} (x,2g(x) - y_n) f (x,y_n) \, dy_n \, dx
\]

\[
+ 2 \int_{U^-} \frac{\partial \phi}{\partial x_n} \left( x, \frac{3}{2}g(x) - \frac{y_n}{2} \right) f (x,y_n) \, dy_n \, dx.
\]

For \( (x,y_n) \in U^- \), and defining

\[
\psi (x,y_n) \equiv \phi (x,y_n) + 3\phi (x,2g(x) - y_n) - 4\phi \left( x, \frac{3}{2}g(x) - \frac{y_n}{2} \right),
\]

it follows \( \psi = 0 \) when \( y_n = g(x) \) and so

\[
\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_n} \, dx = \int_{U^-} \frac{\partial \psi}{\partial y_n} f (x,y_n) \, dx \, dy_n.
\]

Now from the definition of \( \psi \) given above,

\[
\|\psi\|_{1,2,U^-} \leq C_g \|\phi\|_{1,2,U^-} \leq C_g \|\phi\|_{1,2,\mathbb{R}^n}
\]

and so

\[
\left\| \frac{\partial f}{\partial x_n} \right\|_{-1,2,\mathbb{R}^n} \leq \sup \left\{ \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_n} \, dx : \phi \in C_\infty_c (\mathbb{R}^n), \|\phi\|_{1,2,\mathbb{R}^n} \leq 1 \right\} \leq \left\| \frac{\partial \psi}{\partial x_n} \right\|_{-1,2,U^-} \leq C_g \left\| \frac{\partial \phi}{\partial x_n} \right\|_{-1,2,\mathbb{R}^n}.
\]

(39.1.2)
39.1. A FUNDAMENTAL INEQUALITY

It remains to establish a similar inequality for the case where the derivatives are taken with respect to \( x_i \) for \( i < n \). Let \( \phi \in C^\infty_c (\mathbb{R}^n) \). Then

\[
\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} \, dx = \int_{U_-} f \frac{\partial \phi}{\partial x_i} \, dx
\]

\[
\int_{U_+} \frac{\partial \phi}{\partial x_i} \left[ -3f (x, g(x) - x_n) + 4f (x, 3g(x) - 2x_n) \right] \, dx.
\]

Changing the variables as before, this last integral equals

\[
-3 \int_{U_-} D_i \phi (x, 2g(x) - y_n) f (x, y_n) \, dy_n \, dx
\]

\[
+ 2 \int_{U_-} D_i \phi \left( x, \frac{3}{2} g(x) - \frac{y_n}{2} \right) f (x, y_n) \, dy_n \, dx.
\]  \hspace{1cm} (39.1.3)

Now let

\[ \psi_1 (x, y_n) = \phi (x, 2g(x) - y_n), \psi_2 (x, y_n) = \phi \left( x, \frac{3}{2} g(x) - \frac{y_n}{2} \right). \]

Then

\[ \frac{\partial \psi_1}{\partial x_i} = D_i \phi (x, 2g(x) - y_n) + D_n \phi (x, 2g(x) - y_n) 2D_i g(x), \]

\[ \frac{\partial \psi_2}{\partial x_i} = D_i \phi \left( x, \frac{3}{2} g(x) - \frac{y_n}{2} \right) + D_n \phi \left( x, \frac{3}{2} g(x) - \frac{y_n}{2} \right) \frac{3}{2} D_i g(x). \]

Also

\[ \frac{\partial \psi_1}{\partial y_n} (x, y_n) = -D_n \phi (x, 2g(x) - y_n), \]

\[ \frac{\partial \psi_2}{\partial y_n} (x, y_n) = \left( \frac{-1}{2} \right) D_n \phi \left( x, \frac{3}{2} g(x) - \frac{y_n}{2} \right). \]

Therefore,

\[ \frac{\partial \psi_1}{\partial x_i} (x, y_n) = D_i \phi (x, 2g(x) - y_n) - 2 \frac{\partial \psi_1}{\partial y_n} (x, y_n) D_i g(x), \]

\[ \frac{\partial \psi_2}{\partial x_i} (x, y_n) = D_i \phi \left( x, \frac{3}{2} g(x) - \frac{y_n}{2} \right) - 3 \frac{\partial \psi_2}{\partial y_n} (x, y_n) D_i g(x). \]

Using this in (39.1.3), the integrals in this expression equal

\[ -3 \int_{U_-} \left[ \frac{\partial \psi_1}{\partial x_i} (x, y_n) + 2 \frac{\partial \psi_1}{\partial y_n} (x, y_n) D_i g(x) \right] f (x, y_n) \, dy_n \, dx + \]

\[ 2 \int_{U_-} \left[ \frac{\partial \psi_2}{\partial x_i} (x, y_n) + 3 \frac{\partial \psi_2}{\partial y_n} (x, y_n) D_i g(x) \right] f (x, y_n) \, dy_n \, dx \]
\[
\int_{U^-} \left[ -3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] f(x, y_n) \, dy_n \, dx.
\]
Therefore,
\[
\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} f \, dx = \int_{U^-} \left[ \frac{\partial \phi}{\partial x_i} - 3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] f \, dx \, dy_n
\]
and also
\[
\phi(x, g(x)) - 3\psi_1(x, g(x)) + 2\psi_2(x, g(x)) = \phi(x, g(x)) - 3\phi(x, g(x)) + 2\phi(x, g(x)) = 0
\]
and so \( \phi - 3\psi_1 + 2\psi_2 \in H^1_0(U^-) \). It also follows from the definition of the functions \( \psi_i \) and the assumption that \( g \) is Lipschitz, that
\[
||\psi_i||_{1,2,U^-} \leq C_g \||\phi||_{1,2,U^-} \leq C_g \||\phi||_{1,2,\mathbb{R}^n}.
\]
Therefore,
\[
\left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} \equiv \sup \left\{ \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} \, dx : ||\phi||_{1,2,\mathbb{R}^n} \leq 1 \right\}
\]
\[
= \sup \left\{ \int_{U^-} f \left[ \frac{\partial \phi}{\partial x_i} - 3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] \, dx : ||\phi||_{1,2,\mathbb{R}^n} \leq 1 \right\}
\]
\[
\leq C_g \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n}
\]
where \( C_g \) is a constant which depends on \( g \). This inequality along with \( 39.1.2 \) yields
\[
\sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} \leq C_g \left( \sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} \right).
\]
The inequality,
\[
||f||_{-1,2,\mathbb{R}^n} \leq C_g ||f||_{-1,2,\mathbb{R}^n}
\]
follows from \( 39.1.3 \) and the equation,
\[
\int_{\mathbb{R}^n} f \phi \, dx = \int_{U^-} f \phi \, dx - 3 \int_{U^-} f(x, y_n) \psi_1(x, y_n) \, dx \, dy_n
\]
\[
+ 2 \int_{U^-} f(x, y_n) \psi_2(x, y_n) \, dx \, dy_n
\]
which results in the same way as before by changing variables using the definition of \( f \) off \( U^- \). This proves the lemma.

The next lemma is a simple application of Fourier transforms.

**Lemma 39.1.3** If \( f \in L^2(\mathbb{R}^n) \), then the following formula holds.

\[
C_n \||f||_{0,2,\mathbb{R}^n} = \sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} + ||f||_{-1,2,\mathbb{R}^n}
\]
Proof: For $\phi \in C^\infty_c(\mathbb{R}^n)$

$$||\phi||_{1,2,\mathbb{R}^n} \equiv \left(\int_{\mathbb{R}^n} \left(1 + |t|^2\right) |F\phi|^2 \, dt\right)^{1/2}$$

is an equivalent norm to the usual Sobolev space norm for $H^1_0(\mathbb{R}^n)$ and is used in the following argument which depends on Plancherel’s theorem and the fact that $F\left(\frac{\partial \phi}{\partial x_i}\right) = t_i F(\phi)$.

$$\left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2,\mathbb{R}^n} \equiv \sup \left\{ \left|\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} f \, dx \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup \left\{ \left|\int_{\mathbb{R}^n} t_i (F\phi) (FF)dt \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup \left\{ \int_{\mathbb{R}^n} t_i (F\phi) \left(\frac{1 + |t|^2}{1 + |t|^2}\right)^{1/2} (FF)dt \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \left(\int_{\mathbb{R}^n} \frac{|Ff|^2 t_i^2}{(1 + |t|^2)} \, dt\right)^{1/2}$$

(39.1.5)

Also,

$$||f||_{-1,2} \equiv \sup \left\{ \left|\int_{\mathbb{R}^n} \phi f \, dx \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup \left\{ \int_{\mathbb{R}^n} (F\phi) (FF) \, dx \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup \left\{ \int_{\mathbb{R}^n} F\phi \left(\frac{1 + |t|^2}{1 + |t|^2}\right)^{1/2} (FF)dt \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \left(\int_{\mathbb{R}^n} \frac{|Ff|^2}{(1 + |t|^2)} \, dt\right)^{1/2}$$

This along with (39.1.5) yields the conclusion of the lemma because

$$\sum_{i=1}^n \left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2}^2 + ||f||_{-1,2}^2 = C_n \int_{\mathbb{R}^n} |Ff|^2 \, dx = C_n ||f||_{0,2}^2.$$

Now consider Theorem 39.1.1. First note that by Lemma 39.1.2 and $U^-$ defined there, Lemma 39.1.3 implies that for $f$ extended as in Lemma 39.1.2,

$$||f||_{0,2,U^-} \leq ||f||_{0,2,\mathbb{R}^n} = C_n \left(||f||_{-1,2,\mathbb{R}^n} + \sum_{i=1}^n \left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2,\mathbb{R}^n}\right)$$
CHAPTER 39. KORN’S INEQUALITY

\[ \leq C_{gm} \left( \| f \|_{-1,2,\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right). \]  (39.1.6)

Let \( \Omega \) be a bounded open set having Lipschitz boundary which lies locally on one side of its boundary. Let \( \{Q_j\}_{j=0}^{p} \) be cubes of the sort used in the proof of the divergence theorem such that \( Q_0 \subseteq \Omega \) and the other cubes cover the boundary of \( \Omega \). Let \( \{\psi_j\} \) be a \( C^\infty \) partition of unity with \( \text{spt} (\psi_j) \subseteq Q_j \) and let \( f \in L^2(\Omega) \). Then for \( \phi \in C^\infty_c(\Omega) \) and \( \psi \) one of these functions in the partition of unity,

\[ \left\| \frac{\partial (f \psi)}{\partial x_i} \right\|_{-1,2,\Omega} \leq \sup_{|\phi|_{1,2} \leq 1} \left| \int_{\Omega} f \frac{\partial}{\partial x_i} (\psi \phi) \, dx \right| + \sup_{|\phi|_{1,2} \leq 1} \left| \int_{\Omega} f \phi \frac{\partial \psi}{\partial x_i} \, dx \right| \]

Now if \( |\phi|_{1,2} \leq 1 \), then for a suitable constant, \( C_\psi \),

\[ |\psi|_{1,2} \leq C_\psi |\phi|_{1,2} \leq C_\psi, \quad \left\| \frac{\partial \psi}{\partial x_i} \right\|_{1,2} \leq C_\psi. \]

Therefore,

\[ \left\| \frac{\partial (f \phi)}{\partial x_i} \right\|_{-1,2,\Omega} \leq \sup_{|\phi|_{1,2} \leq 1} \left| \int_{\Omega} f \frac{\partial \eta}{\partial x_i} \, dx \right| + \sup_{|\eta|_{1,2} \leq 1} \left| \int_{\Omega} f \eta \, dx \right| \]

\[ \leq C_\psi \left( \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} + \| f \|_{-1,2,\Omega} \right). \]  (39.1.7)

Now using 39.1.7 and 39.1.6

\[ \| f \psi_j \|_{0,2,\Omega} \leq C_g \left( \| f \psi_j \|_{-1,2,\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial (f \psi_j)}{\partial x_i} \right\|_{-1,2,\Omega} \right) \]

\[ \leq C_\psi C_g \left( \| f \|_{-1,2,\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right). \]

Therefore, letting \( C = \sum_{j=1}^{p} C_\psi C_g \),

\[ \| f \|_{0,2,\Omega} \leq \sum_{j=1}^{p} \| f \psi_j \|_{0,2,\Omega} \leq C \left( \| f \|_{-1,2,\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right). \]  (39.1.8)

This proves the hard half of the inequality of Theorem 39.1.1.

To complete the proof, let \( \overline{f} \) denote the zero extension of \( f \) off \( \Omega \). Then

\[ \| f \|_{-1,2,\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \leq \| \overline{f} \|_{-1,2,\Omega} + \sum_{i=1}^{n} \left\| \frac{\partial \overline{f}}{\partial x_i} \right\|_{-1,2,\Omega} \]

\[ \leq C_n \| \overline{f} \|_{0,2,\Omega} = C_n \| f \|_{0,2,\Omega}. \]

This along with 39.1.8 proves Theorem 39.1.1.
39.2 Korn’s Inequality

The inequality in this section is known as Korn’s second inequality. It is also known as coercivity of strains. For \( u \) a vector valued function in \( \mathbb{R}^n \), define

\[
\varepsilon_{ij}(u) \equiv \frac{1}{2} (u_{i,j} + u_{j,i})
\]

This is known as the strain or small strain. Korn’s inequality says that the norm given by,

\[
||| u ||| = \left( \sum_{i=1}^{n} ||u_i||^2_{0,2,\Omega} + \sum_{i=1}^{n} \sum_{j=1}^{n} ||\varepsilon_{ij}(u)||^2_{0,2,\Omega} \right)^{1/2}
\]

(39.2.9)

is equivalent to the norm,

\[
|| u || = \left( \sum_{i=1}^{n} ||u_i||^2_{0,2,\Omega} + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\| \frac{\partial u_i}{\partial x_j} \right\|^2_{0,2,\Omega} \right)^{1/2}
\]

(39.2.10)

It is very significant because it is the strain as just defined which occurs in many of the physical models proposed in continuum mechanics. The inequality is far from obvious because the strains only involve certain combinations of partial derivatives.

**Theorem 39.2.1 (Korn’s second inequality)** Let \( \Omega \) be any domain for which the conclusion of Theorem 39.1.1 holds. Then the two norms in 39.2.9 and 39.2.10 are equivalent.

**Proof:** Let \( u \) be such that \( u_i \in H^1(\Omega) \) for each \( i = 1, \ldots, n \). Note that

\[
\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j} (\varepsilon_{ik}(u)) + \frac{\partial}{\partial x_k} (\varepsilon_{ij}(u)) - \frac{\partial}{\partial x_i} (\varepsilon_{jk}(u)).
\]

Therefore, by Theorem 39.1.4.

\[
\left\| \frac{\partial u_i}{\partial x_j} \right\|_{0,2,\Omega} \leq C \left[ \left\| \frac{\partial u_i}{\partial x_j} \right\|_{-1,2,\Omega} + \sum_{k=1}^{n} \left\| \frac{\partial^2 u_i}{\partial x_j \partial x_k} \right\|_{-1,2,\Omega} \right]
\]

\[
\leq C \left[ \left\| \frac{\partial u_i}{\partial x_j} \right\|_{-1,2,\Omega} + \sum_{r,s,p} \left\| \frac{\partial \varepsilon_{rs}(u)}{\partial x_p} \right\|_{-1,2,\Omega} \right]
\]

\[
\leq C \left[ \left\| \frac{\partial u_i}{\partial x_j} \right\|_{-1,2,\Omega} + \sum_{r,s} \left\| \varepsilon_{rs}(u) \right\|_{0,2,\Omega} \right].
\]

But also by this theorem,

\[
||u_i||_{-1,2,\Omega} + \sum_p \| \frac{\partial u_i}{\partial x_p} \|_{-1,2,\Omega} \leq C \|u_i\|_{0,2,\Omega}
\]
and so
\[ \left\| \frac{\partial u_i}{\partial x_j} \right\|_{0,2,\Omega} \leq C \left[ \left\| u_i \right\|_{0,2,\Omega} + \sum_{r,s} \left\| \varepsilon_{rs}(u) \right\|_{0,2,\Omega} \right] \]

This proves the theorem.

Note that \( \Omega \) did not need to be bounded. It suffices to be able to conclude the result of Theorem 39.1.1 which would hold whenever the boundary of \( \Omega \) can be covered with finitely many boxes of the sort to which Lemma 39.1.2 can be applied.
Chapter 40

Elliptic Regularity And Nirenberg Differences

40.1 The Case Of A Half Space

Regularity theorems are concerned with obtaining more regularity given a weak solution. This extra regularity is essential in order to obtain error estimates for various problems. In this section a regularity is given for weak solutions to various elliptic boundary value problems. To save on notation, I will use the repeated index summation convention. Thus you sum over repeated indices. Consider the following picture.

Here $V$ is an open set,

$$U \equiv \{ y \in V : y_n < 0 \}, \Gamma \equiv \{ y \in V : y_n = 0 \}$$

and $U_1$ is an open set as shown for which $U_1 \subseteq V \cap U$. Assume also that $V$ is
bounded. Suppose

\[ f \in L^2(U), \]
\[ \alpha^{rs} \in C^{0,1}(U), \quad (40.1.1) \]
\[ \alpha^{rs}(y) v_r v_s \geq \delta |v|^2, \quad \delta > 0. \quad (40.1.2) \]

The following technical lemma gives the essential ideas.

**Lemma 40.1.1** Suppose

\[ w \in H^1(U), \quad (40.1.3) \]
\[ \alpha^{rs} \in C^{0,1}(U), \quad (40.1.4) \]
\[ h_s \in H^1(U), \quad (40.1.5) \]
\[ f \in L^2(U). \quad (40.1.6) \]

and

\[ \int_U \alpha^{rs}(y) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy + \int_U h_s(y) \frac{\partial z}{\partial y^s} dy = \int_U f z dy \quad (40.1.7) \]

for all \( z \in H^1(U) \) having the property that \( \text{spt}(z) \subseteq V \). Then \( w \in H^2(U_1) \) and for some constant \( C \), independent of \( f, w, \) and \( g \), the following estimate holds.

\[ ||w||^2_{H^2(U_1)} \leq C \left( ||w||^2_{H^1(U)} + ||f||^2_{L^2(U)} + \sum_s ||h_s||^2_{H^1(U)} \right). \quad (40.1.8) \]

**Proof:** Define for small real \( h \),

\[ D^h_k l(y) \equiv \frac{1}{h} \left( l(y + h e_k) - l(y) \right). \]

Let \( U_1 \subseteq \overline{U} \subseteq W \subseteq \overline{W} \subseteq V \) and let \( \eta \in C_c^\infty(W) \) with \( \eta(y) \in [0,1] \), and \( \eta = 1 \) on \( U_1 \) as shown in the following picture.
40.1. THE CASE OF A HALF SPACE

For \( h \) small \((3h < \text{dist}(\mathcal{W}, \mathcal{V}^C))\), let

\[
\begin{align*}
    z(y) & \equiv \frac{1}{h} \left\{ \eta^2 (y - he_k) \left[ \frac{w(y) - w(y - he_k)}{h} \right] \\
    & \quad - \eta^2 (y) \left[ \frac{w(y + he_k) - w(y)}{h} \right] \right\} \\
    & \equiv -D_k^{-h} (\eta^2 D_k^h w),
\end{align*}
\]

(40.1.9)

where here \( k < n \). Thus \( z \) can be used in equation (40.1.7). Begin by estimating the left side of (40.1.7).

\[
\int_{U} \alpha^{rs}(y) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy = \frac{1}{h} \int_{U} \alpha^{rs}(y + he_k) \frac{\partial w}{\partial y^r} (y + he_k) \frac{\partial (\eta^2 D_k^h w)}{\partial y^s} dy \\
\quad - \frac{1}{h} \int_{U} \alpha^{rs}(y) \frac{\partial w}{\partial y^r} (\eta^2 D_k^h w) \frac{\partial (\eta^2 D_k^h w)}{\partial y^s} dy \\
= \frac{1}{h} \int_{U} (\alpha^{rs}(y + he_k) - \alpha^{rs}(y)) \frac{\partial w}{\partial y^r} (\eta^2 D_k^h w) \frac{\partial (\eta^2 D_k^h w)}{\partial y^s} dy \\
    \quad + \frac{1}{h} \int_{U} (\alpha^{rs}(y + he_k) - \alpha^{rs}(y)) \frac{\partial w}{\partial y^r} (\eta^2 D_k^h w) \frac{\partial (\eta^2 D_k^h w)}{\partial y^s} dy + \int_{U} \alpha^{rs}(y + he_k) \frac{\partial (D_k^h w)}{\partial y^r} \frac{\partial (\eta^2 D_k^h w)}{\partial y^s} dy
\]

(40.1.11)

Now

\[
\frac{\partial (\eta^2 D_k^h w)}{\partial y^s} = 2\eta \frac{\partial \eta}{\partial y^s} D_k^h w + \eta^2 \frac{\partial (D_k^h w)}{\partial y^s}.
\]

(40.1.12)

therefore,

\[
\quad + \frac{1}{h} \int_{U} (\alpha^{rs}(y + he_k) - \alpha^{rs}(y)) \frac{\partial w}{\partial y^r} (\eta^2 D_k^h w) \frac{\partial (\eta^2 D_k^h w)}{\partial y^s} dy \equiv A. + \{B.\}.
\]

(40.1.13)

Now consider these two terms. From (40.1.2),

\[
A. \geq \delta \int_U \eta^2 |\nabla D_k^h w|^2 dy.
\]

(40.1.14)

Using the Lipschitz continuity of \( \alpha^{rs} \) and (40.1.3),

\[
B. \leq C (\eta, \text{Lip} (\alpha), \alpha) \left\{ \left\| D_k^h w \right\|_{L^2(\mathcal{W} \cap U)} \left\| \eta \nabla D_k^h w \right\|_{L^2(\mathcal{W} \cap U; \mathbb{R}^n)} + \right\}
\]

(40.1.15)
\[ ||\eta \nabla w||_{L^2(U;\mathbb{R}^n)} ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)} \]
\[ + ||\eta \nabla w||_{L^2(U;\mathbb{R}^n)} ||D_k^h w||_{L^2(U;\mathbb{R}^n)} \] (40.1.15)
\[ \leq C (\eta, \text{Lip} (\alpha), \alpha) C_\varepsilon \left( ||D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 + ||\eta \nabla w||_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \right) \] (40.1.16)

Now
\[ ||D_k^h w||_{L^2(U;\mathbb{R}^n)} \leq ||\nabla w||_{L^2(U;\mathbb{R}^n)} \] (40.1.17)

To see this, observe that if \( w \) is smooth, then
\[ \left( \int_W \left| \frac{w(y + he_k) - w(y)}{h} \right|^2 dy \right)^{1/2} \leq \left( \int_W \left| \frac{1}{h} \int_0^h \nabla w(y + te_k) \cdot e_k \, dt \right|^2 dy \right)^{1/2} \]
\[ \leq \left( \int_0^h \left( \int_W |\nabla w(y + te_k) \cdot e_k|^2 dy \right)^{1/2} \frac{dt}{h} \right) \leq ||\nabla w||_{L^2(U;\mathbb{R}^n)} \]
so by density of such functions in \( H^1 (U) \), (40.1.15) holds. Therefore, changing \( \varepsilon \), yields
\[ B. \leq C_\varepsilon (\eta, \text{Lip} (\alpha), \alpha) ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \] (40.1.18)

With (40.1.15) and (40.1.17) established, consider the other terms of (40.1.16)
\[ \left| \int_U f \, dz \, dy \right| \]
\[ \leq \left| \int_U f \left( -D_k^{-h} \eta^2 D_k^h w \right) \, dy \right| \]
\[ \leq \left( \int_U |f|^2 \, dy \right)^{1/2} \left( \int_U |D_k^{-h} (\eta^2 D_k^h w)|^2 \, dy \right)^{1/2} \]
\[ \leq \|f\|_{L^2(U)} \|\nabla (\eta^2 D_k^h w)\|_{L^2(U;\mathbb{R}^n)} \]
\[ \leq \|f\|_{L^2(U)} \left( ||2\nabla \eta D_k^h w||_{L^2(U;\mathbb{R}^n)} + ||\eta^2 \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)} \right) \]
\[ \leq C \|f\|_{L^2(U)} \|\nabla w\|_{L^2(U;\mathbb{R}^n)} + \|f\|_{L^2(U)} \|\eta \nabla D_k^h w\|_{L^2(U;\mathbb{R}^n)} \]
\[ \leq C_\varepsilon \left( \|f\|_{L^2(U)}^2 + \|\nabla w\|_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \right) \] (40.1.19)
\[ \begin{align*}
\left| \int_U h_s(y) \frac{\partial z}{\partial y_s} dy \right| & \leq \left| \int_U h_s(y) \frac{\partial}{\partial y_s} \left( \eta^{-h} (\eta^2 D_k^h w) \right) dy \right| \\
& \leq \left| \int_U D_k^h h_s(y) \frac{\partial}{\partial y_s} (\eta^2 D_k^h w) dy \right|
\end{align*} \]

The following inequalities in 40.1.14, 40.1.15, 40.1.16 and 40.1.19 are summarized here.

A. \( \geq \delta \int_U \eta^2 |\nabla D_k^h w|^2 \, dy \),

B. \( \leq C_\varepsilon (\eta, \text{Lip}(\alpha), \alpha) ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon \left( \frac{1}{2} ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 + \frac{1}{2} ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon \right) ||\nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \),

\[ \left| \int_U f z dy \right| \leq C_\varepsilon \left( ||f||_{L^2(U)}^2 + ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 \right) + \varepsilon \left( ||\nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \right) \]

Therefore,

\[ \begin{align*}
\delta ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 & \leq C_\varepsilon (\eta, \text{Lip}(\alpha), \alpha) ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon \left( ||\nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \right) \\
& + C_\varepsilon \sum_s ||h_s||_{H^1(U)} ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 + \varepsilon \left( ||\nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \right) \\
& + C_\varepsilon \left( ||f||_{L^2(U)}^2 + ||\nabla w||_{L^2(U;\mathbb{R}^n)}^2 \right) + \varepsilon \left( ||\nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \right)
\end{align*} \]

Letting \( \varepsilon \) be small enough and adjusting constants yields

\[ \begin{align*}
||\nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 & \leq ||\eta \nabla D_k^h w||_{L^2(U;\mathbb{R}^n)}^2 \\
& \leq C \left( ||w||_{H^1(U)}^2 + ||f||_{L^2(U)}^2 + \varepsilon \sum_s ||h_s||_{H^1(U)}^2 \right)
\end{align*} \]
where the constant, $C$, depends on $\eta, \text{Lip}(\alpha), \alpha, \delta$. Since this holds for all $h$ small enough, it follows that \( \frac{\partial w}{\partial y^k} \in H^1(U_1) \) and

\[
\left\| \nabla \frac{\partial w}{\partial y^k} \right\|_{L^2(U_1, \mathbb{R}^n)}^2 \leq C \left( ||w||_{H^1(U)}^2 + ||f||_{L^2(U)}^2 + C_\varepsilon \sum_s ||h_s||_{H^1(U)}^2 \right) \tag{40.1.21}
\]

for each $k < n$. It remains to estimate \( \frac{\partial^2 w}{\partial y^2} \). To do this return to (40.1.7) which must hold for all $z \in C^\infty_c(U_1)$. Therefore, using (40.1.7) it follows that for all $z \in C^\infty_c(U_1)$,

\[
\int_U \alpha^{rs}(y) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy = -\int_U \frac{\partial h_s}{\partial y^s} z dy + \int_U f z dy.
\]

Now from the Lipschitz assumption on $\alpha^{rs}$, it follows

\[
F \equiv \sum_{r,s \leq n-1} \frac{\partial}{\partial y^s} \left( \alpha^{rs} \frac{\partial w}{\partial y^r} \right) + \sum_{s \leq n-1} \frac{\partial}{\partial y^s} \left( \alpha^{ns} \frac{\partial w}{\partial y^n} \right) - \sum_s \frac{\partial h_s}{\partial y^s} + f
\]

\[
\in L^2(U_1)
\]

and

\[
||F||_{L^2(U_1)} \leq C \left( ||w||_{H^1(U)}^2 + ||f||_{L^2(U)}^2 + C_\varepsilon \sum_s ||h_s||_{H^1(U)}^2 \right). \tag{40.1.22}
\]

Therefore, from density of $C^\infty_c(U_1)$ in $L^2(U_1)$,

\[
-\frac{\partial}{\partial y^n} \left( \alpha^{nn}(y) \frac{\partial w}{\partial y^n} \right) = F, \text{ no sum on } n
\]

and so

\[
-\frac{\partial \alpha^{nn}}{\partial y^n} \frac{\partial w}{\partial y^n} - \alpha^{nn} \frac{\partial^2 w}{\partial (y^n)^2} = F
\]

By (40.1.7) $\alpha^{nn}(y) \geq \delta$ and so it follows from (40.1.22) that there exists a constant $C$ depending on $\delta$ such that

\[
\left\| \frac{\partial^2 w}{\partial (y^n)^2} \right\|_{L^2(U_1)} \leq C \left( ||F||_{L^2(U_1)} + ||w||_{H^2(U_1)} \right)
\]
which with ~[1427] and ~[1427] implies the existence of a constant, $C$ depending on $\delta$ such that

$$||w||_{H^2(U_1)}^2 \leq C \left( ||w||_{H^1(U)}^2 + ||f||_{L^2(U)}^2 + C_\varepsilon \sum_s ||h_s||_{H^1(U)}^2 \right),$$

proving the lemma.

What if more regularity is known for $f$, $h_s$, $\alpha^{rs}$ and $w$? Could more be said about the regularity of the solution? The answer is yes and is the content of the next corollary.

First here is some notation. For $\alpha$ a multi-index with $|\alpha| = k - 1$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ define

$$D^h_{\alpha} (y) \equiv \prod_{k=1}^n (D^h_k)^{\alpha_k} I(y).$$

Also, for $\alpha$ and $\tau$ multi indices, $\tau < \alpha$ means $\tau_i < \alpha_i$ for each $i$.

**Corollary 40.1.2** Suppose in the context of Lemma 40.1.1 on Page 1422 the following for $k \geq 1$.

$$
\begin{align*}
    w & \in H^k(U), \\
    \alpha^{rs} & \in C^{k-1,1}(U), \\
    h_s & \in H^k(U), \\
    f & \in H^{k-1}(U),
\end{align*}
$$

and

$$\int_U \alpha^{rs} (y) \frac{\partial w}{\partial y^r} \frac{\partial}{\partial y^s} dy + \int_U h_s (y) \frac{\partial}{\partial y^s} dy = \int_U f z dy \quad (40.1.23)$$

for all $z \in H^1(U)$ or $H^1_0(U)$ such that $\text{spt} (z) \subseteq V$. Then there exists $C$ independent of $w$ such that

$$||w||_{H^{k+1}(U)} \leq C \left( ||f||_{H^{k-1}(U)} + \sum_s ||h_s||_{H^k(U)} + ||w||_{H^k(U)} \right). \quad (40.1.24)$$

**Proof:** The proof involves the following claim which is proved using the conclusion of Lemma 40.1.1 on Page 1422.

**Claim:** If $\alpha = (\alpha',0)$ where $|\alpha'| \leq k - 1$, then there exists a constant independent of $w$ such that

$$||D^\alpha w||_{H^{k+1}(U)} \leq C \left( ||f||_{H^{k-1}(U)} + \sum_s ||h_s||_{H^k(U)} + ||w||_{H^k(U)} \right). \quad (40.1.25)$$

**Proof of claim:** First note that if $|\alpha| = 0$, then ~[1427] follows from Lemma 40.1.1 on Page 1422. Now suppose the conclusion of the claim holds for all $|\alpha| \leq j-1$ where $j < k$. Let $|\alpha| = j$ and $\alpha = (\alpha',0)$. Then for $z \in H^1(U)$ having compact support in $V$, it follows that for $h$ small enough,

$$D^{-h}_{\alpha} z \in H^1(U), \text{ spt} (D^{-h}_{\alpha} z) \subseteq V.$$
Therefore, you can replace $z$ in \(40.1.23\) with $D^{-h}z$. Now note that you can apply the following manipulation.

\[
\int_U p(y) D^{-h}z(y) \, dy = \int_U D^h p(y) z(y) \, dy
\]

and obtain

\[
\int_U \left( D^h \left( \alpha^{rs} \frac{\partial w}{\partial y^r} + D^h (h_s) \frac{\partial z}{\partial y^s} \right) \right) \, dy = \int_U \left( (D^h f) z \right) \, dy. \quad (40.1.26)
\]

Letting $h \to 0$, this gives

\[
\int_U \left( D^\alpha \left( \alpha^{rs} \frac{\partial w}{\partial y^r} + D^\alpha (h_s) \frac{\partial z}{\partial y^s} \right) \right) \, dy = \int_U \left( (D^\alpha f) z \right) \, dy.
\]

Now

\[
D^\alpha \left( \alpha^{rs} \frac{\partial w}{\partial y^r} \right) = \alpha^{rs} \frac{\partial (D^\alpha w)}{\partial y^r} + \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r}
\]

where $C(\tau)$ is some coefficient. Therefore, from \(40.1.26\),

\[
\int_U \alpha^{rs} \frac{\partial (D^\alpha w)}{\partial y^r} \frac{\partial z}{\partial y^s} \, dy + \int_U \left( \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r} + D^\alpha (h_s) \right) \frac{\partial z}{\partial y^s} \, dy
\]

\[
= \int_U (D^\alpha f) zd\, dy. \quad (40.1.27)
\]

Let $\hat{U}_1$ be as indicated in the following picture.

Now apply the induction hypothesis to $\hat{U}_1$ in order to write

\[
\| \frac{\partial (D^\tau w)}{\partial y^r} \|_{H^1(\hat{U}_1)} \leq \| D^\tau w \|_{H^2(\hat{U}_1)}
\]
40.1. THE CASE OF A HALF SPACE

\[ \leq C \left( |f|_{H^{k-1}(U)} + \sum_s |h_s|_{H^k(U)} + |w|_{H^k(U)} \right). \]

Since \( \alpha^{rs} \in C^{k-1,1}(\overline{U}) \), it follows that each term from the sum in satisfies an inequality of the form

\[ \left| \int C(\tau) D^{\alpha-\tau}(\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r} \right|_{H^1(\overline{U}_1)} \leq \]

\[ C \left( |f|_{H^{k-1}(U)} + \sum_s |h_s|_{H^k(U)} + |w|_{H^k(U)} \right) \]

and consequently,

\[ \left| \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau}(\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r} + D^\alpha(h) \right|_{H^1(\overline{U}_1)} \leq \]

\[ C \left( |f|_{H^{k-1}(U)} + \sum_s |h_s|_{H^k(U)} + |w|_{H^k(U)} \right). \]

Now consider. The equation remains true if you replace \( U \) with \( \widehat{U}_1 \) and require that \( \text{spt}(z) \subseteq \widehat{U}_1 \). Therefore, by Lemma on Page there exists a constant, \( C \) independent of \( w \) such that

\[ |D^\alpha w|_{H^2(U_1)} \leq C \left( |D^\alpha f|_{L^2(\overline{U}_1)} + |D^\alpha w|_{H^1(\overline{U}_1)} \right) \]

\[ + \sum_s \left| \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau}(\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r} + D^\alpha(h) \right|_{H^1(\overline{U}_1)} \]

and by this implies

\[ |D^\alpha w|_{H^2(U_1)} \leq C \left( |f|_{H^{k-1}(U)} + |w|_{H^k(U)} + \sum_s |h_s|_{H^k(U)} \right) \]

which proves the Claim.

To establish it only remains to verify that if \( |\alpha| \leq k + 1 \), then

\[ |D^\alpha w|_{L^2(U_1)} \leq C \left( |f|_{H^{k-1}(U)} + |w|_{H^k(U)} + \sum_s |h_s|_{H^k(U)} \right). \]

If \( |\alpha| < k + 1 \), there is nothing to show because it is given that \( w \in H^k(U) \).

Therefore, assume \( |\alpha| = k + 1 \). If \( \alpha_n \) equals 0 the conclusion follows from the claim.
because in this case, you can subtract 1 from a pair of positive \( \alpha \) and obtain a new multi index, \( \beta \) such that \( |\beta| = k - 1 \) and \( \beta_n = 0 \) and then from the claim,

\[
\|D^{\alpha}w\|_{L^2(U_1)} \leq \|D^{\beta}w\|_{H^2(U_1)} \leq C \left( \|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right).
\]

If \( \alpha_n = 1 \), then subtract 1 from some positive \( \alpha_i \) and consider

\[
\beta = (\alpha_1, \ldots, \alpha_i - 1, \alpha_i + 1, \ldots, \alpha_n - 1, 0).
\]

Then from the claim,

\[
\|D^{\alpha}w\|_{L^2(U_1)} \leq \|D^{\beta}w\|_{H^2(U_1)} \leq C \left( \|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right).
\]

Suppose \(40.1.29\) holds for \( \alpha_n \leq j - 1 \) where \( j - 1 \geq 1 \) and consider \( \alpha \) for which \( |\alpha| = k + 1 \) and \( \alpha_n = j \). Let

\[
\beta \equiv (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 2).
\]

Thus \( D^\alpha = D^{\beta}D^2_n \). Restricting \( 40.1.23 \) to \( z \in C^\infty_c(U_1) \) and using the density of this set of functions in \( L^2(U_1) \), it follows that

\[
-\frac{\partial}{\partial y^s} \left( \alpha^{rs} (y) \frac{\partial w}{\partial y^r} \right) - \frac{\partial h_s}{\partial y^s} = f.
\]

Therefore, from the product rule,

\[
\frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \frac{\partial h_s}{\partial y^s} = -f
\]

and so

\[
\alpha^{nn} D^2_n w = \left( \frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \sum_s \alpha^{ns} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn} \frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f \right).
\]

As noted earlier, the condition, \( 40.1.2 \) implies \( \alpha^{nn}(y) \geq \delta > 0 \) and so

\[
D^2_n w = -\frac{1}{\alpha^{nn}} \left( \frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \sum_s \alpha^{ns} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn} \frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f \right).
\]
It follows from $D^\alpha = D^\beta D^n_\alpha$ that

$$D^\alpha w = D^\beta \left[ -\frac{1}{\alpha^{nmn}} \left( \frac{\partial \alpha^{\alpha_n}}{\partial y^*} \frac{\partial w}{\partial y^*} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{r,s} \frac{\partial^2 w}{\partial y^s \partial y^r} + \sum_{s} \alpha^{n,s} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_{r} \alpha^{r,n} \frac{\partial^2 w}{\partial y^r \partial y^n} + \frac{\partial h_s}{\partial y^s} + f \right) \right].$$

Now you note that terms like $D^\beta \left( \frac{\partial^2 w}{\partial y^s \partial y^n} \right)$ have $\alpha_n = j - 1$ and so, from the induction hypothesis along with the assumptions on the given functions,

$$||D^\alpha w||_{L^2(U,)} \leq C \left( ||f||_{H^{k-1}(U)} + ||w||_{H^k(U)} + \sum_s ||h_s||_{H^k(U)} \right).$$

This proves the corollary.

### 40.2 The Case Of Bounded Open Sets

The main interest in all this is in the application to bounded open sets. Recall the following definition.

**Definition 40.2.1** A bounded open subset, $\Omega$, of $\mathbb{R}^n$ has a $C^{m,1}$ boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \Omega \setminus \Omega$, there exists an open set, $W$, containing $p$, an open interval $(0, b)$, a bounded open box $U' \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, $R_W$ consisting of a distance preserving linear transformation followed by a translation such that

$$R_W W = U' \times (0, b),$$

$$R_W (W \cap \Omega) = \{ u \in \mathbb{R}^n : u^* \in U', \; 0 < u_n < \phi_W (u^*) \}$$

where $\phi_W \in C^{m,1} (U')$ meaning $\phi_W$ is the restriction to $U'$ of a function, still denoted by $\phi_W$ which is in $C^{m,1} (\mathbb{R}^{n-1})$ and

$$\inf \{ \phi_W (u^*) : u^* \in U' \} > 0.$$

The following picture depicts the situation.
CHAPTER 40. ELLIPTIC REGULARITY AND NIRENBERG DIFFERENCES

For the situation described in the above definition, let \( h_W : U' \to \Gamma \cap W \) be defined by

\[
\begin{align*}
    h_W (u') &\equiv R_W^{-1} (u', \phi_W (u') - u_n), \\
    g_W (x) &\equiv (R_W x)', \\
    H_W (u) &\equiv R_W^{-1} (u', \phi_W (u') - u_n).
\end{align*}
\]

where \( x' \equiv (x_1, \cdots, x_{n-1}) \) for \( x = (x_1, \cdots, x_n) \). Thus \( g_W \circ h_W = \text{id} \) on \( U' \) and \( h_W \circ g_W = \text{id} \) on \( \Gamma \cap W \). Also note that \( H_W \) is defined on all of \( \mathbb{R}^n \) is \( C^{m,1} \), and has an inverse with the same properties. To see this, let \( G_W (u) = (u', \phi_W (u') - u_n) \). Then \( H_W = R_W^{-1} \circ G_W \) and \( G_W^{-1} = (u', \phi_W (u') - u_n) \) and so \( H_W^{-1} = G_W^{-1} \circ R_W \).

Note also that as indicated in the picture,

\[
R_W (W \cap \Omega) = \{ u \in \mathbb{R}^n : u' \in U' \text{ and } 0 < u_n < \phi_W (u') \}.
\]

Since \( \Gamma = \partial \Omega \) is compact, there exist finitely many of these open sets, \( W_i \), denoted by \( \{ W_i \}_{i=1}^q \) such that \( \Gamma \subseteq \bigcup_{i=1}^q W_i \). Let the corresponding sets, \( U_i' \) be denoted by \( U_i' \) and let the functions, \( \phi \) be denoted by \( \phi_i \). Also let \( h_i = h_{W_i}, G_{W_i} = G_i \) etc. Now let

\[
\Phi_i : G_i R_i (\Omega \cap W) \equiv V_i \to \Omega \cap W_i
\]

be defined by

\[
\Phi_i (y) \equiv R_i^{-1} \circ G_i^{-1} (y).
\]

Thus \( \Phi_i, \Phi_i^{-1} \in C^{m,1} (\mathbb{R}^n) \). The following picture might be helpful.
40.2. THE CASE OF BOUNDED OPEN SETS

Therefore, by Lemma \[\text{Page 1379}\] on Page \[\text{Page 1379}\], it follows that for \(t \in [m, m + 1)\),
\[\Phi^*_i \in \mathcal{L}(H^1(W_i \cap \Omega), H^1(V_i)).\]

Assume
\[a^{ij}(x)v_i v_j \geq \delta |v|^2.\] (40.2.32)

Lemma 40.2.2 Let \(W\) be one of the sets described in the above definition and let \(m \geq 1\). Let \(W_1 \subseteq W_2 \subseteq W\) where \(W_1\) is an open set. Suppose also that
\[u \in H^1(\Omega),\]
\[\alpha^{rs} \in C^{0,1} (\Omega),\]
\[f \in L^2(\Omega),\]
\[h_k \in H^1(\Omega),\]
and that for all \(v \in H^1(\Omega \cap W)\) such that \(\text{spt}(v) \subseteq \Omega \cap W)\),
\[\int_{\Omega} a^{ij}(x) u_i(x) v_j(x) \, dx + \int_{\Omega} h_k(x) v_k(x) \, dx = \int_{\Omega} f(x) v(x) \, dx.\] (40.2.33)
Then there exists a constant, \(C\), independent of \(f, u,\) and \(g\) such that
\[||u||^2_{H^2(\Omega \cap W_1)} \leq C \left(||f||^2_{L^2(\Omega)} + ||u||^2_{H^1(\Omega)} + \sum_k ||h_k||^2_{H^1(\Omega)}\right).\] (40.2.34)

Proof: Let \(E \equiv \{v \in H^1(\Omega \cap W) : \text{spt}(v) \subseteq W\}\)
\(u\) restricted to \(W \cap \Omega\) is in \(H^1(\Omega \cap W)\) and
\[\int_{\Omega \cap W} a^{ij}(x) u_i(x) v_j(x) \, dx + \int_{\Omega} h_k(x) v_k(x) \, dx = \int_{\Omega} f(x) v(x) \, dx \text{ for all } v \in E.\] (40.2.35)

Now let \(\Phi_i(y) = x\). For this particular \(W\), denote \(\Phi_i\) more simply by \(\Phi\), \(U_i \equiv \Phi_i(\Omega \cap W_i)\) by \(U\), and \(V_i\) by \(V\). Denoting the coordinates of \(V\) by \(y\), and letting \(u(x) \equiv w(y)\) and \(v(x) \equiv z(y)\), it follows that in terms of the new coordinates, \(\Phi\) takes the form
\[
\int_U a^{ij}(\Phi(y)) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} |\det D\Phi(y)| \, dy
+ \int_U h_k(\Phi(y)) \frac{\partial z}{\partial y^l} |\det D\Phi(y)| \, dx
= \int_U f(\Phi(y)) z(y) |\det D\Phi(y)| \, dy
\]
Let
\[\alpha^{rs}(y) \equiv a^{ij}(\Phi(y)) \frac{\partial y^r}{\partial x^i} \frac{\partial y^s}{\partial x^j} |\det D\Phi(y)|,\] (40.2.36)
CHAPTER 40. ELLIPTIC REGULARITY AND NIRENBERG DIFFERENCES

\( \vec{h}_l(y) \equiv h_k(\Phi(y)) \frac{\partial y^l}{\partial x^k} |\det D\Phi(y)|, \) \hspace{1cm} (40.2.37)

and

\( \vec{f}(y) \equiv \Phi^* f |\det D\Phi(y)| \equiv f(\Phi(y)) |\det D\Phi(y)|. \) \hspace{1cm} (40.2.38)

Now the function on the right in \( 40.2.36 \) is in \( C^{0,1}(\Omega) \). This is because of the assumption that \( m \geq 1 \) in the statement of the lemma. This function is therefore a finite product of bounded functions in \( C^{0,1}(\Omega) \).

The function \( \vec{h}_l \) defined in \( 40.2.37 \) is in \( H^1(U) \) and

\( \|\vec{h}_l\|_{H^1(U)} \leq C \sum_k ||h_k||_{H^1(\Omega \cap \omega)} \),

again because \( m \geq 1 \).

Finally, the right side of \( 40.2.38 \) is a function in \( L^2(U) \) by Lemma \( 37.3.3 \) on Page 1379 and the observation that \( |\det D\Phi(\cdot)| \in C^{0,1}(\Omega) \) which follows from the assumption of the lemma that \( m \geq 1 \) so \( \Phi \in C^{1,1}(\mathbb{R}^n) \). Also

\( \|\vec{f}\|_{L^2(U)} \leq C \|f\|_{L^2(\Omega \cap \omega)}. \)

Therefore, \( 40.2.35 \) is of the form

\[ \int_U \alpha^{rs}(y) w_{r,s} dy + \int_U \vec{h}_l z_l dy = \int_U \vec{f} z dy, \] \hspace{1cm} (40.2.39)

for all \( z \in H^1(U) \) having support in \( V \).

**Claim:** There exists \( r > 0 \) independent of \( y \in U \) such that for all \( y \in U \),

\( \alpha^{rs}(y) v_r v_s \geq r |v|^2. \)

**Proof of the claim:** If this is not so, there exist vectors, \( v^n, |v^n| = 1, \) and \( y_n \in \Omega \) such that \( \alpha^{rs}(y_n) v^n_r v^n_s \leq \frac{1}{n} \). Taking a subsequence, there exists \( y \in \Omega \) and \( |v| = 1 \) such that \( \alpha^{rs}(y) v_r v_s = 0 \) contradicting \( 40.2.32 \).

Therefore, by Lemma \( 40.1.1 \) there exists a constant, \( C \), independent of \( f, g, \) and \( w \) such that

\[ \|w\|_{H^2(\Phi^{-1}(W \cap \Omega))} \leq C \left( \|\vec{f}\|_{L^2(U)}^2 + \|w\|_{H^1(U)}^2 + \sum_l \|\vec{h}_l\|_{H^1(U)}^2 \right). \]

Therefore,

\[ \|u\|_{H^2(W \cap \Omega)}^2 \leq C \left( \|f\|_{L^2(\Omega \cap \omega)}^2 + \|w\|_{H^1(\Omega \cap \omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega \cap \omega)}^2 \right) \leq C \left( \|f\|_{L^2(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right), \]

which proves the lemma.

With this lemma here is the main result.
40.2. THE CASE OF BOUNDED OPEN SETS

Theorem 40.2.3 Let $\Omega$ be a bounded open set with $C^{1,1}$ boundary as in Definition 40.2.1, let $f \in L^2(\Omega), h_k \in H^1(\Omega),$ and suppose that for all $x \in \bar{\Omega},$

$$a^{ij}(x)v_iv_j \geq \delta |v|^2.$$ Suppose also that $u \in H^1(\Omega)$ and

$$\int_{\Omega} a^{ij}(x)u_i(x)v_j(x) \, dx + \int_{\Omega} h_k(x)v_k(x) \, dx = \int_{\Omega} f(x)v(x) \, dx$$

for all $v \in H^1(\Omega).$ Then $u \in H^2(\Omega)$ and for some $C$ independent of $f, g,$ and $u,$

$$\|u\|^2_{H^2(\Omega)} \leq C \left( \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} + \sum_k \|h_k\|^2_{H^1(\Omega)} \right).$$

Proof: Let the $W_i$ for $i = 1, \ldots, l$ be as described in Definition 40.2.1. Thus $\partial\Omega \subseteq \cup_{i=1}^l W_i.$ Then let $C_1 = \partial\Omega \setminus \bigcup_{i=2}^l W_i,$ a closed subset of $W_1.$ Let $D_1$ be an open set satisfying

$$C_1 \subseteq D_1 \subseteq \overline{D_1} \subseteq W_1.$$ Then $D_1, W_2, \ldots, W_l$ cover $\partial\Omega.$ Let $C_2 = \partial\Omega \setminus \left( D_1 \cup \bigcup_{i=3}^l W_i \right).$ Then $C_2$ is a closed subset of $W_2.$ Choose an open set, $D_2$ such that

$$C_2 \subseteq D_2 \subseteq \overline{D_2} \subseteq W_2.$$ Thus $D_1, D_2, W_3 \ldots, W_l$ covers $\partial\Omega.$ Continue in this way to get $\overline{D_i} \subseteq W_i,$ and $\partial\Omega \subseteq \bigcup_{i=1}^l D_i,$ and $D_i$ is an open set. Now let

$$D_0 \equiv \Omega \setminus \bigcup_{i=1}^l \overline{D_i}.$$ Also, let $\overline{D_i} \subseteq V_i \subseteq \overline{V_i} \subseteq W_i.$ Therefore, $D_0, V_1, \ldots, V_l$ covers $\Omega.$ Then the same estimation process used above yields

$$\|u\|^2_{H^2(D_0)} \leq C \left( \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} + \sum_k \|h_k\|^2_{H^1(\Omega)} \right).$$

From Lemma 40.2.2,

$$\|u\|^2_{H^2(V_i \cap \Omega)} \leq C \left( \|f\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} + \sum_k \|h_k\|^2_{H^1(\Omega)} \right)$$

also. This proves the theorem since

$$\|u\|^2_{H^2(\Omega)} \leq \sum_{i=1}^l \|u\|^2_{H^2(V_i \cap \Omega)} + \|u\|^2_{H^2(D_0)}.$$ What about the Dirichlet problem? The same differencing procedure as above yields the following.
Theorem 40.2.4 Let $\Omega$ be a bounded open set with $C^{1,1}$ boundary and motion-theorem as in Definition 40.4.1, let $f \in L^2(\Omega)$, $h_k \in H^1(\Omega)$, and suppose that for all $x \in \Omega$, 
\[ a^{ij}(x) v_i v_j \geq \delta |v|^2. \]

Suppose also that $u \in H^1_0(\Omega)$ and 
\[ \int_{\Omega} a^{ij}(x) u_i(x) v_j(x) \, dx + \int_{\Omega} h_k(x) v_k(x) \, dx = \int_{\Omega} f(x) v(x) \, dx \]
for all $v \in H^1_0(\Omega)$. Then $u \in H^2(\Omega)$ and for some $C$ independent of $f$, $u$, and $h$,
\[ \|u\|_{H^2(\Omega)} \leq C \left( \|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right). \]

What about higher regularity?

Lemma 40.2.5 Let $W$ be one of the sets described in Definition 40.2.1 and let $m \geq k$. Let $W_1 \subseteq \overline{W} \subseteq W$ where $W_1$ is an open set. Suppose also that
\[ u \in H^k(\Omega), \]
\[ \alpha^{rs} \in C^{k-1,1}(\overline{\Omega}), \]
\[ f \in H^{k-1}(\Omega), \]
\[ h_s \in H^k(\Omega), \]
and that for all $v \in H^1(\Omega \cap W)$ such that $\text{spt}(v) \subseteq \Omega \cap W$,
\[ \int_{\Omega} a^{ij}(x) u_i(x) v_j(x) \, dx + \int_{\Omega} h_s(x) v_s(x) \, dx = \int_{\Omega} f(x) v(x) \, dx. \] (40.2.40)

Then there exists a constant, $C$, independent of $f$, $u$, and $g$ such that
\[ \|u\|_{H^{k+1}(\Omega \cap W)}^2 \leq C \left( \|f\|_{H^{k+1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_s \|h_s\|_{H^k(\Omega)}^2 \right). \] (40.2.41)

Proof: Let 
\[ E \equiv \{ v \in H^k(\Omega \cap W) : \text{spt}(v) \subseteq W \} \]
u restricted to $W \cap \Omega$ is in $H^k(\Omega \cap W)$ and 
\[ \int_{\Omega \cap W} a^{ij}(x) u_i(x) v_j(x) \, dx + \int_{\Omega} h_s(x) v_s(x) \, dx \]
\[ = \int_{\Omega} f(x) v(x) \, dx \] for all $v \in E$. (40.2.42)

Now let $\Phi_i(y) = x$. For this particular $W$, denote $\Phi_i$ more simply by $\Phi$, $U_i \equiv \Phi_i(\Omega \cap W_i)$ by $U$, and $V_i$ by $V$. Denoting the coordinates of $V$ by $y$, and letting
40.2. THE CASE OF BOUNDED OPEN SETS

\[ u(x) \equiv w(y) \text{ and } v(x) \equiv z(y), \]

it follows that in terms of the new coordinates, \( 40.2.35 \) takes the form

\[
\int_U a^{ij}(\Phi(y)) \frac{\partial w}{\partial y^r} \frac{\partial w}{\partial y^s} |\text{det } D\Phi(y)| \, dy
+ \int_U h_k(\Phi(y)) \frac{\partial z}{\partial y^l} |\text{det } D\Phi(y)| \, dx
= \int_U f(\Phi(y)) z(y) |\text{det } D\Phi(y)| \, dy
\]

Let

\[
\alpha^{rs}(y) \equiv a^{ij}(\Phi(y)) \frac{\partial y^r}{\partial x^i} \frac{\partial y^s}{\partial x^j} |\text{det } D\Phi(y)|, \quad (40.2.43)
\]

\[
h_l(y) \equiv h_k(\Phi(y)) \frac{\partial y^l}{\partial x^k} |\text{det } D\Phi(y)|, \quad (40.2.44)
\]

and

\[
f(y) \equiv \Phi^* f |\text{det } D\Phi(y)| \equiv f(\Phi(y)) |\text{det } D\Phi(y)|. \quad (40.2.45)
\]

Now the function on the right in \( 40.2.43 \) is in \( C^{k,1}(U) \). This is because of the assumption that \( m \geq k \) in the statement of the lemma. This function is therefore a finite product of bounded functions in \( C^{k,1}(U) \).

The function \( \tilde{h}_l \) defined in \( 40.2.44 \) is in \( H^{k-1}(U) \) and

\[
\left\| \tilde{h}_l \right\|_{H^k(U)} \leq C \sum_s \|h_s\|_{H^{k}(\Omega\cap W)}
\]

again because \( m \geq k \).

Finally, the right side of \( 40.2.43 \) is a function in \( H^{k-1}(U) \) by Lemma 37.3.3 on Page 1379 and the observation that \( |\text{det } D\Phi(\cdot)| \in C^{k-1,1}(U) \) which follows from the assumption of the lemma that \( m \geq k \) so \( \Phi \in C^{k-1,1}(\mathbb{R}^n) \). Also

\[
\left\| \tilde{f} \right\|_{H^{k-1}(U)} \leq C \|f\|_{H^{k-1}(\Omega\cap W)}.
\]

Therefore, \( 40.2.46 \) is of the form

\[
\int_U \alpha^{rs}(y) w_r z_s \, dy + \int_U \tilde{h}_l z_l \, dy = \int_U \tilde{f} z \, dy, \quad (40.2.46)
\]

for all \( z \in H^1(U) \) having support in \( V \).

**Claim:** There exists \( r > 0 \) independent of \( y \in \overline{U} \) such that for all \( y \in \overline{U} \),

\[
\alpha^{rs}(y) v_r v_s \geq r |v|^2.
\]

**Proof of the claim:** If this is not so, there exist vectors, \( v^n, |v^n| = 1 \), and \( y_n \in \overline{U} \) such that \( \alpha^{rs}(y_n) v^n_r v^n_s \leq \frac{1}{n} \). Taking a subsequence, there exists \( y \in \overline{U} \) and \( |v| = 1 \) such that \( \alpha^{rs}(y) v_r v_s = 0 \) contradicting \( 40.2.32 \).
Therefore, by Corollary 40.1.2, there exists a constant, $C$, independent of $f, g,$ and $w$ such that

$$||w||_{H^{k+1}(\Phi^{-1}(W_1 \cap \Omega))}^2 \leq C \left( ||f||_{H^{k-1}(W)}^2 + ||w||_{H^k(W)}^2 + \sum_l ||h_l||_{H^k(W)}^2 \right).$$

Therefore,

$$||u||_{H^{k+1}(W_1 \cap \Omega)}^2 \leq C \left( ||f||_{H^{k-1}(W \cap \Omega)}^2 + ||w||_{H^k(W \cap \Omega)}^2 + \sum_s ||h_s||_{H^k(W \cap \Omega)}^2 \right).$$

which proves the lemma.

Now here is a theorem which generalizes the one above in the case where more regularity is known.

**Theorem 40.2.6** Let $\Omega$ be a bounded open set with $C^{k,1}$ boundary as in Definition 40.2.1, let $f \in H^{k-1}(\Omega)$, $h_s \in H^k(\Omega)$, and suppose that for all $x \in \Omega$,

$$a^{ij}(x) v_i v_j \geq \delta |v|^2.$$

Suppose also that $u \in H^k(\Omega)$ and

$$\int_{\Omega} a^{ij}(x) u_i(x) v_j(x) \, dx + \int_{\Omega} h_k(x) v_k(x) \, dx = \int_{\Omega} f(x) v(x) \, dx$$

for all $v \in H^k(\Omega)$. Then $u \in H^{k+1}(\Omega)$ and for some $C$ independent of $f, g,$ and $u$,

$$||u||_{H^{k+1}(\Omega)}^2 \leq C \left( ||f||_{H^{k-1}(\Omega)}^2 + ||w||_{H^k(\Omega)}^2 + \sum_s ||h_s||_{H^k(\Omega)}^2 \right).$$

**Proof:** Let the $W_i$ for $i = 1, \cdots, l$ be as described in Definition 40.2.1. Thus $\partial \Omega \subseteq \bigcup_{j=1}^l W_j$. Then let $C_1 \equiv \partial \Omega \setminus \bigcup_{i=2}^l W_i$, a closed subset of $W_1$. Let $D_1$ be an open set satisfying

$$C_1 \subseteq D_1 \subseteq \overline{D_1} \subseteq W_1.$$

Then $D_1, W_2, \cdots, W_l$ cover $\partial \Omega$. Let $C_2 = \partial \Omega \setminus \left( D_1 \cup \left( \bigcup_{i=3}^l W_i \right) \right)$. Then $C_2$ is a closed subset of $W_2$. Choose an open set, $D_2$ such that

$$C_2 \subseteq D_2 \subseteq \overline{D_2} \subseteq W_2.$$

Thus $D_1, D_2, W_3, \cdots, W_l$ covers $\partial \Omega$. Continue in this way to get $\overline{D_l} \subseteq W_1$, and $\partial \Omega \subseteq \bigcup_{i=1}^l D_i$, and $D_i$ is an open set. Now let

$$D_0 \equiv \Omega \setminus \bigcup_{i=1}^l \overline{D_i}.$$
Also, let $D_i \subseteq V_i \subseteq W_i$. Therefore, $D_0, V_1, \cdots, V_l$ covers $\Omega$. Then the same estimation process used above yields

$$\|u\|_{H^{k+1}(D_0)} \leq C \left( \|f\|_{H^{k-1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_k \|h_k\|_{H^k(\Omega)}^2 \right).$$

From Lemma 40.2.5

$$\|u\|_{H^{k+1}(V \cap \Omega)} \leq C \left( \|f\|_{H^{k-1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_k \|h_k\|_{H^k(\Omega)}^2 \right)$$

also. This proves the theorem since

$$\|u\|_{H^{k+1}(\Omega)} \leq \sum_{i=1}^l \|u\|_{H^{k+1}(V_i \cap \Omega)} + \|u\|_{H^{k+1}(D_0)}.$$
Chapter 41

Interpolation In Banach Space

41.1 Some Standard Techniques In Evolution Equations

41.1.1 Weak Vector Valued Derivatives

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 41.1.1 Suppose \( g \in L^1 ([a,b] ; X) \) where \( X \) is a Banach space. Then if
\[
\int_a^b g(t) \phi(t) \, dt = 0
\]
for all \( \phi \in C_c^\infty (a,b) \), then \( g(t) = 0 \) a.e.

Proof: Let \( E \) be a measurable subset of \((a,b)\) and let \( K \subseteq E \subseteq V \subseteq (a,b) \) where \( K \) is compact, \( V \) is open and \( m (V \setminus K) < \varepsilon \). Let \( K < h < V \) as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging \( K \) slightly and convolving with a mollifier, it can be assumed \( h \in C^\infty_c (a,b) \). Then
\[
\left| \int_a^b X_E (t) g(t) \, dt \right| = \left| \int_a^b (X_E (t) - h (t)) g(t) \, dt \right|
\leq \int_a^b |X_E (t) - h(t)| \| g(t) \| \, dt
\leq \int_{V \setminus K} \| g(t) \| \, dt.
\]

Now let \( K_n \subseteq E \subseteq V_n \) with \( m (V_n \setminus K_n) < 2^{-n} \). Then from the above,
\[
\left| \int_a^b X_E (t) g(t) \, dt \right| \leq \int_a^b X_{V_n \setminus K_n} (t) \| g(t) \| \, dt
\]
and the integrand of the last integral converges to 0 a.e. as \( n \to \infty \) because 
\( \sum_{n} m(V_{n} \setminus K_{n}) < \infty \). By the dominated convergence theorem, this last integral 
converges to 0. Therefore, whenever \( E \subseteq (a, b) \),
\[
\int_{a}^{b} X_{E}(t) g(t) \, dt = 0.
\]
Since the endpoints have measure zero, it also follows that for any measurable \( E \), 
the above equation holds.

Now \( g \in L^{1}([a, b]; X) \) and so it is measurable. Therefore, \( g([a, b]) \) is separable. 
Let \( D \) be a countable dense subset and let \( E \) denote the set of linear combinations of 
the form \( \sum a_{i}d_{i} \) where \( a_{i} \) is a rational point of \( \mathbb{P} \) and \( d_{i} \in D \). Thus \( E \) is countable. 
Denote by \( Y \) the closure of \( E \) in \( X \). Thus \( Y \) is a separable closed subspace of \( X \) 
which contains all the values of \( g \).

Now let \( S_{n} \equiv g^{-1}(B(y_{n}, \|y_{n}\|/2)) \) where \( E = \{y_{n}\}_{n=1}^{\infty} \). Therefore, \( \cup_{n} S_{n} = 
\{y_{n}\}_{n=1}^{\infty}(X \setminus \{0\}) \). This follows because if \( x \in Y \) and \( x \neq 0 \), then in \( B \left(x, \frac{\|x\|}{4}\right) \) there
is a point of \( E, y_{n} \). Therefore, \( \|y_{n}\| > \frac{3}{4} \|x\| \) and so \( \frac{\|y_{n}\|}{2} > \frac{3}{8} \frac{\|x\|}{4} \) so \( x \in B \left(y_{n}, \|y_{n}\|/2\right) \). It follows that if each \( S_{n} \) has measure zero, then \( g(t) = 0 \) for a.e. \( t \). Suppose then that for some \( n \), the set, \( S_{n} \) has positive mesure. Then from what 
was shown above,
\[
\|y_{n}\| = \left\| \frac{1}{m(S_{n})} \int_{S_{n}} g(t) \, dt - y_{n} \right\| = \left\| \frac{1}{m(S_{n})} \int_{S_{n}} g(t) \, dt - y_{n} \right\| 
\leq \frac{1}{m(S_{n})} \int_{S_{n}} \|g(t) - y_{n}\| \, dt \leq \frac{1}{m(S_{n})} \int_{S_{n}} \|y_{n}\|/2 \, dt = \|y_{n}\|/2
\]
and so \( y_{n} = 0 \) which implies \( S_{n} = \emptyset \), a contradiction to \( m(S_{n}) > 0 \). This contradic-
tion shows each \( S_{n} \) has measure zero and so as just explained, \( g(t) = 0 \) a.e.

**Definition 41.1.2** For \( f \in L^{1}(a, b; X) \), define an extension, \( \overline{f} \) defined on
\([2a - b, 2b - a] = [a - (b - a), b + (b - a)]\)
as follows.

\[
\overline{f}(t) = \begin{cases} 
 f(t) & \text{if } t \in [a, b] \\
 f(2a - t) & \text{if } t \in [2a - b, a] \\
 f(2b - t) & \text{if } t \in [b, 2b - a]
\end{cases}
\]

**Definition 41.1.3** Also if \( f \in L^{p}(a, b; X) \) and \( h > 0 \), define for \( t \in [a, b] \), 
\( f_{h}(t) = \overline{f}(t - h) \) for all \( h < b - a \). Thus the map \( f \to f_{h} \) is continuous 
and linear on \( L^{p}(a, b; X) \). It is continuous because
\[
\int_{a}^{b} \|f_{h}(t)\|^{p} \, dt = \int_{a}^{a+h} \|f(2a - t + h)\|^{p} \, dt + \int_{a}^{b-h} \|f(t)\|^{p} \, dt 
= \int_{a}^{a+h} \|f(t)\|^{p} \, dt + \int_{a}^{b-h} \|f(t)\|^{p} \, dt \leq 2 \|f\|_{p}^{p}.
\]
41.1. SOME STANDARD TECHNIQUES IN EVOLUTION EQUATIONS

The following lemma is on continuity of translation in $L^p(a,b;X)$.

**Lemma 41.1.4** Let $\mathcal{F}$ be as defined in Definition 37.5. Then for $f \in L^p(a,b;X)$ for $p \in [1, \infty)$,

$$\lim_{\delta \to 0} \int_a^b \|\mathcal{F}(t - \delta) - f(t)\|^p_X dt = 0.$$

**Proof:** Regarding the measure space as $(a,b)$ with Lebesgue measure, by Lemma 19.5.9 there exists $g \in C^c_c(a,b;X)$ such that $\|f - g\|_p < \varepsilon$. Here the norm is the norm in $L^p(a,b;X)$. Therefore,

$$\|fh - f\|_p \leq \|fh - gh\|_p + \|gh - g\|_p + \|g - f\|_p$$

$$\leq \left(2^{1/p} + 1\right)\|f - g\|_p + \|gh - g\|_p$$

$$< \left(2^{1/p} + 1\right)\varepsilon + \varepsilon$$

whenever $h$ is sufficiently small. This is because of the uniform continuity of $g$. Therefore, since $\varepsilon > 0$ is arbitrary, this proves the lemma. ■

**Definition 41.1.5** Let $f \in L^1(a,b;X)$. Then the distributional derivative in the sense of $X$ valued distributions is given by

$$f'(\phi) = - \int_a^b f(t) \phi'(t) dt$$

Then $f' \in L^1(a,b;X)$ if there exists $h \in L^1(a,b;X)$ such that for all $\phi \in C^\infty_c(a,b)$,

$$f'(\phi) = \int_a^b h(t) \phi(t) dt.$$

Then $f'$ is defined to equal $h$. Here $f$ and $f'$ are considered as vector valued distributions in the same way as was done for scalar valued functions.

**Lemma 41.1.6** The above definition is well defined.

**Proof:** Suppose both $h$ and $g$ work in the definition. Then for all $\phi \in C^\infty_c(a,b)$,

$$\int_a^b (h(t) - g(t)) \phi(t) dt = 0.$$ 

Therefore, by Lemma 41.1.4, $h(t) - g(t) = 0$ a.e. ■

The other thing to notice about this is the following lemma. It follows immediately from the definition.

**Lemma 41.1.7** Suppose $f, f' \in L^1(a,b;X)$. Then if $[c,d] \subseteq [a,b]$, it follows that $(f|_{[c,d]})' = f'|_{[c,d]}$. This notation means the restriction to $[c,d]$. 


Recall that in the case of scalar valued functions, if you had both \( f \) and its weak derivative, \( f' \) in \( L^1(a,b) \), then you were able to conclude that \( f \) is almost everywhere equal to a continuous function, still denoted by \( f \) and

\[
    f(t) = f(a) + \int_a^t f'(s) \, ds.
\]

In particular, you can define \( f(a) \) to be the initial value of this continuous function.

It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if \( f' = 0 \) where the derivative is taken in the sense of \( X \) valued distributions, then \( f \) equals a constant.

**Lemma 41.1.8** Suppose \( f \in L^1(a,b; X) \) and for all \( \phi \in C^\infty_c(a,b) \),

\[
    \int_a^b f(t) \phi'(t) \, dt = 0.
\]

Then there exists a constant, \( a \in X \) such that \( f(y) = a \) a.e.

**Proof:** Let \( \phi_0 \in C^\infty_c(a,b) \), \( \int_a^b \phi_0(x) \, dx = 1 \) and define for \( \phi \in C^\infty_c(a,b) \)

\[
    \psi_\phi(x) = \int_a^x [\phi(t) - \left( \int_a^b \phi(y) \, dy \right) \phi_0(t)] \, dt
\]

Then \( \psi_\phi \in C^\infty_c(a,b) \) and \( \psi_\phi' = \phi - \left( \int_a^b \phi(y) \, dy \right) \phi_0 \). Then

\[
    \int_a^b f(t) (\phi(t)) \, dt = \int_a^b f(t) \left( \psi_\phi'(t) + \left( \int_a^b \phi(y) \, dy \right) \phi_0(t) \right) \, dt
\]

\[
    = \int_a^b f(t) \psi_\phi'(t) \, dt + \left( \int_a^b \phi(y) \, dy \right) \int_a^b f(t) \phi_0(t) \, dt
\]

\[
    = \left( \int_a^b \int_a^b f(t) \phi_0(t) \, dt \right) \phi(y) \, dy.
\]

It follows that for all \( \phi \in C^\infty_c(a,b) \),

\[
    \int_a^b \left( f(y) - \left( \int_a^b f(t) \phi_0(t) \, dt \right) \right) \phi(y) \, dy = 0
\]

and so by Lemma [41.1.8].

\[
    f(y) - \left( \int_a^b f(t) \phi_0(t) \, dt \right) = 0 \text{ a.e. } y
\]
41.1. SOME STANDARD TECHNIQUES IN EVOLUTION EQUATIONS

Theorem 41.1.9 Suppose \( f, f' \) both are in \( L^1 (a, b; X) \) where the derivative is taken in the sense of \( X \) valued distributions. Then there exists a unique point of \( X \), denoted by \( f(a) \) such that the following formula holds a.e. \( t \).

\[
f(t) = f(a) + \int_a^t f'(s) \, ds
\]

Proof:

\[
\int_a^b \left( f(t) - \int_a^t f'(s) \, ds \right) \phi'(t) \, dt = \int_a^b f(t) \phi'(t) \, dt - \int_a^b \int_a^t f'(s) \phi'(t) \, ds \, dt.
\]

Now consider \( \int_a^b \int_a^t f'(s) \, ds \, dt \). Let \( \Lambda \in X' \). Then it is routine from approximating \( f' \) with simple functions to verify

\[
\Lambda \left( \int_a^b \int_a^t f'(s) \, ds \, dt \right) = \int_a^b \int_a^t \Lambda (f'(s)) \phi'(t) \, ds \, dt.
\]

Now the ordinary Fubini theorem can be applied to obtain

\[
= \int_a^b \int_s^b \Lambda (f'(s)) \phi'(t) \, dt \, ds
\]

\[
= \Lambda \left( \int_a^b \int_s^b f'(s) \phi'(t) \, dt \, ds \right).
\]

Since \( X' \) separates the points of \( X \), it follows

\[
\int_a^b \int_s^b f'(s) \phi'(t) \, ds \, dt = \int_a^b \int_s^b f'(s) \phi'(t) \, dt \, ds.
\]

Therefore,

\[
\int_a^b \left( f(t) - \int_a^t f'(s) \, ds \right) \phi'(t) \, dt
\]

\[
= \int_a^b f(t) \phi'(t) \, dt - \int_a^b \int_s^b f'(s) \phi'(t) \, dt \, ds
\]

\[
= \int_a^b f(t) \phi'(t) \, dt - \int_a^b f'(s) \int_s^b \phi'(t) \, dt \, ds
\]

\[
= \int_a^b f(t) \phi'(t) \, dt + \int_a^b f'(s) \phi(s) \, ds = 0.
\]

Therefore, by Lemma [41.1.8], there exists a constant, denoted as \( f(a) \) such that

\[
f(t) - \int_a^t f'(s) \, ds = f(a) \]

The integration by parts formula is also important.
Corollary 41.1.10 Suppose \( f, f' \in L^1(a, b; X) \) and suppose \( \phi \in C^1([a, b]) \). Then the following integration by parts formula holds.

\[
\int_a^b f(t) \phi'(t) \, dt = f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) \, dt.
\]

Proof: From Theorem 41.1.9

\[
\int_a^b f(t) \phi'(t) \, dt = \int_a^b f(a) + \int_a^t f'(s) \, ds \phi'(t) \, dt = f(a)(\phi(b) - \phi(a)) + \int_a^b \int_a^b f'(s) \phi'(t) \, ds \, dt
\]

\[
= f(a)(\phi(b) - \phi(a)) + \int_a^b f'(s) \, ds \phi'(t) \, dt
\]

\[
= f(a)(\phi(b) - \phi(a)) + \int_a^b f'(s) (\phi(b) - \phi(s)) \, ds
\]

\[
= f(a)(\phi(b) - \phi(a)) - \int_a^b f'(s) \phi(s) \, ds + (f(b) - f(a)) \phi(b)
\]

\[
= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(s) \phi(s) \, ds.
\]

The interchange in order of integration is justified as in the proof of Theorem 41.1.9.

There is an interesting theorem which is easy to present at this point.

Definition 41.1.11 Let

\( H^1(0, T, X) \)

denote the functions \( f \in L^2(0, T, X) \) whose weak derivative \( f' \) is also in \( L^2(0, T, X) \).

Proposition 41.1.12 Let \( f \in H^1(0, T, X) \). Then \( f \in C^{0,1/2}([0, T], X) \) and the inclusion map is continuous.

Proof: First note that

\[
f(t) - f(s) = \int_s^t f'(r) \, dr
\]

and so

\[
\|f(t) - f(s)\|_X \leq \int_s^t \|f'(r)\|_X \, dr \leq \|f\|_{H^1} |t - s|^{1/2}
\]

It follows that

\[
\sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|}{|t - s|^{1/2}} \leq \|f\|_{H^1}
\]
41.1. SOME STANDARD TECHNIQUES IN EVOLUTION EQUATIONS

Also

\[ f(t) = f(0) + \int_0^t f'(s) \, ds \]

so

\[ \|f(t)\| \leq \|f(0)\| + \int_0^t |f'(s)| \, ds \leq \|f(0)\| + T^{1/2} \|f\|_{L^1} \]

Now consider \( \|f(0)\| \). Then integrating by parts yields

\[ \int_0^T (T-t) f'(t) \, dt = (T-t) f(t) \bigg|_0^T + \int_0^T f(t) \, dt \]

and so

\[ T \|f(0)\| \leq \int_0^T \|f(t)\| \, dt + T \int_0^T \|f'(t)\| \, dt \leq C(T) \|f\|_{L^1} \cdot \]

Hence

\[ \sup_{t \in [0,T]} \|f(t)\| \leq C(T) \|f\|_{L^1} \]

Therefore, this has shown that

\[ \|f\|_{C^{0,1/2}([0,T],X)} \equiv \sup_{t \in [0,T]} \|f(t)\| + \sup_{0 \leq s < t \leq T} \frac{\|f(t) - f(s)\|}{|t-s|^{1/2}} \leq C(T) \|f\|_{L^1} \]

You could imagine that other interesting versions of this are available with similar proof for the case where the function and its weak derivative are in \( L^p(0,T,X) \) for \( p > 1 \).

With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define \( \mathcal{F} \) as in Definition [41.1.2].

**Lemma 41.1.13** Let \( \mathcal{F} \) be given in Definition [41.1.2] and suppose \( f, f' \in L^1(a,b;X) \). Then \( \mathcal{F}, \mathcal{F}' \in L^1(2a-b,2b-a;X) \) also and

\[
\mathcal{F}'(t) \equiv \begin{cases} 
  f'(t) & \text{if } t \in [a,b] \\
  -f'(2a-t) & \text{if } t \in [2a-b,a] \\
  -f'(2b-t) & \text{if } t \in [b,2b-a]
\end{cases} \quad (41.1.1)
\]

**Proof:** It is clear from the definition of \( \mathcal{F} \) that \( \mathcal{F} \in L^1(2a-b,2b-a;X) \) and that in fact

\[ \|\mathcal{F}\|_{L^1(2a-b,2b-a;X)} \leq 3 \|f\|_{L^1(a,b;X)} \cdot \]

(41.1.2)
Let \( \phi \in C_c^\infty (2a - b, 2b - a) \). Then from the integration by parts formula,

\[
\int_{2a-b}^{2b-a} f(t) \phi'(t) \, dt = \int_a^b f(t) \phi'(t) \, dt + \int_0^{2b-a} f(2b - t) \phi'(t) \, dt + \int_{2a-b}^a f(2a - t) \phi'(t) \, dt
\]

\[
= \int_a^b f(t) \phi'(t) \, dt + \int_a^b f(u) \phi'(2b - u) \, du + \int_a^b f(u) \phi'(2a - u) \, du
\]

\[
= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) \, dt - f(b) \phi(b) + f(a) \phi(2b - a)
\]

\[
+ \int_a^b f'(u) \phi(2b - u) \, du - f(b) \phi(2a - b)
\]

\[
+ f(a) \phi(a) + \int_a^b f'(u) \phi(2a - u) \, du
\]

\[
= - \int_a^b f'(t) \phi(t) \, dt + \int_a^b f'(u) \phi(2b - u) \, du + \int_a^b f'(u) \phi(2a - u) \, du
\]

\[
= - \int_a^b f'(t) \phi(t) \, dt - \int_0^{2b-a} -f'(2b - t) \phi(t) \, dt - \int_{2a-b}^a -f'(2a - t) \phi(t) \, dt
\]

\[
= - \int_{2a-b}^{2b-a} f'(t) \phi(t) \, dt
\]

where \( f'(t) \) is given in [1111].

**Definition 41.1.14** Let \( V \) be a Banach space and let \( H \) be a Hilbert space. (Typically \( H = L^2(\Omega) \)). Suppose \( V \subseteq H \) is dense in \( H \) meaning that the closure in \( H \) of \( V \) gives \( H \). Then it is often the case that \( H \) is identified with its dual space, and then because of the density of \( V \) in \( H \), it is possible to write

\[
V \subseteq H = H' \subseteq V'
\]

When this is done, \( H \) is called a pivot space. Another notation which is often used is \( \langle f, g \rangle \) to denote \( f(g) \) for \( f \in V' \) and \( g \in V \). This may also be written as \( \langle f, g \rangle_{V', V} \). Another term is that \( V \subseteq H = H' \subseteq V' \) is called a Gelfand triple.

The next theorem is an example of a trace theorem. In this theorem, \( f \in L^p(0, T; V) \) while \( f' \in L^p(0, T; V') \). It makes no sense to consider the initial values of \( f \) in \( V \) because it is not even continuous with values in \( V \). However, because of the derivative of \( f \) it will turn out that \( f \) is continuous with values in a larger space and so it makes sense to consider initial values of \( f \) in this other space. This other space is called a trace space.

**Theorem 41.1.15** Let \( V \) and \( H \) be a Banach space and Hilbert space as described in Definition 41.1.14. Suppose \( f \in L^p(0, T; V) \) and \( f' \in L^p(0, T; V') \). Then \( f \) is
a.e. equal to a continuous function mapping \([0, T]\) to \(H\). Furthermore, there exists \(f(0) \in H\) such that

\[
\frac{1}{2} |f(t)|_H^2 - \frac{1}{2} |f(0)|_H^2 = \int_0^t \langle f'(s), f(s) \rangle \, ds, \tag{41.1.3}
\]

and for all \(t \in [0, T]\),

\[
\int_0^t f'(s) \, ds \in H, \tag{41.1.4}
\]

and for a.e. \(t \in [0, T]\),

\[
f(t) = f(0) + \int_0^t f'(s) \, ds \text{ in } H, \tag{41.1.5}
\]

Here \(f'\) is being taken in the sense of \(V'\) valued distributions and \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(p \geq 2\).

**Proof:** Let \(\Psi \in C_\infty_c (\mathbb{R}, 2T)\) satisfy \(\Psi(t) = 1\) if \(t \in [-T/2, 3T/2]\) and \(\Psi(t) \geq 0\). For \(t \in \mathbb{R}\), define

\[
\tilde{f}(t) = \begin{cases} \tilde{T}(t) \Psi(t) & \text{if } t \in [-T, 2T] \\ 0 & \text{if } t \notin [-T, 2T] \end{cases}
\]

and

\[
f_n(t) = \int_{-1/n}^{1/n} \tilde{f}(t-s) \phi_n(s) \, ds \tag{41.1.6}
\]

where \(\phi_n\) is a mollifier having support in \((-1/n, 1/n)\). Then by Minkowski’s inequality

\[
\left\| f_n - \tilde{f} \right\|_{L^p(\mathbb{R}; V)} = \left( \int_{\mathbb{R}} \left\| \int_{-1/n}^{1/n} \tilde{f}(t-s) \phi_n(s) \, ds \right\|_V^p \, dt \right)^{1/p} 
\]

\[
= \left( \int_{\mathbb{R}} \left( \int_{-1/n}^{1/n} \left\| \tilde{f}(t-s) \phi_n(s) \right\|_V^p \, dt \right)^{1/p} \right)^{1/p} 
\]

\[
\leq \left( \int_{\mathbb{R}} \left( \int_{-1/n}^{1/n} \left\| \tilde{f}(t-s) \right\|_V \phi_n(s) \, ds \right)^p \, dt \right)^{1/p} 
\]

\[
\leq \int_{-1/n}^{1/n} \phi_n(s) \left( \int_{\mathbb{R}} \left\| \tilde{f}(t-s) \right\|_V^p \, dt \right)^{1/p} \, ds 
\]

\[
\leq \int_{-1/n}^{1/n} \phi_n(s) \, ds = \varepsilon
\]

provided \(n\) is large enough. This follows from continuity of translation in \(L^p\) with Lebesgue measure. Since \(\varepsilon > 0\) is arbitrary, it follows \(f_n \to \tilde{f}\) in \(L^p(\mathbb{R}; V)\). Similarly,
$f_n \to f$ in $L^2(\mathbb{R}; H)$. This follows because $p \geq 2$ and the norm in $V$ and norm in $H$ are related by $|x|_H \leq C||x||_V$ for some constant, $C$. Now

$$\tilde{f}(t) = \begin{cases} 
\Psi(t) f(t) & \text{if } t \in [0, T], \\
\Psi(t) f(2T - t) & \text{if } t \in [T, 2T], \\
\Psi(t) f(-t) & \text{if } t \in [0, T], \\
0 & \text{if } t \notin [-T, 2T].
\end{cases}$$

An easy modification of the argument of Lemma 41.1.3 yields

$$\tilde{f}'(t) = \begin{cases} 
\Psi'(t) f(t) + \Psi(y) f'(t) & \text{if } t \in [0, T], \\
\Psi'(t) f(2T - t) - \Psi(t) f'(2T - t) & \text{if } t \in [T, 2T], \\
\Psi'(t) f(-t) - \Psi(t) f'(-t) & \text{if } t \in [-T, 0], \\
0 & \text{if } t \notin [-T, 2T].
\end{cases}$$

Recall

$$f_n(t) = \int_{-1/n}^{1/n} \tilde{f}(t - s) \phi_n(s) \, ds = \int_{-1/n}^{1/n} \tilde{f}(t) \phi_n(s) \, ds$$

Therefore,

$$f_n'(t) = \int_{-1/n}^{1/n} \tilde{f}'(t) \phi_n(s) \, ds = \int_{-1/n}^{1/n} \tilde{f}(t) \phi_n'(s) \, ds$$

and it follows from the first line above that $f_n'$ is continuous with values in $V$ for all $t \in \mathbb{R}$. Also note that both $f_1'$ and $f_n$ equal zero if $t \notin [-T, 2T]$ whenever $n$ is large enough. Exactly similar reasoning to the above shows that $f_n' \to f'$ in $L^p(\mathbb{R}; V')$.

Now let $\phi \in C_c^\infty(0, T)$. Then

$$\int_{-1/n}^{1/n} |f_n(t)|_H^2 \phi'(t) \, dt = \int_{-1/n}^{1/n} (f_n(t), f_n(t))_H \phi'(t) \, dt \quad (41.1.7)$$

$$= -\int_{-1/n}^{1/n} 2 \langle f_n'(t), f_n(t) \rangle \phi(t) \, dt = -\int_{-1/n}^{1/n} 2 \langle f_n'(t), f_n(t) \rangle \phi(t) \, dt$$

Now

$$\left| \int_{-1/n}^{1/n} \langle f_n'(t), f_n(t) \rangle \phi(t) \, dt - \int_{-1/n}^{1/n} \langle f'(t), f(t) \rangle \phi(t) \, dt \right|$$

$$\leq \int_{-1/n}^{1/n} (|f_n'(t) - f'(t), f_n(t)| + |f'(t) - f(t), f(t)|) \phi(t) \, dt.$$


From the first part of this proof which showed that \( f_n \to \hat{f} \) in \( L^p (\mathbb{R}; V) \) and \( f'_n \to \hat{f}' \) in \( L^{p'} (\mathbb{R}; V') \), an application of Holder’s inequality shows the above converges to 0 as \( n \to \infty \). Therefore, passing to the limit as \( n \to \infty \) in the \( 41.1.8 \)

\[
\int_{\mathbb{R}} |\hat{f}(t)|^2_H \phi'(t) \, dt = -\int_{\mathbb{R}} 2 \langle \hat{f}'(t), \hat{f}(t) \rangle \phi(t) \, dt
\]

which shows \( t \to |\hat{f}(t)|^2_H \) equals a continuous function a.e. and it also has a weak derivative equal to \( 2 \langle \hat{f}', \hat{f} \rangle \).

It remains to verify that \( \hat{f} \) is continuous on \([0, T]\). Of course \( \hat{f} = f \) on this interval. Let \( N \) be large enough that \( f_n (-T) = 0 \) for all \( n > N \). Then for \( m, n > N \) and \( t \in [-T, 2T] \)

\[
|f_n (t) - f_m (t)|^2_H = 2 \int_{-T}^{t} (f'_n (s) - f'_m (s), f_n (s) - f_m (s)) \, ds \\
= 2 \int_{-T}^{t} (f'_n (s) - f'_m (s), f_n (s) - f_m (s))_{V', V} \, ds \\
\leq 2 \int_{-T}^{t} |f'_n (s) - f'_m (s)|_{V'} |f_n (s) - f_m (s)|_{V} \, ds \\
\leq 2 ||f_n - f_m||_{L^p(\mathbb{R}; V')} ||f'_n - f'_m||_{L^p(\mathbb{R}; V')}
\]

which shows from the above that \( \{f_n\} \) is uniformly Cauchy on \([-T, 2T]\) with values in \( H \). Therefore, there exists \( g \) a continuous function defined on \([-T, 2T]\) having values in \( H \) such that

\[
\lim_{n \to \infty} \max \{|f_n (t) - g (t)|^2_H ; t \in [-T, 2T]\} = 0.
\]

However, \( g = \hat{f} \) a.e. because \( f_n \) converges to \( f \) in \( L^p (0, T; V) \). Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that \( V \subseteq H = H' \subseteq V' \) and Theorem 41.1.9, there exists \( f (0) \in V' \) such that for a.e. \( t \),

\[
f (t) = f (0) + \int_{0}^{t} f'(s) \, ds \text{ in } V'
\]

Now \( g = f \) a.e. and \( g \) is continuous with values in \( H \) hence continuous with values in \( V' \) and so

\[
g (t) = f (0) + \int_{0}^{t} f'(s) \, ds \text{ in } V'
\]

for all \( t \). Since \( g \) is continuous with values in \( H \) it is continuous with values in \( V' \). Taking the limit as \( t \downarrow 0 \) in the above, \( g (a) = \lim_{t \to 0^+} g (t) = f (0) \), showing that \( f (0) \in H \). Therefore, for a.e. \( t \),

\[
f (t) = f (0) + \int_{0}^{t} f'(s) \, ds \text{ in } H, \int_{0}^{t} f'(s) \, ds \in H.
\]
Note that if \( f \in \mathcal{L}^p (0, T; V) \) and \( f' \in \mathcal{L}^p (0, T; V') \), then you can consider the initial value of \( f \) and it will be in \( H \). What if you start with something in \( H \)? Is it an initial condition for a function \( f \in \mathcal{L}^p (0, T; V) \) such that \( f' \in \mathcal{L}^p (0, T; V') \)? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if \( V \) is a closed subspace of the Sobolev space, \( W^{1,p} (\Omega) \) which contains \( W^{1,2}_0 (\Omega) \) for \( p \geq 2 \) and \( H = \mathcal{L}^2 (\Omega) \) the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

### 41.2 An Important Formula

It is not necessary to have \( p > 2 \) in order to do the sort of thing just described. Here is a major result which will have a much more difficult stochastic version presented later. First is a simple version of an approximation theorem of Doob.

**Lemma 41.2.1** Let \( Y : [0, T] \to E \) be \( \mathcal{B} ([0, T]) \) measurable and suppose

\[
Y \in \mathcal{L}^p (0, T; E) \equiv K, \ p \geq 1
\]

Then there exists a sequence of nested partitions, \( \mathcal{P}_k \subseteq \mathcal{P}_{k+1} \),

\[
\mathcal{P}_k = \{ t^k_0, \cdots, t^k_{m_k} \}
\]

such that the step functions given by

\[
Y^r_k (t) = \sum_{j=1}^{m_k} Y (t^k_j) \chi_{(t^k_j, t^k_{j+1})} (t)
\]

\[
Y^l_k (t) = \sum_{j=1}^{m_k} Y (t^k_{j-1}) \chi_{(t^k_{j-1}, t^k_j)} (t)
\]

both converge to \( Y \) in \( K \) as \( k \to \infty \) and

\[
\lim_{k \to \infty} \max \{|t^k_j - t^k_{j+1}| : j \in \{0, \cdots, m_k\} \} = 0.
\]

Also, each \( Y (t^k_j), Y (t^k_{j-1}) \) is in \( E \). One can also assume that \( Y (0) = 0 \). The mesh points \( \{t^k_j\}_{j=0}^{m_k} \) can be chosen to miss a given set of measure zero. In addition to this, we can assume that

\[
|t^k_j - t^k_{j-1}| = 2^{-n_k}
\]

except for the case where \( j = 1 \) or \( j = m_n_k \) when this might not be so. In the case of the last subinterval defined by the partition, we can assume

\[
|t^k_m - t^k_{m-1}| = |T - t^k_{m-1}| \geq 2^{-(n_k + 1)}
\]
41.2. AN IMPORTANT FORMULA

Proof: For \( t \in \mathbb{R} \) let \( \gamma_n(t) \equiv k/2^n, \delta_n(t) \equiv (k + 1)/2^n \), where \( t \in (k/2^n, (k + 1)/2^n] \), and \( 2^{-n} < T/4 \). Also suppose \( Y \) is defined to equal 0 on \([0, T]^{C} \). Then \( t \to \| Y(t) \| \) is in \( L^p(\mathbb{R}) \). Therefore by continuity of translation, as \( n \to \infty \) it follows that for \( t \in [0, T] \),

\[
\int_{\mathbb{R}} \| Y(\gamma_n(t) + s) - Y(t + s) \|_E^p \, ds \to 0
\]

The above is dominated by

\[
\int_{\mathbb{R}} 2^{p-1} (||Y(s)||^p + ||Y(s)||^p) \mathcal{X}_{[-2T, 2T]}(s) \, ds
\]

\[
= \int_{-2T}^{2T} 2^{p-1} (||Y(s)||^p + ||Y(s)||^p) \, ds < \infty
\]

Therefore,

\[
\lim_{n \to \infty} \int_{-2T}^{2T} \left( \int_{\mathbb{R}} \| Y(\gamma_n(t) + s) - Y(t + s) \|_E^p \, ds \right) \, dt = 0
\]

by the dominated convergence theorem. Now Fubini. The above equals

\[
\int_{\mathbb{R}} \int_{-2T}^{2T} \| Y(\gamma_n(t) + s) - Y(t + s) \|_E^p \, dtds
\]

Change the variables on the inside.

\[
\int_{\mathbb{R}} \int_{-2T}^{2T+\delta} \| Y(\gamma_n(t) - s) + Y(t) \|_E^p \, dtds
\]

Since \( \gamma_n(t - s) + s \) is within \( 2^{-n} \) of \( t \) and \( Y(t) \) vanishes if \( t \notin [0, T] \), this reduces to

\[
\int_{\mathbb{R}} \int_{-2T}^{2T} \| Y(\gamma_n(t) + s) - Y(t) \|_E^p \, dt \, ds.
\]

This converges to 0 as \( n \to \infty \) as was shown above. Therefore,

\[
\int_{0}^{T} \int_{0}^{T} \| Y(\gamma_n(t) + s) - Y(t) \|_E^p \, dtds
\]

also converges to 0 as \( n \to \infty \). The only problem is that \( \gamma_n(t - s) + s \geq t - 2^{-n} \) and so \( \gamma_n(t - s) + s \) could be less than 0 for \( t \in [0, 2^{-n}] \). Since this is an interval whose measure converges to 0 it follows

\[
\int_{0}^{T} \int_{0}^{T} \| Y(\gamma_n(t - s)^+) - Y(t) \|_E^p \, dtds
\]

converges to 0 as \( n \to \infty \). Let

\[
m_n(s) = \int_{0}^{T} \| Y(\gamma_n(t - s)^+) - Y(t) \|_E^p \, dt
\]
Then
\[ P \left( \left\lvert m_n (s) > \lambda \right\rvert \right) \leq \frac{1}{\lambda} \int_0^T m_n (s) \, ds. \]
It follows there exists a subsequence \( n_k \) such that
\[ P \left( \left\lvert m_{n_k} (s) > \frac{1}{k} \right\rvert \right) < 2^{-k} \]
Hence by the Borel Cantelli lemma, there exists a set of measure zero \( N \) such that for \( s \notin N \),
\[ m_{n_k} (s) \leq 1/k \]
for all \( k \) sufficiently large. Picking \( s_k \notin N \),
\[ Y_k^i (t) \equiv Y \left( (\gamma_{n_k} (t - s_k) + s_k)^+ \right) \]
Then \( t \to Y \left( (\gamma_{n_k} (t - s_k) + s_k)^+ \right) \) is a step function of the sort described above. Of course you can always simply define \( Y_k^i (0) \equiv 0 \). This is because the interval affected has length which converges to 0 as \( k \to \infty \). The jumps in \( t \to \gamma_{n_k} (t - s_k) \) determine the mesh points of the partition. By picking \( s_k \) appropriately, you can have each of these mesh points miss a given set of measure zero except for the first and last point. This is because when you slide \( s_k \) it just moves the mesh points of \( P_k \) except for the first point and last point. Let \( N_1 \) be a set of measure zero and let \( (a, b) \subseteq [0, T] \). Now let \( s \) move through \( (a, b) \) and denote by \( A_j \) the corresponding set of points obtained by the \( j \)th mesh point. Thus \( A_j \) has positive measure and so it is not contained in \( N_1 \). Let \( S_j \) be the points of \( (a, b) \) which correspond to \( A_j \cap N_1 \). Thus \( S_j \) has measure 0. Just pick \( s_k \in (a, b) \setminus \cup_j S_j \). You can also choose \( s_k \) such that
\[ T - s_k - \gamma_{n_k} (T - s_k) > 2^{-(n_k + 1)} \]
which will cause the last condition mentioned above to hold.
To get the other sequence of step functions, just use a similar argument with \( \delta_n \) in place of \( \gamma_n \).

**Theorem 41.2.2** Let \( V \subseteq H = H' \subseteq V' \) be a Gelfand triple and suppose \( Y \in L^p' (0, T; V') \equiv K' \) and

\[ X (t) = X_0 + \int_0^t Y (s) \, ds \text{ in } V' \quad (41.2.8) \]

where \( X_0 \in H \), and it is known that \( X \in L^p (0, T, V) \equiv K \) for \( p > 1 \). Then \( t \to X (t) \) is in \( C ([0, T], H) \) and also
\[ \frac{1}{2} \| X (t) \|^2_H = \frac{1}{2} \| X_0 \|^2_H + \int_0^t (Y (s), X (s)) \, ds \]
41.2. AN IMPORTANT FORMULA

Proof: By Lemma 41.2.1 there exists a sequence of uniform partitions \( (t^n_k)_{k=0}^{m_n} = \mathcal{P}_n, \mathcal{P}_n \subseteq \mathcal{P}_{n+1} \), of \([0, T]\) such that the step functions

\[
\sum_{k=0}^{m_n-1} X(t^n_k) \mathcal{X}_{(t^n_k, t^n_{k+1}]}(t) = X^l(t)
\]

\[
\sum_{k=0}^{m_n-1} X(t^n_{k+1}) \mathcal{X}_{(t^n_{k+1}, t^n_{k+1}]}(t) = X^r(t)
\]

converge to \(X\) in \(K\) and in \(L^2([0, T], H)\).

Lemma 41.2.3 Let \(s < t\). Then for \(X, Y\) satisfying (41.2.8)

\[
|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du - |X(t) - X(s)|^2 \quad (41.2.9)
\]

Proof: It follows from the following computations

\[
X(t) - X(s) = \int_s^t Y(u) \, du
\]

\[
-|X(t) - X(s)|^2 = -|X(t)|^2 + 2 \langle X(t), X(s) \rangle - |X(s)|^2
\]

\[
= -|X(t)|^2 + 2 \left( X(t), X(t) - \int_s^t Y(u) \, du \right) - |X(s)|^2
\]

\[
= -|X(t)|^2 + 2 |X(t)|^2 - 2 \left( \int_s^t Y(u) \, du, X(t) \right) - |X(s)|^2
\]

Hence

\[
|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du - |X(t) - X(s)|^2 \]

Lemma 41.2.4 In the above situation,

\[
\sup_{t \in [0, T]} |X(t)|_H \leq C(\|Y\|_{K'}, \|X\|_K)
\]

Also, \(t \to X(t)\) is weakly continuous with values in \(H\).

Proof: From the above formula applied to the \(k^{th}\) partition of \([0, T]\) described above,

\[
|X(t_m)|^2 - |X_0|^2 = \sum_{j=0}^{m-1} |X(t_{j+1})|^2 - |X(t_j)|^2
\]
\[ = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) \rangle \, du - \|X(t_{j+1}) - X(t_j)\|_H^2 \]

\[ = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X_r^j(u) \rangle \, du - \|X(t_{j+1}) - X(t_j)\|_H^2 \]

Thus, discarding the negative terms and denoting by \( P_k \) the \( k^{th} \) of these partitions,

\[
\sup_{t_j \in P_k} \|X(t_j)\|_H^2 \leq \|X_0\|_H^2 + 2 \int_0^T \|Y(u)\|_V \|X_r^j(u)\|_V \, du
\]

\[
\leq \|X_0\|_H^2 + 2 \left( \int_0^T \|Y(u)\|^{p'}_V \, du \right)^{1/p'} \left( \int_0^T \|X_r^j(u)\|^{p}_V \, du \right)^{1/p} \leq C (\|Y\|_{K'}, \|X\|_K)
\]

because these partitions are chosen such that

\[
\lim_{k \to \infty} \left( \int_0^T \|X_r^j(u)\|^{p}_V \, du \right)^{1/p} = \left( \int_0^T \|X(u)\|^{p}_V \, du \right)^{1/p}
\]

and so these are bounded. This has shown that for the dense subset of \([0, T] \), \( D = \cup_k P_k \),

\[
\sup_{t \in D} \|X(t)\| < C (\|Y\|_{K'}, \|X\|_K)
\]

Now let \( \{g_k\}_{k=1}^\infty \) be linearly independent vectors of \( V \) whose span is dense in \( V \). This is possible because \( V \) is separable. Then let \( \{e_j\}_{j=1}^\infty \) be an orthonormal basis for \( H \) such that \( e_k \in \text{span} \{g_1, \ldots, g_k\} \) and each \( g_k \in \text{span} \{e_1, \ldots, e_k\} \). This is done with the Gram Schmidt process. Then it follows that span \( \{e_k\}_{k=1}^\infty \) is dense in \( V \).

I claim

\[ |y|^2_H = \sum_{j=1}^\infty |\langle y, e_j \rangle|^2. \]

This is certainly true if \( y \in H \) because

\[ \langle y, e_j \rangle = \langle y, e_j \rangle_H \]

If \( y \notin H \), then the series must diverge since otherwise, you could consider the infinite sum

\[ \sum_{j=1}^\infty \langle y, e_j \rangle e_j \in H \]

because

\[ \left| \sum_{j=p}^q \langle y, e_j \rangle e_j \right|^2 \leq \sum_{j=p}^q |\langle y, e_j \rangle|^2 \to 0 \text{ as } p, q \to \infty. \]
Letting \( z = \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \), it follows that \( \langle y, e_j \rangle \) is the \( j \)th Fourier coefficient of \( z \) and that
\[
\langle z - y, v \rangle = 0
\]
for all \( v \in \text{span} \{ e_k \}_{k=1}^{\infty} \) which is dense in \( V \). Therefore, \( z = y \) in \( V' \) and so \( y \in H \).

It follows
\[
|X(t)|^2 = \sup_{n} \sum_{j=1}^{n} |\langle X(t), e_j \rangle|^2
\]
which is just the sup of continuous functions of \( t \). Therefore, \( t \to |X(t)|^2 \) is lower semicontinuous. It follows that for any \( t \), letting \( t_j \to t \) for \( t_j \in D \),
\[
|X(t)|^2 \leq \lim_{j \to \infty} \inf |X(t_j)|^2 \leq C (\|Y\|_{K'}, \|X\|_K)
\]
This proves the first claim of the lemma.

Consider now the claim that \( t \to X(t) \) is weakly continuous. Letting \( v \in V \),
\[
\lim_{t \to s} \langle X(t), v \rangle = \lim_{t \to s} \langle X(t), v \rangle = \langle X(s), v \rangle = \langle X(s), v \rangle
\]
Since it was shown that \( |X(t)| \) is bounded independent of \( t \), and since \( V \) is dense in \( H \), the claim follows.

Now
\[
- \sum_{j=0}^{m-1} |X(t_{j+1}) - X(t_j)|^2_H = |X(t_m)|^2 - |X_0|^2 - \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X_k(u) \rangle du
\]
\[
= |X(t_m)|^2 - |X_0|^2 - 2 \int_{0}^{t_m} \langle Y(u), X_k(u) \rangle du
\]
Thus, since the partitions are nested, eventually \( |X(t_m)|^2 \) is constant for all \( k \) large enough and the integral term converges to
\[
\int_{0}^{t_m} \langle Y(u), X(u) \rangle du
\]
It follows that the term on the left does converge to something. It just remains to consider what it does converge to. However, from the equation solved by \( X \),
\[
X(t_{j+1}) - X(t_j) = \int_{t_j}^{t_{j+1}} Y(u) du
\]
Therefore, this term is dominated by an expression of the form
\[
\sum_{j=0}^{m-1} \left( \int_{t_j}^{t_{j+1}} Y(u) du, X(t_{j+1}) - X(t_j) \right)
\]
\[
= \sum_{j=0}^{m-1} \left( \int_{t_j}^{t_{j+1}} Y(u) du, X(t_{j+1}) - X(t_j) \right)
\]
\[
= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) - X(t_j) \rangle du
\]
\[
\sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} (Y(u), X(t_{j+1})) - \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} (Y(u), X(t_j))
\]

\[
= \int_0^T \langle Y(u), X^r(u) \rangle \, du - \int_0^T \langle Y(u), X^l(u) \rangle \, du
\]

However, both \(X^r\) and \(X^l\) converge to \(X\) in \(K = L^p(0, T, V)\). Therefore, this term must converge to 0. Passing to a limit, it follows that for all \(t \in D\), the desired formula holds. Thus, for such \(t\),

\[
|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle \, ds
\]

It remains to verify that this holds for all \(t\). Let \(t \notin D\) and let \(t(k) \in \mathcal{P}_k\) be the largest point of \(\mathcal{P}_k\) which is less than \(t\). Suppose \(t(m) \leq t(k)\) so that \(m \leq k\). Then

\[
X(t(m)) = X_0 + \int_{t(m)}^t Y(s) \, ds,
\]

a similar formula for \(X(t(k))\). Thus for \(t > t(m)\),

\[
X(t) - X(t(m)) = \int_{t(m)}^t Y(s) \, ds
\]

which is the same sort of thing already looked at except that it starts at \(t(m)\) rather than at 0 and \(X_0 = 0\). Therefore, since \(V\) is dense in \(H\), it follows that \(\xi(t) = X(t)\).

Now for every \(t \in D\), it was shown above that

\[
|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle \, ds
\]
41.3. THE IMPLICIT CASE

Thus, using what was just shown, if \( t \notin D \) and \( t_k \to t \),

\[
|X(t)|^2 = \lim_{k \to \infty} |X(t_k)|^2 = \lim_{k \to \infty} \left( |X_0|^2 + 2 \int_0^{t_k} \langle Y(s), X(s) \rangle \, ds \right)
\]

\[
= |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle \, ds
\]

which proves the desired formula. From this it follows right away that \( t \to X(t) \) is continuous into \( H \) because it was just shown that \( t \to |X(t)| \) is continuous and \( t \to X(t) \) is weakly continuous. Since Hilbert space is uniformly convex, this implies the \( t \to X(t) \) is continuous. To see this in the special case of Hilbert space,

\[
|X(t) - X(s)|^2 = |X(t)|^2 - 2 \langle X(s), X(t) \rangle + |X(s)|^2
\]

Then \( \lim_{t \to s} \left( |X(t)|^2 - 2 \langle X(s), X(t) \rangle + |X(s)|^2 \right) = 0 \) by weak convergence of \( X(t) \) to \( X(s) \) and the convergence of \( |X(t)|^2 \) to \( |X(s)|^2 \).

41.3 The Implicit Case

The above theorem can be generalized to the case where the formula is of the form

\[
BX(t) = BX_0 + \int_0^t Y(s) \, ds
\]

This involves an operator \( B \in \mathcal{L}(W, W') \) and \( B \) satisfies

\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle
\]

for

\[
V \subseteq W, \quad W' \subseteq V'
\]

Where \( V \) is dense in the Hilbert space \( W \). Before giving the theorem, here is a technical lemma.

**Lemma 41.3.1** Suppose \( V, W \) are separable Banach spaces, \( W \) also a Hilbert space such that \( V \) is dense in \( W \) and \( B \in \mathcal{L}(W, W') \) satisfies

\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle, \quad B \neq 0.
\]

Then there exists a countable set \( \{e_i\} \) of vectors in \( V \) such that

\[
\langle Be_i, e_j \rangle = \delta_{ij}
\]

and for each \( x \in W \),

\[
\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,
\]

\[
\langle Be_i, e_j \rangle = \delta_{ij}
\]

and for each \( x \in W \),

\[
\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,
\]

\[
\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,
\]
and also
\[ Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i, \]
the series converging in \(W'\).

**Proof:** Let \(\{g_k\}_{k=1}^{\infty}\) be linearly independent vectors of \(V\) whose span is dense in \(V\). This is possible because \(V\) is separable. Thus, their span is also dense in \(W\). Let \(n_1\) be the first index such that \(\langle Bg_{n_1}, g_{n_1} \rangle \neq 0\).

**Claim:** If there is no such index, then \(B = 0\).

**Proof of claim:** First note that if there is no such first index, then if \(x = \sum_{i=1}^{k} a_i g_i |\langle Bx, x \rangle| = \sum_{i \neq j} |a_i| |a_j| \langle Bg_i, g_j \rangle \leq \sum_{i \neq j} |a_i||a_j| \langle Bg_i, g_j \rangle^{1/2} \langle Bg_j, g_j \rangle^{1/2} = 0\)

Therefore, if \(x\) is given, you could take \(x_k\) in the span of \(\{g_1, \cdots, g_k\}\) such that \(\|x_k - x\|_W \to 0\). Then
\[ |\langle Bx, y \rangle| = \lim_{k \to \infty} |\langle Bx_k, y \rangle| \leq \lim_{k \to \infty} \langle Bx_k, x_k \rangle^{1/2} \langle By, y \rangle^{1/2} = 0 \]
because \(\langle Bx_k, x_k \rangle\) is zero by what was just shown.

Thus assume there is such a first index. Let
\[ e_1 = \frac{g_{n_1}}{\langle Bg_{n_1}, g_{n_1} \rangle^{1/2}} \]
Then \(\langle Be_1, e_1 \rangle = 1\). Now if you have constructed \(e_j\) for \(j \leq k\),
\[ e_j \in \text{span} \{g_{n_1}, \cdots, g_n\}, \quad \langle Be_i, e_j \rangle = \delta_{ij}, \]
g_{n_{j+1}} being the first for which
\[ \left\langle Bg_{n_{j+1}} - \sum_{i=1}^{j} \langle Bg_{n_{j+1}}, e_i \rangle Be_i, g_{n_{j+1}} - \sum_{i=1}^{j} \langle Bg_{n_{j+1}}, e_i \rangle e_i \right\rangle \neq 0, \]
and
\[ \text{span} \{g_{n_1}, \cdots, g_n\} = \text{span} \{e_1, \cdots, e_k\}, \]
let \(g_{n_{k+1}}\) be such that \(g_{n_{k+1}}\) is the first in the list \(\{g_n\}\) such that
\[ \left\langle Bg_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle Be_i, g_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle e_i \right\rangle \neq 0 \]
Note the difference between this and the Gram Schmidt process. Here you don’t necessarily use all of the $g_k$ due to the possible degeneracy of $B$.

**Claim:** If there is no such first $g_{nk+1}$, then $B(\text{span}(e_i, \ldots, e_k)) = BW$ so in this case, $\{Be_i\}_{i=1}^k$ is actually a basis for $BW$.

**Proof:** Let $x \in W$. Let $x_r \in \text{span}(g_1, \ldots, g_r)$, $r > n_k$ such that \(\lim_{r \to \infty} x_r = x\) in $W$. Then

$$x_r = \sum_{i=1}^k c_i^r e_i + \sum_{i \notin \{n_1, \ldots, n_k\}}^r d_i^r g_i \equiv y_r + z_r \quad (41.3.10)$$

If $l \notin \{n_1, \ldots, n_k\}$, then by the construction and the above assumption, for some $j \leq k$

$$\left\langle Bg_l - \sum_{i=1}^j (Bg_i, e_i) Be_i, g_l - \sum_{i=1}^j (Bg_i, e_i) e_i \right\rangle = 0 \quad (41.3.11)$$

If $l < n_k$, this follows from the construction. If the above is nonzero all $j \leq k$, then $l$ would have been chosen but it wasn’t. Thus

$$Bg_l = \sum_{i=1}^j (Bg_i, e_i) Be_i$$

If $l > n_k$, then by assumption, (41.3.11) holds for $j = k$. Thus, in any case, it follows that for each $l \notin \{n_1, \ldots, n_k\}$,

$$Bg_l \in B(\text{span}(e_i, \ldots, e_k)).$$

Now it follows from (41.3.11) that

$$Bx_r = \sum_{i=1}^k c_i^r Be_i + \sum_{i \notin \{n_1, \ldots, n_k\}}^r d_i^r Bg_i$$

$$= \sum_{i=1}^k c_i^r Be_i + \sum_{i \notin \{n_1, \ldots, n_k\}}^r d_i^r \sum_{j=1}^k c_j^r Be_j$$

and so $Bx_r \in B(\text{span}(e_i, \ldots, e_k))$. Then $Bx = \lim_{r \to \infty} Bx_r = \lim_{r \to \infty} By_r$ where $y_r \in \text{span}(e_i, \ldots, e_k)$. Say

$$Bx_r = \sum_{i=1}^k a_i^r Be_i$$

It follows easily that $\langle Bx_r, e_j \rangle = a_j^r$. (Act on $e_j$ by both sides and use $\langle Be_i, e_j \rangle = \delta_{ij}$.) Now since $x_r$ is bounded, it follows that these $a_j^r$ are also bounded. Hence, defining $y_r = \sum_{i=1}^k a_i^r e_i$, it follows that $y_r$ is bounded in $\text{span}(e_i, \ldots, e_k)$ and so, there exists a subsequence, still denoted by $r$ such that $y_r \to y \in \text{span}(e_i, \ldots, e_k)$. Therefore, $Bx = \lim_{r \to \infty} By_r = By$. In other words, $BW = B(\text{span}(e_i, \ldots, e_k))$ as claimed. This proves the claim.
If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

As long as the process does not stop, let

\[ e_{k+1} \equiv \frac{g_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle e_i}{\langle B \left( g_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle e_i \right), g_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle e_i \rangle}^{1/2} \]

Thus, as in the usual argument for the Gram Schmidt process, \( \langle Be_i, e_j \rangle = \delta_{ij} \) for \( i, j \leq k + 1 \). This is already known for \( i, j \leq k \). Letting \( l \leq k \), and using the orthogonality already shown,

\[
\langle Be_{k+1}, e_l \rangle = C \left( \langle Bg_{k+1}, e_l \rangle - \langle Bg_{n_{k+1}}, e_l \rangle \right) = 0
\]

Consider

\[
\langle Bg_p - B \left( \sum_{i=1}^{k} \langle Bg_p, e_i \rangle e_i \right), g_p - \sum_{i=1}^{k} \langle Bg_p, e_i \rangle e_i \rangle
\]

Either this equals 0 because \( p \) is never one of the \( n_k \) or eventually it equals 0 for some \( k \) because \( g_p = g_{n_k} \) for some \( n_k \) and so, from the construction, \( g_{n_k} = g_p \in \text{span } (e_1, \ldots, e_k) \) and therefore,

\[
g_p = \sum_{j=1}^{k} a_j e_j
\]

which requires easily that

\[
Bg_p = \sum_{i=1}^{k} \langle Bg_p, e_i \rangle Be_i,
\]

the above holding for all \( k \) large enough. It follows that for any \( x \in \text{span } \{ g_k \}_{k=1}^{\infty} \), (finite linear combination of vectors in \( \{ g_k \}_{k=1}^{\infty} \))

\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \tag{41.3.12}
\]

because for all \( k \) large enough,

\[
Bx = \sum_{i=1}^{k} \langle Bx, e_i \rangle Be_i
\]
Also note that for such \( x \in \text{span} \left( \{g_k\}_{k=1}^{\infty} \right) \),
\[
\langle Bx, x \rangle = \left\langle \sum_{i=1}^{k} \langle Bx, e_i \rangle B e_i, x \right\rangle = \sum_{i=1}^{k} \langle Bx, e_i \rangle \langle Bx, e_i \rangle = \sum_{i=1}^{k} |\langle Bx, e_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2
\]

Now for \( x \) arbitrary, let \( x_k \rightarrow x \) in \( W \) where \( x_k \in \text{span} \left( \{g_k\}_{k=1}^{\infty} \right) \). Then by Fatou’s lemma,
\[
\sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \leq \liminf_{k \rightarrow \infty} \sum_{i=1}^{\infty} |\langle Bx_k, e_i \rangle|^2 = \liminf_{k \rightarrow \infty} \langle Bx_k, x_k \rangle = \langle Bx, x \rangle \quad (41.3.13)
\]

Thus the series on the left converges. Then also, from the above inequality,
\[
\left| \left\langle \sum_{i=p}^{q} \langle Bx, e_i \rangle B e_i, y \right\rangle \right| \leq \sum_{i=p}^{q} |\langle Bx, e_i \rangle| |\langle Be_i, y \rangle|
\]
\[
\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=p}^{q} |\langle By, e_i \rangle|^2 \right)^{1/2}
\]
\[
\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |\langle By, e_i \rangle|^2 \right)^{1/2}
\]

By (41.3.13),
\[
\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \|B\| \|y\|_W^2 \right)^{1/2}
\]
\[
\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \|B\|^{1/2} \|y\|_W
\]

It follows that
\[
\sum_{i=1}^{\infty} \langle Bx, e_i \rangle B e_i \quad (41.3.14)
\]
converges in $W'$ because it was just shown that
\[
\left\| \sum_{i=p}^{q} \langle Bx, e_i \rangle Be_i \right\|_{W'} \leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \|B\|^{1/2}
\]
and it was shown above that \( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 < \infty \), so the partial sums of the series are a Cauchy sequence in $W'$. Also, the above estimate shows that for $\|y\| = 1$,
\[
\left\| \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i, y \right\| \leq \left( \sum_{i=1}^{\infty} |\langle By, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \|B\|^{1/2}
\]
and so
\[
\left\| \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \right\|_{W'} \leq \left( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \|B\|^{1/2} \tag{41.3.15}
\]
Now for $x$ arbitrary, let $x_k \in \text{span} \left\{ g_j \right\}_{j=1}^{\infty}$ and $x_k \to x$ in $W$. Then for a fixed $k$ large enough,
\[
\|Bx - \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \| \leq \|Bx - Bx_k\|
\]
\[
+ \left\| Bx_k - \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle Be_i \right\| + \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle Be_i - \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i
\]
\[
\leq \varepsilon + \sum_{i=1}^{\infty} \langle B(x_k - x), e_i \rangle Be_i,
\]
the term
\[
\left\| Bx_k - \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle Be_i \right\|
\]
equaling 0 by \textit{41.3.12}. From \textit{41.3.15} and \textit{41.3.13},
\[
\leq \varepsilon \|B\|^{1/2} \left( \sum_{i=1}^{\infty} |\langle B(x_k - x), e_i \rangle|^2 \right)^{1/2}
\]
\[
\leq \varepsilon \|B\|^{1/2} \langle B(x_k - x), x_k - x \rangle^{1/2} < 2\varepsilon
\]
whenever $k$ is large enough. Therefore,
\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i
\]
41.3. THE IMPLICIT CASE

in \( W' \). It follows that

\[
(Bx, x) = \lim_{k \to \infty} \left( \sum_{i=1}^{k} (Bx, e_i) Be_i, x \right) = \lim_{k \to \infty} \sum_{i=1}^{k} |(Bx, e_i)|^2 \equiv \sum_{i=1}^{\infty} |(Bx, e_i)|^2 \n
\]

**Theorem 41.3.2** Let \( V \subseteq W, W' \subseteq V' \) be separable Banach spaces, \( W \) a separable Hilbert space, and let \( Y \in L^p (0, T; V') \equiv K' \) and

\[
BX (t) = BX_0 + \int_0^t Y (s) \, ds \, in \, V' \quad (41.3.16)
\]

where \( X_0 \in W \), and it is known that \( X \in L^p (0, T, V) \equiv K \) for \( p > 1 \). Also assume \( X \in L^2 (0, T, W) \). Then \( t \to BX (t) \) is in \( C ([0, T], W') \) and also

\[
\frac{1}{2} \langle BX (t), X (t) \rangle = \frac{1}{2} \langle BX_0, X_0 \rangle + \int_0^t \langle Y (s), X (s) \rangle \, ds
\]

**Proof:** By Lemma 41.2.2 there exists a sequence of uniform partitions \( \{ t^n_k \}_{k=0}^{m_n} = \mathcal{P}_n, \mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \) of \([0, T]\) such that the step functions

\[
\sum_{k=0}^{m_n-1} X (t^n_k) \mathcal{X}_{(t^n_k, t^n_{k+1})} (t) = X^l (t)
\]

\[
\sum_{k=0}^{m_n-1} X (t^n_{k+1}) \mathcal{X}_{(t^n_k, t^n_{k+1})} (t) = X^r (t)
\]

converge to \( X \) in \( K \) and also \( BX^l, BX^r \to BX \) in \( L^2 ([0, T], W') \).

**Lemma 41.3.3** Let \( s < t \). Then for \( X, Y \) satisfying (41.3.17)

\[
\langle BX (t), X (t) \rangle = \langle BX (s), X (s) \rangle + 2 \int_s^t \langle Y (u), X (t) \rangle \, du - \langle B (X (t) - X (s)), (X (t) - X (s)) \rangle \quad (41.3.17)
\]

**Proof:** It follows from the following computations

\[
B (X (t) - X (s)) = \int_s^t Y (u) \, du
\]

and so

\[
2 \int_s^t \langle Y (u), X (t) \rangle \, du - \langle B (X (t) - X (s)), (X (t) - X (s)) \rangle
\]

\[
= 2 \langle B (X (t) - X (s)), X (t) \rangle - \langle B (X (t) - X (s)), (X (t) - X (s)) \rangle
\]

\[
= 2 \langle BX (t), X (t) \rangle - 2 \langle BX (s), X (t) \rangle - \langle BX (t), X (t) \rangle
\]

\[
+ 2 \langle BX (s), X (t) \rangle - \langle BX (s), X (s) \rangle
\]

\[
= 2 \langle BX (t), X (t) \rangle - 2 \langle BX (s), X (t) \rangle - \langle BX (t), X (t) \rangle
\]

\[
+ 2 \langle BX (s), X (t) \rangle - \langle BX (s), X (s) \rangle
\]
Thus
\[ \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle = 2 \int_s^t \langle Y(u), X(t) \rangle du - \langle B (X(t) - X(s)), (X(t) - X(s)) \rangle \]

**Lemma 41.3.4** In the above situation,
\[ \sup_{t \in [0, T]} \langle BX(t), X(t) \rangle \leq C \left( \|Y\|_{K'}, \|X\|_K \right) \]

Also, \( t \to BX(t) \) is weakly continuous with values in \( W' \).

**Proof:** From the above formula applied to the \( k \)-th partition of \([0, T]\) described above,
\[ \langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle = \sum_{j=0}^{m-1} \langle BX(t_{j+1}), X(t_{j+1}) \rangle - \langle BX(t_j), X(t_j) \rangle \]
\[ = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) \rangle du - \langle B (X(t_{j+1}) - X(t_j)), X(t_{j+1}) - X(t_j) \rangle \]
\[ = \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X_k^r(u) \rangle du - \langle B (X(t_{j+1}) - X(t_j)), X(t_{j+1}) - X(t_j) \rangle \]

Thus, discarding the negative terms and denoting by \( P_k \) the \( k \)-th of these partitions,
\[ \sup_{t_j \in P_k} \langle BX(t_j), X(t_j) \rangle \leq \langle BX_0, X_0 \rangle + 2 \int_0^T \|Y(u)\|_{V'} \|X_k^r(u)\|_V du \]
\[ \leq \langle BX_0, X_0 \rangle + 2 \left( \int_0^T \|Y(u)\|_{V'}^p du \right)^{1/p'} \left( \int_0^T \|X_k^r(u)\|_V^p du \right)^{1/p} \]
\[ \leq C \left( \|Y\|_{K'}, \|X\|_K \right) \]

because these partitions are chosen such that
\[ \lim_{k \to \infty} \left( \int_0^T \|X_k^r(u)\|_V^p du \right)^{1/p} = \left( \int_0^T \|X(u)\|_{V'}^p du \right)^{1/p} \]
and so these are bounded. This has shown that for the dense subset of \([0, T]\),
\[ D \equiv \bigcup_k P_k, \]
\[ \sup_{t \in D} \langle BX(t), X(t) \rangle < C \left( \|Y\|_{K'}, \|X\|_K \right) \]

From Lemma H.3.1 above, there exists \( \{e_i\} \subseteq V \) such that \( \langle Be_i, e_j \rangle = \delta_{ij} \) and
\[ \langle BX(t), X(t) \rangle = \sum_{k=1}^{\infty} |\langle BX(t), e_i \rangle|^2 = \sup_m \sum_{k=1}^{m} |\langle BX(t), e_i \rangle|^2 \]

Since each \( e_i \in V \), and since \( t \to BX(t) \) is continuous into \( V' \) thanks to the formula H.3.16 it follows that \( t \to \sum_{k=1}^{m} |\langle BX(t), e_i \rangle| \) is continuous and so \( t \to \langle BX(t), X(t) \rangle \) is the sup of continuous functions. Therefore, this function of \( t \) is lower semicontinuous. Since \( D \) is dense in \([0, T]\), it follows that for all \( t \),
\[ \langle BX(t), X(t) \rangle \leq C \left( \|Y\|_{K'}, \|X\|_K \right) \]

It only remains to verify the claim about weak continuity.
Consider now the claim that \( t \to BX(t) \) is weakly continuous. Letting \( v \in V \),
\[ \lim_{t \to s} \langle BX(t), v \rangle = \langle BX(s), v \rangle = \langle BX(s), v \rangle \quad \text{(41.3.18)} \]

The limit follows from the formula H.3.16 which implies \( t \to BX(t) \) is continuous into \( V' \). Now
\[ \|BX(t)\| = \sup_{\|v\| \leq 1} |\langle BX(t), v \rangle| \leq \langle Bv, v \rangle^{1/2} \langle BX(t), X(t) \rangle^{1/2} \]
which was shown to be bounded for \( t \in [0, T] \). Now let \( w \in W \). Then
\[ |\langle BX(t), w \rangle - \langle BX(s), w \rangle| \leq |\langle BX(t) - BX(s), w - v \rangle| + |\langle BX(t) - BX(s), v \rangle| \]

Then the first term is less than \( \varepsilon \) if \( v \) is close enough to \( w \) and the second converges to 0 so H.3.16 holds for all \( v \in W \) and so this shows the weak continuity. ■

Now pick \( t \in D \), the union of all the mesh points. Then for all \( k \) large enough, \( t \in P_k \). Say \( t = t_m \). From Lemma H.3.8

\[ - \sum_{j=0}^{m-1} \langle B(X(t_{j+1}) - X(t_j)), (X(t_{j+1}) - X(t_j)) \rangle = \]
\[ \langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle - 2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X_k(u) \rangle \, du \]

Thus, \( \langle BX(t_m), X(t_m) \rangle \) is constant for all \( k \) large enough and the integral term converges to
\[ \int_0^{t_m} \langle Y(u), X(u) \rangle \, du \]
It follows that the term on the left does converge to something as $k \to \infty$. It just remains to consider what it does converge to. However, from the equation solved by $X$,

$$BX (t_{j+1}) - BX (t_j) = \int_{t_j}^{t_{j+1}} Y (u) \, du$$

Therefore, this term is dominated by an expression of the form

$$\sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y (u) \, du, X (t_{j+1}) - X (t_j) \right)$$

$$= \sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y (u) \, du, X (t_{j+1}) - X (t_j) \right)$$

$$= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y (u), X (t_{j+1}) - X (t_j) \rangle \, du$$

$$= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y (u), X (t_{j+1}) \rangle - \int_{t_j}^{t_{j+1}} \langle Y (u), X (t_j) \rangle \, du$$

$$= \int_{t_m}^{t_k} \langle Y (u), X^r (u) \rangle \, du - \int_{t_m}^{t_k} \langle Y (u), X^l (u) \rangle \, du$$

However, both $X^r$ and $X^l$ converge to $X$ in $K = L^p (0, T, V)$. Therefore, this term must converge to 0. Passing to a limit, it follows that for all $t \in D$, the desired formula holds. Thus, for such $t \in D$,

$$\langle BX (t), X (t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y (u), X (u) \rangle \, du$$

It remains to verify that this holds for all $t$. Let $t \not\in D$ and let $t (k) \in P_k$ be the largest point of $P_k$ which is less than $t$. Suppose $t (m) \leq t (k)$ so that $m \leq k$. Then

$$BX (t (m)) = BX_0 + \int_0^{t (m)} Y (s) \, ds,$$

a similar formula for $X (t (k))$. Thus for $t > t (m)$,

$$BX (t) - BX (t (m)) = \int_{t (m)}^t Y (s) \, ds$$

which is the same sort of thing already looked at except that it starts at $t (m)$ rather than at 0 and $X_0 = 0$. Therefore,

$$\langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle = 2 \int_{t (m)}^{t (k)} \langle Y (s), X (s) - X (t (m)) \rangle \, ds$$
Thus, for \( m \leq k \)
\[
\lim_{m,k \to \infty} \langle B (X (t (k))) - X (t (m)), X(t (k)) - X (t (m)) \rangle = 0 \quad (41.3.19)
\]

Hence \( \{BX (t (k))\}_{k=1}^{\infty} \) is a convergent sequence in \( W' \) because
\[
|\langle B (X (t (k))) - X (t (m)), y \rangle| \\
\leq \langle B (X (t (k))) - X (t (m)), X (t (k)) - X (t (m)) \rangle^{1/2} |B|_{y}^{1/2} \\
\leq \langle B (X (t (k))) - X (t (m)), X (t (k)) - X (t (m)) \rangle^{1/2} \|B\|^{1/2} \|y\|_{W}
\]

Does it converge to \( BX (t) \)? Let \( \xi (t) \in W' \) be what it does converge to. Let \( v \in V \). Then
\[
\langle \xi (t), v \rangle = \lim_{k \to \infty} \langle BX (t (k)), v \rangle = \lim_{k \to \infty} \langle BX (t (k)), v \rangle = \langle BX (t), v \rangle
\]
because it is known that \( t \to BX (t) \) is continuous into \( V' \). It is also known that \( BX (t) \in W' \subseteq V' \) and that the \( BX (t) \) for \( t \in [0, T] \) are uniformly bounded in \( W' \). Therefore, since \( V \) is dense in \( W \), it follows that \( \xi (t) = BX (t) \).

Now for every \( t \in D \), it was shown above that
\[
\langle BX (t), X (t) \rangle = \langle BX_{0}, X_{0} \rangle + 2 \int_{0}^{t} \langle Y (u), X (u) \rangle du
\]
Also it was just shown that \( BX (t (k)) \to BX (t) \). Then
\[
|\langle BX (t (k)), X (t (k)) \rangle - \langle BX (t), X (t) \rangle| \\
\leq |\langle BX (t (k)), X (t (k)) - X (t) \rangle| + |\langle BX (t (k)) - BX (t), X (t) \rangle|
\]
Then the second term converges to 0. The first equals
\[
|\langle BX (t (k)) - BX (t), X (t (k)) \rangle| \\
\leq \langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle^{1/2} \langle BX (t (k)), X (t (k)) \rangle^{1/2}
\]
From the above, this is dominated by an expression of the form
\[
\langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle^{1/2} C
\]
Then using the lower semicontinuity of \( t \to \langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle \) which follows from the above, this is no larger than
\[
\lim_{m \to \infty} \inf_{k} \langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle^{1/2} C < \varepsilon
\]
provided \( k \) is large enough. This follows from \( \|X\| \). Since \( \varepsilon \) is arbitrary, it follows that
\[
\lim_{k \to \infty} |\langle BX (t (k)), X (t (k)) \rangle - \langle BX (t), X (t) \rangle| = 0
\]
CHAPTER 41. INTERPOLATION IN BANACH SPACE

Then from the formula,

\[ \langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y(u), X(u) \rangle \, du \]

valid for \( t \in D \), it follows that the same formula holds for all \( t \). This formula implies \( t \to \langle BX(t), X(t) \rangle \) is continuous. Also recall that \( t \to BX(t) \) was shown to be weakly continuous into \( W' \). Then

\[ \langle B(X(t) - X(s)), X(t) - X(s) \rangle = \langle BX(t), X(t) \rangle - 2 \langle BX(t), X(s) \rangle + \langle BX(s), X(s) \rangle \]

From this, it follows that \( t \to BX(t) \) is continuous into \( W' \) because \( \lim_{t \to s} \) of the right side gives 0 and so the same is true of the left. Hence,

\[
\| (B(X(t) - X(s)), y) \| \\
\leq \langle By, y \rangle^{1/2} \langle (B(X(t) - X(s)), X(t) - X(s)) \rangle^{1/2} \\
\leq \| B \|^{1/2} \langle (B(X(t) - X(s)), X(t) - X(s)) \rangle^{1/2} \| y \|
\]

so

\[ \| B(X(t) - X(s)) \|_{W'} \leq \| B \|^{1/2} \langle (B(X(t) - X(s)), X(t) - X(s)) \rangle^{1/2} \]

which converges to 0 as \( t \to s \). ■

41.4 Some Implicit Inclusions

Let \( B \in \mathcal{L}(W, W') \) and \( B \) satisfies

\[ \langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle \]

for

\[ V \subseteq W, \quad W' \subseteq V' \]

Where \( V \) is dense in the Hilbert space \( W \). Now let

\[ D(L) \equiv \{ u \in V : (Bu)' \in V', Bu(0) = 0 \}, \quad Lu \equiv (Bu)' \]

(41.4.20)

Then clearly \( D(L) \) is dense in \( V \). Here \( V \equiv L^p ([0, T], V) \) where \( p \geq 2 \) for simplicity. Now let

\[ D(T) \equiv \{ u \in V : u' \in V \text{ and } u(T) = 0 \}, \quad Tu \equiv -B(u') \]

(41.4.21)

The idea is to show that \( L = T^* \) and that \( T \) is monotone. Then this will imply using Proposition 43.5.2 that \( L^* \) is monotone. This is done by showing that \( \mathcal{G}(L^*) = \mathcal{G}'(T) \).

**Lemma 41.4.1** \( T \) is monotone, \( T^* = L \) and \( L^*, L \) are both monotone.
41.4. SOME IMPLICIT INCLUSIONS

**Proof:** First, why is $T$ monotone?

\[
\int_0^T \langle -Bu', u \rangle \, dt = \int_0^T -\langle Bu, u' \rangle \, dt = -\langle Bu, u \rangle|_0^T + \int_0^T \langle (Bu)', u \rangle \, dt
\]

\[
= \langle Bu(0), u(0) \rangle + \int_0^T \langle Bu', u \rangle \, dt
\]

and so

\[
2 \langle Tu, u \rangle = 2 \int_0^T \langle -Bu', u \rangle \, dt = \langle Bu(0), u(0) \rangle \geq 0
\]

Next, why is $T^* = L$? Let $u \in D(L)$. Then for $v \in D(T)$,

\[
\langle Tv, u \rangle = \int_0^T \langle -Bv', u \rangle \, dt = \int_0^T \langle -Bu, v' \rangle \, dt = \langle -Bu, v \rangle|_0^T + \int_0^T \langle (Bu)', v \rangle \, ds
\]

\[
= \langle Lu, v \rangle
\]

Hence $u \in D(T^*)$ and $T^* u = Lu$ since $D(T)$ is dense. Thus $D(L) \subseteq D(T^*)$ and on $D(L)$, these two are equal. Next suppose $u \in D(T^*)$. Then for all $v \in D(T)$,

\[
\langle Tv, u \rangle \leq C \|v\|_V
\]

Thus, by density and the Riesz representation theorem, there exists a unique $g^* \in \mathcal{V}'$ such that

\[
\langle Tv, u \rangle = \int_0^T \langle g^*, v \rangle \, dt = \int_0^T \langle -Bv', u \rangle \, dt = -\int_0^T \langle Bu, v' \rangle \, dt
\]

In particular, it follows from the definition of weak $V'$ valued distributions that $g^* = (Bu)'$. Simply specialize to letting $v(t) = v\phi(t)$ where $\phi \in \mathcal{C}^\infty_c(0, T)$. Thus in particular $(Bu)' \in \mathcal{V}'$ and the above reduces to

\[
\langle Tv, u \rangle = \int_0^T \langle (Bu)', v \rangle \, dt
\]

Thus also $(Bu)(0) = 0$. Hence $D(T^*) \subseteq D(L)$ and this shows that $L = T^*$ as claimed.

Why is $L$ monotone? From the material on weak derivatives,

\[
Bu(t) = \int_0^t (Bu)'(s) \, ds
\]
and now use Theorem 41.3.2 to obtain
\[
0 \leq \langle Bu(t), u(t) \rangle = 2 \int_0^t \langle (Bu)', u \rangle \, ds.
\]
In particular, this holds for \( t = T \) and so \( \langle Lu, u \rangle \geq 0 \).

Why is \( L^* \) monotone? This follows from Proposition 23.8.2 and the fact that \( L = T^* \) shown above.

Consider \((\tau S)^\perp\). To say that \((x, y^*) \in (\tau S)^\perp\) is to say that if \((a, b^*) \in S\), then \(\langle (x, y^*), (-b^*, a) \rangle = 0\) or in other words, \(\langle x, b^* \rangle = \langle y^*, a \rangle\). To say that \((x, y^*) \in \tau(S^\perp)\) is to say that \((x, y^*) = (-c, d^*)\) where

\[
\langle (d^*, c), (a, b^*) \rangle = 0
\]

for all \((a, b^*) \in S\). That is, \(\langle (y^*, -x), (a, b^*) \rangle = 0\) for all \((a, b^*) \in S\). In other words \(\langle y^*, a \rangle = \langle x, b^* \rangle\) for all \((a, b^*) \in S\). Thus \((\tau S)^\perp = \tau(S^\perp)\). Now \(\tau \tau(M) = M\) if \(M\) is a subspace. and \((M^\perp)^\perp = M\) if \(M\) is a subspace. Hence

\[
G(L^*) = (\tau G(L))^\perp = \tau \left( (G(L))^\perp \right) = \tau \left( (\tau (T))^\perp \right) = (G(T))^\perp = G(T)
\]

Now it follows that, since \(T\) is monotone, it follows that \(L^*\) is also monotone. 

Note that as part of this argument, we have proved that for \(T\) a densely defined linear operator, \(G(T^{**}) = G(T)\).

Now recall Theorem 41.4.1 on Page 906 which is listed next.

**Theorem 41.4.2** Let \(L : D(L) \subseteq V \rightarrow V'\) where \(D(L)\) is dense, \(L\) is monotone, \(L\) is closed, and \(L^*\) is monotone, \(L\) a linear map. Let \(T : V \rightarrow P(V')\) be \(L\) pseudomonotone, bounded, coercive. Then \(L + T\) is onto. Here \(V\) is a reflexive Banach space such that the norms for \(V\) and \(V'\) are strictly convex.

To apply this theorem, let \(B\) be as above and \(V \rightarrow V \equiv L^p([0, T], V)\). Letting \(u_0 \in V\), let
\[
T(u) \equiv A(u + u_0)
\]
where \(A : V \rightarrow P(V')\). Suppose that \(T\) just defined is set valued pseudomonotone and coercive. Let \(Lu = (Bu)'\) as described above in 41.4.20. Then from Theorem 41.4.1 and if \(f \in V'\), there exists a solution \(u\) to
\[
Lu + A(u + u_0) \ni f
\]
Thus there exists \(\xi \in A(u + u_0)\) such that
\[
Lu + \xi = f \text{ in } V'
\]
Then letting \( w = u + u_0 \), it follows that \( \xi \in A(w) \) and
\[
L(w - w_0) + \xi = f
\]
Thus,
\[
(Bw)' + \xi = f, \quad (Bw)(0) = Bw_0
\]
Written in terms of \( A \),
\[
(Bw)' + A(w) \ni f \text{ in } V', \quad (Bw)(0) = Bu_0.
\]
This proves the following theorem about the existence of solutions to implicit evolution inclusions.

**Theorem 41.4.3** Suppose \( u \to A(u + u_0) \) is set valued pseudomonotone and coercive for \( u_0 \in V \). Also let
\[
V \subseteq W, \quad W' \subseteq V'
\]
where \( W \) is a Hilbert space, \( V \) is a reflexive Banach space dense in \( W \). Suppose \( B : W \to W' \) is self adjoint and nonnegative. Then there exists a solution \( w \in V \) to the implicit evolution equation
\[
(Bw)' + A(w) \ni f \text{ in } V', \quad (Bw)(0) = Bu_0.
\]

### 41.5 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling’s lemma.

**Definition 41.5.1** Let \( E, W \) be Banach spaces such that \( E \subseteq W \) and the injection map from \( E \) into \( W \) is continuous. The injection map is said to be compact if every bounded set in \( E \) has compact closure in \( W \). In other words, if a sequence is bounded in \( E \) it has a convergent subsequence converging in \( W \). This is also referred to by saying that bounded sets in \( E \) are precompact in \( W \).

**Theorem 41.5.2** Let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Then for every \( \varepsilon > 0 \) there exists a constant, \( C_\varepsilon \), such that for all \( u \in E \),
\[
||u||_W \leq \varepsilon ||u||_E + C_\varepsilon ||u||_X
\]

**Proof:** Suppose not. Then there exists \( \varepsilon > 0 \) and for each \( n \in \mathbb{N} \), \( u_n \) such that
\[
||u_n||_W > \varepsilon ||u_n||_E + n ||u_n||_X
\]
Now let \( v_n = u_n/||u_n||_E \). Therefore, \( ||v_n||_E = 1 \) and
\[
||v_n||_W > \varepsilon + n ||v_n||_X
\]
It follows there exists a subsequence, still denoted by \( v_n \) such that \( v_n \) converges to \( v \) in \( W \). However, the above inequality shows that \( ||v_n||_X \to 0 \). Therefore, \( v = 0 \). But then the above inequality would imply that \( ||v_n|| > \varepsilon \) and passing to the limit yields \( 0 > \varepsilon \), a contradiction.
Definition 41.5.3 Define $C([a,b];X)$ the space of functions continuous at every point of $[a,b]$ having values in $X$.

You should verify that this is a Banach space with norm

$$\|u\|_{\infty,X} = \max \{|u_{n_k}(t) - u(t)|_X : t \in [a,b]\}.$$ 

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. 

Theorem 41.5.4 Let $q > 1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $S$ be defined by

$$\{ u \text{ such that } \|u(t)\|_E + \|u'\|_{L^q([a,b];X)} \leq R \text{ for all } t \in [a,b] \}.$$ 

Then $S \subseteq C([a,b];W)$ and if $\{u_n\} \subseteq S$, there exists a subsequence, $\{u_{n_k}\}$ which converges to a function $u \in C([a,b];W)$ in the following way.

$$\lim_{k \to \infty} \|u_{n_k} - u\|_{\infty,W} = 0.$$ 

Proof: First consider the issue of $S$ being a subset of $C([a,b];W)$. By Theorem 41.1.9 on Page 1445 the following holds in $X$ for $u \in S$.

$$u(t) - u(s) = \int_s^t u'(r) \, dr.$$ 

Thus $S \subseteq C([a,b];X)$. Let $\varepsilon > 0$ be given. Then by Theorem 41.3.2 there exists a constant, $C_\varepsilon$ such that for all $u \in W$

$$\|u\|_W \leq \frac{\varepsilon}{4R} \|u\|_E + C_\varepsilon \|u\|_X.$$ 

Therefore, for all $u \in S$,

$$\|u(t) - u(s)\|_W \leq \frac{\varepsilon}{6R} \|u(t) - u(s)\|_E + C_\varepsilon \|u(t) - u(s)\|_X$$

$$\leq \frac{\varepsilon}{3} + C_\varepsilon \int_s^t |u'(r)|_X \, dr$$

$$\leq \frac{\varepsilon}{3} + C_\varepsilon \int_s^t |u'(r)|_X \, dr \leq \frac{\varepsilon}{3} + C_\varepsilon R |t - s|^{1/q}$$

(41.5.22)

Since $\varepsilon$ is arbitrary, it follows $u \in C([a,b];W)$.

Let $D = \{ n \in \mathbb{Q} : n \in [a,b] \}$ so $D$ is a countable dense subset of $[a,b]$. Let $D = \{ t_n \}_{n=1}^\infty$. By compactness of the embedding of $E$ into $W$, there exists a subsequence $u_{n_{(1)}}$ such that as $n \to \infty$, $u_{n_{(1)}}(t_1)$ converges to a point in $W$. Now take a subsequence of this, called $(n,2)$ such that as $n \to \infty$, $u_{n_{(2)}}(t_2)$ converges to a point in $W$. It follows that $u_{n_{(2)}}(t_1)$ also converges to a point of $W$. Continue this way. Now
consider the diagonal sequence, \( u_k \equiv u_{(k,k)} \). This sequence is a subsequence of \( u_{(n,t)} \) whenever \( k > l \). Therefore, \( u_k \) converges for all \( t_j \in D \).

**Claim:** Let \( \{u_k\} \) be as just defined, converging at every point of \( D \equiv \mathbb{Q} \cap [a, b] \). Then \( \{u_k\} \) converges at every point of \([a, b]\).

**Proof of claim:** Let \( \varepsilon > 0 \) be given. Let \( t \in [a, b] \). Pick \( t_m \in D \cap [a, b] \) such that \( |t - t_m| < \varepsilon / 3 \). Then there exists \( N \) such that if \( l, n > N \), then \( ||u_l(t_m) - u_n(t_m)||_X < \varepsilon / 3 \). It follows that for \( l, n > N \),

\[
\|u_l(t) - u_n(t)\|_X \leq ||u_l(t) - u_l(t_m)|| + ||u_l(t_m) - u_n(t_m)|| + ||u_n(t_m) - u_n(t)|| \leq 2\varepsilon/3 + \frac{2\varepsilon}{3} < 2\varepsilon
\]

Since \( \varepsilon \) was arbitrary, this shows \( \{u_k(t)\}_{k=1}^\infty \) is a Cauchy sequence. Since \( W \) is complete, this shows this sequence converges.

Now for \( t \in [a, b] \), it was just shown that if \( \varepsilon > 0 \) there exists \( N_t \) such that if \( n, m > N_t \), then

\[
||u_n(t) - u_m(t)|| < \frac{\varepsilon}{3}.
\]

Now let \( s \neq t \). Then

\[
||u_n(s) - u_m(s)|| \leq ||u_n(s) - u_n(t)|| + ||u_n(t) - u_m(t)|| + ||u_m(t) - u_m(s)||
\]

From \( \text{(41.5.23)} \)

\[
||u_n(s) - u_m(s)|| \leq 2\left(\frac{\varepsilon}{3} + C \varepsilon R |t - s|^{1/q}\right) + ||u_n(t) - u_m(t)||
\]

and so it follows that if \( \delta \) is sufficiently small and \( s \in B(t, \delta) \), then when \( n, m > N_t \)

\[
||u_n(s) - u_m(s)|| < \varepsilon.
\]

Since \([a, b]\) is compact, there are finitely many of these balls, \( \{B(t_i, \delta)\}_{i=1}^p \), such that for \( s \in B(t_i, \delta) \) and \( n, m > N_{t_i} \), the above inequality holds. Let \( N > \max \{N_{t_1}, \ldots, N_{t_p}\} \). Then if \( m, n > N \), and \( s \in [a, b] \) is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

**Claim:** Let \( \varepsilon > 0 \) be given. Then there exists \( N \) such that if \( m, n > N \), then

\[
||u_n - u_m||_W < \varepsilon.
\]

Now let \( u(t) = \lim_{k \to \infty} u_k(t) \).

\[
||u(t) - u(s)||_W \leq ||u(t) - u_n(t)||_W + ||u_n(t) - u_n(s)||_W + ||u_n(s) - u(s)||_W.
\]

Let \( N \) be in the above claim and fix \( n > N \). Then

\[
||u(t) - u_n(t)||_W = \lim_{m \to \infty} ||u_m(t) - u_n(t)||_W \leq \varepsilon
\]

and similarly, \( ||u_n(s) - u(s)||_W \leq \varepsilon \). Then if \( |t - s| \) is small enough, \( \text{(41.5.22)} \) shows the middle term in \( \text{(41.5.23)} \) is also smaller than \( \varepsilon \). Therefore, if \( |t - s| \) is small enough,

\[
||u(t) - u(s)||_W < 3\varepsilon.
\]
Thus \( u \) is continuous. Finally, let \( N \) be as in the above claim. Then letting \( m, n > N \), it follows that for all \( t \in [a, b] \),
\[
\| u_m (t) - u_n (t) \| < \varepsilon.
\]
Therefore, letting \( m \to \infty \), it follows that for all \( t \in [a, b] \),
\[
\| u (t) - u_n (t) \| \leq \varepsilon.
\]
and so \( \| u - u_n \|_{\infty, W} \leq \varepsilon \). Since \( \varepsilon \) is arbitrary, this proves the theorem.

The next theorem is another such imbedding theorem found in \([22]\). It is often used in partial differential equations.

**Theorem 41.5.5** Let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Let \( p \geq 1 \), let \( q > 1 \), and define
\[
S \equiv \{ u \in L^p ([a, b]; E) : u' \in L^q ([a, b]; X) \}
\]
and \( \| u \|_{L^p([a, b]; E)} + \| u' \|_{L^q([a, b]; X)} \leq R \} \). Then \( S \) is precompact in \( L^p ([a, b]; W) \). This means that if \( \{ u_n \}_{n=1}^{\infty} \subseteq S \), it has a subsequence \( \{ u_{n_k} \} \) which converges in \( L^p ([a, b]; W) \).

**Proof:** By Proposition 12.4 on Page 440 it suffices to show \( S \) has an \( \eta \) net in \( L^p ([a, b]; W) \) for each \( \eta > 0 \).

If not, there exists \( \eta > 0 \) and a sequence \( \{ u_n \} \subseteq S \), such that
\[
\| u_n - u_m \| \geq \eta \quad (41.5.24)
\]
for all \( n \neq m \) and the norm refers to \( L^p ([a, b]; W) \). Let
\[
a = t_0 < t_1 < \cdots < t_k = b, \ t_i - t_{i-1} = (b - a) / k.
\]
Now define
\[
\overline{u}_n (t) \equiv \sum_{i=1}^{k} \overline{u}_n \chi_{[t_{i-1}, t_i)} (t), \ \overline{u}_n, \equiv \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n (s) \, ds.
\]
The idea is to show that \( \overline{u}_n \) approximates \( u_n \) well and then to argue that a subsequence of the \( \{ \overline{u}_n \} \) is a Cauchy sequence yielding a contradiction to (41.5.24).

Therefore,
\[
u_n (t) - \overline{u}_n (t) = \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n (t) - u_n (s)) \, ds \chi_{[t_{i-1}, t_i)} (t) .
\]
It follows from Jensen’s inequality that
\[
\| u_n (t) - \overline{u}_n (t) \|_W^p \leq \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n (t) - u_n (s) \|_W^p \, ds \chi_{[t_{i-1}, t_i)} (t)
\]
41.5. SOME IMBEDDING THEOREMS

\[ 41.5.25 \]

From Theorems 11.9.1 and 11.10.1, if \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that
\[
\| u_n(t) - u_n(s) \|^p_{W} \leq \varepsilon \| u_n(t) - u_n(s) \|^p_{L^p} + C_\varepsilon \| u_n(t) - u_n(s) \|^p_X
\]
\[
\leq 2^{p-1}\varepsilon (\| u_n(t) \|^p + \| u_n(s) \|^p) + C_\varepsilon \left( \int_s^t \| u_n'(r) \|_X dr \right)^p
\]
\[
\leq 2^{p-1}\varepsilon (\| u_n(t) \|^p + \| u_n(s) \|^p) + C_\varepsilon \left( \int_s^t \| u_n'(r) \|_X dr \right)^p
\]
\[
+ C_\varepsilon \left( \int_s^t \| u_n'(r) \|^q_X dr \right)^{1/q} |t - s|^{1/q'}
\]
\[
= 2^{p-1}\varepsilon (\| u_n(t) \|^p + \| u_n(s) \|^p) + C_\varepsilon R^{p/q} |t - s|^{p/q'}.
\]

This is substituted in to \[ 41.5.24 \] to obtain
\[
\int_a^b \| (u_n(t) - \pi_n(s)) \|^p_W ds \leq
\]
\[
\sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \left( 2^{p-1}\varepsilon (\| u_n(t) \|^p + \| u_n(s) \|^p) \right)
\]
\[
+ C_\varepsilon R^{p/q} |t - s|^{p/q'} dsdt
\]
\[
= \sum_{i=1}^k 2^p \varepsilon \int_{t_{i-1}}^{t_i} \| u_n(t) \|^p_W + C_\varepsilon R^{p/q} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} |t - s|^{p/q'} dsdt
\]
\[
= 2^p \varepsilon \int_a^b \| u_n(t) \|^p dt + C_\varepsilon R^{p/q} \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})^{p/q'}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} dsdt
\]
\[
= 2^p \varepsilon \int_a^b \| u_n(t) \|^p dt + C_\varepsilon R^{p/q} \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})^{p/q'}} (t_i - t_{i-1})^{2}
\]
\[
\leq 2^p \varepsilon R^p + C_\varepsilon R^{p/q} \sum_{i=1}^k (t_i - t_{i-1})^{1+p/q'} = 2^p \varepsilon R^p + C_\varepsilon R^{p/q} k \left( \frac{b - a}{k} \right)^{1+p/q'}.
\]
Taking \( \varepsilon \) so small that \( 2^p \varepsilon < \frac{\eta^p}{8^p} \) and then choosing \( k \) sufficiently large, it follows

\[
||u_n - \overline{u}_n||_{L^p([a,b];W)} < \frac{\eta}{4}.
\]

Now use compactness of the embedding of \( E \) into \( W \) to obtain a subsequence such that \( \{\overline{u}_n\} \) is Cauchy in \( L^p (a,b;W) \) and use this to contradict 41.5.24. The details follow.

Suppose \( \overline{u}_n (t) = \sum_{i=1}^{k} u^n_i \mathcal{X}_{[t_{i-1},t_i)} (t) \). Thus

\[
||\overline{u}_n (t)||_E = \sum_{i=1}^{k} ||u^n_i||_E \mathcal{X}_{[t_{i-1},t_i)} (t)
\]

and so

\[
R \geq \int_a^b ||\overline{u}_n (t)||_E^p dt = \frac{T}{k} \sum_{i=1}^{k} ||u^n_i||_E^p
\]

Therefore, the \( \{u^n_i\} \) are all bounded. It follows that after taking subsequences \( k \) times there exists a subsequence \( \{u_{nk}\} \) such that \( u_{nk} \) is a Cauchy sequence in \( L^p (a,b;W) \). You simply get a subsequence such that \( u^{nk}_i \) is a Cauchy sequence in \( W \) for each \( i \). Then denoting this subsequence by \( n_k \),

\[
||u_n - u_m||_{L^p(a,b;W)} \leq ||u_n - \overline{u}_n||_{L^p(a,b;W)} + ||\overline{u}_n - \overline{u}_m||_{L^p(a,b;W)} + ||\overline{u}_m - u_m||_{L^p(a,b;W)} \leq \frac{\eta}{4} + ||\overline{u}_n - \overline{u}_m||_{L^p(a,b;W)} + \frac{\eta}{4} < \eta
\]

provided \( m, n \) are large enough, contradicting 41.5.24. This proves the theorem.

### 41.6 The K Method

This considers the problem of interpolating Banach spaces. The idea is to build a Banach space between two others in a systematic way, thus constructing a new Banach space from old ones. The first method of defining intermediate Banach spaces is called the \( K \) method. For more on this topic as well as the other topics on interpolation see [15] which is what I am following. See also [40]. There is far more on these subjects in these books than what I am presenting here! My goal is to present only enough to give an introduction to the topic and to use it in presenting more theory of Sobolev spaces.

In what follows a topological vector space is a vector space in which vector addition and scalar multiplication are continuous. That is \( \cdot : \mathbb{F} \times X \to X \) is continuous and \( + : X \times X \to X \) is also continuous.

A common example of a topological vector space is the dual space, \( X' \) of a Banach space, \( X \) with the weak * topology. For \( S \subseteq X \) a finite set, define

\[
B_S (x^*, r) \equiv \{ y^* \in X' : |y^* (x) - x^*(x)| < r \text{ for all } x \in S \}
\]
Then the $B_S (x^*, r)$ for $S$ a finite subset of $X$ and $r > 0$ form a basis for the topology on $X'$ called the weak $*$ topology. You can check that the vector space operations are continuous.

Definition 41.6.1 Let $A_0$ and $A_1$ be two Banach spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, also written as $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$, and let $X$ be a topological vector space such that $A_i \subseteq X$ for $i = 1, 2$, and the identity map from $A_i$ to $X$ is continuous. For each $t > 0$, define a norm on $A_0 + A_1$ by

$$K (t, a) \equiv \{a\}_{t} \equiv \inf \{\|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = a\}.$$

This is short for $K (t, a, A_0, A_1)$. Thus $K (t, a, A_1, A_0)$ will mean

$$K (t, a, A_1, A_0) \equiv \{\|a_1\|_{A_1} + t \|a_0\|_{A_0} : a_0 + a_1 = a\}$$

but the default is $K (t, a, A_0, A_1)$ if $K (t, a)$ is written.

The following lemma is an interesting exercise.

Lemma 41.6.2 $(A_0 + A_1, K (t, \cdot))$ is a Banach space and all the norms, $K (t, \cdot)$ are equivalent.

Proof: First, why is $K (t, \cdot)$ a norm? It is clear that $K (t, a) \geq 0$ and that if $a = 0$ then $K (t, a) = 0$. Is this the only way this can happen? Suppose $K (t, a) = 0$. Then there exist $a_{0n} \in A_0$ and $a_{1n} \in A_1$ such that $\|a_{0n}\|_0 \to 0$, $\|a_{1n}\|_1 \to 0$, and $a = a_{0n} + a_{1n}$. Since the embedding of $A_1$ into $X$ is continuous and since $X$ is a topological vector space, it follows

$$a = a_{0n} + a_{1n} \to 0$$

and so $a = 0$.

Let $\alpha$ be a nonzero scalar. Then

$$K (t, \alpha a) = \inf \{\|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = \alpha a\}$$

$$= \inf \{\|\frac{a_0}{\alpha}\|_0 + t \|\frac{a_1}{\alpha}\|_1 : \frac{a_0}{\alpha} + \frac{a_1}{\alpha} = a\}$$

$$= |\alpha| \inf \{\frac{\|a_0\|_0}{\alpha} + t \|\frac{a_1}{\alpha}\|_1 : \frac{a_0}{\alpha} + \frac{a_1}{\alpha} = a\}$$

$$= |\alpha| \inf \{\|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = a\} = |\alpha| K (t, a).$$

It remains to verify the triangle inequality. Let $\varepsilon > 0$ be given. Then there exist $a_0, a_1, b_0,$ and $b_1$ in $A_0, A_1, A_0,$ and $A_1$ respectively such that $a_0 + a_1 = a, b_0 + b_1 = b$ and

$$\varepsilon + K (t, a) + K (t, b) > \|a_0\|_0 + t \|a_1\|_1 + \|b_0\|_0 + t \|b_1\|_1$$

$$\geq \|a_0 + b_0\|_0 + t \|b_1 + a_1\|_1 \geq K (t, a + b).$$

$^1$Vector addition is continuous is the property which is used here.
This has shown that $K(t, \cdot)$ is at least a norm. Are all these norms equivalent? If $0 < s < t$ then it is clear that $K(t, a) \geq K(s, a)$. To show there exists a constant, $C$ such that $CK(s, a) \geq K(t, a)$ for all $a$,

$$\frac{t}{s} K(s, a) = \frac{t}{s} \inf \{ ||a_0||_0 + s ||a_1||_1 : a_0 + a_1 = a \}$$

$$= \inf \left\{ \frac{t}{s} ||a_0||_0 + \frac{s}{t} ||a_1||_1 : a_0 + a_1 = a \right\}$$

$$= \inf \left\{ \frac{t}{s} ||a_0||_0 + t ||a_1||_1 : a_0 + a_1 = a \right\}$$

$$\geq \inf \{ ||a_0||_0 + t ||a_1||_1 : a_0 + a_1 = a \} = K(t, a).$$

Therefore, the two norms are equivalent as hoped.

Finally, it is required to verify that $(A_0 + A_1, K(t, \cdot))$ is a Banach space. Since all these norms are equivalent, it suffices to only consider the norm, $K(1, \cdot)$. Let $\{a_{0n} + a_{1n}\}_{n=1}^\infty$ be a Cauchy sequence in $A_0 + A_1$. Then for $m, n$ large enough,

$$K(1, a_{0n} + a_{1n} - (a_{0m} + a_{1m})) < \varepsilon.$$

It follows there exist $x_n \in A_0$ and $y_n \in A_1$ such that $x_n + y_n = 0$ for every $n$ and whenever $m, n$ are large enough,

$$||a_{0n} + x_n - (a_{0m} + x_m)||_0 + ||a_{1n} + y_n - (a_{1m} + y_m)||_1 < \varepsilon$$

Hence $\{a_{1n} + y_n\}$ is a Cauchy sequence in $A_1$ and $\{a_{0n} + x_n\}$ is a Cauchy sequence in $A_0$. Let

$$a_{0n} + x_n \to a_0 \in A_0$$

$$a_{1n} + y_n \to a_1 \in A_1.$$

Then

$$K(1, a_{0n} + a_{1n} - (a_0 + a_1)) = K(1, a_{0n} + x_n + a_{1n} + y_n - (a_0 + a_1))$$

$$\leq ||a_{0n} + x_n - a_0||_0 + ||a_{1n} + y_n - a_1||_1$$

which converges to 0. Thus $A_0 + A_1$ is a Banach space as claimed.

With this, there exists a method for constructing a Banach space which lies between $A_0 \cap A_1$ and $A_0 + A_1$.

**Definition 41.6.3** Let $1 \leq q < \infty, 0 < \theta < 1$. Define $(A_0, A_1)_{\theta, q}$ to be those elements of $A_0 + A_1, a$, such that

$$||a||_{\theta, q} = \left[ \int_0^\infty (t^{-\theta} K(t, a, A_0, A_1))^q \frac{dt}{t} \right]^{1/q} < \infty.$$
41.6. THE K METHOD

Theorem 41.6.4 \((A_0, A_1)_{θ,q}\) is a normed linear space satisfying

\[ A_0 \cap A_1 \subseteq (A_0, A_1)_{θ,q} \subseteq A_0 + A_1, \]

with the inclusion maps continuous, and

\[ \left( (A_0, A_1)_{θ,q}, \|\cdot\|_{θ,q} \right) \text{ is a Banach space.} \]

If \(a \in A_0 \cap A_1\), then

\[ \|a\|_{θ,q} \leq \left( \frac{1}{qθ(1 - θ)} \right)^{1/q} \|a\|^θ_1 \|a\|^{1-θ}_0. \]

If \(A_0 \subseteq A_1\) with \(\|\cdot\|_0 \geq \|\cdot\|_1\), then

\[ A_0 \cap A_1 = A_0 \subseteq (A_0, A_1)_{θ,q} \subseteq A_1 = A_0 + A_1. \]

Also, if bounded sets in \(A_0\) have compact closures in \(A_1\) then the same is true if \(A_1\) is replaced with \((A_0, A_1)_{θ,q}\). Finally, if \(T \in \mathcal{L}(A_0, B_0), T \in \mathcal{L}(A_1, B_1)\), and \(T\) is a linear map from \(A_0 + A_1\) to \(B_0 + B_1\) where the \(A_i\) and \(B_i\) are Banach spaces with the properties described above, then it follows

\[ T \in \mathcal{L} \left( (A_0, A_1)_{θ,q}, (B_0, B_1)_{θ,q} \right) \]

and if \(M\) is its norm, and \(M_0\) and \(M_1\) are the norms of \(T\) as a map in \(\mathcal{L}(A_0, B_0)\) and \(\mathcal{L}(A_1, B_1)\) respectively, then

\[ M \leq M_0^{1-θ} M_1^θ. \]

Proof: Suppose first \(a \in A_0 \cap A_1\). Then

\[ \|a\|^q_{θ,q} = \int_0^r \left( \frac{t^{-θ} K(t,a)}{t} \right)^q dt + \int_r^∞ \left( \frac{t^{-θ} K(t,a)}{t} \right)^q dt \]

\[ \leq \int_0^r \left( t^{-θ} \|a\|_1 \right)^q t dt + \int_r^∞ \left( t^{-θ} \|a\|_0 \right)^q t dt \]

\[ = \|a\|^q_1 \int_0^r t^{q(1-θ)-1} dt + \|a\|^q_0 \int_r^∞ t^{-θq} dt \]

\[ = \|a\|^q_1 \frac{r^{q-qθ}}{q-qθ} + \|a\|^q_0 \frac{r^{-θq}}{-θq} < ∞ \]

Which shows the first inclusion of (41.6.26). The above holds for all \(r > 0\) and in particular for the value of \(r\) which minimizes the expression on the right in (41.6.33), \(r = \|a\|_0 / \|a\|_1\). Therefore, doing some calculus,

\[ \|a\|^q_{θ,q} \leq \frac{1}{θq(1 + θ)} \|a\|^q_0 (1-θ) \|a\|^q_1 \]
which shows \[ \text{This also verifies that the first inclusion map is continuous in } L^q. \] because if \( a_n \to 0 \) in \( A_0 \cap A_1 \), then \( a_n \to 0 \) in \( A_0 \) and in \( A_1 \) and so the above shows \( a_n \to 0 \) in \( (A_0, A_1)_{\theta, q} \).

Now consider the second inclusion in \( L^q \). The inclusion is obvious because \( (A_0, A_1)_{\theta, q} \) is given to be a subset of \( A_0 + A_1 \) defined by

\[
\left( \int_0^\infty \left( t^{-\theta} K(t, a_n) \right)^q \frac{dt}{t} \right)^{1/q} < \infty
\]

It remains to verify the inclusion map is continuous. Therefore, suppose \( a_n \to 0 \) in \( (A_0, A_1)_{\theta, q} \). Since \( a_n \to 0 \) in \( (A_0, A_1)_{\theta, q} \), it follows the function, \( t \to t^{-\theta} K(t, a_n) \) converges to zero in \( L^q(0, \infty) \) with respect to the measure, \( dt/t \). Therefore, taking another subsequence, still denoted as \( a_n \), you can assume this function converges to 0 a.e. Pick such a \( t \) where this convergence takes place. Then \( K(t, a_n) \to 0 \) as \( n \to \infty \) and \( a_n \to 0 \) in \( A_0 + A_1 \). (Recall all these norms \( K(t, \cdot) \) are equivalent.) This shows that if \( a_n \to 0 \) in \( (A_0, A_1)_{\theta, q} \), then there exists a subsequence \( \{a_{n_k}\} \) such that \( a_{n_k} \to 0 \) in \( A_0 + A_1 \). It follows that if \( a_n \to 0 \) in \( (A_0, A_1)_{\theta, q} \), then \( a_n \to 0 \) in \( A_0 + A_1 \). This proves the continuity of the embedding.

What about \( L^q \)? Suppose \( \{a_n\} \) is a Cauchy sequence in \( (A_0, A_1)_{\theta, q} \). Then from what was just shown this is a Cauchy sequence in \( A_0 + A_1 \) and so there exists \( a \in A_0 + A_1 \) such that \( a_n \to a \) in \( A_0 + A_1 \) because \( A_0 + A_1 \) is a Banach space. Thus, \( K(t, a_n) \to K(t, a) \) for all \( t > 0 \). (Recall all these norms \( K(t, \cdot) \) are equivalent.) Therefore, by Fatou’s lemma,

\[
\left( \int_0^\infty \left( t^{-\theta} K(t, a) \right)^q \frac{dt}{t} \right)^{1/q} \leq \liminf_{n \to \infty} \left( \int_0^\infty \left( t^{-\theta} K(t, a_n) \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \max \left\{ ||a_n||_{\theta, q} : n \in \mathbb{N} \right\} < \infty
\]

and so \( a \in (A_0, A_1)_{\theta, q} \). Now

\[
||a - a_n||_{\theta, q} \leq \liminf_{m \to \infty} \left( \int_0^\infty \left( t^{-\theta} K(t, a - a_m) \right)^q \frac{dt}{t} \right)^{1/q} \\
= \liminf_{m \to \infty} ||a_n - a_m||_{\theta, q} < \epsilon
\]

whenever \( n \) is large enough. Thus \( (A_0, A_1)_{\theta, q} \) is complete as claimed.

Next suppose \( A_0 \subseteq A_1 \) and the inclusion map is compact. In this case, \( A_0 \cap A_1 = A_0 \) and so it has been shown above that \( A_0 \subseteq (A_0, A_1)_{\theta, q} \). It remains to show that every bounded subset, \( S \), contained in \( A_0 \) has an \( \eta \) net in \( (A_0, A_1)_{\theta, q} \). Recall the inequality, \[ \text{This inequality, } L^q \]

\[
||a||_{\theta, q} \leq \left( \frac{1}{q^{\theta} (1 - \theta)} \right)^{1/q} ||a||_{1}^{\theta} ||a||_{0}^{1-\theta} \\
= \frac{C}{\varepsilon} ||a||_{1}^{\theta} ||a||_{0}^{1-\theta} .
\]
41.6. **THE K METHOD** 1483

Now this implies

\[
||a||_{\theta,q} \leq \left( \frac{C}{\varepsilon} \right)^{1/\theta} \theta ||a||_1 + \varepsilon^{1/(1-\theta)} (1 - \theta) ||a||_0
\]

By compactness of the embedding of \( A_0 \) into \( A_1 \), it follows there exists an \( \varepsilon^{(1+\theta)/\theta} \) net for \( S \) in \( A_1, \{a_1, \cdots, a_p\} \). Then for \( a \in S \), there exists \( k \) such that \( ||a - a_k||_1 < \varepsilon^{(1+\theta)/\theta} \). It follows

\[
||a - a_k||_{\theta,q} \leq \left( \frac{C}{\varepsilon} \right)^{1/\theta} \theta ||a - a_k||_1 + \varepsilon^{1/(1-\theta)} (1 - \theta) ||a - a_k||_0
\]

\[
\leq \left( \frac{C}{\varepsilon} \right)^{1/\theta} \theta \varepsilon^{(1+\theta)/\theta} + \varepsilon^{1/(1-\theta)} (1 - \theta) 2M
\]

\[
= C^{1/\theta} \theta \varepsilon + \varepsilon^{1/(1-\theta)} (1 - \theta) 2M
\]

where \( M \) is large enough that \( ||a||_0 \leq M \) for all \( a \in S \). Since \( \varepsilon \) is arbitrary, this shows the existence of a \( \eta \) net and proves the compactness of the embedding into \( (A_0, A_1)_{\theta,q} \).

It remains to verify the assertions 41.6.29-41.6.31. Let \( T \in \mathcal{L}(A_0, B_0), T \in \mathcal{L}(A_1, B_1) \) with \( T \) a linear map from \( A_0 + A_1 \) to \( B_0 + B_1 \). Let \( a \in (A_0, A_1)_{\theta,q} \subseteq A_0 + A_1 \) and consider \( Ta \in B_0 + B_1 \). Denote by \( K(t, \cdot) \) the norm described above for both \( A_0 + A_1 \) and \( B_0 + B_1 \) since this will cause no confusion. Then

\[
||Ta||_{\theta,q} = \left( \int_0^\infty (t^{-\theta} K(t, Ta))^q \frac{dt}{t} \right)^{1/q} . \quad (41.6.34)
\]

Now let \( a_0 + a_1 = a \) and so \( Ta_0 + Ta_1 = Ta \)

\[
K(t, Ta) \leq ||Ta_0||_q + t ||Ta_1||_1 \leq M_0 ||a_0||_0 + M_1 t ||a_1||_1
\]

\[
\leq M_0 \left( ||a_0||_0 + t \left( \frac{M_1}{M_0} \right) ||a_1||_1 \right)
\]

and so, taking inf for all \( a_0 + a_1 = a \), yields

\[
K(t, Ta) \leq M_0 K \left( t \left( \frac{M_1}{M_0} \right), a \right)
\]
It follows from (41.6.34) that
\[
||Ta||_{\theta,q} \equiv \left( \int_0^\infty \left( t^{-\theta} K(t, Ta) \right)^q \frac{dt}{t} \right)^{1/q}
\]
\[
\leq \left( \int_0^\infty \left( t^{-\theta} M_0 K(t \left( \frac{M_1}{M_0} \right), a) \right)^q \frac{dt}{t} \right)^{1/q}
\]
\[
= M_0 \left( \int_0^\infty \left( t^{-\theta} K(t \left( \frac{M_1}{M_0} \right), a) \right)^q \frac{ds}{s} \right)^{1/q}
\]
\[
= M_0 \left( \int_0^\infty \left( \frac{M_0}{M_1} s^{-\theta} K(s, a) \right)^q \frac{ds}{s} \right)^{1/q} = M_0 \cdot M_0^{(1-\theta)} ||a||_{\theta,q}.
\]

This shows \( T \in \mathcal{L} \left( (A_0, A_1)_{\theta,q}, (B_0, B_1)_{\theta,q} \right) \) and if \( M \) is the norm of \( T, M \leq M_0^{1-\theta} M_1^\theta \) as claimed. This proves the theorem.

### 41.7 The J Method

There is another method known as the J method. Instead of
\[
K(t, a) \equiv \inf \left\{ ||a_0||_{A_0} + t ||a_1||_{A_1} : a_0 + a_1 = a \right\}
\]
for \( a \in A_0 + A_1 \), this method considers \( a \in A_0 \cap A_1 \) and \( J(t, a) \) defined below gives a norm on \( A_0 \cap A_1 \).

**Definition 41.7.1** For \( A_0 \) and \( A_1 \) Banach spaces as described above, and \( a \in A_0 \cap A_1 \),
\[
J(t, a) \equiv \max \left( ||a||_{A_0}, t ||a||_{A_1} \right).
\]
(41.7.35)

This is short for \( J(t, a, A_0, A_1) \). Thus
\[
J(t, a, A_1, A_0) \equiv \max \left( ||a||_{A_1}, t ||a||_{A_0} \right)
\]
but unless indicated otherwise, \( A_0 \) will come first. Now for \( \theta \in (0,1) \) and \( q \geq 1 \), define a space, \( (A_0, A_1)_{\theta,q,J} \) as follows. The space, \( (A_0, A_1)_{\theta,q,J} \) will consist of those elements, \( a \), of \( A_0 + A_1 \) which can be written in the form
\[
a = \int_0^\infty u(t) \frac{dt}{t} \equiv \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 u(t) \frac{dt}{t} + \lim_{r \to \infty} \int_1^r u(t) \frac{dt}{t}
\]
(41.7.36)

the limits taking place in \( A_0 + A_1 \) with the norm
\[
K(1, a) \equiv \inf_{a = a_0 + a_1} \left( ||a_0||_{A_0} + ||a_1||_{A_1} \right),
\]
where \( u(t) \) is strongly measurable with values in \( A_0 \cap A_1 \) and bounded on every compact subset of \((0, \infty)\) such that

\[
\left( \int_0^\infty \left( t^{-\theta} J(t, u(t), A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q} < \infty.
\]

(41.7.37)

For such \( a \in A_0 + A_1 \), define

\[
||a||_{g,q,J} \equiv \inf_u \left\{ \left( \int_0^\infty \left( t^{-\theta} J(t, u(t), A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q} \right\}
\]

(41.7.38)

where the infimum is taken over all \( u \) satisfying 41.7.36 and 41.7.37.

Note that a norm on \( A_0 \times A_1 \) would be

\[
||a|| \equiv \max \left( ||a||_{A_0}, ||a||_{A_1} \right)
\]

and so \( J(t, \cdot) \) is the restriction of this norm to the subspace of \( A_0 \times A_1 \) defined by \( \{(a,a) : a \in A_0 \cap A_1\} \). Also for each \( t > 0 \) \( J(t, \cdot) \) is a norm on \( A_0 \cap A_1 \) and furthermore, any two of these norms are equivalent. In fact, for \( 0 < t < s \),

\[
J(t, a) = \max \left( ||a||_{A_0}, t ||a||_{A_1} \right)
\]

\[
\geq \max \left( ||a||_{A_0}, s ||a||_{A_1} \right)
\]

\[
= J(s, a)
\]

\[
\geq \max \left( \frac{s}{t} ||a||_{A_0}, s ||a||_{A_1} \right)
\]

\[
= \frac{s}{t} \max \left( ||a||_{A_0}, ||a||_{A_1} \right)
\]

\[
\geq \frac{s}{t} \cdot J(t, a).
\]

The following lemma is significant and follows immediately from the above definition.

**Lemma 41.7.2** Suppose \( a \in (A_0, A_1)_{g,q,J} \) and \( a = \int_0^\infty u(t) \frac{dt}{t} \) where \( u \) is described above. Then letting \( r > 1 \),

\[
u_r(t) = \begin{cases} u(t) & \text{if } t \in \left( \frac{1}{r}, r \right) \\ 0 & \text{otherwise} \end{cases}
\]

it follows that

\[
\int_0^\infty u_r(t) \frac{dt}{t} \in A_0 \cap A_1.
\]

**Proof:** The integral equals \( \int_{1/r}^r u(t) \frac{dt}{t} \cdot \int_{1/r}^r \frac{1}{t} dt = 2 \ln r < \infty \). Now \( u_r \) is measurable in \( A_0 \cap A_1 \) and bounded. Therefore, there exists a sequence of measurable
simple functions, \( \{s_n\} \) having values in \( A_0 \cap A_1 \) which converges pointwise and uniformly to \( u_r \). It can also be assumed \( J(r, s_n(t)) \leq J(r, u_r(t)) \) for all \( t \in [1/r, r] \). Therefore, 
\[
\lim_{n,m \to \infty} \int_{1/r}^{r} J(r, s_m - s_n) \frac{dt}{t} = 0.
\]

It follows from the definition of the Bochner integral that 
\[
\lim_{n \to \infty} \int_{1/r}^{r} s_n \frac{dt}{t} = \int_{1/r}^{r} u_r \frac{dt}{t} \in A_0 \cap A_1.
\]

This proves the lemma.

The remarkable thing is that the two spaces, \((A_0, A_1)_{\theta,q}\) and \((A_0, A_1)_{\theta,q,J}\) coincide and have equivalent norms. The following important lemma, called the fundamental lemma of interpolation theory in [73] is used to prove this. This lemma is really incredible.

**Lemma 41.7.3** Suppose for \( a \in A_0 + A_1 \), \( \lim_{t \to 0^+} K(t,a) = 0 \) and \( \lim_{t \to \infty} \frac{K(t,a)}{t} = 0 \). Then for any \( \varepsilon > 0 \), there is a representation, 
\[
a = \sum_{i=-\infty}^{\infty} u_i = \lim_{n,m \to \infty} \sum_{i=-m}^{n} u_i, \ u_i \in A_0 \cap A_1,
\]

the convergences taking place in \( A_0 + A_1 \), such that
\[
J(2^i, u_i) \leq 3(1 + \varepsilon) K(2^i, a).
\]

**Proof:** For each \( i \), there exist \( a_{0,i} \in A_0 \) and \( a_{1,i} \in A_1 \) such that
\[
a = a_{0,i} + a_{1,i},
\]
and
\[
(1 + \varepsilon) K(2^i, a) \geq ||a_{0,i}||_{A_0} + 2^i ||a_{1,i}||_{A_1}.
\]

This follows directly from the definition of \( K(t,a) \). From the assumed limit conditions on \( K(t,a) \),
\[
\lim_{i \to \infty} ||a_{1,i}||_{A_1} = 0, \ \lim_{i \to -\infty} ||a_{0,i}||_{A_0} = 0.
\]

Then let \( u_i \equiv a_{0,i} - a_{0,i-1} = a_{1,i-1} - a_{1,i} \). The reason these are equal is \( a = a_{0,i} + a_{1,i} = a_{0,i-1} + a_{1,i-1} \). Then
\[
\sum_{i=-m}^{n} u_i = a_{0,n} - a_{0,-(m+1)} = a_{1,-(m+1)} - a_{1,n}.
\]

It follows \( a - \sum_{i=-m}^{n} u_i = a - (a_{0,n} - a_{0,-(m+1)}) = a_{0,-(m+1)} + a_{1,n} \), and both terms converge to zero as \( m \) and \( n \) converge to \( \infty \) by [73]. Therefore,
\[
K(1, a - \sum_{i=-m}^{n} u_i) \leq ||a_{0,-(m+1)}|| + ||a_{1,n}||
\]
and so this shows $a = \sum_{i=-\infty}^{\infty} u_i$ which is one of the claims of the lemma. Also

$$J(2^i, u_i) \equiv \max (||u_i||_{A_0}, 2^i ||u_i||_{A_1}) \leq ||u_i||_{A_0} + 2^i ||u_i||_{A_1}$$

$$\leq 2(||a_{0,i-1}||_{A_0} + 2^{i-1}||a_{1,i-1}||_{A_1})$$

$$\leq (1 + \varepsilon) K(2^i, a) + 2(1 + \varepsilon) K(2^{i-1}, a) \leq 3(1 + \varepsilon) K(2^i, a)$$

because $t \to K(t, a)$ is nondecreasing. This proves the lemma.

**Lemma 41.7.4** If $a \in A_0 \cap A_1$, then $K(t, a) \leq \min \left(1, \frac{t}{s}\right) J(s, a)$.

**Proof:** If $s \geq t$, then $\min \left(1, \frac{t}{s}\right) = \frac{t}{s}$ and so

$$\min \left(1, \frac{t}{s}\right) J(s, a) = \frac{t}{s} \max (||a||_{A_0}, s ||a||_{A_1}) \geq \left(\frac{t}{s}\right) s ||a||_{A_1}$$

$$= t ||a||_{A_1} \geq K(t, a) .$$

Now in case $s < t$, then $\min \left(1, \frac{t}{s}\right) = 1$ and so

$$\min \left(1, \frac{t}{s}\right) J(s, a) = \max (||a||_{A_0}, s ||a||_{A_1}) \geq ||a||_{A_0}$$

$$\geq K(t, a) .$$

This proves the lemma.

**Theorem 41.7.5** Let $A_0, A_1, K$ and $J$ be as described above. Then for all $q \geq 1$ and $\theta \in (0, 1)$,

$$(A_0, A_1)_{\theta,q} = (A_0, A_1)_{\theta,q,J}$$

and furthermore, the norms are equivalent.

**Proof:** Begin with $a \in (A_0, A_1)_{\theta,q}$. Thus

$$||a||_{\theta,q}^q = \int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} < \infty \quad (41.7.43)$$

and it is necessary to produce $u(t)$ as described above,

$$a = \int_0^\infty u(t) \frac{dt}{t}$$

where

$$\int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} < \infty .$$

From $\lim_{t \to 0^+} K(t, a) = 0$ since $t \to K(t, a)$ is nondecreasing and so if its limit is positive, the integrand would have a non integrable singularity like $t^{-\theta q-1}$.

Next consider what happens to $K(t, a)$ as $t \to \infty$.

**Claim:** $t \to \frac{K(t, a)}{t}$ is decreasing.
**Proof of the claim:** Choose \( a_0 \in A_0 \) and \( a_1 \in A_1 \) such that \( a_0 + a_1 = a \) and

\[
K(t, a) + \varepsilon t > \|a_0\|_{A_0} + t\|a_1\|_{A_1}
\]

let \( s > t \). Then

\[
\frac{K(t, a) + ts}{t} \geq \frac{\|a_0\|_{A_0} + t\|a_1\|_{A_1}}{t} \geq \frac{\|a_0\|_{A_0} + s\|a_1\|_{A_1}}{s} = \frac{K(s, a)}{s}.
\]

Since \( \varepsilon \) is arbitrary, this proves the claim.

Let \( r = \lim_{t \to \infty} \frac{K(t, a)}{t} \). Is \( r = 0? \) Suppose to the contrary that \( r > 0 \). Then the integrand of (41.7.43) is at least as large as

\[
t^{-\theta q} K(t, a)^{q-1} \frac{K(t, a)}{t} \geq t^{-\theta q} K(t, a)^{q-1} r
\]

\[
\geq t^{-\theta q} (tr)^{q-1} r \geq r^{q(q(1-\theta))^{-1}}
\]

whose integral is infinite. Therefore, \( r = 0 \).

Lemma 41.7.43 implies there exist \( u_i \in A_0 \cap A_1 \) such that \( a = \sum_{i=-\infty}^{\infty} u_i \), the convergence taking place in \( A_0 + A_1 \) with the inequality of that Lemma holding,

\[
J(2^i, u_i) \leq 3(1 + \varepsilon) K(2^i, a).
\]

For \( i \) an integer and \( t \in [2^{i-1}, 2^i) \), let

\[
u(t) = u_i / \ln 2.
\]

Then

\[
a = \sum_{i=-\infty}^{\infty} u_i = \int_{0}^{\infty} u(t) \frac{dt}{t}. \tag{41.7.44}
\]

Now

\[
||a||_{\theta, q, J}^q \leq \int_{0}^{\infty} \left(t^{-\theta} J(t, u(t))\right)^q \frac{dt}{t}
\]

\[
= \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} J(t, \frac{u_i}{\ln 2})\right)^q \frac{dt}{t}
\]

\[
\leq \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} J(2^i, u_i)\right)^q \frac{dt}{t}
\]

\[
\leq \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} 3(1 + \varepsilon) K(2^i, a)\right)^q \frac{dt}{t}
\]
Using the above claim, \( K(2^i, a) \leq K(2^{i-1}, a) \) and so \( K(2^i, a) \leq 2K(2^{i-1}, a) \). Therefore, the above is no larger than

\[
\leq 2 \left( \frac{1}{\ln 2} \right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} (t^{-\theta} 3(1 + \varepsilon) K(2^{i-1}, a))^q \frac{dt}{t}
\]

\[
\leq 2 \left( \frac{1}{\ln 2} \right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} (t^{-\theta} 3(1 + \varepsilon) K(t, a))^q \frac{dt}{t}
\]

\[
= 2 \left( \frac{3(1 + \varepsilon)}{\ln 2} \right)^q \int_0^\infty \left( t^{-\theta} K(t, a) \right)^q \frac{dt}{t} = 2 \left( \frac{3(1 + \varepsilon)}{\ln 2} \right)^q ||a||_{\theta, q}^q \tag{41.7.45}
\]

This has shown that if \( a \in (A_0, A_1)_{\theta, q} \), then by \[41.7.44\] and \[41.7.45\], \( a \in (A_0, A_1)_{\theta, q, J} \) and

\[
||a||_{\theta, q, J}^q \leq 2 \left( \frac{3(1 + \varepsilon)}{\ln 2} \right)^q ||a||_{\theta, q}^q. \tag{41.7.46}
\]

It remains to prove the other inclusion and norm inequality, both of which are much easier to obtain. Thus, let \( a \in (A_0, A_1)_{\theta, q, J} \) with

\[
a = \int_0^\infty u(t) \frac{dt}{t} \tag{41.7.47}
\]

where \( u \) is a strongly measurable function having values in \( A_0 \cap A_1 \) and for which

\[
\int_0^\infty \left( t^{-\theta} J(t, u(t)) \right)^q dt < \infty. \tag{41.7.48}
\]

\[
K(t, a) = K(t, \int_0^\infty u(s) \frac{ds}{s}) \leq \int_0^\infty K(t, u(s)) \frac{ds}{s}. \tag{41.7.49}
\]

Now by Lemma \[41.7.2\], this is dominated by an expression of the form

\[
\leq \int_0^\infty \min \left( 1, \frac{t}{s} \right) J(s, u(s)) \frac{ds}{s} = \int_0^\infty \min \left( 1, \frac{1}{s} \right) J(ts, u(ts)) \frac{ds}{s} \tag{41.7.50}
\]

where the equation follows from a change of variable. From Minkowski’s inequality and \[41.7.50\],

\[
||a||_{\theta, q} \equiv \left( \int_0^\infty \left( t^{-\theta} K(t, a) \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
\leq \left( \int_0^\infty \left( t^{-\theta} \int_0^\infty \min \left( 1, \frac{1}{s} \right) J(ts, u(ts)) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
\leq \int_0^\infty \left( \int_0^\infty \left( t^{-\theta} \min \left( 1, \frac{1}{s} \right) J(ts, u(ts)) \right)^q \frac{dt}{t} \right)^{1/q} \frac{ds}{s}. \nonumber
\]

Now change the variable in the inside integral to obtain, letting \( t = \tau s \),

\[
\leq \int_0^\infty \left( \int_0^\infty \left( \tau^{-\theta} \min \left( 1, \frac{1}{s} \right) J(\tau s, u(\tau s)) \right)^q \frac{ds}{s} \right)^{1/q} \frac{dt}{t}. \nonumber
\]
$\leq \int_0^\infty \min\left(1, \frac{1}{s}\right) \left(\int_0^\infty \left(t^{-\theta} J(t,s,u(t))\right)^q \frac{dt}{t}\right)^{1/q} \frac{ds}{s}$

$= \left(\int_0^\infty \min\left(1, \frac{1}{s}\right) s^\theta \frac{ds}{s} \left(\int_0^\infty \left(t^{-\theta} J(t,u(t))\right)^q \frac{dt}{t}\right)^{1/q}\right)$

$= \left(\frac{1}{(1-\theta)\theta}\right) \left(\int_0^\infty \left(t^{-\theta} J(t,u(t))\right)^q \frac{dt}{t}\right)^{1/q}$.

This has shown that

$$\|a\|_{\theta,q} \leq \left(\frac{1}{(1-\theta)\theta}\right) \left(\int_0^\infty \left(t^{-\theta} J(t,u(t))\right)^q \frac{dt}{t}\right)^{1/q} < \infty$$

for all $u$ satisfying \[41.7.47\] and \[41.7.48\]. Therefore, taking the infimum it follows

$$a \in (A_0, A_1)_{\theta,q}$$

This proves the theorem.

### 41.8 Duality And Interpolation

In this section it will be assumed that $A_0 \cap A_1$ is dense in $A_i$ for $i = 0, 1$. This is done so that $A_0' \subseteq (A_0 \cap A_1)'$ and the inclusion map is continuous. Thus it makes sense to add something in $A_0'$ to something in $A_1'$.

What is the dual space of $(A_0, A_1)_{\theta,q}$? The answer is based on the following lemma, \[13\]. Remember that

$$J(t,a) = \max (\|a\|_{A_0}, t \|a\|_{A_1})$$

and this is a norm on $A_0 \cap A_1$ and

$$K(t,a) = \inf \{ \|a\|_{A_0} + t \|a\|_{A_1} : a = a_0 + a_1 \}.$$

As mentioned above, $A_0' + A_1' \subseteq (A_0 \cap A_1)'$. In fact these two are equal. This is the first part of the following lemma.

**Lemma 41.8.1** Suppose $A_0 \cap A_1$ is dense in $A_i$, $i = 0, 1$. Then

$$A_0' + A_1' = (A_0 \cap A_1)' \quad \text{and} \quad (A_0 \cap A_1) '$$

and for $a' \in A_0' + A_1' = (A_0 \cap A_1)'$,

$$K(t,a') = \sup_{a \in A_0 \cap A_1} \frac{|a'(a)|}{J(t^{-1},a)}.$$
Thus $K(t, \cdot)$ is an equivalent norm to the usual operator norm on $(A_0 \cap A_1)'$ taken with respect to $J(t^{-1}, \cdot)$. If, in addition to this, $A_i$ is reflexive, then for $a' \in A_0' \cap A_1'$, and $a \in A_0 \cap A_1$,

$$J(t, a') K(t^{-1}, a) \geq |a'(a)| .$$

(41.8.53)

**Proof:** First consider the claim that $A_0' + A_1' = (A_0 \cap A_1)'$. As noted above, $\subseteq$ is clear. Define a norm on $A_0 \times A_1$ as follows.

$$\|(a_0, a_1)\|_{A_0 \times A_1} \equiv \max (\|a_0\|_{A_0}, t^{-1} \|a_1\|_{A_1}) .$$

(41.8.54)

Let $a' \in (A_0 \cap A_1)'$. Let

$$E \equiv \{(a, a) : a \in A_0 \cap A_1\}$$

with the norm $J(t^{-1}, a) \equiv \max (\|a\|_{A_0}, t^{-1} \|a\|_{A_1})$. Now define $\lambda$ on $E$, the subspace of $A_0 \times A_1$ by

$$\lambda ((a, a)) \equiv a'(a) .$$

Thus $\lambda$ is a continuous linear map on $E$ and in fact,

$$|\lambda ((a, a))| = |a'(a)| \leq \|a'\| J(t^{-1}, a) .$$

By the Hahn-Banach theorem there exists an extension of $\lambda$ to all of $A_0 \times A_1$. This extension is of the form $(a_0', a_1') \in A_0' \times A_1'$. Thus

$$(a_0', a_1') ((a, a)) = a_0'(a) + a_1'(a) = a'(a)$$

and therefore, $a_0' + a_1' = a'$ provided $a_0' + a_1'$ is continuous. But

$$\|(a_0' + a_1')(a)\| = |a_0'(a) + a_1'(a)| \leq |a_0'(a)| + |a_1'(a)|$$

$$\leq \|a_0'\| \|a\|_{A_0} + \|a_1'\| \|a\|_{A_1}$$

$$\leq \|a_0'\| \|a\|_{A_0} + t \|a_1'\| t^{-1} \|a\|_{A_1}$$

$$\leq \left( (\|a_0'\| + t \|a_1'\|) J(t^{-1}, a) \right)$$

which shows that $a_0' + a_1'$ is continuous and in fact

$$\|a_0' + a_1'\|_{(A_0 \cap A_1)'} \leq (\|a_0'\| + t \|a_1'\|) .$$

This proves the first part of the lemma.

**Claim:** With this definition of the norm in (41.8.54), the operator norm of $(a_0', a_1') \in (A_0 \times A_1)' = A_0' \times A_1'$ is

$$\|(a_0', a_1')\|_{(A_0 \times A_1)'} = \|a_0'\|_{A_0'} + t \|a_1'\|_{A_1'} .$$

(41.8.55)

**Proof of the claim:** $$(\|a_0'\| + t \|a_1'\|) \|a_0\| = (\|a_0'\| + t \|a_1'\|) \max (\|a_0\|, t^{-1} \|a_1\|) .$$
It follows that for all \( \lambda \) is an extension of \( \lambda \). By the Hahn Banach theorem, there exists an extension of \( \lambda \) so that

\[
\text{is continuous on the subspace,}
\]

\[
\text{Thus, from (41.8.56),}
\]

\[
\text{is equality achieved? Let } a_{0n} \text{ and } a_{1n} \text{ be points of } A_0 \text{ and } A_1 \text{ respectively such that } \|a_{0n}\|, \|a_{1n}\| \leq 1 \text{ and } \lim_{n \to \infty} a', (a_{1n}) = \|a'\|. \text{ Then}
\]

\[
(a', a'_1) (a_{0n}, ta_{1n}) \to \|a'|| + t \|a'|',
\]

and also, \( (a_{0n}, ta_{1n}) |_{A_0 \times A_1} = \|a_{0n}\|, t^{-1} t |a_{1n}| |A_1| \leq 1 \). Therefore, equality is indeed achieved and this proves the claim.

Consider (HIL5.13). Take \( a' \in A'_0 + A'_1 = (A_0 \cap A_1)' \) and let

\[
E \equiv \{(a, a) \in A_0 \times A_1 : a \in A_0 \cap A_1\}.
\]

Now define a linear map, \( \lambda \) on \( E \) as before.

\[
\lambda ((a, a)) \equiv a' (a).
\]

If \( a' = \tilde{a}'_0 + \tilde{a}'_1 \),

\[
|\lambda ((a, a))| \leq \|\tilde{a}'_0\|_{A'_0} |a|_{A_0} + \|\tilde{a}'_1\|_{A'_1} |a|_{A_1}
\]

\[
= \|\tilde{a}'_0\|_{A'_0} |a|_{A_0} + t \|\tilde{a}'_1\|_{A'_1} t^{-1} |a|_{A_1}
\]

\[
= \|\tilde{a}'_0\| + t \|\tilde{a}'_1\| \|a\|_{A_0 \times A_1}
\]

so \( \lambda \) is continuous on the subspace, \( E \) of \( A_0 \times A_1 \) and

\[
\|\lambda\|_{E'} \leq \|\tilde{a}'_0\| + t \|\tilde{a}'_1\|.
\]

By the Hahn Banach theorem, there exists an extension of \( \lambda \) defined on all of \( A_0 \times A_1 \) with the same norm. Thus, from (HIL5.13), there exists \( (a'_0, a'_1) \in (A_0 \times A_1)' \) which is an extension of \( \lambda \) such that

\[
\|\lambda\|_{E'} = \|\tilde{a}'_0\|_{A'_0} + t \|\tilde{a}'_1\|_{A'_1} = \|\lambda\|_{E'}
\]

and for all \( a \in A_0 \cap A_1 \),

\[
a'_0 (a) + a'_1 (a) = \lambda ((a, a)) = a' (a).
\]

It follows that \( a'_0 + a'_1 = a' \) in \( (A_0 \cap A_1)' \). Therefore, from (HIL5.14),

\[
\|\lambda\|_{E'} \leq \inf \{\|\tilde{a}'_0\|_{A'_0} + t \|\tilde{a}'_1\|_{A'_1} : a' = \tilde{a}'_0 + \tilde{a}'_1\} \equiv K (t, a') \tag{41.8.57}
\]

\[
\leq \|a'_0\|_{A'_0} + t \|a'_1\|_{A'_1} = \|\lambda\|_{E'} \equiv \sup_{a \in A_0 \cap A_1} \frac{|a' (a)|}{J (t^{-1}, a)} \tag{41.8.58}
\]
41.8. DUALITY AND INTERPOLATION

because on \( E \), \( J (t^{-1}, a) = \|(a, a)\|_{A_0 \times A_1} \) which proves \( \text{Lemma 41.8.3} \).

To obtain \( \text{Lemma 41.8.2} \) in the case that \( A_i \) is reflexive, apply \( \text{Lemma 41.8.2} \) to the case where \( A_i'' \) plays the role of \( A_i \) in \( \text{Lemma 41.8.2} \). Thus, for \( a'' \in A_0'' + A_1'' \),

\[
K (t, a'') = \sup_{a' \in A_0'' \cap A_1''} \left| \frac{a'' (a')}{} \right| J (t^{-1}, a').
\]

Now \( a'' = a_1'' + a_0'' = \eta_1 (a_1 + \eta_0 a_0) \) where \( \eta_1 \) is the map from \( A_i \) to \( A_i'' \) which is onto and preserves norms, given by \( \eta a' (a) \equiv a' (a) \). Therefore, letting \( a_1 + a_0 = a \)

\[
K (t, a) = K (t, a'') = \sup_{a' \in A_0'' \cap A_1''} \left| \frac{a'' (a')}{\left| J (t^{-1}, a') \right|} \right| = \sup_{a' \in A_0'' \cap A_1''} \left| \frac{\eta_1 (a_1 + \eta_0 a_0) (a')}{\left| J (t^{-1}, a') \right|} \right|
\]

and so

\[
K (t, a) = \sup_{a' \in A_0'' \cap A_1''} \left| \frac{a' (a)}{\left| J (t^{-1}, a') \right|} \right|
\]

Changing \( t \to t^{-1} \),

\[
K (t^{-1}, a) J (t, a') \geq |a' (a)|.
\]

which proves the lemma.

Consider \( (A_0, A_1)_{\theta, q} \).

**Definition 41.8.2** Let \( q \geq 1 \). Then \( \lambda^{\theta, q} \) will denote the sequences, \( \{\alpha_i\}_{i=-\infty}^{\infty} \) such that

\[
\sum_{i=-\infty}^{\infty} |\alpha_i| 2^{-i\theta} < \infty.
\]

For \( \alpha \in \lambda^{\theta, q} \),

\[
\|\alpha\|_{\lambda^{\theta, q}} \equiv \left( \sum_{i=-\infty}^{\infty} |\alpha_i| 2^{-i\theta} \right)^{1/q}.
\]

Thus \( \alpha \in \lambda^{\theta, q} \) means \( \{\alpha, 2^{-i\theta}\} \in l_q \).

**Lemma 41.8.3** Let \( f (t) \geq 0 \), and let \( f (t) = \alpha_i \) for \( t \in [2^i, 2^{i+1}] \) where \( \alpha \in \lambda^{\theta, q} \). Then there exists a constant, \( C \), such that

\[
\|t^{-\theta} f\|_{L^q (0, \infty; \frac{\varphi}{2})} \leq C \|\alpha\|_{\lambda^{\theta, q}}. \tag{41.8.59}
\]

Also, if whenever \( \alpha \in \lambda^{\theta, q} \), and \( \alpha_i \geq 0 \) for all \( i \),

\[
\sum_{i} f (2^i) 2^{-i} \alpha_i \leq C \|\alpha\|_{\lambda^{\theta, q}}, \tag{41.8.60}
\]

then

\[
\left\{ f (2^i) \right\}_{i=-\infty}^{\infty} \|_{\lambda^{\theta, q}} \leq C. \tag{41.8.61}
\]
\textbf{Proof:} Consider \textit{41.8.59.}

$$\int_0^\infty \left( t^{-\theta} f (t) \right)^q \frac{dt}{t} = \sum_i \int_{2^i}^{2^{i+1}} t^{-\theta} \alpha_i^q \frac{dt}{t}$$

$$\leq \sum_i \int_{2^i}^{2^{i+1}} (2^{-i} \alpha_i)^q \frac{dt}{t} = \ln 2 \| \alpha \|_{\lambda_{\theta,q}}^q.$$

\textit{41.8.60.} is next. By \textit{41.8.60}, whenever $\alpha \in \lambda_{\theta,q}$,

$$\left| \sum_i \{ f (2^i)^{2-(1-\theta)i} \} 2^{-i} \alpha_i \right| \leq C \| \{ f (2^i)^{2-(1-\theta)i} \} \|_{l_q}.$$

It follows from the Riesz representation theorem that $\{ f (2^i)^{2-(1-\theta)i} \}$ is in $l_{q'}$ and

$$\left| \sum_i \left( f (2^i)^{2-(1-\theta)i} \right) 2^{-i} \alpha_i \right| \leq C \| \{ f (2^i)^{2-(1-\theta)i} \} \|_{l_{q'}} \leq C.$$

This proves the lemma.

The dual space of $(A_0, A_1)_{\theta,q,J}$ is discussed next.

\textbf{Lemma 41.8.4} \textit{Let $\theta \in (0, 1)$ and let $q \geq 1$. Then,}

$$(A_0, A_1)_{\theta,q,J}' \subseteq (A_1', A_0')_{1-\theta, q'}$$

\textit{and the inclusion map is continuous.}

\textbf{Proof:} Let $a' \in (A_0, A_1)_{\theta,q,J}'$. Now

$$A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta,q,J}$$

and if

$$a \in (A_0, A_1)_{\theta,q,J},$$

then $a$ has a representation of the form

$$a = \int_0^\infty u (t) \frac{dt}{t}$$

where

$$\int_0^\infty \left( t^{-\theta} J (t, u (t)) \right)^q \frac{dt}{t} < \infty$$

where

$$J (t, u (t)) = \max \{ ||u (t)||_{A_0}, t ||u (t)||_{A_1} \}$$

for $u (t) \in A_0 \cap A_1$. Now let

$$u_r (t) = \begin{cases} u (t) & \text{if } t \in \left( \frac{1}{r}, r \right) \\ 0 & \text{otherwise} \end{cases}.$$
41.8. DUALITY AND INTERPOLATION

Then \( \int_0^\infty \left( t^{-\theta} \int J(t, u_r(t)) \right)^q \frac{dt}{t} < \infty \) and

\[
a_r \equiv \int_0^\infty u_r(t) \frac{dt}{t} \in A_0 \cap A_1
\]

by Lemma 41.7.2. Also

\[
|a - a_r|_{\theta, q, J}^q \leq \int_0^1 \left( t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t} + \int_r^\infty \left( t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t}
\]

which is small whenever \( r \) is large enough thanks to the dominated convergence theorem. Therefore, \( A_0 \cap A_1 \) is dense in \( (A_0, A_1)_{\theta, q, J} \) and so

\[
(A_0, A_1)'_{\theta, q, J} \subseteq (A_0 \cap A_1)' = A_0' + A_1',
\]

the equality following from Lemma 41.8.1.

It follows \( a' \in A_0' + A_1' \) and so, by Lemma 41.8.1, there exists \( b_i \in A_0 \cap A_1 \) such that

\[
K \left( 2^{-i}, a', A_0', A_1' \right) - \varepsilon \min \left( 1, 2^{-i} \right) \leq \frac{a'(b_i)}{J(2^i, b_i, A_0, A_1)}.
\]

Now let \( \alpha \in \lambda^{\theta, q} \) with \( \alpha_i \geq 0 \) for all \( i \) and let

\[
a_{\infty} \equiv \sum_i J(2^i, b_i, A_0, A_1)^{-1} b_i \alpha_i. \quad (41.8.62)
\]

Consider first whether \( a_{\infty} \) makes sense before proceeding further.

\[
a_{\infty} \equiv \sum_i \max \left( ||b_i||_{A_0}, 2^i ||b_i||_{A_1} \right) 2^{-i\theta} \alpha_i.
\]

Now

\[
\left\| \frac{b_i 2^{i\theta}}{\max \left( ||b_i||_{A_0}, 2^i ||b_i||_{A_1} \right)} \right\|_{A_0 + A_1} \leq \begin{cases} 2^{i\theta} & \text{if } i < 0 \\ 2^{-i(1-\theta)} & \text{if } i \geq 0 \end{cases}
\]

(41.8.63)

This is fairly routine to verify. Consider the case where \( i \geq 0 \). Then

\[
\left\| \frac{b_i 2^{i\theta}}{\max \left( ||b_i||_{A_0}, 2^i ||b_i||_{A_1} \right)} \right\|_{A_0 + A_1} \leq \left\| \frac{b_i 2^{i\theta}}{2^i ||b_i||_{A_1}} \right\|_{A_0 + A_1} \leq 2^{-i(1-\theta)}
\]

because \( ||b_i||_{A_1} \geq ||b_i||_{A_0 + A_1} \). Therefore,

\[
\sum_{i=0}^{M} \left\| \frac{b_i 2^{i\theta}}{\max \left( ||b_i||_{A_0}, 2^i ||b_i||_{A_1} \right)} 2^{-i\theta} \alpha_i \right\|_{A_0 + A_1} \leq
\]
\[
\sum_{i=0}^{M} 2^{-i(1-\theta)} 2^{-i\theta} \alpha_i \leq \left( \sum_{i=0}^{\infty} 2^{-i(1-\theta)q'} \right)^{1/q'} \left( \sum_{i=0}^{\infty} 2^{-iq\theta} \alpha_i \right)^{1/q} < \infty
\]

and similarly,
\[
\sum_{i=-\infty}^{0} \left\| \frac{b_i 2^{i\theta}}{\max \left( \|b_i\|_{A_0}, 2^{i} \|b_i\|_{A_1} \right)} 2^{-i\theta} \alpha_i \right\|_{A_0 + A_1} < \infty
\]

converges. Therefore, \( a_{\infty} \) makes sense in \( A_0 + A_1 \) and also from (41.8.63), we see that
\[
\left\{ \frac{\|b_i\|_{A_0 + A_1} 2^{i\theta}}{J(2^i, b_i)} \right\} \in \lambda^{(1-\theta)q'}
\]

Now let
\[
u(t) = \frac{\alpha_i b_i}{J(2^i, b_i) \ln 2}\text{ on } [2^{i-1}, 2^i].
\]

Then
\[
\int_0^\infty \nu(t) \frac{dt}{t} = \sum_i \int_{2^{i-1}}^{2^i} \frac{\alpha_i b_i}{J(2^i, b_i) \ln 2} \frac{dt}{t}
\]
\[
= \sum_i \frac{\alpha_i b_i}{J(2^i, b_i)} = a_{\infty}.
\]

Also
\[
\int_0^\infty (t^{-\theta} J(t, \nu(t)))^q \frac{dt}{t} \leq \sum_i \int_{2^{i-1}}^{2^i} \left( 2^{(1-i)\theta} J(2^i, \nu(2^{i-1})) \right) \frac{dt}{t}
\]
\[
\leq \sum_i \left[ 2^{- (i-1)\theta} J(2^i, \nu(2^{i-1})) \right]^q \ln 2
\]
\[
= \sum_i \left[ 2^{- (i-1)\theta} \frac{J(2^i, b_i) \alpha_i}{J(2^i, b_i) \ln 2} \right]^q \ln 2
\]
\[
= C \sum_i (2^{-i\theta} |\alpha_i|)^q < \infty \tag{41.8.64}
\]

and so \( \|a_{\infty}\|_{\theta, q, J} < \infty \). Now for \( a' \) as above, \( a' \in (A_0, A_1)'_{\theta, q, J} \subseteq (A_0 + A_1)' \), and so since the sum for \( a_{\infty} \) converges in \( A_0 + A_1 \), we have
\[
a'(a_{\infty}) = \sum_i J(2^i, b_i)^{-1} \alpha_i a'(b_i).
\]

Therefore,
\[
a'(a_{\infty}) \geq \sum_i \left[ K(2^{-i}, a') - \varepsilon \min(1, 2^{-i}) \right] \alpha_i
\]
\[
= \sum_i K(2^{-i}, a') \alpha_i - \sum_i \varepsilon \min(1, 2^{-i}) \alpha_i
\]
\[
= \sum_i K(2^{-i}, a') \alpha_i - O(\varepsilon) \tag{41.8.65}
\]
41.8. DUALITY AND INTERPOLATION

The reason for this is that $\alpha \in \lambda^{\theta,q}$ so $\{\alpha_i 2^{-i\theta}\} \in l_q$. Therefore,

\[
\sum_i \varepsilon \min \left(1, 2^{-i}\right) \alpha_i = \varepsilon \left(\sum_{i=0}^{\infty} 2^{-i} \alpha_i + \sum_{i=-\infty}^{-1} \alpha_i\right) \\
= \varepsilon \left(\sum_{i=0}^{\infty} 2^{-i \theta} 2^{(\theta-1) i} \alpha_i + \sum_{i=-\infty}^{-1} \alpha_i 2^{-i \theta} 2^{i \theta}\right) \\
\leq \varepsilon \left(\left(\sum_i |\alpha_i 2^{-i \theta}|^q\right)^{1/q} \left(\sum_{i=0}^{\infty} \left(2^{(\theta-1) i}\right)^{q'}\right)^{1/q'} \right) \\
+ \left(\sum_i |\alpha_i 2^{-i \theta}|^q\right)^{1/q} \left(\sum_{i=0}^{\infty} \left(2^{i \theta}\right)^{q'}\right)^{1/q'} \\
< C \varepsilon.
\]

Also

\[
|a' (a_\infty)| \leq ||a'||_{(A_0,A_1)_{\theta,q,J}} ||a_\infty||_{(A_0,A_1)_{\theta,q,J}}.
\]

Now from the definition of $K$,

\[
K \left(2^{-i}, a', A'_0, A'_1\right) = 2^{-i} K \left(2^i, a', A'_0, A'_1\right)
\]

and so from Lemma 41.8.3,

\[
\sum_i 2^{-i} K \left(2^i, a', A'_1, A'_0\right) \alpha_i - O(\varepsilon) \leq a' (a_\infty) \\
\leq ||a'||_{(A_0,A_1)_{\theta,q,J}} C_\theta ||\alpha||_{\lambda^{\theta,q}}.
\]

Since $\varepsilon$ is arbitrary, it follows that whenever, $\alpha \in \lambda^{\theta,q}, \alpha_i \geq 0$,

\[
\sum_i 2^{-i} K \left(2^i, a', A'_1, A'_0\right) \alpha_i \leq ||a'||_{(A_0,A_1)_{\theta,q,J}} C_\theta ||\alpha||_{\lambda^{\theta,q}}.
\]

By Lemma 41.8.3, $\{K \left(2^i, a', A'_1, A'_0\right)\} \in \lambda^{1-\theta,q'}$ and

\[
||\{K \left(2^i, a', A'_1, A'_0\right)\}||_{\lambda^{1-\theta,q'}} \leq ||a'||_{(A_0,A_1)_{\theta,q,J}} C_\theta.
\]
Therefore,

\[
\left( \frac{1}{\ln 2} \int_0^\infty (K(t, a', A_1', A_0') t^{-(1-\theta)q'} \frac{dt}{t})^{1/q'} \right) = \left( \sum_i \frac{1}{\ln 2} \int_{2^i}^{2^{i+1}} (K(t, a', A_1', A_0') t^{-(1-\theta)q'} \frac{dt}{t})^{1/q'} \right) \leq \left( \sum_i \left( \frac{2^{-i(1-\theta)}}{K(0, a', A_1', A_0')} \right)^{1/q'} \right) \leq ||a'||_{(A_0, A_1)'_{\theta,q,J}} C_\theta.
\]

Thus

\[
||a'||_{(A_1', A_0)'_{1-\theta,q'}} = \left| \left| t^{-(1-\theta)K(t, a', A_1', A_0')} \right|_{L^{q'}(0, \infty, \frac{dt}{t})} \leq C ||a'||_{(A_0, A_1)'_{\theta,q,J}}
\]

which shows that \((A_0, A_1)'_{\theta,q,J} \subseteq (A_1', A_0)'_{1-\theta,q'}\) with the inclusion map continuous. This proves the lemma.

**Lemma 41.8.5** If \(A_i\) is reflexive for \(i = 0, 1\) and if \(A_0 \cap A_1\) is dense in \(A_i\), then

\[
(A_1', A_0)'_{1-\theta,q',J} \subseteq (A_0, A_1)'_{\theta,q,J}
\]

and the inclusion map is continuous.

**Proof:** Let \(a' \in (A_1', A_0)'_{1-\theta,q',J}\). Thus, there exists \(u^*\) bounded on compact subsets of \((0, \infty)\) and measurable with values in \(A_0 \cap A_1\) and

\[
a' = \int_0^\infty u^* \left( \frac{dt}{t} \right), \quad (41.8.66)
\]

\[
\int_0^\infty \left( t^{-(1-\theta)} J(t, u^* (t)) \right) a' \frac{dt}{t} < \infty.
\]

Then

\[
a' = \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} u^* \left( \frac{dt}{t} \right) = \sum_{i=-\infty}^{\infty} a'_i
\]

where \(a'_i \in A'_i \cap A'_0\), the convergence taking place in \(A'_1 + A'_0\). Now let \(a \in A_0 \cap A_1\).
41.8. DUALITY AND INTERPOLATION

From Lemma 41.8.1

\[ |a'(a)| \leq \sum_{i=-\infty}^{\infty} |a'_i(a)| \]
\[ \leq \sum_{i=-\infty}^{\infty} J(2^{-i}, a'_i, A'_0, A'_1) K(2^i, a, A_0, A_1) \]
\[ = \sum_{i=-\infty}^{\infty} 2^{-i} J(2^i, a'_i, A'_0, A'_1) K(2^i, a, A_0, A_1) \]
\[ \leq \left( \sum_{i} \left( 2^{-(1-th)i} J(2^i, a'_i, A'_1, A'_0) \right)^{q'} \right)^{1/q} \cdot \left( \sum_{i} \left( 2^{-\theta i} K(2^i, a, A_0, A_1) \right)^q \right)^{1/q} \]
\[ \leq C \left[ \int_{0}^{\infty} \left( t^{-(1-th)} J(t, u^*(t), A'_1, A'_0) \right)^{q'} \frac{dt}{t} \right]^{1/q'} \cdot \left[ \int_{0}^{\infty} \left( t^{-\theta} K(t, a, A_0, A_1) \right)^q \frac{dt}{t} \right]^{1/q} . \]

In going from the sums to the integrals, express the first sum as a sum of integrals on \([2^i, 2^{i+1})\) and the second sum as a sum of integrals on \([2^{i-1}, 2^i]\).

Taking the infimum over all \(u^*\) representing \(a'\),

\[ |a'(a)| \leq C ||a'||((A'_1, A'_0))_{1-th, q', J} ||a||_{\theta, q}. \]

It follows \(a' \in (A_0, A_1)_{\theta, q}'\) and \(||a'||((A_0, A_1))_{\theta, q} \leq C ||a'||((A'_1, A'_0))_{1-th, q', J}\) which proves the lemma.

With these two lemmas the main result follows.

**Theorem 41.8.6** Suppose \(A_0 \cap A_1\) is dense in \(A_i\) and \(A_i\) is reflexive. Then

\[ (A'_1, A'_0)_{1-th, q'} = (A_0, A_1)'_{\theta, q} \]

and the norms are equivalent.

**Proof:** By Theorem 41.7.9, and the last two lemmas,

\[ (A_0, A_1)'_{\theta, q} = (A_0, A_1)'_{\theta, q, J} \subseteq (A'_1, A'_0)_{1-th, q'} \]
\[ = (A'_1, A'_0)_{1-th, q', J} \subseteq (A_0, A_1)'_{\theta, q} . \]

This proves the theorem.
Chapter 42

Trace Spaces

42.1 Definition And Basic Theory Of Trace Spaces

Another approach to these sorts of problems is to use trace spaces. This allows the consideration of fractional order Sobolev spaces. In so far as the subject of Sobolev spaces is concerned, I will present this material in a manner which is essentially independent of the previous material on interpolation spaces.

As in the case of interpolation spaces, suppose $A_0$ and $A_1$ are two Banach spaces which are continuously embedded in some topological vector space, $X$.

**Definition 42.1.1** Define a norm on $A_0 + A_1$ as follows.

$$||a||_{A_0 + A_1} \equiv \inf \left\{ ||a_0||_{A_0} + ||a_1||_{A_1} : a_0 + a_1 = a \right\}$$  \hfill (42.1.1)

**Lemma 42.1.2** $A_0 + A_1$ with the norm just described is a Banach space.

**Proof**: This was already explained in the treatment of the $K$ method of interpolation. It is just $K(1,a)$.

**Definition 42.1.3** Take $f'$ in the sense of distributions for any $f \in L^1_{loc}(0,\infty; A_0 + A_1)$ as follows.

$$f'(\phi) = \int_0^\infty -f(t) \phi'(t) \, dt$$

whenever $\phi \in C_c^\infty(0,\infty)$. Define a Banach space, $W(A_0, A_1, p, \theta) = W$ where $p \geq 1$, $\theta \in (0, 1)$. Let

$$||f||_W \equiv \max \left( ||t^\theta f||_{L^p(0,\infty; A_0)} : ||t^\theta f'||_{L^p(0,\infty; A_1)} \right)$$  \hfill (42.1.2)

and let $W$ consist of $f \in L^1_{loc}(0,\infty; A_0 + A_1)$ such that $||f||_W < \infty$. 

1501
Note that to be in $W$, $f(t) \in A_0$ and $f'(t) \in A_1$.

**Lemma 42.1.4** If $f \in W$, then

$$\text{Trace}(f) \equiv f(0) \equiv \lim_{t \to 0} f(t)$$

exists in $A_0 + A_1$. Also $Z \equiv \{f \in W : f(0) = 0\}$ is a closed subspace of $W$. In addition to this, for every $f \in W$ and $\varepsilon > 0$ there exists a $g \in W$ such that $\|f - g\|_W < \varepsilon$ and $g \in C^\infty(0, \infty; A_0)$ while $g' \in C^\infty(0, \infty; A_1)$.

**Proof:** Let $0 < s < t$. Let $\nu + \frac{1}{p} = \theta$. Then for a generic $g$,

$$\int_0^\infty \|\tau\nu g(\tau)\|_p^p d\tau = \int_0^\infty \|\tau\theta g(\tau)\|_p^p d\tau$$

so that $t'' f' \in L^p(0, \infty; A_1)$, the measure in this case being usual Lebesgue measure. Then

$$f(t) - f(s) = \int_s^t f'(\tau) d\tau = \int_s^t \tau\nu f'(\tau) \tau^{-\nu} d\tau.$$ 

For $\frac{1}{p} + \frac{1}{p'} = 1$, $\nu p' = \left(\theta - \frac{1}{p}\right) p' < 1$ because $\theta < 1 = \frac{1}{p} + \frac{1}{p}$. Therefore,

$$\|f(t) - f(s)\|_{A_0 + A_1} \leq \int_s^t \|f'(\tau)\|_{A_0 + A_1} d\tau \leq \int_s^t \|f'(\tau)\|_{A_1} d\tau = \int_s^t \|\tau\nu f'(\tau)\|_{A_1} \tau^{-\nu} d\tau \leq \left(\int_s^t \|\tau\nu f'(\tau)\|_{A_1}^p d\tau\right)^{1/p} \left(\int_s^t \tau^{-\nu} d\tau\right)^{1/p'} \leq \|f\|_W \left(t\nu' - \frac{s\nu'}{1 - \nu'}\right)$$

(42.1.3)

which converges to 0 as $t \to 0$. This shows that $\lim_{t \to 0^+} f(t)$ exists in $A_0 + A_1$.

Clearly $Z$ is a subspace. Let $f_n \to f$ in $W$ and suppose $f_n \in Z$. Then since $f \in W$, (42.1.3) implies $f$ is continuous. Using (42.1.3) and replacing $f$ with $f_n - f_m$ and then taking a limit as $s \to 0$,

$$\|f_n(t) - f_m(t)\|_{A_0 + A_1} \leq \|f_n - f_m\|_W C_\nu t^{1 - \nu p'}$$

Taking a subsequence, it can be assumed $f_n(t)$ converges to $f(t)$ a.e. But the above inequality shows that $f_n(t)$ is a Cauchy sequence in $C([0, \beta]; A_0 + A_1)$ for all $\beta < \infty$. Therefore, $f_n(t) \to f(t)$ for all $t$. Also,

$$\|f_n(t)\|_{A_0 + A_1} \leq C_\nu \|f_n\|_W t^{1 - \nu p'} \leq K t^{1 - \nu p'}$$
42.1. **Definition and Basic Theory of Trace Spaces**

for some $K$ depending on $\max \{ \| f_n \| : n \geq 1 \}$ and so

$$\| f(t) \|_{A_0 + A_1} \leq K t^{1-\nu p}$$

which implies $f(0) = 0$. Thus $Z$ is closed.

Consider the last claim. For a generic $t^\theta g \in L^p \left(0, \infty, \frac{dt}{t}; A\right)$, changing variables $t = e^\tau$,

$$\int_0^\infty t^\theta |g(t)|^p \frac{dt}{t} = \int_{-\infty}^{\infty} e^{\tau \theta p} |g(e^\tau)|^p \, d\tau$$

Let $\tilde{g}(\tau) \equiv g(e^\tau)$. Thus $\tau \to e^{\tau \theta} \tilde{g}(\tau)$ is $L^p \left(\mathbb{R}; A\right)$ and $\tilde{g} \in L^1_{1, \text{loc}}(\mathbb{R})$. Now let $\psi_\delta$ be a mollifier and consider

$$e^{\theta \tau} \psi_\delta(\tau) = \int_{-\infty}^{\infty} e^{\theta \sigma} \tilde{g}(\tau - \sigma) \psi_\delta(\sigma) \, d\sigma$$

so that

$$\tilde{g}_\delta(\tau) = \int_{-\infty}^{\infty} e^{-\theta \sigma} \tilde{g}(\tau - \sigma) \psi_\delta(\sigma) \, d\sigma$$

Thus $\tilde{g}_\delta \in C^\infty(\mathbb{R}; A)$ and using Minkowski's inequality,

$$\left( \int_{-\infty}^{\infty} \left\| e^{\theta \tau} \psi_\delta(\tau) - e^{\theta \tau} \tilde{g}(\tau) \right\|^p \, d\tau \right)^{1/p} \leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \left| e^{\theta (\tau - \sigma)} \tilde{g}(\tau - \sigma) - e^{\theta \tau} \tilde{g}(\tau) \right|^p \, d\sigma \right)^{1/p} \, d\tau$$

provided $\delta$ is small enough due to continuity of translation in $L^p$. Thus changing variables in $\tilde{g}_\delta$, letting $\tau = \ln(t)$ and $g_\delta(t) \equiv \tilde{g}_\delta(\ln(t))$, it follows $g_\delta \in C^\infty(0, \infty; A)$ and this integral equals

$$\left( \int_0^\infty e^{\theta p} |g_\delta(t) - g(t)|^p \frac{dt}{t} \right)^{1/p}$$

This result applied to $f$ and $f'$ with $A = A_0$ and then $A = A_1$ shows the last claim.

This proves the lemma.
Definition 42.1.5 Let $W$ be a Banach space and let $Z$ be a closed subspace. Then the quotient space, denoted by $W/Z$, consists of the set of equivalence classes $[x]$ where the equivalence relation is defined by $x \sim y$ means $x - y \in Z$. Then $W/Z$ is a vector space if the operations are defined by $\alpha [x] \equiv [\alpha x]$ and $[x] + [y] \equiv [x + y]$ and these vector space operations are well defined. The norm on the quotient space is defined as $\| [x] \| = \inf \{ \| x + z \| : z \in Z \}$.

The verification of the algebraic claims made in the above definition is left to the reader. It is routine. What is not as routine is the following lemma. However, it is similar to some topics in the presentation of the $K$ method of interpolation.

Lemma 42.1.6 Let $W$ be a Banach space and let $Z$ be a closed subspace of $W$. Then $W/Z$ with the norm described above is a Banach space.

Proof: That $W/Z$ is a vector space is left to the reader. Why is $\| \cdot \|$ a norm? Suppose $\alpha \neq 0$. Then

\[
\| \alpha [x] \| = \| \alpha x \| = \inf \{ \| \alpha x + z \| : z \in Z \} = \inf \{ \| \alpha x + \alpha z \| : z \in Z \} = |\alpha| \inf \{ \| x + z \| : z \in Z \} = |\alpha| \| [x] \|.
\]

Now let $\| [x] \| \geq \| x + z_1 \| - \varepsilon$ and let $\| [y] \| \geq \| y + z_2 \| - \varepsilon$ where $z_1, z_2 \in Z$. Then

\[
\| [x] + [y] \| \leq \| [x + y] \| \leq \| x + y + z_1 + z_2 \| \leq \| x + z_1 \| + \| y + z_2 \| \leq \| [x] \| + \| [y] \| + 2\varepsilon.
\]

Since $\varepsilon$ is arbitrary, this shows the triangle inequality. Clearly, $\| [x] \| \geq 0$. It remains to show that the only way $\| [x] \| = 0$ is for $x \in Z$. Suppose then that $\| [x] \| = 0$. This means there exist $z_n \in Z$ such that $\| x + z_n \| \to 0$. Therefore, $-x$ is a limit of a sequence of points of $Z$ and since $Z$ is closed, this requires $-x \in Z$. Hence $x \in Z$ also because $Z$ is a subspace. This shows $\| \cdot \|$ is a norm on $W/Z$. It remains to verify that $W/Z$ is a Banach space.

Suppose $\{ [x_n] \}$ is a Cauchy sequence in $W/Z$ and suppose $\| [x_n] - [x_{n+1}] \| < \frac{1}{n+1}$. Let $x'_1 = x_1$. If $x'_n$ has been chosen let $x'_{n+1} = x_{n+1} + z_{n+1}$ where $z_{n+1} \in Z$ be such that

\[
\| x'_{n+1} - x'_n \| \leq \| [x_{n+1} - x_n] \| + \frac{1}{2(n+1)} \leq \| [x_n] - [x_{n+1}] \| + \frac{1}{2(n+1)} < \frac{1}{2n}.
\]

It follows $\{ x'_n \}$ is a Cauchy sequence in $W$ and so it must converge to some $x \in W$. Now

\[
\| [x] - [x_n] \| = \| [x - x_n] \| = \| [x - x'_n] \| \leq \| x - x'_n \|
\]

which converges to 0. Now if $\{ [x_n] \}$ is just a Cauchy sequence, there exists a subsequence satisfying $\| [x_{n_k} - x_{n_{k+1}}] \| < \frac{1}{2^{k+1}}$ and so from the first part, the subsequence converges to some $[x] \in W/Z$ and so the original Cauchy sequence also converges. Therefore, $W/Z$ is a Banach space as claimed.
42.1. DEFINITION AND BASIC THEORY OF TRACE SPACES

Definition 42.1.7 Define $T (A_0, A_1, p, \theta) = T$, to consist of

$$
\left\{ a \in A_0 + A_1 : a = \lim_{t \to 0^+} f(t) \text{ for some } f \in W (A_0, A_1, p, \theta) \right\},
$$

the limit taking place in $A_0 + A_1$. Let $\gamma f$ be defined for $f \in W$ by $\gamma f \equiv \lim_{t \to 0^+} f(t)$. Thus $T = \gamma (W)$. As above $Z \equiv \{ f \in W : \gamma f = 0 \} = \ker (\gamma)$.

Lemma 42.1.8 $T$ is a Banach space with norm given by

$$
\|a\|_T \equiv \inf \{ \|f\|_W : f(0) = a \}. \quad (42.1.5)
$$

Proof: Define a mapping, $\psi : W/Z \to T$ by

$$
\psi ([f]) \equiv \gamma f.
$$

Then $\psi$ is one to one and onto. Also

$$
\|f\| \equiv \inf \{ \|f + g\| : g \in Z \} = \inf \{ \|h\|_W : \gamma h = \gamma f \} = \|\gamma (f)\|_T.
$$

Therefore, the Banach space, $W/Z$ and $T$ are isometric and so $T$ must be a Banach space since $W/Z$ is.

The following is an important interpolation inequality.

Theorem 42.1.9 If $a \in T$, then

$$
\|a\|_T = \inf \left\{ \|t^\theta f\|_{L^p,(0,\infty),A_0}^{1-\theta} \|t^\theta f'\|_{L^p,(0,\infty),A_1}^\theta \right\} \quad (42.1.6)
$$

where the infimum is taken over all $f \in W$ such that $a = f(0)$. Also, if $a \in A_0 \cap A_1$, then $a \in T$ and

$$
\|a\|_T \leq K \|a\|_{A_1}^{1-\theta} \|a\|_{A_0}^\theta \quad (42.1.7)
$$

for some constant $K$. Also

$$
A_0 \cap A_1 \subseteq T (A_0, A_1, p, \theta) \subseteq A_0 + A_1 \quad (42.1.8)
$$

and the inclusion maps are continuous.

Proof: First suppose $f(0) = a$ where $f \in W$. Then letting $f_\lambda(t) \equiv f(\lambda t)$, it follows that $f_\lambda(0) = a$ also and so

$$
\|a\|_T \leq \max \left( \|t^\theta f_\lambda\|_{L^p,(0,\infty),A_0} \|t^\theta f'_\lambda\|_{L^p,(0,\infty),A_1} \right)
= \max \left( \lambda^{-\theta} \|t^\theta f\|_{L^p,(0,\infty),A_0} \|t^\theta f'\|_{L^p,(0,\infty),A_1} \right)
= \max \left( \lambda^{-\theta} R, \lambda^{1-\theta} S \right).
$$

Now choose $\lambda = R/S$ to obtain

$$
\|a\|_T \leq R^{1-\theta} S^\theta = \|t^\theta f\|_{L^p,(0,\infty),A_0}^{1-\theta} \|t^\theta f'\|_{L^p,(0,\infty),A_1}^\theta.
$$
Thus
\[ \|a\|_T \leq \inf \left\{ \|t^\theta f\|_{L^p(0, \infty, \mathcal{G}; A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \mathcal{G}; A_1)}^{\theta} \right\}. \]

Next choose \( f \in W \) such that \( f(0) = a \) and \( \|f\|_W \approx \|a\|_T \). More precisely, pick \( f \in W \) such that \( f(0) = a \) and \( \|a\|_T > -\varepsilon + \|f\|_W \). Also let
\[ R \equiv \|t^\theta f\|_{L^p(0, \infty, \mathcal{G}; A_0)}, \quad S \equiv \|t^\theta f'\|_{L^p(0, \infty, \mathcal{G}; A_1)}. \]

Then as before,
\[ \|t^\theta f\|_{L^p(0, \infty, \mathcal{G}; A_0)} = \lambda^{-\theta} R, \quad \|t^\theta (f\lambda)'\|_{L^p(0, \infty, \mathcal{G}; A_1)} = \lambda^{1-\theta} S. \] (42.1.9)

so that \( \|f\|_W = \max(R, S) \). Then, changing the variables, letting \( \lambda = R/S \),
\[ \|t^\theta f\|_{L^p(0, \infty, \mathcal{G}; A_0)} = \|t^\theta (f\lambda)'\|_{L^p(0, \infty, \mathcal{G}; A_1)} = R^{1-\theta} S^\theta \] (42.1.10)
Since \( f\lambda(0) = a, f\lambda \in W \), and it is always the case that for positive \( R, S, R^{1-\theta} S^\theta \leq \max(R, S) \), this shows that
\[ \|a\|_T \leq \max \left( \|t^\theta f\|_{L^p(0, \infty, \mathcal{G}; A_0)}, \|t^\theta (f\lambda)'\|_{L^p(0, \infty, \mathcal{G}; A_1)} \right) \]
\[ = R^{1-\theta} S^\theta \leq \max(R, S) = \|f\|_W < \|a\|_T + \varepsilon, \]
the first inequality holding because \( \|a\|_T \) is the infimum of such things on the right. This shows \( \text{[2.1.5]} \).

It remains to verify \( \text{[2.1.6]} \). To do this, let \( \psi \in C^\infty([0, \infty)) \), with \( \psi(0) = 1 \) and \( \psi(t) = 0 \) for all \( t > 1 \). Then consider the special \( f \in W \) which is given by \( f(t) \equiv a\psi(t) \) where \( a \in A_0 \cap A_1 \). Thus \( f \in W \) and \( f(0) = a \) so \( a \in T \). From the first part, there exists a constant, \( K \) such that
\[ \|a\|_T \leq \|t^\theta f\|_{L^p(0, \infty, \mathcal{G}; A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \mathcal{G}; A_1)}^{\theta} \leq K \|a\|_{A_0 \cap A_1}^{1-\theta} \|a\|_{A_1}^{\theta}. \]

This shows \( \text{[2.1.6]} \) the first inclusion in \( \text{[2.1.8]} \). From the inequality just obtained,
\[ \|a\|_T \leq K \left( (1 - \theta) \|a\|_{A_0} + \theta \|a\|_{A_1} \right) \]
\[ \leq K \|a\|_{A_0 \cap A_1}. \]

This shows the first inclusion map of \( \text{[2.1.8]} \) is continuous.

Now take \( a \in T \). Let \( f \in W \) be such that \( a = f(0) \) and
\[ \|a\|_T + \varepsilon > \|f\|_W \geq \|a\|_T. \]

By \( \text{[2.1.6]} \),
\[ \|a - f(t)\|_{A_0 + A_1} \leq C_m t^{1-\nu p'} \|f\|_W \]
where $\frac{1}{\theta} + \nu = \theta$, and so

$$||a||_{A_0 + A_1} \leq ||f(t)||_{A_0 + A_1} + C_\nu t^{1-vp'} ||f||_W .$$

Now $||f(t)||_{A_0 + A_1} \leq ||f(t)||_{A_0}$.

$$||a||_{A_0 + A_1} \leq t^\nu ||f(t)||_{A_0 + A_1} t^{-\nu} + C_\nu t^{1-vp'} ||f||_W \leq t^\nu ||f(t)||_{A_0} t^{-\nu} + C_\nu t^{1-vp'} ||f||_W .$$

Therefore, recalling that $vp' < 1$, and integrating both sides from 0 to 1,

$$||a||_{A_0 + A_1} \leq C_\nu ||f||_W \leq C_\nu (||a||_T + \varepsilon) .$$

To see this,

$$\int_0^1 t^\nu ||f(t)||_{A_0} t^{-\nu} dt \leq \left( \int_0^1 (t^\nu ||f(t)||_{A_0})^p dt \right)^{1/p} \left( \int_0^1 t^{-vp'} dt \right)^{1/p'} \leq C ||f||_W .$$

Since $\varepsilon > 0$ is arbitrary, this verifies the second inclusion and continuity of the inclusion map completing the proof of the theorem.

The interpolation inequality, is very significant. The next result concerns bounded linear transformations.

**Theorem 42.1.10** Now suppose $A_0, A_1$ and $B_0, B_1$ are pairs of Banach spaces such that $A_i$ embeds continuously into a topological vector space, $X$ and $B_i$ embeds continuously into a topological vector space, $Y$. Suppose also that $L \in \mathcal{L}(A_0, B_0)$ and $L \in \mathcal{L}(A_1, B_1)$ where the operator norm of $L$ in these spaces is $K_i, i = 0, 1$. Then

$$L \in \mathcal{L}(A_0 + A_1, B_0 + B_1) \quad (42.1.11)$$

with

$$||La||_{B_0 + B_1} \leq \max(K_0, K_1) ||a||_{A_0 + A_1} \quad (42.1.12)$$

and

$$L \in \mathcal{L}(T(A_0, A_1, p, \theta), T(B_0, B_1, p, \theta)) \quad (42.1.13)$$

and for $K$ the operator norm,

$$K \leq K_0^{1-\theta} K_1^\theta . \quad (42.1.14)$$

**Proof:** To verify , let $a \in A_0 + A_1$ and pick $a_0 \in A_0$ and $a_1 \in A_1$ such that

$$||a||_{A_0 + A_1} + \varepsilon > ||a_0||_{A_0} + ||a_1||_{A_1} .$$

Then

$$||L(a)||_{B_0 + B_1} = ||La_0 + La_1||_{B_0 + B_1} \leq ||La_0||_{B_0} + ||La_1||_{B_1} .$$
\[ \leq K_0 \|a_0\|_{A_0} + K_1 \|a_1\|_{A_1} \leq \max (K_0, K_1) (\|a\|_{A_0 + A_1} + \varepsilon). \]

This establishes (42.1.12). Now consider the other assertions.

Let \( a \in T(A_0, A_1, p, \theta) \) and pick \( f \in W(A_0, A_1, p, \theta) \) such that \( \gamma f = a \) and
\[
\|a\|_{T(A_0, A_1, p, \theta)} + \varepsilon > \left\| t^\theta f \right\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}^{1-\theta} \left\| t^\theta f' \right\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}^\theta.
\]

Then consider \( Lf \). Since \( L \) is continuous on \( A_0 + A_1 \),
\[ Lf (0) = La \]
and \( Lf \in W(B_0, B_1, p, \theta) \). Therefore, by Theorem (42.1.11),
\[
\|La\|_{T(B_0, B_1, p, \theta)} \leq \left\| t^\theta Lf \right\|_{L^p(0, \infty, \frac{dt}{t}; B_0)}^{1-\theta} \left\| t^\theta Lf' \right\|_{L^p(0, \infty, \frac{dt}{t}; B_1)}^\theta
\leq K_0^{1-\theta} K_1^\theta \left\| t^\theta f \right\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}^{1-\theta} \left\| t^\theta f' \right\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}^\theta
\leq K_0^{1-\theta} K_1^\theta \left\| a \right\|_{T(A_0, A_1, p, \theta)} + \varepsilon.
\]

and since \( \varepsilon > 0 \) is arbitrary, this proves the theorem.

### 42.2 Trace And Interpolation Spaces

Trace spaces are equivalent to interpolation spaces. In showing this, a more general sort of trace space than that presented earlier will be used.

**Definition 42.2.1** Define for \( m \) a positive integer, \( V^m = V^m(A_0, A_1, p, \theta) \) to be the set of functions, \( u \) such that
\[ t \to t^\theta u(t) \in L^p \left( 0, \infty, \frac{dt}{t}; A_0 \right) \]

and
\[ t \to t^{\theta + m-1} u^{(m)}(t) \in L^p \left( 0, \infty, \frac{dt}{t}; A_1 \right). \]

\( V^m \) is a Banach space with the norm
\[ \|u\|_{V^m} \equiv \max \left( \left\| t^\theta u(t) \right\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, \left\| t^{\theta + m-1} u^{(m)}(t) \right\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} \right). \]

Thus \( V^m \) equals \( W \) in the case when \( m = 1 \). More generally, as in (42.2.7) different exponents are used for the two \( L^p \) spaces, \( p_0 \) in place of \( p \) for the space corresponding to \( A_0 \) and \( p_1 \) in place of \( p \) for the space corresponding to \( A_1 \).

**Definition 42.2.2** Denote by \( T^m(A_0, A_1, p, \theta) \) the set of all \( a \in A_0 + A_1 \) such that for some \( u \in V^m \),
\[ a = \lim \limits_{t \to 0^+} u(t) \equiv \text{trace} (u), \]

the limit holding in \( A_0 + A_1 \). For the norm
\[ \|a\|_{T^m} \equiv \inf \{ \|u\|_{V^m} : \text{trace} (u) = a \}. \]
The case when $m = 1$ was discussed in Section 42.1. Note it is not known at this point whether $\lim_{t \to 0^+} u(t)$ even exists for every $u \in V_m$. Of course, if $m = 1$ this was shown earlier but it has not been shown for $m > 1$. The following theorem is absolutely amazing. Note the lack of dependence on $m$ of the right side!

**Theorem 42.2.3** The following hold.

$$T^m(A_0, A_1, p, \theta) = (A_0, A_1)_{\theta, p, J} = (A_0, A_1)_{\theta, p}.$$ (42.2.19)

**Proof:** It is enough to show the first equality because of Theorem 41.7.5 which identifies $(A_0, A_1)_{\theta, p, J}$ and $(A_0, A_1)_{\theta, p}$. Let $a \in T_m$. Then there exists $u \in V_m$ such that

$$a = \lim_{t \to 0^+} u(t) \text{ in } A_0 + A_1.$$

The first task is to modify this $u(t)$ to get a better one which is more usable in order to show $a \in (A_0, A_1)_{\theta, p, J}$. Remember, it is required to find $w(t) \in A_0 \cap A_1$ for all $t \in (0, \infty)$ and $a = \int_0^\infty w(t) \frac{dt}{t}$, a representation which is not known at this time. To get such a thing, let

$$\phi \in C_c^\infty (0, \infty), \text{spt } (\phi) \subseteq [\alpha, \beta]$$ (42.2.20)

with $\phi \geq 0$ and

$$\int_0^\infty \phi(t) \frac{dt}{t} = 1.$$ (42.2.21)

Then define

$$\tilde{u}(t) = \int_0^\infty \phi \left( \frac{t}{\tau} \right) u(\tau) \frac{d\tau}{\tau} = \int_0^\infty \phi(s) u \left( \frac{t}{s} \right) \frac{ds}{s}. $$ (42.2.22)

**Claim:** $\lim_{t \to 0^+} \tilde{u}(t) = a$ and $\lim_{t \to \infty} \tilde{u}^{(k)}(t) = 0$ in $A_0 + A_1$ for all $k \leq m$.

**Proof of the claim:** From 42.2.22 and 42.2.21 it follows that for $\| \cdot \|$ referring to $\| \cdot \|_{A_0 + A_1}$,

$$\| \tilde{u}(t) - a \| \leq \int_0^\infty \left\| u \left( \frac{t}{s} \right) - a \right\| \phi(s) \frac{ds}{s}$$

$$= \int_0^\infty \| u(\tau) - a \| \phi \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau}$$

$$= \int_{t/\beta}^{t/\alpha} \| u(\tau) - a \| \phi \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau}$$

$$\leq \int_{t/\beta}^{t/\alpha} \varepsilon \phi \left( \frac{t}{\tau} \right) \frac{d\tau}{\tau} = \varepsilon \int_{t/\alpha}^{t/\beta} \phi(s) \frac{ds}{s} = \varepsilon$$

whenever $t$ is small enough due to the convergence of $u(t)$ to $a$ in $A_0 + A_1$.

Now consider what occurs when $t \to \infty$. For $\| \cdot \|$ referring to the norm in $A_0$,

$$\tilde{u}^{(k)}(t) = \int_0^\infty \phi^{(k)} \left( \frac{t}{\tau} \right) \frac{1}{\tau^k} u(\tau) \frac{d\tau}{\tau}.$$
and so
\[ \left\| \overline{u}^{(k)}(t) \right\|_{A_0} \leq C_k \int_{t/\beta}^{t/\alpha} \left\| u(\tau) \right\|_{A_0} \frac{d\tau}{\tau} \]
\leq C \left( \int_{t/\beta}^{t/\alpha} \frac{d\tau}{\tau} \right)^{1/p'} \left( \int_{t/\beta}^{t/\alpha} \left\| u(\tau) \right\|_{A_0}^{p} \frac{d\tau}{\tau} \right)^{1/p}.

Now \( \left( \frac{\beta}{\alpha} \right)^{\theta \tau} \geq 1 \) for \( \tau \geq t/\beta \) and so the above expression
\[ \leq C \left( \ln \frac{\beta}{\alpha} \right)^{1/p'} \left( \frac{\beta}{\alpha} \right)^{\theta \tau} \left( \int_{t/\beta}^{t/\alpha} \left\| u(\tau) \right\|_{A_0}^{p} \frac{d\tau}{\tau} \right)^{1/p} \]
and so \( \lim_{t \to \infty} \left\| \overline{u}^{(k)}(t) \right\|_{A_0} = 0 \) and therefore, this also holds in \( A_0 + A_1 \). This proves the claim.

Thus \( \overline{u} \) has the same properties as \( u \) in terms of having \( a \) as its trace. \( \overline{u} \) is used to build the desired \( v \), representing \( a \) as an integral. Define
\[ v(t) = \frac{(-1)^m t^m}{(m-1)!} \overline{u}^{(m)}(t) = \frac{(-1)^m}{(m-1)!} \int_{0}^{\infty} t^m \phi^{(m)}(t) u(\tau) \frac{d\tau}{\tau} \]
\[ = \frac{(-1)^m}{(m-1)!} \int_{0}^{\infty} s^m \phi^{(m)}(s) u\left( \frac{t}{s} \right) \frac{ds}{s}. \quad (42.2.23) \]
Then from the claim, and integration by parts in the last step,
\[ \int_{0}^{\infty} v\left( \frac{1}{t} \right) \frac{dt}{t} = \int_{0}^{\infty} v(t) \frac{dt}{t} = \frac{(-1)^m}{(m-1)!} \int_{0}^{\infty} t^{m-1} \overline{u}^{(m)}(t) dt = a. \quad (42.2.24) \]

Thus \( v\left( \frac{1}{t} \right) \) represents \( a \) in the way desired for \( (A_0, A_1)_{\theta, p, J} \) if it is also true that \( v\left( \frac{1}{t} \right) \in A_0 \cap A_1 \) and \( t \to t^{-\theta} v\left( \frac{1}{t} \right) \) is in \( L^p \left( 0, \infty; \frac{dt}{t}; A_0 \right) \) and \( t \to t^{1-\theta} v\left( \frac{1}{t} \right) \) is in \( L^p \left( 0, \infty; \frac{dt}{t}; A_1 \right) \). First consider whether \( v(t) \in A_0 \cap A_1 \). \( v(t) \in A_0 \) for each \( t \) from \( \overline{\mathfrak{tr}} \) and the assumption that \( u \in L^p \left( 0, \infty; \frac{dt}{t}; A_0 \right) \). To verify \( v(t) \in A_1 \), integrate by parts in \( \overline{\mathfrak{tr}} \) to obtain
\[ v(t) = \frac{(-1)^m}{(m-1)!} \int_{0}^{\infty} \phi^{(m)}(s) \left( s^{m-1} u\left( \frac{t}{s} \right) \right) ds \quad (42.2.25) \]
\[ = \frac{1}{(m-1)!} \int_{0}^{\infty} \phi(s) \frac{d^m}{ds^m} \left( s^{m-1} u\left( \frac{t}{s} \right) \right) ds \]
\[ = \frac{(-1)^m}{(m-1)!} \int_{0}^{\infty} \phi(s) \frac{t^m}{s^{m+1}} u^{(m)}\left( \frac{t}{s} \right) ds \in A_1. \]
The last step may look very mysterious. If so, consider the case where \( m = 2 \).

\[
\phi(s) \left( su \left( \frac{t}{s} \right) \right)'' \\
= \phi(s) \left( -\frac{t}{s} u' \left( \frac{t}{s} \right) + u \left( \frac{t}{s} \right) \right)'
\]

\[
= \phi(s) \left( \left( -\frac{t}{s} u'' \left( \frac{t}{s} \right) \left( -\frac{t}{s^2} \right) + \frac{t}{s^2} u' \left( \frac{t}{s} \right) - \frac{t}{s^2} u' \left( \frac{t}{s} \right) \right) \right)
\]

\[
= \phi(s) \frac{t^2}{s^3} u'' \left( \frac{t}{s} \right).
\]

You can see the same pattern will take place for other values of \( m \).

Now

\[
\|a\|_{\theta,p,J} \leq \left( \int_0^\infty \left( t^{-\theta} J(t, v \left( \frac{1}{t} \right)) \right)^p \frac{dt}{t} \right)^{1/p}
\]

\[
\leq C_p \left\{ \left( \int_0^\infty \left( t^{-\theta} \left\| v \left( \frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} + \left( \int_0^\infty \left( t^{1-\theta} \left\| v \left( \frac{1}{t} \right) \right\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \right\}.
\]

The first term equals

\[
\left( \int_0^\infty \left( t^{-\theta} \left\| v \left( \frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} = \left( \int_0^\infty \left( t^{\theta} \left\| v \left( \frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p}
\]

\[
= \left( \int_0^\infty \left( t^{\theta} \left\| v \left( \frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p}
\]

\[
\leq \left( \int_0^\infty \left( t^\theta \left\| s^m \phi^{(m)} \left( s \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} ds
\]

\[
\leq \int_0^\infty s^m \left| \phi^{(m)} \left( s \right) \right| \left( \int_0^\infty \left( t^\theta \left\| u \left( \frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} ds
\]

\[
\leq \int_0^\infty s^m \left| \phi^{(m)} \left( s \right) \right| \left( \int_0^\infty \left( t^\theta \left\| u \left( \frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} ds
\]
= \int_0^{\infty} s^{\theta + m} \left| \phi^{(m)} (s) \right| \frac{ds}{s} \left( \int_0^{\infty} (\tau^\theta \|u (\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p} \\
= C \left( \int_0^{\infty} (\tau^\theta \|u (\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p}. \quad (42.2.27)

The second term equals

\begin{align*}
\left( \int_0^{\infty} \left( t^{1-\theta} \left\| \frac{1}{t} \right\|_{A_1} \right) \frac{dt}{t} \right)^{1/p} &= \left( \int_0^{\infty} \left( t^{\theta - 1} \|v (t)\|_{A_1} \right) \frac{dt}{t} \right)^{1/p} \\
&= \left( \int_0^{\infty} \left( \int_0^{\infty} \phi (s) \left\| u^{(m)} \left( \frac{t}{s} \right) \right\|_{A_1} \frac{ds}{s} \right) \frac{dt}{t} \right)^{1/p} \\
&\leq \int_0^{\infty} \left( \frac{\phi (s)}{s^m} \int_0^{\infty} \left( t^{\theta + m - 1} \left\| u^{(m)} \left( \frac{t}{s} \right) \right\|_{A_1} \right) \frac{dt}{t} \right)^{1/p} \frac{ds}{s} \\
&= \int_0^{\infty} \left( \phi (s) \int_0^{\infty} \left( t^{\theta + m - 1} \left\| u^{(m)} \left( \frac{t}{s} \right) \right\|_{A_1} \right) \frac{dt}{t} \right)^{1/p} \frac{ds}{s} \\
&= \int_0^{\infty} \left( \frac{\phi (s)}{s^m} s^{\theta + m - 1} \int_0^{\infty} \left( t^{\theta + m - 1} \left\| u^{(m)} (\tau) \right\|_{A_1} \right) \frac{d\tau}{\tau} \right)^{1/p} \frac{ds}{s} \\
&= C \left( \int_0^{\infty} \left( \tau^{\theta + m - 1} \left\| u^{(m)} (\tau) \right\|_{A_1} \right) \frac{d\tau}{\tau} \right)^{1/p}. \quad (42.2.28)
\end{align*}

Now from the estimates on the two terms in (42.2.26) found in (42.2.27) and (42.2.28), and the simple estimate,

\[ 2 \max (\alpha, \beta) \geq \alpha + \beta, \]

it follows

\[ \|a\|_{\theta, p, J} \leq C \max \left( \int_0^{\infty} (\tau^\theta \|u (\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p}, \quad (42.2.29) \]

\[ \left( \int_0^{\infty} (\tau^{\theta + m - 1} \|u^{(m)} (\tau)\|_{A_1})^p \frac{d\tau}{\tau} \right)^{1/p} \right)^{1/p} \]

\[ \left( \int_0^{\infty} (\tau^{\theta + m - 1} \|u^{(m)} (\tau)\|_{A_1})^p \frac{d\tau}{\tau} \right)^{1/p} \right)^{1/p} \]

\[ \left( \int_0^{\infty} (\tau^{\theta + m - 1} \|u^{(m)} (\tau)\|_{A_1})^p \frac{d\tau}{\tau} \right)^{1/p} \right)^{1/p} \]

which shows that after taking the infimum over all \( u \) whose trace is \( a \), it follows

\[ a \in (A_0, A_1)_{\theta, p, J}. \]

\[ \|a\|_{\theta, p, J} \leq C \|a\|_{T^m}. \quad (42.2.30) \]

Thus \( T^m (A_0, A_1, \theta, p) \subseteq (A_0, A_1)_{\theta, p, J}. \)
Is \((A_0, A_1)_{p, J} \subseteq T^m(A_0, A_1, \theta, p)\)? Let \(a \in (A_0, A_1)_{p, J}\). There exists \(u\) having values in \(A_0 \cap A_1\) such that
\[
a = \int_0^\infty u(t) \frac{dt}{t} = \int_0^\infty u\left(\frac{1}{t}\right) \frac{dt}{t},
\]
in \(A_0 + A_1\) such that
\[
\int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^p dt < \infty, \quad \text{where } J(t, a) = \max\left(||a||_{A_0}, t ||a||_{A_1}\right).
\]
Then let
\[
w(t) = \int_t^\infty \left(1 - \frac{t}{\tau}\right)^{m-1} u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau} = \int_t^1 \left(1 - \tau\right)^{m-1} u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau}. \tag{42.2.33}
\]
It is routine to verify from (42.2.33) that
\[
w^{(m)}(t) = (m-1)! (-1)^m \frac{u\left(\frac{1}{t}\right)}{t^m}. \tag{42.2.35}
\]
For example, consider the case where \(m = 2\).
\[
\left(\int_t^\infty \left(1 - \frac{t}{\tau}\right) u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau}\right)'' = \left(0 + \int_t^\infty \left(- \frac{1}{\tau}\right) u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau}\right)'
\]
\[
= \frac{1}{t^2} u\left(\frac{1}{t}\right).
\]
Also from (42.2.33), it follows that \(\text{trace}(w) = a\). It remains to verify \(w \in V^m\).
From (42.2.33)
\[
\left(\int_0^\infty \left(t^{\theta-1} \left\|w^{(m)}(t)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} = C_m \left(\int_0^\infty \left(t^{1-\theta} \left\|u(t)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p}
\]
\[
\leq C_m \left(\int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{42.2.36}
\]
It remains to consider \(\left(\int_0^\infty \left(t^\theta \left\|w(t)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p}\). From (42.2.34)
\[
\left(\int_0^\infty \left(t^\theta \left\|w(t)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} = \left(\int_0^\infty \left(t^\theta \int_0^1 (1 - \tau)^{m-1} u\left(\frac{\tau}{t}\right) \frac{d\tau}{\tau}\right)^p \frac{dt}{t}\right)^{1/p}
\]
\[ = \left( \int_0^\infty \left( t^{-\theta} \left\| \int_0^1 (1 - \tau)^{m-1} u(\tau t) \frac{d\tau}{\tau} \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \]
\[
\leq \int_0^1 \left( \int_0^\infty \left( t^{-\theta} (1 - \tau)^{m-1} \|u(\tau t)\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \frac{d\tau}{\tau} \]
\[
= \int_0^1 \tau^\theta (1 - \tau)^{m-1} \left( \int_0^\infty \left( s^{-\theta} \|u(s)\|_{A_0} \right)^p \frac{ds}{s} \right)^{1/p} \frac{d\tau}{\tau} \]
\[
= \left( \int_0^1 \tau^{\theta-1} (1 - \tau)^{m-1} d\tau \right) \left( \int_0^\infty \left( s^{-\theta} \|u(s)\|_{A_0} \right)^p \frac{ds}{s} \right)^{1/p} \]
\[
\leq C \left( \int_0^\infty \left( s^{-\theta} \|u(s)\|_{A_0} \right)^p \frac{ds}{s} \right)^{1/p} \]
\[
\leq C \left( \int_0^\infty (t^{-\theta} J(t, u(t)))^p \frac{dt}{t} \right)^{1/p} < \infty. \tag{42.2.37} \]

It follows that
\[
\|w\|_{V_m} = \max \left( \left( \int_0^\infty \left( t^{\theta} \|w(t)\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p}, \left( \int_0^\infty \left( t^{\theta+m-1} \|w^{(m)}(t)\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \right) \]
\[
\leq C \left( \int_0^\infty (t^{-\theta} J(t, u(t)))^p \frac{dt}{t} \right)^{1/p} < \infty \]

which shows that \(a \in T^m(A_0, A_1, \theta, p)\). Taking the infimum,
\[
\|a\|_{T^m} \leq C \|a\|_{\theta, p, J} \cdot \]

This together with \(42.2.32\) proves the theorem.

By Theorem \(42.2.3\) and Theorem \(41.8.6\), we obtain the following important corollary describing the dual space of a trace space.

**Corollary 42.2.4** Let \(A_0 \cap A_1\) be dense in \(A_i\) for \(i = 0, 1\) and suppose that \(A_i\) is reflexive for \(i = 0, 1\). Then for \(\infty > p \geq 1\),
\[
T^m(A_0, A_1, \theta, p)' = T^m(A_1', A_0', 1 - \theta, p') \]
Chapter 43

Traces Of Sobolev Spaces
And Fractional Order Spaces

43.1 Traces Of Sobolev Spaces On The Boundary
Of A Half Space

In this section consider the trace of $W^{m,p}(\mathbb{R}^n_+)$ onto a Sobolev space of functions defined on $\mathbb{R}^{n-1}$. This latter Sobolev space will be defined in terms of the following theory in such a way that the trace map is continuous. The trace map is continuous as a map from $W^{m,p}(\mathbb{R}^n_+)$ to $W^{m-1,p}(\mathbb{R}^{n-1})$ but here I will give a better conclusion using the above theory.

Definition 43.1.1 Let $\theta \in (0,1)$ and let $\Omega$ be an open subset of $\mathbb{R}^m$. We define

$$W^{\theta,p}(\Omega) \equiv T(W^{1,p}(\Omega),L^p(\Omega),p,1-\theta).$$

Thus, from the above general theory, $W^{1,p}(\Omega) \hookrightarrow W^{\theta,p}(\Omega) \hookrightarrow L^p(\Omega) = L^p(\Omega) + W^{1,p}(\Omega)$. Now we consider the trace map for Sobolev space.

Lemma 43.1.2 Let $\phi \in C^\infty(\mathbb{R}^n_+)$. Then $\gamma \phi(\mathbf{x}') \equiv \phi(\mathbf{x}',0)$. Then $\gamma : C^\infty(\mathbb{R}^n_+) \to L^p(\mathbb{R}^{n-1})$ is continuous as a map from $W^{1,p}(\mathbb{R}^n_+)$ to $L^p(\mathbb{R}^{n-1})$.

Proof: We know

$$\phi(\mathbf{x}',x_n) = \gamma \phi(\mathbf{x}') + \int_0^{x_n} \frac{\partial \phi(\mathbf{x}',t)}{\partial t} dt.$$
Then by Jensen's inequality,
\[
\int_{\mathbb{R}^{n-1}} |\gamma \phi (x')|^p \, dx' \\
= \int_0^1 \int_{\mathbb{R}^{n-1}} |\gamma \phi (x')|^p \, dx' \, dx_n \\
\leq C \int_0^1 \int_{\mathbb{R}^{n-1}} |\phi (x', x_n)|^p \, dx' \, dx_n \\
+ C \int_0^1 \int_{\mathbb{R}^{n-1}} \left| \int_0^x \frac{\partial \phi (x', t)}{\partial t} \, dt \right|^p \, dx' \, dx_n \\
\leq C \|\phi\|_{0,p,\mathbb{R}_+^n}^p + C \int_0^1 x_n^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^x \frac{\partial \phi (x', t)}{\partial t} \, dt \, dx' \, dx_n \\
\leq C \|\phi\|_{0,p,\mathbb{R}_+^n}^p + C \int_0^1 x_n^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\partial \phi (x', t)}{\partial t} \, dt \, dx' \, dx_n \\
\leq C \|\phi\|_{0,p,\mathbb{R}_+^n}^p + \frac{C}{p} \int_{\mathbb{R}^{n-1}} \int_0^\infty \frac{\partial \phi (x', t)}{\partial t} \, dt \, dx' \\
\leq C \|\phi\|_{1,p,\mathbb{R}_+^n}^p.
\]
This proves the lemma.

**Definition 43.1.3** We define the trace, \( \gamma : W^{1,p} (\mathbb{R}_+^n) \rightarrow L^p (\mathbb{R}^{n-1}) \) as follows. \( \gamma \phi (x') \equiv \phi (x', 0) \) whenever \( \phi \in C^\infty (\mathbb{R}_+^n) \). For \( u \in W^{1,p} (\mathbb{R}_+^n) \), we define \( \gamma u \equiv \lim_{k \rightarrow \infty} \gamma \phi_k \) in \( L^p (\mathbb{R}^{n-1}) \) where \( \phi_k \rightarrow u \) in \( W^{1,p} (\mathbb{R}_+^n) \). Then the above lemma shows this is well defined.

Also from this lemma we obtain a constant, \( C \) such that
\[
\|\phi\|_{0,p,\mathbb{R}^{n-1}} \leq C \|\phi\|_{1,p,\mathbb{R}_+^n}
\]
and the same constant holds for all \( u \in W^{1,p} (\mathbb{R}_+^n) \).

From the definition of the norm in the trace space, if \( f \in C^\infty (\mathbb{R}_+^n) \), and letting \( \theta = 1 - \frac{1}{p} \), it follows from the definition
\[
\|\gamma f\|_{1-\frac{1}{p}, \mathbb{R}^{n-1}} \\
\leq \max \left( \left( \int_0^\infty \left( t^{1/p} \| f (t) \|_{1,p,\mathbb{R}^{n-1}} \right)^p \frac{dt}{t} \right)^{1/p} \right) \\
\leq C \|f\|_{1,p,\mathbb{R}_+^n}.
\]
Thus, if \( f \in W^{1,p} (\mathbb{R}_+^n) \), define \( \gamma f \in W^{1-\frac{1}{p}, \mathbb{R}^{n-1}} \) according to the rule,
\[
\gamma f = \lim_{k \rightarrow \infty} \gamma \phi_k.
\]
43.1. TRACES OF SOBOLEV SPACES ON THE BOUNDARY OF A HALF SPACE

where \( \phi_k \to f \) in \( W^{1,p}(\mathbb{R}^n_+) \) and \( \phi_k \in C^\infty(\mathbb{R}^n_+) \). This shows the continuity part of the following lemma.

**Lemma 43.1.4** The trace map, \( \gamma \), is a continuous map from \( W^{1,p}(\mathbb{R}^n_+) \) onto 
\[ W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}). \]
Furthermore, for \( f \in W^{1,p}(\mathbb{R}^n_+) \),
\[ \gamma f = f(0) = \lim_{t \to 0^+} f(t) \]
the limit taking place in \( L^p(\mathbb{R}^{n-1}) \).

**Proof:** It remains to verify \( \gamma \) is onto along with the displayed equation. But by definition, things in \( W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \) are of the form \( \lim_{t \to 0^+} f(t) \) where \( f \in L^p(0, \infty; W^{1,p}(\mathbb{R}^{n-1})) \), and \( f' \in L^p(0, \infty; L^p(\mathbb{R}^{n-1})) \), the limit taking place in 
\[ W^{1,p}(\mathbb{R}^{n-1}) + L^p(\mathbb{R}^{n-1}) = L^p(\mathbb{R}^{n-1}), \]
and
\[ \left( \int_0^\infty ||f(t)||^p_{1,p,\mathbb{R}^{n-1}} dt \right)^{1/p} + \left( \int_0^\infty ||f'(t)||^p_{0,\mathbb{R}^{n-1}} dt \right)^{1/p} < \infty. \]
Then taking a measurable representative, we see \( f \in W^{1,p}(\mathbb{R}^n_+) \) and \( f, x_n = f' \). Also, as an equation in \( L^p(\mathbb{R}^{n-1}) \), the following holds for all \( t > 0 \).
\[ f(\cdot, t) = f(0) + \int_0^t f, x_n(\cdot, s) ds \]
But also, for a.e. \( x' \), the following equation holds for a.e. \( t > 0 \).
\[ f(x', t) = \gamma f(x') + \int_0^t f, x_n(x', s) ds, \quad (43.1.1) \]
showing that
\[ \gamma f = f(0) \in W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \equiv T \left( W^{1,p}(\Omega), L^p(\Omega), p, \frac{1}{p} \right). \]
To see that \( 43.1.1 \) holds, approximate \( f \) with a sequence from \( C^\infty(\mathbb{R}^n_+) \) and finally obtain an equation of the form
\[ \int_{\mathbb{R}^{n-1}} \int_0^\infty \left[ f(x', t) - \gamma f(x') - \int_0^t f, x_n(x', s) ds \right] \psi(x', t) dt dx' = 0, \]
which holds for all \( \psi \in C_c^\infty(\mathbb{R}^n_+) \). This proves the lemma.
Thus taking the trace on the boundary loses exactly \( \frac{1}{p} \) derivatives.
43.2 A Right Inverse For The Trace For A Half Space

It is also important to show there is a continuous linear function,

\[ R : W^{1-\frac{1}{p}, p} (\mathbb{R}^{n-1}) \to W^{1-p} (\mathbb{R}^n) \]

which has the property that \( \gamma (Rg) = g \). Define this function as follows.

\[ Rg (x', x_n) \equiv \int_{\mathbb{R}^{n-1}} g (y') \phi \left( \frac{x' - y'}{x_n} \right) \frac{1}{x_n^{n-1}} dy' \quad (43.2.2) \]

where \( \phi \) is a mollifier having support in \( B (0, 1) \).

**Lemma 43.2.1** Let \( R \) be defined in (43.2.2). Then \( Rg \in W^{1-p} (\mathbb{R}^n) \) and is a continuous linear map from \( W^{1-\frac{1}{p}, p} (\mathbb{R}^{n-1}) \) to \( W^{1-p} (\mathbb{R}^n) \) with the property that \( \gamma Rg = g \).

**Proof:** Let \( f \in W^{1-p} (\mathbb{R}^n) \) be such that \( \gamma f = g \). Let \( \psi (x_n) \equiv (1 - x_n)_+ \) and assume \( f \) is Borel measurable by taking a Borel measurable representative. Then for a.e. \( x' \) we have the following formula holding for a.e. \( x_n \).

\[ Rg (x', x_n) = \int_{\mathbb{R}^{n-1}} [\psi (x_n) f (y', \psi (x_n)) - \int_0^{\psi (x_n)} (\psi f)_n (y', t) dt] \phi \left( \frac{x' - y'}{x_n} \right) x_n^{1-n} dy' . \]

Using the repeated index summation convention to save space, we obtain that in terms of weak derivatives,

\[ Rg_n (x', x_n) \]

\[ = \int_{\mathbb{R}^{n-1}} [\psi (x_n) f (y', \psi (x_n)) - \int_0^{\psi (x_n)} (\psi f)_n (y', t) dt] \cdot \left[ \phi, k \left( \frac{x' - y'}{x_n} \right) \frac{y_k - x_k}{x_n^a} + \phi \left( \frac{x' - y'}{x_n} \right) \frac{1 - n}{x_n^a} \right] dy' \]

\[ = \int_{\mathbb{R}^{n-1}} [\psi (x_n) f (x' - x_n z', \psi (x_n)) - \int_0^{\psi (x_n)} (\psi f)_n (x' - x_n z', t) dt] \cdot \left[ \phi, k (z') \left( \frac{y_k - x_k}{x_n^a} \right) z_k + \phi (z') \frac{1 - n}{x_n^a} \right] x_n^a dz' \]

and so

\[ |Rg_n (x', x_n)| \leq C (\phi) \int_{B (0, 1)} \left| [\psi (x_n) f (x' - x_n z', \psi (x_n)) - \int_0^{\psi (x_n)} (\psi f)_n (x' - x_n z', t) dt] \right| \]
43.2. A RIGHT INVERSE FOR THE TRACE FOR A HALF SPACE

\[
\leq \frac{C(\phi)}{x_{n-1}^p} \left\{ \int_{B(0,x_n)} |\psi(x_n) f(x' + y', \psi(x_n))| \, dy' + \int_{B(0,x_n)} \int_0^{\psi(x_n)} \left| (\psi f)_n (x' + y', t) \right| \, dtdy' \right\}
\]

Therefore,
\[
\left( \int_0^\infty \int_{\mathbb{R}^{n-1}} |Rg_n (x', x_n)|^p \, dx' dx_n \right)^{1/p} \leq C(\phi) \left( \int_0^\infty \int_{\mathbb{R}^{n-1}} \left( \int_{B(0,x_n)} |\psi(x_n) f(x' + y', \psi(x_n))| \, dy' \right)^p \, dx' dx_n \right)^{1/p} + C(\phi) \left( \int_0^\infty \int_{\mathbb{R}^{n-1}} \left( \int_{B(0,x_n)} \int_0^{\psi(x_n)} \left| (\psi f)_n (x' + y', t) \right| \, dtdy' \right)^p \, dx' dx_n \right)^{1/p}
\]

Consider the first term on the right. We change variables, letting \( y' = z' x_n \). Then this term becomes
\[
C(\phi) \left( \int_0^1 \int_{\mathbb{R}^{n-1}} \left( \int_{B(0,1)} |\psi(x_n) f(x' + x_n z', \psi(x_n))| \, dz' \right)^p \, dx' dx_n \right)^{1/p} \leq C(\phi) \int_{B(0,1)} \left( \int_0^1 \int_{\mathbb{R}^{n-1}} |\psi(x_n) f(x' + x_n z', \psi(x_n))|^p \, dx' dx_n \right)^{1/p} \, dz'
\]

Now we change variables, letting \( t = \psi(x_n) \). This yields
\[
= C(\phi) \int_{B(0,1)} \left( \int_0^1 \int_{\mathbb{R}^{n-1}} |tf(x' + x_n z', t)|^p \, dx' dt \right)^{1/p} \, dz' \leq C(\phi) ||f||_{0,p,\mathbb{R}^n_+}.
\]

Now we consider the second term on the right in (43.2.3). Using the same arguments which were used on the first term involving Minkowski’s inequality and changing the variables, we obtain the second term
\[
\leq C(\phi) \int_{B(0,1)} \left( - \int_0^1 \int_{\mathbb{R}^{n-1}} \left| (\psi f)_n (x' + x_n z', t) \right|^p \, dx' dx_n \right)^{1/p} \, dt \, dy' \leq C(\phi) ||f||_{1,p,\mathbb{R}^n_+}.
\]

It is somewhat easier to verify that
\[
||Rg||_{0,p,\mathbb{R}^n_+} \leq C(\phi) ||f||_{1,p,\mathbb{R}^n_+}.
\]

Therefore, we have shown that whenever \( \gamma f = f(0) = g \),
\[
||Rg||_{1,p,\mathbb{R}^n_+} \leq C(\phi) ||f||_{1,p,\mathbb{R}^n_+}.
\]
Taking the infimum over all such \( f \) and using the definition of the norm in \( W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \),

it follows

\[ ||Rg||_{1,p,\mathbb{R}^{n}} \leq C(\phi) ||g||_{1-\frac{1}{p},p,\mathbb{R}^{n-1}}, \]

showing that this map, \( R \), is continuous as claimed. It is obvious that

\[ \lim_{x_n \to 0} Rg(x_n) = g, \]

the convergence taking place in \( L^p(\mathbb{R}^{n-1}) \) because of general results about convolution with mollifiers. This proves the lemma.

### 43.3 Intrinsic Norms

The above presentation is very abstract, involving the trace of a function in

\[ W(A_0, A_1, p, \theta) \]

and a norm which was the infimum of norms of functions in \( W \) which have trace equal to the given function. It is very useful to have a description of the norm in these fractional order spaces which is defined in terms of the function itself rather than functions which have the given function as trace. This leads to something called an intrinsic norm. I am following Adams [1].

The following interesting lemma is called Young’s inequality. It holds more generally than stated.

**Lemma 43.3.1** Let \( g = f \ast h \) where \( f \in L^1(\mathbb{R}), \ h \in L^p(\mathbb{R}), \) and \( f, h \) are all Borel measurable, \( p \geq 1 \). Then \( g \in L^p(\mathbb{R}) \) and

\[ ||g||_{L^p(\mathbb{R})} \leq ||f||_{L^1(\mathbb{R})} ||h||_{L^p(\mathbb{R})} \]

**Proof:** First of all it is good to show \( g \) is well defined. Using Minkowski’s inequality

\[ \left( \int \left( \int |h(t-s)f(s)| ds \right)^p dt \right)^{1/p} \]

\[ \leq \int \left( \int |h(t-s)|^p |f(s)|^p dt \right)^{1/p} ds \]

\[ = \int |f(s)| \left( \int |h(t-s)|^p dt \right)^{1/p} ds \]

\[ = ||f||_{L^p} ||h||_{L^p} \]
Therefore, for a.e. $t$,
\[
\int |h(t-s)f(s)| \, ds = \int |h(s)f(t-s)| \, ds < \infty
\]
and so for all such $t$ the convolution $f \ast h(t)$ makes sense. The above also shows
\[
\|g\|_{L^p} \equiv \left( \int \left| \int f(t-s)h(s) \, ds \right|^p \, dt \right)^{1/p} \leq \|f\|_{L^1} \|h\|_{L^p}
\]
and this proves the lemma.

The following is a very interesting inequality of Hardy Littlewood and Pólya.

**Lemma 43.3.2** Let $f$ be a real valued function defined a.e. on $[0, \infty)$ and let $\alpha \in (-\infty, 1)$ and
\[
g(t) = \frac{1}{t} \int_0^t f(\xi) \, d\xi \quad (43.3.6)
\]
For $1 \leq p < \infty$
\[
\int_0^\infty t^{\alpha p} |g(t)|^p \, dt \leq \frac{1}{(1-\alpha)p} \int_0^\infty t^{\alpha p} |f(t)|^p \, dt \quad (43.3.7)
\]

**Proof:** First it can be assumed the right side of (43.3.7) is finite since otherwise there is nothing to show. Changing the variables letting $t = e^\tau$, the above inequality takes the form
\[
\int_{-\infty}^\infty e^{\tau \alpha} |g(e^\tau)|^p \, d\tau \leq \frac{1}{(1-\alpha)p} \int_{-\infty}^\infty e^{\tau \alpha} |f(e^\tau)|^p \, d\tau
\]
Now from the definition of $g$ it follows
\[
g(e^\tau) = e^{-\tau} \int_{-\infty}^{e^\tau} f(\xi) \, d\xi = e^{-\tau} \int_{-\infty}^{\tau} f(e^\sigma) \, e^{\sigma} \, d\sigma
\]
and so the left side equals
\[
\int_{-\infty}^\infty e^{\tau \alpha (\alpha-1)} \left| \int_{-\infty}^{\tau} f(e^\sigma) \, e^\sigma \, d\sigma \right|^p \, d\tau
\]
\[
= \int_{-\infty}^\infty \left| \int_{-\infty}^{\tau} e^{\tau \sigma} f(e^\sigma) \, e^{-(\tau-\sigma)} \, d\sigma \right|^p \, d\tau
\]
\[
= \int_{-\infty}^\infty \left| \int_{-\infty}^{\tau} e^{(\tau-\sigma)\alpha} e^{-(\tau-\sigma)} f(e^\sigma) \, d\sigma \right|^p \, d\tau
\]
and by Lemma 43.3.1,
\[
\leq \left( \int_{-\infty}^{0} e^{(\alpha-1)u} du \right)^{p} \int_{-\infty}^{\infty} e^{p\sigma\alpha} |f(e^{\sigma})|^{p} d\sigma
\]
\[
= \left( \frac{1}{1-\alpha} \right)^{p} \int_{-\infty}^{\infty} e^{p\sigma\alpha} |f(e^{\sigma})|^{p} d\sigma
\]
which was to be shown. This proves the lemma.

Next consider the case where $G(t), t > 0$ is a continuous semigroup on $A_1$ and $A_0 \equiv D(\Lambda)$ where $\Lambda$ is the generator of this semigroup. Recall that from Proposition 17.12.5 on Page 558 $\Lambda$ is a closed densely defined operator and so $A_0$ is a Banach space if the norm is given by

$$
||u||_{A_0} \equiv ||u||_{A_1} + ||\Lambda u||_{A_1}
$$

Also assume $||G(t)||$ is uniformly bounded for $t \in [0, \infty)$. I have in mind the case where $A_1 = L^p(\mathbb{R}^n)$ and $G(t)u(x) = u(x + te_i)$ but it is notationally easier to discuss this in the general case. First here is a simple lemma.

**Lemma 43.3.3** Let $A_0 = D(\Lambda)$ as just described. Then for $u \in A_1$

$$
||u||_{A_1 + A_0} = ||u||_{A_1}
$$

**Proof:** $D(\Lambda) \subseteq A_1$. Now let $u \in A_1$.

$$
||u||_{A_1 + A_0} \equiv \inf \{ ||u_0||_{A_1} + ||\Lambda u_0||_{A_1} + ||u_1||_{A_1} : u = u_0 + u_1 \}
$$

To make this as small as possible you should clearly take $u_1 = u$ because

$$
||u_0||_{A_1} + ||\Lambda u_0||_{A_1} + ||u_1||_{A_1} \geq ||u_0 + u_1||_{A_1} + ||\Lambda u_0|| = ||u||_{A_1} + ||\Lambda u_0||
$$

Therefore, the result of the lemma follows.

**Lemma 43.3.4** Let $\Lambda$ be the generator of $G(t)$ and let $t \to g(t)$ be in $C^1(0, \infty; A_1)$. Then there exists a unique solution to the initial value problem

$$
y' - \Lambda y = g, \, y(0) = y_0 \in D(\Lambda)
$$

and it is given by

$$
y(t) = G(t)y_0 + \int_{0}^{t} G(t-s)g(s) \, ds. \quad (43.3.8)
$$

This solution is continuous having continuous derivative and has values in $D(\Lambda)$. 
**Proof:** First I show the following claim.

**Claim:** \( \int_0^t G(t-s)g(s)\,ds \in D(\Lambda) \) and

\[
\Lambda \left( \int_0^t G(t-s)g(s)\,ds \right) = G(t)g(0) - g(t) + \int_0^t G(t-s)g'(s)\,ds
\]

**Proof of the claim:**

\[
\frac{1}{h} \left( G(h) \int_0^t G(t-s)g(s)\,ds - \int_0^t G(t-s)g(s)\,ds \right)
\]

\[
= \frac{1}{h} \left( \int_0^t G(t-s+h)g(s)\,ds - \int_0^t G(t-s)g(s)\,ds \right)
\]

\[
= \frac{1}{h} \left( \int_{t-h}^{t-h} G(t-s)g(s+h)\,ds - \int_0^t G(t-s)g(s)\,ds \right)
\]

\[
= \frac{1}{h} \int_{t-h}^0 G(t-s)g(s+h)\,ds + \int_0^{t-h} G(t-s)\frac{g(s+h) - g(s)}{h}
\]

\[
- \frac{1}{h} \int_{t-h}^t G(t-s)g(s)\,ds
\]

Using the estimate in Theorem 17.12.3 on Page 557 and the dominated convergence theorem the limit as \( h \to 0 \) of the above equals

\[
G(t)g(0) - g(t) + \int_0^t G(t-s)g'(s)\,ds
\]

which proves the claim.

Since \( y_0 \in D(\Lambda) \),

\[
G(t)\Lambda y_0 = G(t) \lim_{h \to 0} \frac{G(h)y_0 - y_0}{h}
\]

\[
= \lim_{h \to 0} \frac{G(t+h) - G(t)}{h}y_0
\]

\[
= \lim_{h \to 0} \frac{G(h)G(t)y_0 - G(t)y_0}{h}
\]

(43.3.9)

Since this limit exists, the last limit in the above exists and equals

\[
\Lambda G(t)y_0
\]

(43.3.10)

and \( G(t)y_0 \in D(\Lambda) \). Now consider

\[
\frac{y(t+h) - y(t)}{h} = \frac{G(t+h) - G(t)}{h}y_0 +
\]
\[
\frac{1}{h} \left( \int_0^{t+h} G(t-s+h)g(s)\,ds - \int_0^t G(t-s)g(s)\,ds \right)
\]
\[
= \frac{G(t+h) - G(t)}{h} y_0 + \frac{1}{h} \int_t^{t+h} G(t-s+h)g(s)\,ds
\]
\[
+ \frac{1}{h} \left( G(h) \int_0^t G(t-s)g(s)\,ds - \int_0^t G(t-s)g(s)\,ds \right)
\]

From the claim and \[43.3.9\], \[43.3.10\] the limit of the right side is
\[
\Lambda G(t) y_0 + g(t) + \Lambda \left( \int_0^t G(t-s)g(s)\,ds \right)
\]
\[
= \Lambda \left( G(t) y_0 + \int_0^t G(t-s)g(s)\,ds \right) + g(t)
\]

Hence
\[
y'(t) = \Lambda y(t) + g(t)
\]
and from the formula, \( y' \) is continuous since by the claim and \[43.3.10\] it also equals
\[
G(t) \Lambda y_0 + g(t) + G(t) g(0) - g(t) + \int_0^t G(t-s) g'(s)\,ds
\]
which is continuous. The claim and \[43.3.10\] also shows \( y(t) \in D(\Lambda) \). This proves the existence part of the lemma.

It remains to prove the uniqueness part. It suffices to show that if
\[
y' - \Lambda y = 0, \ y(0) = 0
\]
and \( y \) is \( C^1 \) having values in \( D(\Lambda) \), then \( y = 0 \). Suppose then that \( y \) is this way. Letting \( 0 < s < t \),
\[
\frac{d}{ds} \left( G(t-s) y(s) \right)
\]
\[
= \lim_{h \to 0} \frac{G(t-s-h) y(s+h) - y(s)}{h}
\]
\[
- \frac{G(t-s) y(s) - G(t-s-h) y(s)}{h}
\]
provided the limit exists. Since \( y' \) exists and \( y(s) \in D(\Lambda) \), this equals
\[
G(t-s) y'(s) - G(t-s) \Lambda y(s) = 0.
\]
Let \( y^* \in A_1^t \). This has shown that on the open interval \((0,t)\) the function \( s \to y^*(G(t-s) y(s)) \) has a derivative equal to 0. Also from continuity of \( G \) and \( y \), this function is continuous on \([0,t]\). Therefore, it is constant on \([0,t]\) by the mean value theorem. At \( s = 0 \), this function equals 0. Therefore, it equals 0 on \([0,t]\). Thus for fixed \( s > 0 \) and letting \( t \to s \), \( y^* \left( G(t-s) y(s) \right) = 0 \). Now let \( t \) decrease toward \( s \). Then \( y^*(y(s)) = 0 \) and since \( y^* \) was arbitrary, it follows \( y(s) = 0 \). This proves uniqueness.
**Definition 43.3.5** Let $G(t)$ be a uniformly bounded continuous semigroup defined on $A_1$ and let $\Lambda$ be its generator. Let the norm on $D(\Lambda)$ be given by

$$\|u\|_{D(\Lambda)} \equiv \|u\|_{A_1} + \|\Lambda u\|_{A_1}$$

so that by Lemma 43.3.3 the norm on $A_1 + D(\Lambda)$ is just $\|\cdot\|_{A_1}$. Let

$$T_0 \equiv \left\{ u \in A_1 : \|u\|^p_{A_1} + \int_0^\infty e^{tp} \left\| \frac{G(t) u - u}{t} \right\|^p_{A_1} \frac{dt}{t} \equiv \|u\|^p_{T_0} < \infty \right\}$$

**Theorem 43.3.6** $T_0 = T(D(\Lambda), A_1, p, \theta) \equiv T$ and the two norms are equivalent.

**Proof:** Take $u \in T(D(\Lambda), A_1, p, \theta)$. I will show $\|u\|^p_{T_0} \leq C(\theta, p) \|u\|^p_T$. By the definition of the norm in $T$, there exists $f \in W(D(\Lambda), A_1, p, \theta)$ such that

$$\|u\|^p_T + \delta > \|f\|^p_W, \quad f(0) = u.$$ 

Now by Lemma 42.1.4 there exists $g_r \in W$ such that $\|g_r - f\|^p_W < r$ and $g_r \in C^\infty(0, \infty; D(\Lambda))$ and $g'_r \in C^\infty(0, \infty; A_1)$. Thus for each $\varepsilon > 0$, $g_r(\varepsilon) \in D(\Lambda)$ although possibly $g_r(0) \notin D(\Lambda)$. Then letting $h_r(t)$ be defined by

$$g'_r(t) - \Lambda g_r(t) = h_r(t)$$

it follows $h_r \in C^1(0, \infty; A_1)$ and applying Lemma 43.3.3 on $[\varepsilon, \infty)$ it follows

$$g_r(t) = G(t - \varepsilon) g_r(\varepsilon) + \int_\varepsilon^t G(t - s) h_r(s) ds. \quad (43.3.11)$$

By Lemma 42.1.4 again, $g_r(\varepsilon)$ converges to $g_r(0)$ in $A_1$. Thus

$$\int_\varepsilon^t \|G(t-s) h_r(s)\|_{A_1} ds \leq C$$

for some constant independent of $\varepsilon$. Thus $s \to G(t-s) h_r(s)$ is in $L^1(0, t; A_1)$ and it is possible to pass to the limit in $43.3.11$ as $\varepsilon \to 0$ to conclude

$$g_r(t) = G(t) g_r(0) + \int_0^t G(t-s) h_r(s) ds$$

Now

$$\frac{G(t) g_r(0) - g_r(0)}{t} = \frac{1}{t} \int_0^t g'_r(s) ds - \frac{1}{t} \int_0^t G(t-s) h_r(s) ds$$

and so using the assumption that $G(t)$ is uniformly bounded,

$$\left\| \frac{G(t) g_r(0) - g_r(0)}{t} \right\| \leq \frac{1}{t} \int_0^t \|g'_r\|_{A_1} + M \|h_r\|_{A_1}$$
\[
\begin{align*}
&\leq \frac{1}{t} \int_0^t \|g_r\| (M + 1) + M \|Ag_r\| ds \\
&\leq \frac{M + 1}{t} \int_0^t \|g_r\|_{A_1} + \|g_r\|_{D(A)} ds
\end{align*}
\]
Therefore, from Lemma 43.3.2
\[
\begin{align*}
&\int_0^\infty t^{p\theta - p} \left\| G(t) g_r(0) - g_r(0) \right\|_{A_1}^p \frac{dt}{t} \\
&= \int_0^\infty t^p \left\| \frac{G(t) g_r(0) - g_r(0)}{t} \right\|_{A_1}^p \frac{dt}{t} \\
&\leq \int_0^\infty t^p \left( \frac{M + 1}{t} \int_0^t \|g_r\|_{A_1} + \|g_r\|_{D(A)} ds \right)^p dt/t \\
&\leq (M + 1)^p 2^{p-1} \left( \frac{1}{1 - \theta} \right)^p \int_0^\infty t^p \left( \|g_r\|_{A_1}^p + \|g_r\|_{D(A)}^p \right)
\end{align*}
\]
Now since \(g_r \to f\) in \(W\), it follows from Lemma 43.4.1 that \(g_r(0) \to u\) in \(T\) and hence by Theorem 42.1.9 this also in \(A_1\). Therefore, using Fatou’s lemma in the above along with the convergence of \(g_r\) to \(f\),
\[
\begin{align*}
&\int_0^\infty t^{p\theta - p} \left\| G(t) u - u \right\|_{A_1}^p \frac{dt}{t} \\
&\leq (M + 1)^p 2^{p-1} \left( \frac{1}{1 - \theta} \right)^p \int_0^\infty t^p \left( \|f\|_{A_1}^p + \|f\|_{D(A)}^p \right) \\
&\leq (M + 1)^p 2^{p-1} \left( \frac{1}{1 - \theta} \right)^p (\|u\|_{T}^p + \delta)
\end{align*}
\]
Since \(u \in T\), Theorem 43.4.1 implies \(u \in A_1\) and \(\|u\|_{A_1} \leq C \|u\|_{T}\). Therefore, since \(\delta\) was arbitrary, this has shown that \(u \in T_0\) and
\[
\|u\|_{T_0} \leq C (\theta, p) \|u\|_{T}.
\]
This shows \(T \subseteq T_0\) with continuous inclusion.

Now it is necessary to take \(u \in T_0\) and show it is in \(T\). Since \(u \in T_0\)
\[
\int_0^\infty t^{p\theta - p} \left\| G(t) u - u \right\|_{A_1}^p \frac{dt}{t} = \|u\|_{T_0}^p
\]
Let \(\phi\) be a nonnegative decreasing infinitely differentiable function such that \(\phi(0) = 1\) and \(\phi(t) = 0\) for all \(t > 1\). Then define
\[
f(t) \equiv \phi(t) \frac{1}{t} \int_0^t G(\tau) u d\tau.
\]
It is easy to see that \( f(t) \in D(\Lambda) \). In fact, changing variables as needed,

\[
\frac{1}{h} \left( G(h) \int_0^t G(\tau) \, d\tau - \int_0^t G(\tau) \, d\tau \right) = \frac{1}{h} \int_h^{t+h} G(\tau) \, d\tau - \frac{1}{h} \int_0^t G(\tau) \, d\tau = \frac{1}{h} \int_0^{t+h} G(\tau) \, d\tau - \frac{1}{h} \int_0^h G(\tau) \, d\tau
\]

which converges to \( G(t) \, u \) and so

\[\Lambda \int_0^t G(\tau) \, d\tau = G(t) \, u - u.\] (43.3.12)

Thus

\[
\int_0^t t^\theta ||\Lambda f||_{A_1}^p \, dt \leq \int_0^t t^\theta \left|\frac{G(t) \, u - u}{t}\right| A_1^p \, dt \\
\leq ||u||_{p, T_0}^p
\]

Next it is necessary to consider

\[
\int_0^t t^\theta ||f'||_{A_1}^p \, dt.
\]

\[
f'(t) = \phi'(t) \frac{1}{t} \int_0^t G(\tau) \, d\tau + \phi(t) \left( -\frac{1}{t^2} \int_0^t G(\tau) \, d\tau + \frac{1}{t} G(t) \, u \right)
\]

and so there is a constant \( C \) depending on \( \phi \) and the uniform bound on \( ||G(t)|| \) such that

\[
||f'(t)||_{A_1} \leq C \chi_{[0,1]}(t) \left( ||u||_{A_1} + \frac{1}{t^2} \int_0^t ||G(t - \tau) \, u - u|| \, d\tau \right) = C \chi_{[0,1]}(t) \left( ||u||_{A_1} + \frac{1}{t^2} \int_0^t ||G(\tau) \, u - u|| \, d\tau \right)
\]

Now

\[
\int_0^\infty \chi_{[0,1]}(t) t^\theta ||u||_{A_1}^p \, dt \leq C(p, \theta) ||u||_{T_0}^p
\]
and using Lemma 43.3.2,

\[
\int_0^\infty \mathcal{X}_{[0,1]}(t) \left| \frac{1}{t^\theta} \int_0^t ||G(\tau) u - u|| \, d\tau \right|^p \frac{dt}{t^\theta} \\
\leq \int_0^\infty \left| \frac{1}{t} \int_0^t ||G(\tau) u - u|| \, d\tau \right|^p \frac{dt}{t^\theta(\theta - 1)} \\
\leq \frac{1}{(1 - (\theta - 1))p} \int_0^\infty ||G(\tau) u - u||^p \frac{dt}{t^\theta(\theta - 1)} \\
= \frac{1}{(2 - \theta)p} \int_0^\infty \left\| \frac{G(\tau) u - u}{t} \right\|^p \frac{dt}{t^\theta} \leq C(\theta, p) ||u||_{T_0}^p
\]

This proves the theorem.

Of course the case of most interest here is where \( A_1 = L^p(\mathbb{R}^n) \) and

\[
G(t) u(x) \equiv u(x + te_i)
\]

Thus \( \Lambda u = \partial u/\partial x_i \), the weak derivative. The trace space \( T(D(\Lambda), L^p(\mathbb{R}^n), p, 1 - \theta) \) then is a space of functions in \( L^p(\mathbb{R}^n) \) which have a fractional order partial derivative with respect to \( x_i \).

Recall from Definition 43.1.1 that for \( \theta \in (0, 1) \),

\[
W^{\theta, p}(\mathbb{R}^n) \equiv T(W^{1, p}(\mathbb{R}^n), L^p(\mathbb{R}^n), p, 1 - \theta)
\]

Let \( f \in W(W^{1, p}(\mathbb{R}^n), L^p(\mathbb{R}^n), p, 1 - \theta) \). Then

\[
||f||_W \equiv \max \left( \int_0^\infty t^{(1-\theta)p} ||f(t)||_{W^{1, p}} \frac{dt}{t}, \int_0^\infty t^{(1-\theta)p} ||f'(t)||_{L^p} \frac{dt}{t} \right)
\]

Letting \( G_i(t) u(x) \equiv u(x + te_i) \) and \( \Lambda_i \) its generator,

\[
W^{1, p}(\Omega) = \cap_{i=1}^n DA_i \cap L^p(\mathbb{R}^n)
\]

with the norm given by

\[
||u||^p = ||u||_{L^p}^p + \sum_{i=1}^n ||\Lambda_i u||_{L^p}^p
\]

which is equivalent to the norm

\[
||u||^p = \sum_{i=1}^n ||u||_{D(\Lambda_i)}^p.
\]

Then by considering each of the \( G_i \) and repeating the above argument in Theorem 43.3.6, it follows an equivalent intrinsic norm is

\[
||u||_{W^{\theta, p}(\mathbb{R}^n)}^p = ||u||_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left\| \frac{G_i(t) u - u}{t} \right\|_{L^p}^p \frac{dt}{t}
\]
43.3. INTRINSIC NORMS

\[ ||u||_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left\| \frac{u \left( t + \theta e_i \right) - u \left( \cdot \right)}{t} \right\|_{L^p}^p \frac{dt}{t} \]  \tag{43.3.13}

and \( u \in W^{\theta,p}(\mathbb{R}^n) \) when this norm is finite. The only new detail is that in showing that for \( u \in T_0 \) it follows it is in \( T \), you use the function

\[ f(t) \equiv \phi(t) \frac{1}{t^n} \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u d\tau_1 \cdots d\tau_n \]

and the fact that these semigroups commute. To get this started, note that

\[ g(t) \equiv \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u d\tau_1 \cdots d\tau_n \in D(\Lambda_i) \]

for each \( i \). This follows from writing it as

\[ \int_0^t G_i(\tau_i)(w_i) \, d\tau_i \]

for \( w_i \in L^p \) coming from the other integrals and then repeating the earlier argument to get

\[ \Lambda_i g(t) = G_i(t) w_i - w_i \]

and then

\[ \int_0^\infty t^{p(1-\theta)} \left\| \Lambda_i f \right\|_{L^p}^p \frac{dt}{t} \]

\[ \leq \int_0^\infty t^{p(1-\theta)} \left\| G_i(t) w_i - w_i \right\|_{L^p}^p \frac{dt}{t} \]

\[ \leq C \int_0^\infty t^{p(1-\theta)} \left\| G_i(t) u - u \right\|_{L^p}^p \frac{dt}{t} \leq C ||u||_{T_0}^p \]

Thus all is well as far as \( f \) is concerned and the proof will work as it did earlier in Theorem 43.3.6. What about \( f' \)? As before, the only term which is problematic is

\[ \phi(t) \left( \frac{1}{t^n} \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u d\tau_1 \cdots d\tau_n \right)' \]

After enough massaging, it becomes

\[ \sum_{i=1}^n \prod_{j \neq i} \frac{1}{t} \int_0^t G_j(\tau_j) d\tau_j \frac{1}{t^n} \int_0^t (G_i(t) u - G_i(\tau_i) u) \, d\tau_i \]

where the operator \( \sum_{i=1}^n \prod_{j \neq i} \frac{1}{t} \int_0^t G_j(\tau_j) d\tau_j \) is bounded. Thus similar arguments to those of Theorem 43.3.6 will work, the only difference being a sum.
Theorem 43.3.7 An equivalent norm for \( W^{\theta,p}(\mathbb{R}^n) \) for \( \theta \in (0,1) \) is
\[
||u||_{W^{\theta,p}(\mathbb{R}^n)}^p = ||u||_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left( \left| \frac{G_i(t) u - \frac{u}{t}}{t} \right|_{L^p} \right)^p \frac{dt}{t} \]
(43.3.14)

Note it is obvious from 43.3.13 that a Lipschitz map takes \( W^{\theta,p}(\mathbb{R}^n) \) to \( W^{\theta,p}(\mathbb{R}^n) \) and is continuous.

The above description in Theorem 43.3.7 also makes possible the following corollary.

Corollary 43.3.8 \( W^{\theta,p}(\mathbb{R}^n) \) is reflexive.

Proof: Let \( u \in W^{\theta,p}(\mathbb{R}^n) \). For each \( i = 1, 2, \cdots, n \), define for \( t > 0 \),
\[
\Delta_i u(t)(x) = \frac{u(x + te_i) - u(x)}{t}
\]
Then by Theorem 43.3.7,
\[
\Delta_i u \in L^p((0,\infty); L^p(\mathbb{R}^n), \mu) \equiv Y
\]
where
\[
\mu(E) = \int_E t^{(1-\theta)p}t^{-1}dt.
\]
Clearly the measure space is \( \sigma \) finite and so \( Y \) is reflexive by Corollary 19.8.9 on Page 676. Also \( \Delta_i \) is a closed operator whose domain is \( W^{\theta,p}(\mathbb{R}^n) \). To see this, suppose \( u_n \in W^{\theta,p}(\mathbb{R}^n) \) and \( u_n \to u \) in \( L^p(\mathbb{R}^n) \) while \( \Delta_i u_n \to g \) in \( Y \). Then in particular \( ||\Delta_i u_n||_Y \) is bounded. Now by Fatou’s lemma,
\[
\int_0^\infty t^{(1-\theta)p} \left( \left| \frac{u(x + te) - u(x)}{t} \right|_{L^p(\mathbb{R}^n)} \right)^p \frac{dt}{t} \leq \lim inf_{n \to \infty} \int_0^\infty t^{(1-\theta)p} \left( \left| \frac{u_n(x + te) - u_n(x)}{t} \right|_{L^p(\mathbb{R}^n)} \right)^p \frac{dt}{t} < \infty.
\]

Letting \( \bar{\Delta} \equiv (\Delta_1, \Delta_2, \cdots, \Delta_n) \), it follows from similar reasoning that \( \bar{\Delta} \) is a closed operator mapping \( W^{\theta,p}(\mathbb{R}^n) \) to \( Y^n \). Therefore
\[
\left( \text{id}, \bar{\Delta} \right)(W^{\theta,p}(\mathbb{R}^n)) \subseteq L^p(\mathbb{R}^n) \times Y^n
\]
and is a closed subspace of the reflexive space \( L^p(\mathbb{R}^n) \times Y^n \). With the norm in \( L^p(\mathbb{R}^n) \times Y^n \) given as the sum of the norms of the components, it follows the
mapping \((\text{id}, \tilde{\Delta})\) is a norm preserving isomorphism between \(W^{\theta,p}(\mathbb{R}^n)\) and this closed subspace of \(L^p(\mathbb{R}^n) \times Y^n\). Since \(L^p(\mathbb{R}^n)\) and \(Y\) are reflexive, their product is reflexive. By Lemma 19.2.7 on Page 641 it follows \((\text{id}, \tilde{\Delta})(W^{\theta,p}(\mathbb{R}^n))\) and hence \(W^{\theta,p}(\mathbb{R}^n)\) is reflexive. This proves the theorem.

One can generalize this to find an intrinsic norm for \(W^{\theta,p}(\Omega)\). The version given above will not do because it requires the function to be defined on all of \(\mathbb{R}^n\) in order to make sense of the shift operators \(G_i\). However, you can give a different version of this intrinsic norm which will make sense for \(\Omega \neq \mathbb{R}^n\).

**Lemma 43.3.9** Let \(t \neq 0\) be a number. Then there is a constant \(C(n,\theta,p)\) depending on the indicated quantities such that

\[
\int_{\mathbb{R}^n} \frac{1}{(t^2 + |s|^2)^{\frac{1}{2}(n+p\theta)}} ds = C(n,\theta,p) \frac{|t|}{|t|^{1+p\theta}}
\]

**Proof:** Change the integral to polar coordinates. Thus the integral equals

\[
\int_{S^{n-1}} \int_0^\infty \frac{\rho^{n-2}}{(t^2 + \rho^2)^{\frac{1}{2}(n+p\theta)}} d\rho d\sigma
\]

Now change the variables, \(\rho = |t| u\). Then the above integral becomes

\[
C_n \int_0^\infty \frac{|t|^{n-2} u^{n-2} |t|}{|t|^{n+p\theta}} \frac{u^{n-2}}{(1 + u^2)^{\frac{1}{2}(n+p\theta)}} du = C(n,\theta,p) \frac{|t|}{|t|^{1+p\theta}}.
\]

This proves the lemma.

Now let \(u \in W^{\theta,p}(\mathbb{R}^n)\). This means the norm of \(||u||_{W^{\theta,p}}\) can be taken as

\[
||u||_{W^{\theta,p}}^p + \frac{1}{2} \sum_{i=1}^n \int_0^\infty |t|^{(1-\theta)p} \left| \int_{\mathbb{R}^n} \frac{|u(x + te_i) - u(x)|^p}{t} dx \right| dt
\]

That integral over \(\mathbb{R}^n\) can be massaged and one obtains the above equal to

\[
||u||_{L^p}^p + \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^n} \int_0^\infty |t|^{1+p\theta} \left| u(x_1, \ldots, x_i + t, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n) \right|^p dx_i dy_n dt dt
\]
Lemma 4.3.9

Then if you Fubini again, it reduces to the expression

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x_1, \ldots, x_i-1, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dx \, dy \]

Next let

\[ (s_1, \ldots, s_{n-1}) = (y_1 - x_1, \ldots, y_{i-1} - x_{i-1}, x_{i+1} - y_{i+1}, \ldots, x_n - y_n) \]

where the new variables of integration in the integral corresponding to \( ds \) are \( y_1, \ldots, y_{i-1} \) and \( x_{i+1}, \ldots, x_n \). Then changing the variables, the above reduces to

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x_1, \ldots, x_i-1, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dy \, dx \]

Then if you Fubini again, it reduces to the expression

\[ \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \frac{1}{(t^2 + |s|^2)^{\frac{1}{2}(n+p\theta)}} |u(x_1, \ldots, x_i-1, x_i + t, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dt \, ds \, dx \, dy 

\]

where \( dx_i \) refers to the first \( i \) entries and \( dy_{n-i} \) refers to the remaining entries. From Lemma 4.3.3, the complicated expression above equals

\[ \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \frac{1}{(t^2 + |s|^2)^{\frac{1}{2}(n+p\theta)}} |u(x_1, \ldots, x_i-1, x_i + t, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dt \, ds \, dx \, dy 

\]

Changing the variable in the inside integral to \( t = y_i - x_i \), this equals

\[ \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \frac{1}{(y_i - x_i)^2 + |s|^2}^{\frac{1}{2}(n+p\theta)} |u(x_1, \ldots, x_i-1, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dy_i \, dx_i \, dy_{n-i} 

\]

Next let

\[ (s_1, \ldots, s_{n-1}) = (y_1 - x_1, \ldots, y_{i-1} - x_{i-1}, x_{i+1} - y_{i+1}, \ldots, x_n - y_n) \]

where the new variables of integration in the integral corresponding to \( ds \) are \( y_1, \ldots, y_{i-1} \) and \( x_{i+1}, \ldots, x_n \). Then changing the variables, the above reduces to

\[ \frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^{n} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^{i}} \frac{1}{|x - y|^{n+p\theta}} |u(x_1, \ldots, x_i-1, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dy_i \, dx_i \, dy_{n-i} \]

Then if you Fubini again, it reduces to the expression

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x_1, \ldots, x_i-1, y_i, y_{i+1}, \ldots, y_n) - u(x_1, \ldots, x_i, y_{i+1}, \ldots, y_n)|^p \, dx \, dy 

\]
43.3. INTRINSIC NORMS

Now taking the sum inside and adjusting the constants yields

$$\geq C(n, \theta, p) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy$$

Thus there exists a constant $C(n, \theta, p)$ such that

$$||u||_{W^{\theta,p} (\mathbb{R}^n)} \geq C(n, \theta, p) \left( ||u||_{L^p} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \right)^{1/p}.$$

Next start with the right side of the above. It suffices to consider only the complicated term. First note that for a vector,

$$\left( \sum_{i=1}^{n} a_i^2 \right)^{p/2} \geq |a_i|^p$$

and so

$$\sum_{i=1}^{n} |a_i|^p \leq n \left( \sum_{i=1}^{n} a_i^2 \right)^{p/2} = n |a|^p$$

from which it follows

$$\frac{1}{n} \sum_{i=1}^{n} |a_i|^p \leq |a|^p$$

Then it follows

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \geq \frac{1}{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{|u(x_1, \ldots, x_i-1, y_i, y_i+1, \ldots, y_n) - u(x_1, \ldots, x_i, y_i+1, \ldots, y_n)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy$$

Consider the $i^{th}$ term. By Fubini’s theorem it equals

$$\frac{1}{n} \int \cdots \int \frac{1}{\left( (y_i - x_i)^2 + \sum_{j \neq i} (y_j - x_j)^2 \right)^{\frac{4}{2} \left( (n+p\theta) \right)}}$$

$$|u(x_1, \ldots, x_i-1, y_i, y_i+1, \ldots, y_n) - u(x_1, \ldots, x_i, y_i, y_i+1, \ldots, y_n)|^p$$

$$dy_1 \, dx_1 \cdots dx_{i-1} \, dy_i \cdots dy_n \, dy_1 \cdots dy_{i-1} \, dx_i \cdots dx_n$$

Let $t = y_i - x_i$. Then it reduces to

$$\frac{1}{n} \int \cdots \int \frac{1}{\left( t^2 + \sum_{j \neq i} (y_j - x_j)^2 \right)^{\frac{4}{2} \left( (n+p\theta) \right)}}$$

$$|u(x_1, \ldots, x_i-1, x_i + t, y_i+1, \ldots, y_n) - u(x_1, \ldots, x_i, y_i+1, \ldots, y_n)|^p$$

$$dt \, dx_1 \cdots dx_{i-1} \, dy_{i+1} \cdots dy_n \, dy_1 \cdots dy_{i-1} \, dx_i \cdots dx_n$$
Now let
\[(s_1,\ldots,s_{n-1}) = (x_1 - y_1,\ldots,x_{i-1} - y_{i-1},y_{i+1} - x_{i+1},\ldots,y_n - x_n)\]
on the next \(n-1\) iterated integrals. Then using Fubini’s theorem again and changing
the variables, it equals
\[
\frac{1}{n} \int \cdots \int \frac{1}{(t^2 + |s|^2)^{(n+p\theta)}}. \\
|u(s_1+y_1,\ldots,y_{i-1}+s_{i-1},x_i + t, x_{i+1} + s_i,\ldots,x_n + s_{n-1}) - u(s_1+y_1,\ldots,y_{i-1}+s_{i-1},x_i,x_{i+1} + s_i,\ldots,x_n + s_{n-1})|^p \\
dy_1 \cdots dy_{i-1} dx_i \cdots dx_n ds_1 \cdots ds_{i-1} ds_i \cdots ds_{n-1} dt
\]
By translation invariance of the measure, the inside integrals corresponding to
\[dy_1 \cdots dy_{i-1} dx_i \cdots dx_n\]
simplify and the expression can be written as
\[
\frac{1}{n} \int \cdots \int \frac{1}{(t^2 + |s|^2)^{(n+p\theta)}}. \\
|u(x_1,\ldots,x_{i-1},x_i + t, x_{i+1},\ldots,x_n) - u(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)|^p \\
dx_1 \cdots dx_{i-1} dx_i \cdots dx_n ds_1 \cdots ds_{i-1} ds_i \cdots ds_{n-1} dt
\]
where I just renamed the variables. Use Fubini’s theorem again to get
\[
\frac{1}{n} \int \cdots \int \frac{1}{(t^2 + |s|^2)^{(n+p\theta)}}. \\
|u(x_1,\ldots,x_{i-1},x_i + t, x_{i+1},\ldots,x_n) - u(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)|^p \\
ds_1 \cdots ds_{i-1} ds_i \cdots ds_{n-1} dx_1 \cdots dx_{i-1} dx_i \cdots dx_n dt
\]
Now from Lemma 43.3.9, the inside \(n-1\) integrals corresponding to \(ds_1 \cdots ds_{i-1} ds_i \cdots ds_{n-1}\)
can be replaced with
\[
C(n,\theta,p) \\
\frac{C(n,\theta,p)}{|t|^{1+p\theta}}
\]
and this yields
\[
C(n,\theta,p) \int_{R} \frac{1}{|t|^{p\theta}} \int_{R^n} |u(x + te_i) - u(x)|^p dx dt \\
= \frac{1}{2} C(n,\theta,p) \int_{0}^{\infty} t^{p(1-\theta)} \frac{|u(\cdot + te_i) - u(\cdot)|^p_{L^p(R^n)}}{t} dt
\]
Applying this to each term of the sum in \(43.3.15\) and adjusting the constant, it follows

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \geq C(n, \theta, p) \sum_{i=1}^{n} \int_0^\infty t^{p(1-\theta)} \frac{||u(\cdot + t e_i) - u(\cdot)||^p_{L^p(\mathbb{R}^n)}}{t^p} \, dt
\]

Therefore,

\[
||u||_{W^{\theta,p}(\mathbb{R}^n)} \geq C(n, \theta, p) \left( ||u||^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \right)^{1/p}
\]

This has proved most of the following theorem about the intrinsic norm.

**Theorem 43.3.10** An equivalent norm for \(W^{\theta,p}(\mathbb{R}^n)\) is

\[
||u|| = \left( ||u||^p_{L^p(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \right)^{1/p}.
\]

Also for any open subset of \(\mathbb{R}^n\)

\[
||u|| = \left( ||u||^p_{L^p(\Omega)} + \int_{\Omega} \int_{\Omega} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \right)^{1/p}.
\]

(43.3.16)

is a norm.

**Proof:** It only remains to verify this is a norm. Recall the \(l_p\) norm on \(\mathbb{R}^2\) given by

\[
|(x, y)|_{l_p} = (|x|^p + |y|^p)^{1/p}
\]

For \(u, v \in W^{\theta,p}\) denote by \(\rho(u)\) the expression

\[
\left( \int_{\Omega} \int_{\Omega} \frac{|u(y) - u(x)|^p}{|x - y|^{(n+p\theta)}} \, dx \, dy \right)^{1/p}
\]

a similar definition holding for \(v\). Then it follows from the usual Minkowski inequality that \(\rho(u + v) \leq \rho(u) + \rho(v)\). Then from

\[
||u + v|| = (||u + v||^p_{L^p} + \rho(u + v)^p)^{1/p}
\]

\[
\leq ((||u||^p_{L^p} + ||v||^p_{L^p})^p + (\rho(u) + \rho(v))^p)^{1/p}
\]
CHAPTER 43. TRACES OF SOBOLEV SPACES AND FRACTIONAL ORDER SPACES

\[ = \|\|u\|_{L^p,\rho}\| + \|\|v\|_{L^p,\rho}\| \]
\[ \leq \|\|u\|_{L^p,\rho}\| + \|\|v\|_{L^p,\rho}\| \]
\[ = (\|u\|_{L^p}^p + \rho(u)^p)^{1/p} + (\|v\|_{L^p}^p + \rho(v)^p)^{1/p} \]
\[ = \|u\| + \|v\| \]

The other properties of a norm are obvious. This proves the theorem.

As pointed out in the above theorem, this is a norm in \( L^p(\Omega) \). One could define a set of functions for which this norm is finite. In the case where \( \Omega = \mathbb{R}^n \) the conclusion of Theorem 43.3.10 is that this space of functions is the same as \( W^{\theta,p}(\mathbb{R}^n) \) and the norms are equivalent. Does this happen for other open subsets of \( \mathbb{R}^n \)?

**Definition 43.3.11** Denote by \( \widetilde{W^{\theta,p}}(U) \) the functions in \( L^p(U) \) for which the norm of Theorem 43.3.10 is finite. Here \( \theta \in (0,1) \).

**Proposition 43.3.12** Let \( U \) be a bounded open set which has Lipschitz boundary and \( \theta \in (0,1) \). Then for each \( p \geq 1 \), there exists \( E \in \mathcal{L}\left(\widetilde{W^{\theta,p}}(U), W^{\theta,p}(\mathbb{R}^n)\right) \) such that \( Eu(x) = u(x) \ a.e. \ x \in U \).

**Proof:** In proving this, I will use the equivalent norm of Theorem 43.3.10 as the norm of \( W^{\theta,p}(\mathbb{R}^n) \) Consider the following picture.

![Diagram](attachment:image.png)

The wavy line signifies a part of the boundary of \( U \) and \( \text{spt}(u) \) is contained in the circle as shown. It is drawn as a circle but this is not important. Denote by \( U^+ \) the region above the part of the boundary which is shown. Also let the boundary be given by \( x_n = g(\tilde{x}) \) for \( \tilde{x} \in B \equiv B(\tilde{y}_0, r) \subseteq \mathbb{R}^{n-1} \). Of course \( u \) is only defined on \( U \) so actually the support of \( u \) is contained in the intersection of the circle with \( U \). Let the Lipschitz constant for \( g \) be very small and denote it by \( K \). In fact, assume \( 8K^2 < 1 \). I will first show how to extend when this condition holds and then I will remove it with a simple trick. Define

\[ Eu(\tilde{x}, x_n) \equiv \begin{cases} u(\tilde{x}, x_n) & \text{if } x_n \leq g(\tilde{x}) \\ u(\tilde{x}, 2g(\tilde{x}) - x_n) & \text{if } x_n > g(\tilde{x}) \\ 0 & \text{if } \tilde{x} \notin B \end{cases} \]
I will write $U$ instead of $U \cap B \times (a, b)$ to save space but this does not matter because $u$ is assumed to be zero outside the indicated region. Then

\[
\frac{1}{\int \int_{\mathbb{R}^2} |E u (\bar{x}, x_n) - E u (\bar{y}, y_n)|^p}{\int \int_{\mathbb{R}^2} (\bar{x} - \bar{y})^2 + (x_n - y_n)^2)^{(1/2)(n+p)} dxdy} = \int \int_{U} \frac{|u (\bar{x}, x_n) - u (\bar{y}, y_n)|^p}{\int \int_{\mathbb{R}^2} (\bar{x} - \bar{y})^2 + (x_n - y_n)^2)^{(1/2)(n+p)} dxdy} + \int \int_{U^c} \frac{|E u (\bar{x}, x_n) - E u (\bar{y}, y_n)|^p}{\int \int_{\mathbb{R}^2} (\bar{x} - \bar{y})^2 + (x_n - y_n)^2)^{(1/2)(n+p)} dxdy}.
\]

Consider the second of the integrals on the right of the equal sign. Using Fubini’s theorem, it equals

\[
\int \int_{U} \frac{|u (\bar{x}, x_n) - u (\bar{y}, 2g (\bar{y}) - y_n)|^p}{\int \int_{\mathbb{R}^2} (\bar{x} - \bar{y})^2 + (x_n - y_n)^2)^{(1/2)(n+p)} dxdy} = \int \int_{U \times B} \frac{|u (\bar{x}, x_n) - u (\bar{y}, 2g (\bar{y}) - y_n)|^p}{\int \int_{\mathbb{R}^2} (\bar{x} - \bar{y})^2 + (x_n - y_n)^2)^{(1/2)(n+p)} dydx}.
\]

I need to estimate $|x_n - z_n|$. 

\[
|x_n - z_n| \leq |x_n - g (\bar{x})| + |g (\bar{x}) - g (\bar{y})| + |g (\bar{y}) - z_n| 
\leq g (\bar{x}) - x_n + K |\bar{x} - \bar{y}| + y_n - g (\bar{y}) 
\leq |y_n - x_n| + 2K |\bar{x} - \bar{y}|
\]

and so

\[
|x_n - z_n|^2 \leq 8K^2 |\bar{x} - \bar{y}|^2 + 2 |y_n - x_n|^2 
\leq |\bar{x} - \bar{y}|^2 + 2 |y_n - x_n|^2
\]
Thus,

\[(y_n - x_n)^2 \geq \frac{1}{2} |x_n - z_n|^2 - \frac{1}{2} |\bar{x} - \bar{y}|^2\]

and so, the above change of variables results in an expression which is dominated by

\[\int_U \int_U \frac{|u(\bar{x}, x_n) - u(\bar{y}, z_n)|^p}{\left| \frac{1}{2} |\bar{x} - \bar{y}|^2 + \frac{1}{2} (x_n - z_n)^2 \right|^{(1/2)(n+\theta)}} dxdy\]

where \(y\) refers to \((\bar{y}, z_n)\) in the above formula. Hence there is a constant \(C(n, \theta)\) such that \((3.3.19)\) is dominated by \(C(n, \theta) ||u||^p_{W^{n, \theta}(U)}\). A similar inequality holds for the third term. Finally consider \((3.3.19)\). This equals

\[\int_{U^+} \int_{U^+} \frac{|u(\bar{x}, 2g(\bar{x}) - x_n) - u(\bar{y}, 2g(\bar{y}) - y_n)|^p}{\left| \frac{1}{2} |\bar{x} - \bar{y}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+\theta)}} dxdy\]

Changing variables, \(x'_n = 2g(\bar{x}) - x_n, y'_n = 2g(\bar{y}) - y_n\), it equals

\[\int_{U^+} \int_{U^+} \frac{|u(\bar{x}, x'_n) - u(\bar{y}, y'_n)|^p}{\left| \frac{1}{2} |\bar{x} - \bar{y}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+\theta)}} dx'dy', \quad (3.3.20)\]

each of \(x_n, y_n\) being a function of \(x'_n, y'_n\) where an estimate needs to be obtained on \(|x'_n - y'_n|\) in terms of \(|x_n - y_n|\).

\[\left( x'_n - y'_n \right)^2 = (2g(\bar{x}) - g(\bar{y}))) + y_n - x_n \right)^2\]

\[= (y_n - x_n)^2 + 4(g(\bar{x}) - g(\bar{y}))(y_n - x_n) + 4(g(\bar{x}) - g(\bar{y}))^2\]

\[\leq (y_n - x_n)^2 + 2(g(\bar{x}) - g(\bar{y}))^2 + 2(y_n - x_n)^2 + 4(g(\bar{x}) - g(\bar{y}))^2\]

and so

\[\left( x'_n - y'_n \right)^2 \leq 3(y_n - x_n)^2 + 6K^2 |\bar{x} - \bar{y}|^2\]

which implies

\[(y_n - x_n)^2 \geq \frac{1}{3} (x'_n - y'_n)^2 - 2K^2 |\bar{x} - \bar{y}|^2\]

Then substituting this in to \((3.3.20)\), a short computation shows \((3.3.19)\) is dominated by an expression of the form \(C(n, \theta) ||u||^p_{W^{n, \theta}(U)}\) and this proves the existence of an extension operator provided the Lipschitz constant is small enough. It is clear \(E\) is linear where \(E\) is defined above.

Now this assumption on the smallness of \(K\) needs to be removed. For \((\bar{x}, x_n) \in U\) define

\[U' \equiv \left\{ \hat{\bar{x}} = \lambda(\bar{x} - \bar{b}_0) : \bar{x} \in U \right\}\]
here this is centering at 0 and stretching B since λ will be large. Let h be the name of this mapping. Thus
\[
\begin{align*}
h(x) & = \lambda \left( \tilde{x} - \tilde{b}_0 \right), \\
k(x) & = h^{-1}(x) = \frac{1}{\lambda} \tilde{x} + \tilde{b}_0
\end{align*}
\]
These mappings are defined on all of \( \mathbb{R}^n \). Now let \( u' \) be defined on \( U' \) as follows.
\[
u'(\tilde{x}, x_n) \equiv k^* u(\tilde{x}, x_n).
\]
Also let
\[
g'(\tilde{x}) \equiv k^* g(\tilde{x}).
\]
Thus
\[
g'(\tilde{x}) = g \left( \frac{1}{\lambda} \tilde{x} + \tilde{b}_0 \right) = g \left( k(\tilde{x}) \right).
\]
Then choosing λ large enough the Lipschitz condition for \( g' \) is as small as desired. Always assume λ has been chosen this large and also \( \lambda \geq 1 \). Furthermore, \( g'(\tilde{x}') = x'_n \) describes the boundary in the same way as \( x_n = g(\tilde{x}) \).

Now I need to consider whether \( u' \in \tilde{W}^{\theta,p}(U') \). Consider
\[
\begin{align*}
\int_{U'} \int_{U'} & \frac{|u'(\tilde{x}', x_n) - u'(\tilde{y}', y_n)|^p}{\left| \tilde{x}' - \tilde{y}' \right|^2 + (x_n - y_n)^2} \, dx'dy' \\
& = \int_{U'} \int_{U'} \frac{|k^* u(\tilde{x}', x_n) - k^* u(\tilde{y}', y_n)|^p}{\left| \tilde{x}' - \tilde{y}' \right|^2 + (x_n - y_n)^2} \, dx'dy'
\end{align*}
\]
Then change the variables \( \tilde{x}' = \lambda \left( \tilde{x} - \tilde{b}_0 \right) = h(\tilde{x}) \) with a similar change for \( \tilde{y}' \), the above expression equals
\[
(\lambda^{n-1})^2 \int_{U} \int_{U} \frac{|u(\tilde{x}, x_n) - u(\tilde{y}, y_n)|^p}{\left( \lambda^2 |\tilde{x} - \tilde{y}|^2 + (x_n - y_n)^2 \right)^{p+n\theta}} \, dxdy
\]
Thus
\[
||u'||_{\tilde{W}^{\theta,p}(U')} \leq \lambda^{n-1} ||u||_{\tilde{W}^{\theta,p}(U)} < \infty \quad \text{(43.3.21)}
\]
and \( k^*: W^{\theta,p}(U) \to W^{\theta,p}(U') \) is continuous and linear. Similar reasoning shows that \( h^* \) is continuous and linear mapping \( W^{\theta,p}(\mathbb{R}^n) \) to \( W^{\theta,p}(\mathbb{R}^n) \). By the first part of the argument there exists a continuous linear map
\[
E': \tilde{W}^{\theta,p}(U') \to \tilde{W}^{\theta,p}(\mathbb{R}^n)
\]
Now define
\[ Eu \equiv h^* E'(k^* u) \]
Say \( x_n \leq g(\tilde{x}) \). Then \( (\tilde{x}, x_n) \in U \) and so \( (h(\tilde{x}), x_n) \in U' \). Thus
\[
Eu(\tilde{x}, x_n) = h^* E'(k^* u)(\tilde{x}, x_n) \\
= E'(k^* u)(h(\tilde{x}), x_n)
\]
(43.3.22)
Now \( g(\tilde{x}) \geq x_n \) and so \( g'(h(\tilde{x})) \geq x_n \) because
\[ g'(h(\tilde{x})) = k^* g(h(\tilde{x})) = g(\tilde{x}) \]
and so 43.3.22 equals
\[ (k^* u)(h(\tilde{x}), x_n) = u(\tilde{x}, x_n) \]
Thus \( E \) leaves \( u \) unchanged at points \((\tilde{x}, x_n)\) where \( x_n \leq g(\tilde{x}) \). Also
\[
\|Eu\|_{W^{\theta,p}(\mathbb{R}^n)} = \|h^* E' k^* u\|_{W^{\theta,p}(\mathbb{R}^n)} \leq \|h^*\| \|E' k^* u\|_{W^{\theta,p}(\mathbb{R}^n)} \\
\leq \|h^*\| \|E'\| \|k^* u\|_{W^{\theta,p}(U)} = C \|u\|_{W^{\theta,p}(U)}
\]
To complete the proof, cover \( U \) with finitely many sets of this sort oriented with respect to one of the coordinate axes as this one was along with an open set whose closure is contained in \( U \) and then use a smooth partition of unity to localize the function to the situation of the sort just discussed and one whose support is contained in \( U \). Extend that one to equal zero off its support and treat the others as above. This proves the proposition.

Recall Theorem 43.2.7 which provides an extension operator from \( W^{1,p}(U) \) to \( W^{1,p}(\mathbb{R}^n) \) also denoted by \( E \). Now it is not hard to see that \( \mathring{W}^{\theta,p}(U) = W^{\theta,p}(U) \) and the two norms are equivalent.

**Theorem 43.3.13** Let \( U \) be a bounded open set which has Lipschitz boundary and \( \theta \in (0,1) \). Then \( W^{\theta,p}(U) = \mathring{W}^{\theta,p}(U) \) and the two norms are equivalent.

**Proof:** Let \( u \in \mathring{W}^{\theta,p}(U) \). Letting \( E \) be the extension operator of Lemma 43.1.2, there is a constant \( C \) such that
\[
C \|u\|_{\mathring{W}^{\theta,p}(U)} \geq \|Eu\|_{\mathring{W}^{\theta,p}(\mathbb{R}^n)} = \|E u\|_{W^{\theta,p}(\mathbb{R}^n)} \geq \|u\|_{W^{\theta,p}(U)}.
\]
Thus if \( u \in \mathring{W}^{\theta,p}(U) \), then \( u \in W^{\theta,p}(U) \) and the inclusion map is continuous.

Next suppose \( u \in W^{\theta,p}(U) \) and let \( \gamma f = u \) for
\[
f \in W \left( W^{1,p}(U) , L^p(U) , p , 1 - \theta \right) \equiv W_U
\]
Then from Theorem 43.2.7 and the definition of the norm in \( W_U \)
\[
Ef \in W \left( W^{1,p}(\mathbb{R}^n) , L^p(\mathbb{R}^n) , p , 1 - \theta \right) \equiv W
\]
where this $E$ pertains to extending $W^{1,p}(U)$.

$$C_1 \|f\|_{W^1} \geq \|Ef\|_W \geq \|Eu\|_{W^{\sigma,p}(\mathbb{R}^n)} \geq \|u\|_{W^{\sigma,p}(U)}$$

Since this is true for every $f \in W^1$, it follows $u \in W^{\sigma,p}(U)$ and

$$\|u\|_{W^{\sigma,p}(U)} \leq C_1 \|u\|_{W^{\sigma,p}(U)}.$$

This proves the theorem.

**Corollary 43.3.14** Let $U$ be a bounded open set with Lipschitz boundary. Then $W^{\theta,p}(U)$ is reflexive.

**Proof:** From Proposition 43.3.12 and Theorem 43.3.13, there exists an extension operator $E : W^{\theta,p}(U) \to W^{\theta,p}(\mathbb{R}^n)$ which is continuous. This operator is one to one and continuous. Furthermore, $\|Eu\|_{W^{\theta,p}(\mathbb{R}^n)} \geq \|u\|_{W^{\theta,p}(U)}$ and so $E(W^{\theta,p}(U))$ is closed. Therefore, by Corollary on Page 641 and the fact $W^{\theta,p}(\mathbb{R}^n)$ is reflexive which was shown in Corollary 43.3.8, it follows $W^{\theta,p}(U)$ is reflexive. This proves the corollary.

There may be other sets $U$ for which the intrinsic norm is an equivalent norm for $W^{\theta,p}(U)$ but this much will suffice. It should be routine to verify that this works for $U$ a half space for example and the extension argument should be much easier than that presented above. More generally, the assumption that $U$ was bounded in the above extension argument of Proposition 43.3.12 was never needed except for giving finitely many of those special sets covering the boundary. If you just assumed this at the outset instead of an assumption the set is bounded, the same sort of extension would work.

### 43.4 Fractional Order Sobolev Spaces

Now it is time to define fractional order Sobolev spaces between $W^{m,p}$ and $W^{m+1,p}$.

**Definition 43.4.1** Let $m$ be a nonnegative integer and let $s = m + \sigma$ where $\sigma \in (0,1)$. Then $W^{s,p}(\Omega)$ will consist of those elements of $W^{m,p}(\Omega)$ for which $D^\alpha u \in W^{\sigma,p}(\Omega)$ for all $|\alpha| = m$. The norm is given by the following.

$$\|u\|_{s,p,\Omega} \equiv \left( \|u\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \|D^\alpha u\|_{\sigma,p,\Omega}^p \right)^{1/p}.$$

**Corollary 43.4.2** The space, $W^{s,p}(\Omega)$ is a reflexive Banach space whenever $p > 1$.

**Proof:** From the theory of interpolation spaces, $W^{s,p}(\Omega)$ is reflexive. This is because it is an interpolation space for the two reflexive spaces, $L^p(\Omega)$ and $L^{1,p}(\Omega)$. (Alternatively, you could use Corollary 43.3.14 in the case where $\Omega$ is a bounded
open set with Lipschitz boundary or you could use Corollary 43.3.8 in case \( \Omega = \mathbb{R}^n \).

In addition, the same ideas would work if \( \Omega \) were any space for which there was a continuous extension map from \( W^{\sigma,p}(\Omega) \) to \( W^{\sigma,p}(\mathbb{R}^n) \).

Now the formula for the norm of an element in \( W^{s,p}(\Omega) \) shows this space is isometric to a closed subspace of \( W^{m,p}(\Omega) \times W^{\sigma,p}(\Omega)^k \) for suitable \( k \). Therefore, from Corollary 43.3.8 on Page 641, \( W^{s,p}(\Omega) \) is also reflexive.

**Theorem 43.4.3** The trace map, \( \gamma : W^{m,p}(\mathbb{R}^n_+) \rightarrow W^{m-\frac{1}{p},p}(\mathbb{R}^{n-1}) \) is continuous.

**Proof:** Let \( f \in \mathcal{S} \), the Schwartz class. Let \( \sigma = 1 - \frac{1}{p} \) so that \( m - \left( \frac{1}{p} \right) = m - 1 + \sigma \).

Then from the definition and using \( f \in \mathcal{S} \),

\[
||\gamma f||_{m-\frac{1}{p},p,\mathbb{R}^{n-1}} = \left( ||\gamma f||_{m-1,p,\mathbb{R}^{n-1}}^p + \sum_{|\alpha| = m-1} ||D^\alpha \gamma f||_{1,p,\mathbb{R}^{n-1}}^p \right)^{1/p} 
\]

and from Lemma 43.3.1 and the fact that the trace is continuous as a map from \( W^{m,p}(\mathbb{R}^n_+) \) to \( W^{m-1,p}(\mathbb{R}^{n-1}) \),

\[
||\gamma f||_{m-\frac{1}{p},p,\mathbb{R}^{n-1}} \leq \left( C_1 ||f||_{m,p,\mathbb{R}^n_+}^p + C_2 \sum_{|\alpha| = m-1} ||D^\alpha f||_{1,p,\mathbb{R}^n}^p \right)^{1/p} 
\]

Then using density of \( \mathcal{S} \) this implies the desired result.

With the definition of \( W^{s,p}(\Omega) \) for \( s \) not an integer, here is a generalization of an earlier theorem.

**Theorem 43.4.4** Let \( h : U \rightarrow V \) where \( U \) and \( V \) are two open sets and suppose \( h \) is bilipschitz and that \( D^\alpha h \) and \( D^\alpha h^{-1} \) exist and are Lipschitz continuous if \( |\alpha| \leq m \) where \( m = 0, 1, \ldots \) and \( s = m + \sigma \) where \( \sigma \in (0, 1) \). Then

\[
h^* : W^{s,p}(V) \rightarrow W^{s,p}(U)
\]

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being \( (h^{-1})^* \).

**Proof:** In case \( m = 0 \), the conclusion of the theorem is immediate from the general theory of trace spaces. Therefore, assume \( m \geq 1 \). It follows from the definition that

\[
||h^* u||_{m+\sigma,p,U} = \left( ||h^* u||_{m,p,U}^p + \sum_{|\alpha| = m} ||D^\alpha (h^* u)||_{\sigma,p,U}^p \right)^{1/p}
\]
Consider the case when $m = 1$. Then it is routine to verify that
$$D_j h^* u(x) = u_{k,j} (h (x)) h_{k,j} (x).$$

Let $L_k : W^{1,p} (V) \to W^{1,p} (U)$ be defined by
$$L_k v = h^* (v) h_{k,j}.$$

Then $L_k$ is continuous as a map from $W^{1,p} (V)$ to $W^{1,p} (U)$ and as a map from $L^p (V)$ to $L^p (U)$ and therefore, it follows that $L_k$ is continuous as a map from $W^{\sigma,p} (V)$ to $W^{\sigma,p} (U)$. Therefore,
$$||L_k (v)||_{\sigma,p,U} \leq C_k ||v||_{\sigma,p,U}$$

and so
$$||D_j (h^* u)||_{\sigma,p,U} \leq \sum_k ||L_k (u,k)||_{\sigma,p,U} \leq \sum_k C_k ||D_k u||_{\sigma,p,V} \leq C \left( \sum_k ||D_k u||_{\sigma,p,V}^p \right)^{1/p}.$$

Therefore, it follows that
$$||h^* u||_{1+\sigma,p,U} \leq \left[ ||h^* u||_{1,p,U}^p + \sum_j C^\sigma \sum_k ||D_k u||_{\sigma,p,V}^p \right]^{1/p} \leq C \left[ ||u||_{1,p,V}^p + \sum_k ||D_k u||_{\sigma,p,V}^p \right]^{1/p} = C ||u||_{1+\sigma,p,V}.$$

The general case is similar except for the use of a more complicated linear operator in place of $L_k$. This proves the theorem.

It is interesting to prove this theorem using Theorem 43.3.13 and the intrinsic norm.

Now we prove an important interpolation inequality for Sobolev spaces.

**Theorem 43.4.5** Let $\Omega$ be an open set in $\mathbb{R}^n$ which has the segment property and let $f \in W^{m+1,p} (\Omega)$ and $\sigma \in (0,1)$. Then for some constant, $C$, independent of $f$,
$$||f||_{m+\sigma,p,\Omega} \leq C \left( ||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^\sigma \right).$$

Also, if $L \in \mathcal{L} (W^{m,p} (\Omega), W^{m,p} (\Omega))$ for all $m = 0, 1, \cdots$, and $L \circ D^\alpha = D^\alpha \circ L$ on $C^\infty (\bar{\Omega})$, then $L \in \mathcal{L} (W^{m+\sigma,p} (\Omega), W^{m+\sigma,p} (\Omega))$ for any $m = 0, 1, \cdots$. 
Proof: Recall from above, $W^{1-\delta,p}(\Omega) \equiv T \left( W^{1,p}(\Omega), L^p(\Omega), p, \theta \right)$. Therefore, from Theorem 42.1.9, if $f \in W^{1,p}(\Omega)$, $||f||_{1-\theta,p,\Omega} \leq K ||f||_{\theta,0,\Omega}^{1-\theta}$.

Therefore,

$$||f||_{m+\sigma,p,\Omega} \leq \left( ||f||_{m,p,\Omega}^p + \sum_{|\alpha|=m} K \left( ||D^\alpha f||_{1,p,\Omega}^{1-\sigma} ||D^\alpha f||_{0,\Omega}^\sigma \right)^p \right)^{1/p} \leq C \left( ||f||_{m+1,p,\Omega}^p + \left( ||f||_{m,p,\Omega}^{1-\sigma} \right)^p \right)^{1/p} \leq C \left( ||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^\sigma \right)^{1/p}.$$

This proves the first part. Now consider the second. Let $\phi \in C^\infty(\Omega)$

$$||L\phi||_{m+\sigma,p,\Omega} = \left( ||L\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} ||D^\alpha L\phi||_{\sigma,p,\Omega}^p \right)^{1/p} \leq \left( ||L\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} ||LD^\alpha \phi||_{T(W^{1,p},L^p,1-\sigma)}^p \right)^{1/p} \leq \left( ||L\phi||_{m,\Omega}^p + \sum_{|\alpha|=m} \left[ \inf \left( ||t^{1-\sigma} Lf_\alpha||_1 \left( ||t^{1-\sigma} Lf'_\alpha||_2 \right)^{1-\sigma} \right) \right]^{1/p} \right) \quad (43.4.23)$$

where

$$\inf \left( ||t^{1-\sigma} Lf_\alpha||_1 \left( ||t^{1-\sigma} Lf'_\alpha||_2 \right)^{1-\sigma} \right) = \inf \left( ||t^{1-\sigma} Lf_\alpha||_{L^p(0,\infty; W^{1,\sigma}(\Omega))} \left( ||t^{1-\sigma} Lf'_\alpha||_{L^p(0,\infty; L^p(\Omega))} \right)^{1-\sigma} \right),$$

$f_\alpha(0) \equiv \lim_{t \to 0} f_\alpha(t) = D^\alpha \phi$ in $W^{1,p}(\Omega) + L^p(\Omega)$, and the infimum is taken over all such functions. Therefore, from 43.4.23, and letting $||L||_1$ denote the operator...
43.4. FRACTIONAL ORDER SOBOLEV SPACES

norm of \( L \) in \( W^{1,p}(\Omega) \) and \( ||L||_2 \) denote the operator norm of \( L \) in \( L^p(\Omega) \),

\[
||L\phi||_{m+\sigma,p,\Omega} \leq \left( ||L\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} \left[ \inf \left( ||L||_1^q ||L||_2^{1-\sigma} ||t^{1-\sigma} f_\alpha||_1^q ||t^{1-\sigma} f'_\alpha||_2^{1-\sigma} \right) \right]^p \right)^{1/p}
\]

\[
\leq \left( ||L\phi||_{m,p,\Omega}^p + \left( ||L||_1^q ||L||_2^{1-\sigma} \right)^p \sum_{|\alpha|=m} \left[ \inf \left( ||t^{1-\sigma} f_\alpha||_1^q ||t^{1-\sigma} f'_\alpha||_2^{1-\sigma} \right) \right]^p \right)^{1/p}
\]

\[
\leq C \left( ||\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} \left[ ||D^\alpha \phi||_{\sigma,p,\Omega} \right]^p \right)^{1/p} = C ||\phi||_{m+\sigma,p,\Omega}.
\]

Since \( C^\infty(\Omega) \) is dense in all the Sobolev spaces, this inequality establishes the desired result.

**Definition 43.4.6** Define for \( s \geq 0, W^{-s,p'}(\mathbb{R}^n) \) to be the dual space of \( W^{s,p}(\mathbb{R}^n) \).

Here \( \frac{1}{p} + \frac{1}{p'} = 1 \).

Note that in the case of \( m = 0 \) this is consistent with the Riesz representation theorem for the \( L^p \) spaces.
Chapter 44

Sobolev Spaces On Manifolds

44.1 Basic Definitions

Consider the following situation. There exists a set, $\Gamma \subseteq \mathbb{R}^m$ where $m > n$, mappings, $h_i : U_i \rightarrow \Gamma_i = \Gamma \cap W_i$ for $W_i$ an open set in $\mathbb{R}^m$ with $\Gamma \subseteq \bigcup_{i=1}^l W_i$ and $U_i$ is an open subset of $\mathbb{R}^n$ which $h_i$ one to one and onto. Assume $h_i$ is of the form

$$h_i(x) = H_i(x, 0)$$

(44.1.1)

where for some open set, $O_i$, $H_i : U_i \times O_i \rightarrow W_i$ is bilipschitz having bilipschitz inverse such that for $G = H_i$ or $H_i^{-1}$, $D^\alpha G$ is Lipschitz for $|\alpha| \leq k$.

For example, let $m = n + 1$ and let

$$H_i(x, y) = \begin{pmatrix} x \\ \phi(x) + y \end{pmatrix}$$

where $\phi$ is a Lipschitz function having $D^\alpha \phi$ Lipschitz for all $|\alpha| \leq k$. This is an example of the sort of thing just described if $x \in U_i \subseteq \mathbb{R}^n$ and $O_i = \mathbb{R}$, because it is obvious the inverse of $H_i$ is given by

$$H_i^{-1}(x, y) = \begin{pmatrix} x \\ y - \phi(x) \end{pmatrix}.$$  

Also let $\{\psi_i\}_{i=1}^l$ be a partition of unity subordinate to the open cover $\{W_i\}$ satisfying $\psi_i \in C^\infty_c (W_i)$. Then the definition of $W^{s,p}(\Gamma)$ follows.

**Definition 44.1.1** $u \in W^{s,p}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, h_i, H_i\}_{i=1}^l$ is described above with $h_i \in C^{k,1}$, $h_i^*(u \psi_i) \in W^{s,p}(U_i)$. The norm is given by

$$||u||_{s,p,\Gamma} \equiv \sum_{i=1}^l ||h_i^*(u \psi_i)||_{s,p,U_i}$$
It is not at all obvious this norm is well defined. What if
\[
\left\{ W_i, \phi_i, \Gamma_i, V_i, g_i, G_i \right\}_{i=1}^r
\]
is as described above. Would the two norms be equivalent? To begin with consider the following lemma which involves a particular choice for \( \{ W_i, \psi_i, \Gamma_i, U_i, h_i, H_i \}_{i=1}^l \).

**Lemma 44.1.2** \( W^{s,p} (\Gamma) \) as just described, is a Banach space. If \( p > 1 \) then it is reflexive.

**Proof:** Let \( L : W^{s,p} (\Gamma) \to \prod_{i=1}^l W^{s,p} (U_i) \) be defined by \( (Lu)_i = h_i^* (u\psi_i) \). Let \( \{ u_j \}_{j=1}^\infty \) be a Cauchy sequence in \( W^{s,p} (\Gamma) \). Then \( \{ h_i^* (u_j \psi_i) \}_{j=1}^\infty \) is a Cauchy sequence in \( W^{s,p} (U_i) \) for each \( i \). Therefore, for each \( i \), there exists \( w_i \in W^{s,p} (U_i) \) such that
\[
\lim_{j \to \infty} h_i^* (u_j \psi_i) = w_i \text{ in } W^{s,p} (U_i)
\]
But also, there exists a subsequence, still denoted by \( j \) such that for each \( i \)
\[
\{ h_i^* (u_j \psi_i) (x) \}_{j=1}^\infty
\]
is a Cauchy sequence for a.e. \( x \). Since \( h_i \) is given to be Lipschitz, it maps sets of measure 0 to sets of \( n \) dimensional Hausdorff measure zero. Therefore,
\[
\{ u_j \psi_i (y) \}_{j=1}^\infty
\]
is a Cauchy sequence for \( \mu \) a.e. \( y \in W_i \cap \Gamma \) where \( \mu \) denotes the \( n \) dimensional Hausdorff measure. It follows that for \( \mu \) a.e. \( y \), \( \{ u_j (y) \}_{j=1}^\infty \) is a Cauchy sequence and so it converges to a function denoted as \( u (y) \).
\[
u_j (y) \to u (y) \text{ } \mu \text{ a.e.}
\]
Therefore, \( w_i (x) = h_i^* (u\psi_i) (x) \) a.e. and this shows \( h_i^* (u\psi_i) \in W^{s,p} (U_i) \). Thus \( u \in W^{s,p} (\Gamma) \) showing completeness. It is clear \( ||| \cdot |||_{s,p,\Gamma} \) is a norm. Thus \( L \) is an isometry of \( W^{s,p} (\Gamma) \) and a closed subspace of \( \prod_{i=1}^l W^{s,p} (U_i) \). By Corollary \ref{cor:reflexive}, \( W^{s,p} (U_i) \) is reflexive which implies the product is reflexive. Closed subspaces of reflexive spaces are reflexive by Lemma \ref{lemma:reflexive} on Page \ref{page:reflexive} and so \( W^{s,p} (\Gamma) \) is also reflexive. This proves the lemma.

I now show that any two such norms are equivalent.

Suppose \( \{ W'_i, \phi_i, \Gamma_j, V_j, g_j, G_j \}_{i=1}^r \) and \( \{ W_i, \psi_i, \Gamma_i, U_i, h_i, H_i \}_{i=1}^l \) both satisfy the conditions described above. Let \( ||| \cdot |||_{s,p,\Gamma} \) denote the norm defined by
\[
||| u |||_{s,p,\Gamma}^1 \equiv \sum_{j=1}^r ||| g_j^* (u\phi_j) \cdot |||_{s,p,V_j}
\]
\[
\leq \sum_{j=1}^r ||| g_j^* \left( \sum_{i=1}^l u\phi_j \psi_i \right) \cdot |||_{s,p,V_j} \leq \sum_{j,i} ||| g_j^* (u\phi_j \psi_i) \cdot |||_{s,p,V_j}
\]
44.2. THE TRACE ON THE BOUNDARY OF AN OPEN SET

\[ \sum_{j,i} g^*_j (u \phi_j \psi_i) \bigg|_{s,p,g^{-1}_j(W_i \cap W'_j)} \]  

(44.1.2)

Now define a new norm \( \|u\|_{s,p,\Gamma}^1 \) by the formula

\[ \{ W'_j \cap W_i, \psi_i \phi_j, \Gamma_j \cap \Gamma_i, V_j, g_{i,j}, G_{i,j} \} \]

This norm is determined by the identity map is continuous from \((W_{s,p}(\Gamma), \|\cdot\|_{s,p,\Gamma}^1)\) to \((W_{s,p}(\Gamma), \|\cdot\|_{s,p,\Gamma}^1)\). It follows the two norms, \( \|\cdot\|_{s,p,\Gamma}^1 \) and \( \|\cdot\|_{s,p,\Gamma}^2 \), are equivalent by the open mapping theorem. In a similar way, the norms, \( \|\cdot\|_{s,p,\Gamma}^2 \) and \( \|\cdot\|_{s,p,\Gamma}^2 \) are equivalent where

\[ \|u\|_{s,p,\Gamma}^2 \equiv \sum_{j=1}^l \|h^*_j (u \psi_i)\|_{s,p,U_i} \]

and

\[ \|u\|_{s,p,\Gamma}^2 \equiv \sum_{j,i} \|h^*_j (u \phi_j \psi_i)\|_{s,p,U_i} = \sum_{j,i} \|h^*_j (u \phi_j \psi_i)\|_{s,p,h^{-1}_i(W_i \cap W'_j)} \]

But from the assumptions on \( h \) and \( g \), in particular the assumption that these are restrictions of functions which are defined on open subsets of \( \mathbb{R}^m \) which have Lipschitz derivatives up to order \( k \) along with their inverses, Theorem 43.4.4 implies, there exist constants \( C_1 \), independent of \( u \) such that

\[ \|h^*_j (u \phi_j \psi_i)\|_{s,p,h^{-1}_i(W_i \cap W'_j)} \leq C_1 \|g^*_j (u \phi_j \psi_i)\|_{s,p,g^{-1}_j(W_i \cap W'_j)} \]

and

\[ \|g^*_j (u \phi_j \psi_i)\|_{s,p,g^{-1}_j(W_i \cap W'_j)} \leq C_2 \|h^*_j (u \phi_j \psi_i)\|_{s,p,h^{-1}_i(W_i \cap W'_j)} \]

Therefore, the two norms, \( \|\cdot\|_{s,p,\Gamma}^1 \) and \( \|\cdot\|_{s,p,\Gamma}^2 \) are equivalent. It follows that the norms, \( \|\cdot\|_{s,p,\Gamma}^2 \) and \( \|\cdot\|_{s,p,\Gamma}^2 \) are equivalent. This proves the following theorem.

**Theorem 44.1.3** Let \( \Gamma \) be described above. Then any two norms for \( W_{s,p}(\Gamma) \) as in Definition 37.6.3 are equivalent.

44.2 The Trace On The Boundary Of An Open Set

Next is a generalization of earlier theorems about the loss of \( \frac{1}{p} \) derivatives on the boundary.

**Definition 44.2.1** Define

\[ \mathbb{R}^{n-1}_k \equiv \{ x \in \mathbb{R}^n : x_k = 0 \} , \hat{x}_k \equiv (x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n) . \]

An open set, \( \Omega \) is \( C^{m,1} \) if there exist open sets, \( W_i, i = 0, 1, \cdots, l \) such that

\[ \Omega = \bigcup_{i=0}^l W_i \]
with \( \overline{W}_0 \subseteq \Omega \), open sets \( U_i \subseteq \mathbb{R}^{n-1}_k \) for some \( k \), and open intervals, \((a_i, b_i)\) containing \( 0 \) such that for \( i \geq 1 \),

\[
\partial \Omega \cap W = \{ \tilde{x}_k + \phi_i (\tilde{x}_k) e_k : \tilde{x}_k \in U_i \},
\]

\[
\Omega \cap W_i = \{ \tilde{x}_k + (\phi_i (\tilde{x}_k) + x_k) e_k : (\tilde{x}_k, x_k) \in U_i \times I_i \},
\]

where \( \phi_i \) is Lipschitz with partial derivatives up to order \( m \) also Lipschitz. Here \( I_i = (a_i, 0) \) or \((0, b_i)\). The case of \((a_i, 0)\) is shown in the picture.

Assume \( \Omega \) is \( C^{m-1,1} \). Define

\[
h_i (\tilde{x}_k) = \tilde{x}_k + \phi_i (\tilde{x}_k) e_k, \quad H_i (x) = \tilde{x}_k + (\phi_i (\tilde{x}_k) + x_k) e_k,
\]

and let \( \psi_i \in C^\infty_c (W_i) \) with \( \sum_{i=0}^l \psi_i(x) = 1 \) on \( \overline{\Omega} \). Thus

\[
\{ W_i, \psi_i, \partial \Omega \cap W, U_i, h_i, H_i \}_{i=1}^l
\]
satisfies all the conditions for defining \( W^{s,p} (\partial \Omega) \) for \( s \leq m \). Let \( u \in C^\infty (\overline{\Omega}) \) and let \( h_i \) be as just described. The trace, denoted by \( \gamma \) is that operator which evaluates functions in \( C^\infty (\overline{\Omega}) \) on \( \partial \Omega \). Thus for \( u \in C^\infty (\overline{\Omega}) \), and \( y \in \partial \Omega \),

\[
u (y) = \sum_{i=1}^l (u \psi_i) (y)
\]

and so using the notation to suppress the reference to \( y \),

\[
\gamma u = \sum_{i=1}^l \gamma (u \psi_i)
\]

It is necessary to show this is a continuous map. Letting \( u \in W^{m,p} (\Omega) \), it follows from Theorem 43.3.5.3, and Theorem 36.0.16

\[
\| \gamma u \|_{m-\frac{1}{p},p,\partial \Omega} = \sum_{i=1}^l \| h_i^* (\gamma (\psi_i u)) \|_{m-\frac{1}{p},p,U_i}
\]
44.2. The Trace on the Boundary of an Open Set

\[
\begin{align*}
44.2.1 & \quad = \sum_{i=1}^{l} ||h_i \gamma (\psi_i u)||_{m-\frac{1}{p}, p, \mathbb{R}^{n-1}} \leq C \sum_{i=1}^{l} ||H_i^* (\psi_i u)||_{m, p, \mathbb{R}^n} \\
& \leq C \sum_{i=1}^{l} ||H_i^* (\psi_i u)||_{m, p, U_i \times (a_i, 0)} \leq C \sum_{i=1}^{l} ||(\psi_i u)||_{m, p, W_i \cap \Omega} \\
& \leq C \sum_{i=1}^{l} ||(\psi_i u)||_{m, p, \Omega} \leq C \sum_{i=1}^{l} ||u||_{m, p, \Omega} \leq C \|u\|_{m, p, \Omega} \\
\end{align*}
\]

Now use the density of \( C^\infty (\overline{\Omega}) \) in \( W^{m, p} (\Omega) \) to see that \( \gamma \) extends to a continuous linear map defined on \( W^{m, p} (\Omega) \) still called \( \gamma \) such that for all \( u \in W^{m, p} (\Omega) \),

\[
||\gamma u||_{m-\frac{1}{p}, p, \partial \Omega} \leq C ||u||_{m, p, \Omega}.
\]

(44.2.3)

Also, it can be shown that \( \gamma \) maps \( W^{m, p} (\Omega) \) onto \( W^{m-\frac{1}{p}} (\partial \Omega) \). Let \( g \in W^{m-\frac{1}{p}} (\partial \Omega) \). By definition, this means

\[
h_i^* (\psi_i g) \in W^{m-\frac{1}{p}} (U_i), \text{ each } i
\]

and so, using a cutoff function, there exists \( w_i \in W^{m, p} (U_i \times I_i) \) such that

\[
\gamma w_i = h_i^* (\psi_i g) = h_i^* (\gamma \psi_i g)
\]

Thus \( (H_i^{-1})^* w_i \in W^{mp} (\Omega \cap W_i) \). Let

\[
w \equiv \sum_{i=1}^{l} \psi_i (H_i^{-1})^* w_i \in W^{mp} (\Omega)
\]

then

\[
\gamma w = \sum_{i} \gamma \psi_j (H_i^{-1})^* w_i = \sum_{i} \gamma \psi_j (H_i^{-1})^* \gamma w_i
\]

\[
= \sum_{i} \gamma \psi_j (H_i^{-1})^* h_i^* (\gamma \psi_i g) = g
\]

In addition to this, in the case where \( m = 1 \), Lemma [43.2.1] implies there exists a linear map, \( R \), from \( W^{1-\frac{1}{p}} (\partial \Omega) \) to \( W^{1, p} (\Omega) \) which has the property that \( \gamma R g = g \) for every \( g \in W^{1-\frac{1}{p}} (\partial \Omega) \). I show this now. Letting \( g \in W^{1-\frac{1}{p}} (\partial \Omega) \),

\[
g = \sum_{i=1}^{l} \psi_i g.
\]

Then also,

\[
h_i^* (\psi_i g) \in W^{1-\frac{1}{p}} (\mathbb{R}^{n-1})
\]
if extended to equal 0 off $U_i$. From Lemma 43.2.1 there exists an extension of this to $W^{1,p}(\mathbb{R}^n)$. Without loss of generality, assume that $R\psi_i g \in W^{1,p}(U_i \times (a_i, 0))$. If not so, multiply by a suitable cut off function in the definition of $R$. Then the extension is

$$R g = \sum_{i=1}^l (H_i^{-1})^* R\psi_i g.$$ 

This works because from the definition of $\gamma$ on $C^\infty(\overline{\Omega})$ and continuity of the map established above, $\gamma$ and $(H_i^{-1})^*$ commute and so

$$\gamma R g = \sum_{i=1}^l \gamma (H_i^{-1})^* R\psi_i g$$

$$= \sum_{i=1}^l (H_i^{-1})^* \gamma R\psi_i g$$

$$= \sum_{i=1}^l (H_i^{-1})^* h_i \psi_i g = g.$$ 

This proves the following theorem about the trace.

**Theorem 44.2.2** Let $\Omega \in C^{m-1,1}$. Then there exists a constant, $C$ independent of $u \in W^{m,p}(\Omega)$ and a continuous linear map, $\gamma : W^{m,p}(\Omega) \to W^{m-\frac{1}{p},p}(\partial \Omega)$ such that $44.2.3$ holds. This map satisfies $\gamma u(x) = u(x)$ for all $u \in C^\infty(\overline{\Omega})$ and $\gamma$ is onto. In the case where $m = 1$, there exists a continuous linear map, $R : W^{1,\frac{n}{n-1}}(\partial \Omega) \to W^{1,p}(\Omega)$ which has the property that $\gamma R g = g$ for all $g \in W^{1,\frac{n}{n-1}}(\partial \Omega)$.

Of course more can be proved but this is all to be presented here.
Part IV

Multifunctions
Chapter 45

The Yankov von Neumann Aumann theorem

The Yankov von Neumann Aumann theorem deals with the projection of a product measurable set. It is a very difficult but interesting theorem. The material of this chapter is taken from [28], [29], [9], and [63]. We use the standard notation that for \( S \) and \( F \) \( \sigma \) algebras, \( S \times F \) is the \( \sigma \) algebra generated by the measurable rectangles, the product measure \( \sigma \) algebra. The next result is fairly easy and the proof is left for the reader.

**Lemma 45.0.3** Let \((X,d)\) be a metric space. Then if \( d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \), it follows that \( d_1 \) is a metric on \( X \) and the basis of open balls taken with respect to \( d_1 \) yields the same topology as the basis of open balls taken with respect to \( d \).

**Theorem 45.0.4** Let \((X_i,d_i)\) denote a complete metric space and let \( X = \prod_{i=1}^{\infty} X_i \). Then \( X \) is also a complete metric space with the metric

\[
\rho(x,y) = \sum_{i=1}^{\infty} 2^{-i} \frac{d_i(x_i,y_i)}{1 + d_i(x_i,y_i)}.
\]

Also, if \( X_i \) is separable for each \( i \) then so is \( X \).

**Proof:** It is clear from the above lemma that \( \rho \) is a metric on \( X \). We need to verify \( X \) is complete with this metric. Let \( \{x^n\} \) be a Cauchy sequence in \( X \). Then it is clear from the definition that \( \{x^n_i\} \) is a Cauchy sequence for each \( i \) and converges to \( x_i \in X_i \). Therefore, letting \( \varepsilon > 0 \) be given, we choose \( N \) such that

\[
\sum_{k=N}^{\infty} 2^{-k} < \frac{\varepsilon}{2},
\]

we choose \( M \) large enough that for \( n > M \),

\[
2^{-i} \frac{d_i(x^n_i,x_i)}{1 + d_i(x^n_i,x_i)} < \frac{\varepsilon}{2(N+1)}
\]
for all \( i = 1, 2, \ldots, N \). Then letting \( x = \{ x_i \} \)

\[
\rho(x, x^n) \leq \frac{\varepsilon N}{2(N + 1)} + \sum_{k=N}^{\infty} 2^{-k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

We need to verify that \( X \) is separable. Let \( D \) denote a countable dense set in \( X \), \( D \equiv \{ r^i_k \}_{k=1}^\infty \). Then let

\[
D_k \equiv D_1 \times \cdots \times D_k \times \{ r^1_{k+1} \} \times \{ r^2_{k+2} \} \times \cdots
\]

Thus \( D_k \) is a countable subset of \( X \). Let \( D \equiv \bigcup_{k=1}^{\infty} D_k \). Then \( D \) is countable and we can see \( D \) is dense in \( X \) as follows. The projection of \( D_k \) onto the first \( k \) entries is dense in \( \prod_{i=1}^{k} X_i \) and for \( k \) large enough the remaining component’s contribution to the metric, \( \rho \) is very small. Therefore, obtaining \( d \in D \) close to \( x \in X \) may be accomplished by finding \( d \in D \) such that \( d \) is close to \( x \) in the first \( k \) components for \( k \) large enough. Note that we do not use \( \prod_{k=1}^{\infty} D_k \)!

**Definition 45.0.5** A complete separable metric space is called a polish space.

**Theorem 45.0.6** Let \( X \) be a polish space. Then there exists \( f : N^N \rightarrow X \) which is onto and continuous. Here \( N^N \equiv \prod_{i=1}^{\infty} N \) and a metric is given according to the above theorem. Thus for \( n, m \in N^N \),

\[
\rho(n, m) = \sum_{i=1}^{\infty} 2^{-i} \frac{|n_i - m_i|}{1 + |n_i - m_i|}.
\]

**Proof:** Since \( X \) is polish, there exists a countable covering of \( X \) by closed sets having diameters no larger than \( 2^{-1}, \{ B(i) \}_{i=1}^{\infty} \). Each of these closed sets is also a polish space and so there exists a countable covering of \( B(i) \) by a countable collection of closed sets, \( \{ B(i, j) \}_{j=1}^{\infty} \) each having diameter no larger than \( 2^{-2} \) where \( B(i, j) \subseteq B(i) \neq \emptyset \) for all \( j \). Continue this way. Thus

\[
B(n_1, n_2, \ldots, n_m) = \bigcup_{i=1}^{\infty} B(n_1, n_2, \ldots, n_m, i)
\]

and each of \( B(n_1, n_2, \ldots, n_m, i) \) is a closed set contained in \( B(n_1, n_2, \ldots, n_m) \) whose diameter is at most half of the diameter of \( B(n_1, n_2, \ldots, n_m) \). Now we define our mapping from \( N^N \) to \( X \). If \( n = \{ n_k \}_{k=1}^{\infty} \in N^N \), we let \( f(n) \equiv \cap_{m=1}^{\infty} B(n_1, n_2, \ldots, n_m) \).

Since the diameters of these sets converge to \( 0 \), there exists a unique point in this countable intersection and this is \( f(n) \).

We need to verify \( f \) is continuous. Let \( n \in N^N \) be given and suppose \( m \) is very close to \( n \). The only way this can occur is for \( n_k \) to coincide with \( m_k \) for many \( k \). Therefore, both \( f(n) \) and \( f(m) \) must be contained in \( B(n_1, n_2, \ldots, n_m) \) for some fairly large \( m \). This implies, from the above construction that \( f(m) \) is as close to \( f(n) \) as \( 2^{-m} \), proving \( f \) is continuous. To see that \( f \) is onto, note that from the construction, if \( x \in X \), then \( x \in B(n_1, n_2, \ldots, n_m) \) for some choice of \( n_1, \ldots, n_m \) for each \( m \). Note nothing is said about \( f \) being one to one. It probably is not one to one.
**Definition 45.0.7** We call a topological space $X$ a Suslin space if $X$ is a Hausdorff space and there exists a polish space, $Z$ and a continuous function $f$ which maps $Z$ onto $X$.

$$Z \xrightarrow{f \text{ onto}} X$$

These Suslin spaces are also called analytic sets in some contexts but we will use the term Suslin space in referring to them.

**Corollary 45.0.8** $X$ is a Suslin space, if and only if there exists a continuous mapping from $\mathbb{N}^\mathbb{N}$ onto $X$.

**Proof:** We know there exists a polish space $Z$ and a continuous function, $h : Z \to X$ which is onto. By the above theorem there exists a continuous map, $g : \mathbb{N}^\mathbb{N} \to Z$ which is onto. Then $h \circ g$ is a continuous map from $\mathbb{N}^\mathbb{N}$ onto $X$. The “if” part of this theorem is accomplished by noting that $\mathbb{N}^\mathbb{N}$ is a polish space.

**Lemma 45.0.9** Let $X$ be a Suslin space and suppose $X_i$ is a subspace of $X$ which is also a Suslin space. Then $\bigcup_{i=1}^\infty X_i$ and $\bigcap_{i=1}^\infty X_i$ are also Suslin spaces. Also every Borel set in $X$ is a Suslin space.

**Proof:** Let $f_i : Z_i \to X_i$ where $Z_i$ is a polish space and $f_i$ is continuous and onto. Without loss of generality we may assume the spaces $Z_i$ are disjoint because if not, we could replace $Z_i$ with $Z_i \times \{i\}$. Now we define a metric, $\rho$, for $Z \equiv \bigcup_{i=1}^\infty Z_i$ as follows.

$$\rho(x, y) \equiv 1 \text{ if } x \in Z_i, y \in Z_k, i \neq k$$

$$\rho(x, y) \equiv \frac{d_i(x, y)}{1 + d_i(x, y)} \text{ if } x, y \in Z_i.$$ 

Here $d_i$ is the metric on $Z_i$. It is easy to verify $\rho$ is a metric and that $(Z, \rho)$ is a polish space. Now we define $f : Z \to \bigcup_{i=1}^\infty X_i$ as follows. For $x \in Z_i$, $f(x) \equiv f_i(x)$. This is well defined because the $Z_i$ are disjoint. If $y$ is very close to $x$ it must be that $x$ and $y$ are in the same $Z_i$ otherwise this could not happen. Therefore, continuity of $f$ follows from continuity of $f_i$. This shows countable unions of Suslin subspaces of a Suslin space are Suslin spaces.

If $H \subseteq X$ is a closed subset, then, letting $f : Z \to X$ be onto and continuous, it follows $f : f^{-1}(H) \to H$ is onto and continuous. Since $f^{-1}(H)$ is closed, it follows $f^{-1}(H)$ is a polish space. Therefore, $H$ is a Suslin space.

Now we show countable intersections of Suslin spaces are Suslin. It is clear that $\theta : \prod_{i=1}^\infty Z_i \to \prod_{i=1}^\infty X_i$ given by $\theta(z) \equiv x = \{x_i\}$ where $x_i = f_i(z_i)$ is continuous and onto, this with respect to the usual product topology. Note that $\prod_{i=1}^\infty Z_i$ is a polish space because of the assumption that each $Z_i$ is and the above considerations. Therefore, $\prod_{i=1}^\infty X_i$ is a Suslin space. Now let $P \equiv \{y \in \prod_{i=1}^\infty f_i(Z_i) : y_i = y_j$ for all $i, j\}$. (This is how you get it on the intersection. I guess this must be the case where each $X_i \subseteq X$). Then $P$ is a closed subspace of a Suslin space and so it is Suslin. Then we define $h : P \to \bigcap_{i=1}^\infty X_i$ by $h(y) \equiv f_i(y_i)$. This shows $\bigcap_{i=1}^\infty X_i$ is Suslin because $h$ is continuous and onto.
(h \circ \theta : \theta^{-1}(P) \to \cap_{i=1}^{\infty} X_i \text{ is continuous and } \theta^{-1}(P) \text{ being a closed subset of a polish space is polish.})

Next let $U$ be an open subset of $X$. Then $f^{-1}(U)$, being an open subset of a polish space, can be obtained as an increasing limit of closed sets, $K_n$. Therefore, $U = \cup_{n=1}^{\infty} f(K_n)$. Each $f(K_n)$ is a Suslin space because it is the continuous image of a polish space, $K_n$. Therefore, by the first part of the lemma, $U$ is a Suslin space. Now let

$$F \equiv \{ E \subseteq X : \text{ both } E^C \text{ and } E \text{ are Suslin} \}.$$ 

We see that $F$ is closed with respect to taking complements. The first part of this lemma shows $F$ is closed with respect to countable unions. Therefore, $F$ is a $\sigma$ algebra and so, since it contains the open sets, must contain the Borel sets.

It turns out that Suslin spaces tend to be measurable sets. In order to develop this idea, we need a technical lemma.

**Lemma 45.0.10** Let $(\Omega, F, \mu)$ be a measure space and denote by $\mu^*$ the outer measure generated by $\mu$. Thus

$$\mu^*(S) \equiv \inf \{ \mu(E) : E \supseteq S, E \in F \}.$$ 

Then $\mu^*$ is regular, meaning that for every $S$, there exists $E \in F$ such that $E \supseteq S$ and $\mu(E) = \mu^*(S)$. If $S_n \uparrow S$, it follows that $\mu^*(S_n) \uparrow \mu^*(S)$. Also if $\mu(\Omega) < \infty$, then a set, $E$, is measurable if and only if

$$\mu^*(\Omega) \geq \mu^*(E) + \mu^*(\Omega \setminus E).$$

**Proof:** First we verify that $\mu^*$ is regular. If $\mu^*(S) = \infty$, let $E = \Omega$. Then $\mu^*(S) = \mu(E)$ and $E \supseteq S$. On the other hand, if $\mu^*(S) < \infty$, then we can obtain $E_n \in F$ such that $\mu^*(S) + \frac{1}{n} \geq \mu(E_n)$ and $E_n \supseteq S$. Now let $F_n = \cap_{i=1}^{n} E_i$. Then $F_n \supseteq S$ and so $\mu^*(S) + \frac{1}{n} \geq \mu(F_n) \geq \mu^*(S)$. Therefore, letting $F = \cap_{k=1}^{\infty} F_k \in F$, it follows $\mu(F) = \lim_{n \to \infty} \mu(F_n) = \mu^*(S)$.

Let $E_n \supseteq S_n$ be such that $E_n \in F$ and $\mu(E_n) = \mu^*(S_n)$. Also let $E_{\infty} \supseteq S$ such that $\mu(E_{\infty}) = \mu^*(S)$ and $E_{\infty} \in F$. Now consider $B_n \equiv \cup_{k=1}^{n} E_k$. We claim

$$\mu(B_n) = \mu(S_n). \tag{45.0.1}$$

Here is why:

$$\mu(E_1 \setminus E_2) = \mu(E_1) - \mu(E_1 \cap E_2) = \mu^*(S_1) - \mu^*(S_1) = 0.$$ 

Therefore,

$$\mu(B_2) = \mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2) = \mu(E_2) = \mu^*(S_2).$$

Continuing in this way we see that $\mu^*(S) \leq \mu(C) = \lim_{n \to \infty} \mu(C_n) = \lim_{n \to \infty} \mu^*(S_n) \leq \mu^*(S)$. Therefore, $\mu^*(S) = \mu(C) = \lim_{n \to \infty} \mu(C_n) = \lim_{n \to \infty} \mu^*(S_n) \leq \mu^*(S)$. 


Now we verify the second claim of the lemma. It is clear the formula holds whenever $E$ is measurable. Suppose now that the formula holds. Let $S$ be an arbitrary set. We need to verify that

$$
\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S \setminus E).
$$

Let $F \supseteq S$, $F \in \mathcal{F}$, and $\mu(F) = \mu^*(S)$. Then since $\mu^*$ is subadditive,

$$
\mu^*(\Omega \setminus F) \leq \mu^*(E \setminus F) + \mu^*(\Omega \cap E^C \cap F^C), \tag{45.0.2}
$$

Since $F$ is measurable,

$$
\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \setminus F) \tag{45.0.3}
$$

and

$$
\mu^*(\Omega \setminus E) = \mu^*(F \setminus E) + \mu^*(\Omega \cap E^C \cap F^C) \tag{45.0.4}
$$

and by the hypothesis,

$$
\mu^*(\Omega) \geq \mu^*(E) + \mu^*(\Omega \setminus E). \tag{45.0.5}
$$

Therefore,

$$
\mu(\Omega) \geq \mu^*(E) + \mu^*(\Omega \setminus E) = \mu^*(E \cap F) + \mu^*(E \setminus F) + \mu^*(\Omega \setminus E) = \mu^*(E \cap F) + \mu^*(E \setminus F) + \mu^*(F \setminus E) + \mu^*(\Omega \cap E^C \cap F^C) \geq \mu^*(E \cap F) + \mu^*(E \setminus F) + \mu^*(F \setminus E) + \mu^*(\Omega \setminus F) \geq \mu^*(\Omega \setminus F) + \mu^*(F) = \mu(\Omega),
$$

showing that all the inequalities must be equal signs. Hence, referring to the top and fourth lines above,

$$
\mu(\Omega) = \mu^*(\Omega \setminus F) + \mu^*(F \setminus E) + \mu^*(E \cap F).
$$

Subtracting $\mu^*(\Omega \setminus F) = \mu(\Omega \setminus F)$ from both sides gives

$$
\mu^*(S) = \mu(F) = \mu^*(F \setminus E) + \mu^*(E \cap F) \geq \mu^*(S \setminus E) + \mu^*(E \cap S),
$$

This proves the lemma.

The next theorem is a major result. It states that the Suslin subsets are measurable under appropriate conditions. This is sort of interesting because something being a Suslin subset has to do with topology and this topological condition implies that the set is measurable.

**Theorem 45.0.11** Let $\Omega$ be a metric space and let $(\Omega, \mathcal{F}, \mu)$ be a complete Borel measure space with $\mu(\Omega) < \infty$. Denote by $\mu^*$ the outer measure generated by $\mu$. Then if $A$ is a Suslin subset of $\Omega$, it follows that $A$ is $\mu^*$ measurable. Since the original measure space is complete, it follows that the completion produces nothing new and so in fact $A$ is in $\mathcal{F}$. See Proposition [10.1.5].
**Proof:** We need to verify that

\[ \mu^* (\Omega) \geq \mu^* (A) + \mu^* (\Omega \setminus A) . \]

We know from Corollary 45.0.8 there exists a continuous map, \( f : \mathbb{N}^\mathbb{N} \to A \) which is onto. Let

\[ E'(k) = \{ n \in \mathbb{N}^\mathbb{N} : n_1 \leq k \} . \]

Then \( E'(k) \uparrow \mathbb{N}^\mathbb{N} \) and so from Lemma 45.0.10 we know \( \mu^* (f(E'(k))) \uparrow \mu^* (A) \). Therefore, there exists \( m_1 \) such that

\[ \mu^* (f(E'(m_1))) > \mu^* (A) - \frac{\varepsilon}{2} . \]

Now \( E'(k) \) is clearly not compact but it is trying to be as far as the first component is concerned. Now we let

\[ E(m_1, k) = \{ n \in \mathbb{N}^\mathbb{N} : n_1 \leq m_1 \text{ and } n_2 \leq k \} . \]

Thus \( E(m_1, k) \uparrow E(m_1) \) and so we can pick \( m_2 \) such that

\[ \mu^* (f(E(m_1, m_2))) > \mu^* (f(E(m_1))) - \frac{\varepsilon}{2} . \]

We continue in this way obtaining a decreasing list of sets, \( f(E(m_1, m_2, \ldots, m_{k-1}, m_k)) \), such that

\[ \mu^* (f(E(m_1, m_2, \ldots, m_{k-1}, m_k))) > \mu^* (f(E(m_1, m_2, \ldots, m_{k-1}))) - \frac{\varepsilon}{2^k} . \]

Therefore,

\[ \mu^* (f(E(m_1, m_2, \ldots, m_{k-1}, m_k))) - \mu^* (A) > \sum_{l=1}^{k} - \left( \frac{\varepsilon}{2^l} \right) > -\varepsilon . \]

Now define a closed set,

\[ C = \bigcap_{k=1}^{\infty} f(E(m_1, m_2, \ldots, m_{k-1}, m_k)) . \]

The sets \( f(E(m_1, m_2, \ldots, m_{k-1}, m_k)) \) are decreasing as \( k \to \infty \) and so

\[ \mu^* (C) = \lim_{k \to \infty} \mu^* \left( f(E(m_1, m_2, \ldots, m_{k-1}, m_k)) \right) \geq \mu^* (A) - \varepsilon . \]

We wish to verify that \( C \subseteq A \). If we can do this we will be done because \( C \), being a closed set, is measurable and so

\[ \mu^* (\Omega) = \mu^* (C) + \mu^* (\Omega \setminus C) \geq \mu^* (A) - \varepsilon + \mu^* (\Omega \setminus A) . \]

Since \( \varepsilon \) is arbitrary, this will conclude the proof. Therefore, we only need to verify that \( C \subseteq A \).
What we know is that each $f(E(m_1, m_2, \ldots, m_{k-1}, m_k))$ is contained in $A$. We do not know their closures are contained in $A$. We let $m \equiv \{m_i\}_{i=1}^\infty$ where the $m_i$ are defined above. Then letting

$$K \equiv \{n \in \mathbb{N}^\infty : n_i \leq m_i \text{ for all } i\},$$

we see that $K$ is a closed, hence complete subset of $\mathbb{N}^\infty$ which is also totally bounded due to the definition of the distance. Therefore, $K$ is compact and so $f(K)$ is also compact, hence closed due to the assumption that $\Omega$ is a Hausdorff space and we know that $f(K) \subseteq A$. We verify that $C = f(K)$. We know $f(K) \subseteq C$. Suppose therefore, $p \in C$. From the definition of $C$, we know there exists $r^k \in E(m_1, m_2, \ldots, m_{k-1}, m_k)$ such that $d(f(r^k), p) < \frac{1}{k}$. Denote by $r^k$ the element of $\mathbb{N}^\infty$ which consists of modifying $r^k$ by taking all components after the $k^{th}$ equal to one. Thus $r^k \in K$. Now $\{r^k\}$ is in a compact set and so taking a subsequence we can have $r^k \to r \in K.$ But from the metric on $\mathbb{N}^\infty$, it follows that $\rho\left(r^k, r^\ell\right) < \frac{1}{k}$. Therefore, $r^k \to r$ also and so $f(r^k) \to f(r) = p$. Therefore, $p \in f(K)$ and this proves the theorem.

Note we could have proved this under weaker assumptions. If we had assumed only that every point has a countable basis (first axiom of countability) and $\Omega$ is Hausdorff, the same argument would work. We will need the following definition.

**Definition 45.0.12** Let $\mathcal{F}$ be a $\sigma$ algebra of sets from $\Omega$ and let $\mu$ denote a finite measure defined on $\mathcal{F}$. We let $\mathcal{F}_\mu$ denote the completion of $\mathcal{F}$ with respect to $\mu$. Thus we let $\mu^*$ be the outer measure determined by $\mu$ and $\mathcal{F}_\mu$ will be the $\sigma$ algebra of $\mu^*$ measurable subsets of $\Omega$. We also define $\tilde{\mathcal{F}}$ by

$$\tilde{\mathcal{F}} \equiv \cap \{\mathcal{F}_\mu : \mu \text{ is a finite measure defined on } \mathcal{F}\}.$$

Also, if $X$ is a topological space, we will denote by $B(X)$ the Borel sets of $X$.

With this notation, we can give the following simple corollary of Theorem 45.0.11. This is really quite amazing.

**Corollary 45.0.13** Let $\Omega$ be a compact metric space and let $A$ be a Suslin subset of $\Omega$. Then $A \in B(\Omega)$.

**Proof:** Let $\mu$ be a finite measure defined on $B(\Omega)$. By Theorem 45.0.11, $A \in B(\Omega)_\mu$. Since this is true for every finite measure, $\mu$, it follows $A \in B(\Omega)$ as claimed. This proves the corollary.

We give another technical lemma about the completion of measure spaces.

**Lemma 45.0.14** Let $\mu$ be a finite measure on a $\sigma$ algebra, $\Sigma$. Then $A \in \Sigma_\mu$ if and only if there exists $A_1 \in \Sigma$ and $N_1$ such that $A = A_1 \cup N_1$ where there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $N_1 \subseteq N$. 
**Proof:** Suppose first \( A = A_1 \cup N_1 \) where these sets are as described. Let \( S \in \mathcal{P}(\Omega) \) and let \( \mu^* \) denote the outer measure determined by \( \mu \). Then since \( A_1 \in \Sigma \subseteq \Sigma_\mu \)

\[
\mu^* (S) \leq \mu^* (S \setminus A) + \mu^* (S \cap A) \\
\leq \mu^* (S \setminus A_1) + \mu^* (S \cap A_1) + \mu^* (N_1) \\
= \mu^* (S \setminus A_1) + \mu^* (S \cap A_1) = \mu^* (S)
\]

showing that \( A \in \Sigma_\mu \).

Now suppose \( A \in \Sigma_\mu \). Then there exists \( B_1 \supseteq A \) such that \( \mu^* (B_1) = \mu^* (A) \), and \( B_1 \in \Sigma \). Also there exists \( A_1^C \in \Sigma \) with \( A_1^C \supseteq A^C \) and \( \mu (A_1^C) = \mu^* (A^C) \). Then \( A_1 \subseteq A \subseteq B_1 \)

\[
A \subseteq A_1 \cup (B_1 \setminus A_1).
\]

Now

\[
\mu (A_1) + \mu^* (A^C) = \mu (A_1) + \mu (A_1^C) = \mu (\Omega)
\]

and so

\[
\mu (B_1 \setminus A_1) = \mu^* (B_1 \setminus A_1) \\
= \mu^* (B_1 \setminus A) + \mu^* (A \setminus A_1) \\
= \mu^* (B_1) - \mu^* (A) + \mu^* (A) - \mu^* (A_1) \\
= \mu^* (A) (\mu (\Omega) - \mu^* (A^C)) = 0
\]

because \( A \in \Sigma_\mu \) implying \( A = A_1 \cup (B_1 \setminus A_1) \cap A \) and \( N_1 \subseteq N \equiv (B_1 \setminus A_1) \in \Sigma \) with \( \mu (N) = 0 \). This proves the lemma.

Next we need another definition.

**Definition 45.0.15** We say \((\Omega, \Sigma)\), where \( \Sigma \) is a \( \sigma \) algebra of subsets of \( \Omega \), is separable if there exists a sequence \( \{ A_n \}_{n=1}^\infty \subseteq \Sigma \) such that \( \sigma (\{ A_n \}) = \Sigma \) and if \( w \neq w' \), then there exists \( A \in \Sigma \) such that \( X_A (w) \neq X_A (w') \). This last condition is referred to by saying \( \{ A_n \} \) separates the points of \( \Omega \). Given two measure spaces, \((\Omega, \Sigma) \) and \((\Omega', \Sigma')\), we say they are isomorphic if there exists a function, \( f : \Omega \to \Omega' \) which is one to one and \( f (E) \in \Sigma' \) whenever \( E \in \Sigma \) and \( f^{-1} (F) \in \Sigma \) whenever \( F \in \Sigma' \).

The interesting thing about separable measure spaces is that they are isomorphic to a very simple sort of measure space in which topology plays a significant role.

**Lemma 45.0.16** Let \((\Omega, \Sigma)\) be separable. Then there exists \( E \in \{0, 1\}^N \) such that \((\Omega, \Sigma) \) and \((E, B (E))\) are isomorphic.

**Proof:** First we show \( \{ A_n \} \) separates the points. Here \( \sigma (\{ A_n \}) = \Sigma \). We already know \( \Sigma \) separates the points but now we show the smaller set does so also.
If this is not so, there exists \( \omega, \omega_1 \in \Omega \) such that for all \( n \), \( \mathcal{X}_{A_n}(\omega) = \mathcal{X}_{A_n}(\omega_1) \).

Then let
\[
\mathcal{F} \equiv \{ F \in \Sigma : \mathcal{X}_F(\omega) = \mathcal{X}_F(\omega_1) \}
\]

Thus \( A_n \in \mathcal{F} \) for all \( n \). It is also clear that \( \mathcal{F} \) is a \( \sigma \) algebra and so \( \mathcal{F} = \Sigma \) contradicting the assumption that \( \Sigma \) separates points. Now we define a function from \( \Omega \) to \( \{0, 1\}^\mathbb{N} \) as follows.

\[
f(\omega) \equiv \{ \mathcal{X}_{A_n}(\omega) \}_{n=1}^\infty
\]

We also let \( E \equiv f(\Omega) \). Since the \( \{A_n\} \) separate the points, we see that \( f \) is one to one. A subbasis for the topology of \( \{0, 1\}^\mathbb{N} \) consists of sets of the form \( \prod_{i=1}^\infty H_i \) where \( H_i = \{0, 1\} \) for all \( i \) except one, when \( i = j \) and \( H_j \) equals either \( \{0\} \) or \( \{1\} \).

Therefore, \( f^{-1}(\text{subbasic open set}) \in \Sigma \) because if \( H_j \) is the exceptional set then this equals \( A_j \) if \( H_j = \{1\} \) and \( A_j^c \) if \( H_j = \{0\} \). Intersections of these subbasic sets with \( E \) gives a countable subbasis for \( E \) and so the inverse image of all sets in a countable subbasis for \( E \) are in \( \Sigma \), showing that \( f^{-1}(\text{open set}) = \Sigma \). Now we consider \( f(A_n) \).

\[
f(A_n) \equiv \{ \{ \lambda_k \}_{k=1}^\infty : \lambda_n = 1 \} \cap E,
\]

an open set in \( E \). Hence \( f(A_n) \in B(E) \). Now letting
\[
\mathcal{F} \equiv \{ G \subseteq \Omega : f(G) \in B(E) \},
\]

we see that \( \mathcal{F} \) is a \( \sigma \) algebra which contains \( \{A_n\}_{n=1}^\infty \) and so \( \mathcal{F} \supseteq \sigma(\{A_n\}) = \Sigma \). Thus \( f(F) \in B(E) \) for all \( A \in \Sigma \). This proves the lemma.

**Lemma 45.0.17** Let \( \phi : (\Omega_1, \Sigma_1) \to (\Omega_2, \Sigma_2) \) where \( \phi^{-1}(U) \in \Sigma_1 \) for all \( U \in \Sigma_2 \).

Then if \( F \subseteq \widehat{\Sigma}_2 \), it follows \( \phi^{-1}(F) \in \widehat{\Sigma}_1 \).

**Proof:** Let \( \mu \) be a finite measure on \( \Sigma_1 \) and define a measure \( \phi(\mu) \) on \( \Sigma_2 \) by the rule
\[
\phi(\mu)(F) \equiv \mu(\phi^{-1}(F)).
\]

Now let \( A \in \Sigma_2^{\phi(\mu)} \). Then by Lemma 45.0.14, \( A = A_1 \cup N_1 \) where there exists \( N \in \Sigma_2 \) with \( \phi(\mu)(N) = 0 \) and \( A_1 \in \Sigma_2 \). Therefore, from the definition of \( \phi(\mu) \), we have \( \mu(\phi^{-1}(N)) = 0 \) and therefore, \( \phi^{-1}(A) = \phi^{-1}(A_1) \cup \phi^{-1}(N_1) \) where \( \phi^{-1}(N_1) \subseteq \phi^{-1}(N) \in \Sigma_1 \) and \( \mu(\phi^{-1}(N)) = 0 \). Therefore, \( \phi^{-1}(A) \in \Sigma_1^{\mu} \) and so if \( F \subseteq \widehat{\Sigma}_2 \), then
\[
F \in \cap \{ \Sigma_2^\nu : \nu \text{ is a finite measure on } \Sigma_2 \} \subseteq \cap \{ \Sigma_2^{\phi(\mu)} : \mu \text{ is a finite measure on } \Sigma_1 \},
\]

and so \( \phi^{-1}(F) \in \Sigma_1^{\mu} \). Since \( \mu \) is arbitrary, this shows \( \phi^{-1}(F) \in \widehat{\Sigma}_1 \).

The next lemma is a special case of the Yankov von Neumann Aumann projection theorem. It contains the main idea of the proof of the more general theorem.

**Definition 45.0.18** Let \( (\Omega, \Sigma) \) be a measurable space and let \( G \in \Sigma \times B(\mathbb{X}) \) the product measurable sets resulting from the Borel sets \( B(\mathbb{X}) \) for \( \mathbb{X} \) a Suslan space. Then
\[
\text{proj}_\Omega(G) \equiv \{ \omega \in \Omega : \text{there exists } x \in \mathbb{X} \text{ with } (\omega, x) \in G \}.
\]
Let \( \text{proj}_\Omega (G) = \{ \omega \in \Omega : A(\omega) \cap X \neq \emptyset \} \)

Of course, you might ask whether this particular \( G \) is in \( \Sigma \times B(X) \). For fixed \( \omega \) the \( \omega \) section is closed which is Borel. To say that \( \text{proj}_\Omega (G) \in \Sigma \) would be to say that each \( y \) section is in \( \Sigma \) so this \( G \) is in \( \Sigma \times B(X) \) exactly when \( \text{proj}_\Omega (G) \in \Sigma \). Later this will be defined as saying that \( A \) is a measurable multifunction. It will be measurable when \( X \) is open, strongly measurable when \( X \) is closed.

**Lemma 45.0.19** Let \((\Omega, \Sigma)\) be separable and let \( X \) be a Suslin space. Let \( G \in \Sigma \times B(X) \). (Recall \( \Sigma \times B(X) \) is the \( \sigma \) algebra of product measurable sets, the smallest \( \sigma \) algebra containing the measurable rectangles.) Then

\[
\text{proj}_\Omega (G) \in \bar{\Sigma}.
\]

**Proof:** Let \( f : (\Omega, \Sigma) \to (E, B(E)) \) be the isomorphism of Lemma 45.11. We have the following claim.

**Claim:** \( f \times \text{id}_X \) maps \( \Sigma \times B(X) \) to \( B(E) \times B(X) \).

**Proof of the claim:** First of all, assume \( A \times B \) is a measurable rectangle where \( A \in \Sigma \) and \( B \in B(X) \). Then by the assumption that \( f \) is an isomorphism, \( f(A) \in B(E) \) and so

\[
f \times \text{id}_X (A \times B) \in B(E) \times B(X).
\]

Now let

\[
\mathcal{F} \equiv \{ P \in \Sigma \times B(X) : f \times \text{id}_X (P) \in B(E) \times B(X) \}.
\]

Then we see that \( \mathcal{F} \) is a \( \sigma \) algebra and contains the elementary sets. (\( \mathcal{F} \) is closed with respect to complements because \( f \) is one to one.) Therefore, \( \mathcal{F} = \Sigma \times B(X) \) and this proves the claim.

Therefore, since \( G \in \Sigma \times B(X) \), we see

\[
f \times \text{id}_X (G) \in B(E) \times B(X) \subseteq B(E \times X).
\]

The set inclusion follows from the observation that if \( A \in B(E) \) and \( B \in B(X) \) then \( A \times B \) is in \( B(E \times X) \) and the collection of sets in \( B(E) \times B(X) \) which are in \( B(E \times X) \) is a \( \sigma \) algebra.

Therefore, there exists \( D \), a Borel set in \( E \times X \) such that \( f \times \text{id}_X (G) = D \cap (E \times X) \). Now from this it follows from Lemma 45.11.14 that \( D \) is a Suslin space. Letting \( Y = \{0, 1\}^\mathbb{N} \), it follows that \( \text{proj}_Y(D) \) is a Suslin space in \( Y \). By Corollary 45.11.14, we see that \( \text{proj}_Y(D) \in B(Y) \). Now

\[
\text{proj}_\Omega (G) = \{ \omega \in \Omega : \text{there exists } x \in X \text{ with } (\omega, x) \in G \}
\]

\[
= \{ \omega \in \Omega : \text{there exists } x \in X \text{ with } (f(\omega), x) \in f \times \text{id}_X (G) \}
\]
\[ f^{-1}(\{y \in Y : \text{there exists } x \in X \text{ with } (y, x) \in D\}) \]
\[ = f^{-1}(\text{proj}_Y(D)) \]

Now \( \text{proj}_Y(D) \in \overline{B(Y)} \) and so Lemma 45.0.17 shows \( f^{-1}(\text{proj}_Y(D)) \in \hat{\Sigma} \). This proves the lemma.

Now we are ready to prove the Yankov von Neumann Aumann projection theorem. First we must present another technical lemma.

**Lemma 45.0.20** Let \( X \) be a Hausdorff space and let \( G \in \Sigma \times B(X) \) where \( \Sigma \) is a \( \sigma \) algebra of sets of \( \Omega \). Then there exists \( \Sigma_0 \subseteq \Sigma \) a countably generated \( \sigma \) algebra such that \( G \in \Sigma_0 \times B(X) \).

**Proof:** First suppose \( G \) is a measurable rectangle, \( G = A \times B \) where \( A \in \Sigma \) and \( B \in B(X) \). Letting \( \Sigma_0 \) be the finite \( \sigma \) algebra, \( \{\emptyset, A, A^c, \Omega\} \), we see that \( G \in \Sigma_0 \times B(X) \). Similarly, if \( G \) equals an elementary set, then the conclusion of the lemma holds for \( G \). Let
\[ F \equiv \{H \in \Sigma \times B(X) : H \in \Sigma_0 \times B(X)\} \]
for some countably generated \( \sigma \) algebra, \( \Sigma_0 \). We just saw that \( F \) contains the elementary sets. If \( H \in F \), then \( H^c \in \Sigma_0 \times B(X) \) for the same \( \Sigma_0 \) and so \( F \) is closed with respect to complements. Now suppose \( H_n \in F \). Then for each \( n \), there exists a countably generated \( \sigma \) algebra, \( \Sigma_{0n} \) such that \( H_n \in \Sigma_{0n} \times B(X) \). Then \( \bigcup_{n=1}^\infty H_n \in \sigma(\{\Sigma_{0n} \times B(X)\}) \). We will be done when we show
\[ \sigma(\{\Sigma_{0n} \times B(X)\}_{n=1}^\infty) \subseteq \sigma(\{\Sigma_{0n}\}_{n=1}^\infty) \times B(X) \]
because it is clear that \( \sigma(\{\Sigma_{0n}\}_{n=1}^\infty) \) is countably generated. We see that
\[ \sigma(\{\Sigma_{0n} \times B(X)\}_{n=1}^\infty) \]
is generated by sets of the form \( A \times B \) where \( A \in \Sigma_{0n} \) and \( B \in B(X) \). But each such set is also contained in \( \sigma(\{\Sigma_{0n}\}_{n=1}^\infty) \times B(X) \) and so the desired inclusion is obtained. Therefore, \( F \) is a \( \sigma \) algebra and so since \( F \) was shown to contain the measurable rectangles, this verifies \( F = \Sigma \times B(X) \) and this proves the lemma.

**Theorem 45.0.21** Let \( (\Omega, \Sigma) \) be a measure space and let \( G \in \hat{\Sigma} \times B(X) \) where \( X \) is a Suslin space. Then
\[ \text{proj}_{\Omega\Sigma}(G) \in \hat{\Sigma}. \]

**Proof:** By the previous lemma, \( G \in \Sigma_0 \times B(X) \) where \( \Sigma_0 \) is countably generated. If \( (\Omega, \Sigma_0) \) were separable, we could then apply Lemma 45.0.18 and be done. Unfortunately, we don’t know \( \Sigma_0 \) separates the points of \( \Omega \). Therefore, we define an equivalence class on the points of \( \Omega \) as follows. We say \( \omega \sim \omega_1 \) if and only if \( \mathcal{X}_A(\omega) = \mathcal{X}_A(\omega_1) \) for all \( A \in \Sigma_0 \). Now the nice thing to notice about this equivalence relation is that if \( \omega \in A \in \Sigma_0 \), and if \( \omega \sim \omega_1 \), then \( 1 = \mathcal{X}_A(\omega) = \mathcal{X}_A(\omega_1) \) implying \( \omega_1 \in A \) also. Therefore, every set of \( \Sigma_0 \) is the union of equivalence classes.
It follows that for $A \in \Sigma_0$, and $\pi$ the map given by $\pi \omega \equiv [\omega]$ where $[\omega]$ is the equivalence class determined by $\omega$,

$$\pi (A) \cap \pi (\Omega \setminus A) = \emptyset.$$

Suppose now that $H_n \in \Sigma_0 \times B(X)$. If $([\omega], x) \in \cap_{n=1}^{\infty} \pi \times id_X (H_n)$, then for each $n$,

$$( [\omega], x ) = ( \pi w_n, x )$$

for some $(\omega_n, x) \in H_n$. But this implies $\omega \sim \omega_n$ and so from the above observation that the sets of $\Sigma_0$ are unions of equivalence classes, it follows that $(\omega, x) \in H_n$. Therefore, $(\omega, x) \in \cap_{n=1}^{\infty} H_n$ and so $([\omega], x) = \pi \times id_X (\omega, x)$ where $(\omega, x) \in \cap_{n=1}^{\infty} H_n$.

This shows that

$$\pi \times id_X (\cap_{n=1}^{\infty} H_n) \supset \cap_{n=1}^{\infty} \pi \times id_X (H_n).$$

In fact these two sets are equal because the other inclusion is obvious. We will denote by $\Omega_1$ the set of equivalence classes and $\Sigma_1$ will be the subsets, $S_1$, of $\Omega_1$ such that $S_1 = \{ [\omega]: \omega \in S \in \Sigma_0 \}$. Then $(\Omega_1, \Sigma_1)$ is clearly a measure space which is separable. Let

$$F \equiv \{ H \in \Sigma_0 \times B(X) : \pi \times id_X (H), \pi \times id_X (H^c) \in \Sigma_1 \times B(X) \}.$$

We see that the measurable rectangles, $A \times B$ where $A \in \Sigma_0$ and $B \in B(X)$ are in $F$, that from the above observation on countable intersections, $F$ is closed with respect to countable unions and closed with respect to complements. Therefore, $F$ is a $\sigma$ algebra and so $F = \Sigma_0 \times B(X)$. By Lemma 15.0.14 $(\Omega_1, \Sigma_1)$ is isomorphic to $(E, B(E))$ where $E$ is a subspace of $\{0, 1\}^N$. Denoting the isomorphism by $h$, it follows as in Lemma 15.0.14 that $h \times id_X$ maps $\Sigma_1 \times B(X)$ to $B(E) \times B(X)$. Therefore, we see $f \equiv h \circ \pi$ is a mapping from $\Omega$ to $E$ which has the property that $f \times id_X$ maps $\Sigma_0 \times B(X)$ to $B(E) \times B(X)$. Now from the proof of Lemma 15.0.14 starting with the claim, we see that $G \in \hat{\Sigma}_0$. However, if $\mu$ is a finite measure on $\hat{\Sigma}$, then $(\hat{\Sigma})_\mu = \Sigma_\mu$ and so $\hat{\Sigma}_0 \subseteq (\hat{\Sigma}) \subseteq \hat{\Sigma}$. This proves the theorem.
Chapter 46

Multifunctions And Their Measurability

46.1 The General Case

Let $X$ be a separable complete metric space and let $(\Omega, \mathcal{C}, \mu)$ be a set, a $\sigma$ algebra of subsets of $\Omega$, and a measure $\mu$ such that this is a complete $\sigma$ finite measure space. Also let $\Gamma : \Omega \rightarrow \mathcal{P}_c(X)$, the closed subsets of $X$.

Definition 46.1.1 We define $\Gamma^-(S) \equiv \{ \omega \in \Omega : \Gamma(\omega) \cap S \neq \emptyset \}$

We will consider a theory of measurability of set valued functions. The following theorem is the main result in the subject. In this theorem the third condition is what we will refer to as measurable.

Theorem 46.1.2 The following are equivalent in case of a complete $\sigma$ finite measure space. However 3 and 4 are equivalent for any measurable space consisting only of a set $\Omega$ and a $\sigma$ algebra $\mathcal{C}$.

1. For all $B$ a Borel set in $X$, $\Gamma^-(B) \in \mathcal{C}$.
2. For all $F$ closed in $X$, $\Gamma^-(F) \in \mathcal{C}$
3. For all $U$ open in $X$, $\Gamma^-(U) \in \mathcal{C}$
4. There exists a sequence, $\{\sigma_n\}$ of measurable functions satisfying $\sigma_n(\omega) \in \Gamma(\omega)$ such that for all $\omega \in \Omega$,

$$\Gamma(\omega) = \{\sigma_n(\omega) : n \in \mathbb{N}\}$$

These functions are called measurable selections.
5. For all $x \in X$, $\omega \rightarrow \text{dist}(x, \Gamma(\omega))$ is a measurable real valued function.
6. \( G(\Gamma) \equiv \{(\omega, x) : x \in \Gamma(\omega)\} \subseteq C \times B(X) \).

**Proof:** It is obvious that 1.) \(\Rightarrow\) 2.). To see that 2.) \(\Rightarrow\) 3.) note that \(\Gamma^- (\cup_{i=1}^\infty F_i) = \cup_{i=1}^\infty \Gamma^- (F_i)\). Since any open set in \(X\) can be obtained as a countable union of closed sets, this implies 2.) \(\Rightarrow\) 3.).

Now we verify that 3.) \(\Rightarrow\) 4.). For convenience, drop the assumption that \(\Gamma(\omega)\) is closed in this part of the argument. It will just be set valued and satisfy the measurability condition. A measurable selection will be obtained in \(G(\omega)\). Let \(\{x_n\}_{n=1}^\infty\) be a countable dense subset of \(X\). For \(\omega \in \Omega\), let \(\psi_1(\omega) = x_n\) where \(n\) is the smallest integer such that \(\Gamma(\omega) \cap B(x_n, 1) \neq \emptyset\). Therefore, \(\psi_1(\omega)\) has countably many values, \(x_{n_1}, x_{n_2}, \ldots\) where \(n_1 < n_2 < \ldots\). Now

\[
\{\omega : \psi_1 = x_n\} = \\
\{\omega : \Gamma(\omega) \cap B(x_n, 1) \neq \emptyset\} \cap \{|\Omega \setminus \cup_{k<n} \{\omega : \Gamma(\omega) \cap B(x_k, 1) \neq \emptyset\}\} \in C.
\]

Thus we see that \(\psi_1\) is measurable and \(\text{dist}(\psi_1(\omega), \Gamma(\omega)) < 1\). Let

\[\Omega_n \equiv \{\omega \in \Omega : \psi_1(\omega) = x_n\}\] .

Then \(\Omega_n \subseteq C \) and \(\Omega_n \cap \Omega_m = \emptyset\) for \(n \neq m\) and \(\bigcup_{n=1}^\infty \Omega_n = \Omega\). Let

\[D_n \equiv \{x_k : x_k \in B(x_n, 1)\}\] .

Now for each \(n\), and \(\omega \in \Omega_n\), let \(\psi_2(\omega) = x_k\) where \(k\) is the smallest index such that \(x_k \in D_n\) and \(B(x_k, \frac{1}{2}) \cap \Gamma(\omega) \neq \emptyset\). Thus \(\text{dist}(\psi_2(\omega), \Gamma(\omega)) < \frac{1}{2}\) and

\[d(\psi_2(\omega), \psi_1(\omega)) < 1\] .

Continue this way obtaining \(\psi_k\) a measurable function such that

\[\text{dist}(\psi_k(\omega), \Gamma(\omega)) < \frac{1}{2k-1}, \ d(\psi_k(\omega), \psi_{k+1}(\omega)) < \frac{1}{2k-2}\] .

Then for each \(\omega\), \(\{\psi_k(\omega)\}\) is a Cauchy sequence converging to a point, \(\sigma(\omega) \in \Gamma(\omega)\).

This has shown that if \(\Gamma\) is measurable, there exists a measurable selection, \(\sigma(\omega) \in \Gamma(\omega)\). Of course, if \(\Gamma(\omega)\) is closed, then \(\sigma(\omega) \in \Gamma(\omega)\). Note that this had nothing to do with the measure. It remains to show there exists a sequence of these measurable selections \(\sigma_n\) such that the conclusion of 4.) holds. To do this we define for \(\Gamma(\omega)\) closed and measurable,

\[
\Gamma_n(\omega) \equiv \begin{cases} 
\Gamma(\omega) \cap B(x_n, 2^{-i}) & \text{if } \Gamma(\omega) \cap B(x_n, 2^{-i}) \neq \emptyset \\
\Gamma(\omega) & \text{otherwise.}
\end{cases}
\]

Thus

\[
\Gamma(\omega) \cap B(x_n, 2^{-(i+1)}) \subseteq \Gamma_n(\omega) \subseteq \Gamma(\omega) \cap B(x_n, 2^{-i}).
\]

First we show that \(\Gamma_n(\omega)\) is measurable. Let \(U\) be open. Then

\[
\{\omega : \Gamma_n(\omega) \cap U \neq \emptyset\} = \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \cap U \neq \emptyset\} \cup
\]

\[
\{\omega : \Gamma(\omega) \cap U = \emptyset\}.
\]

Next we show that \(\Gamma_n(\omega)\) is measurable. Let \(U\) be open. Then

\[
\{\omega : \Gamma_n(\omega) \cap U \neq \emptyset\} = \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \cap U \neq \emptyset\} \cup
\]

\[
\{\omega : \Gamma(\omega) \cap U = \emptyset\}.
\]
separable, there exists $B$ (\(\omega\), $x$) and so $\sigma(x,\omega)$ is measurable and $\phi(\omega, x) \equiv \text{dist}(x, \Gamma(\omega))$ is measurable. On $X$ due to the continuity of $x \rightarrow \text{dist}(x, \Gamma(\omega))$ and so we must argue that $\phi$ is product measurable. On

$$E_n \equiv \Omega \times (B(x_n, 2^{-k}) \setminus \cup_{m<n} B(x_m, 2^{-k})),$$

we wish to show that $(\omega, x) \rightarrow \text{dist}(x, \Gamma(\omega))$ is product measurable because then $G(\Gamma)$ being the inverse image of $\{0\}$ will be product measurable. Let $\{x_k\}$ be a countable dense set in $X$ and let

$$\phi_k(\omega, x) \equiv \text{dist}(x_n, \Gamma(\omega))$$

where $n$ is the first index such that $x \in B(x_n, 2^{-k})$. Then $\phi_k(x, \omega) \rightarrow \text{dist}(x, \Gamma(\omega))$ due to the continuity of $x \rightarrow \text{dist}(x, \Gamma(\omega))$ and so we must argue that $\phi_k$ is product measurable. On

$$[\{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) = \emptyset\} \cap \{\omega : \Gamma(\omega) \setminus U \neq \emptyset\}]$$

$$= \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \setminus U \neq \emptyset\} \cup \{(\Omega \setminus \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \neq \emptyset\}) \cap \{\omega : \Gamma(\omega) \setminus U \neq \emptyset\}\},$$

the product measurable. By what was just shown, there exists $\sigma_{n_i}$, a measurable function such that $\sigma_{n_i}(\omega) \in \Gamma_{n_i}(\omega) \subseteq \Gamma(\omega)$ for all $\omega \in \Omega$. If $x \in \Gamma(\omega)$, then

$$x \in \overline{B(x_n, 2^{-(i+2)})}$$

whenever $x_n$ is close enough to $x$. Thus both $x, \sigma_{n(i+2)}(\omega)$ are in $B(x_n, 2^{-(i+2)})$ and so $|\sigma_{n(i+1)}(\omega) - x| < 2^{-i}$. It follows that condition 4.) holds. Note that this had nothing to do with the measure.

Now we verify that 4.) $\Rightarrow$ 3.). Suppose there exist measurable selections $\sigma_n(\omega) \in \Gamma(\omega)$ satisfying condition 4.). Let $U$ be open. Then

$$\{\omega : \Gamma(\omega) \cap U \neq \emptyset\} = \cup_{i=1}^{\infty} \Gamma_{n_i}(\omega) \in C.$$

Now we verify that 4.) $\Rightarrow$ 5.). Let $F(\omega) \equiv \text{dist}(\omega, \Gamma(\omega))$. Then letting $U$ be an open set in $[0, \infty)$, $F(\omega) \in U$ if and only if $d(x, \sigma_n(\omega)) \in U$ for some $\sigma_n(\omega)$. Let $h_n(\omega) \equiv d(x, \sigma_n(\omega))$. Then $h_n$ is measurable and $F^{-1}(U) = \cup_{i=1}^{\infty} h_n^{-1}(U) \in C$. This shows that for all $x \in X$, $\omega \rightarrow \text{dist}(x, \Gamma(\omega))$ is measurable and this proves 5.).

Now we verify that 5.) $\Rightarrow$ 4.). We know $\text{dist}(x, \Gamma(\omega))$ is measurable and we show $\{\omega : \Gamma(\omega) \cap U \neq \emptyset\} \in C$ whenever $U$ is open and then use 3.) $\Rightarrow$ 4.). Since $X$ is separable, there exists $B(x_i, r_i)$ such that $U = \cup_{i=1}^{\infty} B(x_i, r_i)$. Then

$$\{\omega : \Gamma(\omega) \cap U \neq \emptyset\} = \cup_{i=1}^{\infty} \{\omega : \Gamma(\omega) \cap B(x_i, r_i) \neq \emptyset\} = \cup_{i=1}^{\infty} \{\omega : \text{dist}(x_i, \Gamma(\omega)) < r_i\} \in C.$$
\[ \phi_k(\omega, x) = \text{dist}(x_n, \Gamma(\omega)). \] Thus, on this set, \( \phi_k \) equals a measurable function of \( \omega \) and does not depend on \( x \) on \( E_n \). It follows that there are measurable simple \( C \) measurable functions, \( s_m(\omega) \) which increase pointwise to \( \text{dist}(x_n, \Gamma(\omega)) \) on \( E_n \). Thus \( s_m(\omega) X_{E_n}(x) \) increases to \( \phi_k(\omega, x) \) on \( E_n \) showing that \( \phi_k X_{E_n} \) is product measurable with respect to \( C \times \sigma(\tau) \) since \( E_n \) is a measurable rectangle with respect to \( C \) and \( \sigma(\tau) \). Therefore, \( \phi_k \) is product measurable and so \( (\omega, x) \rightarrow \text{dist}(x, \Gamma(\omega)) \) is also product measurable.

It remains to prove 6.) \( \Rightarrow \) 1.). This follows from Theorem 45.1.3

\[ \Gamma^{-}(B) \equiv \{ \omega : \Gamma(\omega) \cap B \neq \emptyset \} \]
\[ = \text{proj}_{\Omega} (G(\Gamma) \cap (\Omega \cap B)). \]

But from Theorem 45.1.3, \( \text{proj}_{\Omega} (G(\Gamma) \cap (\Omega \cap B)) \in \mathcal{C}_\mu = \mathcal{C}. \]

The last part results from \((\Omega, \mathcal{C}_\mu)\) being a complete measure space. Note that without this assumption we could not draw the conclusion desired. This required consideration of the measure. The following theorem is like part of the above but without an assumption that \( \Gamma(\omega) \) is closed.

**Theorem 46.1.3** The following are equivalent for any measurable space consisting only of a set \( \Omega \) and a \( \sigma \) algebra \( \mathcal{C} \). Here nothing is known about \( \Gamma(\omega) \) other than that is a nonempty set.

1. For all \( U \) open in \( X, \Gamma^{-}(U) \in \mathcal{C} \)
   \[ \Gamma^{-}(U) \equiv \{ \omega : \Gamma(\omega) \cap U \neq \emptyset \} \]

2. There exists a sequence, \( \{\mathcal{C}_n\} \) of measurable functions satisfying \( \mathcal{C}_n(\omega) \in \Gamma(\omega) \) such that for all \( \omega \in \Omega \),
   \[ \overline{\mathcal{C}(\omega)} = \{ \mathcal{C}_n(\omega) : n \in \mathbb{N} \} \]
   These functions are called measurable selections.

**Proof:** First 1.) \( \Rightarrow \) 2.). A measurable selection will be obtained in \( \overline{\mathcal{C}(\omega)} \). Let \( \{x_n\}_{n=1}^{\infty} \) be a countable dense subset of \( X \). For \( \omega \in \Omega \), let \( \psi_1(\omega) = x_n \) where \( n \) is the smallest integer such that \( \Gamma(\omega) \cap B(x_n, 1) \neq \emptyset \). Therefore, \( \psi_1(\omega) \) has countably many values, \( x_{n_1}, x_{n_2}, \cdots \) where \( n_1 < n_2 < \cdots \). Now
\[ \{ \omega : \psi_1 = x_n \} = \{ \omega : \Gamma(\omega) \cap B(x_n, 1) \neq \emptyset \} \cap [\Omega \setminus \bigcup_{k<n} \{ \omega : \Gamma(\omega) \cap B(x_k, 1) \neq \emptyset \}] \in \mathcal{C}. \]
Thus we see that \( \psi_1 \) is measurable and \( \text{dist}(\psi_1(\omega), \Gamma(\omega)) < 1 \). Let
\[ \Omega_n \equiv \{ \omega \in \Omega : \psi_1(\omega) = x_n \}. \]
Then \( \Omega_n \in \mathcal{C} \) and \( \Omega_n \cap \Omega_m = \emptyset \) for \( n \neq m \) and \( \bigcup_{n=1}^{\infty} \Omega_n = \Omega \). Let
\[ D_n \equiv \{ x_k : x_k \in B(x_n, 1) \}. \]
46.1. THE GENERAL CASE

Now for each \( n \), and \( \omega \in \Omega_n \), let \( \psi_2 (\omega) = x_k \) where \( k \) is the smallest index such that \( x_k \in D_n \) and \( B (x_k, \frac{1}{2}) \cap \Gamma (\omega) \neq \emptyset \). Thus \( \text{dist} (\psi_2 (\omega), \Gamma (\omega)) < \frac{1}{2} \) and

\[
d (\psi_2 (\omega), \psi_1 (\omega)) < 1.
\]

Continue this way obtaining \( \psi_k \) a measurable function such that

\[
\text{dist} (\psi_k (\omega), \Gamma (\omega)) < \frac{1}{2^{k-1}}, \quad d (\psi_k (\omega), \psi_{k+1} (\omega)) < \frac{1}{2^{k-2}}.
\]

Then for each \( \omega \), \( \{ \psi_k (\omega) \} \) is a Cauchy sequence converging to a point, \( \sigma (\omega) \in \overline{\Gamma (\omega)} \).

This has shown that if \( \Gamma \) is measurable, there exists a measurable selection, \( \sigma (\omega) \in \overline{\Gamma (\omega)} \). Of course, if \( \Gamma (\omega) \) is closed, then \( \sigma (\omega) \in \Gamma (\omega) \). Note that this had nothing to do with a measure.

It remains to show there exists a sequence of these measurable selections \( \sigma_n \) such that the conclusion of 2.) holds. To do this we define

\[
\Gamma_n (\omega) \equiv \begin{cases} \Gamma (\omega) \cap B (x_n, 2^{-i}) & \text{if } \Gamma (\omega) \cap B (x_n, 2^{-i}) \neq \emptyset \\ \Gamma (\omega) & \text{otherwise} \end{cases}
\]

First we show that \( \Gamma_n \) is measurable. Let \( U \) be open. Then

\[
\{ \omega : \Gamma_n (\omega) \cap U \neq \emptyset \} = \{ \omega : \Gamma (\omega) \cap B (x_n, 2^{-i}) \cap U \neq \emptyset \} \cup
\]

\[
\{ \omega : \Gamma (\omega) \cap B (x_n, 2^{-i}) = \emptyset \} \cap \{ \omega : \Gamma (\omega) \cap U \neq \emptyset \}
\]

\[
\cap \{ \omega : \Gamma (\omega) \cap U \neq \emptyset \}.
\]

a measurable set. By what was just shown, there exists \( \sigma_{ni} \), a measurable function such that \( \sigma_{ni} (\omega) \in \overline{\Gamma_n (\omega)} \subseteq \overline{\Gamma (\omega)} \) for all \( \omega \in \Omega \). If \( x \in \overline{\Gamma (\omega)} \), then

\[
x \in B (x_n, 2^{-(i+2)})
\]

whenever \( x_n \) is close enough to \( x \). Thus both \( x, \sigma_{ni+2} (\omega) \) are in \( B (x_n, 2^{-(i+2)}) \) and so \( |\sigma_{ni+2} (\omega) - x| < 2^{-i} \). It follows that condition 2.) holds. Note that this had nothing to do with a measure.

Now consider why 2.) \( \Rightarrow \) 1.). We have \( \{ \sigma_n (\omega) \} \subseteq \Gamma (\omega) \) and \( \sigma_n \) is measurable and \( \cap_n \sigma_n (\omega) \) equals \( \overline{\Gamma (\omega)} \). Why is \( \Gamma \) a measurable multifunction? Let \( U \) be an open set

\[
\Gamma (U) \equiv \{ \omega : \Gamma (\omega) \cap U \neq \emptyset \} = \{ \omega : \overline{\Gamma (\omega)} \cap U \neq \emptyset \} \subseteq \cup_n \sigma_n^{-1} (U) \in \mathcal{C}.
\]

For much more on multifunctions, you should see the book by Hu and Papageorgiou. The above proof follows the presentation in this book.
46.1.1 A Special Case Which Is Easier

The above is a pretty long and difficult argument to show that $\Gamma^{-}(U) \in \mathcal{C}$ for all $U$ open is equivalent to $\Gamma^{-}(F)$ for all $F$ closed. However, there is a special case for which this is much easier to show. Suppose $\Gamma(\omega)$ is not just closed but is also compact. Then as above, if $\Gamma^{-}(F) \in \mathcal{C}$ for all $F$ closed, then $\Gamma^{-}(U) = \bigcup_{n} \Gamma^{-}(F_{n})$ where $F_{n}$ is an increasing sequence of closed sets whose union is $U$. This follows from the observation that $\Gamma(\omega) \cap U = \bigcup_{n} \Gamma(\omega) \cap F_{n}$ and so to say the set on the left is nonempty is to say that at least one of the sets on the right is nonempty. Thus if $\Gamma^{-}(F) \in \mathcal{C}$ for all $F$ closed, then $\Gamma^{-}(U) \in \mathcal{C}$ for all $U$ open. This requires no special considerations.

Now suppose $\Gamma(\omega)$ is compact for every $\omega$ and that $\Gamma^{-}(U) \in \mathcal{C}$ for every $U$ open. Then let $F$ be a closed set and let $\{U_{n}\}$ be a decreasing sequence of open sets whose intersection equals $F$ such that also, for all $n$, $U_{n} \supseteq \overline{U_{n+1}}$. Then

$$\Gamma(\omega) \cap F = \bigcap_{n} \Gamma(\omega) \cap U_{n} = \bigcap_{n} \Gamma(\omega) \cap \overline{U_{n}}$$

Now because of compactness, the set on the left is nonempty if and only if each set on the right is also nonempty. Thus $\Gamma^{-}(F) = \bigcap_{n} \Gamma^{-}(U_{n}) \in \mathcal{C}$. Thus in this special case, it is much easier to see that these two conditions for measurability are equivalent. Note that there is no condition on measures or completeness or any such thing. This proves the following proposition.

**Proposition 46.1.4** Let $X$ be a Polish space and let $\Gamma : X \to \mathcal{P}(X)$ have compact values. Then $\Gamma$ is measurable if and only if it is strongly measurable, the latter being the statement that $\Gamma^{-}(C)$ is measurable whenever $C$ is closed.

Let $\Gamma$ be strongly measurable. Let $\mathcal{G}$ be the sets $G$ such that $\Gamma^{-}(G)$ and $\Gamma^{-}(G^{c})$ are both in $\mathcal{C}$. Then clearly $\mathcal{G}$ is closed with respect to complements. If $G \in \mathcal{G}$ is $G^{c}$? Is $\Gamma^{-}(G^{c})$ and $\Gamma^{-}(G^{c})^{c}$ in $\mathcal{C}$? I guess this is just the definition of what it means to be in $\mathcal{G}$. Also if you have $\{G_{i}\} \subseteq \mathcal{G}$, Then

$$\Gamma^{-}\left(\bigcup_{i} G_{i}\right) = \bigcup_{i} \Gamma^{-}(G_{i}) \in \mathcal{C}$$

and so $\mathcal{G}$ is closed with respect to countable unions. Hence $\mathcal{G}$ must contain the Borel sets because the strong measurability implies that the closed sets and hence open sets are in $\mathcal{G}$. Thus $\Gamma^{-}(G) \in \mathcal{C}$ whenever $G$ is Borel.

46.1.2 Other Measurability Considerations

Here are some general considerations about measurable multifunctions.

**Lemma 46.1.5** Suppose $f : K \times \Omega \to X, K \subseteq X$. Here $X$ is Polish space, separable complete metric space, and $(\Omega, \mathcal{F})$ is a measurable space. Also suppose $\omega \to f(x, \omega)$ is measurable and $x \to f(x, \omega)$ is continuous. Also suppose that $K(\omega) := f(K, \omega)$ is a compact set for each $\omega$. Then you can conclude that $\omega \to K(\omega)$ is a strongly measurable multifunction.
46.1. THE GENERAL CASE

Proof: Let \( \{x_n\} \) be a countable dense subset of \( X \). Then if \( U \) is open,
\[
\{ \omega : K(\omega) \cap U \neq \emptyset \} = \bigcup_{n=1}^{\infty} f(x_n, \cdot)^{-1}(U)
\]
and each of the sets in the union is assumed to be measurable. The reason for the equality is as follows. It is clear that the right side is contained in the left. Now if \( K(\omega) \cap U \neq \emptyset \), then by definition, \( f(x, \omega) \in U \) for some \( x \) but then by continuity, \( f(x_n, \omega) \in U \) also for some \( x_n \) close to \( x \). Thus the two sets are actually equal. \( K(\omega) \) has compact values and is measurable so it will be strongly measurable. \( \blacksquare \)

Then from standard results on measurable multifunctions, [22], there is a countable dense \( \{ f(x_n(\omega), \omega) \} \) such that these are measurable functions of \( \omega \) and for each \( \omega \), this sequence of measurable selections is dense in \( K(\omega) \).

This lemma gives an easy example of a measurable multifunction having compact values. In fact this is the one of most interest in what follows. However, we also have the following general result. It gives the existence of a measurable \( \varepsilon \) net. This is formulated in Banach space because it is convenient to add. It could also be formulated in Polish space with a little more difficulty. One just defines things a little differently.

Proposition 46.1.6 Let \( \omega \to K(\omega) \) be a measurable multifunction where \( K(\omega) \) is a compact set. Then for each \( \varepsilon > 0 \), there exists \( N(\omega) \) and measurable functions \( y_j, j = 1, 2, \cdots, N(\omega), y_j(\omega) \in K(\omega), \) such that
\[
\bigcup_{j=1}^{N} B(y_j(\omega), \varepsilon) \supseteq K(\omega)
\]
for each \( \omega \). Also \( \omega \to N(\omega) \) is measurable.

Proof: Suppose that \( \omega \to K(\omega) \) is a measurable multifunction having compact values in \( X \) a Banach space. Let \( \{\sigma_n(\omega)\} \) be the measurable selections such that for each \( \omega, \{\sigma_n(\omega)\}_{n=1}^{\infty} \) is dense in \( K(\omega) \). Let \( y_1(\omega) := \sigma_1(\omega) \). Now let \( 2(\omega) \) be the first index after 1 such that \( \|\sigma_{2(\omega)}(\omega) - \sigma_1(\omega)\| > \varepsilon/2 \). Thus \( 2(\omega) = k \) on the measurable set
\[
\left\{ \omega \in \Omega : \|\sigma_k(\omega) - \sigma_1(\omega)\| > \frac{\varepsilon}{2} \right\} \cap \left\{ \omega \in \Omega : \cap_{j=1}^{k-1} \|\sigma_j(\omega) - \sigma_1(\omega)\| \leq \frac{\varepsilon}{2} \right\}
\]
Suppose \( 1(\omega), 2(\omega), \cdots, (m-1)(\omega) \) have been chosen such that this is a strictly increasing sequence for each \( \omega \), each is a measurable function, and for \( i, j \leq m-1, \)
\[
\|\sigma_{i(\omega)}(\omega) - \sigma_{j(\omega)}(\omega)\| > \frac{\varepsilon}{2}
\]
Each \( \omega \to \sigma_{i(\omega)}(\omega) \) is measurable because it equals
\[
\sum_{k=1}^{\infty} \mathcal{K}_{[i(\omega) = k]}(\omega) \sigma_k(\omega).
\]
Then \( m(\omega) \) will be the first index larger than \( (m-1)(\omega) \) such that
\[
\|\sigma_{m(\omega)}(\omega) - \sigma_{j(\omega)}(\omega)\| > \frac{\varepsilon}{2}
\]
for all \( j(\omega) < m(\omega) \). Thus \( \omega \to m(\omega) \) is also measurable because it equals \( k \) on the measurable set
\[
\left( \cap \left\{ \omega : \| \sigma_k(\omega) - \sigma_{j(\omega)}(\omega) \| > \frac{\varepsilon}{2}, \ j \leq m - 1 \right\} \right) \cap \{ \omega : (m - 1)(\omega) < k \}
\]
\[
\cap \left( \cup \left\{ \omega : \| \sigma_{k-1}(\omega) - \sigma_{j(\omega)}(\omega) \| \leq \frac{\varepsilon}{2}, \ j \leq m - 1 \right\} \cup \{ \omega : (m - 1)(\omega) \geq k - 1 \} \right)
\]
The top line says it does what is wanted and the second says it is the first after \((m - 1)(\omega)\) which does so. Since \( \mathcal{K}(\omega) \) is a compact set, it follows that the above measurable set will be empty for all \( m(\omega) \) sufficiently large called \( N(\omega) \), also a measurable function, and so the process ends. Let \( y_i(\omega) := \sigma_{i(\omega)}(\omega) \). Then this gives the desired measurable \( \varepsilon \) net. The fact that
\[
\cup_{i=1}^{N(\omega)} B(y_i(\omega), \varepsilon) \supseteq \mathcal{K}(\omega)
\]
follows because if there exists \( z \in \mathcal{K}(\omega) \setminus \left( \cup_{i=1}^{N(\omega)} B(y_i(\omega), \varepsilon) \right) \), then \( B(z, \frac{\varepsilon}{2}) \) would have empty intersection with all of the balls \( B(y_i(\omega), \frac{\varepsilon}{2}) \) and by density of the \( \sigma_i(\omega) \) in \( \mathcal{K}(\omega) \), there would be some \( \sigma_i(\omega) \) contained in \( B(z, \frac{\varepsilon}{2}) \) for arbitrarily large \( l \) and so the process would not have ended as shown above.

The existence of such measurable nets allows the generalization of Kuratowski’s theorem to the following. The proof goes the same way.

Recall the Kuratowski theorem. We can give a generalization to the following theorem in which \( E \) is replaced with \( E(\omega) \) where \( \omega \to E(\omega) \) is measurable.

**Theorem 46.1.7** Let \( E(\omega) \) be a compact metric space and let \((\Omega, \mathcal{F})\) be a measure space. Suppose \( \psi : E \times \Omega \to \mathbb{R} \) has the property that \( x \to \psi(x, \omega) \) is continuous and \( \omega \to \psi(x, \omega) \) is measurable. Then there exists a measurable function, \( f \) having values in \( E(\omega) \) such that
\[
\psi(f(\omega), \omega) = \sup_{x \in E} \psi(x, \omega).
\]
Furthermore, \( \omega \to \psi(f(\omega), \omega) \) is measurable.

**Proof:** Let \( C_1(\omega) \) be a \( 2^{-1} \) net of \( E(\omega) \). Suppose \( C_1, \ldots, C_m(\omega) \) is measurable have been chosen such that \( C_k(\omega) \) is a \( 2^{-k} \) net and \( C_{i+1}(\omega) \supseteq C_i(\omega) \) for all \( i \). Say \( C_i(\omega) = \{ y_{i1}(\omega) \}_{i=1}^{r_i(\omega)} \) where \( \omega \to r_i(\omega) \) is measurable. That such exists follows from Proposition 16.1.18. Then consider \( E(\omega) \setminus \cup \{ B(x, 2^{-(m+1)}) : x \in C_m(\omega) \} \). If this set is empty, let \( C_{m+1}(\omega) = C_m(\omega) \). If it is nonempty, let \( \{ y_{i1}(\omega) \}_{i=1}^{r(\omega)} \) be a \( 2^{-(m+1)} \) net for this compact set, \( r(\omega) \) measurable. Then let \( C_{m+1}(\omega) = C_m(\omega) \cup \{ y_i \}_{i=1}^{r(\omega)} \).

It follows \( \{ C_m(\omega) \}_{m=1}^{\infty} \) satisfies \( C_m(\omega) \) is a \( 2^{-m} \) net and \( C_m(\omega) \subseteq C_{m+1}(\omega) \).

Let \( \{ x_{1k} \}_{k=1}^{m(1,\omega)} \) equal \( C_1, \omega \to m(1,\omega) \) measurable. Let
\[
A_1 \equiv \left\{ \omega : \psi(x_{1k}, \omega) = \max_k \psi(x_{1k}, \omega) \right\}
\]
46.1. THE GENERAL CASE

For \( \omega \in A_1 \), define \( s_1(\omega) \equiv x_1^1 \). Next let

\[
A_1^1 \equiv \left\{ \omega \not\in A_1^1 : \psi(x_2^1, \omega) = \max_k \psi(x_k^1, \omega) \right\}
\]

and let \( s_1(\omega) \equiv x_1^1 \) on \( A_1^1 \). Continue in this way to obtain a simple function, \( s_1 \) such that

\[
\psi(s_1(\omega), \omega) = \max \{ \psi(x, \omega) : x \in C_1(\omega) \}
\]

and \( s_1 \) has values in \( C_1(\omega) \).

Suppose \( s_1(\omega), s_2(\omega), \ldots, s_m(\omega) \) are simple functions with the property that if \( m > 1 \),

\[
d(s_k(\omega), s_{k+1}(\omega)) < 2^{-k},
\]

\[
\psi(s_k(\omega), \omega) = \max \{ \psi(x, \omega) : x \in C_k(\omega) \}
\]

for each \( k + 1 \leq m \), only the second and third assertions holding if \( m = 1 \). Letting

\[
C_m(\omega) = \left\{ x_k(\omega) \right\}_{k=1}^N_{\omega},
\]

it follows \( s_m(\omega) \) is of the form

\[
s_m(\omega) = \sum_{k=1}^N x_k(\omega) \chi_{A_k}(\omega), \quad A_i \cap A_j = \emptyset.
\]

meaning that \( s_m(\omega) \) has value \( x_k(\omega) \) on \( A_k \). Denote by \( \{ y_{1i}(\omega) \}_{i=1}^{n_1(\omega)} \) those points of \( C_{m+1}(\omega) \) which are contained in \( B(x_1(\omega), 2^{-m}) \). Letting \( A_k \) play the role of \( \Omega \) in the first step in which \( s_1 \) was constructed, for each \( \omega \in A_1 \) let \( s_{m+1}(\omega) \) be a simple function which has one of the values \( y_{1i} \) and satisfies

\[
\psi(s_{m+1}(\omega), \omega) = \max_{i \leq n_1} \psi(y_{1i}(\omega), \omega)
\]

for each \( \omega \in A_1 \). Next let \( \{ y_{2i}(\omega) \}_{i=1}^{n_2(\omega)} \) be those points of \( C_{m+1}(\omega) \) different than \( \{ y_{1i}(\omega) \}_{i=1}^{n_1(\omega)} \) which are contained in \( B(x_2(\omega), 2^{-m}) \). Then define \( s_{m+1}(\omega) \) on \( A_2 \) to have values taken from \( \{ y_{2i}(\omega) \}_{i=1}^{n_2(\omega)} \) and

\[
\psi(s_{m+1}(\omega), \omega) = \max_{i \leq n_2(\omega)} \psi(y_{2i}(\omega), \omega)
\]

for each \( \omega \in A_2 \). Continuing this way defines \( s_{m+1} \) on all of \( \Omega \) and it satisfies

\[
d(s_m(\omega), s_{m+1}(\omega)) < 2^{-m} \text{ for all } \omega \in \Omega
\]

It remains to verify

\[
\psi(s_{m+1}(\omega), \omega) = \max \{ \psi(x, \omega) : x \in C_{m+1}(\omega) \}.
\]

To see this is so, pick \( \omega \in \Omega \). Let

\[
\max \{ \psi(x, \omega) : x \in C_{m+1}(\omega) \} = \psi(y_{rj}(\omega), \omega)
\]

(46.1.4)
where \( y_{r_j} (\omega) \in \{ y_{r_i} (\omega) \}_{i=1}^{n_r(\omega)} \subseteq \mathcal{C}_{m+1} (\omega) \) and out of all the balls \( B (x_j (\omega), 2^{-m_j}) \), let the first one which contains \( y_{r_j} (\omega) \) be \( B (x_k (\omega), 2^{-m_k}) \). Then by the construction, \( s_{m+1} (\omega) = y_{r_j} (\omega) \) because \( \psi (y_{r_j} (\omega), \omega) \) is at least as large as \( \psi (y_{s_k} (\omega), \omega) \) for all the other \( y_{s_k} (\omega) \). This and (46.1.4) verifies (46.1.4).

From (46.1.4) it follows \( s_m (\omega) \) converges uniformly on \( \Omega \) to a measurable function, \( f (\omega). \) Then from the construction, \( \psi (f (\omega), \omega) \geq \psi (s_m (\omega), \omega) \) for all \( m \) and \( \omega. \) Now pick \( \omega \in \Omega \) and let \( z (\omega) \) be such that \( \psi (z (\omega), \omega) = \max_{x \in E (\omega)} \psi (x, \omega). \) Letting \( y_{k(\omega)} (\omega) \to z (\omega) \) where \( y_{k(\omega)} (\omega) \in \mathcal{C}_{k (\omega)}, \) it follows from continuity of \( \psi \) in the first argument that

\[
\max_{x \in E (\omega)} \psi (x, \omega) = \psi (z (\omega), \omega) = \lim_{k \to \infty} \psi (y_{k(\omega)} (\omega), \omega) \\
\leq \lim_{m \to \infty} \psi (s_m (\omega), \omega) = \psi (f (\omega), \omega) \leq \max_{x \in E} \psi (x, \omega).
\]

To show \( \omega \to \psi (f (\omega), \omega) \) is measurable, note that since \( E (\omega) \) is compact, there exists a countable dense subset, \( D (\omega) \) consisting of the union of measurable nets as in Proposition 46.1.6. Then using continuity of \( \psi \) in the first argument,

\[
\psi (f (\omega), \omega) = \sup_{x \in E (\omega)} \psi (x, \omega) \\
= \sup_{x \in D (\omega)} \psi (x, \omega)
\]

which equals a measurable function of \( \omega \) because \( D (\omega) \) is countable. 

### 46.2 Existence Of Measurable Fixed Points

#### 46.2.1 Simplices And Labeling

First define an \( n \) simplex, denoted by \([x_0, \ldots, x_n]\), to be the convex hull of the \( n+1 \) points, \( \{x_0, \ldots, x_n\} \) where \( \{x_i - x_0\}_{i=1}^{n} \) are independent. Thus

\[
[x_0, \ldots, x_n] = \left\{ \sum_{i=0}^{n} t_i x_i : \sum_{i=0}^{n} t_i = 1, \; t_i \geq 0 \right\}.
\]

Since \( \{x_i - x_0\}_{i=1}^{n} \) is independent, the \( t_i \) are uniquely determined. If two of them are

\[
\sum_{i=0}^{n} t_i x_i = \sum_{i=0}^{n} s_i x_i
\]

Then

\[
\sum_{i=0}^{n} t_i (x_i - x_0) = \sum_{i=0}^{n} s_i (x_i - x_0)
\]

so \( t_i = s_i \) for \( i \geq 1 \). Since the \( s_i \) and \( t_i \) sum to 1, it follows that also \( s_0 = t_0 \). If \( n \leq 2 \), the simplex is a triangle, line segment, or point. If \( n \leq 3 \), it is a
46.2. **Existence of Measurable Fixed Points**

46.2.2 **Labeling Vertices**

Next is a way to label the vertices. Let \( p_0, \ldots, p_n \) be the first \( n + 1 \) prime numbers. All vertices of a simplex \( S = [x_0, \ldots, x_n] \) having \( \{x_k - x_0\}_{k=1}^n \) independent will be labeled with one of these primes. In particular, the vertex \( x_k \) will be labeled as \( p_k \) if the simplex is \([x_0, \ldots, x_n]\). The value of a simplex will be the product of its labels. Triangulate this \( S \). Consider a 1 simplex coming from the original simplex...
[\mathbf{x}_k, \mathbf{x}_{k+1}]$, label one end as \(p_k\) and the other as \(p_{k+1}\). Then label all other vertices of this triangulation which occur on \([\mathbf{x}_k, \mathbf{x}_{k+1}]\) either \(p_k\) or \(p_{k+1}\). Then obviously there will be an odd number of simplices in this triangulation having value \(p_k p_{k+1}\), that is a \(p_k\) at one end and a \(p_{k+1}\) at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on \([\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}]\),

\[
\{\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}\} \subseteq \{\mathbf{x}_0, \ldots, \mathbf{x}_n\}
\]

any \(k\) simplex for \(k \leq n-1\), and there is an odd number of simplices from the triangulation having value equal to \(\prod_{i=1}^{k+1} p_{j_i}\). Consider \(\hat{S} = [\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}]\). Then by induction, there is an odd number of \(k\) simplices on the \(k+1\)th face \([\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}]\) having value \(\prod_{i \neq j} p_{j_i}\). In particular the face \([\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}]\) has an odd number of simplices with value \(\prod_{i \leq k+1} p_{j_i}\). Now no simplex in any other face of \(\hat{S}\) can have this value by uniqueness of prime factorization. Label the “interior” vertices, those \(u\) having all \(s_i > 0\) in \(u = \sum_{i=1}^{k+2} s_i \mathbf{x}_{j_i}\), (These have not yet been labeled,) with any of the \(p_{j_1}, \ldots, p_{j_{k+2}}\). Pick a simplex on the face \([\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}]\) which has value \(\prod_{i \leq k+1} p_{j_i}\) and cross this simplex into \(\hat{S}\). Continue crossing simplices having value \(\prod_{i \leq k+1} p_{j_i}\) which have not been crossed till the process ends. It must end because there are an odd number of these simplices having value \(\prod_{i \leq k+1} p_{j_i}\).

If the process leads to the outside of \(\hat{S}\), then one can always enter it again because there are an odd number of simplices with value \(\prod_{i \leq k+1} p_{j_i}\) available and you will have used up an even number. When the process ends, the value of the simplex must be \(\prod_{i=1}^{k+2} p_{j_i}\) because it will have the additional label \(p_{j_{k+2}}\) on a vertex since if not, there will be another way out of the simplex. This identifies a simplex in the triangulation with value \(\prod_{i=1}^{k+2} p_{j_i}\). Then repeat the process with \(\prod_{i \leq k+1} p_{j_i}\) valued simplices on \([\mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_{k+1}}, \mathbf{x}_{j_{k+2}}]\) which have not been crossed. Repeating the process, entering from the outside, cannot deliver a \(\prod_{i=1}^{k+2} p_{j_i}\) valued simplex encountered earlier. This is because you cross faces labeled \(\prod_{i \leq k+1} p_{j_i}\). If the remaining vertex is labeled \(p_{j_i}\) where \(i \neq k+2\), then this yields exactly one other face to cross. There are two, the one with the first vertex \(p_{j_i}\) and the next one with the new vertex labeled \(p_{j_i}\) substituted for the first vertex having this label. Thus there is either one route in to a simplex or two. Thus, starting at a simplex labeled \(\prod_{i \leq k+1} p_{j_i}\), one can cross faces having this value till one is led to the \(\prod_{i \leq k+1} p_{j_i}\) valued simplex on the selected face of \(\hat{S}\). In other words, the process is one to one in selecting a \(\prod_{i \leq k+1} p_{j_i}\), vertex from crossing such a vertex on the selected face of \(\hat{S}\). Continue doing this, crossing a \(\prod_{i \leq k+1} p_{j_i}\) simplex on the face of \(\hat{S}\) which has not been crossed previously. This identifies an odd number of simplices having value \(\prod_{i=1}^{k+2} p_{j_i}\). These are the ones which are “accessible” from the outside using this process. If there are any which are not accessible from outside, applying the same process starting inside one of these, leads to exactly one other inaccessible simplex with value \(\prod_{i=1}^{k+2} p_{j_i}\). Hence these inaccessible simplices occur in pairs and so there are an odd number of simplices in the triangulation having value \(\prod_{i=1}^{k+2} p_{j_i}\). We refer to this procedure of labeling as Sperner’s lemma. The system of labeling is well defined thanks to the assumption that \(\{\mathbf{x}_k - \mathbf{x}_0\}_{k=1}^n\) is independent which implies
46.2. EXISTENCE OF MEASURABLE FIXED POINTS

that \( \{x_k - x_i\}_{k \neq i} \) is also linearly independent. The following is a description of the system of labeling the vertices.

**Lemma 46.2.1** Let \([x_0, \ldots, x_n]\) be an \( n \) simplex with \( \{x_k - x_0\}_{k=1}^n \) independent, and let the first \( n+1 \) primes be \( p_0, p_1, \ldots, p_n \). Label \( x_k \) as \( p_k \) and consider a triangulation of this simplex. Labeling the vertices of this triangulation which occur on \([x_{k_1}, \ldots, x_{k_s}]\) with any of \( p_{k_1}, \ldots, p_{k_s} \), beginning with all 1 simplices \([x_{k_1}, x_{k_2}]\) and then 2 simplices and so forth, there are an odd number of simplices \([y_{k_1}, \ldots, y_{k_s}]\) of the triangulation contained in \([x_{k_1}, \ldots, x_{k_s}]\) which have value \( p_{k_1}, \ldots, p_{k_s} \). This for \( s = 1, 2, \ldots, n \).

**Another way To Explain The Labeling**

We now give a brief discussion of the system of labeling for Sperner’s lemma from the point of view of counting numbers of faces rather than obtaining them with an algorithm. Let \( p_0, \ldots, p_n \) be the first \( n+1 \) prime numbers. All vertices of a simplex \( S = [x_0, \ldots, x_n] \) having \( \{x_k - x_0\}_{k=1}^n \) independent will be labeled with one of these primes. In particular, the vertex \( x_k \) will be labeled as \( p_k \). The value of a simplex will be the product of its labels. Triangulate this \( S \). Consider a 1 simplex coming from the original simplex \([x_{k_1}, x_{k_2}]\), label one end as \( p_{k_1} \) and the other as \( p_{k_2} \). Then label all other vertices of this triangulation which occur on \([x_{k_1}, x_{k_2}]\) either \( p_{k_1} \) or \( p_{k_2} \). Then obviously there will be an odd number of simplices in this triangulation having value \( p_{k_1}p_{k_2} \), that is a \( p_{k_j} \) at one end and a \( p_{k_i} \) at the other. Suppose that the labeling has been done for all vertices of the triangulation which are on \([x_{j_1}, \ldots, x_{j_{k+1}}]\),

\[ \{x_{j_1}, \ldots, x_{j_{k+1}}\} \subseteq \{x_0, \ldots, x_n\} \]

any \( k \) simplex for \( k \leq n-1 \), and there is an odd number of simplices from the triangulation having value equal to \( \prod_{j=1}^{k+1} p_{j_i} \). Consider \( \tilde{S} = [x_{j_1}, \ldots, x_{j_{k+1}}, x_{j_{k+2}}] \). Then by induction, there is an odd number of \( k \) simplices on the \( s^{th} \) face

\[ [x_{j_1}, \ldots, x_{j_s}, \ldots, x_{j_{k+1}}] \]

having value \( \prod_{i \neq s} p_{j_i} \). In particular the face \([x_{j_1}, \ldots, x_{j_{k+1}}, x_{j_{k+2}}]\) has an odd number of simplices with value \( \prod_{i=1}^{k+2} p_{j_i} := \tilde{P}_k \). We want to argue that some simplex in the triangulation which is contained in \( \tilde{S} \) has value \( \tilde{P}_{k+1} := \prod_{i=1}^{k+2} p_{j_i} \). Let \( Q \) be the number of \( k+1 \) simplices from the triangulation contained in \( \tilde{S} \) which have two faces with value \( \tilde{P}_k \) (A \( k+1 \) simplex has either 1 or 2 \( \tilde{P}_k \) faces.) and let \( R \) be the number of \( k+1 \) simplices from the triangulation contained in \( \tilde{S} \) which have exactly one \( \tilde{P}_k \) face. These are the ones we want because they have value \( \tilde{P}_{k+1} \). Thus the number of faces having value \( \tilde{P}_k \) which is described here is \( 2Q + R \). All interior \( \tilde{P}_k \) faces being counted twice by this number. Now we count the total number of \( \tilde{P}_k \) faces another way. There are \( P \) of them on the face \([x_{j_1}, \ldots, x_{j_{k+2}}]\) and by induction, \( P \) is odd. Then there are \( O \) of them which are not on this face. These faces got counted twice. Therefore,

\[ 2Q + R = P + 2O \]
and so, since $P$ is odd, so is $R$. Thus there is an odd number of $P_{k+1}$ simplices in $\tilde{S}$.

We refer to this procedure of labeling as Sperner’s lemma. The system of labeling is well defined thanks to the assumption that $\{x_k - x_0\}_{k=1}^n$ is independent which implies that $\{x_k - x_i\}_{k \neq i}$ is also linearly independent. Sperner’s lemma is now a consequence of this discussion.

### 46.2.3 Measurability Of Brouwer Fixed Points

First, here is a nice measurable selection theorem.

**Lemma 46.2.2** Let $U$ be a separable reflexive Banach space. Suppose there is a sequence $\{u_j(\omega)\}_{j=1}^\infty$ in $U$, where each $\omega \to u_j(\omega)$ is measurable and for each $\omega$, $\sup_i \|u_i(\omega)\| < \infty$. Then, there exists $u(\omega) \in U$ such that $\omega \to u(\omega)$ is measurable, and a subsequence $n(\omega)$, that depends on $\omega$, such that the weak limit

$$\lim_{n(\omega) \to \infty} u_n(\omega)(\omega) = u(\omega)$$

holds.

**Proof:** Let $\{z_i\}_{i=1}^\infty$ be a countable dense subset of $U'$. Let $h : U \to \prod_{i=1}^\infty \mathbb{R}$ be defined by

$$h(u) = \prod_{i=1}^\infty \langle z_i, u \rangle.$$  

Let $X = \prod_{i=1}^\infty \mathbb{R}$ with the product topology. Then, this is a Polish space with the metric defined as $d(x, y) = \sum_{i=1}^\infty \frac{|x_i - y_i|}{1 + |x_i - y_i|} 2^{-i}$. By compactness, for a fixed $\omega$, the $h(u_n(\omega))$ are contained in a compact subset of $X$. Next, define

$$\Gamma_n(\omega) = \bigcup_{k \geq n} h(u_k(\omega)),$$

which is a nonempty compact subset of $X$. Moreover, $\Gamma_n(\omega)$ is a measurable multifunction into $X$.

Next, we claim that $\omega \to \Gamma_n(\omega)$ is a measurable multifunction.

The proof of the claim is as follows. It is necessary to show that $\Gamma_n^-(O)$ defined as $\{\omega : \Gamma_n(\omega) \cap O \neq \emptyset\}$ is measurable whenever $O$ is open. It suffices to verify this for $O$ a basic open set in the topology of $X$. Thus let $O = \prod_{i=1}^\infty O_i$ where each $O_i$ is a proper open subset of $\mathbb{R}$ only for $i \in \{j_1, \ldots, j_m\}$. Then,

$$\Gamma_n^-(O) = \bigcup_{k \geq n} \cap_{i=1}^{j_m} \{\omega : \langle z_{j_i}, u_k(\omega) \rangle \in O_{j_i}\},$$

which is a measurable set since $u_k$ is measurable.

Then, it follows that $\omega \to \Gamma_n(\omega)$ is strongly measurable because it has compact values in $X$, thanks to Tychonoff’s theorem. Thus $\Gamma_n^-(H) = \{\omega : H \cap \Gamma_n(\omega) \neq \emptyset\}$ is measurable whenever $H$ is a closed set. Now, let $\Gamma(\omega)$ be defined as $\cap_n \Gamma_n(\omega)$ and then for $H$ closed,

$$\Gamma^-(H) = \cap_n \Gamma_n^-(H).$$
and each set in the intersection is measurable, so this shows that \( \omega \to \Gamma (\omega) \) is also measurable. Therefore, it has a measurable selection \( g (\omega) \). It follows from the definition of \( \Gamma (\omega) \) that there exists a subsequence \( n (\omega) \) such that

\[
g (\omega) = \lim_{n (\omega) \to \infty} h (u_{n(\omega)} (\omega)) \quad \text{in } X.
\]

In terms of components, we have

\[
g_i (\omega) = \lim_{n (\omega) \to \infty} \langle z_i, u_{n(\omega)} (\omega) \rangle.
\]

Furthermore, there is a further subsequence, still denoted with \( n (\omega) \), such that \( u_{n(\omega)} (\omega) \to u (\omega) \) weakly. This means that for each \( i \),

\[
g_i (\omega) = \lim_{n (\omega) \to \infty} \langle z_i, u_{n(\omega)} (\omega) \rangle = \langle z_i, u (\omega) \rangle.
\]

Thus, for each \( z_i \) in a dense set, \( \omega \to \langle z_i, u (\omega) \rangle \) is measurable. Since the \( z_i \) are dense, this implies \( \omega \to \langle z, u (\omega) \rangle \) is measurable for every \( z \in U' \) and so by the Pettis theorem, \( \omega \to u (\omega) \) is measurable. ■

Now we consider the case of fixed points for simplices.

Suppose \( f (\cdot, \omega) : S \to S \) for \( S \) a simplex. Then from the Brouwer fixed point theorem, there is a fixed point \( x (\omega) \) provided \( f (\cdot, \omega) \) is continuous. Can it be arranged to have \( \omega \to x (\omega) \) also measurable? In fact, it can, and this is shown here. In other words, if \( P (\omega) \) are the fixed points of \( f (\cdot, \omega) \), there exists a measurable selection in \( P (\omega) \).

\( S \equiv [x_0, \cdots, x_n] \) is a simplex in \( \mathbb{R}^n \). Assume \( \{x_i - x_0\}_{i=1}^n \) are linearly independent. Thus a typical point of \( S \) is of the form

\[
\sum_{i=0}^n t_i x_i
\]

where the \( t_i \) are uniquely determined and the map \( x \to t \) is continuous from \( S \) to the compact set \( \{ t \in \mathbb{R}^{n+1} : \sum t_i = 1, t_i \geq 0 \} \).

To see this, suppose \( x^k \to x \) in \( S \). Let \( x^k \equiv \sum_{i=0}^n t_i^k x_i \) with \( x \) defined similarly with \( t_i^k \) replaced with \( t_i \), \( x \equiv \sum_{i=0}^n t_i x_i \). Then

\[
x^k - x_0 = \sum_{i=0}^n t_i x_i - \sum_{i=0}^n t_i^k x_0 = \sum_{i=0}^n t_i (x_i - x_0)
\]

Thus

\[
x^k - x_0 = \sum_{i=1}^n t_i^k (x_i - x_0), \quad x - x_0 = \sum_{i=1}^n t_i (x_i - x_0)
\]

Say \( t_i^k \) fails to converge to \( t_i \) for all \( i \geq 1 \). Then there exists a subsequence, still denoted with superscript \( k \) such that for each \( i = 1, \cdots, n \), it follows that \( t_i^k \to s_i \) where \( s_i \geq 0 \) and some \( s_i \neq t_i \). But then, taking a limit, it follows that

\[
x - x_0 = \sum_{i=1}^n s_i (x_i - x_0) = \sum_{i=1}^n t_i (x_i - x_0)
\]
which contradicts independence of the $x_i - x_0$. It follows that for all $i \geq 1, t_i^k \to t_i$. Since they all sum to 1, this implies that also $t_i^0 \to t_0$. Thus the claim about continuity is verified.

Let $f(\cdot, \omega) : S \to S$ be continuous such that $\omega \to f(x, \omega)$ is measurable. When doing $f(\cdot, \omega)$ to a point $\sum_{i=0}^n t_i x_i$, one obtains another point of $S$ denoted as $\sum_{i=0}^n s_i(\omega) x_i$. The coefficients $s_i$ must be measurable functions. This is because

$$
\omega \to f(x, \omega) = \sum_{i=0}^n s_i(\omega) x_i
$$

and the left side is measurable so it follows the right is also. Now as noted above, the map which takes a point of $S$ to its coefficients is continuous and so each $s_i$ is measurable as a function of $\omega$. Note also that this new method of labeling does not contradict the original labels placed on the vertices obviously a measurable function. Note also that this new method of labeling does not contradict the original labels placed on the vertices $x_i$. This is because for $x_i, t_i = 1$ and all other $t_j = 0$ so the only ratio that is finite will be $s_i/t_i$. All others are $\infty$ by definition. As for the vertices which are on the $k^{th}$ face $[x_0, \cdots, x_k, \cdots, x_n]$, these will be labeled from the list $\{p_0, \cdots, p_k, \cdots, p_n\}$ because $t_k = 0$ for each of these and so $r_k(\omega) = \infty$. By the Spnern's lemma procedure described above, there are an odd number of simplices having value $\prod_{i \neq k} p_i$ on the $k^{th}$ face and an odd number of simplices in the triangulation of $S$ for which the product of the labels on their vertices equals $p_0 p_1 \cdots p_n = P_n$. We call this the value of the simplex. Thus if $[y_0, \cdots, y_n]$ is one of these simplices, and $p(y_i, \omega)$ is the label for $y_i$, a measurable function of $\omega$,

$$
\prod_{i=0}^n p(y_i, \omega) = \prod_{i=0}^n p_i \equiv P_n
$$

For $\omega \in F_k$, what is $r_k(\omega)$? Could it be larger than 1? $r_k(\omega)$ is certainly finite because at least some $t_j \neq 0$ since they sum to 1. Thus, if $r_k(\omega) > 1$, you would
have \(s_k(\omega) > t_k\). The \(s_j\) sum to 1 and so some \(s_j(\omega) < t_j\) since otherwise, the sum of the \(t_j\) equalling 1 would require the sum of the \(s_j\) to be larger than 1. Hence \(r_k(\omega)\) was not really the smallest so \(\omega \notin F_k\). Thus \(r_k(\omega) \leq 1\). Hence \(s_k(\omega) \leq t_k\).

Let \(S\) denote those simplices whose value is \(P_n\) for some \(\omega\). In other words, if \(\{y_0, \cdots, y_n\}\) are the vertices of one of these simplices, and

\[
y_s = \sum_{i=0}^n t^k_i x_i
\]

then for some \(\omega\), \(r_{k_i}(\omega) \leq r_j(\omega)\) for all \(j \neq k\) and \(\{k_0, \cdots, k_n\} = \{0, \cdots, n\}\). There are finitely many of these simplices, so \(S = \{S_1, \cdots, S_m\}\). Let \(F_1 \subseteq \Omega\) be defined by

\[
F_1 = \left\{ \omega : \prod_{i=0}^n p(y_i, \omega) = P_n \right\} = [y_0, \cdots, y_n] = S_1
\]

If \(F_1 = \Omega\), then stop. If not, let

\[
F_2 = \left\{ \omega \notin F_1 : \prod_{i=0}^n p(y_i, \omega) = P_n \right\} = [y_0, \cdots, y_n] = S_2
\]

Continue this way obtaining disjoint measurable sets \(F_j\) whose union is all of \(\Omega\). The union is \(\Omega\) because every \(\omega\) is associated with at least one of the \(S_i\). Now for \(\omega \in F_k\) and \([y_0, \cdots, y_n] = S_k\), it follows that \(\prod_{i=0}^n p(y_i, \omega) = P_n\). For \(\omega \in F_k\), let \(b(\omega)\) denote the barycenter of \(S_k\). Thus \(\omega \rightarrow b(\omega)\) is a measurable function, being constant on a measurable set. Thus we let \(b(\omega) = \sum_{i=1}^m \chi_{F_i}(\omega) b_i\) where \(b_i\) is the barycenter of \(S_i\).

Now do this for a sequence \(\varepsilon_k \rightarrow 0\) where \(b_k(\omega)\) is a barycenter as above. By Lemma 46.2.2 there exists \(x(\omega)\) such that \(\omega \rightarrow x(\omega)\) is measurable and a sequence \(\{b_{k(\omega)}\}_{k(\omega)=1}^\infty, \lim_{k(\omega) \rightarrow \infty} b_{k(\omega)}(\omega) = x(\omega)\). This \(x(\omega)\) is also a fixed point.

Consider this last claim. \(x(\omega) = \sum_{i=0}^n t_i(\omega) x_i\) and after applying \(f(\cdot, \omega)\), the result is \(\sum_{i=0}^n s_i(\omega) x_i\). Then \(b_{k(\omega)}(\omega) \in \sigma_k(\omega)\) where \(\sigma_k(\omega)\) is a simplex having vertices \(\{y^k_0(\omega), \cdots, y^k_n(\omega)\}\) and the value of \([y^k_0(\omega), \cdots, y^k_n(\omega)]\) is \(P_n\). Re ordering these if necessary, we can assume that the label for \(y^k_i(\omega) = p_i\) which implies that, as noted above,

\[
\frac{s_i(\omega)}{t_i(\omega)} \leq 1, \quad s_i(\omega) \leq t_i(\omega)
\]

the \(i^{th}\) coordinate of \(f(y^k_i(\omega), \omega)\) with respect to the original vertices of \(S\) decreases and each \(i\) is represented for \(i \in \{0, 1, \cdots, n\}\). Thus

\[
y^k_i(\omega) \rightarrow x(\omega)
\]

and so the \(i^{th}\) coordinate of \(y^k_i(\omega), t^k_i(\omega)\) must converge to \(t_i(\omega)\). Hence if the \(i^{th}\) coordinate of \(f(y^k_i(\omega), \omega)\) is denoted by \(s^k_i(\omega)\),

\[
s^k_i(\omega) \leq t^k_i(\omega)
\]
By continuity of \( f \), it follows that \( s_k^b (\omega) \to s_i (\omega) \). Thus the above inequality is preserved on taking \( k \to \infty \) and so

\[
0 \leq s_i (\omega) \leq t_i (\omega)
\]

this for each \( i \). But these \( s_i \) add to 1 as do the \( t_i \) and so in fact, \( s_i (\omega) = t_i (\omega) \) for each \( i \) and so \( f (x (\omega) , \omega) = x (\omega) \). This proves the following theorem which gives the existence of a measurable fixed point.

**Theorem 46.2.3** Let \( S \) be a simplex \([x_0, \ldots , x_n] \) such that \( \{x_i - x_0\}_{i=1}^n \) are independent. Also let \( f (\cdot , \omega) : S \to S \) be continuous for each \( \omega \) and \( \omega \to f (x , \omega) \) is measurable, meaning inverse images of sets open in \( S \) are in \( \mathcal{F} \) where \((\Omega , \mathcal{F})\) is a measurable space. Then there exists \( x (\omega) \in S \) such that \( \omega \to x (\omega) \) is measurable and \( f (x (\omega) , \omega) = x (\omega) \).

**Proof:** Let \( S \) be a large simplex containing \( K \) and let \( P \) be the projection map onto \( K \). Consider \( g (x , \omega) \equiv f (P (x) , \omega) \). Then \( g \) satisfies the necessary conditions for Theorem 46.2.3 and so there exists \( x (\omega) \in S \) such that \( \omega \to x (\omega) \) is measurable and \( g (x (\omega) , \omega) = x (\omega) \). But this says \( x (\omega) \in K \) and so \( g (x (\omega) , \omega) = f (x (\omega) , \omega) \).

**Much shorter proof**

The above gives a proof of a measurable fixed point as part of a proof of the Brouwer fixed point theorem directly but it is a lot easier if you simply begin with the existence of the Brouwer fixed point and show it is measurable. We sent the above to be considered for publication and the referee pointed this out. I totally missed it because I had forgotten about the Kuratowski selection theorem. The functions \( f : \Omega \times E \to \mathbb{R} \) in which \( f (\cdot , \omega) \) is continuous are called Carathéodory functions.

Kuratowski [2] which is presented next. It is Theorem 46.2.4 in this collection.

**Theorem 46.2.5** Let \( E \) be a compact metric space and let \((\Omega , \mathcal{F})\) be a measure space. Suppose \( \psi : E \times \Omega \to \mathbb{R} \) has the property that \( x \to \psi (x , \omega) \) is continuous and \( \omega \to \psi (x , \omega) \) is measurable. Then there exists a measurable function, \( f \) having values in \( E \) such that

\[
\psi (f (\omega) , \omega) = \sup_{x \in E} \psi (x , \omega).
\]

Furthermore, \( \omega \to \psi (f (\omega) , \omega) \) is measurable.

**Theorem 46.2.6** Let \( K \) be a closed convex bounded subset of \( \mathbb{R}^n \). Let \( f (\cdot , \omega) : K \to K \) be continuous for each \( \omega \) and \( \omega \to f (x , \omega) \) is measurable, meaning inverse images of sets open in \( K \) are in \( \mathcal{F} \) where \((\Omega , \mathcal{F})\) is a measurable space. Then there exists \( x (\omega) \in K \) such that \( \omega \to x (\omega) \) is measurable and \( f (x (\omega) , \omega) = x (\omega) \).
46.2. EXISTENCE OF MEASURABLE FIXED POINTS

Proof: Simply consider $E = K$ and $\psi (x, \omega) \equiv -|x - f(x, \omega)|$. It has a maximum $x(\omega)$ for each $\omega$ thanks to continuity of $f(\cdot, \omega)$. Thanks to the Brouwer fixed point theorem, this $x(\omega)$ must be a fixed point. By the above Kuratowski theorem, one of these $x(\omega)$ is measurable. Obviously, by continuity of $f(\cdot, \omega)$, $\omega \to f(x(\omega), \omega)$ is measurable.

### 46.2.4 Measurability Of Schauder Fixed Points

Now we consider the Schauder fixed point theorem. Let $\omega \to K(\omega)$ be a measurable multifunction having compact convex values. Here $K(\omega) \subseteq X$ a separable Banach space. Also assume $f(\cdot, \omega) : K(\omega) \to K(\omega)$, $\omega \to f(x(\omega), \omega)$ is measurable.

**Lemma 46.2.7** $K(\omega) \equiv \overline{f(K(\omega), \omega)}$ is compact valued and measurable.

Proof: It is given to be compact valued since $f$ is continuous. We need to check that it is measurable. For $U$ open, is

$$\left\{ \omega : \overline{f(K(\omega), \omega)} \cap U \neq \emptyset \right\}$$

a measurable set? Since $U$ is open and $x \to f(x, \omega)$ is continuous, the set is the same as

$$\left\{ \omega : f(K(\omega), \omega) \cap U \neq \emptyset \right\}$$

Let $\{k_j(\omega)\}$ be a dense countable subset of $K(\omega)$ where $\omega \to k_j(\omega)$ is measurable. Then this last set is of the form

$$\left\{ \omega : f(K(\omega), \omega) \cap U \neq \emptyset \right\} = \bigcup_{i=1}^{\infty} f(k_i(\cdot, \cdot)^{-1}(U) \text{ which is measurable.} \blacklozenge$$

Next we have the following approximation result.

**Lemma 46.2.8** Let $f$ be as above and $f(K(\omega), \omega) \subseteq K(\omega)$ for $K(\omega)$ compact and $\omega \to K(\omega)$ a measurable multifunction. For each $r > 0$ and $\omega$, there exists a finite set of points

$$\{ y_1(\omega), \ldots, y_n(\omega) \} \subseteq f(K(\omega), \omega), \omega \to y_i(\omega) \text{ measurable}$$

and continuous functions $\psi_i(\cdot, \omega)$ defined on $f(K(\omega), \omega)$ such that for $y \in f(K(\omega), \omega)$,

$$\sum_{i=1}^{n(\omega)} \psi_i(y, \omega) = 1, \quad (46.2.5)$$

$$\psi_i(y, \omega) = 0 \text{ if } y \notin B(y_i(\omega), r), \quad \psi_i(y, \omega) > 0 \text{ if } y \in B(y_i(\omega), r).$$
CHAPTER 46. MULTIFUNCTIONS AND THEIR MEASURABILITY

If

\[ f_r(x, \omega) \equiv \sum_{i=1}^{n(\omega)} y_i(\omega) \psi_i(f(x, \omega), \omega), \]  

(46.2.6)

then whenever \( x \in K(\omega) \),

\[ \|f(x, \omega) - f_r(x, \omega)\| \leq r. \]

**Proof:** Using the compactness of \( \overline{f(K(\omega), \omega)} \), Proposition 46.1.6 says there exist measurable functions \( y_i(\omega) \)

\[ \{y_1(\omega), \cdots, y_n(\omega)\} \subseteq \overline{f(K(\omega), \omega)} \subseteq K(\omega) \]

such that

\[ \{B(y_i(\omega), r)\}_{i=1}^{n} \]

covers \( \overline{f(K, \omega)} \). Let

\[ \phi_i(y, \omega) \equiv (r - \|y - y_i(\omega)\|)^+ \]

Thus \( \phi_i \) is continuous in \( y \) and measurable in \( \omega \) for fixed \( y \). Also \( \phi_i(y, \omega) > 0 \) if \( y \in B(y_i(\omega), r) \) and \( \phi_i(y, \omega) = 0 \) if \( y \notin B(y_i(\omega), r) \). For \( y \in \overline{f(K, \omega)} \), let

\[ \psi_i(y, \omega) \equiv \phi_i(y, \omega) \left( \sum_{j=1}^{n(\omega)} \phi_j(y, \omega) \right)^{-1}. \]

From the formula, \( \omega \rightarrow \psi_i(y, \omega) \) is measurable. Also 46.2.6 is satisfied. Indeed the denominator is not zero because \( y \) is in one of the \( B(y_i(\omega), r) \). Thus it is obvious that the sum of these equals 1 on \( f(K, \omega) \). Now let \( f_r \) be given by 46.2.6 for \( x \in K \). For such \( x \),

\[ f(x, \omega) - f_r(x, \omega) = \sum_{i=1}^{n} (f(x, \omega) - y_i(\omega)) \psi_i(f(x, \omega), \omega) \]

Thus

\[ f(x, \omega) - f_r(x, \omega) = \sum_{\{i: f(x) \in B(y_i(\omega), r)\}} (f(x, \omega) - y_i(\omega)) \psi_i(f(x, \omega), \omega) \]

\[ + \sum_{\{i: f(x, \omega) \notin B(y_i(\omega), r)\}} (f(x, \omega) - y_i(\omega)) \psi_i(f(x, \omega), \omega) \]

\[ = \sum_{\{i: f(x, \omega) - y_i(\omega) \in B(0, r)\}} (f(x, \omega) - y_i(\omega)) \psi_i(f(x, \omega), \omega) \epsilon B(0, r) \]

\[ \sum_{\{i: f(x, \omega) - y_i(\omega) \in B(0, r)\}} (f(x, \omega) - y_i(\omega)) \psi_i(f(x, \omega), \omega) \epsilon B(0, r) \]
46.2. EXISTENCE OF MEASURABLE FIXED POINTS

because \( 0 \in B(0, r) \), \( B(0, r) \) is convex, and \( f(x, \omega) - f_r(x, \omega) \) is a convex combination of vectors in \( B(0, r) \).

We think of \( f_r(\cdot, \omega) \) as an approximation to \( f(\cdot, \omega) \). In fact it is uniformly within \( r \) of \( f(\cdot, \omega) \) on \( K(\omega) \). The next lemma shows that this \( f_r(\cdot, \omega) \) has a fixed point. This is the main result and comes from the Brouwer fixed point theorem in \( \mathbb{R}^n \). It is an approximate fixed point.

**Lemma 46.2.9** For each \( r > 0 \), there exists \( x_r(\omega) \in \text{convex hull of } f(K(\omega), \omega) \subseteq K(\omega) \) such that

\[
      f_r(x_r(\omega), \omega) = x_r(\omega), \quad \| f_r(x, \omega) - f(x, \omega) \| < r \quad \text{for all } x \in K(\omega)
\]

and \( \omega \to x_r(\omega) \) is measurable.

**Proof:** The upper limit in the sum of the above lemma \( n(\omega) \) is a measurable function. One can partition the measure space according to the value of \( n(\omega) \). This gives a countable set of disjoint measurable subsets \( \{ \Omega_n \}_{n=1}^\infty \) in the partition such that on the measurable set \( \Omega_n \), \( n(\omega) = n \). Specializing to the measurable space consisting of \( \Omega_n \), we will assume here that \( n(\omega) = n \) and show that there exists a measurable fixed point \( x_r(\omega) \in K(\omega) \) for \( \omega \in \Omega_n \). Then the result follows by letting \( x_r(\omega) \) be that which has been obtained on \( \Omega_n \). Thus, from now on, simply denote as \( n \) the upper limit and let \( \omega \in \Omega_n \). If \( f_r(x_r, \omega) = x_r \) and

\[
      x_r = \sum_{i=1}^n a_i y_i(\omega)
\]

for \( \sum_{i=1}^n a_i = 1 \) and the \( y_i \) described in the above lemma, we need

\[
      f_r(x_r, \omega) = \sum_{i=1}^n y_i(\omega) \psi_i(f(x_r, \omega), \omega) = \sum_{j=1}^n y_j(\omega) \psi_j(f\left(\sum_{i=1}^n a_i y_i, \omega\right), \omega) = \sum_{j=1}^n a_j y_j(\omega) = x_r.
\]

Also, if this is satisfied, then we have the desired fixed point.

This will be satisfied if for each \( j = 1, \cdots, n \),

\[
      a_j = \psi_j\left(f\left(\sum_{i=1}^n a_i y_i, \omega\right), \omega\right) \quad (46.2.7)
\]

so, let

\[
      \Sigma_{n-1} = \left\{ a \in \mathbb{R}^n : \sum_{i=1}^n a_i = 1, \ a_i \geq 0 \right\}
\]

and let \( h(\cdot, \omega) : \Sigma_{n-1} \to \Sigma_{n-1} \) be given by

\[
      h(a, \omega)_j = \psi_j\left(f\left(\sum_{i=1}^n a_i y_i, \omega\right), \omega\right)
\]
Can we obtain a fixed point \( a (\omega) \) such that \( \omega \to a (\omega) \) is measurable? Since \( h (., \omega) \) is a continuous function of \( a \) and \( \omega \to h (x, \omega) \) is measurable, such a measurable fixed point exists thanks to Theorem 46.2.6 above. Then \( x_r (\omega) = \sum_{i=1}^n a_i (\omega) y_i (\omega) \) so \( x_r \) is measurable.

The following is the Schauder fixed point theorem for measurable fixed points.

**Theorem 46.2.10** Let \( \omega \to K (\omega) \) be a measurable multifunction which has convex and compact values in a separable Banach space. Let \( f (., \omega) : K (\omega) \to K (\omega) \) be continuous and \( \omega \to f (x, \omega) \) is measurable. Then \( f (., \omega) \) has a fixed point \( x (\omega) \) such that \( \omega \to x (\omega) \) is measurable.

**Proof:** Recall that \( f (x_r (\omega), \omega) - f_r (x_r (\omega), \omega) \in B (0, r) \) and \( f_r (x_r (\omega), \omega) = x_r (\omega) \) with \( x_r (\omega) \in \text{convex hull of } f (K (\omega), \omega) \subseteq K (\omega) \). Here \( x_r \) is measurable. By Lemma 46.2.2 there is a measurable function \( x (\omega) \) which equals the weak \( \lim_{r(\omega) \to 0} x_r (\omega) \). However, since \( f (K (\omega), \omega) \) is compact, there is a subsequence still denoted with \( r (\omega) \) such that \( f (x_r (\omega), \omega) \) converges strongly to some \( x \in \overline{f (K (\omega), \omega)} \). It follows that \( f_r (\omega) (x_r (\omega), \omega) \) also converges to \( x \) strongly. But this equals \( x_r (\omega) \) which shows that \( x_r (\omega) \) converges strongly to the measurable \( x (\omega) \). Therefore,

\[
\begin{align*}
    f (x (\omega), \omega) &= \lim_{r (\omega) \to 0} f (x_r (\omega), \omega) \\
    &= \lim_{r (\omega) \to 0} f_r (x_r (\omega), \omega) \\
    &= \lim_{r (\omega) \to 0} x_r (\omega) = x (\omega).
\end{align*}
\]

As a special case of the above, here is a corollary which generalizes the earlier result on the Brouwer fixed point theorem.

**Corollary 46.2.11** Let \( (\Omega, F) \) be a measurable space and let \( K (\omega) \) be a convex and compact set in \( \mathbb{R}^n \) and \( \omega \to K (\omega) \) is a measurable multifunction. Also let \( f (., \omega) : K (\omega) \to K (\omega) \) be continuous and for fixed \( x \in \mathbb{R}^n \), \( \omega \to f (x, \omega) \) is measurable. Then there exists a fixed point \( x (\omega) \) for \( f (., \omega) \) such that \( \omega \to x (\omega) \) is measurable.

Note that in all of these considerations, there is no loss of generality in assuming \( f (., \omega) \) is defined on the whole space \( X \) thanks to the theorem which says that a continuous function defined on a convex closed set can be extended to a continuous function defined on the whole space.

In the case of a single set, the following corollary is also obtained.

**Corollary 46.2.12** Let \( X \) be a Banach space and let \( K \) be a compact convex subset. Let \( f : K \times \Omega \to K \) satisfy

\[
\begin{align*}
    x &\to f (x, \omega) \text{ is continuous} \\
    \omega &\to f (x, \omega) \text{ is measurable}
\end{align*}
\]

Then \( f (., \omega) \) has a fixed point \( x (\omega) \) such that \( \omega \to x (\omega) \) is measurable.
46.3. A Set Valued Browder Lemma With Measurability

Proof: The set $K$ has a countable dense subset $\{k_i\}$. You could consider $Y$ as the closure in $X$ of the span of these $k_i$. Thus $Y$ is a separable Banach space which contains $K$. Now apply the above result. ■

If $X$ is only a normed linear space, you could just consider its completion and apply the above result. Since $K$ is compact, it is automatically complete with respect to the norm on $X$.

46.3 A Set Valued Browder Lemma With Measurability

A simple application is a measurable version of the Browder lemma which is also valid for upper semicontinuous set valued maps. In what follows, we do not assume that $A(\cdot, \omega)$ is a set valued measurable multifunction, only that it has a measurable selection which is a weaker assumption. First is a general result on upper set valued maps $u, \omega$ valued maps ($\mathcal{K}$ being in $X$ is compact, it is automatically complete with respect to the norm on $X$.

**Theorem 46.3.1** Let $V$ be a reflexive separable Banach space. Suppose $\omega \rightarrow A(u, \omega)$ has a measurable selection in $V'$, for each $u \in V$ and $\omega \in \Omega$ the set $A(u, \omega)$ is closed and convex in $V'$ and $u \rightarrow A(u, \omega)$ is bounded. Also, suppose $u \rightarrow A(u, \omega)$ is upper-semicontinuous from the strong topology of $V$ to the weak topology of $V'$. That is, if $u_n \rightarrow u$ in $V$ strongly, then if $O$ is a weakly open set containing $A(u, \omega)$, it follows that $A(u_n, \omega) \in O$ for all $n$ large enough. Conclusion: Then, whenever $\omega \rightarrow y(\omega)$ is measurable into $V$ there is a measurable selection for $\omega \rightarrow A(u(\omega), \omega)$ into $V'$.

Proof: Let $\omega \rightarrow u(\omega)$ be measurable into $V$, and let $u_n(\omega) \rightarrow u(\omega)$ in $V$ where $u_n$ is a simple function

$$u_n(\omega) = \sum_{k=1}^{m_n} c_{k}^{n}X_{E_{k}^{n}}(\omega), \text{ the } E_{k}^{n} \text{ disjoint, } \Omega = \bigcup_{k} E_{k}^{n},$$

each $c_{k}^{n}$ being in $V$. We can assume that $\|u_n(\omega)\| \leq 2\|u(\omega)\|$ for all $\omega$. Then, by assumption, there is a measurable selection for $\omega \rightarrow A(c_{k}^{n}, \omega)$ denoted as $\omega \rightarrow y_{k}^{n}(\omega)$. Thus, $\omega \rightarrow y_{k}^{n}(\omega)$ is measurable into $V'$ and $y_{k}^{n}(\omega) \in A(c_{k}^{n}, \omega)$ for all $\omega \in \Omega$.

Consider now

$$y^{n}(\omega) = \sum_{k=1}^{m_n} y_{k}^{n}(\omega)X_{E_{k}^{n}}(\omega).$$

It is measurable and for $\omega \in E_{k}^{n}$ it equals $y_{k}^{n}(\omega) \in A(c_{k}^{n}, \omega) = A(u_n(\omega), \omega)$. Thus, $y^{n}$ is a measurable selection of $\omega \rightarrow A(u_n(\omega), \omega)$. By the assumption $A(\cdot, \omega)$ is bounded, for each $\omega$ these $y^{n}(\omega)$ lie in a bounded subset of $V'$. The bound might depend on $\omega$ of course. It follows now from Lemma that there is a subsequence $\{y^{n}(\omega)\}$ that converges weakly to $y(\omega)$, where $\omega \rightarrow y(\omega)$ is measurable. But,

$$y^{n}(\omega) \in A(u_n(\omega), \omega)$$
is a convex closed set for which \( u \to A(u, \omega) \) is upper-semicontinuous and \( u_n(\omega) \to u \), hence, \( y(\omega) \in A(u(\omega), \omega) \). This is the claimed measurable selection. ■

The next lemma is about the projection map onto a set valued map whose values are closed convex sets.

**Lemma 46.3.2** Let \( \omega \to K(\omega) \) be measurable into \( \mathbb{R}^n \) where \( K(\omega) \) is closed and convex. Then \( \omega \to P_{K(\omega)}u(\omega) \) is also measurable into \( \mathbb{R}^n \) if \( \omega \to u(\omega) \) is measurable. Here \( P_{K(\omega)} \) is the projection map giving the closest point.

**Proof:** It follows from standard results on measurable multi-functions also in Theorem 46.1.2 above that there is a countable collection \( \{w_n(\omega)\} \), \( \omega \to w_n(\omega) \) being measurable and \( w_n(\omega) \in K(\omega) \) for each \( \omega \) such that for each \( \omega \), \( K(\omega) = \bigcup_n w_n(\omega) \). Let

\[
d_n(\omega) = \min \{\|u(\omega) - w_k(\omega)\|, k \leq n\}
\]

Let \( u_1(\omega) \equiv w_1(\omega) \). Let

\[
u_2(\omega) = u_1(\omega)
\]

on the set

\[
\{\omega : \|u(\omega) - w_1(\omega)\| < \{\|u(\omega) - w_2(\omega)\|\}\}
\]

and

\[
u_2(\omega) \equiv w_2(\omega)
\]

off the above set.

Thus \( \|u_2(\omega) - u(\omega)\| = d_2 \). Let

\[
u_3(\omega) = \begin{cases} u_1(\omega) & \text{on } \left\{ \omega : \|u(\omega) - w_1(\omega)\| < \|u(\omega) - w_j(\omega)\|, j = 2, 3 \right\} \\ S_1 \end{cases}
\]

\[
u_3(\omega) = \begin{cases} u_2(\omega) & \text{on } S_1 \cap \left\{ \omega : \|u(\omega) - w_1(\omega)\| < \|u(\omega) - w_j(\omega)\|, j = 3 \right\} \\ S_2 \end{cases}
\]

\[
u_3(\omega) = w_3(\omega) \text{ on the remainder of } \Omega
\]

Thus \( \|u_3(\omega) - u(\omega)\| = d_3 \). Continue this way, obtaining \( u_n(\omega) \) such that

\[
\|u_n(\omega) - u(\omega)\| = d_n(\omega)
\]

and \( u_n(\omega) \in K(\omega) \) with \( u_n \) measurable. Thus, in effect one picks the closest of all the \( w_k(\omega) \) for \( k \leq n \) as the value of \( u_n(\omega) \) and \( u_n \) is measurable and by density in \( K(\omega) \) of \( \{w_n(\omega)\} \) for each \( \omega \), \( \{u_n(\omega)\} \) must be a minimizing sequence for

\[
\lambda(\omega) \equiv \inf \{\|u(\omega) - z\| : z \in K(\omega)\}
\]

Then it follows that \( u_n(\omega) \to P_{K(\omega)}u(\omega) \) weakly in \( \mathbb{R}^n \). Here is why: Suppose it fails to converge to \( P_{K(\omega)}u(\omega) \). Since it is minimizing, it is a bounded sequence. Thus there would be a subsequence, still denoted as \( u_n(\omega) \) which converges to some \( q(\omega) \neq P_{K(\omega)}u(\omega) \). Then

\[
\lambda(\omega) = \lim_{n \to \infty} \|u(\omega) - u_n(\omega)\| \geq \|u(\omega) - q(\omega)\|
\]
because convex and lower semicontinuous is weakly lower semicontinuous. But this implies \( q(\omega) = P_{K(\omega)}(u(\omega)) \) because the projection map is well defined thanks to strict convexity of the norm used. This is a contradiction. Hence \( P_{K(\omega)} u(\omega) = \lim_{n \to \infty} u_n(\omega) \) and so is a measurable function. It follows that \( \omega \to P_{K(\omega)}(u(\omega), \omega) \) is measurable into \( \mathbb{R}^n \). ■

One way to prove the following Theorem in simpler cases is to use a measurable version of the Kakutani fixed point theorem. It is done this way in [22] without the dependence on \( \omega \). See also [22] for a measurable version of this fixed point theorem. However, one can also prove it by a generalization of the proof Browder gave for a single valued case and this is summarized here. We want to include the case where \( A \) is a sum of two set valued operators and this involves careful consideration of the details. Such a situation occurs when one considers operators which are a sum, one dependent on the boundary of a region, and the other from a partial differential inclusion. Also, we will need to consider finite dimensional subspaces which depend on \( \omega \) which further complicates the considerations.

**Theorem 46.3.3 Assumptions:** Let \( B(\cdot, \omega) : V \to \mathcal{P}(V')\), \( C(\cdot, \omega) : V \to \mathcal{P}(V') \) for \( V \) a separable Banach space. Suppose that \( \omega \to B(x, \omega), \omega \to C(x, \omega) \) each has a measurable selection and \( x \to B(x, \omega), x \to C(x, \omega) \) each is upper semicontinuous from strong to weak topologies. Also let \( E(\omega) \) be an \( n \) dimensional subspace of \( V \) which has a basis \( \{b_1(\omega), \ldots, b_n(\omega)\} \) each of which is a measurable function into \( V \), and that \( K(\omega) \subseteq E(\omega) \) where \( K(\omega) \) is a measurable multifunction which has convex closed bounded values. Also let \( y(\omega) \) be given, a measurable function into \( V' \). **Conclusion:** There exist measurable functions \( w_B(\omega), w_C(\omega) \) and \( x(\omega) \) with \( w_B(\omega) \in B(x(\omega), \omega), w_C(\omega) \in C(x(\omega), \omega) \), and \( x(\omega) \in K(\omega) \) such that for all \( z \in K(\omega) \),

\[
\langle y(\omega) - (w_B(\omega) + w_C(\omega)), z - x(\omega) \rangle \leq 0
\]

**Proof:** The argument will refer to the following commutative diagram.

\[
\begin{array}{ccc}
E(\omega)' & \xrightarrow{\theta(\omega)^*} & \mathbb{R}^n \\
i(\omega)^* A(\cdot, \omega) & & \uparrow \theta(\omega)^* \quad \uparrow \theta(\omega)^* A(\theta(\omega) \cdot, \omega) \\
E(\omega) & \xleftarrow{\theta(\omega)} & \mathbb{R}^n
\end{array}
\]

where \( A(\cdot, \omega) \) will be either \( B(\cdot, \omega) \) or \( C(\cdot, \omega) \). Here \( \theta(\omega) e_i = b_i(\omega) \) and extended linearly. Then it is clear that \( \theta(\omega) \) maps measurable functions to measurable functions.

What of \( \theta(\omega)^{-1} ? \) Is \( \omega \to \theta(\omega)^{-1} h(\omega) \) measurable into \( \mathbb{R}^n \) whenever \( h \) is measurable into \( V' \)? Let \( h(\omega) \) have values in \( E(\omega) \) and be measurable into \( V' \). Thus

\[
h(\omega) = \sum_i a_i(\omega) b_i(\omega)
\]

The question reduces to whether the \( a_i \) are measurable. To see that these are measurable, consider first \( \|h(\omega)\| < M \) for all \( \omega \). Let \( S_r = \{ \omega : \inf_{|n| > r} \| \sum_i a_i b_i(\omega) \| > M \} \).

Thus this is a measurable set. Also every \( \omega \) is in some \( S_r \) because if not, you could...
get a sequence \(|a^n| \to \infty\) and yet \(||\sum_i a_i b_i(\omega)|| \leq M\). But then, dividing by \(|a^n|\) and taking a suitable subsequence, one can obtain \(\sum_i a_i b_i(\omega) = 0\) for some \(|a| = 1\). Also the \(S_r\) are increasing in \(r\). Now for \(\omega \in S_r\), define \(\Phi(a, \omega) = -||\sum_i a_i b_i(\omega) - h(\omega)||\) where we will let \(|a| \leq r + 1\). Since \(\{b_i(\omega)\}\) is a basis, there exists \(a(\omega)\) such that \(\Phi(a(\omega), \omega) = 0\). This \(a\) must satisfy \(|a| \leq r + 1\) because if not, then you would have \(||\sum_i a_i b_i(\omega)|| \geq M\) since \(\omega \in S_r\). But \(||\sum_i a_i b_i(\omega)|| = ||h(\omega)|| < M\). Thus the maximum of \(a \to \Phi(a, \omega)\) occurs on the compact set \(|a| \leq r + 1\) and is 0. By Kuratowski’s theorem, we have \(\omega \to a(\omega)\) is measurable where \(h(\omega) = \sum_i a_i(\omega) b_i(\omega)\) on \(S_r\). Thus, since every \(\omega\) is in some \(S_r\), we must have \(\omega \to a_i(\omega)\) is measurable in case \(||h(\omega)|| \leq M\) for all \(\omega\). In the general case, let \(a_i^m(\omega)\) be the measurable function which goes with \(h_m(\omega)\) where \(h_m(\omega)\) is given by a truncation of \(h\) so that \(||h_m(\omega)|| \leq m\). For each \(\omega, h_m(\omega)\) is eventually smaller than \(m\), so \(h(\omega) = h_m(\omega)\). Thus if \(a_i^m(\omega)\) go with \(h_m(\omega)\), these are constant for all \(m\) large enough. Thus letting \(a_i(\omega) = \lim_{m \to \infty} a_i^m(\omega)\), \(a_i\) is measurable and

\[
h(\omega) = \lim_{m \to \infty} h_m(\omega) = \lim_{m \to \infty} \sum_i a_i^m(\omega) b_i(\omega) = \sum_i a_i(\omega) b_i(\omega)
\]

and so the \(a_i\) are indeed measurable. Thus the \(\theta(\omega)^{-1} h(\omega) = \sum_i a_i(\omega) e_i\) which shows that \(\theta(\omega)^{-1}\) does map measurable functions to measurable functions. In particular, \(\theta(\omega)^{-1} K(\omega)\) is indeed a closed, bounded, convex, and measurable multifunction which can be seen by considering a sequence \(\{k_i(\omega)\}_{i=1}^\infty\) of measurable functions dense in \(K(\omega)\).

Define for \(A = B\) or \(C\),

\[
\hat{A}(\cdot, \omega) = \theta(\omega)^* i(\omega)^* A(\theta(\omega) \cdot, \omega), y(\omega) = \theta(\omega)^* i(\omega)^* y(\omega).
\]

We claim that \(\omega \to \hat{A}(x, \omega)\) has a measurable selection and for fixed \(\omega\) this is upper semicontinuous in \(x\). The second condition for fixed \(\omega\) is obvious. Consider the first. It was shown above that \(\theta(\omega)^* x\) is measurable into \(V\). Thus, by Theorem 46.3.1, it follows that \(\omega \to A(\theta(\omega)^* x, \omega)\) has a measurable selection into \(V'\). Therefore, it suffices to show that if \(z(\omega)\) is measurable into \(V'\) then \(\theta(\omega)^* i(\omega)^* z(\omega)\) is measurable into \(\mathbb{R}^n\). Let \(w \in \mathbb{R}^n\). Then

\[
(\theta(\omega)^* i(\omega)^* z(\omega), w)_{\mathbb{R}^n} = \langle i(\omega)^* z(\omega), \theta(\omega) w \rangle
\]

\[
= \langle i(\omega)^* z(\omega), \sum_i w_i b_i(\omega) \rangle
\]

\[
= \langle z(\omega), \sum_i w_i b_i(\omega) \rangle_{V', V}
\]

which is measurable. By the Pettis theorem, \(\omega \to \theta(\omega)^* i(\omega)^* z(\omega)\) is measurable. Thus \(\hat{A}(\cdot, \omega)\) has the properties claimed.

Now tile \(\mathbb{R}^n\) with \(n\) simplices, each having diameter less than \(\varepsilon < 1\), the set of simplices being locally finite. Define for \(A = B\) or \(C\) the single valued function \(A_\varepsilon\)
46.3. A SET VALUED BROUWER LEMMA WITH MEASURABILITY

on all of \( \mathbb{R}^n \) by the following rule. If

\[
x \in [x_0, \ldots, x_n],
\]

so \( x = \sum_{i=0}^{n} t_i x_i, t_i \geq 0, \sum_i t_i = 1 \), then let \( \hat{A}_\varepsilon (x_k, \omega) \) be a measurable selection from \( \hat{A}(x_k, \omega) \) for each \( x_k \) a vertex of the simplex. However, we chose \( \hat{A}_\varepsilon (x_k, \omega) \) in the obvious way. It is \( \theta(\omega)^* i(\omega)^* w_k^B(\omega) \) where \( w_k^B(\omega) \) is a measurable selection of \( B(\theta(\omega) x_k, \omega) \), measurable into \( V' \) when \( A = B \) and \( \theta(\omega)^* i(\omega)^* w_k^C(\omega) \) where \( w_k^C(\omega) \) is a measurable selection of \( C(\theta(\omega) x_k, \omega), \) measurable into \( V' \) when \( A = C \). Then

\[
\hat{B}_\varepsilon (x_k, \omega) = \theta(\omega)^* i(\omega)^* w_k^B(\omega), \omega \rightarrow w_k^B(\omega) \text{ measurable}
\]

with a similar definition holding for \( \hat{C}_\varepsilon \).

Define single valued maps as follows. For \( x = \sum_{i=0}^{n} t_i x_i, \sum_i t_i = 1, t_i \geq 0, [x_0, \ldots, x_n] \) in the tiling,

\[
\hat{B}_\varepsilon (x, \omega) = \sum_{k=0}^{n} t_k \left( \hat{B}_\varepsilon (x_k, \omega) \right), \hat{C}_\varepsilon (x, \omega) = \sum_{k=0}^{n} t_k \left( \hat{C}_\varepsilon (x_k, \omega) \right)
\]

\[
\hat{A}_\varepsilon (x, \omega) \equiv \hat{B}_\varepsilon (x, \omega) + \hat{C}_\varepsilon (x, \omega)
\]

Thus \( \hat{A}_\varepsilon (\cdot, \omega) \) is a continuous map defined on \( \mathbb{R}^n \) thanks to the local finiteness of the tiling, and \( \omega \rightarrow \hat{A}_\varepsilon (x, \omega) \) is measurable.

Let \( P_{\theta(\omega)^{-1} K(\omega)} \) denote the projection onto the closed convex set \( \theta(\omega)^{-1} K(\omega) \). This is a continuous mapping by Hilbert space considerations. Therefore,

\[
x \rightarrow P_{\theta(\omega)^{-1} K(\omega)} \left( y(\omega) - \hat{A}_\varepsilon (x, \omega) + x \right)
\]

is continuous and by Lemma 10.3.4, \( \omega \rightarrow P_{\theta(\omega)^{-1} K(\omega)} \left( y(\omega) - \hat{A}_\varepsilon (x, \omega) + x \right) \) is measurable, and for each \( \omega \), this function of \( x \) maps into \( \theta(\omega)^{-1} K(\omega) \). Therefore by Corollary 10.3.11, there exists a fixed point \( x_\varepsilon (\omega) \in \theta(\omega)^{-1} K(\omega) \) such that

\[
x_\varepsilon (\omega) = y(\omega) - \left( \hat{B}_\varepsilon (x_\varepsilon (\omega), \omega) + \hat{C}_\varepsilon (x_\varepsilon (\omega), \omega) \right) + x_\varepsilon (\omega)
\]

This requires

\[
\left( y(\omega) - \left( \hat{B}_\varepsilon (x_\varepsilon (\omega), \omega) + \hat{C}_\varepsilon (x_\varepsilon (\omega), \omega) \right) , z - x_\varepsilon (\omega) \right) \in \mathbb{R}^n \leq 0
\]

for all \( z \in \theta(\omega)^{-1} K(\omega) \). Note that this implies \( \omega \rightarrow \hat{B}_\varepsilon (x_\varepsilon (\omega), \omega), \hat{C}_\varepsilon (x_\varepsilon (\omega), \omega) \) are measurable because of the continuity in first argument and measurability of \( \omega \rightarrow x_\varepsilon (\omega) \).

We have

\[
x_\varepsilon (\omega) = \sum_{k=0}^{n} t_k^\varepsilon (\omega) x_k^\varepsilon (\omega)
\]
where the \( x^k_\varepsilon (\omega) \) are vertices of the tiling corresponding to \( \varepsilon \).

**Claim:** The vertices \( x^k_\varepsilon (\omega) \) and coordinates \( t^k_\varepsilon (\omega) \) can be considered measurable.

**Proof of claim:** Let the simplices in the tiling be \( \{ \sigma_k \}_{k=1}^\infty \) and let the vertices of simplices in the tiling be \( \{ z_j \}_{j=1}^\infty \). Say the vertices of \( \sigma_k \) are \( \{ x^k_0, \ldots, x^k_n \} \) listed in the order of the given enumeration of vertices of simplices in the tiling. Let \( F_k \) and \( E_k \) be defined as follows.

\[
F_k \equiv x^{-1}_\varepsilon (\sigma_k), \quad E_1 \equiv F_1, \ldots, E_k \equiv F_k \setminus \bigcup_{i=1}^k F_i
\]

Then each \( \omega \) is in exactly one of these measurable sets \( E_k \) which partition \( \Omega \). For \( \omega \in E_k, x_\varepsilon (\omega) \in \sigma_k (\omega) \). Thus \( \sigma_k (\omega) \) is the first simplex which contains \( x_\varepsilon (\omega) \) and the ordered vertices of this simplex are constant on the measurable set \( E_k \).

These vertices are determined this way on a measurable set \( \omega \) are similar for \( \hat{x}_\varepsilon (\omega) \) be measurable. This shows the claim.

Recall [46.3.1] Let \( W^\varepsilon (\omega) \) be defined as follows.

\[
W^\varepsilon (\omega) \equiv \left( \frac{t^k_\varepsilon (\omega)}{x_\varepsilon (\omega)}, \ldots, \frac{t^n_\varepsilon (\omega)}{x_n(\omega)}, \frac{x^0_\varepsilon (\omega)}{x_\varepsilon (\omega)}, \ldots, \frac{x^n_\varepsilon (\omega)}{x_\varepsilon (\omega)} \right)
\]

This is in \( \mathbb{R}^{2(n+1)} \times (V')^{2(n+1)} \). Then by Theorem [46.2.2], since \( W^\varepsilon (\omega) \) is bounded in a reflexive separable Banach space, there is a subsequence \( \varepsilon (\omega) \to 0 \) such that \( W^\varepsilon (\omega) \to W(\omega) \) weakly and given by

\[
W(\omega) \equiv \left( \frac{t_0(\omega)}{x(\omega)}, \ldots, \frac{t_n(\omega)}{x_n(\omega)}, \frac{x_\varepsilon (\omega)}{x(\omega)}, \ldots, \frac{x^n_\varepsilon (\omega)}{x_n(\omega)} \right)
\]

where each of these components is measurable into the appropriate space. Of course, in the finite dimensional components, the convergence is strong because strong and weak convergence is the same in finite dimensions. Since the diameter of the simplex containing the fixed point \( x_\varepsilon (\omega) \) converges to 0, it follows that

\[
\lim_{\varepsilon (\omega) \to 0} x^\varepsilon (\omega) (\omega) = x(\omega)
\]

By upper semicontinuity, for \( A = B, C \), it follows that \( \hat{A} \left( x^\varepsilon (\omega) (\omega), \omega \right) \subseteq \hat{A} (x(\omega), \omega) + B(0, r) \) for all \( \varepsilon (\omega) \) small enough. Since, by the construction,

\[
\hat{B}_{\varepsilon (\omega)} \left( x^\varepsilon (\omega) (\omega), \omega \right) = \theta (\omega)^* i(\omega)^* w^\varepsilon (\omega)B(\omega) \subseteq \hat{B} \left( x^\varepsilon (\omega) (\omega), \omega \right),
\]

a similar statement for \( \hat{C} \), it follows that \( \hat{B}_{\varepsilon (\omega)} \left( x^\varepsilon (\omega) (\omega), \omega \right) = \theta(\omega)^* i(\omega)^* w^\varepsilon (\omega)B(\omega) \) is within \( r \) of the closed convex bounded set \( \hat{B} (x(\omega), \omega) \) whenever \( \varepsilon (\omega) \) is small enough, similar for \( \hat{C} \)

Thus

\[
\theta (\omega)^* i(\omega)^* w^B_k (\omega) \subseteq \hat{B} (x(\omega), \omega)
\]
similar for \( \hat{C} \). Since this last set is convex, it follows that
\[
\theta(\omega)^* i(\omega)^* \sum_{k} t_k(\omega) w_k^B(\omega) \in \hat{B}(x(\omega), \omega)
\]
similar for \( \hat{C} \).

Now recall (46.3.10) and the inequality (46.3.11) which imply that for \( z \in \theta(\omega)^{-1} K(\omega) \),
\[
\left( y(\omega) - \theta(\omega)^* i(\omega)^* \left( \sum_{k=0}^{n} t_k(\omega) w_k^B(\omega) + \sum_{k=0}^{n} t_k(\omega) w_k^C(\omega) \right), z - x(\omega) \right) 
\]
\[
= \lim_{\varepsilon(\omega) \to 0} \left( y(\omega) - \left( \sum_{k=0}^{n} t_k^\varepsilon(\omega) \theta(\omega)^* i(\omega)^* w_k^\varepsilon(\omega)B(\omega) + \sum_{k=0}^{n} t_k^\varepsilon(\omega) \theta(\omega)^* i(\omega)^* w_k^\varepsilon(\omega)C(\omega) \right), z - x_\varepsilon(\omega)(\omega) \right) 
\]
\[
= \lim_{\varepsilon(\omega) \to 0} \left( y(\omega) - \left( \sum_{k=0}^{n} t_k^\varepsilon(\omega) \theta(\omega)^* i(\omega)^* \theta(\omega)^* i(\omega)^* w_k^\varepsilon(\omega)B(\omega) + \sum_{k=0}^{n} t_k^\varepsilon(\omega) \theta(\omega)^* i(\omega)^* w_k^\varepsilon(\omega)C(\omega) \right), z - x_\varepsilon(\omega)(\omega) \right) 
\]
Recall (46.3.10) and (46.3.11) which imply from the above conventions that the sum in the above equals \( \hat{B}_\varepsilon(\omega)(x_\varepsilon(\omega), \omega) + \hat{C}_\varepsilon(\omega)(x_\varepsilon(\omega), \omega) \). Thus the above equals
\[
\lim_{\varepsilon(\omega) \to 0} \left( y(\omega) - \left( \hat{B}_\varepsilon(\omega)(x_\varepsilon(\omega), \omega) + \hat{C}_\varepsilon(\omega)(x_\varepsilon(\omega), \omega) \right), z - x_\varepsilon(\omega)(\omega) \right) \leq 0
\]
Now \( u_k^{\varepsilon(\omega)B}(\omega) \in B(\theta(\omega)x_\varepsilon(\omega)(\omega), \omega) \) and the weak uppersemicontinuity must then imply that \( w_k^B(\omega) \in B(\theta(\omega)x(\omega), \omega) \), a similar statement holding for \( C \). By convexity,
\[
w_B(\omega) \equiv \sum_{k=0}^{n} t_k(\omega) w_k^B(\omega) \in B(\theta(\omega)x(\omega), \omega),
\]
similar for \( C \). Then from (46.3.10),
\[
\left( y(\omega) - \theta(\omega)^* i(\omega)^* \left( \sum_{k=0}^{n} t_k(\omega) w_k^B(\omega) + \sum_{k=0}^{n} t_k(\omega) w_k^C(\omega) \right), z - x(\omega) \right) 
\]
\[
= \left( \theta(\omega)^* i(\omega)^* (y(\omega) - (w_B(\omega) + w_C(\omega))), z - x(\omega) \right) \leq 0
\]
It follows that if \( x(\omega) \equiv \theta(\omega)x(\omega) \),
\[
\langle y(\omega) - (w_B(\omega) + w_C(\omega)), \theta(\omega)z - x(\omega) \rangle \leq 0
\]
each of \( x(\omega), w_B(\omega), w_C(\omega) \) are measurable and \( w_B(\omega) \in B(x(\omega), \omega) \), \( w_C(\omega) \in C(x(\omega), \omega) \). Since \( \theta(\omega)z \) is a generic element of \( K(\omega) \), this proves the theorem. □

Obviously one could have any finite sum of operators having the same properties as \( B, C \) above and one could get a similar result.
46.4 A Measurable Kakutani Theorem

Recall the Kakutani theorem, Theorem 46.3.1.

**Theorem 46.4.1** Let $K$ be a compact convex subset of $\mathbb{R}^n$ and let $A : K \to \mathcal{P}(K)$ such that $Ax$ is a closed convex subset of $K$ and $A$ is upper semicontinuous. Then there exists $x$ such that $x \in Ax$. This is the “fixed point”.

Here is a measurable version of this theorem. It is just like the proof of the above Browder lemma.

**Theorem 46.4.2** Let $K(\omega)$ be compact, convex, and $\omega \to K(\omega)$ a measurable multifunction. Let $A(\cdot, \omega) : K(\omega) \to K(\omega)$ be upper semicontinuous, and let $\omega \to A(x, \omega)$ have a measurable selection for each $x \in \mathbb{R}^n$. Then there exists $x(\omega) \in K(\omega) \cap A(K(\omega), \omega)$ such that $\omega \to x(\omega)$ is measurable.

**Proof:** Tile $\mathbb{R}^n$ with $n$ simplices such that the collection is locally finite and each simplex has diameter less than $\varepsilon < 1$. This collection of simplices is determined by a countable collection of vertices so there exists a one to one and onto map from $\mathbb{N}$ to the collection of vertices. By assumption, for each vertex $x$, there exists $A_\varepsilon(x, \omega) \in A(P_{K(\omega)}x, \omega)$. By Lemma 46.3.2 $\omega \to P_{K(\omega)}x$ is measurable and by Theorem 46.3.3 there is a measurable selection for $\omega \to A(P_{K(\omega)}x, \omega)$ which is denoted as $A_\varepsilon(x, \omega)$. By local finiteness, this function is continuous in $x$ on the set of vertices. Define $A_\varepsilon(x)$ on all of $\mathbb{R}^n$ by the following rule. If

$$x \in [x_0, \ldots, x_n],$$

so $x = \sum_{i=0}^n t_i x_i$, then

$$A_\varepsilon(x, \omega) = \sum_{k=0}^n t_k A_\varepsilon(x_k, \omega).$$

By local finiteness, this function satisfies $\omega \to A_\varepsilon(x, \omega)$ is measurable and also $x \to A_\varepsilon(x, \varepsilon)$ is continuous. It also maps $\mathbb{R}^n$ to $K(\omega)$. By Corollary 46.4.4 there is a measurable fixed point $x_\varepsilon(\omega)$ satisfying $x_\varepsilon(\omega) \in K(\omega)$ and $A_\varepsilon(x_\varepsilon(\omega), \omega) = x_\varepsilon(\omega)$.

Suppose $x_\varepsilon(\omega) \in [x_0^\varepsilon(\omega), \ldots, x_n^\varepsilon(\omega)]$ so $x_\varepsilon(\omega) = \sum_{k=0}^n t_k^\varepsilon(\omega) x_k^\varepsilon(\omega)$.

**Claim:** The vertices $x_k^\varepsilon(\omega)$ can be considered measurable also as is $t_k^\varepsilon(\omega)$.

**Proof of claim:** Let the simplices in the tiling be $\{\sigma_k\}_{k=1}^\infty$ and let the vertices of simplices in the tiling be $\{z_j^k\}_{j=1}^\infty$. Let

$$F_k := x_k^{-1}(\sigma_k), E_1 := F_1, \ldots, E_k := F_k \setminus \bigcup_{i=1}^k F_i$$

Then $\omega$ is in exactly one of these measurable sets $E_k$. These measurable sets partition $\Omega$. Let $\sigma_k(\omega)$ be the unique simplex for $\omega \in E_k$. Thus $x_\varepsilon(\omega) \in \sigma_k(\omega)$ on the measurable set $E_k$. Its vertices are $z_{i_0}(\omega), z_{i_1}(\omega), \ldots, z_{i_n}(\omega)$. These are $x_0^\varepsilon(\omega), \ldots, x_n^\varepsilon(\omega)$ in order. They are determined in this way on a measurable set so they are measurable $\mathbb{R}^n$ valued functions. Then $\omega \to t_k^\varepsilon(\omega)$ is also measurable because there is a continuous mapping to these scalars from $x_\varepsilon(\omega)$. 


46.4. A MEASURABLE KAKUTANI THEOREM

Then since \( x_\varepsilon (\omega) \) is contained in \( K(\omega) \), a compact set, and the diameter of each simplex is less than 1, it follows that \( A_\varepsilon (x_k^\varepsilon (\omega), \omega) \) is contained in

\[
A(K(\omega) + B(0,1), \omega)
\]

which is a compact set. Let \( W^\varepsilon (\omega) \in \mathbb{R}^{2n + 2n^2} \) be defined as follows.

\[
W^\varepsilon (\omega) := \left( t_1^\varepsilon (\omega), \ldots, t_n^\varepsilon (\omega), x_0^\varepsilon (\omega), \ldots, x_n^\varepsilon (\omega), x_\varepsilon (\omega), \ldots, A_\varepsilon (x_1^\varepsilon (\omega), \omega) \cdots A_m(\omega, x_n^\varepsilon (\omega), \omega) \right)
\]

Thus \( W^\varepsilon \) has values in a compact subset of \( \mathbb{R}^{2n + 2n^2} \) and is measurable. By Lemma

there exists a subsequence \( \varepsilon (\omega) \to 0 \) and a measurable function \( \omega \to W(\omega) \) such that

\[
W^{\varepsilon (\omega)} (\omega) \to W (\omega) = \left( t_1 (\omega), \ldots, t_n (\omega), x_0 (\omega), \ldots, x_n (\omega), x_\omega, \ldots, y_n (\omega) \right)
\]

as \( \varepsilon (\omega) \to 0 \). Recall also that

\[
A_\varepsilon (x_k^\varepsilon (\omega), \omega) \subseteq A(P_{K(\omega)}x_k^\varepsilon, \omega)
\]

Now

\[
|P_{K(\omega)}x_k^\varepsilon (\omega) - x_\varepsilon (\omega)| = |P_{K(\omega)}x_k^\varepsilon (\omega) - P_{K(\omega)}x_\varepsilon (\omega)| < |x_k^\varepsilon - x_\varepsilon| < \varepsilon
\]

Both \( x_k^{\varepsilon (\omega)} (\omega) \) and \( x^{\varepsilon (\omega)} (\omega) \) converge to \( x(\omega) \) and so the above shows that also, \( P_{K(\omega)}x_k^\varepsilon (\omega) \to x(\omega) \). Therefore,

\[
A_\varepsilon (x_k^{\varepsilon (\omega)} (\omega), \omega) \subseteq A(x(\omega), \omega) + B(0, r)
\]

whenever \( \varepsilon (\omega) \) is small enough. Since \( A(x(\omega), \omega) \) is closed, this implies \( y_k (\omega) \in A(x(\omega), \omega) \). Since \( A(x(\omega), \omega) \) is convex,

\[
\sum_{k=1}^{n} t_k (\omega) y_k (\omega) \in A(x(\omega), \omega)
\]

Also, from the construction,

\[
x_\varepsilon (\omega) = A_\varepsilon (x_\varepsilon (\omega), \omega) = \sum_{k=0}^{n} t_k^\varepsilon (\omega) A_\varepsilon (x_k^\varepsilon (\omega), \omega)
\]

so passing to the limit as \( \varepsilon (\omega) \to 0 \), we get

\[
x(\omega) = \sum_{k=0}^{n} t_k (\omega) y_k (\omega) \in A(x(\omega), \omega)
\]

and this is the measurable fixed point. \( \blacksquare \)
46.5 Some Variational Inequalities

In the following, $V$ will be a reflexive separable Banach space. Following [46.3.17], here is a definition of a pseudomonotone operator. Actually, we will consider a slight generalization of the usual definition in [46.3.17] which involves an assumption that there exists a subsequence such that the lim inf condition holds rather than use the original sequence.

**Definition 46.5.1** Let $V$ be a reflexive Banach space. Then $A : V \to P (V')$ is pseudomonotone if the following conditions hold.

1. $Au$ is closed, nonempty, convex. \hspace{1cm} (46.5.14)
2. If $F$ is a finite dimensional subspace of $V$, then if $u \in F$ and $W \supseteq Au$ for $W$ a weakly open set in $V'$, then there exists $\delta > 0$ such that $v \in B(u, \delta) \cap F$ implies $Av \subseteq W$. \hspace{1cm} (46.5.15)
3. If $u_k \rightharpoonup u$ and if $u^*_k \in Au_k$ is such that $\limsup_{k \to \infty} \langle u^*_k, u_k - u \rangle \leq 0$,
   \begin{equation}
   \liminf_{k \to \infty} \langle u^*_k, u_k - v \rangle \geq \langle u^*(v), (u - v) \rangle. \hspace{1cm} (46.5.16)
   \end{equation}
4. We say $A$ is coercive if
   \begin{equation}
   \lim_{\|v\| \to \infty} \inf \left\{ \frac{\langle z^*, v \rangle}{\|v\|} : z^* \in Av \right\} = \infty. \hspace{1cm} (46.5.17)
   \end{equation}

If one assumes $A$ is bounded, then the weak upper semicontinuity condition \hspace{1cm} (46.5.15) can be proved from the other conditions. It has been known for a long time that these operators are useful in the study of variational inequalities. In this section, we give a short example to show how one can obtain measurable solutions to variational inequalities from the measurable Browder lemma given above. This is the following theorem which gives a measurable version of old results of Brezis dating from the late 1960s. This will involve the following assumptions.

1. **Measurability condition**
   For each $u \in V$, there is a measurable selection $z(\omega)$ such that $z(\omega) \in A(u, \omega)$.

2. **Values of $A$**
   $A(\cdot, \omega) : V \to P (V')$ has bounded, closed, nonempty, convex values. $A(\cdot, \omega)$ maps bounded sets to bounded sets.
3. Limit conditions

If \( u_n \rightarrow u \) and \( \limsup_{n \rightarrow \infty} \langle z_n, u_n - u \rangle \leq 0, \ z_n \in A(u_n, \omega) \)

then for given \( v \), there exists \( z(v) \in A(u, \omega) \) such that

\[
\liminf_{k \rightarrow \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle
\]

Thus, for fixed \( \omega \), \( A(\cdot, \omega) \) is a set valued bounded pseudomonotone operator. Recall that the sum of two of these is also a set valued bounded pseudomonotone operator.

By Theorem 46.3.1 if \( \omega \rightarrow u(\omega) \) is measurable, then \( A(u(\omega), \omega) \) has a measurable selection. Also, the limit condition implies that \( A(\cdot, \omega) \) is upper semicontinuous from the strong to the weak topology. The overall approach to the following theorem is well known. The new ingredients are Lemma 46.2.2 and Theorem 46.3.3 which are what allows us to obtain measurable solutions. First is a standard result on the sum of two pseudomonotone bounded operators. See Theorem 23.5.1 on Page 836.

**Theorem 46.5.2** Say \( A, B \) are set valued bounded pseudomonotone operators. Then their sum is also a set valued bounded pseudomonotone operator. Also, if \( u_n \rightarrow u \) weakly, \( z_n \rightarrow z \), \( z_n \in A(u_n) \), and \( w_n \rightarrow w \) weakly with \( w_n \in A(u_n) \), then if

\[
\limsup_{n \rightarrow \infty} \langle z_n + w_n, u_n - u \rangle \leq 0,
\]

it follows that

\[
\liminf_{n \rightarrow \infty} \langle z_n + w_n, u_n - v \rangle \geq \langle z(v) + w(v), u - v \rangle, \ z(v) \in A(u), w(v) \in B(u),
\]

and \( z \in A(u), w \in B(u) \).

**Theorem 46.5.3** Let \( V \) be a reflexive separable Banach space. Let \( \omega \rightarrow K(\omega) \) be a measurable multifunction, \( K(\omega) \) convex, closed, and bounded. Also for \( A = B, C \) let \( A(\cdot, \cdot) \) satisfy \( \mathbb{A} - \mathbb{B} \). Let \( \omega \rightarrow f(\omega) \) be measurable with values in \( V' \). Then there exists measurable \( \omega \rightarrow u(\omega) \in K(\omega) \) and \( \omega \rightarrow w_B(\omega), \omega \rightarrow w_C(\omega) \) with \( w_B(\omega) \in B(u(\omega), \omega), w_C(\omega) \in C(u(\omega), \omega) \) such that

\[
\langle f(\omega) - (w_B(\omega) + w_C(\omega)), z - u(\omega) \rangle \leq 0
\]

for all \( z \in K(\omega) \). If it is only known that \( K(\omega) \) is closed and convex, the same conclusion can be obtained if it is also known that for some \( z(\omega) \in K(\omega), B(\cdot, \omega) + C(\cdot, \omega) \) is coercive meaning

\[
\lim_{\|v\| \rightarrow \infty} \inf \left\{ \frac{\langle z^*, v - z \rangle}{\|v\|} : z^* \in B(v, \omega) + C(v, \omega) \right\} = \infty.
\]
Proof: Let $V_n = V_n(\omega)$ denote an increasing sequence of finite dimensional subspaces whose union is dense in $V$. Let $V_n(\omega)$ contain the first $n$ vectors of $\{d_k(\omega)\}_{k=1}^{\infty}$ where the closure of this sequence equals $K(\omega)$ for each $\omega$, each function being measurable. Let

$$V_n(\omega) = \text{span}(v_1, \ldots, v_n, d_1(\omega), \ldots, d_n(\omega)).$$

where $\{v_k\}_{k=1}^{\infty}$ is dense in $V$. Also let

$$\gamma_n(\omega) = \max\{|d_1(\omega)|, \ldots, |d_n(\omega)|, |v_1|, \ldots, |v_n|\}$$

and $B_n(\omega)$ defined as $B(0, \gamma_n(\omega))$. Then let

$$K_n(\omega) = V_n \cap K(\omega) \cap B_n(\omega) \neq \emptyset \quad \forall_n K_n(\omega) \text{ dense in } K(\omega)$$

Now each $K_n(\omega)$ is a set valued compact convex subset of $V_n(\omega)$ which is a measurable multifunction. It is a measurable multifunction because the linear combinations of the measurable functions $\{v_1, \ldots, v_n, d_1(\omega), \ldots, d_n(\omega)\}$ having a subset of the rational numbers as coefficients is a dense subset of $K_n(\omega)$. Then by Theorem 46.3.3, there exist measurable functions

$$u_n(\omega) \in K_n(\omega), w_n^B(\omega) \in B(u_n(\omega), \omega), w_n^C(\omega) \in C(u_n(\omega), \omega)$$

such that

$$(f(\omega) - (w_n^B(\omega) + w_n^C(\omega)), z - u_n(\omega)) \leq 0 \quad (*)$$

for all $z \in K_n(\omega)$.

Thus for all $w \in K_n(\omega)$,

$$(w_n^B(\omega) + w_n^C(\omega), u_n(\omega) - w) \leq (f(\omega), u_n(\omega) - w).$$

These $u_n(\omega)$ are bounded because $K(\omega)$ is a bounded set or in the other case, one can pick the special $z(\omega)$ in the definition for coercivity to obtain that these $u_n(\omega)$ are bounded. Thus, since $A(\cdot, \omega)$ is assumed to be bounded for $A = B, C$, each of $u_n(\omega)$ and $w_n^B(\omega), w_n^C(\omega)$ are bounded for each $\omega$.

By Lemma 46.3.3, there is a subsequence $n(\omega)$ such that

$$\left(u_{n(\omega)}(\omega), w_{n(\omega)}^B(\omega), w_{n(\omega)}^C(\omega)\right)$$

converges weakly to $(u(\omega), w_B(\omega), w_C(\omega))$ in $V \times V' \times V'$ and

$$\omega \to (u(\omega), w_B(\omega), w_C(\omega))$$

is measurable into $V \times V' \times V'$. It is now only a matter of verifying the desired variational inequality for each $\omega$.

By convexity, $u(\omega) \in K(\omega)$. Now for fixed $\omega$, let $\hat{u}_{n(\omega)} \to u(\omega)$ strongly in $V$ where $\hat{u}_{n(\omega)} \in K_n(\omega)$. Then

$$\lim_{n(\omega) \to \infty} \sup \left(w_{n(\omega)}^B(\omega) + w_{n(\omega)}^C(\omega), u_{n(\omega)}(\omega) - u(\omega)\right)$$

$$= \lim_{n(\omega) \to \infty} \sup \left(w_{n(\omega)}^B(\omega) + w_{n(\omega)}^C(\omega), u_{n(\omega)}(\omega) - \hat{u}_{n(\omega)}\right)$$
Then that \( \hat{\text{condition}} \) holds for this subsequence, there exists for any \( v \) we will consider this subsequence or a further subsequence. Then since the \( \lim \sup \) it follows that

\[
\lim_{n(\omega) \to \infty} \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle \leq 0.
\]

By Theorem \[46.5.3\],

\[
w_B(\omega) \in B(u(\omega), \omega), w_C(\omega) \in C(u(\omega), \omega).
\]

Also, there is a subsequence, still denoted with \( n(\omega) \) such that

\[
\lim_{n(\omega) \to \infty} \inf \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle \geq \langle w(u(\omega)), u(\omega) - u(\omega) \rangle = 0
\]

for some \( w(u(\omega)) \in B(u(\omega), \omega) + C(u(\omega), \omega) \) because the sum of pseudomonotone operators is pseudomonotone. Thus for this subsequence, since

\[
\lim_{n(\omega) \to \infty} \sup \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle \leq 0 \leq \lim_{n(\omega) \to \infty} \inf \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle,
\]

it follows that

\[
\lim_{n(\omega) \to \infty} \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle = 0.
\]

We will consider this subsequence or a further subsequence. Then since the \( \lim \sup \) condition holds for this subsequence, there exists for any \( v \) \( w_B(v) \in B(u(\omega), \omega) \), \( w_C(v) \in C(u(\omega), \omega) \) such that

\[
\langle w_B(\omega) + w_C(\omega), u(\omega) - v \rangle \geq \lim_{n(\omega) \to \infty} \inf \left( \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle \right)
\]

\[
= \lim_{n(\omega) \to \infty} \inf \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - v \rangle
\]

\[
\geq \langle w_B(v) + w_C(v), u(\omega) - v \rangle
\]

Finally, let \( v \in K(\omega) \). Then it follows that there exists a sequence \( \{\hat{v}_n\} \) such that \( \hat{v}_n \in K_n(\omega) \) which converges strongly to \( v \). Thus

\[
\langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - \hat{v}_n \rangle \leq \langle f(\omega), u_n(\omega) - \hat{v}_n \rangle
\]

Then

\[
\lim_{n(\omega) \to \infty} \sup \langle w^B_n(\omega) + w^C_n(\omega), u(\omega) - v \rangle = \lim_{n(\omega) \to \infty} \sup \left( \langle w^B_n(\omega) + w^C_n(\omega), u_n(\omega) - u(\omega) \rangle \right)
\]

\[
+ \langle w^B_n(\omega) + w^C_n(\omega), u(\omega) - v \rangle
\]

\[
= \langle w_B(\omega) + w_C(\omega), u(\omega) - v \rangle
\]

\[
= 0
\]
We can let \( B \) be a measurable multifunction. Here both \( B \) and \( C \) are measurable. If \( A \) is the sum of \( B,C \) and \( u \to B(u,t) \) and \( u \to C(u,t) \), these each satisfying the conditions. Then one can conclude that \( w \in L^p ([0,T] ; V') \) and \( u \in L^p ([0,T] ; V) \). This paper has resolved the only difficult issue which is existence of a measurable solution with no monotonicity or uniqueness assumptions on the problem for fixed \( t \).

**Example 46.5.5** We can let \( \Omega = [0, T] \) and let the measurable sets be the Lebesgue measurable sets, \( t \to f(t) \) measurable into \( V' \). Then for \( A(\cdot , \cdot) \) satisfying \( B - K \) the above theorem gives the solution \( u,w(t), w(t) \in K(t) \) to variational inequalities of the form

\[
\langle f(t) - w(t), z - u(t) \rangle \leq 0, \ w(t) \in A(u(t), t)
\]

for all \( z \in K(t) \) where \( K(t) \) is a closed bounded convex subset of \( V \) for \( t \to K(t) \) a measurable multifunction. Here both \( u \) and \( w \) are measurable. If \( u \to A(u,t) \) is coercive, this allows for \( K(t) \) only closed and convex. If \( A \) is the sum of \( B,C \) and \( u \to B(u,t) \) and \( u \to C(u,t) \), these each satisfying the conditions. If \( w \in L^p ([0,T] ; V') \) and \( u \in L^p ([0,T] ; V) \) then one can conclude that \( w \in L^p ([0,T] ; V') \) and \( u \in L^p ([0,T] ; V) \). This paper has resolved the only difficult issue which is existence of a measurable solution with no monotonicity or uniqueness assumptions on the problem for fixed \( t \).

**Example 46.5.6** In the case of a filtration \( \{ F_t \} \), one could let the \( \sigma \) algebra consist of the progressively measurable sets and obtain the same conclusions. Thus the variational inequality would be of the form

\[
\langle f(t,\omega) - w(t,\omega), z - u(t,\omega) \rangle \leq 0, \ w(t,\omega) \in A(u(t,\omega), t, \omega), z \in K(t,\omega)
\]

This result is quite interesting because it is describing a situation where there is no uniqueness or monotonicity and all that is required are conditions of measurability on \( f \). Also, one only needs to check the limit conditions on \( u \to A(u,\omega) \) for fixed \( \omega \) so all Sobolev embedding theorems are available. Nor is it necessary to assume that \( \omega \to A(u,\omega) \) is a measurable multifunction as is often done. It suffices to check that it has a measurable selection. This is a strictly more general condition.
46.6 An Example

Let \( \sigma (r, t) \) be a continuous function of \( r \) which satisfies
\[
\begin{align*}
    r &\rightarrow \sigma (r, t) \text{ is continuous, } t \rightarrow \sigma (r, t) \text{ is measurable,} \\
    0 &< \delta (t) \leq \sigma (r, t) \leq 1/\delta (t)
\end{align*}
\]
There is no uniform lower bound needed for \( \delta (t) \). Then we will let \( V \) be a closed subspace of \( H^1 (\Omega) \) where \( \Omega \) is a bounded open set with Lipschitz boundary.

\[
V \equiv \{ u \in H^1 (\omega) : \gamma u = 0 \text{ on } \Sigma_0 \}
\]
where \( \alpha (\Sigma_0) > 0 \) for \( \alpha \) the surface measure, \( \Sigma_0 \) a closed subset of \( \partial \Omega \) and \( \gamma \) is the trace map. Thus an equivalent norm for \( V \) is
\[
\| u \|_V^2 = \int_\Omega |\nabla u|^2 \, dx
\]
Also let \( H = L^2 (\Omega) \) and let \( H = H' \subseteq V' \). Then let \( A (\cdot, t) : V \rightarrow V' \) be defined by
\[
\langle A (u, t), v \rangle \equiv \int_\Omega \sigma (u, t) \nabla u \cdot \nabla v
\]
Is this a bounded pseudomonotone map? It is clearly bounded thanks to the bounds on \( \sigma \). Suppose then that \( u_n \rightarrow u \) weakly in \( V \) and
\[
\limsup_{n \rightarrow \infty} \langle A (u_n, t), u_n - u \rangle \leq 0
\]
Does the \( \lim \inf \) condition hold? If not, then there exists a subsequence and \( v \in V \) such that
\[
\lim_{n \rightarrow \infty} \langle A (u_n, t), u_n - v \rangle < \langle A (u, t), u - v \rangle
\]
By compactness, there is a further subsequence still denoted with \( n \) such that \( u_n \rightarrow u \) strongly in \( L^2 (\Omega) \) and pointwise. Consider
\[
\int_\Omega \sigma (u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v)
\]
Now by the dominated convergence theorem,
\[
\int_\Omega |\sigma (u_n, t) - \sigma (u, t)|^2 \rightarrow 0
\]
and so in fact \( \sigma (u_n, t) \nabla u_n \rightarrow \sigma (u, t) \nabla u \) weakly in \( H^3 \). Then
\[
\int_\Omega \sigma (u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v) = \int_\Omega \sigma (u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla u) + \int_\Omega \sigma (u_n, t) \nabla u_n \cdot (\nabla u - \nabla v)
\]
\[ \geq \int_{\Omega} \sigma(u_n, t) \nabla u \cdot (\nabla u_n - \nabla u) + \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u - \nabla v) \]

and so, taking lim inf of both sides using the estimates to justify the use of Fatou’s lemma,

\[ \liminf_{n \to \infty} \int_{\Omega} \sigma(u_n, t) \nabla u_n \cdot (\nabla u_n - \nabla v) \geq \int_{\Omega} \sigma(u, t) \nabla u \cdot (\nabla u - \nabla v) \]

This is a contradiction. Thus the lim inf condition must hold.

Next consider another operator. Let \( \Sigma_1 \) be \( \partial \Omega \setminus \Sigma_0 \) and has positive surface measure. Let \( r \to a(r, t) \) be lower semicontinuous and \( r \to b(r, t) \) be upper semicontinuous. Let \( 0 < \delta(t) \leq a(r, t) \leq b(r, t) \leq \frac{1}{\delta(t)} \). Also let both of these functions be measurable in \( t \). Now \( \gamma : V \to L^2(\Sigma_1) \) and so \( \gamma^* : L^2(\Sigma_1) \to V' \) defined in the usual way. Then \( z \in B(u, t) \) will mean \( z = \gamma^* w \) for some \( w \in L^2(\Sigma_1) \) with

\[ w(x) \in [a(\gamma u(x), t), b(\gamma u(x), t)] \]

for a.e. \( x \) such that

\[ \langle z, v \rangle = \int_{\Sigma_1} w(x) \gamma v(x) \]

Using Sobolev embedding theorems, if \( u_n \to u \) weakly in \( V \), then from the Sobolev embedding theorem \( u_n \to u \) strongly in a suitable Sobolev space of fractional order such that the embedding of \( V \) into this space is compact and the trace map is still continuous. Thus there is a subsequence such that \( \gamma u_n(x) \to \gamma u(x) \) pointwise a.e. and \( w_n \to w \) in \( L^2(\Sigma_1) \). Then by the semicontinuity properties of \( a, b \) we obtain from routine considerations that \( w(x) \in [a(\gamma u(x), t), b(\gamma u(x), t)] \) a.e. To see how you can do this, let

\[ E = \left\{ x : w(x) \geq b(\gamma u(x), t) + \frac{1}{k} \right\} . \]

Then

\[ \int_{\Sigma_1} \chi_E(x) (-b(\gamma u_n(x), t)) \leq \int_{\Sigma_1} \chi_E(x) (-w_n(x)) \to \int_{\Sigma_1} \chi_E(x) (-w(x)) \]

\[ \leq \int_{\Sigma_1} \chi_E(x) \left(-b(\gamma u(x), t) - \frac{1}{k}\right) \]

By lower semicontinuity of \(-b(\cdot, t)\) and the boundedness assumption, we can use Fatou’s lemma to take lim inf of both sides and conclude that

\[ \int_{\Sigma_1} \chi_E(x) (-b(\gamma u(x), t)) \leq \int_{\Sigma_1} \chi_E(x) \left(-b(\gamma u(x), t) - \frac{1}{k}\right) \]

an obvious contradiction unless \( \alpha(E) = 0 \). Then taking the union of the exceptional sets for all \( k \), it follows that \( w(x) \leq b(\gamma u(x), t) \) a.e. The other side of the inequality can be shown similarly. Letting \( z_n \in B(u_n, t) \) and \( v \in V \), is it true that

\[ \liminf_{n \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u_n - v \rangle \]
for some \( z(v) \in B(u,t) \)? Suppose not. Then from the above, there is a subsequence such that the limit equals the \( \liminf \) but which has the inequality turned around for some \( v \) and all \( z \in B(u,t) \). Then from what was just shown, letting \( w_n \) go with \( z_n \), there is a further subsequence such that \( w_n \to w \) weakly in \( L^2(\Sigma_1) \) and and \( \gamma u_n \to \gamma u \) strongly in \( L^2(\Sigma_1) \) and

\[
\int_{\Sigma_1} w_n (x) (\gamma u_n (x) - \gamma v (x)) \to \int_{\Sigma_1} w (x) (\gamma u (x) - \gamma v (x)) = (z, u - v)
\]

where \( w \in B(u,t) \) and \( z = \gamma^* w \) so the \( \liminf \) condition holds. Thus this second operator is pseudomonotone.

Do these have measurable selections? This is obvious. Letting \( u \in V, t \to \gamma^* a(\gamma u (x),t) \) is measurable into \( V' \) and is in \( B(u,t) \). Similarly \( t \to A(u,t) \) is measurable into \( V' \). Note that on the second operator, it was really only necessary to assume that there exists \( t \to c(r,t) \) measurable with \( c(r,t) \in [a(r,t), b(r,t)] \) and totally eliminate the assumption that either \( a \) or \( b \) is measurable in \( t \).

Now let \( t \to f(t) \) be measurable into \( V' \). Say

\[
\langle f(t), v \rangle = \int_\Omega h(t) v dx + \int_{\Sigma_1} \beta(t) v d\alpha
\]

and let \( K(t) \subseteq V \) be a closed convex subset of \( V \). There is obviously a coercivity condition holding for the sum of these two operators \( A(u,t) + B(u,t) \) and so there exists \( u(t) \in K(t) \) such that for all \( v \in K(t) \)

\[
\langle f(t) - (A(u(t),t) + z(t)), v - u(t) \rangle \leq 0 \tag{*}
\]

where \( t \to u(t) \) is measurable into \( V \), \( t \to z(t) \) measurable into \( V' \), \( t \to A(u(t),t) \) measurable into \( V' \). Is \( t \to w(t) \) measurable where \( z(t) = \gamma^* w(t) \)? Let \( \phi \in V \). Then

\[
\langle z(t), \phi \rangle_{V',V} = \langle w(t), \gamma \phi \rangle_{L^2(\Sigma_1)}
\]

and is given to be a measurable function of \( t \). However, since \( \Sigma_1 \) is open, the image of the trace is dense in \( L^2(\Sigma_1) \) and so by this density and Pettis theorem, \( t \to w(t) \) must be measurable into \( L^2(\Sigma_1) \). Thus the variational inequality * is of the form

\[
\left( (h(t), v - u(t))_H + (\beta(t) - w(t), \gamma v - \gamma u(t))_{L^2(\Sigma_1)} \right) - (\sigma(u(t),t) \nabla u(t) , \nabla v)_{H^3} \leq 0
\]

for all \( v \in K(t) \). If the inequality which gives coercivity were eliminated, we would still have the above if \( K(t) \) were assumed bounded.

What equation is satisfied if \( K(t) = V \)? We would have

\[
\int_\Omega \sigma(u(t),t) \nabla u(t) \cdot \nabla v dx + \int_{\Sigma_1} w(t) v dx = \int_\Omega \langle f(t), v \rangle dx,
\]
Let \( F \) are points of density. Therefore, there exists a sequence \( h_k \to 0 \) such that \( t_n - h_k \to t_n \) as \( k \to \infty \) and \( (t_n - h_k) \in F(t_1, \ldots, t_{n-1}) \). Otherwise there would be some open set about \( t_n \) which excludes points of \( F(t_1, \ldots, t_{n-1}) \) which would imply that \( t_n \) is not actually a point of density. Then using the fundamental theorem of calculus, we get for such points which are points of \( F(t_1, \ldots, t_{n-1}) \) the fact that \( u_{x_n}(t_1, \ldots, t_{n-1}, t_n) = 0 \). Thus for a.e. \( t_n \in F(t_1, \ldots, t_{n-1}), u_{x_n}(t_1, \ldots, t_{n-1}, t_n) = 0 \). Thus \( u_{x_n}(t_1, \ldots, t_{n-1}, t_n) = 0 \) for a.e. \( t_n \) in \( F(t_1, \ldots, t_{n-1}) \). Similar reasoning holds for differentiation with respect to the other variables. Thus \( \nabla u = 0 \) a.e. on \( F \).

**Lemma 46.6.2** Let \( V \) be a closed subset of \( W^{1,p}(\Omega) \), \( p > 1 \) and let \( k \in V \). Then \( \max (k, u) \in V \) and if \( u_n \to u \) in \( V \), then \( \max (u_n, k) \to \max (u, k) \) in \( V \).
46.6. AN EXAMPLE

Proof: We consider \( \psi (r) = |r| , \psi_\varepsilon (r) = \sqrt{\varepsilon + r^2} \). Then for \( \phi \in C_c^\infty (\Omega) \),

\[
\int_\Omega \psi (u (x)) \phi_{,x_k} (x) = \lim_{\varepsilon \to 0} \int_\Omega \psi_\varepsilon (u (x)) \phi_{,x_k} (x) \\
= - \lim_{\varepsilon \to 0} \int_\Omega u (x) \sqrt{\varepsilon + u^2} \phi_{,x_k} (x) \\
= - \int_\Omega \xi (u (x)) u_{,x_k} (x) \phi (x)
\]

where \( \xi (r) = 1 \) if \( r > 0 \), \( -1 \) if \( r < 0 \) and \( 0 \) if \( r = 0 \). Thus \( \psi (u)_{,x_k} = \xi (u (x)) u_{,x_k} (x) \) a.e. and this is clearly in \( W^{1,p} (\Omega) \). Of course \( \max (u, k) = \frac{|k - u| + (k + u)}{2} \) so this shows that \( \max (u, k) \) is in \( W^{1,p} (\Omega) \).

Next suppose \( u_n \to u \) in \( W^{1,p} (\Omega) \). Does

\[
\xi (u_n) u_{n,x_k} \to \xi (u) u_{,x_k} ?
\]

Let \( G = \{ x : u (x) \neq 0 \} \). A subsequence, still denoted by \( u_n \), converges pointwise a.e. to \( u \) and \( u_{n,x_k} \to u_{,x_k} \) pointwise a.e. Therefore, off a set of measure zero, \( \xi (u_n (x)) = \xi (u (x)) \) for all \( n \) large enough. Thus

\[
\begin{align*}
\left( \int_\Omega |\xi (u_n) u_{n,x_k} - \xi (u) u_{,x_k}|^p \right)^{1/p} &\leq \left( \int_\Omega |\xi (u_n) u_{n,x_k} - \xi (u_n) u_{,x_k}|^p \right)^{1/p} \\
+ \left( \int_\Omega |\xi (u_n) - \xi (u)|^p |u_{,x_k}|^p \right)^{1/p} &\leq 2 \left( \int_\Omega |\xi (u_n) - \xi (u)|^p |u_{,x_k}|^p \right)^{1/p}
\end{align*}
\]

That second term on the right converges to 0. It equals

\[
\left( \int_G |\xi (u_n) - \xi (u)|^p |u_{,x_k}|^p + \int_G |\xi (u_n) - \xi (u)|^p |u_{,x_k}|^p \right)^{1/p}
\]

Now on \( \mathcal{G}_{+} , u_{,x_k} (x) = 0 \) a.e. by the above lemma and so the first term in the parentheses is 0. The second converges to 0 by the dominated convergence theorem. Then this shows that the second term in * converges to 0. The first obviously converges to 0 from the convergence of \( u_{n,x_k} \) to \( u_{,x_k} \). Now consider whether \( \psi (u_n)_{,x_k} \) converges to \( \psi (u)_{,x_k} \). Those functions \( \psi (u_n)_{,x_k} \) are bounded in \( L^p (\Omega) \) from the above description and so if it fails to converge to \( \psi (u)_{,x_k} \) in \( L^p \) a subsequence converges weakly to \( \zeta \neq \psi (u)_{,x_k} \). But then, the above argument shows that a further subsequence does converge strongly to \( \psi (u)_{,x_k} \) contrary to \( \zeta \neq \psi (u)_{,x_k} \).

Now, from the description of the maximum of two functions given above, we obtain that \( \max (u_n, k) \to \max (u, k) \) in \( V \) provided \( u_n \to u \) in \( V \).

What of the convex set \( K (t) \)? Consider \( k (t, x) \) where \( k (t, \cdot) \) is continuous into \( V \). Thus \( k (t, \cdot) \) is clearly measurable into \( V \). Also \( C ([0, T] ; V) \) is separable, hence completely separable. It follows that

\[
\{ u \in C ([0, T] ; V) : \text{ for all } t, u (t, x) \geq k (t, x) \text{ a.e. } x \}
\]
CHAPTER 46. MULTIFUNCTIONS AND THEIR MEASURABILITY

is also separable. Let \( \{b_i(t)\}_{i=1}^{\infty} \) be a dense countable subset of \( K(t) \), each \( b_i \) measurable. One could obtain such by letting \( b_i(t,x) = \max (k(t,x), d_i(t,x)) \) where \( \{d_i\}_{i=1}^{\infty} \) is a countable dense subset of \( C([0,T];V) \). Then let \( K(t) \) be the closure of \( b_i(t) \) in \( V \). It follows that \( t \to K(t) \) is a measurable multifunction with values in \( V \). If \( u \in K(t) \), there exists a sequence \( b_{i_k}(t) \to u \) in \( V \). Therefore, a subsequence converges to \( u \) pointwise a.e. and it follows that \( u(x) \geq k(t,x) \) a.e. \( x \). On the other hand, if \( u(x) \geq k(t,x) \) for a.e. \( x \), does it follow that it is the limit of a subsequence of the \( b_i(t) \) in \( V \)? Consider \( \max (u(x), k(s,x)) \equiv l(s,x) \). Then \( l \) is in \( C([0,T];V) \) and is always as large as \( k(s,x) \) for all \( s \). Therefore, there is a subsequence \( b_{i_k}(t,\cdot) \) which converges to \( l(t,\cdot) \) pointwise a.e. and in \( V \). Thus \( b_{i_k}(t) \) converges to \( u \) pointwise also for a.e. \( x \). It follows that \( t \to K(t) \) is measurable into \( V \) and \( K(t) = \{u \in V : u(x) \geq k(t,x) \text{ a.e.} \} \). As an example, you could simply take \( k \) to be the restriction to \( \Omega \times [0,T] \) of a smooth function. This is an example of an obstacle problem in which the obstacle changes in \( t \) and there is no uniqueness even though there exists a measurable solution to the variational inequality for each \( t \).

One could also replace \( \sigma(u,t) \) with a graph having a jump as in the second of the two operators and get similar results by beginning with the above solutions and then using Lemma 46.2.2, and the arguments used in the second operator to pass to a limit.

The next section is an interesting result on the pseudomonotone condition for Nemytskii operators defined in this section.

46.7 Limit Conditions For Nemytskii Operators

This is about the following problem. You know

\[ u \to A(u,t) \]

is pseudomonotone. You can also define \( \hat{A} : V \to V' \) by

\[ \hat{A}(u)(t) \equiv A(u(t),t) \text{ a.e.} \]

Then when can you obtain a useable limit condition for \( \hat{A} \)? I think the earliest solution to this problem was given in [10]. These ideas were extended to set valued maps in [17] and to another situation in [18].

Define \( V \equiv V_p \) by

\[ V = L^p([0,T];V), \quad p > 1, \]

where \( V \) is a separable Banach space and \( H \) is a Hilbert space such that

\[ V \subseteq H = H' \subseteq V' \]

with each space dense in the following one. The measure space is chosen to be \( ([0,T],\mathcal{B}([0,T]),m) \) where \( m \) is the Lebesgue measure and \( \mathcal{B}([0,T]) \) consists of all the Borel sets, although one could use the \( \sigma \) algebra of Lebesgue measurable sets as
well. We denote by \( V \) or \( \mathcal{V} \) the above space. If \( U \) is a Banach space, \( \mathcal{U} \) will denote \( L^r ([0, T], U) \).

We will assume the following measurability condition. For each \( u \in V \),

\[
t \to A(u(t), t) \text{ is a measurable multifunction} \tag{46.7.18}
\]

In the case when \( A(\cdot, t) \) is single-valued, bounded and pseudomonotone, this measurability condition is satisfied and so it is measurable. Thus, this definition is a generalization of what would be expected for single-valued operators. We use the following lemma.

**Lemma 46.7.1** Let \( U \) be a separable reflexive Banach space. Suppose there is a sequence \( \{ u_j(\omega) \}_{j=1}^{\infty} \) in \( U \), where each \( \omega \to u_j(\omega) \) is measurable and for each \( \omega \),

\[
\sup_i \| u_i(\omega) \| < \infty.
\]

Then, there exists \( u(\omega) \in U \) such that \( \omega \to u(\omega) \) is measurable, and a subsequence \( n(\omega) \), that depends on \( \omega \), such that the weak limit

\[
\lim_{n(\omega) \to \infty} u_n(\omega)(\omega) = u(\omega)
\]

holds.

**Proof.** Let \( \{ z_i \}_{i=1}^{\infty} \) be a countable dense subset of \( U' \). Let \( h: U \to \prod_{i=1}^{\infty} \mathbb{R} \) be defined by

\[
h(u) = \prod_{i=1}^{\infty} \langle z_i, u \rangle.
\]

Let \( X = \prod_{i=1}^{\infty} \mathbb{R} \) with the product topology. Then, this is a Polish space with the metric defined as \( d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{1 + |x_i - y_i|} 2^{-i} \). By compactness, for a fixed \( \omega \), the \( h(u_n(\omega)) \) are contained in a compact subset of \( X \). Next, define

\[
\Gamma_n(\omega) = \bigcup_{k \geq n} h(u_k(\omega)),
\]

which is a nonempty compact subset of \( X \).

Next, we claim that \( \omega \to \Gamma_n(\omega) \) is a measurable multifunction.

The proof of the claim is as follows. It is necessary to show that \( \Gamma_n^-(O) \) defined as \( \{ \omega : \Gamma_n(\omega) \cap O \neq \emptyset \} \) is measurable whenever \( O \) is open. It suffices to verify this for \( O \) a basic open set in the topology of \( X \). Thus let \( O = \prod_{i=1}^{\infty} O_i \) where each \( O_i \) is a proper open subset of \( \mathbb{R} \) only for \( i \in \{ j_1, \cdots, j_m \} \). Then,

\[
\Gamma_n^-(O) = \bigcup_{k \geq n} \bigcap_{i=1}^{m} \{ \omega : \langle z_j, u_k(\omega) \rangle \in O_j \},
\]

which is a measurable set since \( u_k \) is measurable.

Then, it follows that \( \omega \to \Gamma_n(\omega) \) is strongly measurable because it has compact values in \( X \), thanks to Tychonoff’s theorem. Thus \( \Gamma_n^-(H) = \{ \omega : H \cap \Gamma_n(\omega) \neq \emptyset \} \) is measurable whenever \( H \) is a closed set. Now, let \( \Gamma(\omega) \) be defined as \( n, \Gamma_n(\omega) \) and then for \( H \) closed,

\[
\Gamma^-(H) = \cap_n \Gamma_n^-(H)
\]
and each set in the intersection is measurable, so this shows that \( \omega \to \Gamma (\omega) \) is also measurable. Therefore, it has a measurable selection \( g(\omega) \). It follows from the definition of \( \Gamma (\omega) \) that there exists a subsequence \( n(\omega) \) such that

\[
g(\omega) = \lim_{n(\omega) \to \infty} h(u_{n(\omega)}(\omega)) \quad \text{in } X.
\]

In terms of components, we have

\[
g_i(\omega) = \lim_{n(\omega) \to \infty} \langle z_i, u_{n(\omega)}(\omega) \rangle.
\]

Furthermore, there is a further subsequence, still denoted with \( n(\omega) \), such that

\[
u_n(\omega)(\omega) \to u(\omega) \quad \text{weakly.}
\]

This means that for each \( i \),

\[
g_i(\omega) = \lim_{n(\omega) \to \infty} \langle z_i, u_{n(\omega)}(\omega) \rangle = \langle z_i, u(\omega) \rangle.
\]

Thus, for each \( z_i \) in a dense set, \( \omega \to \langle z_i, u(\omega) \rangle \) is measurable. Since the \( z_i \) are dense, this implies \( \omega \to \langle z, u(\omega) \rangle \) is measurable for every \( z \in U' \) and so by the Pettis theorem, \( \omega \to u(\omega) \) is measurable.

Also is a definition.

**Definition 46.7.2** Let \( A(\cdot, t) : V \to \mathcal{P}(V') \). Then, the Nemytskii operator associated with \( A \),

\[
\hat{A} : L^p([0,T];V) \to \mathcal{P} \left( L^{p'}([0,T];V') \right),
\]

is given by

\[
z \in \hat{A}(u) \quad \text{if and only if } z \in L^{p'}([0,T];V') \text{ and } z(t) \in A(u(t), t) \quad \text{a.e. } t.
\]

**Growth and coercivity**

The next three conditions on the operator \( A \) are similar to the conditions proposed by Bian and Webb, [17] See also Berkovitz and Mustonen [16] which seems to be the paper where these ideas originated. These specific and reasonable conditions, together with a fourth one we add below, allow us to prove an appropriate limit condition that is based on the assumption that \( u \to A(u,t) \) is a set-valued, bounded and pseudomonotone map from \( V \) to \( \mathcal{P}(V') \) and \( t \to A(u,t) \) has a measurable selection.

Our aim is to provide reasonable conditions under which an assumption of pseudomonotonicity on \( u \to A(u,t) \) transfers to a usable limit condition for the operator \( \hat{A} \) defined on \( V = L^p([0,T];V) \).

It is obvious that \( \hat{A}u \) is convex because this is true of \( A(u,t) \). It is also closed. To see this, suppose \( z_n \in \hat{A}(u) \). Then \( z_n(t) \in A(u(t), t) \) for a.e.t. Taking the union of the exceptional sets, we can assume this inclusion holds off a single set of measure zero for all \( n \). If you have \( z_n \to w \) strongly in \( V' \), then a subsequence converges pointwise a.e. Therefore, by upper semicontinuity of the pointwise operator \( u \to A(u,t) \), it follows that \( w(t) \in A(u(t), t) \) for a.e. \( t \). Thus \( \hat{A}u \) is convex and strongly closed.

We assume the following conditions on \( A \).
1. \( A(\cdot, t) : V \to \mathcal{P}(V') \) is pseudomonotone and bounded: \( A(u, t) \) is a closed convex set for each \( t, u \to A(u, t) \) is bounded, and if
\[
\lim_{n \to \infty} \langle A(u_n, t), u_n - u \rangle \leq 0
\]
then for any \( v \in V \),
\[
\liminf_{n \to \infty} \langle A(u_n, t), u_n - v \rangle \geq \langle z(v), u - v \rangle \text{ some } z(v) \in A(u, t)
\]
2. \( A(\cdot, t) \) satisfies the estimates: There exists \( b_1 \geq 0 \) and \( b_2 \geq 0 \), such that
\[
||z||_{V'} \leq b_1 ||u||_{V}^{p-1} + b_2(t),
\]
(46.7.19)
for all \( z \in A(u, t) \), \( b_2(\cdot) \in L^p([0, T]) \).
3. There exist a positive constant \( b_3 \) and a nonnegative function \( b_4 \) that is \( B([0, T]) \) measurable and also \( b_4(\cdot) \in L^1([0, T]) \), such that
\[
\inf_{z \in A(u, t)} \langle z, u \rangle \geq b_3 ||u||_V^p - b_4(t) - \lambda |u|_H^2 .
\]
(46.7.20)
4. The operators \( t \to A(u(t), t) \) are measurable in the sense that
\[
t \to A(u(t), t)
\]
is a measurable multifunction with respect to \( \mathcal{F} \) where \( \mathcal{F} \) will be the \( \sigma \) algebra of Lebesgue measurable sets whenever \( t \to u(t) \) is in \( \mathcal{V}_p \).
5. For \( u \in \mathcal{V}_p \), we define \( \hat{A}(u) \in \mathcal{P}(\mathcal{V}_p') \) as follows: \( z \in \hat{A}(u) \) means that \( z(t) \in A(u(t), t) \ a.e. \ t \). Thus this is the Nemytskii operator for \( A(\cdot, t) \).

In the following theorem and in arguments which take place below, \( U \) will be a Hilbert space dense in \( V \) with the inclusion map compact. Such a Hilbert space always exists and is important in probability theory where \( (i, U, V) \) is an abstract Wiener space. However, in most applications from partial differential equations, it suffices to take \( U \) as a suitable Sobolev space.

**Theorem 46.7.3** Suppose conditions 1-5 hold. Also, suppose
\[
V \subseteq H = H' \subseteq V', \text{ where } V \text{ is dense in } H,
\]
Then, the operator \( \hat{A} \) satisfies the following.

**Hypotheses:**
\[
u_n \to u \text{ weakly in } V, \quad \limsup_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} \leq 0,
\]
for \( z_n \in \hat{A}u_n \), and there exists a set of zero measure \( \Sigma \) such that for \( t \notin \Sigma \), every subsequence of \( \{u_n\} \) has a further subsequence, possibly depending on \( t \notin \Sigma \) such that
\[
u_n(t) \to u(t) \text{ weakly in } U',
\]
where $U$ is a Banach space dense in $V$. In (46.7.23) if $\lambda > 0$, then assume also that
\[
\sup_{n} \sup_{t \in [0, T]} |u_n(t)|_H < \infty. \tag{46.7.21}
\]

**Conclusion:** If the above conditions hold, then there exists $z(v)$ with
\[
\lim_{n \to \infty} \langle z_n, u_n - v \rangle_{\mathcal{V}', \mathcal{V}} \geq \langle z(v), u - v \rangle_{\mathcal{V}', \mathcal{V}}.
\]
where $z(v) \in \hat{A}(u)$. Furthermore, $\hat{A}u$ is a nonempty, closed and convex set in $\mathcal{V}'$.

**Proof:** It was argued above that $\hat{A}(u)$ is closed and convex. Enlarge the set of measure zero $\Sigma$, if needed, so that for each $n$,
\[
z_n(t) \in A(u_n(t), t)
\]
for each $t \notin \Sigma$.

Next, we claim that if $t \notin \Sigma$, then
\[
\lim_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle \geq 0.
\]

**Proof of the claim:** Let $t \notin \Sigma$ be fixed and suppose to the contrary that
\[
\lim_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle < 0. \tag{46.7.22}
\]
Then, there exists a subsequence $\{n_k\}$, which may depend on $t$, such that
\[
\lim_{k \to \infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t) \rangle = \lim_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle < 0. \tag{46.7.23}
\]
Now, condition (3) implies that for all $k$ large enough,
\[
b_1 |u_{n_k}(t)||\mathbf{\nu} - b_3(t) - \lambda |u_{n_k}(t)||H < z_{n_k}(t)||\mathbf{\nu} - ||u(t)||\mathbf{\nu}
\]
\[
\leq \left(b_1 |u_{n_k}(t)||\mathbf{\nu} + b_2(t) \right)|u(t)||\mathbf{\nu},
\]
therefore, $|u_{n_k}(t)||\mathbf{\nu}$ and consequently $z_{n_k}(t)||\mathbf{\nu}$ are bounded. This follows from (46.7.24) in case $\lambda > 0$. Note that $|z_{n_k}(t)||\mathbf{\nu}$ is bounded independently of $n_k$ because of the assumption that $A(\cdot, t)$ is bounded and we just showed that $|u_{n_k}(t)||\mathbf{\nu}$ is bounded.

Taking a further subsequence if necessary, let $u_{n_k}(t) \to u(t)$ weakly in $U'$ and $u_{n_k}(t) \to \xi$ weakly in $V$. Thus, by density considerations, $\xi = u(t)$. Now, (46.7.26) and the limit conditions for pseudomonotone operators imply that the limit condition holds. There exists $z_\infty \in A(u(t), t)$ such that
\[
\lim_{k \to \infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t) \rangle \geq \langle z_\infty, u(t) - u(t) \rangle = 0
\]
\[
> \lim_{k \to \infty} \langle z_{n_k}(t), u_{n_k}(t) - u(t) \rangle,
\]
which is a contradiction. This completes the proof of the claim.

We continue with the proof of the theorem. It follows from this claim that for every \( t \notin \Sigma \),
\[
\liminf_{n \to \infty} \langle z_n (t), u_n (t) - u (t) \rangle \geq 0. \tag{46.7.24}
\]
Also, it is assumed that
\[
\limsup_{n \to \infty} \langle z_n, u_n - u \rangle_V \leq 0.
\]

Then from the estimates,
\[
\int_0^T \left( b_3 \| u_n (t) \|_V^p - b_4 (t) - \lambda \| u_n (t) \|_H^2 \right) dt \leq \int_0^T \| u (t) \|_V \left( \| u_n \|_{p-1} b_1 + b_2 \right) dt
\]
so it is routine to get \( \| u_n \|_V \) is bounded. This follows from the assumptions, in particular \( \text{(46.7.24)} \).

Now, the coercivity condition \( \text{(4)} \) shows that if \( y \in L^p ([0, T] ; V), \) then
\[
\langle z_n (t), u_n (t) - y (t) \rangle \geq b_3 \| u_n (t) \|_V^p - b_4 (t) - \lambda \| u_n (t) \|_H^2
\]
\[
- \left( b_1\| u_n (t) \|_{p-1} + b_2 (t) \right) \| y (t) \|_V.
\]
Using \( p - 1 = \frac{p}{p'} \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \), the right-hand side of this inequality equals
\[
b_3 \| u_n (t) \|_V^p - b_4 (t) - b_1 \| u_n (t) \|_{p/p'} \| y (t) \|_V - b_2 (t) \| y (t) \|_V - \lambda \| u_n (t) \|_H^2,
\]
the last term being bounded independent of \( t, n \) by assumption. Thus there exists \( c \in L^1 (0, T) \) and a positive constant \( C \) such that
\[
\langle z_n (t), u_n (t) - y (t) \rangle \geq - c (t) - C \| y (t) \|_V^p. \tag{46.7.25}
\]
Letting \( y = u \), we use Fatou’s lemma to write
\[
\liminf_{n \to \infty} \int_0^T (\langle z_n (t), u_n (t) - u (t) \rangle + c (t) + C \| u (t) \|_V^p) dt \geq
\]
\[
\int_0^T \liminf_{n \to \infty} (\langle z_n (t), u_n (t) - u (t) \rangle + (c (t) + C \| u (t) \|_V^p)) dt
\]
\[
\geq \int_0^T (c (t) + C \| u (t) \|_V^p) dt.
\]
Here, we added the term \( c (t) + C \| u (t) \|_V^p \) to make the integrand nonnegative in order to apply Fatou’s lemma. Thus,
\[
\liminf_{n \to \infty} \int_0^T (\langle z_n (t), u_n (t) - u (t) \rangle dt \geq 0.
\]
Consequently, using the claim in the last inequality,

\begin{align*}
0 & \geq \limsup_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} \\
& \geq \liminf_{n \to \infty} \int_0^T \langle z_n(t), u_n(t) - u(t) \rangle \, dt \\
& = \liminf_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} \\
& \geq \int_0^T \liminf_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle \, dt \geq 0,
\end{align*}

hence, we find that

\[ \lim_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} = 0. \]  \tag{46.7.26}

We need to show that if \( y \) is given in \( V \) then

\[ \liminf_{n \to \infty} \langle z_n, u_n - y \rangle_{V', V} \geq \langle z, u - y \rangle_{V', V}, \]  \tag{46.7.27}

for all \( z \in \hat{A}u \). Take a subsequence, denoted still with subscript \( n \) such that

\[ \eta = \lim_{n \to \infty} \langle z_n, u_n - y \rangle_{V', V} < \langle z, u - y \rangle_{V', V}. \]  \tag{46.7.28}

We will obtain a contradiction to this. In what follows we continue to use the subsequence just described which satisfies the above inequality.

The estimate \( \ref{10.7.24} \) implies,

\[ 0 \leq \langle z_n(t), u_n(t) - u(t) \rangle \leq c(t) + C \| u(t) \|^p_V, \]  \tag{46.7.29}

where \( c \) is a function in \( L^1(0, T) \). Thanks to \( \ref{10.7.24} \),

\[ \liminf_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle \geq 0, \]

and, therefore, the following pointwise limit exists,

\[ \lim_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle = 0, \]

and so we may apply the dominated convergence theorem using \( \ref{10.7.28} \) and conclude

\[ \lim_{n \to \infty} \int_0^T \langle z_n(t), u_n(t) - u(t) \rangle \, dt = \int_0^T \liminf_{n \to \infty} \langle z_n(t), u_n(t) - u(t) \rangle \, dt = 0. \]
Now, it follows from (46.7.30) and the above equation, that
\[
\lim_{n \to \infty} \int_0^T \langle z_n (t), u_n (t) - u (t) \rangle^+ dt = \lim_{n \to \infty} \int_0^T \langle z_n (t), u_n (t) - u (t) \rangle + \langle z_n (t), u_n (t) - u (t) \rangle^- dt = \lim_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} = 0.
\]

Therefore, both \( \int_0^T \langle z_n (t), u_n (t) - u (t) \rangle^+ dt \) and \( \int_0^T \langle z_n (t), u_n (t) - u (t) \rangle^- dt \) converge to 0, thus,
\[
\lim_{n \to \infty} \int_0^T |\langle z_n (t), u_n (t) - u (t) \rangle| \, dt = 0 \quad (46.7.30)
\]
\[
\lim_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} = 0
\]

From the above, it follows that there exists a further subsequence \( \{n_k\} \) not depending on \( t \) such that
\[
|\langle z_{n_k} (t), u_{n_k} (t) - u (t) \rangle| \to 0 \quad a.e. \ t. \quad (46.7.31)
\]

Therefore, by the pseudomonotone limit condition for \( A \) there exists \( w_t \in A (u (t), t) \) such that for a.e. \( t \),
\[
\alpha (t) = \liminf_{k \to \infty} \langle z_{n_k} (t), u_{n_k} (t) - y (t) \rangle = \liminf_{k \to \infty} \langle z_{n_k} (t), u (t) - y (t) \rangle \geq \langle w_t, u (t) - y (t) \rangle.
\]

Then on the exceptional set, let \( \alpha (t) = \infty \), and consider the set
\[
F (t) \equiv \{ w \in A (u (t), t) : \langle w, u (t) - y (t) \rangle \leq \alpha (t) \},
\]
which then satisfies \( F (t) \neq \emptyset \). Now \( F (t) \) is closed and convex in \( V' \).

**Claim:** \( t \to F (t) \) has a measurable selection off a set of measure zero.

**Proof of claim:** Letting \( B (0, C (t)) \) contain \( A (u (t), t) \), we can assume \( t \to C (t) \) is measurable by using the estimates and the measurability of \( u \). For \( p \in \mathbb{N} \), let \( S_p = \{ t : C (t) < p \} \). If it is shown that \( F \) has a measurable selection on \( S_p \), then it follows that it has a measurable selection. Thus in what follows, assume that \( t \in S_p \).

Define
\[
G (t) \equiv \left\{ w : \langle w, u (t) - y (t) \rangle < \alpha (t) + \frac{1}{n}, t \notin \Sigma \right\} \cap B (0, p)
\]

Thus, it was shown above that this \( G (t) \neq \emptyset \). For \( U \) open,
\[
G^- (U) \equiv \left\{ t \in S_p : \text{for some } w \in U \cap B (0, p), \langle w, u (t) - y (t) \rangle < \alpha (t) + \frac{1}{n} \right\} \quad (*)
\]
Let \( \{w_j\} \) be a dense subset of \( U \cap B(0, p) \). This is possible because \( V' \) is separable. The expression in * equals

\[
\bigcup_{k=1}^{\infty} \left\{ t \in S_p : (w_k, u(t) - y(t)) < \alpha(t) + \frac{1}{n} \right\}
\]

which is measurable. Thus \( G \) is a measurable multifunction.

Since \( t \to G(t) \) is measurable, there is a sequence \( \{w_n(t)\} \) of measurable functions such that \( \bigcup_{n=1}^{\infty} w_n(t) \) equals

\[
G(t) = \left\{ w : (w, u(t) - y(t)) \leq \alpha(t) + \frac{1}{n}, t \notin \Sigma \right\} \cap B(0, p)
\]

As shown above, there exists \( w_t \) in \( A(u(t), t) \) as well as \( G(t) \). Thus there is a sequence of \( w_r(t) \) converging to \( w_t \). Since \( t \to A(u(t), t) \) is a measurable multifunction, it has a countable subset of measurable functions \( \{z_m(t)\} \) which is dense in \( A(u(t), t) \). Let

\[
U_k(t) = \bigcup_{m} B\left(z_m(t), \frac{1}{k}\right) \subseteq A(u(t), t) + B\left(0, \frac{2}{k}\right)
\]

Now define \( A_{1k} = \{t : w_1(t) \in U_k(t)\} \). Then let \( A_{2k} = \{t \notin A_{1k} : w_2(t) \in U_k(t)\} \) and \( A_{3k} = \{t \notin \bigcup_{l=1}^{2} A_{lk} : w_3(t) \in U_k(t)\} \) and so forth. Any \( t \in S_p \) must be contained in one of these \( A_{rk} \) for some \( r \) since if not so, there would not be a sequence \( w_r(t) \) converging to \( w_t \in A(u(t), t) \). These \( A_{rk} \) partition \( S_p \) and each is measurable since the \( \{z_k(t)\} \) are measurable.

Let

\[
\hat{w}_k(t) = \sum_{r=1}^{\infty} A_{rk}(t) w_r(t)
\]

Thus \( \hat{w}_k(t) \) is in \( U_k(t) \) for all \( t \in S_p \) and equals exactly one of the \( w_m(t) \in G(t) \).

Also, by construction, the \( \hat{w}_k(t) \) are bounded in \( L^\infty(S_p; V') \). Therefore, there is a subsequence of these, still called \( \hat{w}_k \) which converges weakly to a function \( w \) in \( L^2(S_p; V') \). Thus \( w \) is a weak limit point of \( \co \left( \bigcup_{j=1}^{\infty} \hat{w}_j \right) \) for each \( k \). Therefore, in the open ball \( B\left(w, \frac{1}{k}\right) \subseteq L^2(S_p; V') \) with respect to the strong topology, there is a convex combination \( \sum_{j=k}^{\infty} c_{jk} \hat{w}_j \) (the \( c_{jk} \) add to 1 and only finitely many are nonzero). Since \( G(t) \) is convex and closed, this convex combination is in \( G(t) \). Off a set of measure zero, we can assume this convergence of \( \sum_{j=k}^{\infty} c_{jk} \hat{w}_j \) as \( k \to \infty \) happens pointwise for a suitable subsequence. However,

\[
\sum_{j=k}^{\infty} c_{jk} \hat{w}_j(t) \in U_k(t) \subseteq A(u(t), t) + B\left(0, \frac{2}{k}\right)
\]

Thus \( w(t) \in A(u(t), t) \) a.e. \( t \) because \( A(u(t), t) \) is a closed set. Since \( w \) is the pointwise limit of measurable functions off a set of measure zero, it can be assumed
You can verify that this is indeed a filtration \( \left\{ \mathcal{F}_t \right\} \). Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.

First, we assume the following conditions on \( A \).

Then

\[
\lim_{n \to \infty} \|z_n - z\|_{L^1} = 0.
\]

This follows from the estimates above.

Now let us consider some special cases.
CHAPTER 46. MULTIFUNCTIONS AND THEIR MEASURABILITY

1. \( A (\cdot, t, \omega) : V \to \mathcal{P}(V') \) is pseudomonotone and bounded: \( A (u, t, \omega) \) is a closed convex set for each \((t, \omega), u \to A (u, t, \omega) \) is bounded, and if \( u_n \to u \) weakly and
\[
\lim \sup_{n \to \infty} \langle A (u_n, t, \omega), u_n - u \rangle \leq 0
\]
then for any \( v \in V, \)
\[
\lim \inf_{n \to \infty} \langle A (u_n, t, \omega), u_n - v \rangle \geq \langle z (v), u - v \rangle \text{ some } z (v) \in A (u, t, \omega)
\]

2. \( A (\cdot, t, \omega) \) satisfies the estimates: There exists \( b_1 \geq 0 \) and \( b_2 \geq 0 \), such that
\[
|||z|||_{V'} \leq b_1 |||u|||_p^{p-1} + b_2 (t, \omega), \quad (46.7.32)
\]
for all \( z \in A (u, t, \omega) \), \( b_2 (\cdot, \cdot) \in L^p ([0, T] \times \Omega) \).

3. There exist a positive constant \( b_3 \) and a nonnegative function \( b_4 \) that is \( B ([0, T]) \times \mathcal{F}_T \) measurable and also \( b_4 (\cdot, \cdot) \in L^1 ([0, T] \times \Omega) \), such that for some \( \lambda \geq 0, \)
\[
\inf_{z \in A (u, t, \omega)} \langle z, u \rangle \geq b_3 |||u|||_p^p - b_4 (t, \omega) - \lambda |u|_H^2. \quad (46.7.33)
\]

4. The mapping \( (t, \omega) \to A (u (t, \omega), t, \omega) \) is measurable in the sense that
\[
(t, \omega) \to A (u (t, \omega), t, \omega)
\]
is a measurable multifunction with respect to \( \mathcal{P} \) whenever \( (t, \omega) \to u (t, \omega) \) is in \( V \equiv V_p \equiv L^p ([0, T] \times \Omega; V, \mathcal{P}) \).

5. For \( u \in V_p \), we define \( \hat{A} (u) \in \mathcal{P} (V'_p) \) as follows: \( z \in \hat{A} (u) \) means that \( z (t, \omega) \in A (u (t, \omega), t, \omega) \) a.e. \((t, \omega) \). Thus this is the Nemytskii operator for \( A (\cdot, t, \omega) \).

In the following theorem and in arguments which take place below, \( U \) will be a Hilbert space dense in \( V \) with the inclusion map compact. Such a Hilbert space always exists and is important in probability theory where \((i, U, V)\) is an abstract Wiener space. However, in most applications from partial differential equations, it suffices to take \( U \) as a suitable Sobolev space. Also for \( S \) a set in \([0, T] \times \Omega, S_\omega \) will denote \( \{t : (t, \omega) \in S\} \).

**Theorem 46.7.4** Suppose conditions 1 - 5 hold. Also, suppose \( U \) is a separable Hilbert space dense in \( V \), a reflexive separable Banach space with the inclusion map compact and \( V \) is dense in a Hilbert space \( H \). Thus
\[
U \subseteq V \subseteq H = H' \subseteq V' \subseteq U',
\]
Then, the operator \( \hat{A} \) in the definition 5 satisfies the following.
46.7. LIMIT CONDITIONS FOR NEMYTSKII OPERATORS

Hypotheses:

\[ u_n \to u \text{ weakly in } V, \quad \limsup_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} \leq 0, \]

for \( z_n \in \hat{A}u_n \). For each \( \omega \), off a set of \( P \) measure zero \( N \), every subsequence of \( u_n (t, \omega) \) has a further subsequence, possibly depending on \( t, \omega \) such that

\[ u_n (t, \omega) \to u(t, \omega) \text{ weakly in } U', \]

Assume also that

\[ \sup_{\omega \notin \Omega \setminus N} \sup_{n} \sup_{t \in [0,T]} \lambda |u_n (t, \omega)|_H < \infty. \quad (46.7.34) \]

Conclusion: If the above conditions hold, then there exists \( z(v) \) with

\[ \lim \inf_{n \to \infty} \langle z_n, u_n - v \rangle_{V', V} \geq \langle z(v), u - v \rangle_{V', V}. \]

where \( z(v) \in \hat{A}(u) \).

Proof: It was argued above that \( \hat{A}(u) \) is closed and convex. Let \( \Sigma \) have measure zero and for each \( (t, \omega) \notin \Sigma \), \( z_n (t, \omega) \in A (u_n (t, \omega), t, \omega) \) for each \( n \). Now \( \Sigma_\omega \) has measure zero for a.e. \( \omega \) since otherwise \( \Sigma \) would not have measure zero. These are the \( \omega \) of interest in the following argument, and we can simply include the exceptional \( \omega \) in the set of measure zero \( N \) which is being ignored since it has measure zero.

First we claim that if \( t \notin \Sigma_\omega \), then

\[ \lim \inf_{n \to \infty} \langle z_n (t, \omega), u_n (t, \omega) - u(t, \omega) \rangle \geq 0. \]

Proof of the claim: Let \( t \notin \Sigma_\omega \) be fixed and suppose to the contrary that

\[ \lim_{n \to \infty} \inf_{n \to \infty} \langle z_n (t, \omega), u_n (t, \omega) - u(t, \omega) \rangle < 0. \quad (46.7.35) \]

Then, there exists a subsequence \( \{n_k\} \), which may depend on \( t, \omega \), such that

\[ \lim_{k \to \infty} \langle z_{n_k} (t, \omega), u_{n_k} (t, \omega) - u(t, \omega) \rangle \]

\[ = \lim_{n \to \infty} \inf_{n \to \infty} \langle z_n (t, \omega), u_n (t, \omega) - u(t, \omega) \rangle < 0. \quad (46.7.36) \]

Now, condition \( \delta \) implies that for all \( k \) large enough,

\[ b_3 |u_{n_k} (t, \omega)||_V^p - b_4 (t, \omega) - \lambda |u_{n_k} (t, \omega)||_H^2 < ||z_{n_k} (t, \omega)||_{V', V}||u(t, \omega)||_V \]

\[ \leq \left( b_1 ||u_{n_k} (t, \omega)||_V^{p-1} + b_2 (t, \omega) \right) ||u(t, \omega)||_V, \]

therefore, \( ||u_{n_k} (t, \omega)||_V \) and consequently \( ||z_{n_k} (t, \omega)||_{V', V} \), are bounded. This follows from \( 46.7.33 \). Note that \( ||z_{n_k} (t, \omega)||_{V', V} \) is bounded independently of \( n_k \) because of
the assumption that \( A(\cdot, t, \omega) \) is bounded and we just showed that \( \|u_{nk}(t, \omega)\|_V \) is bounded.

Taking a further subsequence if necessary, let \( u_{nk}(t, \omega) \to u(t, \omega) \) weakly in \( U' \) and \( u_{nk}(t, \omega) \to \xi \) weakly in \( V \). Thus, by density considerations, \( \xi = u(t, \omega) \). Now, (46.7.36) and the limit conditions for pseudomonotone operators imply that the \( \lim \inf \) condition holds. There exists \( z_\infty \in A(u(t, \omega), t, \omega) \) such that

\[
\liminf_{k \to \infty} \langle z_{nk}(t, \omega), u_{nk}(t, \omega) - u(t, \omega) \rangle \geq \langle z_\infty, u(t, \omega) - u(t, \omega) \rangle = 0
\]

and

\[
\lim_{k \to \infty} \langle z_{nk}(t, \omega), u_{nk}(t, \omega) - u(t, \omega) \rangle,
\]

which is a contradiction. This completes the proof of the claim.

We continue with the proof of the theorem. It follows from this claim that for given \( \omega \), every \( t \notin \Sigma_\omega \),

\[
\liminf_{n \to \infty} \langle z_n(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle \geq 0. \tag{46.7.38}
\]

Also, it is assumed that

\[
\limsup_{n \to \infty} \langle z_n, u_n - u \rangle_V \leq 0.
\]

Then from the estimates,

\[
\int_\Omega \int_0^T \left( b_3 \|u_n(t, \omega)\|_V^p - b_4(t, \omega) - \lambda |u_n(t, \omega)|_H^2 \right) \, dt \, dP
\]

\[
\leq \int_\Omega \int_0^T \|u(t, \omega)\|_V \left( \|u_n(t, \omega)\|_V^{p-1} b_1 + b_2 \right) \, dt \, dP
\]

so it is routine to get \( \|u_n\|_V \) is bounded. This follows from the assumptions, in particular (46.7.39).

Now, the coercivity condition \( \S \) shows that if \( y \in V \), then

\[
\langle z_n(t, \omega), u_n(t, \omega) - y(t, \omega) \rangle \geq b_3 \|u_n(t, \omega)\|_V^p - b_4(t, \omega) - \lambda |u_n(t, \omega)|_H^2 - \left( b_1 \|u_n(t, \omega)\|_V^{p-1} + b_2(t, \omega) \right) \|y(t, \omega)\|_V.
\]

Using \( p - 1 = \frac{p}{p'} \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \), the right-hand side of this inequality equals

\[
b_3 \|u_n(t, \omega)\|_V^p - b_4(t, \omega) - b_1 \|u_n(t, \omega)\|^{p/p'}_V \|y(t, \omega)\|_V
\]

\[
- b_2(t, \omega) \|y(t, \omega)\|_V - \lambda |u_n(t, \omega)|_H^2,
\]

the last term being bounded independent of \( t, n \) by assumption. Thus there exists \( c(\cdot, \cdot) \in L^1([0, T] \times \Omega) \) and a positive constant \( C \) such that

\[
\langle z_n(t, \omega), u_n(t, \omega) - y(t, \omega) \rangle \geq -c(t, \omega) - C \|y(t, \omega)\|_V^p. \tag{46.7.39}
\]
Letting $y = u$, we use Fatou’s lemma to write
\[
\liminf_{n \to \infty} \int_0^T \int_\Omega \left( \langle z_n(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle + c(t, \omega) + C \| u(t, \omega) \|^{p}_V \right) \, dt \, dP \geq \\
\int_0^T \int_\Omega \liminf_{n \to \infty} \langle z_n(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle + (c(t, \omega) + C \| u(t, \omega) \|^{p}_V) \, dt \, dP.
\]
Here, we added the term $c(t, \omega) + C \| u(t, \omega) \|^{p}_V$ to make the integrand nonnegative in order to apply Fatou’s lemma. Thus,
\[
\liminf_{n \to \infty} \int_0^T \int_\Omega \langle z_n(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle \, dt \, dP \geq 0.
\]
Consequently, using the claim in the last inequality,
\[
0 \geq \limsup_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V}
\geq \liminf_{n \to \infty} \int_0^T \int_\Omega \langle z_n(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle \, dt \, dP
= \liminf_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V}
\geq \int_0^T \int_\Omega \liminf_{n \to \infty} \langle z_n(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle \, dt \, dP \geq 0,
\]
hence, we find that
\[
\lim_{n \to \infty} \langle z_n, u_n - u \rangle_{V', V} = 0.
\] (46.7.40)

We need to show that if $y$ is given in $V$ then
\[
\liminf_{n \to \infty} \langle z_n, u_n - y \rangle_{V', V} \geq \langle z(y), u - y \rangle_{V', V}, \quad z(y) \in \hat{A} u
\]
for all $z \in \hat{A} u$. Take a subsequence, denoted still with subscript $n$ such that
\[
\eta = \liminf_{n \to \infty} \langle z_n, u_n - y \rangle_{V', V} < \langle z, u - y \rangle_{V', V},
\] (46.7.41)
for all $z \in \hat{A} u$. Note that this subsequence does not depend on $(t, \omega)$. Thus
\[
\lim_{n \to \infty} \langle z_n, u_n - y \rangle_{V', V} < \langle z, u - y \rangle_{V', V}
\] (46.7.42)
We will obtain a contradiction to this. In what follows, we continue to use the subsequence just described which satisfies the above inequality.
The estimate (46.7.39) implies,
\[ 0 \leq \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle^- \leq c(t,\omega) + C\|u(t,\omega)\|_{V'}^p, \]  
\begin{equation}
(46.7.43)
\end{equation}
where \(c\) is a function in \(L^1(0,T)\). Thanks to (46.7.38),
\[ \liminf_{n \to \infty} \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle \geq 0, \text{ a.e.} \]
and, therefore, the following pointwise limit exists,
\[ \lim_{n \to \infty} \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle^- = 0, \text{ a.e.} \]
and so we may apply the dominated convergence theorem using (46.7.43) and conclude
\[ \lim_{n \to \infty} \int_\Omega \int_0^T \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle^- dt \, dP = 0 \]
Now, it follows from (46.7.40) and the above equation, that
\[ \lim_{n \to \infty} \int_\Omega \int_0^T \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle^+ dt \, dP = 0 \]
Therefore, both
\[ \int_\Omega \int_0^T \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle^+ dt \, dP \]
and
\[ \int_\Omega \int_0^T \langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle^- dt \, dP \]
converge to 0, thus,
\[ \lim_{n \to \infty} \int_\Omega \int_0^T |\langle z_n(t,\omega), u_n(t,\omega) - u(t,\omega) \rangle| \, dt \, dP = 0 \]
\begin{equation}
(46.7.44)
\end{equation}
From the above, it follows that there exists a further subsequence \(\{n_k\}\) not depending on \(t,\omega\) such that
\[ |\langle z_{n_k}(t,\omega), u_{n_k}(t,\omega) - u(t,\omega) \rangle| \to 0 \quad \text{a.e. } (t,\omega). \]
\begin{equation}
(46.7.45)
\end{equation}
46.7. LIMIT CONDITIONS FOR NEMYSKII OPERATORS

Therefore, by the pseudomonotone limit condition for \( A \) there exists \( w_{t,\omega} \in A(u(t,\omega),t,\omega) \) such that for a.e. \((t,\omega)\)

\[
\alpha(t,\omega) = \liminf_{k \to \infty} \langle z_{n_k}(t,\omega), u_{n_k}(t,\omega) - y(t,\omega) \rangle
\]

\[
\geq \liminf_{k \to \infty} \langle z_{n_k}(t,\omega), u(t,\omega) - y(t,\omega) \rangle \geq \langle w_{t,\omega}, u(t,\omega) - y(t,\omega) \rangle.
\]

Then on the exceptional set, let \( \alpha(t,\omega) \equiv \infty \), and consider the set

\[
F(t,\omega) = \{ w \in A(u(t,\omega),t,\omega) : \langle w, u(t,\omega) - y(t,\omega) \rangle \leq \alpha(t,\omega) \},
\]

which then satisfies \( F(t,\omega) \neq \emptyset \). Now \( F(t,\omega) \) is closed and convex in \( V' \).

**Claim**: \( (t,x) \to F(t,\omega) \) has a measurable selection off a set of measure zero.

**Proof of claim**: Letting \( B(0,C(t,\omega)) \) contain \( A(u(t,\omega),t,\omega) \), we can assume \( (t,\omega) \to C(t,\omega) \) is \( \mathcal{P} \) measurable by using the estimates and the measurability of \( u \).

For \( \gamma \in \mathbb{N} \), let \( S_\gamma \equiv \{(t,\omega) : C(t,\omega) < \gamma \} \). If it is shown that \( F \) has a measurable selection on \( S_\gamma \), then it follows that it has a measurable selection. Thus in what follows, assume that \( (t,\omega) \in S_\gamma \).

Define

\[
G(t,\omega) \equiv \left\{ w : \langle w, u(t,\omega) - y(t,\omega) \rangle < \alpha(t,\omega) + \frac{1}{n}, (t,\omega) \notin \Sigma \right\} \cap B(0,\gamma)
\]

Thus, it was shown above that this \( G(t,\omega) \neq \emptyset \) at least for large enough \( \gamma \). For \( U \) open,

\[
G^-(U) \equiv \left\{ (t,\omega) \in S_\gamma : \text{ for some } w \in U \cap B(0,\gamma), \langle w, u(t,\omega) - y(t,\omega) \rangle < \alpha(t,\omega) + \frac{1}{n} \right\}
\]

Let \( \{w_j\} \) be a dense subset of \( U \cap B(0,\gamma) \). This is possible because \( V' \) is separable. The expression in * equals

\[
\bigcup_{k=1}^\infty \left\{ (t,\omega) \in S_\gamma : \langle w_k, u(t,\omega) - y(t,\omega) \rangle < \alpha(t,\omega) + \frac{1}{n} \right\}
\]

which is measurable. Thus \( G \) is a measurable multifunction.

Since \( (t,\omega) \to G(t,\omega) \) is measurable, there is a sequence \( \{w_n(t,\omega)\} \) of measurable functions such that \( \bigcup_{n=1}^\infty w_n(t,\omega) \) equals

\[
G(t,\omega) = \left\{ w : \langle w, u(t,\omega) - y(t,\omega) \rangle \leq \alpha(t,\omega) + \frac{1}{n}, t \notin \Sigma \right\} \cap B(0,\gamma)
\]

As shown above, there exists \( w_{t,\omega} \) in \( A(u(t,\omega),t,\omega) \) as well as \( G(t,\omega) \). Thus there is a sequence of \( w_r(t,\omega) \) converging to \( w_{t,\omega} \). Of course \( r \) will need to depend on \( t,\omega \). Since \( (t,\omega) \to A(u(t,\omega),t,\omega) \) is a measurable multifunction, it has a countable subset of \( \mathcal{P} \) measurable functions \( \{z_k(t,\omega)\} \) which is dense in \( A(u(t,\omega),t,\omega) \). Let

\[
U_k(t,\omega) = \bigcup_m B\left(z_m(t,\omega), \frac{1}{k} \right) \subseteq A(u(t,\omega),t,\omega) + B\left(0, \frac{2}{k} \right)
\]
Now define \( A_{1k} = \{(t, \omega) : w_1(t, \omega) \in U_k(t, \omega)\} \). Then let
\[
A_{2k} = \{(t, \omega) \notin A_{1k} : w_2(t, \omega) \in U_k(t, \omega)\}
\]
and
\[
A_{3k} = \{(t, \omega) \notin \bigcup_{i=1}^{2} A_{ik} : w_3(t, \omega) \in U_k(t, \omega)\}
\]
and so forth. Any \((t, \omega) \in S_\gamma\) must be contained in one of these \(A_r\) for some \(r\) since if not so, there would not be a sequence \(w_r(t, \omega)\) converging to \(w_{t, \omega} \in A(u(t, \omega), t, \omega)\). These \(A_r\) partition \(S_\gamma\) and each is measurable since the \(\{z_k(t, \omega)\}\) are measurable. Let
\[
\hat{w}_k(t, \omega) \equiv \sum_{r=1}^{\infty} \chi_{A_{rk}}(t, \omega) w_r(t, \omega)
\]
Thus \(\hat{w}_k(t, \omega)\) is in \(U_k(t, \omega)\) for all \((t, \omega) \in S_\gamma\) and equals exactly one of the \(w_m(t, \omega) \in G(t, \omega)\).

Also, by construction, the \(\hat{w}_k(\cdot, \cdot)\) are bounded in \(L^\infty(S_\gamma; V')\). Therefore, there is a subsequence of these, still called \(\hat{w}_k\) which converges weakly to a function \(w\) in \(L^2(S_\gamma; V')\). Thus \(w\) is a weak limit point of \(\co{\bigcup_{j=k}^{\infty} \hat{w}_j}\) for each \(k\). Therefore, in the open ball \(B\left(w, \frac{1}{k}\right) \subseteq L^2(S_\gamma; V')\) with respect to the strong topology, there is a convex combination \(\sum_{j=k}^{\infty} c_{jk} \hat{w}_j\) (the \(c_{jk}\) add to 1 and only finitely many are nonzero). Since \(G(t, \omega)\) is convex and closed, this convex combination is in \(G(t, \omega)\). Off a set of \(P\) measure zero, we can assume this convergence of \(\sum_{j=k}^{\infty} c_{jk} \hat{w}_j\) as \(k \to \infty\) happens pointwise a.e. for a suitable subsequence. However,
\[
\sum_{j=k}^{\infty} c_{jk} \hat{w}_j(t, \omega) \in U_k(t, \omega) \subseteq A(u(t, \omega), t, \omega) + B\left(0, \frac{2}{r}\right)
\]
Thus \(w(t, \omega) \in A(u(t, \omega), t, \omega)\) a.e. \((t, \omega)\) because \(A(u(t, \omega), t, \omega)\) is a closed set. Since \(w\) is the pointwise limit of measurable functions off a set of measure zero, it can be assumed measurable and for a.e. \((t, \omega), w(t, \omega) \in A(u(t, \omega), t, \omega) \cap G(t, \omega)\).

Now denote this measurable function \(w_n\). Then
\[
w_n(t, \omega) \in A(u(t, \omega), t, \omega), \langle w_n(t, \omega), u(t, \omega) - y(t, \omega) \rangle \leq \alpha(t, \omega) + \frac{1}{n} \text{ a.e. } (t, \omega)
\]
These \(w_n(t, \omega)\) are bounded for each \((t, \omega)\) off a set of measure zero and so by Lemma 16.7.1 there is a \(P\) measurable function \((t, \omega) \to z(t, \omega)\) and a subsequence \(w_{n(t, \omega)}(t, \omega) \to z(t, \omega)\) weakly as \(n(t, \omega) \to \infty\). Now \(A(u(t, \omega), t, \omega)\) is closed and convex, and \(w_{n(t, \omega)}(t, \omega)\) is in \(A(u(t, \omega), t, \omega)\), and so \(z(t, \omega) \in A(u(t, \omega), t, \omega)\) and
\[
\langle z(t, \omega), u(t, \omega) - y(t, \omega) \rangle \leq \alpha(t, \omega) = \lim_{k \to \infty} \inf \langle z_{nk}(t, \omega), u_{nk}(t, \omega) - y(t, \omega) \rangle \quad (**)
\]
Therefore, \(t \to F(t, \omega)\) has a measurable selection on \(S_\gamma\) excluding a set of measure zero, namely \(z(t, \omega)\) which will be called \(z_\gamma(t, \omega)\) in what follows.
Then $F(t, \omega)$ has a measurable selection on $[0, T] \times \Omega$ other than a set of measure zero. To see this, enlarge $\Sigma$ to include the exceptional sets of measure zero in the above argument for each $\gamma$. Then partition $[0, T] \times \Omega \setminus \Sigma$ as follows. For $\gamma = 1, 2, \cdots$, consider $S_\gamma \setminus S_{\gamma - 1}, \gamma = 1, 2, \cdots$ for $S_0$ defined as $\emptyset$. Then letting $z_\gamma$ be the selection for $(t, \omega) \in S_\gamma$, let $z(t, \omega) = \sum_{\gamma=1}^{\infty} z_\gamma(t, \omega) \chi_{S_\gamma \setminus S_{\gamma - 1}}(t, \omega)$. The estimates imply $z \in \mathcal{V}'$ and so $z \in \hat{A}(u)$.

From the estimates, there exists $h \in L^1([0, T] \times \Omega)$ such that

$$\langle z(t, \omega), u(t, \omega) - y(t, \omega) \rangle \geq -|h(t, \omega)|$$

Thus, from the above inequality,

$$\|h\|_{L^1} + \langle z, u - y \rangle_{\mathcal{V}', \mathcal{V}}$$

$$\leq \int_0^T \int_\Omega \liminf_{k \to \infty} \langle z_{n_k}(t, \omega), u_{n_k}(t, \omega) - y(t, \omega) \rangle + |h(t, \omega)| \, dt \, dP$$

$$\leq \liminf_{k \to \infty} \langle z_{n_k}, u_{n_k} - y \rangle_{\mathcal{V}', \mathcal{V}} + \|h\|_{L^1}$$

$$= \lim_{n \to \infty} \langle z_n, u_n - y \rangle_{\mathcal{V}', \mathcal{V}} + \|h\|_{L^1}$$

which contradicts $\text{[IV.4.31]}$.  

The difficulty with this, is that it is hard to get the hypotheses holding. If you have $u_n \to u$ weakly in $\mathcal{V}$, then how do you get $u_n(t, \omega) \to u(t, \omega)$ weakly in $U'$, for a subsequence, this for each $\omega$ not in a set of measure zero? The weak convergence does not seem to give pointwise weak convergence of the sort you need. More precisely, the pointwise convergence you get, might not be the right thing because in $u_n$ it will be $u_{n(\omega)}$. This is the problem with this theorem. It seems correct but not very useful.
Part V

Complex Analysis
Chapter 47

The Complex Numbers

The reader is presumed familiar with the algebraic properties of complex numbers, including the operation of conjugation. Here a short review of the distance in \( \mathbb{C} \) is presented.

The length of a complex number, referred to as the modulus of \( z \) and denoted by \(|z|\) is given by

\[
|z| \equiv (x^2 + y^2)^{1/2} = (\bar{z}z)^{1/2},
\]

Then \( \mathbb{C} \) is a metric space with the distance between two complex numbers, \( z \) and \( w \) defined as

\[
d(z, w) \equiv |z - w|.
\]

This metric on \( \mathbb{C} \) is the same as the usual metric of \( \mathbb{R}^2 \). A sequence, \( z_n \to z \) if and only if \( x_n \to x \) in \( \mathbb{R} \) and \( y_n \to y \) in \( \mathbb{R} \) where \( z = x + iy \) and \( z_n = x_n + iy_n \). For example if \( z_n = \frac{n}{n+1} + i\frac{1}{n} \), then \( z_n \to 1 + 0i = 1 \).

**Definition 47.0.5** A sequence of complex numbers, \( \{z_n\} \) is a Cauchy sequence if for every \( \varepsilon > 0 \) there exists \( N \) such that \( n, m > N \) implies \( |z_n - z_m| < \varepsilon \).

This is the usual definition of Cauchy sequence. There are no new ideas here.

**Proposition 47.0.6** The complex numbers with the norm just mentioned forms a complete normed linear space.

**Proof:** Let \( \{z_n\} \) be a Cauchy sequence of complex numbers with \( z_n = x_n + iy_n \). Then \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences of real numbers and so they converge to real numbers, \( x \) and \( y \) respectively. Thus \( z_n = x_n + iy_n \to x + iy \). \( \mathbb{C} \) is a linear space with the field of scalars equal to \( \mathbb{C} \). It only remains to verify that \(||\) satisfies the axioms of a norm which are:

\[
|z + w| \leq |z| + |w|
\]

\[
|z| \geq 0 \text{ for all } z
\]
The only one of these axioms of a norm which is not completely obvious is the first one, the triangle inequality. Let \( z = x + iy \) and \( w = u + iv \)

\[
|z + w|^2 = (z + w)(\overline{z} + \overline{w}) = |z|^2 + |w|^2 + 2\, \text{Re}(z\overline{w}) \\
\leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2
\]

and this verifies the triangle inequality.

**Definition 47.0.7** An infinite sum of complex numbers is defined as the limit of the sequence of partial sums. Thus,

\[
\sum_{k=1}^{\infty} a_k \equiv \lim_{n \to \infty} \sum_{k=1}^{n} a_k.
\]

Just as in the case of sums of real numbers, an infinite sum converges if and only if the sequence of partial sums is a Cauchy sequence.

From now on, when \( f \) is a function of a complex variable, it will be assumed that \( f \) has values in \( X \), a complex Banach space. Usually in complex analysis courses, \( f \) has values in \( \mathbb{C} \) but there are many important theorems which don’t require this so I will leave it fairly general for a while. Later the functions will have values in \( \mathbb{C} \). If you are only interested in this case, think \( \mathbb{C} \) whenever you see \( X \).

**Definition 47.0.8** A sequence of functions of a complex variable, \( \{f_n\} \) converges uniformly to a function, \( g \) for \( z \in S \) if for every \( \varepsilon > 0 \) there exists \( N_\varepsilon \) such that if \( n > N_\varepsilon \), then

\[
||f_n(z) - g(z)|| < \varepsilon
\]

for all \( z \in S \). The infinite sum \( \sum_{k=1}^{\infty} f_n \) converges uniformly on \( S \) if the partial sums converge uniformly on \( S \). Here \(||\cdot||\) refers to the norm in \( X \), the Banach space in which \( f \) has its values.

The following proposition is also a routine application of the above definition. Neither the definition nor this proposition say anything new.

**Proposition 47.0.9** A sequence of functions, \( \{f_n\} \) defined on a set \( S \), converges uniformly to some function, \( g \) if and only if for all \( \varepsilon > 0 \) there exists \( N_\varepsilon \) such that whenever \( m, n > N_\varepsilon \),

\[
||f_n - f_m||_\infty < \varepsilon.
\]

Here \(||f||_\infty \equiv \sup \{|f(z)| : z \in S\} \). Just as in the case of functions of a real variable, one of the important theorems is the Weierstrass M test. Again, there is nothing new here. It is just a review of earlier material.
Theorem 47.0.10 Let \( \{f_n\} \) be a sequence of complex valued functions defined on \( S \subseteq \mathbb{C} \). Suppose there exists \( M_n \) such that \( \|f_n\|_\infty < M_n \) and \( \sum M_n \) converges. Then \( \sum f_n \) converges uniformly on \( S \).

\[ \left\| \sum_{k=1}^{n} f_k (z) - \sum_{k=1}^{m} f_k (z) \right\| \leq \sum_{k=m+1}^{n} \|f_k (z)\| \leq \sum_{k=m+1}^{\infty} M_k < \varepsilon \]

whenever \( m \) is large enough. Therefore, the sequence of partial sums is uniformly Cauchy on \( S \) and therefore, converges uniformly to \( \sum_{k=1}^{\infty} f_k (z) \) on \( S \).

47.1 The Extended Complex Plane

The set of complex numbers has already been considered along with the topology of \( \mathbb{C} \) which is nothing but the topology of \( \mathbb{R}^2 \). Thus, for \( z_n = x_n + iy_n, z_n \to z \equiv x + iy \) if and only if \( x_n \to x \) and \( y_n \to y \). The norm in \( \mathbb{C} \) is given by

\[ |x + iy| \equiv ((x + iy)(x - iy))^{1/2} = (x^2 + y^2)^{1/2} \]

which is just the usual norm in \( \mathbb{R}^2 \) identifying \((x, y)\) with \( x + iy \). Therefore, \( \mathbb{C} \) is a complete metric space topologically like \( \mathbb{R}^2 \) and so the Heine Borel theorem that compact sets are those which are closed and bounded is valid. Thus, as far as topology is concerned, there is nothing new about \( \mathbb{C} \).

The extended complex plane, denoted by \( \bar{\mathbb{C}} \), consists of the complex plane, \( \mathbb{C} \) along with another point not in \( \mathbb{C} \) known as \( \infty \). For example, \( \infty \) could be any point in \( \mathbb{R}^3 \). A sequence of complex numbers, \( z_n \), converges to \( \infty \) if, whenever \( K \) is a compact set in \( \mathbb{C} \), there exists a number, \( N \) such that for all \( n > N \), \( z_n \notin K \). Since compact sets in \( \mathbb{C} \) are closed and bounded, this is equivalent to saying that for all \( R > 0 \), there exists \( N \) such that if \( n > N \), then \( z_n \notin B(0, R) \) which is the same as saying \( \lim_{n \to \infty} |z_n| = \infty \) where this last symbol has the same meaning as it does in calculus.

A geometric way of understanding this in terms of more familiar objects involves a concept known as the Riemann sphere.

Consider the unit sphere, \( S^2 \) given by \( (z - 1)^2 + y^2 + x^2 = 1 \). Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, \( p \) in the complex plane to the point \( (0, 0, 2) \) on the top of this sphere and let \( \theta (p) \) denote the point of this sphere which the line intersects. Define \( \theta (\infty) \equiv (0, 0, 2) \).
Then \( \theta^{-1} \) is sometimes called stereographic projection. The mapping \( \theta \) is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that \( \theta^{-1} \) is also continuous. In terms of the extended complex plane, \( \hat{\mathbb{C}} \), a sequence, \( z_n \) converges to \( \infty \) if and only if \( \theta(z_n) \) converges to \( (0, 0, 2) \) and a sequence, \( z_n \) converges to \( z \in \mathbb{C} \) if and only if \( \theta(z_n) \to \theta(z) \).

In fact this makes it easy to define a metric on \( \hat{\mathbb{C}} \).

**Definition 47.1.1** Let \( z, w \in \hat{\mathbb{C}} \) including possibly \( w = \infty \). Then let \( d(x, w) \equiv |\theta(z) - \theta(w)| \) where this last distance is the usual distance measured in \( \mathbb{R}^3 \).

**Theorem 47.1.2** \( (\hat{\mathbb{C}}, d) \) is a compact, hence complete metric space.

**Proof:** Suppose \( \{z_n\} \) is a sequence in \( \hat{\mathbb{C}} \). This means \( \{\theta(z_n)\} \) is a sequence in \( S^2 \) which is compact. Therefore, there exists a subsequence, \( \{\theta z_{n_k}\} \) and a point, \( z \in S^2 \) such that \( \theta z_{n_k} \to \theta z \) in \( S^2 \) which implies immediately that \( d(z_{n_k}, z) \to 0 \). A compact metric space must be complete.

### 47.2 Exercises

1. Prove the root test for series of complex numbers. If \( a_k \in \mathbb{C} \) and \( r \equiv \limsup_{n \to \infty} |a_n|^{1/n} \) then

   \[
   \sum_{k=0}^{\infty} a_k \begin{cases} 
   \text{converges absolutely if } r < 1 \\
   \text{diverges if } r > 1 \\
   \text{test fails if } r = 1.
   \end{cases}
   \]

2. Does \( \lim_{n \to \infty} n \left( \frac{2+1}{3} \right)^n \) exist? Tell why and find the limit if it does exist.

3. Let \( A_0 = 0 \) and let \( A_n = \sum_{k=1}^{n} a_k \) if \( n > 0 \). Prove the partial summation formula,

   \[
   \sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).
   \]

Now using this formula, suppose \( \{b_n\} \) is a sequence of real numbers which converges to 0 and is decreasing. Determine those values of \( \omega \) such that \( |\omega| = 1 \) and \( \sum_{k=1}^{\infty} b_k \omega^k \) converges.
4. Let \( f : U \subseteq \mathbb{C} \rightarrow \mathbb{C} \) be given by \( f(x + iy) = u(x, y) + iv(x, y) \). Show \( f \) is continuous on \( U \) if and only if \( u : U \rightarrow \mathbb{R} \) and \( v : U \rightarrow \mathbb{R} \) are both continuous.
Chapter 48

Riemann Stieltjes Integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. I will base this on the notion of Riemann Stieltjes integrals as in [31], [86], and [59]. The Riemann Stieltjes integral is a generalization of the usual Riemann integral and requires the concept of a function of bounded variation.

**Definition 48.0.1** Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a function. Then $\gamma$ is of bounded variation if

$$
\sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < \cdots < t_n = b \right\} \equiv V(\gamma, [a, b]) < \infty
$$

where the sums are taken over all possible lists, \{a = t_0 < \cdots < t_n = b\}. The set of points $\gamma([a, b])$ will also be denoted by $\gamma^*$.

The idea is that it makes sense to talk of the length of the curve $\gamma([a, b])$, defined as $V(\gamma, [a, b])$. For this reason, in the case that $\gamma$ is continuous, such an image of a bounded variation function is called a rectifiable curve.

**Definition 48.0.2** Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation and let $f : \gamma^* \rightarrow X$. Letting $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, define

$$||\mathcal{P}|| \equiv \max \{ |t_j - t_{j-1}| : j = 1, \cdots, n \}$$

and the Riemann Stieltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^{n} f(\gamma(t_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, $\tau_j$ used. It is understood that this point is arbitrary.) Define
\[ \int_{\gamma} f d\gamma \] as the unique number which satisfies the following condition. For all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( ||P|| \leq \delta \), then
\[ \left| \int_{\gamma} f d\gamma - S(P) \right| < \varepsilon. \]

Sometimes this is written as
\[ \int_{\gamma} f d\gamma \equiv \lim_{||P|| \to 0} S(P). \]

The set of points in the curve, \( \gamma ([a,b]) \) will be denoted sometimes by \( \gamma^* \).

Then \( \gamma^* \) is a set of points in \( \mathbb{C} \) and as \( t \) moves from \( a \) to \( b \), \( \gamma(t) \) moves from \( \gamma(a) \) to \( \gamma(b) \). Thus \( \gamma^* \) has a first point and a last point. If \( \phi : [c,d] \to [a,b] \) is a continuous nondecreasing function, then \( \gamma \circ \phi : [c,d] \to \mathbb{C} \) is also of bounded variation and yields the same set of points in \( \mathbb{C} \) with the same first and last points.

**Theorem 48.0.3** Let \( \phi \) and \( \gamma \) be as just described. Then assuming that
\[ \int_{\gamma} f d\gamma \]
exists, so does
\[ \int_{\gamma \circ \phi} f d(\gamma \circ \phi) \]
and
\[ \int_{\gamma} f d\gamma = \int_{\gamma \circ \phi} f d(\gamma \circ \phi). \] (48.0.1)

**Proof:** There exists \( \delta > 0 \) such that if \( P \) is a partition of \([a,b]\) such that \( ||P|| < \delta \), then
\[ \left| \int_{\gamma} f d\gamma - S(P) \right| < \varepsilon. \]

By continuity of \( \phi \), there exists \( \sigma > 0 \) such that if \( Q \) is a partition of \([c,d]\) with \( ||Q|| < \sigma \), \( Q = \{s_0, \ldots, s_n\} \), then \( ||\phi(s_j) - \phi(s_{j-1})|| < \delta \). Thus letting \( P \) denote the points in \([a,b]\) given by \( \phi(s_j) \) for \( s_j \in Q \), it follows that \( ||P|| < \delta \) and so
\[ \left| \int_{\gamma} f d\gamma - \sum_{j=1}^{n} f(\gamma(\phi(\tau_j))) (\gamma(\phi(s_j)) - \gamma(\phi(s_{j-1}))) \right| < \varepsilon \]
where \( \tau_j \in [s_{j-1}, s_j] \). Therefore, from the definition holds and
\[ \int_{\gamma \circ \phi} f d(\gamma \circ \phi) \]
exists.
This theorem shows that \( \int_{\gamma} f d\gamma \) is independent of the particular \( \gamma \) used in its computation to the extent that if \( \phi \) is any nondecreasing continuous function from another interval, \([c, d]\), mapping to \([a, b]\), then the same value is obtained by replacing \( \gamma \) with \( \gamma \circ \phi \).

The fundamental result in this subject is the following theorem. We have in mind functions which have values in \( C \) but there is no change if the functions have values in any complete normed vector space.

**Theorem 48.0.4** Let \( f : \gamma^* \to X \) be continuous and let \( \gamma : [a, b] \to C \) be continuous and of bounded variation. Then \( \int_{\gamma} f d\gamma \) exists. Also letting \( \delta_m > 0 \) be such that \(|t - s| < \delta_m \) implies \( ||f(\gamma(t)) - f(\gamma(s))|| < \frac{1}{m} \),

\[
\left| \int_{\gamma} f d\gamma - S(\mathcal{P}) \right| \leq \frac{2V(\gamma, [a, b])}{m}
\]

whenever \( ||\mathcal{P}|| < \delta_m \).

**Proof:** The function, \( f \circ \gamma \), is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, \( \{\delta_m\} \) such that if \(|s - t| < \delta_m \), then

\[ |f(\gamma(t)) - f(\gamma(s))| < \frac{1}{m}. \]

Let

\[ F_m \equiv \{S(\mathcal{P}) : ||\mathcal{P}|| < \delta_m\}. \]

Thus \( F_m \) is a closed set. (The symbol, \( S(\mathcal{P}) \) in the above definition, means to include all sums corresponding to \( \mathcal{P} \) for any choice of \( \tau_j \).) It is shown that

\[ \text{diam}(F_m) \leq \frac{2V(\gamma, [a, b])}{m} \]

and then it will follow there exists a unique point, \( I \in \bigcap_{m=1}^{\infty} F_m \). This is because \( X \) is complete. It will then follow \( I = \int_{\gamma} f(t) d\gamma(t) \). To verify \( \text{BSUL} \), it suffices to verify that whenever \( \mathcal{P} \) and \( \mathcal{Q} \) are partitions satisfying \( ||\mathcal{P}|| < \delta_m \) and \( ||\mathcal{Q}|| < \delta_m \),

\[ |S(\mathcal{P}) - S(\mathcal{Q})| \leq \frac{2V(\gamma, [a, b])}{m}. \]

Suppose \( ||\mathcal{P}|| < \delta_m \) and \( \mathcal{Q} \supseteq \mathcal{P} \). Then also \( ||\mathcal{Q}|| < \delta_m \). To begin with, suppose that \( \mathcal{P} \equiv \{t_0, \ldots, t_p, \ldots, t_n\} \) and \( \mathcal{Q} \equiv \{t_0, \ldots, t_{p-1}, t^*, t_p, \ldots, t_n\} \). Thus \( \mathcal{Q} \) contains only one more point than \( \mathcal{P} \). Letting \( S(\mathcal{Q}) \) and \( S(\mathcal{P}) \) be Riemann Steiltjes sums,

\[
S(\mathcal{Q}) \equiv \sum_{j=1}^{p-1} f(\gamma(\sigma_j))(\gamma(t_j) - \gamma(t_{j-1})) + f(\gamma(\sigma_0))(\gamma(t^*) - \gamma(t_{p-1}))
\]

\[
S(\mathcal{P}) \equiv \sum_{j=1}^{p-1} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})) + f(\gamma(\tau_0))(\gamma(t^*) - \gamma(t_{p-1}))
\]
+ f(\gamma(\sigma^*)) (\gamma(t_p) - \gamma(t^*)) + \sum_{j=p+1}^{n} f(\gamma(\sigma_j)) (\gamma(t_j) - \gamma(t_{j-1})),

S(P) \equiv \sum_{j=1}^{p-1} f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) +

\underbrace{f(\gamma(\tau_p)) (\gamma(t^*) - \gamma(t_{p-1})) + f(\gamma(\tau_p)) (\gamma(t_p) - \gamma(t^*))} + \sum_{j=p+1}^{n} f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})).

Therefore,

|S(P) - S(Q)| \leq \sum_{j=1}^{p-1} \frac{1}{m} |\gamma(t_j) - \gamma(t_{j-1})| + \frac{1}{m} |\gamma(t^*) - \gamma(t_{p-1})| +

\frac{1}{m} |\gamma(t_p) - \gamma(t^*)| + \sum_{j=p+1}^{n} \frac{1}{m} |\gamma(t_j) - \gamma(t_{j-1})| \leq \frac{1}{m} V(\gamma, [a, b]). \tag{48.0.4}

Clearly the extreme inequalities would be valid if Q had more than one extra point. You simply do the above trick more than one time. Let S(P) and S(Q) be Riemann Steiltjes sums for which ||P|| and ||Q|| are less than \delta_m and let \mathcal{R} \equiv P \cup Q. Then from what was just observed,

|S(P) - S(Q)| \leq |S(P) - S(\mathcal{R})| + |S(\mathcal{R}) - S(Q)| \leq \frac{2}{m} V(\gamma, [a, b]).

and this shows which proves Theorem 48.0.2. Therefore, there exists a unique complex number, I \in \cap_{m=1}^{\infty} F_m which satisfies the definition of \int_{\gamma} f d\gamma. This proves the theorem.

The following theorem follows easily from the above definitions and theorem.

**Theorem 48.0.5** Let f \in C(\gamma^*) and let \gamma : [a, b] \to \mathbb{C} be of bounded variation and continuous. Let

\[ M \geq \max \{ ||f \circ \gamma(t)|| : t \in [a, b] \}. \tag{48.0.5} \]

Then

\[ \left\| \int_{\gamma} f d\gamma \right\| \leq MV(\gamma, [a, b]). \tag{48.0.6} \]

Also if \{f_n\} is a sequence of functions of C(\gamma^*) which is converging uniformly to the function, f on \gamma^*, then

\[ \lim_{n \to \infty} \int_{\gamma} f_n d\gamma = \int_{\gamma} f d\gamma. \tag{48.0.7} \]
Proof: Let (48.0.6) hold. From the proof of the above theorem, when \(||P|| < \delta_m\),

\[
\left| \int_\gamma f d\gamma - S (P) \right| \leq \frac{2}{m} V (\gamma, [a, b])
\]

and so

\[
\left| \int_\gamma f d\gamma \right| \leq ||S (P)|| + \frac{2}{m} V (\gamma, [a, b])
\]

\[
\leq \sum_{j=1}^{n} M |\gamma (t_j) - \gamma (t_{j-1})| + \frac{2}{m} V (\gamma, [a, b])
\]

\[
\leq MV (\gamma, [a, b]) + \frac{2}{m} V (\gamma, [a, b]).
\]

This proves (48.0.7) since \(m\) is arbitrary. To verify (48.0.8) use the above inequality to write

\[
\left| \int_\gamma f d\gamma - \int f_n d\gamma \right| = \left| \int (f - f_n) d\gamma (t) \right|
\]

\[
\leq \max \{|f \circ \gamma (t) - f_n \circ \gamma (t)| : t \in [a, b]\} V (\gamma, [a, b]).
\]

Since the convergence is assumed to be uniform, this proves (48.0.8).

It turns out to be much easier to evaluate such integrals in the case where \(\gamma\) is also \(C^1 ([a, b])\). The following theorem about approximation will be very useful but first here is an easy lemma.

Lemma 48.0.6 Let \(\gamma : [a, b] \to \mathbb{C}\) be in \(C^1 ([a, b])\). Then \(V (\gamma, [a, b]) < \infty\) so \(\gamma\) is of bounded variation.

Proof: This follows from the following

\[
\sum_{j=1}^{n} |\gamma (t_j) - \gamma (t_{j-1})| = \sum_{j=1}^{n} \left| \int_{t_{j-1}}^{t_j} \gamma' (s) ds \right|
\]

\[
\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |\gamma' (s)| ds
\]

\[
\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} ||\gamma'||_\infty ds
\]

\[
= ||\gamma'||_\infty (b - a).
\]

Therefore it follows \(V (\gamma, [a, b]) \leq ||\gamma'||_\infty (b - a)\). Here \(||\gamma||_\infty = \max \{||\gamma (t)|| : t \in [a, b]\}\).

Theorem 48.0.7 Let \(\gamma : [a, b] \to \mathbb{C}\) be continuous and of bounded variation. Let \(\Omega\) be an open set containing \(\gamma^*\) and let \(f : \Omega \times K \to X\) be continuous for \(K\) a
compact set in $\mathbb{C}$, and let $\varepsilon > 0$ be given. Then there exists $\eta : [a, b] \rightarrow \mathbb{C}$ such that $\eta (a) = \gamma (a)$, $\gamma (b) = \eta (b)$, $\eta \in C^1 ([a, b])$, and

$$||\gamma - \eta|| < \varepsilon,$$  \hspace{1cm} (48.0.8)

$$\left| \int_\gamma f (\gamma, z) d\gamma - \int_\eta f (\gamma, z) d\eta \right| < \varepsilon,$$  \hspace{1cm} (48.0.9)

$$V (\eta, [a, b]) \leq V (\gamma, [a, b]),$$  \hspace{1cm} (48.0.10)

where $||\gamma - \eta|| \equiv \max \{|\gamma (t) - \eta (t)| : t \in [a, b]\}$.

**Proof:** Extend $\gamma$ to be defined on all $\mathbb{R}$ according to $\gamma (t) = \gamma (a)$ if $t < a$ and $\gamma (t) = \gamma (b)$ if $t > b$. Now define

$$\gamma_h (t) \equiv \frac{1}{2h} \int_{-2h + t}^{t + 2h} \gamma (s) ds,$$

where the integral is defined in the obvious way. That is,

$$\int_a^b \alpha (t) + i\beta (t) dt \equiv \int_a^b \alpha (t) dt + i \int_a^b \beta (t) dt.$$

Therefore,

$$\gamma_h (b) = \frac{1}{2h} \int_b^{b + 2h} \gamma (s) ds = \gamma (b),$$

$$\gamma_h (a) = \frac{1}{2h} \int_a^{a - 2h} \gamma (s) ds = \gamma (a).$$

Also, because of continuity of $\gamma$ and the fundamental theorem of calculus,

$$\gamma_h' (t) = \frac{1}{2h} \left\{ \gamma \left( t + \frac{2h}{b - a} (t - a) \right) \left( 1 + \frac{2h}{b - a} \right) - \gamma \left( -2h + t + \frac{2h}{b - a} (t - a) \right) \left( 1 + \frac{2h}{b - a} \right) \right\}$$

and so $\gamma_h \in C^1 ([a, b])$. The following lemma is significant.

**Lemma 48.0.8** $V (\gamma_h, [a, b]) \leq V (\gamma, [a, b]).$

**Proof:** Let $a = t_0 < t_1 < \cdots < t_n = b$. Then using the definition of $\gamma_h$ and changing the variables to make all integrals over $[0, 2h]$,

$$\sum_{j=1}^n |\gamma_h (t_j) - \gamma_h (t_{j-1})| =$$

$$\sum_{j=1}^n \frac{1}{2h} \int_0^{2h} \left[ \gamma \left( s - 2h + t_j + \frac{2h}{b - a} (t_j - a) \right) - \gamma \left( s - 2h + t_{j-1} + \frac{2h}{b - a} (t_{j-1} - a) \right) \right].$$
\[\gamma \left( s - 2h + t_{j-1} + \frac{2h}{b-a} (t_{j-1} - a) \right) \leq \frac{1}{2h} \int_0^{2h} \sum_{j=1}^n \left| \gamma \left( s - 2h + t_j + \frac{2h}{b-a} (t_j - a) \right) - \gamma \left( s - 2h + t_{j-1} + \frac{2h}{b-a} (t_{j-1} - a) \right) \right| ds.\]

For a given \( s \in [0, 2h] \), the points, \( s - 2h + t_j + \frac{2h}{b-a} (t_j - a) \) for \( j = 1, \ldots, n \) form an increasing list of points in the interval \([a - 2h, b + 2h] \) and so the integrand is bounded above by \( V(\gamma, [a - 2h, b + 2h]) = V(\gamma, [a, b]) \). It follows

\[\sum_{j=1}^n |\gamma_h(t_j) - \gamma_h(t_{j-1})| \leq V(\gamma, [a, b])\]

which proves the lemma.

With this lemma the proof of the theorem can be completed without too much trouble. Let \( H \) be an open set containing \( \gamma^* \) such that \( \overline{H} \) is a compact subset of \( \Omega \). Let \( 0 < \varepsilon < \text{dist} (\gamma^*, H^C) \). Then there exists \( \delta_1 \) such that if \( h < \delta_1 \), then for all \( t \),

\[|\gamma(t) - \gamma_h(t)| \leq \frac{1}{2h} \int_{t-2h+t+\frac{2h}{b-a}(t-a)}^{t+\frac{2h}{b-a}(t-a)} |\gamma(s) - \gamma(t)| \, ds \leq \frac{1}{2h} \int_{t-2h+t+\frac{2h}{b-a}(t-a)}^{t+\frac{2h}{b-a}(t-a)} \varepsilon \, ds = \varepsilon \]  

(48.0.11)

due to the uniform continuity of \( \gamma \). This proves **48.0.3**.

From **48.0.2** and the above lemma, there exists \( \delta_2 \) such that if \( ||P|| < \delta_2 \), then for all \( z \in K \),

\[\left| \int_{\gamma} f(\cdot, z) \, d\gamma(t) - S(\gamma)(P) \right| < \frac{\varepsilon}{3}, \quad \left| \int_{\gamma_h} f(\cdot, z) \, d\gamma_h(t) - S_h(\gamma)(P) \right| < \frac{\varepsilon}{3}\]

for all \( h \). Here \( S(\gamma)(P) \) is a Riemann Steiltjes sum of the form

\[\sum_{i=1}^n f(\gamma(t_i), z) (\gamma(t_i) - \gamma(t_{i-1}))\]

and \( S_h(P) \) is a similar Riemann Steiltjes sum taken with respect to \( \gamma_h \) instead of \( \gamma \). Because of **48.0.4** \( \gamma_h(t) \) has values in \( H \subseteq \Omega \). Therefore, fix the partition, \( P \), and choose \( h \) small enough that in addition to this, the following inequality is valid for all \( z \in K \).

\[|S(\gamma)(P) - S_h(\gamma)(P)| < \frac{\varepsilon}{3}\]
This is possible because of Result 48.0.11 and the uniform continuity of \( f \) on \( \overline{H} \times K \). It follows
\[
\left\| \int_{\gamma} f (\cdot, z) \, d\gamma (t) - \int_{\gamma_{h}} f (\cdot, z) \, d\gamma_{h} (t) \right\| \leq \left\| \int_{\gamma} f (\cdot, z) \, d\gamma (t) - S (\mathcal{P}) \right\| + \left\| S (\mathcal{P}) - S_{h} (\mathcal{P}) \right\|
\]
\[
+ \left\| S_{h} (\mathcal{P}) - \int_{\gamma_{h}} f (\cdot, z) \, d\gamma_{h} (t) \right\| < \varepsilon.
\]
Formula 48.0.10 follows from the lemma. This proves the theorem.

Of course the same result is obtained without the explicit dependence of \( f \) on \( z \).

This is a very useful theorem because if \( \gamma \) is \( C^{1} ([a, b]) \), it is easy to calculate \( \int_{\gamma} f d\gamma \) and the above theorem allows a reduction to the case where \( \gamma \) is \( C^{1} \). The next theorem shows how easy it is to compute these integrals in the case where \( \gamma \) is \( C^{1} \).

**Theorem 48.0.9** If \( f : \gamma^{*} \to X \) is continuous and \( \gamma : [a, b] \to \mathbb{C} \) is in \( C^{1} ([a, b]) \), then
\[
\int_{\gamma} f d\gamma = \int_{a}^{b} f (\gamma (t)) \gamma' (t) \, dt.
\]  
\[
(48.0.12)
\]

**Proof:** Let \( \mathcal{P} \) be a partition of \( [a, b] \), \( \mathcal{P} = \{t_{0}, \cdots, t_{n}\} \) and \( ||\mathcal{P}|| \) is small enough that whenever \( |t - s| < ||\mathcal{P}|| \),
\[
|f (\gamma (t)) - f (\gamma (s))| < \varepsilon
\]  
\[
(48.0.13)
\]
and
\[
\left\| \int_{\gamma} f d\gamma - \sum_{j=1}^{n} f (\gamma (\tau_{j})) (\gamma (t_{j}) - \gamma (t_{j-1})) \right\| < \varepsilon.
\]

Now
\[
\sum_{j=1}^{n} f (\gamma (\tau_{j})) (\gamma (t_{j}) - \gamma (t_{j-1})) = \int_{a}^{b} \sum_{j=1}^{n} f (\gamma (\tau_{j})) \chi_{[t_{j-1}, t_{j}]} (s) \gamma' (s) \, ds
\]
where here
\[
\chi_{[a, b]} (s) \equiv \begin{cases} 1 & \text{if } s \in [p, q] \\ 0 & \text{if } s \notin [p, q] \end{cases}.
\]

Also,
\[
\int_{a}^{b} f (\gamma (s)) \gamma' (s) \, ds = \int_{a}^{b} \sum_{j=1}^{n} f (\gamma (s)) \chi_{[t_{j-1}, t_{j}]} (s) \gamma' (s) \, ds
\]
and thanks to

\[
\sum_{j=1}^{n} f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) = \int_a^b \sum_{j=1}^{n} f(\gamma(s)) \delta_{[t_{j-1},t_j]}(s) \gamma'(s) ds
\]

\[
= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma(s)) X_{[t_{j-1},t_j]}(s) \gamma'(s) ds
\]

\[
= \sum_{j=1}^{n} \int_{a}^{b} f(\gamma(s)) \delta_{[t_{j-1},t_j]}(s) \gamma'(s) ds
\]

\[
\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|f(\gamma(s)) - f(\gamma(t_j))\| \gamma'(s) ds \leq \|\gamma'\| \sum_{j} \varepsilon (t_j - t_{j-1})
\]

\[
= \varepsilon \|\gamma'\| (b - a).
\]

It follows that

\[
\left| \int_{\gamma} fd\gamma - \int_{a}^{b} f(\gamma(s)) \gamma'(s) ds \right| \leq \left| \int_{\gamma} fd\gamma - \sum_{j=1}^{n} f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) \right|
\]

\[
+ \left| \sum_{j=1}^{n} f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) - \int_{a}^{b} f(\gamma(s)) \gamma'(s) ds \right| \leq \varepsilon \|\gamma'\| (b - a) + \varepsilon.
\]

Since \(\varepsilon\) is arbitrary, this verifies (48.0.12).

**Definition 48.0.10** Let \(\Omega\) be an open subset of \(\mathbb{C}\) and let \(\gamma : [a,b] \to \Omega\) be a continuous function with bounded variation \(f : \Omega \to X\) be a continuous function. Then the following notation is more customary.

\[
\int_{\gamma} f(z) dz \equiv \int_{\gamma} fd\gamma.
\]

The expression, \(\int_{\gamma} f(z) dz\), is called a contour integral and \(\gamma\) is referred to as the contour. A function \(f : \Omega \to X\) for \(\Omega\) an open set in \(\mathbb{C}\) has a primitive if there exists a function, \(F\), the primitive, such that \(F'(z) = f(z)\). Thus \(F\) is just an antiderivative. Also if \(\gamma_k : [a_k,b_k] \to \mathbb{C}\) is continuous and of bounded variation, for \(k = 1, \ldots, m\) and \(\gamma_k(b_k) = \gamma_{k+1}(a_k)\), define

\[
\int_{\sum_{k=1}^{m} \gamma_k} f(z) dz \equiv \sum_{k=1}^{m} \int_{\gamma_k} f(z) dz. \quad (48.0.14)
\]

In addition to this, for \(\gamma : [a,b] \to \mathbb{C}\), define \(-\gamma : [a,b] \to \mathbb{C}\) by \(-\gamma(t) \equiv \gamma(b + a - t)\). Thus \(\gamma\) simply traces out the points of \(\gamma^*\) in the opposite order.

The following lemma is useful and follows quickly from Theorem (48.0.13).
Lemma 48.0.11  In the above definition, there exists a continuous bounded variation function, \( \gamma \) defined on some closed interval, \([c, d]\), such that \( \gamma ([c, d]) = \bigcup_{k=1}^{m} \gamma_k ([a_k, b_k]) \) and \( \gamma (c) = \gamma_1 (a_1) \) while \( \gamma (d) = \gamma_m (b_m) \). Furthermore,

\[
\int_{\gamma} f (z) \, dz = \sum_{k=1}^{m} \int_{\gamma_k} f (z) \, dz.
\]

If \( \gamma : [a, b] \to \mathbb{C} \) is of bounded variation and continuous, then

\[
\int_{\gamma} f (z) \, dz = - \int_{-\gamma} f (z) \, dz.
\]

Re stating Theorem 48.0.7 with the new notation in the above definition,

Theorem 48.0.12  Let \( K \) be a compact set in \( \mathbb{C} \) and let \( f : \Omega \times K \to X \) be continuous for \( \Omega \) an open set in \( \mathbb{C} \). Also let \( \gamma : [a, b] \to \Omega \) be continuous with bounded variation. Then if \( r > 0 \) is given, there exists \( \eta : [a, b] \to \Omega \) such that \( \eta (a) = \gamma (a) \), \( \eta (b) = \gamma (b) \), \( \eta \) is \( C^1 ([a, b]) \), and

\[
\left| \int_{\gamma} f (z, w) \, dz - \int_{\eta} f (z, w) \, dz \right| < r, \quad \| \eta - \gamma \| < r.
\]

It will be very important to consider which functions have primitives. It turns out, it is not enough for \( f \) to be continuous in order to possess a primitive. This is in stark contrast to the situation for functions of a real variable in which the fundamental theorem of calculus will deliver a primitive for any continuous function. The reason for the interest in such functions is the following theorem and its corollary.

Theorem 48.0.13  Let \( \gamma : [a, b] \to \mathbb{C} \) be continuous and of bounded variation. Also suppose \( F' (z) = f (z) \) for all \( z \in \Omega \), an open set containing \( \gamma^* \) and \( f \) is continuous on \( \Omega \). Then

\[
\int_{\gamma} f (z) \, dz = F (\gamma (b)) - F (\gamma (a)).
\]

Proof:  By Theorem 48.0.12 there exists \( \eta \in C^1 ([a, b]) \) such that \( \gamma (a) = \eta (a) \), and \( \gamma (b) = \eta (b) \) such that

\[
\left\| \int_{\gamma} f (z) \, dz - \int_{\eta} f (z) \, dz \right\| < \varepsilon.
\]

Then since \( \eta \) is in \( C^1 ([a, b]) \),

\[
\int_{\eta} f (z) \, dz = \int_{a}^{b} f (\eta (t)) \eta' (t) \, dt = \int_{a}^{b} \frac{dF (\eta (t))}{dt} \, dt = F (\eta (b)) - F (\eta (a)) = F (\gamma (b)) - F (\gamma (a)).
\]

Therefore,

\[
\left\| (F (\gamma (b)) - F (\gamma (a))) - \int_{\gamma} f (z) \, dz \right\| < \varepsilon
\]

and since \( \varepsilon > 0 \) is arbitrary, this proves the theorem.
Corollary 48.0.14 If \( \gamma : [a, b] \to \mathbb{C} \) is continuous, has bounded variation, is a closed curve, \( \gamma (a) = \gamma (b) \), and \( \gamma^* \subseteq \Omega \) where \( \Omega \) is an open set on which \( F' (z) = f (z) \), then

\[
\int_\gamma f (z) \, dz = 0.
\]

48.1 Exercises

1. Let \( \gamma : [a, b] \to \mathbb{R} \) be increasing. Show \( V (\gamma, [a, b]) = \gamma (b) - \gamma (a) \).

2. Suppose \( \gamma : [a, b] \to \mathbb{C} \) satisfies a Lipschitz condition, \( |\gamma (t) - \gamma (s)| \leq K |s - t| \). Show \( \gamma \) is of bounded variation and that \( V (\gamma, [a, b]) \leq K |b - a| \).

3. \( \gamma : [c_0, c_m] \to \mathbb{C} \) is piecewise smooth if there exist numbers, \( c_k, k = 1, \ldots, m \) such that \( c_0 < c_1 < \cdots < c_{m-1} < c_m \) such that \( \gamma \) is continuous and \( \gamma : [c_k, c_{k+1}] \to \mathbb{C} \) is \( C^1 \). Show that such piecewise smooth functions are of bounded variation and give an estimate for \( V (\gamma, [c_0, c_m]) \).

4. Let \( \gamma : [0, 2\pi] \to \mathbb{C} \) be given by \( \gamma (t) = r (\cos mt + i \sin mt) \) for \( m \) an integer. Find \( \int_\gamma \frac{dz}{z} \).

5. Show that if \( \gamma : [a, b] \to \mathbb{C} \) then there exists an increasing function \( h : [0, 1] \to [a, b] \) such that \( \gamma \circ h ([0, 1]) = \gamma^* \).

6. Let \( \gamma : [a, b] \to \mathbb{C} \) be an arbitrary continuous curve having bounded variation and let \( f, g \) have continuous derivatives on some open set containing \( \gamma^* \). Prove the usual integration by parts formula.

\[
\int_\gamma f g' \, dz = f (\gamma (b)) g (\gamma (b)) - f (\gamma (a)) g (\gamma (a)) - \int_\gamma f' g \, dz.
\]

7. Let \( f (z) \equiv |z|^{-1/2} e^{-i \theta z} \) where \( z = |z| e^{i \theta} \). This function is called the principle branch of \( z^{-1/2} \). Find \( \int_\gamma f (z) \, dz \) where \( \gamma \) is the semicircle in the upper half plane which goes from \((1, 0)\) to \((-1, 0)\) in the counter clockwise direction. Next do the integral in which \( \gamma \) goes in the clockwise direction along the semicircle in the lower half plane.

8. Prove an open set, \( U \) is connected if and only if for every two points in \( U \), there exists a \( C^1 \) curve having values in \( U \) which joins them.

9. Let \( \mathcal{P}, \mathcal{Q} \) be two partitions of \([a, b]\) with \( \mathcal{P} \subseteq \mathcal{Q} \). Each of these partitions can be used to form an approximation to \( V (\gamma, [a, b]) \) as described above. Recall the total variation was the supremum of sums of a certain form determined by a partition. How is the sum associated with \( \mathcal{P} \) related to the sum associated with \( \mathcal{Q} \)? Explain.
10. Consider the curve,

\[ \gamma(t) = \begin{cases} 
  t + it^2 \sin \left( \frac{1}{t} \right) & \text{if } t \in (0, 1] \\
  0 & \text{if } t = 0 
\end{cases} \]

Is \( \gamma \) a continuous curve having bounded variation? What if the \( t^2 \) is replaced with \( t \)? Is the resulting curve continuous? Is it a bounded variation curve?

11. Suppose \( \gamma : [a, b] \to \mathbb{R} \) is given by \( \gamma(t) = t \). What is \( \int_{\gamma} f(t) \, d\gamma \)? Explain.
Chapter 49

Fundamentals Of Complex Analysis

49.1 Analytic Functions

Definition 49.1.1 Let $\Omega$ be an open set in $\mathbb{C}$ and let $f : \Omega \to X$. Then $f$ is analytic on $\Omega$ if for every $z \in \Omega$,

$$
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \equiv f'(z)
$$

exists and is a continuous function of $z \in \Omega$. Here $h \in \mathbb{C}$.

Note that if $f$ is analytic, it must be the case that $f$ is continuous. It is more common to not include the requirement that $f'$ is continuous but it is shown later that the continuity of $f'$ follows.

What are some examples of analytic functions? In the case where $X = \mathbb{C}$, the simplest example is any polynomial. Thus

$$
p(z) \equiv \sum_{k=0}^{n} a_k z^k
$$

is an analytic function and

$$
p'(z) = \sum_{k=1}^{n} a_k k z^{k-1}.
$$

More generally, power series are analytic. This will be shown soon but first here is an important definition and a convergence theorem called the root test.

Definition 49.1.2 Let $\{a_k\}$ be a sequence in $X$. Then $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$ whenever this limit exists. When the limit exists, the series is said to converge.
Theorem 49.1.3 Consider \( \sum_{k=1}^{\infty} a_k \) and let \( \rho \equiv \limsup_{k \to \infty} ||a_k||^{1/k} \). Then if \( \rho < 1 \), the series converges absolutely and if \( \rho > 1 \) the series diverges spectacularly in the sense that \( \lim_{k \to \infty} a_k \neq 0 \). If \( \rho = 1 \) the test fails. Also \( \sum_{k=1}^{\infty} a_k (z-a)^k \) converges on some disk \( B(a,R) \). It converges absolutely if \( |z-a| < R \) and uniformly on \( B(a,r) \) whenever \( r_1 < R \). The function \( f(z) = \sum_{k=1}^{\infty} a_k (z-a)^k \) is continuous on \( B(a,R) \).

Proof: Suppose \( \rho < 1 \). Then there exists \( r \in (\rho,1) \). Therefore, \( ||a_k|| \leq r^k \) for all \( k \) large enough and so by a comparison test, \( \sum_k ||a_k|| \) converges because the partial sums are bounded above. Therefore, the partial sums of the original series form a Cauchy sequence in \( \mathbb{X} \) and so they also converge due to completeness of \( \mathbb{X} \).

Now suppose \( \rho > 1 \). Then letting \( \rho > r > 1 \), it follows \( ||a_k||^{1/k} \geq r \) infinitely often. Thus \( ||a_k|| \geq r^k \) infinitely often. Thus there exists a subsequence for which \( ||a_{n_k}|| \) converges to \( \infty \). Therefore, the series cannot converge.

Now consider \( \sum_{k=1}^{\infty} a_k (z-a)^k \). This series converges absolutely if

\[
\limsup_{k \to \infty} ||a_k||^{1/k} |z-a| < 1
\]

which is the same as saying \( |z-a| < 1/\rho \) where \( \rho \equiv \limsup_{k \to \infty} ||a_k||^{1/k} \). Let \( R = 1/\rho \).

Now suppose \( r_1 < R \). Consider \( |z-a| \leq r_1 \). Then for such \( z \),

\[
||a_k|| |z-a|^k \leq ||a_k|| r_1^k
\]

and

\[
\limsup_{k \to \infty} \left( ||a_k|| r_1^k \right)^{1/k} = \limsup_{k \to \infty} ||a_k||^{1/k} r_1 = \frac{r_1}{R} < 1
\]

so \( \sum_k ||a_k|| r_1^k \) converges. By the Weierstrass M test, \( \sum_{k=1}^{\infty} a_k (z-a)^k \) converges uniformly for \( |z-a| \leq r_1 \). Therefore, \( f \) is continuous on \( B(a,R) \) as claimed because it is the uniform limit of continuous functions, the partial sums of the infinite series.

What if \( \rho = 0 \)? In this case,

\[
\limsup_{k \to \infty} ||a_k||^{1/k} |z-a| = 0 \cdot |z-a| = 0
\]

and so \( R = \infty \) and the series, \( \sum ||a_k|| |z-a|^k \) converges everywhere.

What if \( \rho = \infty \)? Then in this case, the series converges only at \( z = a \) because if \( z \neq a \),

\[
\limsup_{k \to \infty} ||a_k||^{1/k} |z-a| = \infty.
\]

Theorem 49.1.4 Let \( f(z) = \sum_{k=1}^{\infty} a_k (z-a)^k \) be given in Theorem 49.1.3, where \( R > 0 \). Then \( f \) is analytic on \( B(a, R) \). So are all its derivatives.
49.1. ANALYTIC FUNCTIONS

Proof: Consider \( g(z) = \sum_{k=2}^{\infty} a_k (z-a)^{k-1} \) on \( B(a,R) \) where \( R = \rho^{-1} \) as above. Let \( r_1 < r < R \). Then letting \( |z-a| < r_1 \) and \( h < r - r_1 \),

\[
\left\| \frac{f(z+h) - f(z) - g(z)}{h} \right\| \leq \sum_{k=2}^{\infty} ||a_k|| \left| \frac{(z+h - a)^k - (z-a)^k}{h} - k(z-a)^{k-1} \right|
\]

\[
\leq \sum_{k=2}^{\infty} ||a_k|| \left| \frac{1}{h} \left( \sum_{i=0}^{k} \binom{k}{i} (z-a)^{k-i} h^i - (z-a)^k \right) - k(z-a)^{k-1} \right|
\]

\[
= \sum_{k=2}^{\infty} ||a_k|| \left| \frac{1}{h} \left( \sum_{i=1}^{k} \binom{k}{i} (z-a)^{k-i} h^i \right) - k(z-a)^{k-1} \right|
\]

\[
\leq \sum_{k=2}^{\infty} ||a_k|| \left| \sum_{i=2}^{\infty} \binom{k}{i} (z-a)^{k-i} h^{i-1} \right|
\]

\[
\leq |h| \sum_{k=2}^{\infty} ||a_k|| \left| \sum_{i=0}^{k-2} \binom{k}{i+2} |z-a|^{k-2-i} |h|^i \right|
\]

\[
= |h| \sum_{k=2}^{\infty} ||a_k|| \left| \sum_{i=0}^{k-2} \binom{k-2}{i} \frac{k(k-1)}{i+2(i+1)} |z-a|^{k-2-i} |h|^i \right|
\]

\[
\leq |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} \left| \sum_{i=0}^{k-2} \binom{k-2}{i} |z-a|^{k-2-i} |h|^i \right|
\]

\[
= |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} \left( |z-a| + |h| \right)^{k-2} < |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} r^{k-2}.
\]

Then

\[
\lim_{k \to \infty} \sup_{k} \left( ||a_k|| \frac{k(k-1)}{2} r^{k-2} \right)^{1/k} = \rho r < 1
\]

and so

\[
\left| \frac{f(z+h) - f(z) - g(z)}{h} - g(z) \right| \leq C|h|.
\]

Therefore, \( g(z) = f'(z) \). Now by Theorem 49.1.3, it also follows that \( f' \) is continuous. Since \( r_1 < R \) was arbitrary, this shows that \( f'(z) \) is given by the differentiated series above for \( |z-a| < R \). Now a repeat of the argument shows all the derivatives of \( f \) exist and are continuous on \( B(a,R) \).

49.1.1 Cauchy Riemann Equations

Next consider the very important Cauchy Riemann equations which give conditions under which complex valued functions of a complex variable are analytic.
Theorem 49.1.5  Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f: \Omega \to \mathbb{C}$ be a function, such that for $z = x + iy \in \Omega$,

$$f(z) = u(x, y) + iv(x, y).$$

Then $f$ is analytic if and only if $u, v$ are $C^1(\Omega)$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore,

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

**Proof:** Suppose $f$ is analytic first. Then letting $t \in \mathbb{R}$,

$$f'(z) = \lim_{t \to 0} \frac{f(z + t) - f(z)}{t} =$$

$$\lim_{t \to 0} \left( \frac{u(x + t, y) + iv(x + t, y)}{t} - \frac{u(x, y) + iv(x, y)}{t} \right)$$

$$= \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}.$$

But also

$$f'(z) = \lim_{t \to 0} \frac{f(z + it) - f(z)}{it} =$$

$$\lim_{t \to 0} \left( \frac{u(x, y + t) + iv(x, y + t)}{it} - \frac{u(x, y) + iv(x, y)}{it} \right)$$

$$= \frac{1}{i} \left( \frac{\partial u(x, y)}{\partial y} + i\frac{\partial v(x, y)}{\partial y} \right)$$

$$= \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y}.$$

This verifies the Cauchy-Riemann equations. We are assuming that $z \to f'(z)$ is continuous. Therefore, the partial derivatives of $u$ and $v$ are also continuous. To see this, note that from the formulas for $f'(z)$ given above, and letting $z_1 = x_1 + iy_1$

$$\left| \frac{\partial v(x, y)}{\partial y} - \frac{\partial v(x_1, y_1)}{\partial y} \right| \leq |f'(z) - f'(z_1)|,$$

showing that $(x, y) \to \frac{\partial u(x, y)}{\partial y}$ is continuous since $(x_1, y_1) \to (x, y)$ if and only if $z_1 \to z$. The other cases are similar.

Now suppose the Cauchy-Riemann equations hold and the functions, $u$ and $v$ are $C^1(\Omega)$. Then letting $h = h_1 + ih_2$,

$$f(z + h) - f(z) = u(x + h_1, y + h_2)$$
49.1. ANALYTIC FUNCTIONS

We know $u$ and $v$ are both differentiable and so

$$f(z+h) - f(z) = \frac{\partial u}{\partial x} (x,y) h_1 + \frac{\partial u}{\partial y} (x,y) h_2 +$$

$$i \left( \frac{\partial v}{\partial x} (x,y) h_1 + \frac{\partial v}{\partial y} (x,y) h_2 \right) + o(h).$$

Dividing by $h$ and using the Cauchy Riemann equations,

$$f(z+h) - f(z) = \frac{\partial u}{\partial x} (x,y) h_1 + \frac{i \partial v}{\partial y} (x,y) h_2 +$$

$$\frac{i \partial v}{\partial x} (x,y) h_1 + \frac{\partial u}{\partial y} (x,y) h_2 + o(h)\frac{h}{h}$$

$$= \frac{\partial u}{\partial x} (x,y) \frac{h_1 + ih_2}{h} + \frac{i \partial v}{\partial x} (x,y) \frac{h_1 + ih_2}{h} + o(h)\frac{h}{h}$$

Taking the limit as $h \to 0$,

$$f'(z) = \frac{\partial u}{\partial x} (x,y) + i \frac{\partial v}{\partial x} (x,y).$$

It follows from this formula and the assumption that $u, v$ are $C^1(\Omega)$ that $f'$ is continuous.

It is routine to verify that all the usual rules of derivatives hold for analytic functions. In particular, the product rule, the chain rule, and quotient rule.

49.1.2 An Important Example

An important example of an analytic function is $e^z \equiv \exp(z) \equiv e^x (\cos y + i \sin y)$ where $z = x + iy$. You can verify that this function satisfies the Cauchy Riemann equations and that all the partial derivatives are continuous. Also from the above discussion, $(e^z)' = e^x \cos y + ie^x \sin y = e^z$. Later I will show that $e^z$ is given by the usual power series. An important property of this function is that it can be used to parameterize the circle centered at $z_0$ having radius $r$.

Lemma 49.1.6 Let $\gamma$ denote the closed curve which is a circle of radius $r$ centered at $z_0$. Then a parameterization this curve is $\gamma(t) = z_0 + re^{it}$ where $t \in [0,2\pi]$.

Proof: $|\gamma(t) - z_0|^2 = |re^{it}re^{-it}| = r^2$. Also, you can see from the definition of the sine and cosine that the point described in this way moves counter clockwise over this circle.
49.2 Exercises

1. Verify all the usual rules of differentiation including the product and chain rules.

2. Suppose \( f \) and \( f' : U \to \mathbb{C} \) are analytic and \( f(z) = u(x,y) + iv(x,y) \). Verify \( u_{xx} + u_{yy} = 0 \) and \( v_{xx} + v_{yy} = 0 \). This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace’s equation. We say these functions satisfying Laplace’s equation are harmonic functions. If \( u \) is a harmonic function defined on \( B(0,r) \) show that \( v(x,y) = \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt \) is such that \( u + iv \) is analytic.

3. Let \( f : U \to \mathbb{C} \) be analytic and \( f(z) = u(x,y) + iv(x,y) \). Show \( u, v \) and \( uv \) are all harmonic although it can happen that \( u^2 \) is not. Recall that a function, \( w \) is harmonic if \( w_{xx} + w_{yy} = 0 \).

4. Define a function \( f(z) \equiv \bar{z} \equiv x - iy \) where \( z = x + iy \). Is \( f \) analytic?

5. If \( f(z) = u(x,y) + iv(x,y) \) and \( f \) is analytic, verify that

\[ \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'(z)|^2. \]

6. Show that if \( u(x,y) + iv(x,y) = f(z) \) is analytic, then \( \nabla u \cdot \nabla v = 0 \). Recall

\[ \nabla u(x,y) = \langle u_x(x,y), u_y(x,y) \rangle. \]

7. Show that every polynomial is analytic.

8. If \( \gamma(t) = x(t) + iy(t) \) is a \( C^1 \) curve having values in \( U \), an open set of \( \mathbb{C} \), and if \( f : U \to \mathbb{C} \) is analytic, we can consider \( f \circ \gamma \), another \( C^1 \) curve having values in \( \mathbb{C} \). Also, \( \gamma'(t) \) and \((f \circ \gamma)'(t)\) are complex numbers so these can be considered as vectors in \( \mathbb{R}^2 \) as follows. The complex number, \( x + iy \) corresponds to the vector, \( (x,y) \). Suppose that \( \gamma \) and \( \eta \) are two such \( C^1 \) curves having values in \( U \) and that \( \gamma(t_0) = \eta(s_0) = z \) and suppose that \( f : U \to \mathbb{C} \) is analytic. Show that the angle between \( (f \circ \gamma)'(t_0) \) and \((f \circ \eta)'(s_0)\) is the same as the angle between \( \gamma'(t_0) \) and \( \eta'(s_0) \) assuming that \( f'(z) \neq 0 \). Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. . \textbf{Hint}: To make this easy to show, first observe that \( \langle x,y \rangle \cdot \langle a,b \rangle = \frac{1}{2} (x\overline{w} + \overline{x}w) \) where \( z = x + iy \) and \( w = a + ib \).

9. Analytic functions are even better than what is described in Problem 8. In addition to preserving angles, they also preserve orientation. To verify this show that if \( z = x + iy \) and \( w = a + ib \) are two complex numbers, then \( \langle x,y,0 \rangle \) and \( \langle a,b,0 \rangle \) are two vectors in \( \mathbb{R}^3 \). Recall that the cross product, \( \langle x,y,0 \rangle \times \langle a,b,0 \rangle \), yields a vector normal to the two given vectors such that the triple, \( \langle x,y,0 \rangle \), \( \langle a,b,0 \rangle \), and \( \langle x,y,0 \rangle \times \langle a,b,0 \rangle \) satisfies the right hand rule.
49.3. CAUCHY’S FORMULA FOR A DISK

and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product either points in the direction of the positive \( z \) axis or in the direction of the negative \( z \) axis. Thus, either the vectors \( \langle x, y, 0 \rangle, \langle a, b, 0 \rangle, k \) form a right handed system or the vectors \( \langle a, b, 0 \rangle, \langle x, y, 0 \rangle, k \) form a right handed system. These are the two possible orientations. Show that in the situation of Problem 8 the orientation of \( \gamma'(t_0), \eta'(s_0), k \) is the same as the orientation of the vectors \( (f \circ \gamma)'(t_0), (f \circ \eta)'(s_0), k \). Such mappings are called conformal. If \( f \) is analytic and \( f'(z) \neq 0 \), then we know from this problem and the above that \( f \) is a conformal map.

**Hint:** You can do this by verifying that 

\[
(x, y, 0) \times (a, b, 0) = \text{Re} (z \overline{w}) \cdot k.
\]

10. Write the Cauchy Riemann equations in terms of polar coordinates. Recall the polar coordinates are given by 

\[
x = r \cos \theta, \quad y = r \sin \theta.
\]

This means, letting \( u(x, y) = u(r, \theta), v(x, y) = v(r, \theta) \), write the Cauchy Riemann equations in terms of \( r \) and \( \theta \). You should eventually show the Cauchy Riemann equations are equivalent to 

\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.
\]

11. Show that a real valued analytic function must be constant.

### 49.3 Cauchy’s Formula For A Disk

The Cauchy integral formula is the most important theorem in complex analysis. It will be established for a disk in this chapter and later will be generalized to much more general situations but the version given here will suffice to prove many interesting theorems needed in the later development of the theory. The following are some advanced calculus results.

**Lemma 49.3.1** Let \( f : [a, b] \to \mathbb{C} \). Then \( f'(t) \) exists if and only if \( \text{Re} f'(t) \) and \( \text{Im} f'(t) \) exist. Furthermore,

\[
f'(t) = \text{Re} f'(t) + i \text{Im} f'(t).
\]

**Proof:** The if part of the equivalence is obvious.

Now suppose \( f'(t) \) exists. Let both \( t \) and \( t + h \) be contained in \( [a, b] \)

\[
\left| \frac{\text{Re} f(t+h) - \text{Re} f(t)}{h} - \text{Re} (f'(t)) \right| \leq \left| \frac{f(t+h) - f(t)}{h} - f'(t) \right|
\]
and this converges to zero as \( h \to 0 \). Therefore, \( \text{Re} f'(t) = \text{Re} (f'(t)) \). Similarly, \( \text{Im} f'(t) = \text{Im} (f'(t)) \).

**Lemma 49.3.2** If \( g : [a, b] \to \mathbb{C} \) and \( g \) is continuous on \([a, b]\) and differentiable on \((a, b)\) with \( g'(t) = 0 \), then \( g(t) \) is a constant.

**Proof:** From the above lemma, you can apply the mean value theorem to the real and imaginary parts of \( g \).

Applying the above lemma to the components yields the following lemma.

**Lemma 49.3.3** If \( g : [a, b] \to \mathbb{C}^n = X \) and \( g \) is continuous on \([a, b]\) and differentiable on \((a, b)\) with \( g'(t) = 0 \), then \( g(t) \) is a constant.

If you want to have \( X \) be a complex Banach space, the result is still true.

**Lemma 49.3.4** If \( g : [a, b] \to X \) and \( g \) is continuous on \([a, b]\) and differentiable on \((a, b)\) with \( g'(t) = 0 \), then \( g(t) \) is a constant.

**Proof:** Let \( \Lambda \in X' \). Then \( \Lambda g : [a, b] \to \mathbb{C} \). Therefore, from Lemma 49.3.2, for each \( \Lambda \in X' \), \( \Lambda g(s) = \Lambda g(t) \) and since \( X' \) separates the points, it follows \( g(s) = g(t) \) so \( g \) is constant.

**Lemma 49.3.5** Let \( \phi : [a, b] \times [c, d] \to \mathbb{R} \) be continuous and let

\[
g(t) = \int_a^b \phi(s, t) \, ds. \tag{49.3.1}
\]

Then \( g \) is continuous. If \( \frac{\partial \phi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\), then

\[
g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} \, ds. \tag{49.3.2}
\]

**Proof:** The first claim follows from the uniform continuity of \( \phi \) on \([a, b] \times [c, d]\), which uniform continuity results from the set being compact. To establish 49.3.2, let \( t \) and \( t + h \) be contained in \([c, d]\) and form, using the mean value theorem,

\[
\frac{g(t + h) - g(t)}{h} = \frac{1}{h} \int_a^b \left[ \phi(s, t + h) - \phi(s, t) \right] \, ds
\]

\[
= \frac{1}{h} \int_a^b \frac{\partial \phi(s, t + \theta h)}{\partial t} \, ds
\]

\[
= \int_a^b \frac{\partial \phi(s, t + \theta h)}{\partial t} \, ds,
\]

where \( \theta \) may depend on \( s \) but is some number between 0 and 1. Then by the uniform continuity of \( \frac{\partial \phi}{\partial t} \), it follows that 49.3.2 holds.
49.3. CAUCHY'S FORMULA FOR A DISK

Corollary 49.3.6 Let \( \phi : [a, b] \times [c, d] \to \mathbb{C} \) be continuous and let

\[
g(t) = \int_a^b \phi(s, t) \, ds. \tag{49.3.3}
\]

Then \( g \) is continuous. If \( \frac{\partial \phi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\), then

\[
g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} \, ds. \tag{49.3.4}
\]

Proof: Apply Lemma 49.3.5 to the real and imaginary parts of \( \phi \).

Applying the above corollary to the components, you can also have the same result for \( \phi \) having values in \( \mathbb{C}^n \).

Corollary 49.3.7 Let \( \phi : [a, b] \times [c, d] \to \mathbb{C}^n \) be continuous and let

\[
g(t) = \int_a^b \phi(s, t) \, ds. \tag{49.3.5}
\]

Then \( g \) is continuous. If \( \frac{\partial \phi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\), then

\[
g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} \, ds. \tag{49.3.6}
\]

If you want to consider \( \phi \) having values in \( X \), a complex Banach space a similar result holds.

Corollary 49.3.8 Let \( \phi : [a, b] \times [c, d] \to X \) be continuous and let

\[
g(t) = \int_a^b \phi(s, t) \, ds. \tag{49.3.7}
\]

Then \( g \) is continuous. If \( \frac{\partial \phi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\), then

\[
g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} \, ds. \tag{49.3.8}
\]

Proof: Let \( \Lambda \in X' \). Then \( \Lambda \phi : [a, b] \times [c, d] \to \mathbb{C} \) is continuous and \( \frac{\partial \Lambda \phi}{\partial t} \) exists and is continuous on \([a, b] \times [c, d]\). Therefore, from (49.3.5),

\[
\Lambda \left( g'(t) \right) = (\Lambda g)'(t) = \int_a^b \frac{\partial \Lambda \phi(s, t)}{\partial t} \, ds = \Lambda \int_a^b \frac{\partial \phi(s, t)}{\partial t} \, ds
\]

and since \( X' \) separates the points, it follows (49.3.5) holds.

You can give a different proof of this.
Theorem 49.3.9 Let \( \phi : [a, b] \times [c, d] \to X \) be continuous and suppose \( \phi_t \) is continuous. Then
\[
\left( \int_a^b \phi(s, t) \, ds \right)_t = \int_a^b \frac{\partial \phi}{\partial t}(s, t) \, ds
\]
Here \( X \) is a complex Banach space.

**Proof:** Consider the following set \( P \) which is where the ordered pair \((t, h)\) will be. This is so that both \( t \) and \( t + h \) are in \([a, b]\). Then for such an ordered pair, consider
\[
\Delta(s, t, h) = \begin{cases} \frac{\phi(s, t + h) - \phi(s, t)}{h} & \text{if } h \neq 0 \\ \phi_t(s, t) & \text{if } h = 0 \end{cases}
\]

**Claim:** \( \Delta \) is continuous on the compact set \([a, b] \times P\).

**Proof of claim:** It is obvious unless \( h = 0 \). Therefore, consider the point \((s, t, 0)\).
\[
\|\Delta(s', t', h) - \Delta(s, t, 0)\| = \left\| \frac{\phi(s', t' + h) - \phi(s', t')}{h} - \phi_t(s, t) \right\|
\]
\[
= \left\| \frac{1}{h} \int_{t'}^{t' + h} \phi_t(s', r) \, dr - \phi_t(s, t) \right\| \leq \frac{1}{h} \int_{t'}^{t' + h} \|\phi_t(s', r) - \phi_t(s, t)\| \, dr < \varepsilon
\]
provided \(|(s', t', h) - (s, t, 0)|\) is small enough, this by continuity of \( \phi_t \). Therefore, \( \Delta(s, t, h) \) is uniformly continuous.
\[
\left\| \frac{1}{h} \left( \int_a^b \phi(s, t + h) \, ds - \int_a^b \phi(s, t) \, ds \right) - \int_a^b \phi_t(s, t) \, ds \right\|
\]
\[
\leq \int_a^b \left\| \frac{\phi(s, t + h) - \phi(s, t)}{h} - \phi_t(s, t) \right\| \, ds = \int_a^b \|\Delta(s, t, h) - \phi_t(s, t)\| \, ds
\]
Then by uniform continuity, if \( h \) is small enough, the integrand on the right is smaller than \( \varepsilon \).

The following is Cauchy’s integral formula for a disk.

**Theorem 49.3.10** Let \( f : \Omega \to X \) be analytic on the open set, \( \Omega \) and let \( \overline{B(z_0, r)} \subseteq \Omega \).
Let \( \gamma (t) \equiv z_0 + re^{it} \) for \( t \in [0, 2\pi] \). Then if \( z \in B(z_0, r) \),
\[
f (z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw.
\]
(49.3.9)

**Proof:** Consider for \( \alpha \in [0, 1] \),
\[
g (\alpha) = \int_0^{2\pi} f \left( \frac{\alpha (z_0 + re^{it} - z)}{re^{it} + z_0 - z} \right) e^{it} \, dt.
\]
If \( \alpha \) equals one, this reduces to the integral in (49.3.9). The idea is to show \( g \) is a constant and that \( g (0) = 2\pi i f (z) \). First consider the claim about \( g (0) \).
\[
g (0) = \left( \int_0^{2\pi} \frac{re^{it}}{re^{it} + z_0 - z} \, dt \right) if (z)
\]
\[
= if (z) \left( \int_0^{2\pi} \frac{1}{1 - \frac{z - z_0}{re^{it}}} \, dt \right)
\]
\[
= if (z) \sum_{n=0}^{\infty} \frac{r^{-n} e^{-int} (z - z_0)^n}{n!} dt
\]
because \( \left| \frac{z - z_0}{re^{it}} \right| < 1 \). Since this sum converges uniformly you can interchange the sum and the integral to obtain
\[
g (0) = if (z) \sum_{n=0}^{\infty} \frac{r^{-n} (z - z_0)^n}{n!} \int_0^{2\pi} e^{-int} \, dt
\]
\[
= 2\pi if (z)
\]
because \( \int_0^{2\pi} e^{-int} \, dt = 0 \) if \( n > 0 \).

Next consider the claim that \( g \) is constant. By Corollary 49.3.7, for \( \alpha \in (0, 1) \),
\[
g' (\alpha) = \int_0^{2\pi} \frac{f' \left( \frac{\alpha (z_0 + re^{it} - z)}{re^{it} + z_0 - z} \right) \left( \frac{re^{it} + z_0 - z}{re^{it} + z_0 - z} \right) e^{it} \, dt}
\]
\[
= \int_0^{2\pi} f' \left( \frac{\alpha (z_0 + re^{it} - z)}{re^{it} + z_0 - z} \right) e^{it} \, dt
\]
\[
= \int_0^{2\pi} \frac{d}{dt} \left( f \left( \frac{\alpha (z_0 + re^{it} - z)}{\alpha} \right) \right) \, dt
\]
\[
= f \left( \frac{\alpha (z_0 + re^{it} - z)}{\alpha} \right) - f \left( \frac{\alpha (z_0 + re^{it} - z)}{\alpha} \right) \frac{1}{\alpha} = 0.
\]

Now \( g \) is continuous on \([0, 1] \) and \( g' (t) = 0 \) on \((0, 1) \) so by Lemma 49.3.3, \( g \) equals a constant. This constant can only be \( g (0) = 2\pi if (z) \). Thus,
\[
g (1) = \int_{\gamma} \frac{f (w)}{w - z} \, dw = g (0) = 2\pi if (z).
\]
This proves the theorem.

This is a very significant theorem. A few applications are given next.
Theorem 49.3.11 Let $f : \Omega \rightarrow X$ be analytic where $\Omega$ is an open set in $\mathbb{C}$. Then $f$ has infinitely many derivatives on $\Omega$. Furthermore, for all $z \in B(z_0, r)$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}} dw$$

(49.3.10)

where $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$ for $r$ small enough that $B(z_0, r) \subseteq \Omega$.

Proof: Let $z \in B(z_0, r) \subseteq \Omega$ and let $B(z_0, r) \subseteq \Omega$. Then, letting $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$, and $h$ small enough,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw, \quad f(z+h) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z-h} dw$$

Now

$$\frac{1}{w-z-h} - \frac{1}{w-z} = \frac{h}{(-w+z+h)(-w+z)}$$

and so

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \int_{\gamma} \frac{hf(w)}{(-w+z+h)(-w+z)} dw
= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(-w+z+h)(-w+z)} dw.$$ 

Now for all $h$ sufficiently small, there exists a constant $C$ independent of such $h$ such that

$$\left| \frac{1}{(-w+z+h)(-w+z)} - \frac{1}{(-w+z)(-w+z)} \right| = \left| \frac{h}{(w-z-h)(w-z)^2} \right| \leq C|h|$$

and so, the integrand converges uniformly as $h \rightarrow 0$ to

$$= \frac{f(w)}{(w-z)^2}$$

Therefore, the limit as $h \rightarrow 0$ may be taken inside the integral to obtain

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^2} dw.$$ 

Continuing in this way, yields (49.3.10).

This is a very remarkable result. It shows the existence of one continuous derivative implies the existence of all derivatives, in contrast to the theory of functions of a real variable. Actually, more than what is stated in the theorem was shown. The above proof establishes the following corollary.
Corollary 49.3.12 Suppose $f$ is continuous on $\partial B(z_0, r)$ and suppose that for all $z \in B(z_0, r)$,
\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw,
\]
where $\gamma(t) \equiv z_0 + re^{it}$, $t \in [0, 2\pi]$. Then $f$ is analytic on $B(z_0, r)$ and in fact has infinitely many derivatives on $B(z_0, r)$.

Another application is the following lemma.

Lemma 49.3.13 Let $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$, suppose $f_n \to f$ uniformly on $\overline{B}(z_0, r)$, and suppose
\[
f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w-z} dw \tag{49.3.11}
\]
for $z \in B(z_0, r)$. Then
\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw \tag{49.3.12}
\]
implying that $f$ is analytic on $B(z_0, r)$.

Proof: From 49.3.11 and the uniform convergence of $f_n$ to $f$ on $\gamma([0, 2\pi])$, the integrals in 49.3.11 converge to
\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw.
\]
Therefore, the formula 49.3.12 follows.

Uniform convergence on a closed disk of the analytic functions implies the target function is also analytic. This is amazing. Think of the Weierstrass approximation theorem for polynomials. You can obtain a continuous nowhere differentiable function as the uniform limit of polynomials.

The conclusions of the following proposition have all been obtained earlier in Theorem 49.1.4 but they can be obtained more easily if you use the above theorem and lemmas.

Proposition 49.3.14 Let \( \{a_n\} \) denote a sequence in $X$. Then there exists $R \in [0, \infty]$ such that
\[
\sum_{k=0}^{\infty} a_k (z-z_0)^k
\]
converges absolutely if $|z-z_0| < R$, diverges if $|z-z_0| > R$ and converges uniformly on $B(z_0, r)$ for all $r < R$. Furthermore, if $R > 0$, the function,
\[
f(z) \equiv \sum_{k=0}^{\infty} a_k (z-z_0)^k
\]
is analytic on $B(z_0, R)$. 

Proof: The assertions about absolute convergence are routine from the root test if

\[
R \equiv \left( \limsup_{n \to \infty} |a_n|^{1/n} \right)^{-1}
\]

with \( R = \infty \) if the quantity in parenthesis equals zero. The root test can be used to verify absolute convergence which then implies convergence by completeness of \( X \).

The assertion about uniform convergence follows from the Weierstrass \( M \) test and \( M_n \equiv |a_n| r^n. \) \( ( \sum_{n=0}^{\infty} |a_n| r^n < \infty \) by the root test). It only remains to verify the assertion about \( f(z) \) being analytic in the case where \( R > 0 \).

Let \( 0 < r < R \) and define \( f_n(z) \equiv \sum_{k=0}^{n} a_k (z - z_0)^k \). Then \( f_n \) is a polynomial and so it is analytic. Thus, by the Cauchy integral formula above,

\[
f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w - z} dw
\]

where \( \gamma(t) = z_0 + re^{it} \), for \( t \in [0, 2\pi] \). By Lemma 49.3.3 and the first part of this proposition involving uniform convergence,

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw.
\]

Therefore, \( f \) is analytic on \( B(z_0, r) \) by Corollary 49.3.2. Since \( r < R \) is arbitrary, this shows \( f \) is analytic on \( B(z_0, R) \).

This proposition shows that all functions having values in \( X \) which are given as power series are analytic on their circle of convergence, the set of complex numbers, \( z \), such that \( |z - z_0| < R \). In fact, every analytic function can be realized as a power series.

**Theorem 49.3.15** If \( f : \Omega \to X \) is analytic and if \( B(z_0, r) \subseteq \Omega \), then

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n
\]

(49.3.13)

for all \( |z - z_0| < r \). Furthermore,

\[
a_n = \frac{f^{(n)}(z_0)}{n!}.
\]

(49.3.14)

Proof: Consider \( |z - z_0| < r \) and let \( \gamma(t) = z_0 + re^{it}, t \in [0, 2\pi] \). Then for \( w \in \gamma([0, 2\pi]) \),

\[
\frac{|z - z_0|}{|w - z_0|} < 1
\]
and so, by the Cauchy integral formula,

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw \]

\[ = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)\left(1 - \frac{z-z_0}{w-z_0}\right)} \, dw \]

\[ = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{w-z_0} \, dw. \]

Since the series converges uniformly, you can interchange the integral and the sum to obtain

\[ f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} \right) (z-z_0)^n \]

\[ = \sum_{n=0}^{\infty} a_n (z-z_0)^n \]

By Theorem 49.3.11 holds.

Note that this also implies that if a function is analytic on an open set, then all of its derivatives are also analytic. This follows from Theorem 49.1.4 which says that a function given by a power series has all derivatives on the disk of convergence.

49.4 Exercises

1. Show that if \(|e_k| \leq \varepsilon\), then \(\left| \sum_{k=m}^{\infty} e_k (r^k - r^{k+1}) \right| < \varepsilon\) if \(0 \leq r < 1\). Hint: Let \(|\theta| = 1\) and verify that

\[ \theta \sum_{k=m}^{\infty} e_k (r^k - r^{k+1}) = \sum_{k=m}^{\infty} e_k (r^k - r^{k+1}) \left( \sum_{k=m}^{\infty} \frac{\text{Re}(\theta e_k)}{(r^k - r^{k+1})} \right) \]

where \(-\varepsilon < \text{Re}(\theta e_k) < \varepsilon\).

2. Abel’s theorem says that if \(\sum_{n=0}^{\infty} a_n (z-a)^n\) has radius of convergence equal to 1 and if \(A = \sum_{n=0}^{\infty} a_n\), then \(\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n = A\). Hint: Show \(\sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{m} A_k (r^k - r^{k+1})\) where \(A_k\) denotes the \(k^{th}\) partial sum of \(\sum a_j\). Thus

\[ \sum_{k=0}^{\infty} a_k r^k = \sum_{k=m+1}^{\infty} A_k (r^k - r^{k+1}) + \sum_{k=0}^{m} A_k (r^k - r^{k+1}), \]

where \(|A_k - A| < \varepsilon\) for all \(k \geq m\). In the first sum, write \(A_k = A + e_k\) and use Problem 3. Use this theorem to verify that \(\arctan(1) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}\).
3. Find the integrals using the Cauchy integral formula.
   
   (a) $\int_{\gamma} \frac{\sin z}{z-i} dz$ where $\gamma (t) = 2e^{it}$, $t \in [0, 2\pi]$.
   
   (b) $\int_{\gamma} \frac{1}{z} dz$ where $\gamma (t) = a + re^{it}$, $t \in [0, 2\pi]$.
   
   (c) $\int_{\gamma} \frac{\cos z}{z^2} dz$ where $\gamma (t) = e^{it}$, $t \in [0, 2\pi]$.
   
   (d) $\int_{\gamma} \frac{\log (z)}{z^n} dz$ where $\gamma (t) = 1 + \frac{1}{2} e^{it}$, $t \in [0, 2\pi]$, and $n = 0, 1, 2$. In this problem, $\log (z) \equiv \ln |z| + i \arg (z)$ where $\arg (z) \in (-\pi, \pi)$ and $z = |z|e^{i \arg (z)}$. Thus $e^{\log (z)} = z$ and $\log (z)^' = \frac{1}{z}$.

4. Let $\gamma (t) = 4e^{it}$, $t \in [0, 2\pi]$ and find $\int_{\gamma} \frac{z^2 + 4}{z^2 + 1} dz$.

5. Suppose $f (z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < R$. Show that then

   $\frac{1}{2\pi} \int_{0}^{2\pi} |f (re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$

   for all $r \in [0, R]$. **Hint:** Let

   $f_n (z) \equiv \sum_{k=0}^{n} a_k z^k$,

   show

   $\frac{1}{2\pi} \int_{0}^{2\pi} |f_n (re^{i\theta})|^2 d\theta = \sum_{k=0}^{n} |a_k|^2 r^{2k}$

   and then take limits as $n \to \infty$ using uniform convergence.

6. The Cauchy integral formula, marvelous as it is, can actually be improved upon. The Cauchy integral formula involves representing $f$ by the values of $f$ on the boundary of the disk, $B (a, r)$. It is possible to represent $f$ by using only the values of $\text{Re} \ f$ on the boundary. This leads to the Schwarz formula. Supply the details in the following outline.

   Suppose $f$ is analytic on $|z| < R$ and

   $f (z) = \sum_{n=0}^{\infty} a_n z^n \quad (49.4.15)$

   with the series converging uniformly on $|z| = R$. Then letting $|w| = R$,

   $2u (w) = f (w) + \overline{f (w)}$

   and so

   $2u (w) = \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} a_k \overline{w}^k \quad (49.4.16)$
Now letting $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$
\[
\int_{\gamma} \frac{2u(w)}{w} dw = (a_0 + \overline{a_0}) \int_{\gamma} \frac{1}{w} dw = 2\pi i (a_0 + \overline{a_0}).
\]

Thus, multiplying (49.4.16) by $w^{-1}$,
\[
\frac{1}{\pi i} \int_{\gamma} u(w) dw = a_0 + \overline{a_0}.
\]

Now multiply (49.4.16) by $w^{-(n+1)}$ and integrate again to obtain
\[
a_n = \frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w^{n+1}} dw.
\]

Using these formulas for $a_n$ in (49.4.16), we can interchange the sum and the integral (Why can we do this?) to write the following for $|z| < R$.
\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \left( \sum_{k=0}^{\infty} \left( \frac{z}{w} \right)^{k+1} u(w) \right) dw - \overline{a_0}
\]
\[
= \frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w-z} dw - \overline{a_0},
\]
which is the Schwarz formula. Now $\text{Re} a_0 = \frac{1}{2\pi i} \int_{\gamma} \frac{u(w)}{w} dw$ and $\overline{a_0} = \text{Re} a_0 - i \text{Im} a_0$. Therefore, we can also write the Schwarz formula as
\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{u(w)}{w-z} dw + i \text{ Im} a_0.
\] (49.4.17)

7. Take the real parts of the second form of the Schwarz formula to derive the Poisson formula for a disk,
\[
u \left( re^{i\alpha} \right) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{u(Re^{i\theta})}{(R^2 - r^2) \cos (\theta - \alpha)} d\theta.
\] (49.4.18)

8. Suppose that $u(w)$ is a given real continuous function defined on $\partial B (0, R)$ and define $f(z)$ for $|z| < R$ by (49.4.17). Show that $f$, so defined is analytic. Explain why $u$ given in (49.4.18) is harmonic. Show that
\[
\lim_{r \to R^-} u \left( re^{i\alpha} \right) = u \left( Re^{i\alpha} \right).
\]
Thus $u$ is a harmonic function which approaches a given function on the boundary and is therefore, a solution to the Dirichlet problem.
1664  
CHAPTER 49. FUNDAMENTALS OF COMPLEX ANALYSIS

9. Suppose \( f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \) for all \( |z - z_0| < R \). Show that \( f'(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1} \) for all \( |z - z_0| < R \). \textbf{Hint:} Let \( f_n(z) \) be a partial sum of \( f \). Show that \( f'_n \) converges uniformly to some function, \( g \) on \( |z - z_0| \leq r \) for any \( r < R \). Now use the Cauchy integral formula for a function and its derivative to identify \( g \) with \( f' \).

10. Use Problem 4 to find the exact value of \( \sum_{k=0}^{\infty} k^2 \left( \frac{1}{3} \right)^k \).

11. Prove the binomial formula,

\[
(1 + z)^\alpha = \sum_{n=0}^{\infty} \left( \frac{\alpha}{n} \right) z^n
\]

where

\[
\left( \frac{\alpha}{n} \right) = \frac{\alpha \cdots (\alpha - n + 1)}{n!}.
\]

Can this be used to give a proof of the binomial formula,

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k?
\]

Explain.

12. Suppose \( f \) is analytic on \( B(z_0, r) \) and continuous on \( \overline{B(z_0, r)} \) and \( |f(z)| \leq M \) on \( \overline{B(z_0, r)} \). Show that then \( |f^{(n)}(z_0)| \leq \frac{M n!}{r^n} \).

49.5 Zeros Of An Analytic Function

In this section we give a very surprising property of analytic functions which is in stark contrast to what takes place for functions of a real variable.

**Definition 49.5.1** A region is a connected open set.

It turns out the zeros of an analytic function which is not constant on some region cannot have a limit point. This is also a good time to define the order of a zero.

**Definition 49.5.2** Suppose \( f \) is an analytic function defined near a point, \( \alpha \) where \( f(\alpha) = 0 \). Thus \( \alpha \) is a zero of the function, \( f \). The zero is of order \( m \) if \( f(z) = (z - \alpha)^m g(z) \) where \( g \) is an analytic function which is not equal to zero at \( \alpha \).

**Theorem 49.5.3** Let \( \Omega \) be a connected open set (region) and let \( f : \Omega \to X \) be analytic. Then the following are equivalent.

1. \( f(z) = 0 \) for all \( z \in \Omega \)
2. There exists \( z_0 \in \Omega \) such that \( f^{(n)}(z_0) = 0 \) for all \( n \).
3. There exists $z_0 \in \Omega$ which is a limit point of the set,

$$Z \equiv \{ z \in \Omega : f(z) = 0 \}.$$

**Proof:** It is clear the first condition implies the second two. Suppose the third holds. Then for $z$ near $z_0$

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where $k \geq 1$ since $z_0$ is a zero of $f$. Suppose $k < \infty$. Then,

$$f(z) = (z - z_0)^k g(z)$$

where $g(z_0) \neq 0$. Letting $z_n \to z_0$ where $z_n \in Z, z_n \neq z_0$, it follows

$$0 = (z_n - z_0)^k g(z_n)$$

which implies $g(z_n) = 0$. Then by continuity of $g$, we see that $g(z_0) = 0$ also, contrary to the choice of $k$. Therefore, $k$ cannot be less than $\infty$ and so $z_0$ is a point satisfying the second condition.

Now suppose the second condition and let

$$S \equiv \{ z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n \}.$$

It is clear that $S$ is a closed set which by assumption is nonempty. However, this set is also open. To see this, let $z \in S$. Then for all $w$ close enough to $z$,

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} (w - z)^k = 0.$$

Thus $f$ is identically equal to zero near $z \in S$. Therefore, all points near $z$ are contained in $S$ also, showing that $S$ is an open set. Now $\Omega = S \cup (\Omega \setminus S)$, the union of two disjoint open sets, $S$ being nonempty. It follows the other open set, $\Omega \setminus S$, must be empty because $\Omega$ is connected. Therefore, the first condition is verified. This proves the theorem. (See the following diagram.)

Note how radically different this is from the theory of functions of a real variable. Consider, for example the function

$$f(x) = \begin{cases} x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
which has a derivative for all \( x \in \mathbb{R} \) and for which 0 is a limit point of the set, \( Z \), even though \( f \) is not identically equal to zero.

Here is a very important application called Euler’s formula. Recall that

\[
e^z \equiv e^x (\cos(y) + i \sin(y)) \tag{49.5.19}\]

Is it also true that \( e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \)?

**Theorem 49.5.4 (Euler’s Formula)** Let \( z = x + iy \). Then

\[
e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}. \]

**Proof:** It was already observed that \( e^z \) given by (49.5.19) is analytic. So is \( \exp(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!} \). In fact the power series converges for all \( z \in \mathbb{C} \). Furthermore the two functions, \( e^z \) and \( \exp(z) \) agree on the real line which is a set which contains a limit point. Therefore, they agree for all values of \( z \in \mathbb{C} \).

This formula shows the famous two identities,

\[e^{i\pi} = -1 \text{ and } e^{2\pi i} = 1.\]

### 49.6 Liouville’s Theorem

The following theorem pertains to functions which are analytic on all of \( \mathbb{C} \), “entire” functions.

**Definition 49.6.1** A function, \( f : \mathbb{C} \to \mathbb{C} \) or more generally, \( f : \mathbb{C} \to X \) is entire means it is analytic on \( \mathbb{C} \).

**Theorem 49.6.2 (Liouville’s theorem)** If \( f \) is a bounded entire function having values in \( X \), then \( f \) is a constant.

**Proof:** Since \( f \) is entire, pick any \( z \in \mathbb{C} \) and write

\[
f'(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w-z)^2} dw
\]

where \( \gamma_R(t) = z + Re^{it} \) for \( t \in [0, 2\pi] \). Therefore,

\[||f'(z)|| \leq C \frac{1}{R}\]

where \( C \) is some constant depending on the assumed bound on \( f \). Since \( R \) is arbitrary, let \( R \to \infty \) to obtain \( f'(z) = 0 \) for any \( z \in \mathbb{C} \). It follows from this that \( f \) is constant for if \( z, j = 1, 2 \) are two complex numbers, let \( h(t) = f(z_1 + t(z_2 - z_1)) \) for \( t \in [0, 1] \). Then \( h'(t) = f'(z_1 + t(z_2 - z_1))(z_2 - z_1) = 0 \). By Lemmas 49.3.2 - 49.3.4 \( h \) is a constant on \([0, 1]\) which implies \( f(z_1) = f(z_2) \).
49.7. THE GENERAL CAUCHY INTEGRAL FORMULA

With Liouville’s theorem it becomes possible to give an easy proof of the fundamental theorem of algebra. It is ironic that all the best proofs of this theorem in algebra come from the subjects of analysis or topology. Out of all the proofs that have been given of this very important theorem, the following one based on Liouville’s theorem is the easiest.

Theorem 49.6.3 (Fundamental theorem of Algebra) Let

\[ p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \]

be a polynomial where \( n \geq 1 \) and each coefficient is a complex number. Then there exists \( z_0 \in \mathbb{C} \) such that \( p(z_0) = 0 \).

Proof: Suppose not. Then \( p(z)^{-1} \) is an entire function. Also

\[ |p(z)| \geq |z|^n - \left( |a_{n-1}| |z|^{n-1} + \cdots + |a_1| |z| + |a_0| \right) \]

and so \( \lim_{|z| \to \infty} |p(z)| = \infty \) which implies \( \lim_{|z| \to \infty} \left| p(z)^{-1} \right| = 0 \). It follows that, since \( p(z)^{-1} \) is bounded for \( z \) in any bounded set, we must have that \( p(z)^{-1} \) is a bounded entire function. But then it must be constant. However since \( p(z)^{-1} \to 0 \) as \( |z| \to \infty \), this constant can only be 0. However, \( \frac{1}{p(z)} \) is never equal to zero. This proves the theorem.

49.7 The General Cauchy Integral Formula

49.7.1 The Cauchy Goursat Theorem

This section gives a fundamental theorem which is essential to the development which follows and is closely related to the question of when a function has a primitive. First of all, if you have two points in \( \mathbb{C} \), \( z_1 \) and \( z_2 \), you can consider \( \gamma(t) \equiv z_1 + t(z_2 - z_1) \) for \( t \in [0,1] \) to obtain a continuous bounded variation curve from \( z_1 \) to \( z_2 \). More generally, if \( z_1, \cdots, z_m \) are points in \( \mathbb{C} \) you can obtain a continuous bounded variation curve from \( z_1 \) to \( z_m \) which consists of first going from \( z_1 \) to \( z_2 \) and then from \( z_2 \) to \( z_3 \) and so on, till in the end one goes from \( z_{m-1} \) to \( z_m \). We denote this piecewise linear curve as \( \gamma(z_1, \cdots, z_m) \). Now let \( T \) be a triangle with vertices \( z_1, z_2 \) and \( z_3 \) encountered in the counter clockwise direction as shown.

Denote by \( \int_{\partial T} f(z) \, dz \), the expression, \( \int_{\gamma(z_1,z_2,z_3,z_1)} f(z) \, dz \). Consider the fol-
By Lemma 48.0.11

\[ \int_{\partial T} f(z) \, dz = \sum_{k=1}^{4} \int_{\partial T_k} f(z) \, dz. \]  (49.7.20)

On the “inside lines” the integrals cancel as claimed in Lemma 48.0.11 because there are two integrals going in opposite directions for each of these inside lines.

**Theorem 49.7.1 (Cauchy Goursat)** Let \( f : \Omega \to X \) have the property that \( f'(z) \) exists for all \( z \in \Omega \) and let \( T \) be a triangle contained in \( \Omega \). Then

\[ \int_{\partial T} f(w) \, dw = 0. \]

**Proof:** Suppose not. Then

\[ \left\| \int_{\partial T} f(w) \, dw \right\| = \alpha \neq 0. \]

From 49.7.20 it follows

\[ \alpha \leq \sum_{k=1}^{4} \left\| \int_{\partial T_k} f(w) \, dw \right\| \]

and so for at least one of these \( T_k \), denoted from now on as \( T_1 \),

\[ \left\| \int_{\partial T_1} f(w) \, dw \right\| \geq \frac{\alpha}{4}. \]

Now let \( T_1 \) play the same role as \( T \), subdivide as in the above picture, and obtain \( T_2 \) such that

\[ \left\| \int_{\partial T_2} f(w) \, dw \right\| \geq \frac{\alpha}{4^2}. \]

Continue in this way, obtaining a sequence of triangles,

\[ T_k \supseteq T_{k+1}, \text{diam}(T_k) \leq \text{diam}(T) \cdot 2^{-k}, \]
and
\[ \left| \int_{\partial T_k} f(w) \, dw \right| \geq \frac{\alpha}{4^k}. \]
Then let \( z \in \cap_{k=1}^{\infty} T_k \) and note that by assumption, \( f'(z) \) exists. Therefore, for all \( k \) large enough,
\[
\int_{\partial T_k} f(w) \, dw = \int_{\partial T_k} f(z) + f'(z) (w-z) + g(w) \, dw
\]
where \( ||g(w)|| < \varepsilon |w-z| \). Now observe that \( w \to f(z) + f'(z)(w-z) \) has a primitive, namely,
\[
F(w) = f(z)w + f'(z)(w-z)^2/2.
\]
Therefore, by Corollary 48.0.14,
\[
\int_{\partial T_k} f(w) \, dw = \int_{\partial T_k} g(w) \, dw.
\]
From the definition, of the integral,
\[
\frac{\alpha}{4^k} \leq \left| \int_{\partial T_k} g(w) \, dw \right| \leq \varepsilon \operatorname{diam}(T_k) (\text{length of } \partial T_k)
\]
\[
\leq \varepsilon 2^{-k} (\text{length of } T) \operatorname{diam}(T) 2^{-k},
\]
and so
\[
\alpha \leq \varepsilon (\text{length of } T) \operatorname{diam}(T).
\]
Since \( \varepsilon \) is arbitrary, this shows \( \alpha = 0 \), a contradiction. Thus \( \int_{\partial T} f(w) \, dw = 0 \) as claimed.

This fundamental result yields the following important theorem.

**Theorem 49.7.2** (Morera\(^1\)) Let \( \Omega \) be an open set and let \( f'(z) \) exist for all \( z \in \Omega \). Let \( D \equiv B(z_0, r) \subseteq \Omega \). Then there exists \( \varepsilon > 0 \) such that \( f \) has a primitive on \( B(z_0, r + \varepsilon) \).

**Proof:** Choose \( \varepsilon > 0 \) small enough that \( B(z_0, r + \varepsilon) \subseteq \Omega \). Then for \( w \in B(z_0, r + \varepsilon) \), define
\[
F(w) \equiv \int_{\gamma(z_0, w)} f(u) \, du.
\]
Then by the Cauchy Goursat theorem, and \( w \in B(z_0, r + \varepsilon) \), it follows that for \( |h| \) small enough,
\[
\frac{F(w+h) - F(w)}{h} = \frac{1}{h} \int_{\gamma(w, w+h)} f(u) \, du
\]
\[
= \frac{1}{h} \int_0^1 f(w + th) \, hdt = \int_0^1 f(w + th) \, dt.
\]
\(^1\)Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886
which converges to \( f(w) \) due to the continuity of \( f \) at \( w \). This proves the theorem.

The following is a slight generalization of the above theorem which is also referred to as Morera’s theorem.

**Corollary 49.7.3** Let \( \Omega \) be an open set and suppose that whenever

\[
\gamma(z_1, z_2, z_3, z_1)
\]

is a closed curve bounding a triangle \( T \), which is contained in \( \Omega \), and \( f \) is a continuous function defined on \( \Omega \), it follows that

\[
\int_{\gamma(z_1, z_2, z_3, z_1)} f(z) \, dz = 0,
\]

then \( f \) is analytic on \( \Omega \).

**Proof:** As in the proof of Morera’s theorem, let \( \overline{B(z_0, r)} \subseteq \Omega \) and use the given condition to construct a primitive, \( F \) for \( f \) on \( B(z_0, r) \). Then \( F \) is analytic and so by Theorem 49.3.11, it follows that \( F \) and hence \( f \) have infinitely many derivatives, implying that \( f \) is analytic on \( B(z_0, r) \). Since \( z_0 \) is arbitrary, this shows \( f \) is analytic on \( \Omega \).

### 49.7.2 A Redundant Assumption

Earlier in the definition of analytic, it was assumed the derivative is continuous. This assumption is redundant.

**Theorem 49.7.4** Let \( \Omega \) be an open set in \( \mathbb{C} \) and suppose \( f : \Omega \rightarrow X \) has the property that \( f'(z) \) exists for each \( z \in \Omega \). Then \( f \) is analytic on \( \Omega \).

**Proof:** Let \( z_0 \in \Omega \) and let \( B(z_0, r) \subseteq \Omega \). By Morera’s theorem \( f \) has a primitive, \( F \) on \( B(z_0, r) \). It follows that \( F \) is analytic because it has a derivative, \( f \), and this derivative is continuous. Therefore, by Theorem 49.3.11 \( F \) has infinitely many derivatives on \( B(z_0, r) \) implying that \( f \) also has infinitely many derivatives on \( B(z_0, r) \). Thus \( f \) is analytic as claimed.

It follows a function is analytic on an open set, \( \Omega \) if and only if \( f'(z) \) exists for \( z \in \Omega \). This is because it was just shown the derivative, if it exists, is automatically continuous.

The same proof used to prove Theorem 49.7.2 implies the following corollary.

**Corollary 49.7.5** Let \( \Omega \) be a convex open set and suppose that \( f'(z) \) exists for all \( z \in \Omega \). Then \( f \) has a primitive on \( \Omega \).

Note that this implies that if \( \Omega \) is a convex open set on which \( f'(z) \) exists and if \( \gamma : [a, b] \rightarrow \Omega \) is a closed, continuous curve having bounded variation, then letting \( F \) be a primitive of \( f \) Theorem 49.0.13 implies

\[
\int_{\gamma} f(z) \, dz = F(\gamma(b)) - F(\gamma(a)) = 0.
\]
49.7. THE GENERAL CAUCHY INTEGRAL FORMULA

Notice how different this is from the situation of a function of a real variable! It is possible for a function of a real variable to have a derivative everywhere and yet the derivative can be discontinuous. A simple example is the following.

\[ f(x) = \begin{cases} 
  x^2 \sin \left(\frac{1}{x}\right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \]

Then \( f'(x) \) exists for all \( x \in \mathbb{R} \). Indeed, if \( x \neq 0 \), the derivative equals \( 2x \sin \frac{1}{x} - \cos \frac{1}{x} \), which has no limit as \( x \rightarrow 0 \). However, from the definition of the derivative of a function of one variable, \( f'(0) = 0 \).

49.7.3 Classification Of Isolated Singularities

First some notation.

**Definition 49.7.6** Let \( B'(a,r) \equiv \{ z \in \mathbb{C} \text{ such that } 0 < |z-a| < r \} \). Thus this is the usual ball without the center. A function is said to have an isolated singularity at the point \( a \in \mathbb{C} \) if \( f \) is analytic on \( B'(a,r) \) for some \( r > 0 \).

It turns out isolated singularities can be neatly classified into three types, removable singularities, poles, and essential singularities. The next theorem deals with the case of a removable singularity.

**Definition 49.7.7** An isolated singularity of \( f \) is said to be removable if there exists an analytic function, \( g \) analytic at \( a \) and near \( a \) such that \( f = g \) at all points near \( a \).

**Theorem 49.7.8** Let \( f : B'(a,r) \rightarrow X \) be analytic. Thus \( f \) has an isolated singularity at \( a \). Suppose also that

\[ \lim_{z \rightarrow a} f(z)(z-a) = 0. \]

Then there exists a unique analytic function, \( g : B(a,r) \rightarrow X \) such that \( g = f \) on \( B'(a,r) \). Thus the singularity at \( a \) is removable.

**Proof:** Let \( h(z) = (z-a)^2 f(z), h(a) = 0 \). Then \( h \) is analytic on \( B(a,r) \) because it is easy to see that \( h'(a) = 0 \). It follows \( h \) is given by a power series,

\[ h(z) = \sum_{k=2}^{\infty} a_k (z-a)^k \]

where \( a_0 = a_1 = 0 \) because of the observation above that \( h'(a) = h(a) = 0 \). It follows that for \( |z-a| > 0 \)

\[ f(z) = \sum_{k=2}^{\infty} a_k (z-a)^{k-2} \equiv g(z). \]

This proves the theorem.

What of the other case where the singularity is not removable? This situation is dealt with by the amazing Casorati Weierstrass theorem.
Theorem 49.7.9 (Casorati Weierstrass) Let \( a \) be an isolated singularity and suppose for some \( r > 0 \), \( f(B'(a,r)) \) is not dense in \( \mathbb{C} \). Then either \( a \) is a removable singularity or there exist finitely many \( b_1, \cdots, b_M \) for some finite number, \( M \) such that for \( z \) near \( a \),

\[
f(z) = g(z) + \sum_{k=1}^{M} \frac{b_k}{(z-a)^k}
\]

(49.7.21)

where \( g(z) \) is analytic near \( a \).

Proof: Suppose \( B(z_0, \delta) \) has no points of \( f(B'(a,r)) \). Such a ball must exist if \( f(B'(a,r)) \) is not dense. Then for \( z \in B'(a,r), |f(z) - z_0| \geq \delta > 0 \). It follows from Theorem 49.7.8 that \( \frac{1}{f(z) - z_0} \) has a removable singularity at \( a \). Hence, there exists \( h \) an analytic function such that for \( z \) near \( a \),

\[
h(z) = \frac{1}{f(z) - z_0}.
\]

(49.7.22)

There are two cases. First suppose \( h(a) = 0 \). Then \( \sum_{k=1}^{\infty} a_k (z-a)^k = \frac{1}{f(z) - z_0} \) for \( z \) near \( a \). If all the \( a_k = 0 \), this would be a contradiction because then the left side would equal zero for \( z \) near \( a \) but the right side could not equal zero. Therefore, there is a first \( m \) such that \( a_m \neq 0 \). Hence there exists an analytic function, \( k(z) \) which is not equal to zero in some ball, \( B(a, \varepsilon) \) such that

\[
k(z)(z-a)^m = \frac{1}{f(z) - z_0}.
\]

Hence, taking both sides to the \(-1\) power,

\[
f(z) - z_0 = \frac{1}{(z-a)^m} \sum_{k=0}^{\infty} b_k (z-a)^k
\]

and so (49.7.21) holds.

The other case is that \( h(a) \neq 0 \). In this case, raise both sides of (49.7.22) to the \(-1\) power and obtain

\[
f(z) - z_0 = h(z)^{-1},
\]

a function analytic near \( a \). Therefore, the singularity is removable. This proves the theorem.

This theorem is the basis for the following definition which classifies isolated singularities.

Definition 49.7.10 Let \( a \) be an isolated singularity of a complex valued function, \( f \). When (49.7.21) holds for \( z \) near \( a \), then \( a \) is called a pole. The order of the pole in (49.7.21) is \( M \). If for every \( r > 0 \), \( f(B'(a,r)) \) is dense in \( \mathbb{C} \) then \( a \) is called an essential singularity.

In terms of the above definition, isolated singularities are either removable, a pole, or essential. There are no other possibilities.
Theorem 49.7.11 Suppose $f: \Omega \to \mathbb{C}$ has an isolated singularity at $a \in \Omega$. Then $a$ is a pole if and only if
\[
\lim_{z \to a} d(f(z), \infty) = 0
\]
in $\hat{\mathbb{C}}$.

**Proof:** Suppose first $f$ has a pole at $a$. Then by definition, $f(z) = g(z) + \sum_{k=1}^{M} \frac{b_k}{z-a}$ for $z$ near $a$ where $g$ is analytic. Then
\[
|f(z)| \geq \frac{|b_M|}{|z-a|^M} - |g(z)| - \sum_{k=1}^{M-1} \frac{|b_k|}{|z-a|^k}
\]
\[
=\frac{1}{|z-a|^M} \left( |b_M| - \left( |g(z)| |z-a|^M + \sum_{k=1}^{M-1} |b_k| |z-a|^{M-k} \right) \right).
\]
Now $\lim_{z \to a} \left( |g(z)| |z-a|^M + \sum_{k=1}^{M-1} |b_k| |z-a|^{M-k} \right) = 0$ and so the above inequality proves $\lim_{z \to a} |f(z)| = \infty$. Referring to the diagram on Page 1671, you see this is the same as saying
\[
\lim_{z \to a} \left| \theta f(z) - (0,0,2) \right| = \lim_{z \to a} |\theta f(z) - \theta(\infty)| = \lim_{z \to a} d(f(z), \infty) = 0
\]
Conversely, suppose $\lim_{z \to a} d(f(z), \infty) = 0$. Then from the diagram on Page 1673, it follows $\lim_{z \to a} |f(z)| = \infty$ and in particular, $a$ cannot be either removable or an essential singularity by the Casorati-Weierstrass theorem, Theorem 49.7.12. The only case remaining is that $a$ is a pole. This proves the theorem.

**Definition 49.7.12** Let $f: \Omega \to \mathbb{C}$ where $\Omega$ is an open subset of $\mathbb{C}$. Then $f$ is called meromorphic if all singularities are isolated and are either poles or removable and this set of singularities has no limit point. It is convenient to regard meromorphic functions as having values in $\hat{\mathbb{C}}$ where if $a$ is a pole, $f(a) \equiv \infty$. From now on, this will be assumed when a meromorphic function is being considered.

The usefulness of the above convention about $f(a) \equiv \infty$ at a pole is made clear in the following theorem.

**Theorem 49.7.13** Let $\Omega$ be an open subset of $\mathbb{C}$ and let $f: \Omega \to \hat{\mathbb{C}}$ be meromorphic. Then $f$ is continuous with respect to the metric, $d$ on $\hat{\mathbb{C}}$.

**Proof:** Let $z_n \to z$ where $z \in \Omega$. Then if $z$ is a pole, it follows from Theorem 49.7.12 that
\[
d(f(z_n), \infty) = d(f(z_n), f(z)) \to 0.
\]
If $z$ is not a pole, then $f(z_n) \to f(z)$ in $\mathbb{C}$ which implies $|\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \to 0$. Recall that $\theta$ is continuous on $\mathbb{C}$.
49.7.4 The Cauchy Integral Formula

This section presents the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number also called the index, defined in the following theorem. This winding number, along with the earlier results, especially Liouville’s theorem, yields an extremely general Cauchy integral formula.

Definition 49.7.14 Let \( \gamma : [a, b] \rightarrow \mathbb{C} \) and suppose \( z \notin \gamma^* \). The winding number, \( n(\gamma, z) \), is defined by

\[
n(\gamma, z) \equiv \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}.
\]

The main interest is in the case where \( \gamma \) is a closed curve. However, the same notation will be used for any such curve.

Theorem 49.7.15 Let \( \gamma : [a, b] \rightarrow \mathbb{C} \) be continuous and have bounded variation with \( \gamma(a) = \gamma(b) \). Also suppose that \( z \notin \gamma^* \).

Define

\[
n(\gamma, z) \equiv \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}.
\]  

(49.7.23)

Then \( n(\gamma, \cdot) \) is continuous and integer valued. Furthermore, there exists a sequence, \( \eta_k : [a, b] \rightarrow \mathbb{C} \) such that \( \eta_k \in C^1([a, b]) \),

\[
||\eta_k - \gamma|| < \frac{1}{k}, \eta_k(a) = \eta_k(b) = \gamma(a) = \gamma(b),
\]

and \( n(\eta_k, z) = n(\gamma, z) \) for all \( k \) large enough. Also \( n(\gamma, \cdot) \) is constant on every connected component of \( \mathbb{C} \setminus \gamma^* \) and equals zero on the unbounded component of \( \mathbb{C} \setminus \gamma^* \).

Proof: First consider the assertion about continuity.

\[
|n(\gamma, z) - n(\gamma, z_1)| \leq C \left| \int_{\gamma} \left( \frac{1}{w - z} - \frac{1}{w - z_1} \right) \, dw \right|
\]

\[
\leq \tilde{C} \text{(Length of } \gamma) |z_1 - z|
\]

whenever \( z_1 \) is close enough to \( z \). This proves the continuity assertion. Note this did not depend on \( \gamma \) being closed.

Next it is shown that for a closed curve the winding number equals an integer. To do so, use Theorem 42.9.12 to obtain \( \eta_k \), a function in \( C^1([a, b]) \) such that \( z \notin \eta_k([a, b]) \) for all \( k \) large enough, \( \eta_k(x) = \gamma(x) \) for \( x = a, b \), and

\[
\left| \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z} - \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \right| < \frac{1}{k}, \text{ } ||\eta_k - \gamma|| < \frac{1}{k}.
\]

It is shown that each of \( \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \) is an integer. To simplify the notation, write \( \eta \) instead of \( \eta_k \).

\[
\int_{\eta} \frac{dw}{w - z} = \int_{a}^{b} \frac{\eta'(s) \, ds}{\eta(s) - z}.
\]
Define
\[ g(t) = \int_a^t \frac{\eta'(s) \, ds}{\eta(s) - z}. \] (49.7.24)

Then
\[ \left( e^{-g(t)} (\eta(t) - z) \right)' = e^{-g(t)} \eta'(t) - e^{-g(t)} g'(t) (\eta(t) - z) \]
\[ = e^{-g(t)} \eta'(t) - e^{-g(t)} \eta'(t) = 0. \]

It follows that \( e^{-g(t)} (\eta(t) - z) \) equals a constant. In particular, using the fact that \( \eta(a) = \eta(b) \),
\[ e^{-g(b)} (\eta(b) - z) = e^{-g(a)} (\eta(a) - z) = (\eta(a) - z) = (\eta(b) - z) \]
and so \( e^{-g(b)} = 1 \). This happens if and only if \(-g(b) = 2m\pi i\) for some integer \(m\). Therefore, \((49.7.25)\) implies
\[ 2m\pi i = \int_a^b \frac{\eta'(s) \, ds}{\eta(s) - z} = \int_c \frac{dw}{w - z}. \]

Therefore, \( \frac{1}{2\pi i} \int_{\gamma_k} \frac{dw}{w - z} \) is a sequence of integers converging to \( \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z} \equiv n(\gamma, z) \) and so \( n(\gamma, z) \) must also be an integer and \( n(\eta_k, z) = n(\gamma, z) \) for all \(k\) large enough.

Since \( n(\gamma, \cdot) \) is continuous and integer valued, it follows from Corollary 49.7.14 on Page 1674 that it must be constant on every connected component of \( \mathbb{C} \setminus \gamma^* \). It is clear that \( n(\gamma, z) \) equals zero on the unbounded component because from the formula,
\[ \lim_{z \to \infty} |n(\gamma, z)| \leq \lim_{z \to \infty} V(\gamma, [a, b]) \left( \frac{1}{|z|} \right), \]
where \( c \geq \max \{|w| : w \in \gamma^*\} \). This proves the theorem.

**Corollary 49.7.16** Suppose \( \gamma : [a, b] \to \mathbb{C} \) is a continuous bounded variation curve and \( n(\gamma, z) \) is an integer where \( z \notin \gamma^* \). Then \( \gamma(a) = \gamma(b) \). Also \( z \to n(\gamma, z) \) for \( z \notin \gamma^* \) is continuous.

**Proof:** Letting \( \eta \) be a \( C^1 \) curve for which \( \eta(a) = \gamma(a) \) and \( \eta(b) = \gamma(b) \) and which is close enough to \( \gamma \) that \( n(\eta, z) = n(\gamma, z) \), the argument is similar to the above. Let
\[ g(t) = \int_a^t \frac{\eta'(s) \, ds}{\eta(s) - z}. \] (49.7.25)

Then
\[ \left( e^{-g(t)} (\eta(t) - z) \right)' = e^{-g(t)} \eta'(t) - e^{-g(t)} g'(t) (\eta(t) - z) \]
\[ = e^{-g(t)} \eta'(t) - e^{-g(t)} \eta'(t) = 0. \]

Hence
\[ e^{-g(t)} (\eta(t) - z) = c \neq 0. \] (49.7.26)
By assumption
\[ g(b) = \int_{\eta} \frac{1}{w-z} \, dw = 2\pi im \]
for some integer, \( m \). Therefore, from 49.7.26
\[ 1 = e^{2\pi im} = \frac{\eta(b) - z}{c}. \]
Thus \( c = \eta(b) - z \) and letting \( t = a \) in 49.7.26,
\[ 1 = \frac{\eta(a) - z}{\eta(b) - z} \]
which shows \( \eta(a) = \eta(b) \). This proves the corollary since the assertion about continuity was already observed.

It is a good idea to consider a simple case to get an idea of what the winding number is measuring. To do so, consider \( \gamma : [a,b] \to \mathbb{C} \) such that \( \gamma \) is continuous, closed and bounded variation. Suppose also that \( \gamma \) is one to one on \( (a,b) \). Such a curve is called a simple closed curve. It can be shown that such a simple closed curve divides the plane into exactly two components, an “inside” bounded component and an “outside” unbounded component. This is called the Jordan Curve theorem. This is a difficult theorem which requires some very hard topology such as homology theory or degree theory. It won’t be used here beyond making reference to it. For now, it suffices to simply assume that \( \gamma \) is such that this result holds. This will usually be obvious anyway. Also suppose that it is possible to change the parameter to be in \( [0,2\pi] \), in such a way that \( \gamma(t) + \lambda(z + re^{it} - \gamma(t)) - z \neq 0 \) for all \( t \in [0,2\pi] \) and \( \lambda \in [0,1] \). (As \( t \) goes from 0 to \( 2\pi \) the point \( \gamma(t) \) traces the curve \( \gamma([0,2\pi]) \) in the counter clockwise direction.) Suppose \( z \in D \), the inside of the simple closed curve and consider the curve \( \delta(t) = z + re^{it} \) for \( t \in [0,2\pi] \) where \( r \) is chosen small enough that \( B(z,r) \subseteq D \). Then it happens that \( n(\delta,z) = n(\gamma,z) \).

**Proposition 49.7.17** Under the above conditions,
\[ n(\delta,z) = n(\gamma,z) \]
and \( n(\delta,z) = 1 \).

**Proof:** By changing the parameter, assume that \( [a,b] = [0,2\pi] \). From Theorem 49.7.9 it suffices to assume also that \( \gamma \) is \( C^1 \). Define \( h_\lambda(t) \equiv \gamma(t) + \lambda(z + re^{it} - \gamma(t)) \) for \( \lambda \in [0,1] \). (This function is called a homotopy of the curves \( \gamma \) and \( \delta \).) Note that for each \( \lambda \in [0,1] \), \( t \to h_\lambda(t) \) is a closed \( C^1 \) curve. Also,
\[ \frac{1}{2\pi i} \int_{h_\lambda} \frac{1}{w-z} \, dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t) + \lambda(rie^{it} - \gamma(t))}{\gamma(t) + \lambda(z + re^{it} - \gamma(t)) - z} \, dt. \]
This number is an integer and it is routine to verify that it is a continuous function of \( \lambda \). When \( \lambda = 0 \) it equals \( n(\gamma,z) \) and when \( \lambda = 1 \) it equals \( n(\delta,z) \). Therefore,
THE GENERAL CAUCHY INTEGRAL FORMULA

49.7. THE GENERAL CAUCHY INTEGRAL FORMULA

It only remains to compute \( n(\delta, z) \).

\[
n(\delta, z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{r ie^{it}}{re^{it}} dt = 1.
\]

This proves the proposition.

Now if \( \gamma \) was not one to one but caused the point, \( \gamma(t) \) to travel around \( \gamma^* \) twice, you could modify the above argument to have the parameter interval, \([0, 4\pi]\) and still find \( n(\delta, z) = n(\gamma, z) \) only this time, \( n(\delta, z) = 2 \). Thus the winding number is just what its name suggests. It measures the number of times the curve winds around the point. One might ask why bother with the winding number if this is all it does. The reason is that the notion of counting the number of times a curve winds around a point is rather vague. The winding number is precise. It is also the natural thing to consider in the general Cauchy integral formula presented below.

Consider a situation typified by the following picture in which \( \Omega \) is the open set between the dotted curves and \( \gamma_j \) are closed rectifiable curves in \( \Omega \).

The following theorem is the general Cauchy integral formula.

**Definition 49.7.18** Let \( \{\gamma_k\}_{k=1}^n \) be continuous oriented curves having bounded variation. Then this is called a cycle if whenever, \( z \notin \bigcup_{k=1}^n \gamma_k^* \), \( \sum_{k=1}^n n(\gamma_k, z) \) is an integer.

By Theorem 49.7.15 if each \( \gamma_k \) is a closed curve, then \( \{\gamma_k\}_{k=1}^n \) is a cycle.

**Theorem 49.7.19** Let \( \Omega \) be an open subset of the plane and let \( f : \Omega \to X \) be analytic. If \( \gamma_k : [a_k, b_k] \to \Omega, k = 1, \ldots, m \) are continuous curves having bounded variation such that for all \( z \notin \bigcup_{k=1}^m \gamma_k([a_k, b_k]) \)

\[
\sum_{k=1}^m n(\gamma_k, z) \text{ equals an integer}
\]

and for all \( z \notin \Omega \),

\[
\sum_{k=1}^m n(\gamma_k, z) = 0.
\]

Then for all \( z \in \Omega \setminus \bigcup_{k=1}^m \gamma_k([a_k, b_k]) \),

\[
f(z) \sum_{k=1}^m n(\gamma_k, z) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw.
\]
**Proof:** Let $\phi$ be defined on $\Omega \times \Omega$ by

$$\phi(z, w) \equiv \begin{cases} f(w) - f(z) & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases}.$$ 

Then $\phi$ is analytic as a function of both $z$ and $w$ and is continuous in $\Omega \times \Omega$. This is easily seen using Theorem 49.7.8. Consider the case of $w \to \phi(z, w)$.

$$\lim_{w \to z} (w - z) (\phi(z, w) - \phi(z, z)) = \lim_{w \to z} \left( \frac{f(w) - f(z)}{w - z} - f'(z) \right) = 0.$$ 

Thus $w \to \phi(z, w)$ has a removable singularity at $z$. The case of $z \to \phi(z, w)$ is similar.

Define

$$h(z) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w) \, dw.$$ 

Is $h$ analytic on $\Omega$? To show this is the case, verify

$$\int_{\partial T} h(z) \, dz = 0$$ 

for every triangle, $T$, contained in $\Omega$ and apply Corollary 49.7.3. To do this, use Theorem 49.7.27 to obtain for each $k$, a sequence of functions, $\eta_{kn} \in C^1([a_k, b_k])$ such that

$$\eta_{kn}(x) = \gamma_k(x) \text{ for } x \in \{a_k, b_k\}$$

and

$$\eta_{kn}([a_k, b_k]) \subseteq \Omega, \quad ||\eta_{kn} - \gamma_k|| < \frac{1}{n},$$

$$\left|\int_{\eta_{kn}} \phi(z, w) \, dw - \int_{\gamma_k} \phi(z, w) \, dw\right| < \frac{1}{n}, \quad (49.7.27)$$

for all $z \in T$. Then applying Fubini’s theorem,

$$\int_{\partial T} \int_{\eta_{kn}} \phi(z, w) \, dwdz = \int_{\eta_{kn}} \int_{\partial T} \phi(z, w) \, dzdw = 0$$

because $\phi$ is given to be analytic. By 49.7.27,

$$\int_{\partial T} \int_{\gamma_k} \phi(z, w) \, dwdz = \lim_{n \to \infty} \int_{\partial T} \int_{\eta_{kn}} \phi(z, w) \, dwdz = 0$$

and so $h$ is analytic on $\Omega$ as claimed.

Now let $H$ denote the set,

$$H \equiv \left\{ z \in \mathbb{C} \setminus \bigcup_{k=1}^{m} \gamma_k([a_k, b_k]) : \sum_{k=1}^{m} n(\gamma_k, z) = 0 \right\}.$$
49.7. THE GENERAL CAUCHY INTEGRAL FORMULA

$H$ is an open set because $z \to \sum_{k=1}^{m} n(\gamma_k, z)$ is integer valued by assumption and continuous. Define

$$g(z) \equiv \begin{cases} h(z) & \text{if } z \in \Omega \\ \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z} \, dw & \text{if } z \in H \end{cases} \quad (49.7.28)$$

Why is $g(z)$ well defined? For $z \in \Omega \cap H$, $z \not\in \bigcup_{k=1}^{m} \gamma_k([a_k, b_k])$ and so

$$g(z) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w) \, dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} \, dw$$

because $z \in H$. This shows $g(z)$ is well defined. Also, $g$ is analytic on $\Omega$ because it equals $h$ there. It is routine to verify that $g$ is analytic on $H$ also because of the second line of (49.7.28).

By assumption, $\Omega^C \subseteq H$ because it is assumed that $\sum_{k=1}^{m} n(\gamma_k, z) = 0$ for $z \not\in \Omega$ and so $\Omega \cup H = \mathbb{C}$ showing that $g$ is an entire function.

Now note that $\sum_{k=1}^{m} n(\gamma_k, z) = 0$ for all $z$ contained in the unbounded component of $\mathbb{C} \setminus \bigcup_{k=1}^{m} \gamma_k([a_k, b_k])$ which component contains $B(0, r)^C$ for $r$ large enough. It follows that for $|z| > r$, it must be the case that $z \in H$ and so for such $z$, the bottom description of $g(z)$ found in (49.7.28) is valid. Therefore, it follows

$$\lim_{|z|\to\infty} ||g(z)|| = 0$$

and so $g$ is bounded and entire. By Liouville’s theorem, $g$ is a constant. Hence, from the above equation, the constant can only equal zero.

For $z \in \Omega \setminus \bigcup_{k=1}^{m} \gamma_k([a_k, b_k])$,

$$0 = h(z) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w) \, dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} \, dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z} \, dw - f(z) \sum_{k=1}^{m} n(\gamma_k, z) .$$

This proves the theorem.

**Corollary 49.7.20** Let $\Omega$ be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$
for all $z \notin \Omega$. Then if $f : \Omega \to \mathbb{C}$ is analytic,
\[ \sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw = 0. \]

**Proof:** This follows from Theorem 49.7.19 as follows. Let
\[ g(w) = f(w)(w - z) \]
where $z \in \Omega \setminus \bigcup_{k=1}^{m} \gamma_k ([a_k, b_k])$. Then by this theorem,
\[ 0 = 0 \sum_{k=1}^{m} n(\gamma_k, z) = g(z) \sum_{k=1}^{m} n(\gamma_k, z) = \]
\[ \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(w)}{w - z} \, dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw. \]

Another simple corollary to the above theorem is Cauchy’s theorem for a simply connected region.

**Definition 49.7.21** An open set, $\Omega \subseteq \mathbb{C}$ is a region if it is open and connected. A region, $\Omega$ is simply connected if $\mathbb{C} \setminus \Omega$ is connected where $\mathbb{C}$ is the extended complex plane. In the future, the term simply connected open set will be an open set which is connected and $\mathbb{C} \setminus \Omega$ is connected.

**Corollary 49.7.22** Let $\gamma : [a, b] \to \Omega$ be a continuous closed curve of bounded variation where $\Omega$ is a simply connected region in $\mathbb{C}$ and let $f : \Omega \to X$ be analytic. Then
\[ \int_{\gamma} f(w) \, dw = 0. \]

**Proof:** Let $D$ denote the unbounded component of $\mathbb{C} \setminus \gamma^*$. Thus $\infty \in \mathbb{C} \setminus \gamma^*$. Then the connected set, $\mathbb{C} \setminus \Omega$ is contained in $D$ since every point of $\mathbb{C} \setminus \Omega$ must be in some component of $\mathbb{C} \setminus \gamma^*$ and $\infty$ is contained in both $\mathbb{C} \setminus \Omega$ and $D$. Thus $D$ must be the component that contains $\mathbb{C} \setminus \Omega$. It follows that $n(\gamma, \cdot)$ must be constant on $\mathbb{C} \setminus \Omega$, its value being its value on $D$. However, for $z \in D$,
\[ n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} \, dw \]
and so $\lim_{|z| \to \infty} n(\gamma, z) = 0$ showing $n(\gamma, z) = 0$ on $D$. Therefore this verifies the hypothesis of Theorem 49.7.19. Let $z \in \Omega \cap D$ and define
\[ g(w) \equiv f(w)(w - z). \]
Thus $g$ is analytic on $\Omega$ and by Theorem 49.7.19,
\[ 0 = n(z, \gamma) g(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\gamma} f(w) \, dw. \]
This proves the corollary.

The following is a very significant result which will be used later.
Corollary 49.7.23 Suppose \( \Omega \) is a simply connected open set and \( f : \Omega \to X \) is analytic. Then \( f \) has a primitive, \( F \), on \( \Omega \). Recall this means there exists \( F \) such that \( F'(z) = f(z) \) for all \( z \in \Omega \).

Proof: Pick a point, \( z_0 \in \Omega \) and let \( V \) denote those points, \( z \) of \( \Omega \) for which there exists a curve, \( \gamma : [a, b] \to \Omega \) such that \( \gamma \) is continuous, of bounded variation, \( \gamma(a) = z_0 \), and \( \gamma(b) = z \). Then it is easy to verify that \( V \) is both open and closed in \( \Omega \) and therefore, \( V = \Omega \) because \( \Omega \) is connected. Denote by \( \gamma_{z_0, z} \) such a curve from \( z_0 \) to \( z \) and define

\[
F(z) \equiv \int_{\gamma_{z_0, z}} f(w) \, dw.
\]

Then \( F \) is well defined because if \( \gamma_j, j = 1, 2 \) are two such curves, it follows from Corollary 49.7.22 that

\[
\int_{\gamma_1} f(w) \, dw + \int_{-\gamma_2} f(w) \, dw = 0,
\]

implying that

\[
\int_{\gamma_1} f(w) \, dw = \int_{\gamma_2} f(w) \, dw.
\]

Now this function, \( F \) is a primitive because, thanks to Corollary 49.7.22

\[
(F(z + h) - F(z)) \, h^{-1} = \frac{1}{h} \int_{\gamma_{z,z+h}} f(w) \, dw = \frac{1}{h} \int_{0}^{1} f(z + th) \, h \, dt
\]

and so, taking the limit as \( h \to 0 \), \( F'(z) = f(z) \).

49.7.5 An Example Of A Cycle

The next theorem deals with the existence of a cycle with nice properties. Basically, you go around the compact subset of an open set with suitable contours while staying in the open set. The method involves the following simple concept.

Definition 49.7.24 A tiling of \( \mathbb{R}^2 = \mathbb{C} \) is the union of infinitely many equally spaced vertical and horizontal lines. You can think of the small squares which result as tiles. To tile the plane or \( \mathbb{R}^2 = \mathbb{C} \) means to consider such a union of horizontal and vertical lines. It is like graph paper. See the picture below for a representation of part of a tiling of \( \mathbb{C} \).
Theorem 49.7.25 Let $K$ be a compact subset of an open set, $\Omega$. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each $j$, $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,

$$\sum_{k=1}^m n(\Gamma_k, p) = 1.$$  

while for all $z \notin \Omega$

$$\sum_{k=1}^m n(\Gamma_k, z) = 0.$$  

Proof: Let $\delta = \text{dist}(K, \Omega^c)$. Since $K$ is compact, $\delta > 0$. Now tile the plane with squares, each of which has diameter less than $\delta/2$.

Let $S$ denote the set of all the closed squares in this tiling which have nonempty intersection with $K$. Thus, all the squares of $S$ are contained in $\Omega$. First suppose $p$ is a point of $K$ which is in the interior of one of these squares in the tiling. Denote by $\partial S_k$ the boundary of $S_k$ one of the squares in $S$, oriented in the counter clockwise direction and $S_m$ denote the square of $S$ which contains the point, $p$ in its interior.

Let the edges of the square, $S_j$ be $\{\gamma_k\}_{k=1}^4$. Thus a short computation shows $n(\partial S_m, p) = 1$ but $n(\partial S_j, p) = 0$ for all $j \neq m$. The reason for this is that for $z$ in $S_j$, the values $\{z - p : z \in S_j\}$ lie in an open square, $Q$ which is located at a positive distance from 0. Then $\mathbb{C} \setminus Q$ is connected and $1/(z - p)$ is analytic on $Q$. It follows from Corollary 49.7.23 that this function has a primitive on $Q$ and so

$$\int_{\partial S_j} \frac{1}{z - p} \, dz = 0.$$
Similarly, if \( z \notin \Omega \), \( n(\partial S_j, z) = 0 \). On the other hand, a direct computation will verify that \( n(p, \partial S) = 1 \). Thus \( 1 = \sum_{j,k} n(p, \gamma^j_k) = \sum_{S \in \mathcal{S}} n(p, \partial S) \) and if \( z \notin \Omega \), \( 0 = \sum_{j,k} n(z, \gamma^j_k) = \sum_{S \in \mathcal{S}} n(z, \partial S) \).

If \( \gamma^i_* \) coincides with \( \gamma^j_* \), then the contour integrals taken over this edge are taken in opposite directions and so the edge the two squares have in common can be deleted without changing \( \sum_{j,k} n(z, \gamma^j_k) \) for any \( z \) not on any of the lines in the tiling. For example, see the picture.

From the construction, if any of the \( \gamma^i_* \) contains a point of \( K \) then this point is on one of the four edges of \( S_j \) and at this point, there is at least one edge of some \( S_l \) which also contains this point. As just discussed, this shared edge can be deleted without changing \( \sum_{i,j} n(z, \gamma^j_k) \). Delete the edges of the \( S_k \) which intersect \( K \) but not the endpoints of these edges. That is, delete the open edges. When this is done, delete all isolated points. Let the resulting oriented curves be denoted by \( \{ \gamma^j_k \}_{k=1}^m \). Note that you might have \( \gamma^i_k = \gamma^j_l \). The construction is illustrated in the following picture.

Then as explained above, \( \sum_{k=1}^m n(p, \gamma_k) = 1 \). It remains to prove the claim about the closed curves.

Each orientation on an edge corresponds to a direction of motion over that edge. Call such a motion over the edge a route. Initially, every vertex, (corner of a square in \( S \)) has the property there are the same number of routes to and from that vertex. When an open edge whose closure contains a point of \( K \) is deleted,
every vertex either remains unchanged as to the number of routes to and from that vertex or it loses both a route away and a route to. Thus the property of having the same number of routes to and from each vertex is preserved by deleting these open edges. The isolated points which result lose all routes to and from. It follows that upon removing the isolated points you can begin at any of the remaining vertices and follow the routes leading out from this and successive vertices according to orientation and eventually return to that end. Otherwise, there would be a vertex which would have only one route leading to it which does not happen. Now if you have used all the routes out of this vertex, pick another vertex and do the same process. Otherwise, pick an unused route out of the vertex and follow it to return. Continue this way till all routes are used exactly once, resulting in closed oriented curves, $\Gamma_k$. Then

$$\sum_k n(\Gamma_k,p) = \sum_j n(\gamma_j,p) = 1.$$ 

In case $p \in K$ is on some line of the tiling, it is not on any of the $\Gamma_k$ because $\Gamma_k \cap K = \emptyset$ and so the continuity of $z \to n(\Gamma_k,z)$ yields the desired result in this case also. This proves the lemma.

### 49.8 Exercises

1. If $U$ is simply connected, $f$ is analytic on $U$ and $f$ has no zeros in $U$, show there exists an analytic function, $F$, defined on $U$ such that $e^F = f$.

2. Let $f$ be defined and analytic near the point $a \in \mathbb{C}$. Show that then $f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ whenever $|z-a| < R$ where $R$ is the distance between $a$ and the nearest point where $f$ fails to have a derivative. The number $R$, is called the radius of convergence and the power series is said to be expanded about $a$.

3. Find the radius of convergence of the function $\frac{1}{1+z^2}$ expanded about $a = 2$. Note there is nothing wrong with the function, $\frac{1}{1+x^2}$ when considered as a function of a real variable, $x$ for any value of $x$. However, if you insist on using power series, you find there is a limitation on the values of $x$ for which the power series converges due to the presence in the complex plane of a point, $i$, where the function fails to have a derivative.

4. Suppose $f$ is analytic on all of $\mathbb{C}$ and satisfies $|f(z)| < A + B|z|^{1/2}$. Show $f$ is constant.

5. What if you defined an open set, $U$ to be simply connected if $\mathbb{C} \setminus U$ is connected. Would it amount to the same thing? **Hint:** Consider the outside of $B(0,1)$.

6. Let $\gamma(t) = e^{it} : t \in [0,2\pi]$. Find $\int_\gamma \frac{1}{z^n} \, dz$ for $n = 1, 2, \ldots$. 

7. Show \( i \int_0^{2\pi} (2 \cos \theta)^{2n} d\theta = \int_0^1 (z + \frac{1}{z})^{2n} \left(\frac{1}{z}\right) dz \) where \( \gamma(t) = e^{it} : t \in [0, 2\pi] \).

Then evaluate this integral using the binomial theorem and the previous problem.

8. Suppose that for some constants \( a, b \neq 0, a, b \in \mathbb{R}, f(z + ib) = f(z) \) for all \( z \in \mathbb{C} \) and \( f(z + a) = f(z) \) for all \( z \in \mathbb{C} \). If \( f \) is analytic, show that \( f \) must be constant. Can you generalize this? **Hint:** This uses Liouville’s theorem.

9. Suppose \( f(z) = u(x, y) + iv(x, y) \) is analytic for \( z \in U \), an open set. Let \( g(z) = u^*(x, y) + iv^*(x, y) \) where

\[
\begin{pmatrix}
u^* \\
u\end{pmatrix} = Q \begin{pmatrix} u \\
u\end{pmatrix}
\]

where \( Q \) is a unitary matrix. That is \( QQ^* = Q^*Q = I \). When will \( g \) be analytic?

10. Suppose \( f \) is analytic on an open set, \( U \), except for \( \gamma^* \subset U \) where \( \gamma \) is a one to one continuous function having bounded variation, but it is known that \( f \) is continuous on \( \gamma^* \). Show that in fact \( f \) is analytic on \( \gamma^* \) also. **Hint:** Pick a point on \( \gamma^* \), say \( \gamma(t_0) \) and suppose for now that \( t_0 \in (a, b) \). Pick \( r > 0 \) such that \( B = B(\gamma(t_0), r) \subseteq U \). Then show there exists \( t_1 < t_0 \) and \( t_2 > t_0 \) such that \( \gamma([t_1, t_2]) \subseteq B \) and \( \gamma(t_i) \notin B \). Thus \( \gamma([t_1, t_2]) \) is a path across \( B \) going through the center of \( B \) which divides \( B \) into two open sets, \( B_1 \) and \( B_2 \) along with \( \gamma^* \). Let the boundary of \( B_k \) consist of \( \gamma([t_1, t_2]) \) and a circular arc, \( C_k \).

Now letting \( z \in B_k \), the line integral of \( \frac{f(w)}{w-z} \) over \( \gamma^* \) in two different directions cancels. Therefore, if \( z \in B_k \), you can argue that \( f(z) = \frac{1}{2\pi} \int_C \frac{f(w)}{w-z} dw \). By continuity, this continues to hold for \( z \in \gamma((t_1, t_2)) \). Therefore, \( f \) must be analytic on \( \gamma((t_1, t_2)) \) also. This shows that \( f \) must be analytic on \( \gamma((a, b)) \).

To get the endpoints, simply extend \( \gamma \) to have the same properties but defined on \( [a - \varepsilon, b + \varepsilon] \) and repeat the above argument or else do this at the beginning and note that you get \( [a, b] \subseteq (a - \varepsilon, b + \varepsilon) \).

11. Let \( U \) be an open set contained in the upper half plane and suppose that there are finitely many line segments on the \( x \) axis which are contained in the boundary of \( U \). Now suppose that \( f \) is defined, real, and continuous on these line segments and is defined and analytic on \( U \). Now let \( \tilde{U} \) denote the reflection of \( U \) across the \( x \) axis. Show that it is possible to extend \( f \) to a function, \( g \) defined on all of

\[
W \equiv \tilde{U} \cup U \cup \{\text{the line segments mentioned earlier}\}
\]

such that \( g \) is analytic in \( W \). **Hint:** For \( z \in \tilde{U} \), the reflection of \( U \) across the \( x \) axis, let \( g(z) = \overline{f(\overline{z})} \). Show that \( g \) is analytic on \( \tilde{U} \cup U \) and continuous on the line segments. Then use Problem 11 or Morera’s theorem to argue that \( g \) is analytic on the line segments also. The result of this problem is know as the Schwarz reflection principle.
12. Show that rotations and translations of analytic functions yield analytic functions and use this observation to generalize the Schwarz reflection principle to situations in which the line segments are part of a line which is not the $x$ axis. Thus, give a version which involves reflection about an arbitrary line.
Chapter 50

The Open Mapping Theorem

50.1 A Local Representation

The open mapping theorem, is an even more surprising result than the theorem about the zeros of an analytic function. The following proof of this important theorem uses an interesting local representation of the analytic function.

**Theorem 50.1.1 (Open mapping theorem)** Let \( \Omega \) be a region in \( \mathbb{C} \) and suppose \( f : \Omega \to \mathbb{C} \) is analytic. Then \( f(\Omega) \) is either a point or a region. In the case where \( f(\Omega) \) is a region, it follows that for each \( z_0 \in \Omega \), there exists an open set, \( V \) containing \( z_0 \) and \( m \in \mathbb{N} \) such that for all \( z \in V \),

\[
f(z) = f(z_0) + \phi(z)^m \tag{50.1.1}
\]

where \( \phi : V \to B(0, \delta) \) is one to one, analytic and onto, \( \phi(z_0) = 0 \), \( \phi'(z) \neq 0 \) on \( V \) and \( \phi^{-1} \) analytic on \( B(0, \delta) \). If \( f \) is one to one then \( m = 1 \) for each \( z_0 \) and \( f^{-1} : f(\Omega) \to \Omega \) is analytic.

**Proof:** Suppose \( f(\Omega) \) is not a point. Then if \( z_0 \in \Omega \) it follows there exists \( r > 0 \) such that \( f(z) \neq f(z_0) \) for all \( z \in B(z_0, r) \setminus \{z_0\} \). Otherwise, \( z_0 \) would be a limit point of the set,

\[
\{ z \in \Omega : f(z) = f(z_0) = 0 \}
\]

which would imply from Theorem 49.5.3 that \( f(z) = f(z_0) \) for all \( z \in \Omega \). Therefore, making \( r \) smaller if necessary and using the power series of \( f \),

\[
f(z) = f(z_0) + (z - z_0)^m g(z) \left( z \left( (z - z_0) g(z)^{1/m} \right)^m \right)
\]

for all \( z \in B(z_0, r) \), where \( g(z) \neq 0 \) on \( B(z_0, r) \). As implied in the above formula, one wonders if you can take the \( m^{th} \) root of \( g(z) \).

\( \frac{g}{g} \) is an analytic function on \( B(z_0, r) \) and so by Corollary 49.7.4 it has a primitive on \( B(z_0, r) \), \( h \). Therefore by the product rule and the chain rule, \( (ge^{-h})' = 0 \) and
so there exists a constant, \( C = e^{a+ib} \) such that on \( B(z_0, r) \),

\[
ge e^{-h} = e^{a+ib}.
\]

Therefore,

\[
g(z) = e^{h(z)+a+ib}
\]

and so, modifying \( h \) by adding in the constant, \( a+ib \), \( g(z) = e^{h(z)} \) where \( h'(z) = \frac{g'(z)}{g(z)} \) on \( B(z_0, r) \). Letting

\[
\phi(z) = (z-z_0) e^{\frac{h(z)}{m}}
\]

implies formula \([\text{formula}]\) is valid on \( B(z_0, r) \). Now

\[
\phi'(z_0) = e^{\frac{h'(z_0)}{m}} \ne 0.
\]

Shrinking \( r \) if necessary you can assume \( \phi'(z) \ne 0 \) on \( B(z_0, r) \). Is there an open set, \( V \) contained in \( B(z_0, r) \) such that \( \phi \) maps \( V \) onto \( B(0, \delta) \) for some \( \delta > 0 \)?

Let \( \phi(z) = u(x, y) + iv(x, y) \) where \( z = x + iy \). Consider the mapping

\[
\begin{pmatrix}
x \\
y
\end{pmatrix}
\rightarrow
\begin{pmatrix}
u(x, y) \\
v(x, y)
\end{pmatrix}
\]

where \( u, v \) are \( C^1 \) because \( \phi \) is given to be analytic. The Jacobian of this map at \( (x, y) \in B(z_0, r) \) is

\[
\begin{vmatrix}
u_x(x, y) & u_y(x, y) \\
v_x(x, y) & v_y(x, y)
\end{vmatrix}
= \begin{vmatrix}
u_x(x, y) & -v_x(x, y) \\
v_x(x, y) & u_x(x, y)
\end{vmatrix}
\]

\[
= u_x(x, y)^2 + v_x(x, y)^2 = |\phi'(z)|^2 \ne 0.
\]

This follows from a use of the Cauchy Riemann equations. Also

\[
\begin{pmatrix}
u(x_0, y_0) \\
v(x_0, y_0)
\end{pmatrix}
= \begin{pmatrix} 0 \\
0
\end{pmatrix}
\]

Therefore, by the inverse function theorem there exists an open set, \( V \), containing \( z_0 \) and \( \delta > 0 \) such that \( (u, v)^T \) maps \( V \) one to one onto \( B(0, \delta) \). Thus \( \phi \) is one to one onto \( B(0, \delta) \) as claimed. Applying the same argument to other points, \( z \) of \( V \) and using the fact that \( \phi'(z) \ne 0 \) at these points, it follows \( \phi \) maps open sets to open sets. In other words, \( \phi^{-1} \) is continuous.

It also follows that \( \phi^{-m} \) maps \( V \) onto \( B(0, \delta^m) \). Therefore, the formula \([\text{formula}]\) implies that \( f \) maps the open set, \( V \), containing \( z_0 \) to an open set. This shows \( f(\Omega) \) is an open set because \( z_0 \) was arbitrary. It is connected because \( f \) is continuous and \( \Omega \) is connected. Thus \( f(\Omega) \) is a region. It remains to verify that \( \phi^{-1} \) is analytic on \( B(0, \delta) \). Since \( \phi^{-1} \) is continuous,

\[
\lim_{\phi(z_1) \rightarrow \phi(z)} \frac{\phi^{-1}(\phi(z_1)) - \phi^{-1}(\phi(z))}{\phi(z_1) - \phi(z)} = \lim_{z_1 \rightarrow z} \frac{z_1 - z}{\phi(z_1) - \phi(z)} = \frac{1}{\phi'(z)}.
\]
Therefore, \( \phi^{-1} \) is analytic as claimed.

It only remains to verify the assertion about the case where \( f \) is one to one. If \( m > 1 \), then \( e^{2\pi i} \neq 1 \) and so for \( z_1 \in V \),

\[
e^{2\pi i} \phi(z_1) \neq \phi(z_1) . \tag{50.1.2}\]

But \( e^{2\pi i} \phi(z_1) \in B(0, \delta) \) and so there exists \( z_2 \neq z_1 \) (since \( \phi \) is one to one) such that \( \phi(z_2) = e^{2\pi i} \phi(z_1) \). But then

\[
\phi(z_2)^m = \left(e^{2\pi i} \phi(z_1)\right)^m = \phi(z_1)^m
\]

implying \( f(z_2) = f(z_1) \) contradicting the assumption that \( f \) is one to one. Thus \( m = 1 \) and \( f'(z) = \phi'(z) \neq 0 \) on \( V \). Since \( f \) maps open sets to open sets, it follows that \( f^{-1} \) is continuous and so

\[
(f^{-1})'(f(z)) = \lim_{f(z_1) \to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)}
= \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)} .
\]

This proves the theorem.

One does not have to look very far to find that this sort of thing does not hold for functions mapping \( \mathbb{R} \) to \( \mathbb{R} \). Take for example, the function \( f(x) = x^2 \). Then \( f(\mathbb{R}) \) is neither a point nor a region. In fact \( f(\mathbb{R}) \) fails to be open.

**Corollary 50.1.2** Suppose in the situation of Theorem 50.1.1 \( m > 1 \) for the local representation of \( f \) given in this theorem. Then there exists \( \delta > 0 \) such that if \( w \in B(f(z_0), \delta) = f(V) \) for \( V \) an open set containing \( z_0 \), then \( f^{-1}(w) \) consists of \( m \) distinct points in \( V \). (If \( f \) is \( m \) to one on \( V \))

**Proof:** Let \( w \in B(f(z_0), \delta) \). Then \( w = f(\tilde{z}) \) where \( \tilde{z} \in V \). Thus \( f(\tilde{z}) = f(z_0) + \phi(\tilde{z})^m \). Consider the \( m \) distinct numbers, \( \left\{ e^{2k\pi i} \phi(\tilde{z}) \right\}_{k=1}^m \). Then each of these numbers is in \( B(0, \delta) \) and so since \( \phi \) maps \( V \) one to one onto \( B(0, \delta) \), there are \( m \) distinct numbers in \( V \), \( \{z_k\}_{k=1}^m \) such that \( \phi(z_k) = e^{2k\pi i} \phi(\tilde{z}) \). Then

\[
f(z_k) = f(z_0) + \phi(z_k)^m = f(z_0) + e^{2k\pi i} \phi(\tilde{z})^m
= f(z_0) + e^{2k\pi i} \phi(\tilde{z})^m = f(z_0) + \phi(\tilde{z})^m = f(\tilde{z}) = w
\]

This proves the corollary.

### 50.2 Branches Of The Logarithm

The argument used in to prove the next theorem was used in the proof of the open mapping theorem. It is a very important result and deserves to be stated as a theorem.
CHAPTER 50. THE OPEN MAPPING THEOREM

**Theorem 50.2.1** Let \( \Omega \) be a simply connected region and suppose \( f : \Omega \to \mathbb{C} \) is analytic and nonzero on \( \Omega \). Then there exists an analytic function, \( g \) such that \( e^{g(z)} = f(z) \) for all \( z \in \Omega \).

**Proof:** The function, \( f'/f \) is analytic on \( \Omega \) and so by Corollary 49.7.23 there is a primitive for \( f'/f \), denoted as \( g_1 \). Then 
\[
(e^{-g_1} f)' = -\frac{f'}{f} e^{-g_1} f + e^{-g_1} f' = 0
\]
and so since \( \Omega \) is connected, it follows \( e^{-g_1} f \) equals a constant, \( e^{a+ib} \). Therefore, \( f(z) = e^{g_1(z)+a+ib} \). Define \( g(z) \equiv g_1(z) + a + ib \).

The function, \( g \) in the above theorem is called a branch of the logarithm of \( f \) and is written as \( \text{log} \left( f(z) \right) \).

**Definition 50.2.2** Let \( \rho \) be a ray starting at 0. Thus \( \rho \) is a straight line of infinite length extending in one direction with its initial point at 0.

A special case of the above theorem is the following.

**Theorem 50.2.3** Let \( \rho \) be a ray starting at 0. Then there exists an analytic function, \( L(z) \) defined on \( \mathbb{C} \setminus \rho \) such that 
\[
e^{L(z)} = z.
\]
This function, \( L \) is called a branch of the logarithm. This branch of the logarithm satisfies the usual formula for logarithms, \( L(zw) = L(z) + L(w) \) provided \( zw \notin \rho \).

**Proof:** \( \mathbb{C} \setminus \rho \) is a simply connected region because its complement with respect to \( \hat{\mathbb{C}} \) is connected. Furthermore, the function, \( f(z) = z \) is not equal to zero on \( \mathbb{C} \setminus \rho \). Therefore, by Theorem 50.2.1 there exists an analytic function \( L(z) \) such that \( e^{L(z)} = f(z) = z \). Now consider the problem of finding a description of \( L(z) \). Each \( z \in \mathbb{C} \setminus \rho \) can be written in a unique way in the form
\[
z = |z| e^{i \arg_\rho (z)}
\]
where \( \arg_\rho (z) \) is the angle in \( (\theta, \theta + 2\pi) \) associated with \( z \). (You could of course have considered this to be the angle in \( (\theta - 2\pi, \theta) \) associated with \( z \) or in infinitely many other open intervals of length 2\pi. The description of the log is not unique.) Then letting \( L(z) = a + ib \)
\[
z = |z| e^{i \arg_\rho (z)} = e^{L(z)} = e^{a} e^{ib}
\]
and so you can let \( L(z) = \ln |z| + i \arg_\rho (z) \).

Does \( L(z) \) satisfy the usual properties of the logarithm? That is, for \( z, w \in \mathbb{C} \setminus \rho \), is \( L(zw) = L(z) + L(w) \)? This follows from the usual rules of exponents. You know \( e^{z+w} = e^{z} e^{w} \). (You can verify this directly or you can reduce to the case where \( z, w \) are real. If \( z \) is a fixed real number, then the equation holds for all real \( w \). Therefore,
it must also hold for all complex \( w \) because the real line contains a limit point. Now for this fixed \( w \), the equation holds for all \( z \) real. Therefore, by similar reasoning, it holds for all complex \( z \).

Now suppose \( z, w \in \mathbb{C} \setminus \rho \) and \( zw \notin \rho \). Then

\[
e^{L(zw)} = zw, \quad e^{L(z)+L(w)} = e^{L(z)}e^{L(w)} = zw
\]

and so \( L(zw) = L(z) + L(w) \) as claimed. This proves the theorem.

In the case where the ray is the negative real axis, it is called the principal branch of the logarithm. Thus \( \arg(z) \) is a number between \(-\pi\) and \(\pi\).

**Definition 50.2.4** Let \( \log \) denote the branch of the logarithm which corresponds to the ray for \( \theta = \pi \). That is, the ray is the negative real axis. Sometimes this is called the principal branch of the logarithm.

### 50.3 Maximum Modulus Theorem

Here is another very significant theorem known as the maximum modulus theorem which follows immediately from the open mapping theorem.

**Theorem 50.3.1** (maximum modulus theorem) Let \( \Omega \) be a bounded region and let \( f : \Omega \to \mathbb{C} \) be analytic and \( f : \overline{\Omega} \to \mathbb{C} \) continuous. Then if \( z \in \Omega \),

\[
|f(z)| \leq \max \{ |f(w)| : w \in \partial \Omega \}.
\]

(50.3.3)

If equality is achieved for any \( z \in \Omega \), then \( f \) is a constant.

**Proof:** Suppose \( f \) is not a constant. Then \( f(\Omega) \) is a region and so if \( z \in \Omega \), there exists \( r > 0 \) such that \( B(f(z), r) \subseteq f(\Omega) \). It follows there exists \( z_1 \in \Omega \) with \( |f(z_1)| > |f(z)| \). Hence \( \max \{ |f(w)| : w \in \overline{\Omega} \} \) is not achieved at any interior point of \( \Omega \). Therefore, the point at which the maximum is achieved must lie on the boundary of \( \Omega \) and so

\[
\max \{ |f(w)| : w \in \partial \Omega \} = \max \{ |f(w)| : w \in \overline{\Omega} \} > |f(z)|
\]

for all \( z \in \Omega \) or else \( f \) is a constant. This proves the theorem.

You can remove the assumption that \( \Omega \) is bounded and give a slightly different version.

**Theorem 50.3.2** Let \( f : \Omega \to \mathbb{C} \) be analytic on a region, \( \Omega \) and suppose \( \overline{B(a, r)} \subseteq \Omega \). Then

\[
|f(a)| \leq \max \{ |f(a+re^{i\theta})| : \theta \in [0, 2\pi] \}.
\]

Equality occurs for some \( r > 0 \) and \( a \in \Omega \) if and only if \( f \) is constant in \( \Omega \) hence equality occurs for all such \( a, r \).
**Proof:** The claimed inequality holds by Theorem 50.3.1. Suppose equality in the above is achieved for some $B(a, r) \subseteq \Omega$. Then by Theorem 50.3.1 $f$ is equal to a constant, $w$ on $B(a, r)$. Therefore, the function, $f(\cdot) - w$ has a limit point which has a limit point in $\Omega$ and so by Theorem 50.3.1 $f(z) = w$ for all $z \in \Omega$.

Conversely, if $f$ is constant, then the equality in the above inequality is achieved for all $B(a, r) \subseteq \Omega$.

Next is yet another version of the maximum modulus principle which is in Conway [21]. Let $\Omega$ be an open set.

**Definition 50.3.3** Define $\partial_{\infty} \Omega$ to equal $\partial \Omega$ in the case where $\Omega$ is bounded and $\partial \Omega \cup \{\infty\}$ in the case where $\Omega$ is not bounded.

**Definition 50.3.4** Let $f$ be a complex valued function defined on a set $S \subseteq \mathbb{C}$ and let $a$ be a limit point of $S$.

$$\limsup_{z \to a} |f(z)| \equiv \lim_{r \to 0} \left\{ \sup \{f(w): w \in B'(a, r) \cap S\} \right\}. $$

The limit exists because $\{\sup \{f(w): w \in B'(a, r) \cap S\}\}$ is decreasing in $r$. In case $a = \infty$,

$$\limsup_{z \to \infty} |f(z)| \equiv \lim_{r \to \infty} \left\{ \sup \{f(w): |w| > r, w \in S\} \right\}. $$

Note that if $\limsup_{z \to a} |f(z)| \leq M$ and $\delta > 0$, then there exists $r > 0$ such that if $z \in B'(a, r) \cap S$, then $|f(z)| < M + \delta$. If $a = \infty$, there exists $r > 0$ such that if $|z| > r$ and $z \in S$, then $|f(z)| < M + \delta$.

**Theorem 50.3.5** Let $\Omega$ be an open set in $\mathbb{C}$ and let $f: \Omega \to \mathbb{C}$ be analytic. Suppose also that for every $a \in \partial_{\infty} \Omega$,

$$\limsup_{z \to a} |f(z)| \leq M < \infty. $$

Then in fact $|f(z)| \leq M$ for all $z \in \Omega$.

**Proof:** Let $\delta > 0$ and let $H = \{z \in \Omega: |f(z)| > M + \delta\}$. Suppose $H \neq \emptyset$. Then $H$ is an open subset of $\Omega$. I claim that $H$ is actually bounded. If $\Omega$ is bounded, there is nothing to show so assume $\Omega$ is unbounded. Then the condition involving the $\limsup$ implies there exists $r > 0$ such that if $|z| > r$ and $z \in \Omega$, then $|f(z)| \leq M + \delta/2$. It follows $H$ is contained in $B(0, r)$ and so it is bounded. Now consider the components of $\Omega$. One of these components contains points from $H$. Let this component be denoted as $V$ and let $H_V \equiv H \cap V$. Thus $H_V$ is a bounded open subset of $V$. Let $U$ be a component of $H_V$. First suppose $\overline{U} \subseteq V$. In this case, it follows that on $\partial U$, $|f(z)| = M + \delta$ and so by Theorem 50.3.1 $|f(z)| \leq M + \delta$ for all $z \in U$ contradicting the definition of $H$. Next suppose $\partial U$ contains a point of $\partial V, a$. Then in this case, $a$ violates the condition on $\limsup$. Either way you get a contradiction. Hence $H = \emptyset$ as claimed. Since $\delta > 0$ is arbitrary, this shows $|f(z)| \leq M$. 

---

**CHAPTER 50. THE OPEN MAPPING THEOREM**
50.4 Extensions Of Maximum Modulus Theorem

50.4.1 Phragmèn Lindelöf Theorem

This theorem is an extension of Theorem 50.3.5. It uses a growth condition near the extended boundary to conclude that \( f \) is bounded. I will present the version found in Conway [31]. It seems to be more of a method than an actual theorem. There are several versions of it.

**Theorem 50.4.1** Let \( \Omega \) be a simply connected region in \( \mathbb{C} \) and suppose \( f \) is analytic on \( \Omega \). Also suppose there exists a function, \( \phi \) which is nonzero and uniformly bounded on \( \Omega \). Let \( M \) be a positive number. Now suppose \( \partial_{\infty} \Omega = A \cup B \) such that for every \( a \in A \), \( \limsup_{z \to a} |f(z)| \leq M \) and for every \( b \in B \), and \( \eta > 0 \), \( \limsup_{z \to b} |f(z)||\phi(z)|^{\eta} \leq M \). Then \( |f(z)| \leq M \) for all \( z \in \Omega \).

**Proof:** By Theorem 50.2.1 there exists \( \log(\phi(z)) \) analytic on \( \Omega \). Now define \( g(z) \equiv \exp(\eta \log(\phi(z))) \) so that \( g(z) = \phi(z)^{\eta} \). Now also

\[
|g(z)| = |\exp(\eta \log(\phi(z)))| = |\exp(\eta \ln|\phi(z)|)| = |\phi(z)|^{\eta}.
\]

Let \( m \geq |\phi(z)| \) for all \( z \in \Omega \). Define \( F(z) \equiv f(z)g(z)m^{-\eta} \). Thus \( F \) is analytic and for \( b \in B \),

\[
\limsup_{z \to b} |F(z)| = \limsup_{z \to b} |f(z)||\phi(z)|^{\eta}m^{-\eta} \leq Mm^{-\eta}
\]

while for \( a \in A \),

\[
\limsup_{z \to a} |F(z)| \leq M.
\]

Therefore, for \( \alpha \in \partial_{\infty} \Omega \), \( \limsup_{z \to \alpha} |F(z)| \leq \max(M, M\eta^{-\eta}) \) and so by Theorem 50.2.1 \( |f(z)| \leq \left( \frac{m^{\eta}}{|\phi(z)|^{\eta}} \right) \max(M, M\eta^{-\eta}) \). Now let \( \eta \to 0 \) to obtain \( |f(z)| \leq M \).

In applications, it is often the case that \( B = \{\infty\} \).

Now here is an interesting case of this theorem. It involves a particular form for \( \Omega \), in this case \( \Omega = \{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a} \} \) where \( a \geq \frac{1}{2} \).

Then \( \partial \Omega \) equals the two slanted lines. Also on \( \Omega \) you can define a logarithm, \( \log(z) = \ln|z| + i\arg(z) \) where \( \arg(z) \) is the angle associated with \( z \) between \(-\pi\)
and $\pi$. Therefore, if $c$ is a real number you can define $z^c$ for such $z$ in the usual way:

\[ z^c \equiv \exp(c \log(z)) = \exp(c \ln|z| + i \arg(z)) \]

\[ = |z|^c \exp(i \arg(z)) = |z|^c (\cos(c \arg(z)) + i \sin(c \arg(z))). \]

If $|c| < a$, then $|\arg(z)| < \frac{\pi}{2}$ and so $\cos(c \arg(z)) > 0$. Therefore, for such $c$,

\[ |\exp(-z^c)| = |\exp(-|z|^c (\cos(c \arg(z)) + i \sin(c \arg(z))))| \]

\[ = |\exp(-|z|^c (\cos(c \arg(z))))| \]

which is bounded since $\cos(c \arg(z)) > 0$.

**Corollary 50.4.2** Let $\Omega = \{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\}$ where $a \geq \frac{1}{2}$ and suppose $f$ is analytic on $\Omega$ and satisfies $\limsup_{z \to a} |f(z)| \leq M$ on $\partial \Omega$ and suppose there are positive constants, $P, b$ where $b < a$ and

\[ |f(z)| \leq P \exp\left(|z|^b\right) \]

for all $|z|$ large enough. Then $|f(z)| \leq M$ for all $z \in \Omega$.

**Proof:** Let $b < c < a$ and let $\phi(z) \equiv \exp(-z^c)$. Then as discussed above, $\phi(z) \not\equiv 0$ on $\Omega$ and $|\phi(z)|$ is bounded on $\Omega$. Now

\[ |\phi(z)|^0 = |\exp(-|z|^c \eta (\cos(c \arg(z))))| \]

\[ \liminf_{z \to \infty} |f(z)||\phi(z)|^0 \leq \liminf_{z \to \infty} \frac{P \exp\left(|z|^b\right)}{|\exp(|z|^c \eta (\cos(c \arg(z))))|} = 0 \leq M \]

and so by Theorem 50.3.4, $|f(z)| \leq M$.

The following is another interesting case. This case is presented in Rudin [1].

**Corollary 50.4.3** Let $\Omega$ be the open set consisting of $\{z \in \mathbb{C} : a < \text{Re} z < b\}$ and suppose $f$ is analytic on $\Omega$, continuous on $\overline{\Omega}$, and bounded on $\Omega$. Suppose also that $f(z) \geq 1$ on the two lines $\text{Re} z = a$ and $\text{Re} z = b$. Then $|f(z)| \leq 1$ for all $z \in \Omega$.

**Proof:** This time let $\phi(z) = \frac{1}{1+z-a}$. Thus $|\phi(z)| \leq 1$ because $\text{Re}(z-a) > 0$ and $\phi(z) \not\equiv 0$ for all $z \in \Omega$. Also, $\limsup_{z \to \infty} |\phi(z)|^\eta = 0$ for every $\eta > 0$. Therefore, if $a$ is a point of the sides of $\Omega$, $\limsup_{z \to a} |f(z)| \leq 1$ while $\limsup_{z \to \infty} |f(z)| |\phi(z)|^\eta = 0 \leq 1$ and so by Theorem 50.3.4, $|f(z)| \leq 1$ on $\Omega$.

This corollary yields an interesting conclusion.

**Corollary 50.4.4** Let $\Omega$ be the open set consisting of $\{z \in \mathbb{C} : a < \text{Re} z < b\}$ and suppose $f$ is analytic on $\Omega$, continuous on $\overline{\Omega}$, and bounded on $\Omega$. Define

\[ M(x) \equiv \sup\{|f(z) : \text{Re} z = x\} \]

Then for $x \in (a,b)$,

\[ M(x) \leq M(a)^{\frac{\frac{b-x}{b-a}}{\frac{b-a}{a}}} \cdot M(b)^{\frac{\frac{x-a}{b-a}}{\frac{b-a}{a}}} \].
50.4. EXTENSIONS OF MAXIMUM MODULUS THEOREM

Proof: Let \( \varepsilon > 0 \) and define

\[
g(z) = (M(a) + \varepsilon)^{\frac{b-a}{b-a}} (M(b) + \varepsilon)^{\frac{b-a}{b-a}}
\]

where for \( M > 0 \) and \( z \in \mathbb{C} \), \( M^z = \exp(z \ln(M)) \). Thus \( g \neq 0 \) and so \( f/g \) is analytic on \( \Omega \) and continuous on \( \overline{\Omega} \). Also on the left side,

\[
\left| \frac{f(a + iy)}{g(a + iy)} \right| = \left| \frac{f(a + iy)}{(M(a) + \varepsilon)^{\frac{b-a}{b-a}}} \right| = \frac{f(a + iy)}{(M(a) + \varepsilon)^{\frac{b-a}{b-a}}} \leq 1
\]

while on the right side a similar computation shows \( \frac{E}{g} \mid \leq 1 \) also. Therefore, by Corollary \( \text{R.C.W.} \) \( \frac{f}{g} \leq 1 \) on \( \Omega \). Therefore, letting \( x + iy = z \),

\[
|f(z)| \leq |(M(a) + \varepsilon)^{\frac{b-a}{b-a}} (M(b) + \varepsilon)^{\frac{b-a}{b-a}}| = |(M(a) + \varepsilon)^{\frac{b-a}{b-a}} (M(b) + \varepsilon)^{\frac{b-a}{b-a}}|
\]

and so

\[
M(x) \leq (M(a) + \varepsilon)^{\frac{b-a}{b-a}} (M(b) + \varepsilon)^{\frac{b-a}{b-a}}.
\]

Since \( \varepsilon > 0 \) is arbitrary, it yields the conclusion of the corollary.

Another way of saying this is that \( x \to \ln(M(x)) \) is a convex function.

This corollary has an interesting application known as the Hadamard three circles theorem.

50.4.2 Hadamard Three Circles Theorem

Let \( 0 < R_1 < R_2 \) and suppose \( f \) is analytic on \( \{z \in \mathbb{C} : R_1 < |z| < R_2\} \). Then letting \( R_1 < a < b < R_2 \), note that \( g(z) = \exp(z) \) maps the strip \( \{z \in \mathbb{C} : \ln a < \Re z < b\} \) onto \( \{z \in \mathbb{C} : a < |z| < b\} \) and that in fact, \( g \) maps the line \( \ln r + iy \) onto the circle \( re^{i\theta} \). Now let \( M(x) \) be defined as above and \( m \) be defined by

\[
m(r) = \max_{\theta} |f(re^{i\theta})|.
\]

Then for \( a < r < b \), Corollary \( \text{R.C.W.} \) implies

\[
m(r) = \sup_{y} \left| f(e^{\ln r + iy}) \right| = M(\ln r) \leq M(\ln a)^{\frac{\ln b - \ln r}{\ln b - \ln a}} M(\ln b)^{\frac{\ln r - \ln a}{\ln b - \ln a}}
\]

and so

\[
m(r)^{\ln(b/a)} \leq m(a)^{\ln(b/r)} m(b)^{\ln(r/a)}.
\]

Taking logarithms, this yields

\[
\ln \left( \frac{b}{a} \right) \ln(m(r)) \leq \ln \left( \frac{b}{r} \right) \ln(m(a)) + \ln \left( \frac{r}{a} \right) \ln(m(b))
\]

which says the same as \( r \to \ln(m(r)) \) is a convex function of \( \ln r \).

The next example, also in Rudin \( \text{R.C.W.} \) is very dramatic. An unbelievably weak assumption is made on the growth of the function and still you get a uniform bound in the conclusion.
Corollary 50.4.5 Let $\Omega = \{ z \in \mathbb{C} : |\text{Im}(z)| < \frac{\pi}{2} \}$. Suppose $f$ is analytic on $\Omega$, continuous on $\overline{\Omega}$, and there exist constants, $\alpha < 1$ and $A < \infty$ such that

$$|f(z)| \leq \exp(A \exp(|x|))$$

for $z = x + iy$

and

$$|f\left(x \pm i\frac{\pi}{2}\right)| \leq 1$$

for all $x \in \mathbb{R}$. Then $|f(z)| \leq 1$ on $\Omega$.

Proof: This time let $\phi(z) = [\exp(A \exp(\beta z)) \exp(A \exp(-\beta z))]^{-1}$ where $\alpha < \beta < 1$. Then $\phi(z) \neq 0$ on $\Omega$ and for $\eta > 0$

$$|\phi(z)|^\eta = \frac{1}{|\exp(\eta A \exp(\beta z)) \exp(\eta A \exp(-\beta z))|}$$

Now

$$\exp(\eta A \exp(\beta z)) \exp(\eta A \exp(-\beta z))$$

$$= \exp(\eta A (\exp(\beta z) + \exp(-\beta z)))$$

$$= \exp[\eta A (\cos(\beta y) (e^{\beta x} + e^{-\beta x}) + i \sin(\beta y) (e^{\beta x} - e^{-\beta x}))]$$

and so

$$|\phi(z)|^\eta = \frac{1}{\exp[\eta A (\cos(\beta y) (e^{\beta x} + e^{-\beta x}))]}$$

Now $\cos \beta y > 0$ because $\beta < 1$ and $|y| < \frac{\pi}{2}$. Therefore,

$$\limsup_{z \to \infty} |f(z)||\phi(z)|^\eta \leq 0 \leq 1$$

and so by Theorem 50.4.1 $|f(z)| \leq 1$.

50.4.3 Schwarz’s Lemma

This interesting lemma comes from the maximum modulus theorem. It will be used later as part of the proof of the Riemann mapping theorem.

Lemma 50.4.6 Suppose $F : B(0, 1) \to B(0, 1)$, $F$ is analytic, and $F(0) = 0$. Then for all $z \in B(0, 1)$,

$$|F(z)| \leq |z|,$$  \hspace{1cm} (50.4.4)

and

$$|F'(0)| \leq 1.$$  \hspace{1cm} (50.4.5)

If equality holds in (50.4.5) then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$F(z) = \lambda z.$$  \hspace{1cm} (50.4.6)
Proof: First note that by assumption, \( F(z)/z \) has a removable singularity at 0 if its value at 0 is defined to be \( F'(0) \). By the maximum modulus theorem, if \(|z| < r < 1\),
\[
\left| \frac{F(z)}{z} \right| \leq \max_{t \in [0, 2\pi]} \left| \frac{F(re^{it})}{r} \right| \leq \frac{1}{r}.
\]
Then letting \( r \to 1 \),
\[
\left| \frac{F(z)}{z} \right| \leq 1
\]
this shows \(50.4.4\) and it also verifies \(50.4.5\) on taking the limit as \( z \to 0 \). If equality holds in \(50.4.5\), then \( |F(z)/z| \) achieves a maximum at an interior point so \( F(z)/z \) equals a constant, \( \lambda \) by the maximum modulus theorem. Since \( F(z) = \lambda z \), it follows \( F'(0) = \lambda \) and so \(|\lambda| = 1\).

Rudin \[102\] gives a memorable description of what this lemma says. It says that if an analytic function maps the unit ball to itself, keeping 0 fixed, then it must do one of two things, either be a rotation or move all points closer to 0. (This second part follows in case \(|F'(0)| < 1\) because in this case, you must have \(|F(z)| \neq |z|\) and so by \(50.4.5\), \(|F(z)| < |z|\).)

### 50.4.4 One To One Analytic Maps On The Unit Ball

The transformation in the next lemma is of fundamental importance.

**Lemma 50.4.7** Let \( \alpha \in B(0,1) \) and define
\[
\phi_{\alpha}(z) \equiv \frac{z - \alpha}{1 - \overline{\alpha}z}.
\]
Then \( \phi_{\alpha} : B(0,1) \to B(0,1) \), \( \phi_{\alpha} : \partial B(0,1) \to \partial B(0,1) \), and is one to one and onto. Also \( \phi_{-\alpha} = \phi_{\alpha}^{-1} \). Also
\[
\phi'_{\alpha}(0) = 1 - |\alpha|^2, \quad \phi'(\alpha) = \frac{1}{1 - |\alpha|^2}.
\]

**Proof:** First of all, for \(|z| < 1/|\alpha|\),
\[
\phi_{\alpha} \circ \phi_{-\alpha}(z) \equiv \frac{\left( \frac{z + \alpha}{1 + \overline{\alpha}z} \right) - \alpha}{1 - \alpha \left( \frac{z + \alpha}{1 + \overline{\alpha}z} \right)} = z
\]
after a few computations. If I show that \( \phi_{\alpha} \) maps \( B(0,1) \) to \( B(0,1) \) for all \(|\alpha| < 1\), this will have shown that \( \phi_{\alpha} \) is one to one and onto \( B(0,1) \).

Consider \(|\phi_{\alpha}(e^{i\theta})|\). This yields
\[
\left| \frac{e^{i\theta} - \alpha}{1 - \overline{\alpha}e^{i\theta}} \right| = \left| \frac{1 - \alpha e^{-i\theta}}{1 - \overline{\alpha}e^{i\theta}} \right| = 1
\]
where the first equality is obtained by multiplying by \( |e^{-i\theta}| = 1 \). Therefore, \( \phi_\alpha \) maps \( \partial B(0,1) \) one to one and onto \( \partial B(0,1) \). Now notice that \( \phi_\alpha \) is analytic on \( B(0,1) \) because the only singularity, a pole, is at \( z = 1/\alpha \). By the maximum modulus theorem, it follows
\[
|\phi_\alpha(z)| < 1
\]
whenever \( |z| < 1 \). The same is true of \( \phi_{-\alpha} \).

It only remains to verify the assertions about the derivatives. Long division gives
\[
\phi_\alpha(z) = (-\alpha)^{-1} + \left( -\alpha + (\overline{\alpha})^{-1} \right)
\]
and so
\[
\phi'_\alpha(z) = (-1) (1 - \overline{\alpha}z)^{-2} \left( -\alpha + (\overline{\alpha})^{-1} \right) \overline{\alpha}
\]
\[
= \overline{\alpha} (1 - \overline{\alpha}z)^{-2} \left( -\alpha + (\overline{\alpha})^{-1} \right)
\]
\[
= (1 - \overline{\alpha}z)^{-2} \left( -|\alpha|^2 + 1 \right)
\]
Hence the two formulas follow. This proves the lemma.

One reason these mappings are so important is the following theorem.

**Theorem 50.4.8** Suppose \( f \) is an analytic function defined on \( B(0,1) \) and \( f \) maps \( B(0,1) \) one to one and onto \( B(0,1) \). Then there exists \( \theta \) such that
\[
f(z) = e^{i\theta} \phi_\alpha(z)
\]
for some \( \alpha \in B(0,1) \).

**Proof:** Let \( f(\alpha) = 0 \). Then \( h(z) \equiv f \circ \phi_{-\alpha}(z) \) maps \( B(0,1) \) one to one and onto \( B(0,1) \) and has the property that \( h(0) = 0 \). Therefore, by the Schwarz lemma,
\[
|h(z)| \leq |z|
\]
buts it is also the case that \( h^{-1}(0) = 0 \) and \( h^{-1} \) maps \( B(0,1) \) to \( B(0,1) \). Therefore, the same inequality holds for \( h^{-1} \). Therefore,
\[
|h(z)| = |h^{-1}(h(z))| \leq |h(z)|
\]
and so \( |h(z)| = |z| \). By the Schwarz lemma again, \( h(z) \equiv f(\phi_{-\alpha}(z)) = e^{i\theta}z \). Letting \( z = \phi_\alpha \), you get \( f(z) = e^{i\theta} \phi_\alpha(z) \).

### 50.5 Exercises

1. Consider the function, \( g(z) = \frac{\overline{z} + 1}{z + 1} \). Show this is analytic on the upper half plane, \( P+ \) and maps the upper half plane one to one and onto \( B(0,1) \). **Hint:** First show \( g \) maps the real axis to \( \phi B(0,1) \). This is really easy because you end up looking at a complex number divided by its conjugate. Thus \( |g(z)| = 1 \) for \( z \) on \( \partial (P+) \). Now show that \( \limsup_{z \to \infty} |g(z)| = 1 \). Then apply a version of the maximum modulus theorem. You might note that \( g(z) = 1 + \frac{-2i}{z^2} \). This will show \( |g(z)| \leq 1 \). Next pick \( w \in B(0,1) \) and solve \( g(z) = w \). You just have to show there exists a unique solution and its imaginary part is positive.
2. Does there exist an entire function $f$ which maps $\mathbb{C}$ onto the upper half plane?

3. Letting $g$ be the function of Problem 4 show that $(g^{-1})'(0) = 2$. Also note that $g^{-1}(0) = i$. Now suppose $f$ is an analytic function defined on the upper half plane which has the property that $|f(z)| \leq 1$ and $f(i) = \beta$ where $|\beta| < 1$. Find an upper bound to $|f'(i)|$. Also find all functions, $f$ which satisfy the condition, $f(i) = \beta, |f(z)| \leq 1$, and achieve this maximum value. **Hint:** You could consider the function, $h(z) \equiv \phi_\beta \circ f \circ g^{-1}(z)$ and check the conditions for the Schwarz lemma for this function, $h$.

4. This and the next two problems follow a presentation of an interesting topic in Rudin [102]. Let $\phi_\alpha$ be given in Lemma 50.4.7. Suppose $f$ is an analytic function defined on $B(0, 1)$ which satisfies $|f(z)| \leq 1$. Suppose also there are $\alpha, \beta \in B(0, 1)$ and it is required $f(\alpha) = \beta$. If $f$ is such a function, show that $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** To show this consider $g = \phi_\beta \circ f \circ \phi_{-\alpha}$. Show $g(0) = 0$ and $|g(z)| \leq 1$ on $B(0, 1)$. Now use Lemma 50.4.9.

5. In Problem 4 show there exists a function, $f$ analytic on $B(0, 1)$ such that $f(\alpha) = \beta$, $|f(z)| \leq 0$, and $|f'(\alpha)| = \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** You do this by choosing $g$ in the above problem such that equality holds in Lemma 50.4.9. Thus you need $g(z) = \lambda z$ where $|\lambda| = 1$ and solve $g = \phi_\beta \circ f \circ \phi_{-\alpha}$ for $f$.

6. Suppose that $f : B(0, 1) \rightarrow B(0, 1)$ and that $f$ is analytic, one to one, and onto with $f(\alpha) = 0$. Show there exists $\lambda, |\lambda| = 1$ such that $f(z) = \lambda \phi_\alpha(z)$. This gives a different way to look at Theorem 50.4.2. **Hint:** Let $g = f^{-1}$. Then $g'(0)f'(\alpha) = 1$. However, $f(\alpha) = 0$ and $g(0) = \alpha$. From Problem 4 with $\beta = 0$, you can conclude an inequality for $|f'(\alpha)|$ and another one for $|g'(0)|$. Then use the fact that the product of these two equals 1 which comes from the chain rule to conclude that equality must take place. Now use Problem 2 to obtain the form of $f$.

7. In Corollary 50.4.5 show that it is essential that $\alpha < 1$. That is, show there exists an example where the conclusion is not satisfied with a slightly weaker growth condition. **Hint:** Consider $\exp(\exp(z))$.

8. Suppose $\{f_n\}$ is a sequence of functions which are analytic on $\Omega$, a bounded region such that each $f_n$ is also continuous on $\bar{\Omega}$. Suppose that $\{f_n\}$ converges uniformly on $\partial \Omega$. Show that then $\{f_n\}$ converges uniformly on $\Omega$ and that the function to which the sequence converges is analytic on $\Omega$ and continuous on $\bar{\Omega}$.

9. Suppose $\Omega$ is a bounded region and there exists a point $z_0 \in \Omega$ such that $|f(z_0)| = \min \{|f(z)| : z \in \Omega\}$. Can you conclude $f$ must equal a constant?

10. Suppose $f$ is continuous on $B(a, r)$ and analytic on $B(a, r)$ and that $f$ is not constant. Suppose also $|f(z)| = C \neq 0$ for all $|z - a| = r$. Show that there exists $\alpha \in B(a, r)$ such that $f(\alpha) = 0$. **Hint:** If not, consider $f/C$ and $C/f$. Both would be analytic on $B(a, r)$ and are equal to 1 on the boundary.
11. Suppose $f$ is analytic on $B(0,1)$ but for every $a \in \partial B(0,1)$, $\lim_{z \to a} |f(z)| = \infty$. Show there exists a sequence, $\{z_n\} \subseteq B(0,1)$ such that $\lim_{n \to \infty} |z_n| = 1$ and $f(z_n) = 0$.

## 50.6 Counting Zeros

The above proof of the open mapping theorem relies on the very important inverse function theorem from real analysis. There are other approaches to this important theorem which do not rely on the big theorems from real analysis and are more oriented toward the use of the Cauchy integral formula and specialized techniques from complex analysis. One of these approaches is given next which involves the notion of “counting zeros”. The next theorem is the one about counting zeros. It will also be used later in the proof of the Riemann mapping theorem.

**Theorem 50.6.1** Let $\Omega$ be an open set in $\mathbb{C}$ and let $\gamma : [a,b] \to \Omega$ be closed, continuous, bounded variation, and $n(\gamma, z) = 0$ for all $z \notin \Omega$. Suppose also that $f$ is analytic on $\Omega$ having zeros $a_1, \cdots, a_m$ where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on $\gamma^*$. Then

$$
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{m} n(\gamma, a_k).
$$

**Proof:** Let $f(z) = \prod_{j=1}^{m} (z - a_j) g(z)$ where $g(z) \neq 0$ on $\Omega$. Hence

$$
\frac{f'(z)}{f(z)} = \sum_{j=1}^{m} \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}
$$

and so

$$
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{m} n(\gamma, a_j) + \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} \, dz.
$$

But the function, $z \to \frac{g'(z)}{g(z)}$ is analytic and so by Corollary 49.7.20, the last integral in the above expression equals 0. Therefore, this proves the theorem.

The following picture is descriptive of the situation described in the next theorem.

**Theorem 50.6.2** Let $\Omega$ be a region, let $\gamma : [a,b] \to \Omega$ be closed continuous, and bounded variation such that $n(\gamma, z) = 0$ for all $z \notin \Omega$. Also suppose $f : \Omega \to \mathbb{C}$ is
analytic and that \( \alpha \notin f(\gamma^*) \). Then \( f \circ \gamma : [a,b] \to \mathbb{C} \) is continuous, closed, and bounded variation. Also suppose \( \{a_1, \ldots, a_m\} = f^{-1}(\alpha) \) where these points are counted according to their multiplicities as zeros of the function \( f - \alpha \) Then

\[
\n(f \circ \gamma, \alpha) = \sum_{k=1}^{m} n(\gamma, a_k).
\]

**Proof:** It is clear that \( f \circ \gamma \) is continuous. It only remains to verify that it is of bounded variation. Suppose first that \( \gamma^* \subseteq B \subseteq \overline{B} \subseteq \Omega \) where \( B \) is a ball. Then

\[
|f(\gamma(t)) - f(\gamma(s))| = \left| \int_{0}^{1} f'(\gamma(s) + \lambda(\gamma(t) - \gamma(s))) (\gamma(t) - \gamma(s)) d\lambda \right| \leq C |\gamma(t) - \gamma(s)|
\]

where \( C \geq \max \{|f'(z)| : z \in B\} \). Hence, in this case,

\[
V(f \circ \gamma, [a,b]) \leq CV(\gamma, [a,b]).
\]

Now let \( \varepsilon \) denote the distance between \( \gamma^* \) and \( \mathbb{C} \setminus \Omega \). Since \( \gamma^* \) is compact, \( \varepsilon > 0 \). By uniform continuity there exists \( \delta = \frac{b-a}{p} \) for \( p \) a positive integer such that if \( |s-t| < \delta \), then \( |\gamma(s) - \gamma(t)| < \frac{\varepsilon}{p} \). Then

\[
\gamma([t, t+\delta]) \subseteq B\left(\gamma(t), \frac{\varepsilon}{2}\right) \subseteq \Omega.
\]

Let \( C \geq \max \{|f'(z)| : z \in \bigcup_{j=1}^{p} B\left(\gamma(t_j), \frac{\varepsilon}{2}\right)\} \) where \( t_j \equiv \frac{j}{p}(b-a) + a \). Then from what was just shown,

\[
V(f \circ \gamma, [a,b]) \leq \sum_{j=0}^{p-1} V(f \circ \gamma, [t_j, t_{j+1}]) \leq C \sum_{j=0}^{p-1} V(\gamma, [t_j, t_{j+1}]) < \infty
\]

showing that \( f \circ \gamma \) is bounded variation as claimed. Now from Theorem 49.7.15 there exists \( \eta \in C^1([a,b]) \) such that

\[
\eta(a) = \gamma(a) = \gamma(b) = \eta(b), \eta([a,b]) \subseteq \Omega,
\]

and

\[
n(\eta, a_k) = n(\gamma, a_k), n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha) \quad (50.6.7)
\]

for \( k = 1, \ldots, m \). Then

\[
n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha)
\]
= \frac{1}{2\pi i} \oint_{\gamma} \frac{dw}{w - \alpha}
= \frac{1}{2\pi i} \int_{a}^{b} \frac{f'(\eta(t))}{f(\eta(t)) - \alpha} \eta'(t) \, dt
= \frac{1}{2\pi i} \int_{\eta} \frac{f'(z)}{f(z) - \alpha} \, dz
= \sum_{k=1}^{m} n(\gamma, a_k)

By Theorem 50.6.1, this equals \( \sum_{k=1}^{m} n(\gamma, a_k) \) which proves the theorem.

The next theorem is incredible and is very interesting for its own sake. The following picture is descriptive of the situation of this theorem.

\begin{center}
\begin{tikzpicture}
  \node (a1) at (0,0) {a_1};
  \node (a2) at (-1,-1) {a_2};
  \node (a3) at (1,-1) {a_3};
  \node (a4) at (0,-2) {a_4};
  \node (a) at (0,0) {a};
  \node (z) at (2,1) {z};
  \node (alpha) at (2,0) {\alpha};
  \node (Balpha) at (2,-1) {B(\alpha, \delta)};
  \node (Ba) at (0,-2) {B(a, \varepsilon)};
  \draw[->] (a1) edge[bend right] (z);
\end{tikzpicture}
\end{center}

**Theorem 50.6.3** Let \( f : B(a, R) \to \mathbb{C} \) be analytic and let
\[
f(z) - \alpha = (z - a)^m g(z), \quad \infty > m \geq 1
\]
where \( g(z) \neq 0 \) in \( B(a, R) \). \( f(z) - \alpha \) has a zero of order \( m \) at \( z = a \). Then there exist \( \varepsilon, \delta > 0 \) with the property that for each \( z \) satisfying \( 0 < |z - \alpha| < \delta \), there exist points,
\[
\{a_1, \ldots, a_m\} \subseteq B(a, \varepsilon),
\]
such that
\[
f^{-1}(z) \cap B(a, \varepsilon) = \{a_1, \ldots, a_m\}
\]
and each \( a_k \) is a zero of order 1 for the function \( f(\cdot) - z \).

**Proof:** By Theorem 49.5.3, \( f \) is not constant on \( B(a, R) \) because it has a zero of order \( m \). Therefore, using this theorem again, there exists \( \varepsilon > 0 \) such that \( B(a, 2\varepsilon) \subseteq B(a, R) \) and there are no solutions to the equation \( f(z) - \alpha = 0 \) for \( z \in B(a, 2\varepsilon) \) except \( a \). Also assume \( \varepsilon \) is small enough that for \( 0 < |z - a| \leq 2\varepsilon \), \( f'(z) \neq 0 \). This can be done since otherwise, \( a \) would be a limit point of a sequence of points, \( z_n \), having \( f'(z_n) = 0 \) which would imply, by Theorem 49.5.3, that \( f' = 0 \) on \( B(a, R) \), contradicting the assumption that \( f - \alpha \) has a zero of order \( m \) and is therefore not constant. Thus the situation is described by the following picture.
Now pick $\gamma(t) = a + \varepsilon e^{it}$, $t \in [0, 2\pi]$. Then $\alpha \notin f(\gamma^*)$ so there exists $\delta > 0$ with $B(\alpha, \delta) \cap f(\gamma^*) = \emptyset$. (50.6.8)

Therefore, $B(\alpha, \delta)$ is contained on one component of $\mathbb{C} \setminus f(\gamma([0, 2\pi]))$. Therefore, $n(f \circ \gamma, \alpha) = n(f \circ \gamma, z)$ for all $z \in B(\alpha, \delta)$. Now consider $f$ restricted to $B(\alpha, 2\varepsilon)$. For $z \in B(\alpha, \delta)$, $f^{-1}(z)$ must consist of a finite set of points because $f'(w) \neq 0$ for all $w$ in $B(a, 2\varepsilon) \setminus \{a\}$ implying that the zeros of $f(\cdot) - z$ in $B(a, 2\varepsilon)$ have no limit point. Since $B(a, 2\varepsilon)$ is compact, this means there are only finitely many. By Theorem 50.6.2

$$n(f \circ \gamma, z) = \sum_{k=1}^{p} n(\gamma, a_k)$$

(50.6.9)

where $\{a_1, \ldots, a_p\} = f^{-1}(z)$. Each point, $a_k$ of $f^{-1}(z)$ is either inside the circle traced out by $\gamma$, yielding $n(\gamma, a_k) = 1$, or it is outside this circle yielding $n(\gamma, a_k) = 0$ because of 50.6.8. It follows the sum in 50.6.9 reduces to the number of points of $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$. Thus, letting those points in $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$ be denoted by $\{a_1, \ldots, a_r\}$

$$n(f \circ \gamma, z) = r.$$

Also, by Theorem 50.6.4

$m = n(f \circ \gamma, \alpha)$ because $a$ is a zero of $f - \alpha$ of order $m$. Therefore, for $z \in B(\alpha, \delta)$

$$m = n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r.$$

It is required to show $r = m$, the order of the zero of $f - \alpha$. Therefore, $r = m$. Each of these $a_k$ is a zero of order 1 of the function $f(\cdot) - z$ because $f'(a_k) \neq 0$. This proves the theorem.

This is a very fascinating result partly because it implies that for values of $f$ near a value, $\alpha$, at which $f(\cdot) - \alpha$ has a zero of order $m$ for $m > 1$, the inverse image of these values includes at least $m$ points, not just one. Thus the topological properties of the inverse image changes radically. This theorem also shows that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$.

**Theorem 50.6.4 (open mapping theorem)** Let $\Omega$ be a region and $f : \Omega \to \mathbb{C}$ be analytic. Then $f(\Omega)$ is either a point or a region. If $f$ is one to one, then $f^{-1} : f(\Omega) \to \Omega$ is analytic.
**Proof:** If \( f \) is not constant, then for every \( \alpha \in f(\Omega) \), it follows from Theorem 49.5.3 that \( f(\cdot) - \alpha \) has a zero of order \( m < \infty \) and so from Theorem 50.6.3 for each \( a \in \Omega \) there exist \( \varepsilon, \delta > 0 \) such that \( f(B(a, \varepsilon)) \supseteq B(\alpha, \delta) \) which clearly implies that \( f \) maps open sets to open sets. Therefore, \( f(\Omega) \) is open, connected because \( f \) is continuous. If \( f \) is one to one, Theorem 50.6.3 implies that for every \( \alpha \in f(\Omega) \) the zero of \( f(\cdot) - \alpha \) is of order 1. Otherwise, that theorem implies that for \( z \) near \( \alpha \), there are \( m \) points which \( f \) maps to \( z \) contradicting the assumption that \( f \) is one to one. Therefore, \( f'(z) \neq 0 \) and since \( f^{-1} \) is continuous, due to \( f \) being an open map, it follows

\[
(f^{-1})'(f(z)) = \lim_{f(z_1) \to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} = \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}.
\]

This proves the theorem.

### 50.7 An Application To Linear Algebra

Gerschgorin’s theorem gives a convenient way to estimate eigenvalues of a matrix from easy to obtain information. For \( A \) an \( n \times n \) matrix, denote by \( \sigma(A) \) the collection of all eigenvalues of \( A \).

**Theorem 50.7.1** Let \( A \) be an \( n \times n \) matrix. Consider the \( n \) Gerschgorin discs defined as

\[
D_i \equiv \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.
\]

Then every eigenvalue is contained in some Gerschgorin disc.

This theorem says to add up the absolute values of the entries of the \( i \)th row which are off the main diagonal and form the disc centered at \( a_{ii} \) having this radius. The union of these discs contains \( \sigma(A) \).

**Proof:** Suppose \( Ax = \lambda x \) where \( x \neq 0 \). Then for \( A = (a_{ij}) \)

\[
\sum_{j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i.
\]

Therefore, if we pick \( k \) such that \( |x_k| \geq |x_j| \) for all \( x_j \), it follows that \( |x_k| \neq 0 \) since \( |x| \neq 0 \) and

\[
|x_k| \sum_{j \neq k} |a_{kj}| \geq \sum_{j \neq k} |a_{kj}| |x_j| \geq |\lambda - a_{kk}| |x_k|.
\]

Now dividing by \( |x_k| \) we see that \( \lambda \) is contained in the \( k \)th Gerschgorin disc.
50.7. AN APPLICATION TO LINEAR ALGEBRA

More can be said using the theory about counting zeros. To begin with the distance between two \( n \times n \) matrices, \( A = (a_{ij}) \) and \( B = (b_{ij}) \) as follows.

\[
\|A - B\|^2 = \sum_{ij} |a_{ij} - b_{ij}|^2.
\]

Thus two matrices are close if and only if their corresponding entries are close.

Let \( A \) be an \( n \times n \) matrix. Recall the eigenvalues of \( A \) are given by the zeros of the polynomial, \( p_A(z) = \det (zI - A) \) where \( I \) is the \( n \times n \) identity. Then small changes in \( A \) will produce small changes in \( p_A(z) \) and \( p_A'(z) \). Let \( \gamma_k \) denote a very small closed circle which winds around \( z_k \), one of the eigenvalues of \( A \), in the counter clockwise direction so that \( n(\gamma_k, z_k) = 1 \). This circle is to enclose only \( z_k \) and is to have no other eigenvalue on it. Then apply Theorem 50.7.2. According to this theorem

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{p_A'(z)}{p_A(z)} \, dz
\]

is always an integer equal to the multiplicity of \( z_k \) as a root of \( p_A(t) \). Therefore, small changes in \( A \) result in no change to the above contour integral because it must be an integer and small changes in \( A \) result in small changes in the integral. Therefore whenever every entry of the matrix \( B \) is close enough to the corresponding entry of the matrix \( A \), the two matrices have the same number of zeros inside \( \gamma_k \) under the usual convention that zeros are to be counted according to multiplicity. By making the radius of the small circle equal to \( \varepsilon \) where \( \varepsilon \) is less than the minimum distance between any two distinct eigenvalues of \( A \), this shows that if \( B \) is close enough to \( A \), every eigenvalue of \( B \) is closer than \( \varepsilon \) to some eigenvalue of \( A \). The next theorem is about continuous dependence of eigenvalues.

**Theorem 50.7.2** If \( \lambda \) is an eigenvalue of \( A \), then if \( \|B - A\| \) is small enough, some eigenvalue of \( B \) will be within \( \varepsilon \) of \( \lambda \).

Consider the situation that \( A(t) \) is an \( n \times n \) matrix and that \( t \to A(t) \) is continuous for \( t \in [0, 1] \).

**Lemma 50.7.3** Let \( \lambda(t) \in \sigma(A(t)) \) for \( t < 1 \) and let \( \Sigma_t = \cup_{s \geq t} \sigma(A(s)) \). Also let \( K_t \) be the connected component of \( \lambda(t) \) in \( \Sigma_t \). Then there exists \( \eta > 0 \) such that \( K_t \cap \sigma(A(s)) \neq \emptyset \) for all \( s \in [t, t + \eta] \).

**Proof:** Denote by \( D(\lambda(t), \delta) \) the disc centered at \( \lambda(t) \) having radius \( \delta > 0 \), with other occurrences of this notation being defined similarly. Thus

\[
D(\lambda(t), \delta) \equiv \{ z \in \mathbb{C} : |\lambda(t) - z| \leq \delta \}.
\]

Suppose \( \delta > 0 \) is small enough that \( \lambda(t) \) is the only element of \( \sigma(A(t)) \) contained in \( D(\lambda(t), \delta) \) and that \( p_{A(t)} \) has no zeroes on the boundary of this disc. Then by continuity, and the above discussion and theorem, there exists \( \eta > 0, t + \eta < 1 \), such that for \( s \in [t, t + \eta] \), \( p_{A(s)} \) also has no zeroes on the boundary of this disc and that
\[ A(s) \] has the same number of eigenvalues, counted according to multiplicity, in the disc as \( A(t) \). Thus \( \sigma(A(s)) \cap D(\lambda(t), \delta) \neq \emptyset \) for all \( s \in [t, t + \eta] \). Now let

\[
H = \bigcup_{s \in [t, t + \eta]} \sigma(A(s)) \cap D(\lambda(t), \delta).
\]

I will show \( H \) is connected. Suppose not. Then \( H = P \cup Q \) where \( P, Q \) are separated and \( \lambda(t) \in P \). Let

\[
s_0 \equiv \inf \{ s : \lambda(s) \in Q \text{ for some } \lambda(s) \in \sigma(A(s)) \}.
\]

There exists \( \lambda(s_0) \in \sigma(A(s_0)) \cap D(\lambda(t), \delta) \). If \( \lambda(s_0) \notin Q \), then from the above discussion there are

\[
\lambda(s) \in \sigma(A(s)) \cap Q
\]

for \( s > s_0 \) arbitrarily close to \( \lambda(s_0) \). Therefore, \( \lambda(s_0) \in Q \) which shows that \( s_0 > t \) because \( \lambda(t) \) is the only element of \( \sigma(A(t)) \) in \( D(\lambda(t), \delta) \) and \( \lambda(t) \in P \). Now let \( s_n \uparrow s_0 \). Then \( \lambda(s_n) \in P \) for any

\[
\lambda(s_n) \in \sigma(A(s_n)) \cap D(\lambda(t), \delta)
\]

and from the above discussion, for some choice of \( s_n \to s_0 \), \( \lambda(s_n) \to \lambda(s_0) \) which contradicts \( P \) and \( Q \) separated and nonempty. Since \( P \) is nonempty, this shows \( Q = \emptyset \). Therefore, \( H \) is connected as claimed. But \( K_t \supseteq H \) and so \( K_t \cap \sigma(A(s)) \neq \emptyset \) for all \( s \in [t, t + \eta] \). This proves the lemma.

The following is the necessary theorem.

**Theorem 50.7.4** Suppose \( A(t) \) is an \( n \times n \) matrix and that \( t \to A(t) \) is continuous for \( t \in [0, 1] \). Let \( \lambda(0) \in \sigma(A(0)) \) and define \( \Sigma \equiv \bigcup_{t \in [0,1]} \sigma(A(t)) \). Let \( K_{\lambda(0)} = K_0 \) denote the connected component of \( \lambda(0) \) in \( \Sigma \). Then \( K_0 \cap \sigma(A(t)) \neq \emptyset \) for all \( t \in [0, 1] \).

**Proof:** Let \( \mathcal{S} \equiv \{ t \in [0, 1] : K_0 \cap \sigma(A(s)) \neq \emptyset \text{ for all } s \in [0, t] \} \). Then \( 0 \in \mathcal{S} \). Let \( t_0 = \sup(\mathcal{S}) \). Say \( \sigma(A(t_0)) = \lambda_1(t_0), \cdots, \lambda_r(t_0) \). I claim at least one of these is a limit point of \( K_0 \) and consequently must be in \( K_0 \) which will show that \( S \) has a last point. Why is this claim true? Let \( s_n \uparrow t_0 \) so \( s_n \in S \). Now let the discs, \( D(\lambda_i(t_0), \delta), i = 1, \cdots, r \) be disjoint with \( p_{A(t_0)} \) having no zeroes on \( \gamma_i \) the boundary of \( D(\lambda_i(t_0), \delta) \). Then for \( n \) large enough it follows from Theorem 50.6.7 and the discussion following it that \( \sigma(A(s_n)) \) is contained in \( \bigcup_{i=1}^r D(\lambda_i(t_0), \delta) \). Therefore, \( K_0 \cap (\sigma(A(t_0)) + D(0, \delta)) \neq \emptyset \) for all \( \delta \) small enough. This requires at least one of the \( \lambda_i(t_0) \) to be in \( K_0 \). Therefore, \( t_0 \in S \) and \( S \) has a last point.

Now by Lemma 50.4.2, if \( t_0 < 1 \), then \( K_0 \cup K_t \) would be a strictly larger connected set containing \( \lambda(0) \). (The reason this would be strictly larger is that \( K_0 \cap \sigma(A(s)) = \emptyset \) for some \( s \in (t, t + \eta) \) while \( K_t \cap \sigma(A(s)) \neq \emptyset \) for all \( s \in [t, t + \eta] \).) Therefore, \( t_0 = 1 \) and this proves the theorem.

The following is an interesting corollary of the Gershgorin theorem.
Corollary 50.7.5 Suppose one of the Gerschgorin discs, $D_i$, is disjoint from the union of the others. Then $D_i$ contains an eigenvalue of $A$. Also, if there are $n$ disjoint Gerschgorin discs, then each one contains an eigenvalue of $A$.

Proof: Denote by $A(t)$ the matrix $(a'_{ij})$ where if $i \neq j$, $a'_{ij} = ta_{ij}$ and $a'_{ii} = a_{ii}$. Thus to get $A(t)$ we multiply all non diagonal terms by $t$. Let $t \in [0,1]$. Then $A(0) = \text{diag}(a_{11}, \cdots, a_{nn})$ and $A(1) = A$. Furthermore, the map, $t \rightarrow A(t)$ is continuous. Denote by $D'_j$ the Gerschgorin disc obtained from the $j$th row for the matrix, $A(t)$. Then it is clear that $D'_j \subseteq D_j$ the $j$th Gerschgorin disc for $A$. Then $a_{ii}$ is the eigenvalue for $A(0)$ which is contained in the disc, consisting of the single point $a_{ii}$ which is contained in $D_i$. Letting $K$ be the connected component in $\Sigma$ for $\Sigma$ defined in Theorem 50.7.3 which is determined by $a_{ii}$, it follows by Gerschgorin’s theorem that $K \cap \sigma(A(t)) \subseteq \bigcup_{j=1}^{n} D'_{j} \subseteq \bigcup_{j=1}^{n} D_j = D_i \cup (\bigcup_{j \neq i} D_j)$ and also, since $K$ is connected, there are no points of $K$ in both $D_i$ and $(\bigcup_{j \neq i} D_j)$. Since at least one point of $K$ is in $D_i$, $(a_{ii})$ it follows all of $K$ must be contained in $D_i$. Now by Theorem 50.7.4 this shows there are points of $K \cap \sigma(A)$ in $D_i$. The last assertion follows immediately.

Actually, this can be improved slightly. It involves the following lemma.

Lemma 50.7.6 In the situation of Theorem 50.7.3 suppose $\lambda(0) = K_0 \cap \sigma(A(0))$ and that $\lambda(0)$ is a simple root of the characteristic equation of $A(0)$. Then for all $t \in [0,1], \sigma(A(t)) \cap K_0 = \lambda(t)$ where $\lambda(t)$ is a simple root of the characteristic equation of $A(t)$.

Proof: Let $S \equiv \{ t \in [0,1] : K_0 \cap \sigma(A(s)) = \lambda(s), \text{ a simple eigenvalue for all } s \in [0,t] \}$. Then $0 \in S$ so it is nonempty. Let $t_0 = \sup(S)$ and suppose $\lambda_1 \neq \lambda_2$ are two elements of $\sigma(A(t_0)) \cap K_0$. Then choosing $\eta > 0$ small enough, and letting $D_i$ be disjoint discs containing $\lambda_i$ respectively, similar arguments to those of Lemma 50.7.4 imply

$$H_i \equiv \cup_{s \in [t_0-\eta,t_0]} \sigma(A(s)) \cap D_i$$

is a connected and nonempty set for $i = 1, 2$ which would require that $H_i \subseteq K_0$. But then there would be two different eigenvalues of $A(s)$ contained in $K_0$, contrary to the definition of $t_0$. Therefore, there is at most one eigenvalue, $\lambda(t_0) \in K_0 \cap \sigma(A(t_0))$. The possibility that it could be a repeated root of the characteristic equation must be ruled out. Suppose then that $\lambda(t_0)$ is a repeated root of the characteristic equation. As before, choose a small disc, $D$ centered at $\lambda(t_0)$ and $\eta$ small enough that

$$H \equiv \cup_{s \in [t_0-\eta,t_0]} \sigma(A(s)) \cap D$$

is a nonempty connected set containing either multiple eigenvalues of $A(s)$ or else a single repeated root to the characteristic equation of $A(s)$. But since $H$ is connected and contains $\lambda(t_0)$ it must be contained in $K_0$ which contradicts the condition for
CHAPTER 50. THE OPEN MAPPING THEOREM

$s \in S$ for all these $s \in [t_0 - \eta, t_0]$. Therefore, $t_0 \in S$ as hoped. If $t_0 < 1$, there exists a small disc centered at $\lambda(t_0)$ and $\eta > 0$ such that for all $s \in [t_0, t_0 + \eta]$, $A(s)$ has only simple eigenvalues in $D$ and the only eigenvalues of $A(s)$ which could be in $K_0$ are in $D$. (This last assertion follows from noting that $\lambda(t_0)$ is the only eigenvalue of $A(t_0)$ in $K_0$ and so the others are at a positive distance from $K_0$. For $s$ close enough to $t_0$, the eigenvalues of $A(s)$ are either close to these eigenvalues of $A(t_0)$ at a positive distance from $K_0$ or they are close to the eigenvalue, $\lambda(t_0)$ in which case it can be assumed they are in $D$.) But this shows that $t_0$ is not really an upper bound to $S$. Therefore, $t_0 = 1$ and the lemma is proved.

With this lemma, the conclusion of the above corollary can be improved.

**Corollary 50.7.7** Suppose one of the Gerschgorin discs, $D_i$ is disjoint from the union of the others. Then $D_i$ contains exactly one eigenvalue of $A$ and this eigenvalue is a simple root to the characteristic polynomial of $A$.

**Proof:** In the proof of Corollary 50.7.5, first note that $a_{ii}$ is a simple root of $A(0)$ since otherwise the $i$th Gerschgorin disc would not be disjoint from the others. Also, $K$, the connected component determined by $a_{ii}$ must be contained in $D_i$ because it is connected and by Gerschgorin’s theorem above, $K \cap \sigma(A(t))$ must be contained in the union of the Gerschgorin discs. Since all the other eigenvalues of $A(0)$, the $a_{jj}$, are outside $D_i$, it follows that $K \cap \sigma(A(0)) = a_{ii}$. Therefore, by Lemma 50.7.6, $K \cap \sigma(A(1)) = K \cap \sigma(A)$ consists of a single simple eigenvalue. This proves the corollary.

**Example 50.7.8** Consider the matrix,

\[
\begin{pmatrix}
5 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

The Gerschgorin discs are $D(5,1), D(1,2)$, and $D(0,1)$. Then $D(5,1)$ is disjoint from the other discs. Therefore, there should be an eigenvalue in $D(5,1)$. The actual eigenvalues are not easy to find. They are the roots of the characteristic equation, $t^3 - 6t^2 + 3t + 5 = 0$. The numerical values of these are $-0.66966, 1.4231$, and $5.24655$, verifying the predictions of Gerschgorin’s theorem.

**50.8 Exercises**

1. Use Theorem 50.6.1 to give an alternate proof of the fundamental theorem of algebra. **Hint:** Take a contour of the form $\gamma_r = re^{it}$ where $t \in [0, 2\pi]$. Consider $\int_{\gamma_r} \frac{p'(z)}{p(z)} \, dz$ and consider the limit as $r \to \infty$.

2. Let $M$ be an $n \times n$ matrix. Recall that the eigenvalues of $M$ are given by the zeros of the polynomial, $p_M(z) = \det(M - zI)$ where $I$ is the $n \times n$ identity. Formulate a theorem which describes how the eigenvalues depend on small
50.8. EXERCISES

1. Changes in $M$. Hint: You could define a norm on the space of $n \times n$ matrices as $||M|| \equiv \text{tr} \left( MM^* \right)^{1/2}$ where $M^*$ is the conjugate transpose of $M$. Thus

$$||M|| = \left( \sum_{j,k} |M_{jk}|^2 \right)^{1/2}.$$

Argue that small changes will produce small changes in $p_M(z)$. Then apply Theorem 50.6.1 using $\gamma_k$, a very small circle surrounding $z_k$, the $k$th eigenvalue.

3. Suppose that two analytic functions defined on a region are equal on some set, $S$ which contains a limit point. (Recall $p$ is a limit point of $S$ if every open set which contains $p$, also contains infinitely many points of $S$.) Show the two functions coincide. We defined $e^z \equiv e^x (\cos y + i \sin y)$ earlier and we showed that $e^z$, defined this way was analytic on $\mathbb{C}$. Is there any other way to define $e^z$ on all of $\mathbb{C}$ such that the function coincides with $e^z$ on the real axis?

4. You know various identities for real valued functions. For example $\cosh^2 x - \sinh^2 x = 1$. If you define $\cosh z \equiv \frac{e^z + e^{-z}}{2}$ and $\sinh z \equiv \frac{e^z - e^{-z}}{2}$, does it follow that $\cosh^2 z - \sinh^2 z = 1$ for all $z \in \mathbb{C}$? What about $\sin (z + w) = \sin z \cos w + \cos z \sin w$?

Can you verify these sorts of identities just from your knowledge about what happens for real arguments?

5. Was it necessary that $U$ be a region in Theorem 49.5.3? Would the same conclusion hold if $U$ were only assumed to be an open set? Why? What about the open mapping theorem? Would it hold if $U$ were not a region?

6. Let $f : U \to \mathbb{C}$ be analytic and one to one. Show that $f'(z) \neq 0$ for all $z \in U$. Does this hold for a function of a real variable?

7. We say a real valued function, $u$ is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if $u$ is subharmonic on a bounded region, (open connected set) $U$, and continuous on $\overline{U}$ and $u \leq m$ on $\partial U$, then $u \leq m$ on $U$. Hint: If not, $u$ achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_{\varepsilon}(x, y) = \varepsilon x^2 + u(x, y)$ where $\varepsilon$ is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that $u_{\varepsilon}$ also achieves its maximum at some point of $U$ and that therefore, $u_{xx} + u_{yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

8. If $u$ is harmonic on some region, $U$, show that $u$ coincides locally with the real part of an analytic function and that therefore, $u$ has infinitely many
derivatives on $U$. **Hint:** Consider the case where $0 \in U$. You can always reduce to this case by a suitable translation. Now let $B(0, r) \subseteq U$ and use the Schwarz formula to obtain an analytic function whose real part coincides with $u$ on $\partial B(0, r)$. Then use Problem 9.

9. Show the solution to the Dirichlet problem of Problem 8 on Page 1663 is unique. You need to formulate this precisely and then prove uniqueness.
Chapter 51

Residues

Definition 51.0.1 The residue of $f$ at an isolated singularity $\alpha$ which is a pole, written $\text{res}(f, \alpha)$ is the coefficient of $(z - \alpha)^{-1}$ where

$$f(z) = g(z) + \sum_{k=1}^{m} \frac{b_k}{(z - \alpha)^k}.$$ 

Thus $\text{res}(f, \alpha) = b_1$ in the above.

At this point, recall Corollary 49.7.20 which is stated here for convenience.

Corollary 51.0.2 Let $\Omega$ be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \ldots, m$, be closed, continuous and of bounded variation. Suppose also that

$$\sum_{k=1}^{m} (\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \mathbb{C}$ is analytic,

$$\sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw = 0.$$

The following theorem is called the residue theorem. Note the resemblance to Corollary 49.7.23.

Theorem 51.0.3 Let $\Omega$ be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \ldots, m$, be closed, continuous and of bounded variation. Suppose also that

$$\sum_{k=1}^{m} (\gamma_k, z) = 0$$
for all \( z \notin \Omega \). Then if \( f : \Omega \to \hat{\mathbb{C}} \) is meromorphic such that no \( \gamma_k \) contains any poles of \( f \),
\[
\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw = \sum_{\alpha \in A} \text{res}(f, \alpha) \sum_{k=1}^{m} n_\gamma(\alpha_k, \alpha) \tag{51.0.1}
\]
where here \( A \) denotes the set of poles of \( f \) in \( \Omega \). The sum on the right is a finite sum.

**Proof:** First note that there are at most finitely many \( \alpha \) which are not in the unbounded component of \( \mathbb{C} \setminus \bigcup_{k=1}^{m} \gamma_k ([a_k, b_k]) \). Thus there exists a finite set, \( \{\alpha_1, \cdots, \alpha_N\} \subseteq A \) such that these are the only possibilities for which \( \sum_{k=1}^{m} n_\gamma(\alpha_k, \alpha) \) might not equal zero. Therefore, (51.0.1) reduces to
\[
\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w) \, dw = \sum_{j=1}^{N} \text{res}(f, \alpha_j) \sum_{k=1}^{m} n_\gamma(\alpha_j, \alpha_k)
\]
and it is this last equation which is established. Near \( \alpha_j \),
\[
f(z) = g_j(z) + \sum_{r=1}^{m_j} \frac{b^j_r}{(z - \alpha_j)^r} \equiv g_j(z) + Q_j(z).
\]
where \( g_j \) is analytic at and near \( \alpha_j \). Now define
\[
G(z) \equiv f(z) - \sum_{j=1}^{N} Q_j(z).
\]
It follows that \( G(z) \) has a removable singularity at each \( \alpha_j \). Therefore, by Corollary 10.7.20,
\[
0 = \sum_{k=1}^{m} \int_{\gamma_k} G(z) \, dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z) \, dz - \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_k} Q_j(z) \, dz.
\]
Now
\[
\sum_{k=1}^{m} \int_{\gamma_k} Q_j(z) \, dz = \sum_{k=1}^{m} \int_{\gamma_k} \left( \frac{b^j_1}{(z - \alpha_j)} + \sum_{r=2}^{m_j} \frac{b^j_r}{(z - \alpha_j)^r} \right) \, dz
\]
\[
= \sum_{k=1}^{m} \int_{\gamma_k} \frac{b^j_1}{(z - \alpha_j)} \, dz \equiv \sum_{k=1}^{m} n_\gamma(\alpha_j, \alpha_j) \text{res}(f, \alpha_j) (2\pi i).
\]
Therefore,

\[
\sum_{k=1}^{m} \int_{\gamma_k} f(z) \, dz = \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_j} Q_j(z) \, dz \\
= \sum_{j=1}^{N} \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \operatorname{res}(f, \alpha_j) (2\pi i) \\
= 2\pi i \sum_{j=1}^{N} \operatorname{res}(f, \alpha_j) \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \\
= (2\pi i) \sum_{\alpha \in A} \operatorname{res}(f, \alpha) \sum_{k=1}^{m} n(\gamma_k, \alpha)
\]

which proves the theorem.

The following is an important example. This example can also be done by real variable methods and there are some who think that real variable methods are always to be preferred to complex variable methods. However, I will use the above theorem to work this example.

**Example 51.0.4** Find \( \lim_{R \to \infty} \int_{-R}^{R} \frac{\sin(x)}{x} \, dx \)

Things are easier if you write it as

\[
\lim_{R \to \infty} \frac{1}{i} \left( \int_{-R}^{-R^{-1}} \frac{e^{ix}}{x} \, dx + \int_{R^{-1}}^{R} \frac{e^{ix}}{x} \, dx \right).
\]

This gives the same answer because \( \cos(x)/x \) is odd. Consider the following contour in which the orientation involves counterclockwise motion exactly once around.

Denote by \( \gamma_{R^{-1}} \) the little circle and \( \gamma_R \) the big one. Then on the inside of this contour there are no singularities of \( e^{iz}/z \) and it is contained in an open set with the property that the winding number with respect to this contour about any point not in the open set equals zero. By Theorem 49.7.22

\[
\frac{1}{i} \left( \int_{-R}^{-R^{-1}} \frac{e^{iz}}{x} \, dx + \int_{\gamma_{R^{-1}}} \frac{e^{iz}}{z} \, dz + \int_{R^{-1}}^{R} \frac{e^{ix}}{x} \, dx + \int_{\gamma_R} \frac{e^{iz}}{z} \, dz \right) = 0 \quad (51.0.2)
\]
Now
\[
\left| \int_{\gamma_R} \frac{e^{iz}}{z} \, dz \right| = \left| \int_0^\pi e^{R(i \cos \theta - \sin \theta)} \, i \, d\theta \right| \leq \int_0^\pi e^{-R \sin \theta} \, d\theta
\]
and this last integral converges to 0 by the dominated convergence theorem. Now consider the other circle. By the dominated convergence theorem again,
\[
\int_{\gamma_R} \frac{e^{iz}}{z} \, dz = \int_0^{\pi} e^{R^{-1}(i \cos \theta - \sin \theta)} \, i \, d\theta \to -i\pi
\]
as \( R \to \infty \). Then passing to the limit in 51.0.2,
\[
\lim_{R \to \infty} \left( \frac{1}{2} \int_{-R}^R \frac{\sin(x)}{x} \right) dx = \frac{1}{i} (-i\pi) = \pi.
\]

Example 51.0.5 Find \( \lim_{R \to \infty} \int_{-R}^R e^{ixt} \frac{\sin x}{x} \, dx \). Note this is essentially finding the inverse Fourier transform of the function, \( \frac{\sin x}{x} \).

This equals
\[
\lim_{R \to \infty} \int_{-R}^R (\cos(0) + i \sin(0)) \frac{\sin(x)}{x} \, dx
\]
\[
= \lim_{R \to \infty} \int_{-R}^R \cos(x) \frac{\sin(x)}{x} \, dx
\]
\[
= \lim_{R \to \infty} \int_{-R}^R \cos(x) \frac{\sin(x)}{x} \, dx
\]
\[
= \lim_{R \to \infty} \frac{1}{2} \int_{-R}^R \frac{\sin(x(t+1)) + \sin(x(1-t))}{x} \, dx.
\]
Let \( t \neq 1, -1 \). Then changing variables yields
\[
\lim_{R \to \infty} \left( \frac{1}{2} \int_{-R(1+t)}^{R(1+t)} \frac{\sin(u)}{u} \, du + \frac{1}{2} \int_{-R(1-t)}^{R(1-t)} \frac{\sin(u)}{u} \, du \right).
\]
In case \(|t| < 1\) Example 51.0.4 implies this limit is \( \pi \). However, if \( t > 1 \) the limit equals 0 and this is also the case if \( t < -1 \). Summarizing,
\[
\lim_{R \to \infty} \int_{-R}^R e^{ixt} \frac{\sin x}{x} \, dx = \begin{cases} 
\pi & \text{if } |t| < 1 \\
0 & \text{if } |t| > 1 
\end{cases}
\]
51.1. ROUCHE’S THEOREM AND THE ARGUMENT PRINCIPLE

51.1.1 Argument Principle

A simple closed curve is just one which is homeomorphic to the unit circle. The Jordan Curve theorem states that every simple closed curve in the plane divides the plane into exactly two connected components, one bounded and the other unbounded. This is a very hard theorem to prove. However, in most applications the conclusion is obvious. Nevertheless, to avoid using this big topological result and to attain some extra generality, I will state the following theorem in terms of the winding number to avoid using it. This theorem is called the argument principle.

First recall that $f$ has a zero of order $m$ at $\alpha$ if $f(z) = g(z)(z-\alpha)^m$ where $g$ is an analytic function which is not equal to zero at $\alpha$. This is equivalent to having $f(z) = \sum_{k=m}^{\infty} a_k (z-\alpha)^k$ for $z$ near $\alpha$ where $a_m \neq 0$.

Also recall that $f$ has a pole of order $m$ at $\alpha$ if for $z$ near $\alpha$, $f(z)$ is of the form $f(z) = h(z) + \sum_{k=1}^{m} b_k (z-\alpha)^{k}$.

Theorem 51.1.1 (argument principle) Let $f$ be meromorphic in $\Omega$. Also suppose $\gamma^*$ is a closed bounded variation curve containing none of the poles or zeros of $f$ with the property that for all $z \notin \Omega, n(\gamma, z) = 0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the winding number of $\gamma$ about these points equals 1. Let $z_k$ be a zero of order $r_k$ and let $p_k$ be a pole of order $l_k$. Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k$$

Proof: This theorem follows from computing the residues of $f'/f$. It has residues at poles and zeros. I will do this now. First suppose $f$ has a pole of order $p$ at $\alpha$. Then $f$ has the form given in $(51.1.3)$. Therefore,

$$\frac{f'(z)}{f(z)} = \frac{h'(z) - \sum_{k=1}^{p} k b_k (z-\alpha)^{k-1} + \sum_{k=1}^{p} \frac{b_k}{(z-\alpha)^{k}}}{h(z) + \sum_{k=1}^{p} \frac{b_k}{(z-\alpha)^{k}}}$$

$$= \frac{h'(z)(z-\alpha)^p - \sum_{k=1}^{p-1} k b_k (z-\alpha)^{p-1-k+1} + \sum_{k=1}^{p} b_k (z-\alpha)^{p-k}}{h(z)(z-\alpha)^p + \sum_{k=1}^{p-1} b_k (z-\alpha)^{p-k} + b_p}$$

This is of the form

$$= \frac{b_p}{s(z) + b_p} \frac{r(z) - \frac{b_p}{b_p}}{b_p} = \frac{b_p}{s(z) + b_p} \left( \frac{r(z)}{b_p} - \frac{p}{b_p} \right)$$
where \( s(\alpha) = r(\alpha) = 0 \). From this, it is clear \( \text{res}\left(\frac{f'}{f}, \alpha\right) = -p \), the order of the pole.

Next suppose \( f \) has a zero of order \( p \) at \( \alpha \). Then

\[
\frac{f'(z)}{f(z)} = \frac{\sum_{k=p}^{\infty} a_k (z - \alpha)^{k-1}}{\sum_{k=p}^{\infty} a_k (z - \alpha)^k}
\]

and from this it is clear \( \text{res}\left(\frac{f'}{f}, \alpha\right) = p \), the order of the zero. The conclusion of this theorem now follows from Theorem 51.0.3.

One can also generalize the theorem to the case where there are many closed curves involved. This is proved in the same way as the above.

**Theorem 51.1.2 (argument principle)** Let \( f \) be meromorphic in \( \Omega \) and let \( \gamma_k : [a_k, b_k] \to \Omega, \ k = 1, \cdots, m \), be closed, continuous and of bounded variation. Suppose also that

\[
\sum_{k=1}^{m} n(\gamma_k, z) = 0
\]

and for all \( z \notin \Omega \) and for \( z \in \Omega \), \( \sum_{k=1}^{m} n(\gamma_k, z) \) either equals 0 or 1. Now let \( \{p_1, \cdots, p_m\} \) and \( \{z_1, \cdots, z_n\} \) be respectively the poles and zeros for which the above sum of winding numbers equals 1. Let \( z_k \) be a zero of order \( r_k \) and let \( p_k \) be a pole of order \( l_k \). Then

\[
\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k
\]

There is also a simple extension of this important principle which I found in [59].

**Theorem 51.1.3 (argument principle)** Let \( f \) be meromorphic in \( \Omega \). Also suppose \( \gamma^* \) is a closed bounded variation curve with the property that for all \( z \notin \Omega, n(\gamma, z) = 0 \) and for all \( z \in \Omega, n(\gamma, z) \) either equals 0 or 1. Now let \( \{p_1, \cdots, p_m\} \) and \( \{z_1, \cdots, z_n\} \) be respectively the poles and zeros for which the winding number of \( \gamma \) about these points equals \( 1 \) listed according to multiplicity. Thus if there is a pole of order \( m \) there will be this value repeated \( m \) times in the list for the poles. Also let \( g(z) \) be an analytic function. Then

\[
\frac{1}{2\pi i} \oint_{\gamma} g(z) \frac{f'(z)}{f(z)} \, dz = \sum_{k=1}^{n} g(z_k) - \sum_{k=1}^{m} g(p_k)
\]

**Proof:** This theorem follows from computing the residues of \( g \left( \frac{f'}{f} \right) \). It has residues at poles and zeros. I will do this now. First suppose \( f \) has a pole of order
51.1. ROUCHE’S THEOREM AND THE ARGUMENT PRINCIPLE

m at \( \alpha \). Then \( f \) has the form given in 51.1.3. Therefore,

\[
g(z) \frac{f'(z)}{f(z)} = \frac{g(z) \left( h'(z) - \sum_{k=1}^{m} \frac{kb_k}{(z-\alpha)^{k+1}} \right)}{h(z) + \sum_{k=1}^{m} b_k (z-\alpha)^k} = g(z) \frac{h(z)(z-\alpha)^m - \sum_{k=1}^{m-1} kb_k (z-\alpha)^{-k+1+m} - \frac{mb_m}{(z-\alpha)}}{h(z)(z-\alpha)^m + \sum_{k=1}^{m-1} b_k (z-\alpha)^{m-k} + b_m}
\]

From this, it is clear \( \text{res} \left( g \left( \frac{f'}{f} \right), \alpha \right) = -mg(\alpha) \), where \( m \) is the order of the pole. Thus \( \alpha \) would have been listed \( m \) times in the list of poles. Hence the residue at this point is equivalent to adding \( -g(\alpha) m \) times.

Next suppose \( f \) has a zero of order \( m \) at \( \alpha \). Then

\[
g(z) \frac{f'(z)}{f(z)} = g(z) \sum_{k=m}^{\infty} a_k k (z-\alpha)^{k-1} \frac{(z-\alpha)^k}{\sum_{k=m}^{\infty} a_k (z-\alpha)^k} = g(z) \sum_{k=m}^{\infty} a_k k (z-\alpha)^{k-1-m} \frac{(z-\alpha)^k}{\sum_{k=m}^{\infty} a_k (z-\alpha)^k}
\]

and from this it is clear \( \text{res} \left( g \left( \frac{f'}{f} \right) \right) = g(\alpha) m \), where \( m \) is the order of the zero.

The conclusion of this theorem now follows from the residue theorem, Theorem 51.0.3.

The way people usually apply these theorems is to suppose \( \gamma^* \) is a simple closed bounded variation curve, often a circle. Thus it has an inside and an outside, the outside being the unbounded component of \( \mathbb{C} \setminus \gamma^* \). The orientation of the curve is such that you go around it once in the counterclockwise direction. Then letting \( r_k \) and \( l_k \) be as described, the conclusion of the theorem follows. In applications, this is likely the way it will be.

51.1.2 Rouche’s Theorem

With the argument principle, it is possible to prove Rouche’s theorem. In the argument principle, denote by \( Z_f \) the quantity \( \sum_{k=1}^{m} r_k \) and by \( P_f \) the quantity \( \sum_{k=1}^{n} l_k \). Thus \( Z_f \) is the number of zeros of \( f \) counted according to the order of the zero with a similar definition holding for \( P_f \). Thus the conclusion of the argument principle is

\[
\frac{1}{2\pi i} \int_{\gamma^*} \frac{f'(z)}{f(z)} \, dz = Z_f - P_f
\]

Rouche’s theorem allows the comparison of \( Z_h - P_h \) for \( h = f, g \). It is a wonderful and amazing result.

Theorem 51.1.4 (Rouche’s theorem) Let \( f, g \) be meromorphic in an open set \( \Omega \). Also suppose \( \gamma^* \) is a closed bounded variation curve with the property that for all \( z \notin \Omega, n(\gamma, z) = 0 \), no zeros or poles are on \( \gamma^* \), and for all \( z \in \Omega, n(\gamma, z) \) either equals 0 or 1. Let \( Z_f \) and \( P_f \) denote respectively the numbers of zeros and poles of
CHAPTER 51. RESIDUES

Let \( f \), which have the property that the winding number equals 1, counted according to order, with \( Z_g \) and \( P_g \) being defined similarly. Also suppose that for \( z \in \gamma^* \)
\[
|f(z) + g(z)| < |f(z)| + |g(z)|.
\] (51.1.4)

Then
\[
Z_f - P_f = Z_g - P_g.
\]

**Proof:** From the hypotheses,
\[
\left|1 + \frac{f(z)}{g(z)}\right| < 1 + \left|\frac{f(z)}{g(z)}\right|
\]
which shows that for all \( z \in \gamma^* \),
\[
\frac{f(z)}{g(z)} \in \mathbb{C} \setminus [0, \infty).
\]

Letting \( l \) denote a branch of the logarithm defined on \( \mathbb{C} \setminus [0, \infty) \), it follows that \( l\left(\frac{f(z)}{g(z)}\right) \) is a primitive for the function,
\[
\frac{(f/g)'(f/g)}{f/g} = \frac{f'}{f} - \frac{g'}{g}.
\]

Therefore, by the argument principle,
\[
0 = \frac{1}{2\pi i} \int_\gamma \frac{(f/g)'}{f/g} \, dz = \frac{1}{2\pi i} \int_\gamma \left( \frac{f'}{f} - \frac{g'}{g} \right) \, dz = Z_f - P_f - (Z_g - P_g).
\]

This proves the theorem.

Often another condition other than 51.1.4 is used.

**Corollary 51.1.5** In the situation of Theorem 51.1.4 change 51.1.4 to the condition,
\[
|f(z) - g(z)| < |f(z)|
\]
for \( z \in \gamma^* \). Then the conclusion is the same.

**Proof:** The new condition implies \( \left|1 - \frac{g(z)}{f(z)}\right| < \left|\frac{g(z)}{f(z)}\right| \) on \( \gamma^* \). Therefore, \( \frac{g(z)}{f(z)} \notin (-\infty, 0] \) and so you can do the same argument with a branch of the logarithm.

### 51.1.3 A Different Formulation

In [104] I found this modification for Rouche’s theorem concerned with the counting of zeros of analytic functions. This is a very useful form of Rouche’s theorem because it makes no mention of a contour.
Theorem 51.1.6 Let $\Omega$ be a bounded open set and suppose $f, g$ are continuous on $\overline{\Omega}$ and analytic on $\Omega$. Also suppose $|f(z)| < |g(z)|$ on $\partial \Omega$. Then $f$ and $g + f$ have the same number of zeros in $\Omega$ provided each zero is counted according to multiplicity.

Proof: Let $K = \{z \in \overline{\Omega} : |f(z)| \geq |g(z)|\}$. Then letting $\lambda \in [0, 1]$, if $z \notin K$, then $|f(z)| < |g(z)|$ and so

$$0 < |g(z) - f(z)| \leq |g(z)| - \lambda|f(z)| \leq |g(z) + \lambda f(z)|$$

which shows that all zeros of $g + \lambda f$ are contained in $K$ which must be a compact subset of $\Omega$ due to the assumption that $|f(z)| < |g(z)|$ on $\partial \Omega$. By Theorem 19.7.3 on Page 1032 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k \subseteq \Omega \setminus K, \sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then as above, it follows from the residue theorem or more directly, Theorem 51.1.6

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \lambda f'(z) + g'(z) \frac{\lambda f(z) + g(z)}{\lambda f(z) + g(z)} \, dz = \sum_{j=1}^p m_j$$

where $m_j$ is the order of the $j^{th}$ zero of $\lambda f + g$ in $K$, hence in $\Omega$. However,

$$\lambda \rightarrow \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \lambda f'(z) + g'(z) \frac{\lambda f(z) + g(z)}{\lambda f(z) + g(z)} \, dz$$

is integer valued and continuous so it gives the same value when $\lambda = 0$ as when $\lambda = 1$. When $\lambda = 0$ this gives the number of zeros of $g$ in $\Omega$ and when $\lambda = 1$ it is the number of zeros of $f + g$. This proves the theorem.

Here is another formulation of this theorem.

Corollary 51.1.7 Let $\Omega$ be a bounded open set and suppose $f, g$ are continuous on $\overline{\Omega}$ and analytic on $\Omega$. Also suppose $|f(z) - g(z)| < |g(z)|$ on $\partial \Omega$. Then $f$ and $g$ have the same number of zeros in $\Omega$ provided each zero is counted according to multiplicity.

Proof: You let $f - g$ play the role of $f$ in Theorem 19.7.3. Thus $f - g + f = f$ and $g$ have the same number of zeros. Alternatively, you can give a proof of this directly as follows.

Let $K = \{z \in \Omega : |f(z) - g(z)| \geq |g(z)|\}$. Then if $g(z) + \lambda (f(z) - g(z)) = 0$ it follows

$$0 = |g(z) + \lambda (f(z) - g(z))| \geq |g(z)| - \lambda |f(z) - g(z)| \geq |g(z)| - |f(z) - g(z)|$$

and so $z \in K$. Thus all zeros of $g(z) + \lambda (f(z) - g(z))$ are contained in $K$. By Theorem 19.7.3 on Page 1032 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k \subseteq \Omega \setminus K, \sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then by Theorem 51.1.6

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \lambda (f'(z) - g'(z)) + g'(z) \frac{\lambda (f(z) - g(z)) + g(z)}{\lambda (f(z) - g(z)) + g(z)} \, dz = \sum_{j=1}^p m_j$$
where \( m_j \) is the order of the \( j^{th} \) zero of \( \lambda (f - g) + g \) in \( K \), hence in \( \Omega \). The left side is continuous as a function of \( \lambda \) and so the number of zeros of \( g \) corresponding to \( \lambda = 0 \) equals the number of zeros of \( f \) corresponding to \( \lambda = 1 \). This proves the corollary.

51.2 Singularities And The Laurent Series

51.2.1 What Is An Annulus?

In general, when you consider singularities, isolated or not, the fundamental tool is the Laurent series. This series is important for many other reasons also. In particular, it is fundamental to the spectral theory of various operators in functional analysis and is one way to obtain relationships between algebraic and analytical conditions essential in various convergence theorems. A Laurent series lives on an annulus. In all this \( f \) has values in \( X \) where \( X \) is a complex Banach space. If you like, let \( X = \mathbb{C} \).

**Definition 51.2.1** Define \( \text{ann} (a, R_1, R_2) \equiv \{ z : R_1 < |z - a| < R_2 \} \).

Thus \( \text{ann} (a, 0, R) \) would denote the punctured ball, \( B (a, R) \setminus \{0\} \) and when \( R_1 > 0 \), the annulus looks like the following.

The annulus is the stuff between the two circles.

Here is an important lemma which is concerned with the situation described in the following picture.

**Lemma 51.2.2** Let \( \gamma_r (t) \equiv a + re^{it} \) for \( t \in [0, 2\pi] \) and let \( |z - a| < r \). Then \( n (\gamma_r, z) = 1 \). If \( |z - a| > r \), then \( n (\gamma_r, z) = 0 \).

**Proof:** For the first claim, consider for \( t \in [0, 1] \),

\[
f (t) \equiv n (\gamma_r, a + t (z - a)).
\]
Then from properties of the winding number derived earlier, \( f(t) \in \mathbb{Z} \), \( f \) is continuous, and \( f(0) = 1 \). Therefore, \( f(t) = 1 \) for all \( t \in [0, 1] \). This proves the first claim because \( f(1) = n(\gamma_r, z) \).

For the second claim,

\[
\begin{align*}
n(\gamma_r, z) &= \frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{w-z} \, dw \\
&= \frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{w-a-(z-a)} \, dw \\
&= \frac{1}{2\pi i} - \frac{1}{z-a} \int_{\gamma_r} 1 - \frac{1}{w-a} \, dw \\
&= -\frac{1}{2\pi i (z-a)} \int_{\gamma_r} \sum_{k=0}^{\infty} \left( \frac{w-a}{z-a} \right)^k \, dw.
\end{align*}
\]

The series converges uniformly for \( w \in \gamma_r \) because

\[
\left| \frac{w-a}{z-a} \right| = \frac{r}{r+c}
\]

for some \( c > 0 \) due to the assumption that \( |z-a| > r \). Therefore, the sum and the integral can be interchanged to give

\[
n(\gamma_r, z) = -\frac{1}{2\pi i (z-a)} \sum_{k=0}^{\infty} \int_{\gamma_r} \left( \frac{w-a}{z-a} \right)^k \, dw = 0
\]

because \( w \to \left( \frac{w-a}{z-a} \right)^k \) has an antiderivative. This proves the lemma.

Now consider the following picture which pertains to the next lemma.

**Lemma 51.2.3** Let \( g \) be analytic on \( \text{ann } (a, R_1, R_2) \). Then if \( \gamma_r(t) \equiv a + re^{it} \) for \( t \in [0, 2\pi] \) and \( r \in (R_1, R_2) \), then \( \int_{\gamma_r} g(z) \, dz \) is independent of \( r \).

**Proof:** Let \( R_1 < r_1 < r_2 < R_2 \) and denote by \( -\gamma_r(t) \) the curve, \( -\gamma_r(t) \equiv a + re^{i(2\pi-t)} \) for \( t \in [0, 2\pi] \). Then if \( z \in \overline{B(a, R_1)} \), Lemma 51.2.2 implies both
n(\(\gamma_{r_1}, z\)) and n(\(\gamma_{r_2}, z\)) = 1 and so
\[n(-\gamma_{r_1}, z) + n(\gamma_{r_2}, z) = -1 + 1 = 0.\]

Also if \(z \notin B(a, R_2)\), then Lemma 51.2.2 implies \(n(\gamma_{r_j}, z) = 0\) for \(j = 1, 2\). Therefore, whenever \(z \notin \text{ann}(a, R_1, R_2)\), the sum of the winding numbers equals zero. Therefore, by Theorem 51.2.1 applied to the function, \(f(w) = g(z) (w - z)\) and \(z \in \text{ann}(a, R_1, R_2) \setminus \bigcup_{j=1}^{2} \gamma_{r_j} ([0, 2\pi])\),
\[
f(z) (n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)) = 0 \left( n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z) \right) = \frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{g(w)(w - z)}{w - z} \, dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{g(w)(w - z)}{w - z} \, dw = \frac{1}{2\pi i} \int_{\gamma_{r_2}} g(w) \, dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} g(w) \, dw\]
which proves the desired result.

**51.2.2 The Laurent Series**

The Laurent series is like a power series except it allows for negative exponents. First here is a definition of what is meant by the convergence of such a series.

**Definition 51.2.4** \(\sum_{n=-\infty}^{\infty} a_n (z-a)^n\) converges if both the series,
\[
\sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}
\]
converge. When this case, the symbol, \(\sum_{n=-\infty}^{\infty} a_n (z-a)^n\) is defined as
\[
\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}.
\]

**Lemma 51.2.5** Suppose
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n\]
for all \(|z-a| \in (R_1, R_2)|. Then both \(\sum_{n=0}^{\infty} a_n (z-a)^n\) and \(\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}\) converge absolutely and uniformly on \(\{z : r_1 \leq |z-a| \leq r_2\}\) for any \(r_1 < r_2\) satisfying \(R_1 < r_1 < r_2 < R_2\).

**Proof:** Let \(R_1 < |w-a| = r_1 - \delta < r_1\). Then \(\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}\) converges and so
\[
\lim_{n \to \infty} |a_{-n}| |w-a|^{-n} = \lim_{n \to \infty} |a_{-n}| (r_1 - \delta)^{-n} = 0
\]
which implies that for all \( n \) sufficiently large,
\[
|a_n| (r_1 - \delta)^{-n} < 1.
\]
Therefore,
\[
\sum_{n=1}^{\infty} |a_n| |z - a|^{-n} = \sum_{n=1}^{\infty} |a_n| (r_1 - \delta)^{-n} (r_1 - \delta)^n |z - a|^{-n}.
\]
Now for \( |z - a| \geq r_1 \),
\[
|z - a|^{-n} \leq \frac{1}{r_1^n}
\]
and so for all sufficiently large \( n \)
\[
|a_n| |z - a|^{-n} \leq \frac{(r_1 - \delta)^n}{r_1^n}.
\]
Therefore, by the Weierstrass \( M \) test, the series, \( \sum_{n=1}^{\infty} a_n (z - a)^{-n} \) converges absolutely and uniformly on the set
\[
\{ z \in \mathbb{C} : |z - a| \geq r_1 \}.
\]
Similar reasoning shows the series, \( \sum_{n=0}^{\infty} a_n (z - a)^n \) converges uniformly on the set
\[
\{ z \in \mathbb{C} : |z - a| \leq r_2 \}.
\]
This proves the Lemma.

**Theorem 51.2.6** Let \( f \) be analytic on \( \text{ann} (a, R_1, R_2) \). Then there exist numbers, \( a_n \in \mathbb{C} \) such that for all \( z \in \text{ann} (a, R_1, R_2) \),
\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n,
\]
where the series converges absolutely and uniformly on \( \text{ann} (a, r_1, r_2) \) whenever \( R_1 < r_1 < r_2 < R_2 \). Also
\[
a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} dw
\]
where \( \gamma (t) = a + re^{it}, t \in [0, 2\pi] \) for any \( r \in (R_1, R_2) \). Furthermore the series is unique in the sense that if \( 51.2.5, 51.2.6 \) holds for \( z \in \text{ann} (a, R_1, R_2) \), then \( a_n \) is given in \( 51.2.6 \).

**Proof:** Let \( R_1 < r_1 < r_2 < R_2 \) and define \( \gamma_1 (t) = a + (r_1 - \varepsilon) e^{it} \) and \( \gamma_2 (t) = a + (r_2 + \varepsilon) e^{it} \) for \( t \in [0, 2\pi] \) and \( \varepsilon \) chosen small enough that \( R_1 < r_1 - \varepsilon < r_2 + \varepsilon < R_2 \).
Then using Lemma 51.2.2, if \( z \notin \text{ann} (a, R_1, R_2) \) then
\[
n (γ_1, z) + n (γ_2, z) = 0
\]
and if \( z \in \text{ann} (a, r_1, r_2) \),
\[
n (γ_1, z) + n (γ_2, z) = 1.
\]
Therefore, by Theorem 49.7.19, for \( z \in \text{ann} (a, r_1, r_2) \)
\[
f (z) = \sum_{n=0}^{∞} \left( \frac{1}{2πi} \int_{γ_2} \frac{f (w)}{w-a} dw \right) \left( \frac{z-a}{w-a} \right)^n
\]
From the formula 51.2.7, it follows that for \( z \in \text{ann} (a, r_1, r_2) \), the terms in the first sum are bounded by an expression of the form \( C \left( \frac{r_2}{r_1} \right)^n \) while those in the second are bounded by one of the form \( C \left( \frac{r_1-z}{r_1} \right)^n \) and so by the Weierstrass M test, the convergence is uniform and so the integrals and the sums in the above formula may be interchanged and after renaming the variable of summation, this yields
\[
f (z) = \sum_{n=0}^{∞} \left( \frac{1}{2πi} \int_{γ_2} \frac{f (w)}{w-a} dw \right) (z-a)^n
\]
Therefore, by Lemma 51.2.3, for any \( r \in (R_1, R_2) \),

\[
f(z) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} \right) (z-a)^n. \tag{51.2.9}
\]

and so

\[
f_n(z) = \sum_{k=-\infty}^{\infty} \left( \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} \right) (z-a)^k. \tag{51.2.11}
\]

Also if \( k > n \) or if \( k < -n \),

\[
\left( \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} \right) = 0.
\]

and so

\[
f_n(z) = \sum_{k=-n}^{n} \left( \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} \right) (z-a)^k.
\]

which implies from Lemma 51.2.3 that for each \( k \in [-n, n] \),

\[
\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw = a_k
\]

However, from the uniform convergence of the series,

\[
\sum_{n=0}^{\infty} a_n (w-a)^n
\]
and
\[ \sum_{n=1}^{\infty} a^{-n} \]
ensured by Lemma 51.2.5 which allows the interchange of sums and integrals, if \( k \in [-n, n] \),

\[
\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} \, dw
\]
\[ = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{m=0}^{\infty} a_m (w-a)^m + \sum_{m=1}^{\infty} a_{-m} (w-a)^{-m} \, dw \]
\[ = \sum_{m=0}^{\infty} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{(m-(k+1))} \, dw \]
\[ + \sum_{m=1}^{\infty} a_{-m} \int_{\gamma_r} (w-a)^{-(m-(k+1))} \, dw \]
\[ = \sum_{m=0}^{n} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{(m-(k+1))} \, dw \]
\[ + \sum_{m=1}^{n} a_{-m} \int_{\gamma_r} (w-a)^{-(m-(k+1))} \, dw \]
\[ = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} \, dw \]

because if \( l > n \) or \( l < -n \),

\[ \int_{\gamma_r} \frac{a_l (w-a)^l}{(w-a)^{k+1}} \, dw = 0 \]

for all \( k \in [-n, n] \). Therefore,

\[ a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} \, dw \]

and so this establishes uniqueness. This proves the theorem.

### 51.2.3 Contour Integrals And Evaluation Of Integrals

Here are some examples of hard integrals which can be evaluated by using residues. This will be done by integrating over various closed curves having bounded variation.

**Example 51.2.7** The first example we consider is the following integral.

\[ \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx \]
One could imagine evaluating this integral by the method of partial fractions and it should work out by that method. However, we will consider the evaluation of this integral by the method of residues instead. To do so, consider the following picture.

Let \( \gamma_r(t) = re^{it}, t \in [0, \pi] \) and let \( \sigma_r(t) = t : t \in [-r, r] \). Thus \( \gamma_r \) parameterizes the top curve and \( \sigma_r \) parameterizes the straight line from \(-r\) to \(r\) along the \(x\) axis. Denoting by \( \Gamma_r \) the closed curve traced out by these two, we see from simple estimates that

\[
\lim_{r \to \infty} \int_{\gamma_r} \frac{1}{1 + z^4} \, dz = 0.
\]

This follows from the following estimate.

\[
\left| \int_{\gamma_r} \frac{1}{1 + z^4} \, dz \right| \leq \frac{1}{r^4 - 1} \pi r.
\]

Therefore,

\[
\int_{-\infty}^{\infty} \frac{1}{1 + x^4} \, dx = \lim_{r \to \infty} \int_{\Gamma_r} \frac{1}{1 + z^4} \, dz.
\]

We compute \( \int_{\Gamma_r} \frac{1}{1 + z^4} \, dz \) using the method of residues. The only residues of the integrand are located at points, \( z \) where \( 1 + z^4 = 0 \). These points are

\[
\begin{align*}
z &= -\frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2}, \quad z = \frac{1}{2} \sqrt{2} - \frac{1}{2} i \sqrt{2}, \\
z &= \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2}, \quad z = -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2}
\end{align*}
\]

and it is only the last two which are found in the inside of \( \Gamma_r \). Therefore, we need to calculate the residues at these points. Clearly this function has a pole of order one at each of these points and so we may calculate the residue at \( \alpha \) in this list by evaluating

\[
\lim_{z \to \alpha} (z - \alpha) \frac{1}{1 + z^4}
\]
Thus
\[ \text{Res} \left( f, \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right) \]
\[ = \lim_{z \to \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2}} \left( z - \left( \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right) \right) \frac{1}{1 + z^4} \]
\[ = -\frac{1}{8} \sqrt{2} - \frac{1}{8} i \sqrt{2} \]

Similarly we may find the other residue in the same way
\[ \text{Res} \left( f, -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right) \]
\[ = \lim_{z \to -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2}} \left( z - \left( -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right) \right) \frac{1}{1 + z^4} \]
\[ = -\frac{1}{8} i \sqrt{2} + \frac{1}{8} \sqrt{2} \]

Therefore,
\[ \int_{\Gamma_r} \frac{1}{1 + z^4} dz = 2\pi i \left( -\frac{1}{8} i \sqrt{2} + \frac{1}{8} \sqrt{2} - \left( -\frac{1}{8} \sqrt{2} - \frac{1}{8} i \sqrt{2} \right) \right) \]
\[ = \frac{1}{2} \pi \sqrt{2} \]

Thus, taking the limit we obtain
\[ \frac{\pi}{2} \sqrt{2} = \int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx. \]

Obviously many different variations of this are possible. The main idea being that the integral over the semicircle converges to zero as \( r \to \infty \).

Sometimes we don’t blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

**Example 51.2.8** The integral is
\[ \int_0^\pi \frac{\cos \theta}{2 + \cos \theta} d\theta \]

This integrand is even and so it equals
\[ \frac{1}{2} \int_{-\pi}^\pi \frac{\cos \theta}{2 + \cos \theta} d\theta. \]

For \( z \) on the unit circle, \( z = e^{i\theta} \), \( \overline{z} = \frac{1}{z} \) and therefore, \( \cos \theta = \frac{1}{2} (z + \frac{1}{z}) \). Thus \( dz = ie^{i\theta} d\theta \) and so \( d\theta = \frac{dz}{iz} \). Note this is proceeding formally to get a complex integral which reduces to the one of interest. It follows that a complex integral which reduces to the one desired is
\[ \frac{1}{2i} \int_{\gamma} \frac{\frac{1}{2} (z + \frac{1}{z})}{2 + \frac{1}{2} (z + \frac{1}{z})} \frac{dz}{z} = \frac{1}{2i} \int_{\gamma} \frac{z^2 + 1}{z (4z + z^2 + 1)} dz \]
where \( \gamma \) is the unit circle. Now the integrand has poles of order 1 at those points where \( z \left( 4z + z^2 + 1 \right) = 0 \). These points are

\[
0, -2 + \sqrt{3}, -2 - \sqrt{3}.
\]

Only the first two are inside the unit circle. It is also clear the function has simple poles at these points. Therefore,

\[
\text{Res} \left( f, 0 \right) = \lim_{z \to 0} z \left( \frac{z^2 + 1}{z \left( 4z + z^2 + 1 \right)} \right) = 1.
\]

\[
\text{Res} \left( f, -2 + \sqrt{3} \right) = \lim_{z \to -2 + \sqrt{3}} \left( z - \left( -2 + \sqrt{3} \right) \right) \frac{z^2 + 1}{z \left( 4z + z^2 + 1 \right)} = -\frac{2}{3} \sqrt{3}.
\]

It follows

\[
\int_0^{\pi} \frac{\cos \theta}{2 + \cos \theta} d\theta = \frac{1}{2i} \int_{\gamma} \frac{z^2 + 1}{z \left( 4z + z^2 + 1 \right)} dz = \frac{1}{2i} 2\pi i \left( 1 - \frac{2}{3} \sqrt{3} \right) = \pi \left( 1 - \frac{2}{3} \sqrt{3} \right).
\]

Other rational functions of the trig functions will work out by this method also.

Sometimes you have to be clever about which version of an analytic function that reduces to a real function you should use. The following is such an example.

**Example 51.2.9** The integral here is

\[
\int_0^{\infty} \frac{\ln x}{1 + x^4} dx.
\]

The same curve used in the integral involving \( \frac{\sin x}{x} \) earlier will create problems with the log since the usual version of the log is not defined on the negative real axis. This does not need to be of concern however. Simply use another branch of the logarithm. Leave out the ray from 0 along the negative y axis and use Theorem 50.2.3 to define \( L(z) \) on this set. Thus \( L(z) = \ln |z| + i \arg_1(z) \) where \( \arg_1(z) \) will be the angle, \( \theta \), between \( -\frac{\pi}{2} \) and \( \frac{3\pi}{2} \) such that \( z = |z| e^{i\theta} \). Now the only singularities contained in this curve are

\[
\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2}, -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2}
\]

and the integrand, \( f \) has simple poles at these points. Thus using the same procedure as in the other examples,

\[
\text{Res} \left( f, \frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right) =
\]
and
\[
\text{Res} \left( f, -\frac{1}{2} \sqrt{2} + \frac{1}{2} i \sqrt{2} \right) = \frac{3}{32} \sqrt{2} \pi + \frac{3}{32} i \sqrt{2} \pi.
\]

Consider the integral along the small semicircle of radius \( r \). This reduces to
\[
\int_0^\pi \ln |r| + it \frac{1 + (re^{it})^4}{1 + t^4} (rei^t) \, dt
\]
which clearly converges to zero as \( r \to 0 \) because \( r \ln r \to 0 \). Therefore, taking the limit as \( r \to 0 \),
\[
\int_{\text{large semicircle}} \frac{L(z)}{1 + z^4} \, dz + \lim_{r \to 0^+} \int_{-R}^0 \frac{\ln(-t) + i\pi}{1 + t^4} \, dt + \\
\lim_{r \to 0^+} \int_r^R \frac{\ln t}{1 + t^4} \, dt = 2\pi i \left( \frac{3}{32} \sqrt{2} \pi + \frac{3}{32} i \sqrt{2} \pi + \frac{1}{32} \sqrt{2} \pi - \frac{1}{32} i \sqrt{2} \pi \right).
\]

Observing that \( \int_{\text{large semicircle}} \frac{L(z)}{1 + z^4} \, dz \to 0 \) as \( R \to \infty \),
\[
e(R) + 2 \lim_{r \to 0^+} \int_r^R \ln t \frac{1 + t^4}{1 + t^4} \, dt + i\pi \int_{-\infty}^0 \frac{1}{1 + t^4} \, dt = \left( -\frac{1}{8} + \frac{1}{4} i \right) \pi^2 \sqrt{2}
\]
where \( e(R) \to 0 \) as \( R \to \infty \). From an earlier example this becomes
\[
e(R) + 2 \lim_{r \to 0^+} \int_r^R \frac{\ln t}{1 + t^4} \, dt + i\pi \left( \frac{\sqrt{2}}{4} \pi \right) = \left( -\frac{1}{8} + \frac{1}{4} i \right) \pi^2 \sqrt{2}.
\]

Now letting \( r \to 0^+ \) and \( R \to \infty \),
\[
2 \int_0^\infty \frac{\ln t}{1 + t^4} \, dt = \left( -\frac{1}{8} + \frac{1}{4} i \right) \pi^2 \sqrt{2} - i\pi \left( \frac{\sqrt{2}}{4} \pi \right) = -\frac{1}{8} \sqrt{2} \pi^2,
\]
and so
\[
\int_0^\infty \frac{\ln t}{1 + t^4} \, dt = -\frac{1}{16} \sqrt{2} \pi^2,
\]
which is probably not the first thing you would think of. You might try to imagine how this could be obtained using elementary techniques.

The next example illustrates the use of what is referred to as a branch cut. It includes many examples.
Example 51.2.10 Mellin transformations are of the form

\[ \int_0^\infty f(x) x^\alpha \frac{dx}{x}. \]

Sometimes it is possible to evaluate such a transform in terms of the constant, \( \alpha \).

Assume \( f \) is an analytic function except at isolated singularities, none of which are on \((0, \infty)\). Also assume that \( f \) has the growth conditions,

\[ |f(z)| \leq \frac{C}{|z|^b}, \quad b > \alpha \]

for all large \( |z| \) and assume that

\[ |f(z)| \leq \frac{C'}{|z|^{b_1}}, \quad b_1 < \alpha \]

for all \( |z| \) sufficiently small. It turns out there exists an explicit formula for this Mellin transformation under these conditions. Consider the following contour.

In this contour the small semicircle in the center has radius \( \varepsilon \) which will converge to 0. Denote by \( \gamma_R \) the large circular path which starts at the upper edge of the slot and continues to the lower edge. Denote by \( \gamma_\varepsilon \) the small semicircular contour and denote by \( \gamma_{\varepsilon R^+} \) the straight part of the contour from 0 to \( R \) which provides the top edge of the slot. Finally denote by \( \gamma_{\varepsilon R^-} \) the straight part of the contour from \( R \) to 0 which provides the bottom edge of the slot. The interesting aspect of this problem is the definition of \( f(z) z^{\alpha-1} \). Let

\[ z^{\alpha-1} = e^{(\ln|z| + i\arg(z))(\alpha-1)} = e^{(\alpha-1)\log(z)} \]
where \( \text{arg}(z) \) is the angle of \( z \) in \((0, 2\pi)\). Thus you use a branch of the logarithm which is defined on \( \mathbb{C}\setminus(0, \infty) \). Then it is routine to verify from the assumed estimates that
\[
\lim_{R \to \infty} \int_{\gamma_R} f(z) \, z^{\alpha-1} \, dz = 0
\]
and
\[
\lim_{\varepsilon \to 0^+} \int_{\gamma_\varepsilon} f(z) \, z^{\alpha-1} \, dz = 0.
\]
Also, it is routine to verify
\[
\lim_{\varepsilon \to 0^+} \int_{\gamma_R+} f(z) \, z^{\alpha-1} \, dz = \int_0^R f(x) \, x^{\alpha-1} \, dx
\]
and
\[
\lim_{\varepsilon \to 0^+} \int_{\gamma_R-} f(z) \, z^{\alpha-1} \, dz = -e^{i2\pi(\alpha-1)} \int_0^R f(x) \, x^{\alpha-1} \, dx.
\]
Therefore, letting \( \Sigma_R \) denote the sum of the residues of \( f(z) \, z^{\alpha-1} \) which are contained in the disk of radius \( R \) except for the possible residue at 0,
\[
e(R) + \left(1 - e^{i2\pi(\alpha-1)}\right) \int_0^R f(x) \, x^{\alpha-1} \, dx = 2\pi i \Sigma_R
\]
where \( e(R) \to 0 \) as \( R \to \infty \). Now letting \( R \to \infty \),
\[
\lim_{R \to \infty} \int_0^R f(x) \, x^{\alpha-1} \, dx = \frac{2\pi i}{1 - e^{i2\pi(\alpha-1)}} \Sigma = \frac{\pi e^{-\pi i\alpha}}{\sin(\pi \alpha)} \Sigma
\]
where \( \Sigma \) denotes the sum of all the residues of \( f(z) \, z^{\alpha-1} \) except for the residue at 0.

The next example is similar to the one on the Mellin transform. In fact it is a Mellin transform but is worked out independently of the above to emphasize a slightly more informal technique related to the contour.

**Example 51.2.11** \( \int_0^\infty \frac{x^{p-1}}{1+x} \, dx, \quad p \in (0, 1) \).

Since the exponent of \( x \) in the numerator is larger than \(-1\). The integral does converge. However, the techniques of real analysis don’t tell us what it converges to. The contour to be used is as follows: From \((\varepsilon, 0)\) to \((r, 0)\) along the \( x \) axis and then from \((r, 0)\) to \((r, 0)\) counter clockwise along the circle of radius \( r \), then from \((r, 0)\) to \((\varepsilon, 0)\) along the \( x \) axis and from \((\varepsilon, 0)\) to \((\varepsilon, 0)\), clockwise along the circle of radius \( \varepsilon \). You should draw a picture of this contour. The interesting thing about this is that \( z^{p-1} \) cannot be defined all the way around 0. Therefore, use a branch of \( z^{p-1} \) corresponding to the branch of the logarithm obtained by deleting the positive \( x \) axis. Thus
\[
z^{p-1} = e^{i(\ln|z| + iA(z))(p-1)}
\]
where \( z = |z| e^{iA(z)} \) and \( A(z) \in (0, 2\pi) \). Along the integral which goes in the positive direction on the \( x \) axis, let \( A(z) = 0 \) while on the one which goes in the negative direction, take \( A(z) = 2\pi \). This is the appropriate choice obtained by replacing the line from \((\varepsilon, 0)\) to \((r, 0)\) with two lines having a small gap joined by a circle of radius \( \varepsilon \) and then taking a limit as the gap closes. You should verify that the two integrals taken along the circles of radius \( \varepsilon \) and \( r \) converge to 0 as \( \varepsilon \to 0 \) and as \( r \to \infty \). Therefore, taking the limit,

\[
\int_{0}^{\infty} \frac{x^{p-1}}{1+x} dx + \int_{0}^{\infty} \frac{x^{p-1}}{1+x} \left(e^{2\pi i (p-1)}\right) dx = 2\pi i \text{Res} (f, -1).
\]

Calculating the residue of the integrand at \(-1\), and simplifying the above expression,

\[
\left(1 - e^{2\pi i (p-1)}\right) \int_{0}^{\infty} \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)i\pi}.
\]

Upon simplification

\[
\int_{0}^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}.
\]

**Example 51.2.12** The Fresnel integrals are

\[
\int_{0}^{\infty} \cos (x^2) \, dx, \quad \int_{0}^{\infty} \sin (x^2) \, dx.
\]

To evaluate these integrals consider \( f(z) = e^{iz^2} \) on the curve which goes from the origin to the point \( r \) on the \( x \) axis and from this point to the point \( r \left(\frac{1+i}{\sqrt{2}}\right) \) along a circle of radius \( r \), and from there back to the origin as illustrated in the following picture.
Thus the curve to integrate over is shaped like a slice of pie. Denote by $\gamma_r$ the curved part. Since $f$ is analytic,

$$0 = \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{iz^2} dx - \int_0^r e^{i \left( \frac{1 + i}{\sqrt{2}} \right) t} \left( \frac{1 + i}{\sqrt{2}} \right) dt$$

$$= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{iz^2} dx - \int_0^r e^{-t^2} \left( \frac{1 + i}{\sqrt{2}} \right) dt$$

$$= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{iz^2} dx - \frac{\sqrt{\pi}}{2} \left( \frac{1 + i}{\sqrt{2}} \right) + e(r)$$

where $e(r) \to 0$ as $r \to \infty$. Here we used the fact that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. Now consider the first of these integrals.

$$\left| \int_{\gamma_r} e^{iz^2} dz \right| = \left| \int_0^\frac{\pi}{4} e^{i(\sqrt{2}r \sin t)^2} r e^{it} dt \right|$$

$$\leq r \int_0^\frac{\pi}{4} e^{-r^2 \sin^2 t} dt$$

$$= r \int_0^1 e^{-ru^2} \sqrt{1 - u^2} du$$

$$\leq \frac{r}{2} \int_0^{r^{-3/2}} \frac{1}{\sqrt{1 - u^2}} du + \frac{r}{2} \left( \int_0^1 \frac{1}{\sqrt{1 - u^2}} \right) e^{-r^{3/2}}$$

which converges to zero as $r \to \infty$. Therefore, taking the limit as $r \to \infty$,

$$\frac{\sqrt{\pi}}{2} \left( \frac{1 + i}{\sqrt{2}} \right) = \int_0^\infty e^{iz^2} dx$$

and so

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2 dx.$$
The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for \( \cot \pi z \) known as the Mittag-Leffler expansion of \( \cot \pi z \).

Example 51.2.13 Let \( \gamma_N \) be the contour which goes from \(-N - \frac{1}{2} - Ni\) horizontally to \(N + \frac{1}{2} + Ni\) and from there, vertically to \(N + \frac{1}{2} + Ni\) and then horizontally to \(-N - \frac{1}{2} - Ni\) and finally vertically to \(-N - \frac{1}{2} - Ni\). Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction.

Consider the following integral.

\[
I_N = \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z (\alpha^2 - z^2)} \, dz
\]

where \( \alpha \in \mathbb{R} \) is not an integer. This will be used to verify the formula of Mittag-Leffler,

\[
\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \frac{2}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.
\] (51.2.12)

You should verify that \( \cot \pi z \) is bounded on this contour and that therefore, \( I_N \to 0 \) as \( N \to \infty \). Now you compute the residues of the integrand at \( \pm \alpha \) and at \( n \) where \( |n| < N + \frac{1}{2} \) for \( n \) an integer. These are the only singularities of the integrand in this contour and therefore, you can evaluate \( I_N \) by using these. It is left as an exercise to calculate these residues and find that the residue at \( \pm \alpha \) is

\[
-\frac{\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}
\]

while the residue at \( n \) is

\[
\frac{1}{\alpha^2 - n^2}.
\]

Therefore,

\[
0 = \lim_{N \to \infty} I_N = \lim_{N \to \infty} 2\pi i \left[ \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi \alpha}{\alpha} \right]
\]

which establishes the following formula of Mittag-Leffler.

\[
\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.
\]

Writing this in a slightly nicer form, yields 51.2.12.

51.3 Exercises

1. Example 51.2.7 found the integral of a rational function of a certain sort. The technique used in this example typically works for rational functions of the
form \( \frac{f(x)}{g(x)} \) where \( \deg(g(x)) \geq \deg(f(x)) + 2 \) provided the rational function has no poles on the real axis. State and prove a theorem based on these observations.

2. Fill in the missing details of Example 51.2.13 about \( I_N \to 0 \). Note how important it was that the contour was chosen just right for this to happen. Also verify the claims about the residues.

3. Suppose \( f \) has a pole of order \( m \) at \( z = a \). Define \( g(z) \) by
\[
g(z) = (z-a)^m f(z).
\]
Show
\[
\text{Res}(f,a) = \frac{1}{(m-1)!} g^{(m-1)}(a).
\]
\textbf{Hint:} Use the Laurent series.

4. Give a proof of Theorem 51.1.1. \textbf{Hint:} Let \( p \) be a pole. Show that near \( p \), a pole of order \( m \),
\[
\frac{f'(z)}{f(z)} = \frac{-m + \sum_{k=1}^{\infty} b_k (z-p)^k}{(z-p) + \sum_{k=2}^{\infty} c_k (z-p)^k}
\]
Show that \( \text{Res}(f,p) = -m \). Carry out a similar procedure for the zeros.

5. Use Rouche’s theorem to prove the fundamental theorem of algebra which says that if \( p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \), then \( p \) has \( n \) zeros in \( \mathbb{C} \). \textbf{Hint:} Let \( q(z) = -z^n \) and let \( \gamma \) be a large circle, \( \gamma(t) = re^{it} \) for \( r \) sufficiently large.

6. Consider the two polynomials \( z^5 + 3z^2 - 1 \) and \( z^5 + 3z^2 \). Show that on \( |z| = 1 \), the conditions for Rouche’s theorem hold. Now use Rouche’s theorem to verify that \( z^5 + 3z^2 - 1 \) must have two zeros in \( |z| < 1 \).

7. Consider the polynomial, \( z^{11} + 7z^5 + 3z^2 - 17 \). Use Rouche’s theorem to find a bound on the zeros of this polynomial. In other words, find \( r \) such that if \( z \) is a zero of the polynomial, \( |z| < r \). Try to make \( r \) fairly small if possible.

8. Verify that \( \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \). \textbf{Hint:} Use polar coordinates.

9. Use the contour described in Example 51.2.7 to compute the exact values of the following improper integrals.

\begin{align*}
(\text{a}) \quad & \int_{-\infty}^{\infty} \frac{x}{(x^2+2x+13)^2} \, dx \\
(\text{b}) \quad & \int_0^{\infty} \frac{x^2}{(x^2+a^2)^2} \, dx \\
(\text{c}) \quad & \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, \quad a, b > 0
\end{align*}
10. Evaluate the following improper integrals.
   (a) \( \int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} \, dx \)
   (b) \( \int_0^\infty \frac{x \sin x}{(x^2+a^2)^2} \, dx \)

11. Find the Cauchy principle value of the integral
\[
\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} \, dx
\]
defined as
\[
\lim_{\epsilon \to 0^+} \left( \int_{-\infty}^{1-\epsilon} \frac{\sin x}{(x^2+1)(x-1)} \, dx + \int_{1+\epsilon}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} \, dx \right).
\]

12. Find a formula for the integral \( \int_{-\infty}^{\infty} \frac{x \, dx}{(1+x^2)^{n+1}} \) where \( n \) is a nonnegative integer.

13. Find \( \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^4} \, dx \).

14. If \( m < n \) for \( m \) and \( n \) integers, show
\[
\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}} \, dx = \frac{\pi}{2n} \frac{1}{\sin \left( \frac{2m+1}{2n} \pi \right)}.
\]

15. Find \( \int_{-\infty}^{\infty} \frac{1}{(1+x^4)^2} \, dx \).

16. Find \( \int_0^{\infty} \frac{\ln(x)}{1+x^2} \, dx = 0 \).

17. Suppose \( f \) has an isolated singularity at \( \alpha \). Show the singularity is essential if and only if the principal part of the Laurent series of \( f \) has infinitely many terms. That is, show \( f(z) = \sum_{k=0}^{\infty} a_k (z-\alpha)^k + \sum_{k=1}^{\infty} b_k \frac{1}{(z-\alpha)^k} \) where infinitely many of the \( b_k \) are nonzero.

18. Suppose \( \Omega \) is a bounded open set and \( f_n \) is analytic on \( \Omega \) and continuous on \( \overline{\Omega} \). Suppose also that \( f_n \to f \) uniformly on \( \overline{\Omega} \) and that \( f \neq 0 \) on \( \partial \Omega \). Show that for all \( n \) large enough, \( f_n \) and \( f \) have the same number of zeros on \( \Omega \) provided the zeros are counted according to multiplicity.
Chapter 52

Some Important Functional Analysis Applications

52.1 The Spectral Radius Of A Bounded Linear Transformation

As a very important application of the theory of Laurent series, I will give a short description of the spectral radius. This is a fundamental result which must be understood in order to prove convergence of various important numerical methods such as the Gauss Seidel or Jacobi methods.

Definition 52.1.1 Let $X$ be a complex Banach space and let $A \in \mathcal{L}(X, X)$. Then

$$r(A) \equiv \left\{ \lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X, X) \right\}$$

This is called the resolvent set. The spectrum of $A$, denoted by $\sigma(A)$ is defined as all the complex numbers which are not in the resolvent set. Thus

$$\sigma(A) \equiv \mathbb{C} \setminus r(A)$$

Lemma 52.1.2 $\lambda \in r(A)$ if and only if $\lambda I - A$ is one to one and onto $X$. Also if $|\lambda| > \|A\|$, then $\lambda \in \sigma(A)$. If the Neumann series,

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k$$

converges, then

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k = (\lambda I - A)^{-1}.$$
Proof: Note that to be in \( r(A) \), \( \lambda I - A \) must be one to one and map \( X \) onto \( X \) since otherwise, \((\lambda I - A)^{-1} \notin \mathcal{L}(X,X)\).

By the open mapping theorem, if these two algebraic conditions hold, then \((\lambda I - A)^{-1}\) is continuous and so this proves the first part of the lemma. Now suppose \( |\lambda| > ||A|| \). Consider the Neumann series

\[
\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k.
\]

By the root test, Theorem 49.1.3 on Page 1648 this series converges to an element of \( \mathcal{L}(X,X) \) denoted here by \( B \). Now suppose the series converges. Letting \( B_n \equiv \frac{1}{\lambda} \sum_{k=0}^{n} \left( \frac{A}{\lambda} \right)^k \),

\[
(\lambda I - A)B_n = B_n (\lambda I - A) = \sum_{k=0}^{n} \left( \frac{A}{\lambda} \right)^k - \sum_{k=0}^{n} \left( \frac{A}{\lambda} \right)^{k+1}
\]

\[
= I - \left( \frac{A}{\lambda} \right)^{n+1} \rightarrow I
\]

as \( n \to \infty \) because the convergence of the series requires the \( n^{th} \) term to converge to 0. Therefore,

\[
(\lambda I - A)B = B (\lambda I - A) = I
\]

which shows \( \lambda I - A \) is both one to one and onto and the Neumann series converges to \((\lambda I - A)^{-1}\). This proves the lemma.

This lemma also shows that \( \sigma(A) \) is bounded. In fact, \( \sigma(A) \) is closed.

Lemma 52.1.3 \( r(A) \) is open. In fact, if \( \lambda \in r(A) \) and \( |\mu - \lambda| < \left\| (\lambda I - A)^{-1} \right\|^{-1} \), then \( \mu \in r(A) \).

Proof: First note

\[
(\mu I - A) = \left( I - (\lambda - \mu) (\lambda I - A)^{-1} \right)(\lambda I - A) \quad (52.1.1)
\]

\[
= (\lambda I - A) \left( I - (\lambda - \mu) (\lambda I - A)^{-1} \right) \quad (52.1.2)
\]

Also from the assumption about \( |\lambda - \mu| \),

\[
\left\| (\lambda - \mu) (\lambda I - A)^{-1} \right\| \leq |\lambda - \mu| \left\| (\lambda I - A)^{-1} \right\| < 1
\]

and so by the root test,

\[
\sum_{k=0}^{\infty} \left( (\lambda - \mu) (\lambda I - A)^{-1} \right)^k
\]
converges to an element of $\mathcal{L}(X,X)$. As in Lemma 52.1.2,

$$\sum_{k=0}^{\infty} \left( (\lambda - \mu) (\lambda I - A)^{-1} \right)^k = \left( I - (\lambda - \mu) (\lambda I - A)^{-1} \right)^{-1}.$$ 

Therefore, from Lemma 52.1.1,

$$(\mu I - A)^{-1} = (\lambda I - A)^{-1} \left( I - (\lambda - \mu) (\lambda I - A)^{-1} \right)^{-1}.$$ 

This proves the lemma.

**Corollary 52.1.4** $\sigma(A)$ is a compact set.

**Proof:** Lemma 52.1.2 shows $\sigma(A)$ is bounded and Lemma 52.1.3 shows it is closed.

**Definition 52.1.5** The spectral radius, denoted by $\rho(A)$ is defined by

$$\rho(A) \equiv \max \{ |\lambda| : \lambda \in \sigma(A) \}.$$ 

Since $\sigma(A)$ is compact, this maximum exists. Note from Lemma 52.1.2, $\rho(A) \leq ||A||$.

There is a simple formula for the spectral radius.

**Lemma 52.1.6** If $|\lambda| > \rho(A)$, then the Neumann series,

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k$$

converges.

**Proof:** This follows directly from Theorem 51.2.6 on Page 1723 and the observation above that $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k = (\lambda I - A)^{-1}$ for all $|\lambda| > ||A||$. Thus the analytic function, $\lambda \rightarrow (\lambda I - A)^{-1}$ has a Laurent expansion on $|\lambda| > \rho(A)$ by Theorem 51.2.6 and it must coincide with $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k$ on $|\lambda| > ||A||$ so the Laurent expansion of $\lambda \rightarrow (\lambda I - A)^{-1}$ must equal $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left( \frac{A}{\lambda} \right)^k$ on $|\lambda| > \rho(A)$. This proves the lemma.

The theorem on the spectral radius follows. It is due to Gelfand.

**Theorem 52.1.7** $\rho(A) = \lim_{n \to \infty} ||A^n||^{1/n}$.

**Proof:** If $|\lambda| < \lim \sup_{n \to \infty} ||A^n||^{1/n}$
then by the root test, the Neumann series does not converge and so by Lemma 52.1.6 \( |\lambda| \leq \rho(A) \). Thus
\[
\rho(A) \geq \lim{\sup{}}_{n\to\infty} \|A^n\|^{1/n}.
\]

Now let \( p \) be a positive integer. Then \( \lambda \in \sigma(A) \) implies \( \lambda^p \in \sigma(A^p) \) because
\[
\lambda^p I - A^p = (\lambda I - A) (\lambda^{p-1} + \lambda^{p-2} A + \cdots + A^{p-1}) = (\lambda^{p-1} + \lambda^{p-2} A + \cdots + A^{p-1}) (\lambda I - A)
\]
It follows from Lemma 52.1.2 applied to \( A^p \) that for \( \lambda \in \sigma(A), |\lambda^p| \leq \|A^p\| \) and so \( |\lambda| \leq \|A^p\|^{1/p} \). Therefore, \( \rho(A) \leq \|A^p\|^{1/p} \) and since \( p \) is arbitrary,
\[
\lim{\inf{}}_{p\to\infty} \|A^p\|^{1/p} \geq \rho(A) \geq \lim{\sup{}}_{n\to\infty} \|A^n\|^{1/n}.
\]

This proves the theorem.

## 52.2 Analytic Semigroups

### 52.2.1 Sectorial Operators And Analytic Semigroups

With the theory of functions of a complex variable, it is time to consider the notion of analytic semigroups. These are better than continuous semigroups. I am mostly following the presentation in Henry [57]. In what follows \( H \) will be a Banach space unless specified to be a Hilbert space.

**Definition 52.2.1** Let \( \phi < \pi/2 \) and for \( a \in \mathbb{R} \), let \( S_{a\phi} \) denote the sector in the complex plane
\[
\{ z \in \mathbb{C} \setminus \{a\} : |\arg(z - a)| \leq \pi - \phi \}
\]
This sector is as shown below.
A closed, densely defined linear operator, $A$ is called sectorial if for some sector as described above, it follows that for all $\lambda \in S_{a\phi}$,

$$(\lambda I - A)^{-1} \in \mathcal{L}(H,H)$$

and for some $M$ it satisfies

$$\| (\lambda I - A)^{-1} \| \leq \frac{M}{|\lambda - a|}$$

To begin with it is interesting to have a perturbation theorem for sectorial operators. First note that for $\lambda \in S_{a\phi}$,

$$A(\lambda I - A)^{-1} = -I + \lambda (\lambda I - A)^{-1}$$

**Proposition 52.2.2** Suppose $A$ is a sectorial operator as defined above so it is a densely defined closed operator on $D(A) \subseteq H$ which satisfies

$$\| A(\lambda I - A)^{-1} \| \leq C \quad (52.2.3)$$

whenever $|\lambda|, \lambda \in S_{a\phi}$, is sufficiently large and suppose $B$ is a densely defined closed operator such that $D(B) \supseteq D(A)$ and for all $x \in D(A)$,

$$\| Bx \| \leq \varepsilon \| Ax \| + K \| x \| \quad (52.2.4)$$

where $\varepsilon C < 1$. Then $A + B$ is also sectorial.

**Proof:** I need to consider $(\lambda I - (A + B))^{-1}$. This equals

$$\left( \left( I - B(\lambda I - A)^{-1} \right)(\lambda I - A) \right)^{-1}. \quad (52.2.5)$$

The issue is whether this makes any sense for all $\lambda \in S_{b\phi}$ for some $b \in \mathbb{R}$. Let $b > a$ be very large so that if $\lambda \in S_{b\phi}$, then (52.2.3) holds. Then from (52.2.4) it follows that for $\| x \| \leq 1$,

$$\| (\lambda I - A)^{-1} x \| \leq \varepsilon \| A(\lambda I - A)^{-1} x \| + K \| (\lambda I - A)^{-1} x \| \leq \varepsilon C + K/|\lambda - a|$$

and so if $b$ is made still larger, it follows this is less than $r < 1$ for all $\| x \| \leq 1$. Therefore, for such $b$,

$$\left( I - B(\lambda I - A)^{-1} \right)^{-1}$$

exists and so for such $b$, the expression in (52.2.5) makes sense and equals

$$(\lambda I - A)^{-1} \left( I - B(\lambda I - A)^{-1} \right)^{-1}$$
and furthermore,

\[
\left\| \left( \lambda I - A \right)^{-1} \left( I - B \left( \lambda I - A \right)^{-1} \right)^{-1} \right\| \leq \frac{M}{|\lambda - a|} \frac{1}{1 - r} \leq \frac{M'}{|\lambda - b|}
\]

by adjusting the constants because

\[
\frac{M}{|\lambda - a|} \frac{|\lambda - b|}{1 - r}
\]

is bounded for \( \lambda \in S_{b,\phi} \). This proves the proposition.

It is an interesting proposition because when you have compact embeddings, such inequalities tend to hold.

**Definition 52.2.3** Let \( \varepsilon > 0 \) and for a sectorial operator as defined above, let the contour \( \gamma_{\varepsilon,\phi} \) be as shown next where the orientation is also as shown by the arrow.

The little circle has radius \( \varepsilon \) in the above contour.

**Definition 52.2.4** For \( t \in S_{0(\phi + \pi/2)}^0 \) the open sector shown in the following picture,
Define
\[ S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\phi}} e^{\lambda t} (\lambda I - A)^{-1} d\lambda \] (52.2.6)
where \( \varepsilon \) is chosen such that \( t \) is a positive distance from the set of points included in \( \gamma_{\varepsilon,\phi} \). The situation is described by the following picture which shows \( S_{0(\phi+\pi/2)} \) and \( S_{0\phi} \). Note how the dotted line is at right angles to the solid line.

Also define \( S(0) \equiv I \). It isn’t necessary that \( \varepsilon \) be small, just that \( \gamma_{\varepsilon,\phi} \) not contain \( t \). This is because the integrand in (52.2.6) is analytic.

Then it is necessary to show the above definition is well defined.

**Lemma 52.2.5** The above definition is well defined for \( t \in S_{0(\phi+\pi/2)} \). Also there is a constant, \( M_r \) such that
\[ ||S(t)|| \leq M_r \]
for every \( t \in S_{0(\phi+\pi/2)} \) such that \( |\arg t| \leq r < \left( \frac{\pi}{2} - \phi \right) \).
Proof: In the definition of $S(t)$ one can take $\varepsilon = 1/|t|$. Then on the little circle which is part of $\gamma_{\varepsilon,\phi}$ the contour integral equals

$$\frac{1}{2\pi} \int_{\phi - \pi}^{\pi - \phi} e^{i(\theta + \text{arg}(t))} \left( \frac{1}{|t|} e^{i\theta} - A \right)^{-1} \frac{1}{|t|} e^{i\theta} d\theta$$

and by assumption the norm of the integrand is no larger than

$$M \frac{1}{|t|}$$

and so the norm of this integral is dominated by

$$\frac{M}{2\pi} \int_{\phi - \pi}^{\pi - \phi} d\theta = M \frac{2\pi}{2\pi} (2\pi - 2\phi) \leq M$$

which is independent of $t$.

Now consider the part of the contour used to define $S(t)$ which is the top line segment. This equals

$$\frac{1}{2\pi i} \int_{1/|t|}^{\infty} e^{yw} (ywI - A)^{-1} w dy$$

where $w$ is a fixed complex number of unit length which gives a direction for motion along this segment, $\text{arg}(w) = \pi - \phi$. Then the norm of this is dominated by

$$\frac{1}{2\pi} \int_{1/|t|}^{\infty} |e^{yw}| \frac{M}{y} dy = \frac{1}{2\pi} \int_{1/|t|}^{\infty} \exp(y |t| \cos(\text{arg}(w) + \text{arg}(t))) \frac{M}{y} dy$$

By assumption $|\text{arg}(t)| \leq r < (\frac{\pi}{2} - \phi)$ and so

$$\text{arg}(w) + \text{arg}(t) \geq \frac{\pi}{2} + \left( \frac{\pi}{2} - \phi \right) - \frac{\pi}{2} + \delta(r), \quad \delta(r) > 0.$$ 

It follows the integral dominated by

$$\frac{1}{2\pi} \int_{1/|t|}^{\infty} \exp(-c(r) |t| y) \frac{M}{y} dy$$

$$= \frac{1}{2\pi} \int_{1}^{\infty} \exp(-c(r) x) \frac{M |t|}{x} \frac{1}{|t|} dx$$

$$= \frac{1}{2\pi} \int_{1}^{\infty} \exp(-c(r) x) \frac{M}{x} dx$$

where $c(r) < 0$ independent of $|\text{arg}(t)| \leq r$. A similar estimate holds for the integral on the bottom segment. Thus for $|\text{arg}(t)| \leq r, ||S(t)||$ is bounded. This proves the Lemma.
Also note that if the contour is shifted to the right slightly, the integral over the shifted contour, \( \gamma'_{\varepsilon, \phi} \), coincides with the integral over \( \gamma_{\varepsilon, \phi} \) thanks to the Cauchy integral formula and an approximation argument involving truncating the infinite contours and joining them at either end. Also note that in particular, \( ||S(t)|| \) is bounded for all positive real \( t \). The following is the main result.

**Theorem 52.2.6** Let \( A \) be a sectorial operator as defined in Definition 52.2.1 for the sector \( S_{0,\phi} \).

1. Then \( S(t) \) given above in 52.2.6 is analytic for \( t \in S^{0}_{0(\phi+\pi/2)} \).
2. For any \( x \in H \) and \( t > 0 \), then for \( n \) a positive integer,
   \[
   S^{(n)}(t)x = A^nS(t)x
   \]
3. \( S \) is a semigroup on the open sector, \( S^{0}_{0(\phi+\pi/2)} \). That is, for all \( t, s \in S^{0}_{0(\phi+\pi/2)} \),
   \[
   S(t+s) = S(t)S(s)
   \]
4. \( t \to S(t)x \) is continuous at \( t = 0 \) for all \( x \in H \).
5. For some constants \( M, N \) such that if \( t \) is positive and real,
   \[
   ||S(t)|| \leq M
   \]
   \[
   ||AS(t)|| \leq \frac{N}{t}
   \]

**Proof:** Consider the first claim. This follows right away from the formula.

\[
S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} (\lambda I - A)^{-1} d\lambda
\]

The estimates for uniform convergence do not change for small changes in \( t \) and so the formula can be differentiated with respect to the complex variable \( t \) using the dominated convergence theorem to obtain

\[
S'(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} \lambda e^{\lambda t} (\lambda I - A)^{-1} d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} \left( I + A(\lambda I - A)^{-1} \right) d\lambda
\]

\[
= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A(\lambda I - A)^{-1} d\lambda
\]
because of the Cauchy integral theorem and an approximation result. Now approximating the infinite contour with a finite one and then the integral with Riemann sums, one can use the fact \( A \) is closed to take \( A \) out of the integral and write

\[
S'(t) = A \left( \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,0}} e^{\lambda t} (\lambda I - A)^{-1} d\lambda \right) = AS(t)
\]

To get the higher derivatives, note \( S(t) \) has infinitely many derivatives due to \( t \) being a complex variable. Therefore,

\[
S''(t) = \lim_{h \to 0} \frac{S'(t + h) - S'(t)}{h} = \lim_{h \to 0} A \frac{S(t + h) - S(t)}{h}
\]

and

\[
\frac{S(t + h) - S(t)}{h} \to AS(t)
\]

and so since \( A \) is closed, \( AS(t) \in D(A) \) and

\[
A^2 S(t)
\]

Continuing this way yields the claims 1.) and 2.). Note this also implies \( S(t) x \in D(A) \) for each \( t \in S^0_{0(\phi + \pi/2)} \).

Next consider the semigroup property. Let \( s,t \in S^0_{0(\phi + \pi/2)} \) and let \( \varepsilon \) be sufficiently small that \( \gamma_{\varepsilon,0} \) is at a positive distance from both \( s \) and \( t \). As described above let \( \gamma'_{\varepsilon,0} \) denote the contour shifted slightly to the right, still at a positive distance from \( t \). Then

\[
S(t) S(s) = \left( \frac{1}{2\pi i} \right)^2 \int_{\gamma_{\varepsilon,0}} \int_{\gamma'_{\varepsilon,0}} e^{\lambda t} (\lambda I - A)^{-1} e^{\mu s} (\mu I - A)^{-1} d\mu d\lambda
\]

At this point note that

\[
(\lambda I - A)^{-1} (\mu I - A)^{-1} = (\mu - \lambda)^{-1} (\lambda I - A)^{-1} - (\mu I - A)^{-1}
\]

Then substituting this in the integrals above, it equals

\[
\left( \frac{1}{2\pi i} \right)^2 \int_{\gamma_{\varepsilon,0}} \int_{\gamma'_{\varepsilon,0}} e^{\mu s} e^{\lambda t} (\mu - \lambda)^{-1} (\lambda I - A)^{-1} - (\mu I - A)^{-1} d\mu d\lambda
\]

\[
= - \left( \frac{1}{2\pi i} \right)^2 \int_{\gamma_{\varepsilon,0}} e^{\lambda t} \int_{\gamma'_{\varepsilon,0}} e^{\mu s} (\mu - \lambda)^{-1} (\mu I - A)^{-1} d\mu d\lambda
\]

\[
+ \left( \frac{1}{2\pi i} \right)^2 \int_{\gamma_{\varepsilon,0}} \int_{\gamma'_{\varepsilon,0}} e^{\mu s} e^{\lambda t} (\mu - \lambda)^{-1} (\lambda I - A)^{-1} d\mu d\lambda
\]
The order of integration can be interchanged because of the absolute convergence and Fubini's theorem. Then this reduces to

\[
- \left( \frac{1}{2\pi i} \right)^2 \int_{\gamma_{\varepsilon,\phi}} (\mu I - A)^{-1} e^{\mu s} \int_{\gamma_{\varepsilon,\phi}} e^{\lambda t} (\mu - \lambda)^{-1} d\lambda d\mu \\
+ \left( \frac{1}{2\pi i} \right)^2 \int_{\gamma_{\varepsilon,\phi}} (\lambda I - A)^{-1} e^{\lambda t} \int_{\gamma_{\varepsilon,\phi}} e^{\mu s} (\mu - \lambda)^{-1} d\mu d\lambda
\]

Now the following diagram might help in drawing some interesting conclusions.

The first iterated integral equals 0. This can be seen from the above picture. The inner integral taken over \(\gamma_{\varepsilon,\phi}\) is essentially equal to the integral over the closed contour in the above picture provided the radius of the part of the circle in the above closed contour is large enough. This closed contour integral equals 0 by the Cauchy integral theorem. The second iterated integral equals

\[
\frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\phi}} (\lambda I - A)^{-1} e^{\lambda t} e^{\lambda s} d\lambda
\]

This can be seen from considering a similar closed contour, using the Cauchy integral formula. This verifies the semigroup identity.

Next consider 4.), the continuity at \(t = 0\). This requires showing

\[
\lim_{t \to 0^+} S(t) x = x
\]

where the limit is taken through positive real values of \(t\). It suffices to let \(x \in D(A)\) because by Lemma 52.2.5, \(\|S(t)\|\) is bounded for these values of \(t\). Also in doing the computation, let \(\gamma_{\varepsilon,\phi}\) equal \(\gamma_{t^{-1},\phi}\) and it will be assumed \(t < 1\). Then one must estimate

\[
\|S(t) x - x\| = \cdots
\]
By the Cauchy integral formula and approximating the integral with a contour integral over a finite closed contour,

\[ \frac{1}{2\pi i} \int_{\gamma_{t-1, \phi}} e^{\lambda t} \left( (\lambda I - A)^{-1} - I \lambda^{-1} \right) x d\lambda = 1 \]

and so equals

\[ \frac{1}{2\pi i} \int_{\gamma_{t-1, \phi}} e^{\lambda t} \lambda x d\lambda = 1 \]

Changing the variable letting \( \lambda t = \mu \), the above equals

\[ \frac{1}{2\pi i} \int_{\gamma_{t, \phi}} e^\mu ((\mu/t) I - A)^{-1} Ax \frac{1}{t} d\mu \]

which is dominated by

\[ \frac{1}{2\pi} \int_{\gamma_{1, \phi}} e^{\mu/|\cos \psi|} t^2 M \frac{1}{|\mu|^2} ||Ax|| \frac{1}{t} d|\mu| = tC ||Ax||. \]

Therefore, \( \lim_{t \to 0^+} ||S(t)x - x|| = 0 \) whenever \( x \in D(A) \). Since \( S(t) \) is bounded for positive \( t \), if \( y \in H \) is arbitrary, choose \( x \in D(A) \) close enough to \( y \) such that

\[ ||S(t)y - y|| \leq ||S(t)(y - x)|| + ||S(t)x - x|| + ||x - y|| \]
\[ \leq \varepsilon/2 + ||S(t)x - x|| \]

and now if \( t \) is close enough to \( 0 \) it follows the right side is less than \( \varepsilon \). This proves 4).

Finally consider 5.). The first part follows from Lemma 52.2.5. It remains to show the second part. Let \( x \in H \). First suppose \( t \neq 1 \).

\[ ||AS(t)x|| = \left| \frac{1}{2\pi i} \int_{\gamma_{t-1, \phi}} e^{\lambda t} A(\lambda I - A)^{-1} x d\lambda \right| \]

\[ = \left| \frac{1}{2\pi i} \int_{\gamma_{t-1, \phi}} e^{\lambda t} \left( -I + \lambda (\lambda I - A)^{-1} \right) x d\lambda \right| \]

\[ = \left| \frac{1}{2\pi i} \int_{\gamma_{t-1, \phi}} e^{\lambda t} \lambda (\lambda I - A)^{-1} x d\lambda \right| \]

\[ = \left| \frac{1}{2\pi i} \int_{\gamma_{1, \phi}} e^{\mu \frac{\mu}{t}} \left( \frac{\mu}{t} I - A \right)^{-1} x \frac{1}{t} d\mu \right| \]
and this is dominated by
\[
\leq \frac{1}{2\pi} \int_{\gamma_1,\phi} e^{\mu|\cos \psi} Md |\mu| \frac{1}{t} = \frac{N}{t}.
\]
This proves the theorem.

What if $A$ is sectorial in the more general sense with $S_{a\phi}$ taking the place of $S_{0\phi}$ and the resolvent estimate being
\[
\left\| (\lambda I - A)^{-1} \right\| \leq \frac{M}{|\lambda - a|}, \lambda \in S_{a\phi}.
\]
Then $S_{a\phi} = a + S_{0\phi}$ and so for $\lambda \in S_{0\phi}$, $(\lambda + a) \in S_{a\phi}$ and so
\[
(\lambda I - (A - aI))^{-1} = ((\lambda + a) I - A)^{-1} \in \mathcal{L}(H, H)
\]
and
\[
\left\| (\lambda I - (A - aI))^{-1} \right\| = \left\| ((\lambda + a) - A)^{-1} \right\|
\]
\[
\leq \frac{M}{|\lambda + a - a|} = \frac{M}{|\lambda|}
\]
Therefore, letting $A_a \equiv A - aI$, it follows the result of Theorem 52.2.6 holds for this operator. There exists a semigroup, $S_a(t)$ satisfying

1. $S_a(t)$ is analytic for $t \in S_{0(\phi+\pi/2)}$.
2. For any $x \in H$ and $t > 0$, then for $n$ a positive integer,
   \[
   S_a^{(n)}(t) x = A^n_a S_a(t) x
   \]
3. $S_a$ is a semigroup on the open sector, $S_{0(\phi+\pi/2)}$. That is, for all $t, s \in S_{0(\phi+\pi/2)}$,
   \[
   S_a(t + s) = S_a(t) S_a(s)
   \]
4. $t \to S_a(t) x$ is continuous at $t = 0$ for all $x \in H$.
5. For some constants $M, N$ such that if $t$ is positive and real,
   \[
   \| S_a(t) \| \leq M
   \]
   \[
   \| A_a S_a(t) \| \leq \frac{N}{t}
   \]
Define
\[
S(t) \equiv e^{at} S_a(t).
\]
This satisfies the semigroup identity, is continuous at 0, and satisfies the inequality
\[
\| S(t) \| \leq M e^{at}
\]
for all \( t \) real and nonnegative. What is its derivative?

\[
S'(t) = a e^{at} S_a(t) + (A - aI) e^{at} S_a(t)
\]

and by continuing this way using \( A \) is closed as before, it follows

\[
S^{(n)}(t) = A^n S(t).
\]

Also for \( t > 0 \),

\[
||AS(t)|| = ||e^{at} AS_a(t)||
\]

\[
= ||e^{at} (A - aI) S_a(t) + e^{at} aS_a(t)||
\]

\[
= ||e^{at} A_s(t) + e^{at} aS_a(t)||
\]

\[
\leq e^{at} N + ae^{at} M = e^{at} \left( \frac{N}{t} + aM \right).
\]

It is clear \( S(t)x \) is continuous at 0 since this is true of \( S_a(t) \).

This proves the following corollary which is a useful generalization of the above major theorem.

**Corollary 52.2.7** Let \( A \) be sectorial satisfying

\[
(\lambda I - A)^{-1} \in L(H, H) \text{ for } \lambda \in S_{a\phi}
\]

for \( \phi < \pi/2 \) and

\[
|| (\lambda I - A)^{-1} || \leq \frac{M}{|\lambda - a|}
\]

for all \( \lambda \in S_{a\lambda} \). Then there exists a semigroup, \( S(t) \) defined on \( S^0_{0(\phi+\pi/2)} \) which satisfies

1. \( S(t) \) is analytic for \( t \in S^0_{0(\phi+\pi/2)} \).
2. For any \( x \in H \) and \( t > 0 \), then for \( n \) a positive integer,

\[
S^{(n)}(t)x = A^n S(t)x
\]

3. \( S(t) \) is a semigroup on the open sector, \( S^0_{0(\phi+\pi/2)} \). That is, for all \( t, s \in S^0_{0(\phi+\pi/2)} \),

\[
S(t + s) = S(t) S(s)
\]

4. \( t \rightarrow S(t)x \) is continuous at \( t = 0 \) for all \( x \in H \).
5. For some constants \( M, N \) such that if \( t \) is positive and real,

\[
||S(t)|| \leq Me^{at}
\]

\[
||AS(t)|| \leq e^{at} \left( \frac{N}{t} + aM \right)
\]
52.2. THE NUMERICAL RANGE

In Hilbert space, there is a useful easy to check criterion which implies an operator is sectorial.

**Definition 52.2.8** Let $A$ be a closed densely defined operator $A : D(A) \to H$ for $H$ a Hilbert space. The numerical range is the following set.

$$\{(Au, u) : u \in D(A)\}$$

Also recall the resolvent set, $r(A)$ consists of those $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$. Thus, to be in this set $\lambda I - A$ is one to one and onto with continuous inverse.

**Proposition 52.2.9** Suppose the numerical range of $A$, a closed densely defined operator $A : D(A) \to H$ for $H$ a Hilbert space is contained in the set

$$\{z \in \mathbb{C} : |\arg(z)| \geq \pi - \phi\}$$

where $0 < \phi < \pi/2$ and suppose $A^{-1} \in \mathcal{L}(H, H), (0 \in r(A))$. Then $A$ is sectorial with the sector $S_{0,\phi'}$ where $\pi/2 > \phi' > \phi$.

**Proof:** Here is a picture of the situation along with details used to motivate the proof.

\[
\begin{align*}
(Tu, u) & \\
\phi & \\
A & \\
& \\
\end{align*}
\]

In the picture the angle which is a little larger than $\phi$ is $\phi'$. Let $\lambda$ be as shown with $|\arg \lambda| \leq \pi - \phi'$. Then from the picture and trigonometry, if $u \in D(A)$,

$$|\lambda| \sin (\phi' - \phi) < \left| \lambda - \left(A \frac{u}{|u|}, \frac{u}{|u|}\right) \right|$$

and so

$$|u| |\lambda| \sin (\phi' - \phi) < \left| \left(\lambda u - Au, \frac{u}{|u|}\right) \right| \leq \|(\lambda I - A) u\|$$

Hence for all \( \lambda \) such that \( |\arg \lambda| \leq \pi - \phi' \) and \( u \in D(A) \),

\[
|u| < \left( \frac{1}{\sin (\phi' - \phi)} \right) \frac{1}{|\lambda|} |(\lambda I - A)u|
\]

\[
= \frac{M}{|\lambda|} |(\lambda I - A)u|
\]

Thus \( (\lambda I - A) \) is one to one on \( S_{0,\phi'} \) and if \( \lambda \in r(A) \), then

\[
\|((\lambda I - A)^{-1})\| < \frac{M}{|\lambda|}.
\]

By assumption \( 0 \in r(A) \). Now if \( |\mu| \) is small,

\[
(\mu I - A)^{-1}
\]

must exist because it equals

\[
((\mu A^{-1} - I) A)^{-1}
\]

and for \( |\mu| < |A^{-1}| \), \( (\mu A^{-1} - I)^{-1} \in \mathcal{L}(H,H) \) since the infinite series

\[
\sum_{k=0}^{\infty} (-1)^k (\mu A^{-1})^k
\]

converges and must equal to \( (\mu A^{-1} - I)^{-1} \). Therefore, there exists \( \mu \in S_{0,\phi'} \) such that \( \mu \neq 0 \) and \( \mu \in r(A) \). Also if \( \mu \neq 0 \) and \( \mu \in S_{0,\phi'} \), then if \( |\lambda - \mu| < \frac{|\mu|}{\pi M} \), \( (\lambda I - A)^{-1} \) must exist because

\[
(\lambda I - A)^{-1} = \left[ ((\lambda - \mu)(\mu I - A)^{-1} - I) (\mu I - A) \right]^{-1}
\]

where \( ((\lambda - \mu)(\mu I - A)^{-1} - I) \) exists because

\[
\|((\lambda - \mu)(\mu I - A)^{-1})\| = |\lambda - \mu| \left| (\mu I - A)^{-1} \right|
\]

\[
< \frac{|\mu|}{M} \cdot \frac{M}{|\mu|} = 1.
\]

It follows that if \( S \equiv \{ \lambda \in S_{0,\phi'} : \lambda \in r(A) \} \), then \( S \) is open in \( S_{0,\phi'} \). However, \( S \) is also closed because if \( \lambda = \lim_{n \to \infty} \lambda_n \) where \( \lambda_n \in S \), then if \( \lambda = 0 \), it is given \( \lambda \in S \). If \( \lambda \neq 0 \), then for large enough \( n \),

\[
|\lambda - \lambda_n| < \frac{|\lambda_n|}{M}
\]

and so \( \lambda \in S \). Since \( S_{0,\phi'} \) is connected, it follows \( S = S_{0,\phi'} \). This proves the proposition.
Corollary 52.2.10 If for some $a \in \mathbb{R}$, the numerical values of $-aI + A$ are in the set $\{ \lambda : |\lambda| \geq \pi - \phi \}$ where $0 < \phi < \pi/2$, and $a \in r(A)$ then $A$ is sectorial.

Proof: By assumption, $0 \in r(-aI + A)$ and also from Proposition 52.2.9, for $\mu \in S_{0,\phi'}$ where $\pi/2 > \phi' > \phi$,

$$
((−aI + A) − \mu I)^{-1} \in \mathcal{L}(H,H),
$$

$$
\left\|((−aI + A) − \mu I)^{-1}\right\| \leq \frac{M}{|\mu|}
$$

Therefore, for $\mu \in S_{0,\phi'}, \mu + a \in r(A)$. Therefore, if $\lambda \in S_{a,\phi'}, \lambda - a \in S_{0,\phi'}$

$$
\left\|(A - \lambda I)^{-1}\right\| = \left\|(A - aI - (\lambda - a)I)^{-1}\right\| \leq \frac{M}{|\lambda - a|}
$$

This proves the corollary.

52.2.3 An Interesting Example

In this section related to this example, for $V$ a Banach space, $V'$ will denote the space of continuous conjugate linear functions defined on $V$. Usually the symbol $\mathcal{F}$ has meant the space of continuous linear functions but here they will be conjugate linear. That is $f \in V'$ means

$$
f(ax + by) = \overline{a}f(x) + \overline{b}f(y)
$$

and $f$ is continuous.

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and define

$$
V_0 \equiv \{ u \in C^\infty(\overline{\Omega}) : u = 0 \text{ on } \Gamma \}
$$

where $\Gamma$ is some measurable subset of the boundary of $\Omega$ and $C^\infty(\overline{\Omega})$ denotes the restrictions of functions in $C^\infty(\mathbb{R}^n)$ to $\Omega$. By Corollary 13.5.1, $V_0$ is dense in $L^2(\Omega)$.

Now define the following for $u, v \in V_0$.

$$
A_0 u (v) \equiv -a \int_\Omega u \overline{v} dx - \int_\Omega a(x) \nabla u \cdot \nabla \overline{v} dx
$$

where $a > 0$ and $a(x) \geq 0$ is a $C^1(\overline{\Omega})$ function. Also define the following inner product on $V_0$.

$$
(u, v)_1 \equiv \int_\Omega (au \overline{v} + a(x) \nabla u \cdot \nabla \overline{v}) dx
$$

Let $||\cdot||_1$ denote the corresponding norm.

Of course $V_0$ is not a Banach space because it fails to be complete. $u \in V$ will mean that $u \in L^2(\Omega)$ and there exists a sequence $\{u_n\} \subseteq V_0$ such that

$$
\lim_{m,n \to \infty} ||u_n - u_m||_1 = 0
$$
and
\[ \lim_{n \to \infty} |u_n - u|_{L^2(\Omega)} = 0. \]
For \( u \in V \), define \( \nabla u \) to be that element of \( L^2(\Omega; \mathbb{C}^n, a(x) \, dm_n) \), the space of vector valued \( L^2 \) functions taken with respect to the measure \( a(x) \, dm_n \) which satisfies
\[ |\nabla u - \nabla u_n|_{L^2(\Omega; \mathbb{C}^n, a(x) \, dm_n)} \to 0. \]
Denote this space by \( W \) for simplicity of notation.

**Observation 52.2.11** \( V \) is a Hilbert space with inner product given by
\[ (u, v) \equiv \int_{\Omega} (auv + a(x) \nabla u \cdot \nabla v) \, dx \]
Everything is obvious except completeness. Suppose then that \( \{ u_n \} \) is a Cauchy sequence in \( V \). Then there exists a unique \( u \in L^2(\Omega) \) such that \( |u_n - u|_{L^2(\Omega)} \to 0 \). Now let
\[ |w_n - u_n|_{L^2(\Omega)} + |\nabla w_n - \nabla u_n|_W < 1/2^n \]
It follows \( \{ \nabla w_n \} \) is also a Cauchy sequence in \( W \) while \( \{ w_n \} \) is a Cauchy sequence in \( L^2(\Omega) \) converging to \( u \). Thus the thing to which \( \nabla w_n \) converges in \( W \) is the definition of \( \nabla u \) and \( u \in V \). Thus
\[ ||u_n - u||_1 \leq ||u_n - w_n||_1 + ||w_n - u||_1 \]
\[ < \frac{1}{2^n} + ||w_n - u||_1 \]
and the last term converges to 0. Hence \( V \) is complete as claimed.
Then it is clear \( V \) is a Hilbert space. The next observation is a simple one involving the Riesz map.

**Definition 52.2.12** Let \( V \) be a Hilbert space and let \( V' \) be the space of continuous conjugate linear functions defined on \( V \). Then define \( R: V \to V' \) by
\[ Rx(y) \equiv (x, y). \]
This is called the Riesz map.

**Lemma 52.2.13** The Riesz map is one to one and onto and linear.

**Proof:** It is obvious it is one to one and linear. The only challenge is to show it is onto. Let \( z^* \in V' \). If \( z^*(V) = \{0\} \), then letting \( z = 0 \), it follows \( Rz = z^* \). If \( z^*(V) \neq 0 \), then
\[ \ker(z^*) \equiv \{ x \in V : z^*(x) = 0 \} \]
is a closed subspace. It is closed because \( z^* \) is continuous and it is just \( z^{*-1}(0) \). Since \( \ker(z^*) \) is not everything in \( V \) there exists
\[ w \in \ker(z^*)^\perp \equiv \{ x : (x, y) = 0 \ \text{for all} \ y \in \ker(z^*) \} \]
and \( w \neq 0 \). Then
\[
z^* \left( z^* (x) w - z^* (w) x \right) = z^* (x) z^* (w) - z^* (w) z^* (x) = 0
\]
and so \( z^* (x) w - z^* (w) x \in \ker (z^*) \). Therefore, for any \( x \in V \),
\[
0 = \left( w, z^* (x) w - z^* (w) x \right)
= z^* (x) (w, w) - z^* (w) (w, x)
\]
and so
\[
z^* (x) = \left( \frac{z^* (w)}{||w||^2}, x \right)
\]
so let \( z = w/||w||^2 \). Then \( Rz = z^* \) and so \( R \) is onto. This proves the lemma.

Now for the \( V \) described above,
\[
Ru (v) = \int_\Omega (au v + a (x) \nabla u \cdot \nabla v) \, dx
\]
Also, as noted above \( V \) is dense in \( H \equiv L^2 (\Omega) \) and so if \( H \) is identified with \( H' \), it follows
\[
V \subseteq H = H' \subseteq V'.
\]
Let \( A : D (A) \to H \) be given by
\[
D (A) \equiv \{ u \in V : Ru \in H \}
\]
and
\[
A \equiv -R
\]
on \( D (A) \). Then the numerical range for \( A \) is contained in \(( -\infty, -a] \) and so \( A \) is sectorial by Proposition \ref{4.2.9} provided \( A \) is closed and densely defined.

Why is \( D (A) \) dense? It is because it contains \( C_c^\infty (\Omega) \) which is dense in \( L^2 (\Omega) \).
This follows from integration by parts which shows that for \( u, v \in C_c^\infty (\Omega) \),
\[
- \int_\Omega au v dx - \int_\Omega a (x) \nabla u \cdot \nabla v dx
= - \int_\Omega au v dx + \int_\Omega \nabla \cdot (a (x) \nabla u) \nabla v dx
\]
and since \( C_c^\infty (\Omega) \) is dense in \( H \),
\[
Au = -au + \nabla \cdot (a (x) \nabla u) \in L^2 (\Omega) = H.
\]

Why is \( A \) closed? If \( u_n \in D (A) \) and \( u_n \to u \) in \( H \) while \( Au_n \to \xi \) in \( H \), then it follows from the definition that \( Ru_n \to \xi \) and \( \{ u_n \} \) converges to \( u \) in \( V \) so for any \( v \in V \),
\[
Ru (v) = \lim_{n \to \infty} Ru_n (v) = \lim_{n \to \infty} (Ru_n, v)_H = (\xi, v)_H
\]
which shows $R u = -\xi \in H$ and so $u \in D(A)$ and $A u = \xi$. Thus $A$ is closed. This completes the example.

Obviously you could follow identical reasoning to include many other examples of more complexity. What does it mean for $u \in D(A)$? It means that in a weak sense

$$-a u + \nabla \cdot (a(x) \nabla u) \in H.$$  

Since $A$ is sectorial for $S_{-a,\phi}$ for any $0 < \phi < \pi/2$, this has shown the existence of a weak solution to the partial differential equation along with appropriate boundary conditions,

$$-a u + \nabla \cdot (a(x) \nabla u) = f, \ u \in V.$$  

What are these appropriate boundary conditions? $u = 0$ on $\Gamma$ is one. the other would be a variational boundary condition which comes from integration by parts. Letting $v \in V$, formally do the following using the divergence theorem.

$$\langle f, v \rangle_H = \int_{\Omega} (-a u + \nabla \cdot (a(x) \nabla u)) v dx$$

$$= \int_{\Omega} -au dx + \int_{\partial \Omega} (a(x) \nabla u) \cdot n v ds - \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v dx$$

$$= \langle f, v \rangle_H + \int_{\partial \Omega \setminus \Gamma} (a(x) \nabla u) \cdot n v ds$$

and so the other boundary condition is

$$a(x) \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \setminus \Gamma.$$  

To what extent this weak solution is really a classical solution depends on more technical considerations.

### 52.2.4 Fractional Powers Of Sectorial Operators

It will always be assumed in this section that $A$ is sectorial for the sector $S_{-a,\phi}$ where $a > 0$. To begin with, here is a useful lemma which will be used in the presentation of these fractional powers.

**Lemma 52.2.14** The following holds for $\alpha \in (0,1)$ and $\sigma < t$.

$$\int_{\sigma}^{t} (t - s)^{\alpha-1} (s - \sigma)^{-\alpha} ds = \frac{\pi}{\sin(\pi \alpha)}.$$

In particular,

$$\int_{0}^{1} (1 - s)^{\alpha-1} s^{-\alpha} ds = \frac{\pi}{\sin(\pi \alpha)}.$$  

Also for $\alpha, \beta > 0$

$$\Gamma(\alpha) \Gamma(\beta) = \left( \int_{0}^{1} x^{\alpha-1} (1-x)^{\beta-1} dx \right) \Gamma(\alpha + \beta).$$
Proof: First change variables to get rid of the $\sigma$. Let $y = (t - \sigma)^{-1} (s - \sigma)$. Then the integral becomes

$$
\int_0^1 (t - [(t - \sigma) y + \sigma])^{\alpha-1} (t - \sigma)^{-\alpha} y^{-\alpha} (t - \sigma) dy
$$

$$
= \int_0^1 ((t - \sigma) (1 - y))^{\alpha-1} (t - \sigma)^{-\alpha} y^{-\alpha} (t - \sigma) dy
$$

$$
= \int_0^1 (1 - y)^{\alpha-1} y^{-\alpha} dy
$$

Next let $y = x^2$. The integral is

$$
2 \int_0^1 (1 - x^2)^{\alpha-1} x^{1-2\alpha} dx
$$

Next let $x = \sin \theta$

$$
2 \int_0^{\frac{\pi}{2}} (\cos (\theta))^{2\alpha-1} \sin (1-2\alpha) (\theta) d\theta = 2 \int_0^{\frac{\pi}{2}} \left( \frac{\cos (\theta)}{\sin (\theta)} \right)^{2\alpha-1} d\theta
$$

Now change the variable again. Let $u = \cot (\theta)$. Then this yields

$$
2 \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} du
$$

This is fairly easy to evaluate using contour integrals. Consider the following contour called $\Gamma_R$ for large $R$. As $R \to \infty$, the integral over the little circle converges to 0 and so does the integral over the big circle. There is one singularity at $i$.

Thus

$$
\lim_{R \to \infty} \int_{\Gamma_R} \frac{e^{i \arg(z)} (1-2\alpha)}{1 + z^2} dz =
$$

$$
= (1 + \cos (1 - 2\alpha) \pi) \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} du
$$

$$
+ i \sin ((1 - 2\alpha) \pi) \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} du
$$
Then equating the imaginary parts yields

\[
\sin ((1 - 2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} \, du = \pi \sin \left( \frac{\pi}{2} (1 - 2\alpha) \right)
\]

and so using the trig identities for the sum of two angles,

\[
\int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} \, du = \frac{\pi \sin \left( \frac{\pi}{2} (1 - 2\alpha) \right)}{2 \sin \left( \frac{\pi}{2} (1 - 2\alpha) \right) \cos \left( \frac{\pi}{2} (1 - 2\alpha) \right)} = \frac{\pi}{2 \sin (\pi\alpha)}
\]

It remains to verify the last identity.

\[
\Gamma (\alpha) \Gamma (\beta) = \int_0^\infty \int_0^\infty t^{\alpha-1} e^{-t} s^{\beta-1} e^{-s} \, ds dt
\]

\[
= \int_0^\infty t^{\alpha-1} e^{-u} (u - t)^{\beta-1} \, du dt
\]

\[
= \int_0^\infty e^{-u} \int_u^\infty t^{\alpha-1} (u - t)^{\beta-1} \, dt du
\]

\[
= \int_0^1 x^{\alpha-1} (1 - x)^{\beta-1} \, dx \int_0^\infty e^{-u} u^{\alpha+\beta-1} \, du
\]

\[
= \left( \int_0^1 x^{\alpha-1} (1 - x)^{\beta-1} \, dx \right) \Gamma (\alpha + \beta)
\]

This proves the lemma.

If it is not stated otherwise, in all that follows \(\alpha > 0\).

**Definition 52.2.15** Let \(A\) be a sectorial operator corresponding to the sector \(S_{-a\phi}\) where \(-a < 0\). Then define for \(\alpha > 0\),

\[
(-A)^{-\alpha} = \frac{1}{\Gamma (\alpha)} \int_0^\infty t^{\alpha-1} S (t) \, dt
\]

where \(S (t)\) is the analytic semigroup generated by \(A\) as in Corollary 52.2.7. Note that from the estimate, \(\|S (t)\| \leq Me^{-at}\) of this corollary, the integral is well defined and is in \(L (H, H)\).

**Theorem 52.2.16** For \((-A)^{-\alpha}\) as defined in Definition 52.2.15,

\[
(-A)^{-\alpha} (-A)^{-\beta} = (-A)^{-\alpha+\beta}\]

(52.2.8)

Also

\[
(-A)^{-1} (-A) = I, (-A) (-A)^{-1} = I\]

(52.2.9)
and \((-A)^{-\alpha}\) is one to one if \(\alpha \geq 0\), defining \(A^0 \equiv I\).

If \(\alpha < \beta\), then

\[
(-A)^{-\beta} (H) \subseteq (-A)^{-\alpha} (H).
\]

(52.2.10)

If \(\alpha \in (0, 1)\), then

\[
(-A)^{-\alpha} = \frac{\sin(\pi \alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda
\]

(52.2.11)

**Proof:** Consider 52.2.8.

\[
(-A)^{-\alpha} (-A)^{-\beta} = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} S(t + s) ds dt
\]

Changing variables and using Fubini’s theorem which is justified because of the absolute convergence of the iterated integrals, which follows from Corollary 52.2.7, this becomes

\[
\frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} (u - t)^{\beta-1} S(u) du dt
\]

\[
= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^\infty \int_0^u t^{\alpha-1} (u - t)^{\beta-1} S(u) dtdu
\]

\[
= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^\infty S(u) \int_0^1 (ux)^{\alpha-1} (u - ux)^{\beta-1} u dx du
\]

\[
= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \left( \int_0^1 x^{\alpha-1} (1 - x)^{\beta-1} dx \right) \int_0^\infty S(u) u^{\alpha+\beta-1} du
\]

\[
= \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \left( \int_0^1 x^{\alpha-1} (1 - x)^{\beta-1} dx \right) \Gamma(\alpha + \beta) (A)^{-(\alpha+\beta)}
\]

\[
= (-A)^{-(\alpha+\beta)}
\]

This proves the first part of the theorem.

Consider 52.2.9. Since \(A\) is a closed operator, and approximating the integral with an appropriate sequence of Riemann sums, \((-A)\) can be taken inside the integral and so

\[
(-A) \frac{1}{\Gamma(1)} \int_0^\infty t^{1-1} S(t) dt = \int_0^\infty (-A) S(t) dt
\]

\[
= \int_0^\infty -\frac{d}{dt} (S(t)) dt = S(0) = I.
\]

Next let \(x \in D(-A)\). Then

\[
\frac{1}{\Gamma(1)} \int_0^\infty t^{1-1} S(t) dt (-A) x = -\int_0^\infty S(t) Ax dt
\]

\[
= -\int_0^\infty AS(t) x dt = \int_0^\infty -\frac{d}{dt} (S(t)) dt = Ix
\]
This shows that the integral in which \( \alpha = 1 \) deserves to be called \( A^{-1} \) so the definition is not bad notation. Also, by assumption, \( A^{-1} \) is one to one. Thus

\[
(-A)^{-1} (-A)^{-1} x = 0
\]

implies

\[
(-A)^{-1} x = 0
\]

hence \( x = 0 \) so that \( (-A)^{-2} \) is also one to one. Similarly, \( (-A)^{-m} \) is one to one for all positive integers \( m \).

From what was just shown, if \( (-A)^{-\alpha} x = 0 \) for \( \alpha \in (0, 1) \), then

\[
(-A)^{-1} x = (-A)^{-(1-\alpha)} (-A)^{-\alpha} x = 0
\]

and so \( x = 0 \). This shows \( (-A)^{-\alpha} \) is one to one for all \( \alpha \in [0,1] \) if is defined as \((-A)^0 \equiv I\).

What about \( \alpha > 1 \)? For such \( \alpha \), it is of the form \( m + \beta \) where \( \beta \in [0, 1) \) and \( m \) is a positive integer. Therefore, if

\[
(-A)^{-(m+\beta)} x = 0
\]

then

\[
(-A)^{-\beta} ((-A)^{-m}) x = 0
\]

and so from what was just shown,

\[
((-A)^{-m}) x = 0
\]

and now this implies \( x = 0 \) so that \( (-A)^{-\alpha} \) is one to one for all \( \alpha \geq 0 \).

Consider \( \Box \). It was shown above that

\[
(-A)^{-\alpha} (-A)^{-\beta} = (-A)^{-(\alpha+\beta)}
\]

Let \( x = (-A)^{-(\alpha+\beta)} y \). Then

\[
x = (-A)^{-\alpha} (-A)^{-\beta} y \subseteq (-A)^{-\alpha} (-A)^{-\beta} (H) \subseteq (-A)^{-\alpha} (H).
\]

This proves \( \Box \). If \( \alpha < \beta \), \( (-A)^{-\beta} (H) \subseteq (-A)^{-\alpha} (H) \).

Now consider the problem of writing \( (-A)^{-\alpha} \) for \( \alpha \in (0, 1) \) in terms of \( A \), not mentioning \( S(t) \). By Proposition \( \Box \)

\[
(\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt
\]

Then

\[
\int_0^\infty \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda = \int_0^\infty \lambda^{-\alpha} \int_0^\infty e^{-\lambda t} S(t) dt d\lambda
\]

\[
= \int_0^\infty S(t) \int_0^\infty \lambda^{-\alpha} e^{-\lambda t} dt d\lambda
\]

\[
= \int_0^\infty S(t) \int_0^\infty \lambda^{\beta-1} e^{-\lambda t} dt d\lambda
\]
where \( \beta \equiv 1 - \alpha \). Then using Lemma 52.2.14, this equals

\[
\int_0^\infty S(t) \int_0^\infty \mu^{\beta-1} e^{-\mu t} d\mu dt = \int_0^\infty t^{-\beta} S(t) \int_0^\infty \mu^{\beta-1} e^{-\mu t} d\mu dt
\]

\[
= \Gamma (1 - \alpha) \int_0^\infty t^{\alpha-1} S(t) dt = \Gamma (\alpha) \Gamma (1 - \alpha) (-A)^{-\alpha}
\]

\[
= \left( \int_0^1 x^{\alpha-1} (1 - x)^{-\alpha} dx \right) (-A)^{-\alpha} = \frac{\pi}{\sin (\pi \alpha)} (-A)^{-\alpha}
\]

and so this gives the formula

\[
(-A)^{-\alpha} = \frac{\sin (\pi \alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda.
\]

This proves 52.2.11.

**Definition 52.2.17** For \( \alpha \geq 0 \), define \((-A)^\alpha\) on \( D((-A)^\alpha) \equiv (-A)^{-\alpha} (H) \) by

\[
(-A)^\alpha \equiv \left( (-A)^{-\alpha} \right)^{-1}
\]

Note that if \( \alpha, \beta > 0 \), then if \( x \in D((-A)^{\alpha+\beta}) \),

\[
(-A)^{\alpha+\beta} x = \left( (-A)^{-(\alpha+\beta)} \right)^{-1} x = \left( (-A)^{-\alpha} (-A)^{-\beta} \right)^{-1} x = (-A)^{\beta} (-A)^{\alpha} x. \tag{52.2.12}
\]

Next let \( \beta > \alpha > 0 \) and let \( x \in D((-A)^\beta) \). Then from what was just shown,

\[
(-A)^\alpha (-A)^{\beta-\alpha} x = (-A)^\beta x
\]

and so

\[
(-A)^{\beta-\alpha} x = (-A)^{-\alpha} (-A)^\beta x
\]

If \( x \in D((-A)^\beta) \), does it follow that \((-A)^{-\alpha} x \in D((-A)^\beta)\)? Note \( x = (-A)^{-\alpha} y \) and so

\[
(-A)^{-\alpha} x = (-A)^{-\alpha} (-A)^{-\beta} y = (-A)^{-(\alpha+\beta)} y \in D((-A)^{\alpha+\beta}).
\]

Therefore, from 52.2.12,

\[
(-A)^{\beta-\alpha} x = (-A)^{\beta-\alpha} (-A)^{\alpha} (-A)^{-\alpha} x = (-A)^{\beta} (-A)^{-\alpha} x.
\]
CHAPTER 52. SOME IMPORTANT FUNCTIONAL ANALYSIS APPLICATIONS

Theorem 52.2.18 The definition of \((-A)^\alpha\) is well defined and \((-A)^\alpha\) is densely defined and closed. Also for any \(\alpha > 0\),

\[
\|(−A)^\alpha S(t)\| \leq \frac{C_\alpha}{\delta} \frac{1}{t^\alpha} e^{-\delta t} \tag{52.2.13}
\]

where \(-\delta > -a\). Furthermore, \(C_\alpha\) is bounded as \(\alpha \to 0^+\) and is bounded on compact intervals of \((0, \infty)\). Also for \(\alpha \in (0, 1)\) and \(x \in D((-A)^\alpha)\),

\[
\|(S(t) - I)x\| \leq \frac{C_1}{\alpha \delta} \|(−A)^\alpha x\| \tag{52.2.14}
\]

There exists a constant \(C\) independent of \(\alpha \in [0, 1)\) such that for \(x \in D(A)\) and \(\varepsilon > 0\),

\[
\|(−A)^\alpha x\| \leq \varepsilon \|(−A) x\| + C e^{-\alpha/(1-\alpha)} \|x\| \tag{52.2.15}
\]

There exists a constant \(C'\) independent of \(\alpha \in [0, 1]\) such that for \(x \in D(A)\),

\[
\|(−A)^\alpha x\| \leq C' \|(−A) x\|^\alpha \|x\|^{1-\alpha} \tag{52.2.16}
\]

The formula \(\text{(52.2.16)}\) is called an interpolation inequality.

**Proof:** It is obvious \((-A)^\alpha\) is densely defined because its domain is at least as large as \(D(A)\) which was assumed to be dense. It is a closed operator because if \(x_n \in D((-A)^\alpha)\) and \(x_n \to x\), \((-A)^\alpha x_n \to y\),

then

\((-A)^{-\alpha} x_n \to (-A)^{-\alpha} x, x_n = (-A)^{-\alpha} (-A)^\alpha x_n \to (-A)^{-\alpha} y\)

and so

\((-A)^{-\alpha} y = x\)

showing \(x \in D((-A)^\alpha)\) and \(y = (-A)^{-\alpha} x\). Thus \((-A)^\alpha\) is closed and densely defined.

Let \(-\delta > -a\) where the sector for \(A\) was \(S_{-a,\phi}, a > 0\). Then recall from Corollary \(\text{(52.2.14)}\) there is a constant, \(N\) such that

\[
\|(−A) S(t)\| \leq \frac{N}{t} e^{-\delta t}
\]

What about \(\|(−A)^\alpha S(t)\|\)? First note that for \(\alpha \in [0, 1)\) this at least makes sense because \(S(t)\) maps into \(D(A)\). For any \(\alpha > 0\),

\[
S(t) (−A)^{-\alpha} = (−A)^{-\alpha} S(t)
\]

follows from the definition of \((-A)^{-\alpha}\). Therefore,

\[
(−A)^\alpha S(t) (−A)^{-\alpha} = S(t). \tag{52.2.17}
\]
Note this implies that on \( D((−A)^{α}) \),

\[
(−A)^{α} S(t) = S(t)(−A)^{α}.
\]

Also

\[
(−A)^{-1} S(t) = S(t)(−A)^{-1} = S(t)(−A)^{-α}(−A)^{(1−α)}
\]

and so

\[
S(t) = (−A) S(t)(−A)^{-α}(−A)^{(1−α)}
\]

From 52.2.17 it follows

\[
(−A)^{α} S(t) = (−A)(−A)^{α} S(t)(−A)^{-α}(−A)^{(1−α)}
= (−A) S(t)(−A)^{(1−α)}
\]

Then with this formula,

\[
||(-A)^{α} S(t)|| = \left\| S(t)(−A)^{(1−α)} \right\| = \left\| \frac{1}{Γ(1−α)} \int_{0}^{∞} s^{1−α} (−A) S(t+s) ds \right\| \\
\leq \frac{N}{Γ(1−α)} \int_{0}^{∞} \frac{s^{1−α}}{(t+s)} e^{-δ(s+t)} ds
\]

\[
= \frac{N}{Γ(1−α)} \int_{t}^{∞} \frac{(u−t)^{1−α}}{u} e^{-δu} ds
\]

\[
\leq \frac{N}{Γ(1−α)} \int_{t}^{∞} \left( 1−\frac{t}{u} \right)^{1−α} \frac{1}{u^δ} e^{-δu} ds
\]

\[
\leq \frac{N}{Γ(1−α)} \frac{1}{t^α} \int_{t}^{∞} e^{-δu} ds = \frac{N}{Γ(1−α)} \frac{1}{t^α} \frac{1}{e^{δt}}
\]

\[
\equiv C_{α} \frac{1}{δ} \frac{1}{t^α} e^{-δt}.
\]

this establishes the formula when \( α ∈ [0, 1) \). Next suppose \( α = m \), a positive integer.

\[
||A^{m} S(t)|| = \left\| A^{m} S \left( \frac{t}{m} \right) \right\| = \left\| \left( A S \left( \frac{t}{m} \right) \right)^{m} \right\| \leq \frac{N}{t^m} m^m.
\]

This is why the above inequality holds.
If \( \alpha, \beta > 0 \),
\[
\left\| A^{\alpha + \beta} S(t) \right\| = \left\| A^{\alpha + \beta} S\left(\frac{t}{2}\right) S\left(\frac{t}{2}\right) \right\| \\
= \left\| A^{\alpha} S\left(\frac{t}{2}\right) A^{\beta} S\left(\frac{t}{2}\right) \right\| \\
\leq \frac{C_{\alpha} C_{\beta}}{t^{\alpha + \beta}} e^{-2\delta t} = \frac{C}{t^{\alpha + \beta}} e^{-\delta t}
\]

Suppose now that \( \alpha > 0 \). Then
\[
\alpha = m + \beta
\]
where \( \beta \in [0, 1) \). Then from what was just shown,
\[
\left\| A^{m + \beta} S(t) \right\| \leq \frac{C}{t^{m + \beta}} e^{-\delta t}.
\]

Next consider (52.2.14). First note that whenever \( \alpha > 0 \),
\[
(-A)^{-\alpha} S(s) = S(s) (-A)^{-\alpha}
\]
and so on \( D((-A)^{\alpha}) \),
\[
S(s) = (-A)^{\alpha} S(s) (-A)^{-\alpha}, \quad S(s) (-A)^{\alpha} = (-A)^{\alpha} S(s)
\]
Now for \( x \in D((-A)^{\alpha}) \),
\[
\left\| (S(t) - I) x \right\| = \left\| - \int_{0}^{t} (-A) S(s) x ds \right\| \\
= \left\| - \int_{0}^{t} (-A)^{1-\alpha} (-A)^{\alpha} S(s) x ds \right\| \\
= \left\| - \int_{0}^{t} (-A)^{1-\alpha} S(s) (-A)^{\alpha} x ds \right\| \\
\leq \int_{0}^{t} \left\| (-A)^{1-\alpha} S(s) \right\| ds \left\| (-A)^{\alpha} x \right\| \\
\leq \int_{0}^{t} \frac{C_{1-\alpha}}{\delta} \frac{1}{s^{1-\alpha}} e^{-\delta s} ds \left\| (-A)^{\alpha} x \right\| \\
\leq \frac{C_{1-\alpha}}{\delta} \frac{1}{\alpha} \left\| (-A)^{\alpha} x \right\|
\]
and this shows (52.2.15).

Next consider (52.2.16). Let \( x \in H \) and \( \beta \in (0, 1) \). Then
\[
\left\| (-A)^{-\beta} x \right\| = \frac{1}{\Gamma(\beta)} \left\| \int_{0}^{\infty} t^{\beta-1} S(t) x dt \right\|
\[
\begin{align*}
52.2. \text{ANALYTIC SEMIGROUPS} \quad 1767 \\
= & \frac{1}{\Gamma (\beta)} \left| \int_0^\eta t^{\beta - 1} S(t) \, dt + \int_\eta^\infty t^{\beta - 1} S(t) \, dt \right| \\
\leq & \frac{1}{\Gamma (\beta)} \int_0^\eta t^{\beta - 1} \left| S(t) \right| \, dt + \frac{1}{\Gamma (\beta)} \left| \int_\eta^\infty t^{\beta - 1} S(t) \, dt \right| \\
\leq & \frac{C \eta^\beta}{\Gamma (\beta)} \| x \| + \frac{1}{\Gamma (\beta)} \left| \int_\eta^\infty t^{\beta - 1} S(t) \, dt \right| \\
\leq & \frac{C \eta^\beta}{\Gamma (\beta)} \| x \| + \frac{1}{\Gamma (\beta)} \left| \int_\eta^\infty t^{\beta - 1} S(t) \, dt \right| \\
\leq & \frac{C \eta^\beta}{\beta} \| x \| + \frac{1}{\Gamma (\beta)} \left| \int_\eta^\infty t^{\beta - 1} S(t) A^{-1} x \, dt \right| \\
\leq & \frac{C \eta^\beta}{\beta} \| x \| + \frac{1}{\Gamma (\beta)} \left( \left| \int_\eta^\infty t^{\beta - 2} \, dt \right| \right) \\
= & \frac{C \eta^\beta}{\beta} \| x \| + 2 \eta^{\beta - 1} \left| A^{-1} x \right| \\
\end{align*}
\]

Now let \( \delta = C \eta^\beta \) so \( \eta = C^{1/\beta} \delta^{1/\beta} \) and \( \eta^{\beta - 1} = C^{1-\beta} \delta^{(\beta - 1)/\beta} \). Thus for all \( x \in H \),

\[
\left( -A \right)^{-\beta} x \leq \frac{1}{\Gamma (\beta)} \left( \frac{\delta}{\beta} \| x \| + 2C^{1-\beta} \delta^{(\beta - 1)/\beta} \| A^{-1} x \| \right).
\]

Let \( \varepsilon = \frac{\delta}{\beta} \Gamma (\beta) \). Then the above is of the form

\[
\left( -A \right)^{-\beta} x \leq \varepsilon \| x \| + 2C^{1-\beta} \left( \varepsilon \Gamma (1 + \beta) \right)^{(\beta - 1)/\beta} \| A^{-1} x \|
\]

because \( \Gamma \) is decreasing on \((0, 1)\). I need to verify that for \( \beta \in (0, 1) \),

\[
\Gamma (1 + \beta)^{(\beta - 1)/\beta}
\]

is bounded. It is continuous on \((0, 1]\) and so if I can show \( \lim_{\beta \to 0^+} \Gamma (1 + \beta)^{(\beta - 1)/\beta} \) exists, then it will follow the function is bounded. It suffices to show

\[
\lim_{\beta \to 0^+} \frac{\beta - 1}{\beta} \ln \Gamma (1 + \beta) = -\lim_{\beta \to 0^+} \frac{\ln \Gamma (1 + \beta)}{\beta}
\]

exists. Consider this. By L’Hospital’s rule and dominated convergence theorem, this is

\[
\lim_{\beta \to 0^+} \frac{\int_0^\infty \ln (t) t^\beta e^{-t} \, dt}{\Gamma (1 + \beta)} = \lim_{\beta \to 0^+} \int_0^\infty \ln (t) t^\beta e^{-t} \, dt
\]

\[
= \lim_{\beta \to 0^+} \int_0^\infty \ln (t) e^{-t} \, dt.
\]
Thus the function is bounded independent of $\beta \in (0, 1)$. This shows there is a constant $C$ which is independent of $\beta \in (0, 1)$ such that for any $x \in H$,

$$\left\| (-A)^{-\beta} x \right\| \leq \varepsilon \|x\| + C\varepsilon^{(\beta-1)/\beta} \left\| A^{-1} x \right\|. \quad (52.2.19)$$

Now let $y \in D(A) = D((-A))$ and let $x = (-A)y$. Then the above becomes

$$\left\| (-A)^{-\beta} (-A)y \right\| \leq \varepsilon \|(-A)y\| + C\varepsilon^{(\beta-1)/\beta} \|y\|$$

I claim that

$$(-A)^{-\beta} (-A)y = (-A)^{1-\beta}y.$$ 

The reason for this is as follows.

$$(-A)^{1-\beta} (-A)y = (-A)y$$

and so the desired result follows from multiplying on the left by $(-A)^{-\beta}$. Hence

$$\left\| (-A)^{1-\beta} y \right\| \leq \varepsilon \|(-A)y\| + C\varepsilon^{(\beta-1)/\beta} \|y\|$$

Now let $1 - \beta = \alpha$ and obtain

$$\|(-A)^{\alpha} y\| \leq \varepsilon \|(-A)y\| + C\varepsilon^{-\alpha/(1-\alpha)} \|y\|$$

This proves (52.2.15).

Finally choose $\varepsilon$ to minimize the right side of the above expression. Thus let

$$\varepsilon = \left( \frac{\alpha \|y\| C}{\|(-A)y\| (1-\alpha)} \right)^{1-\alpha}$$

Then the above expression becomes

$$\|(-A)^{\alpha} y\| \leq \|(-A)y\| \left( \frac{\alpha \|y\| C}{\|(-A)y\| (1-\alpha)} \right)^{1-\alpha}$$

$$+ C \left( \frac{\alpha \|y\| C}{\|(-A)y\| (1-\alpha)} \right)^{1-\alpha} \|y\|$$

$$= \|(-A)y\|^\alpha \|y\|^{1-\alpha} \left( \frac{\alpha C}{(1-\alpha)} \right)^{1-\alpha}$$

$$+ \|(-A)y\|^\alpha \|y\|^{1-\alpha} \left( \frac{\alpha C}{(1-\alpha)} \right)^{-\alpha}$$

$$= \left( \frac{\alpha C}{(1-\alpha)} \right)^{1-\alpha} + \left( \frac{\alpha C}{(1-\alpha)} \right)^{-\alpha} \|(-A)y\|^\alpha \|y\|^{1-\alpha}$$

$$\leq C' \|(-A)y\|^\alpha \|y\|^{1-\alpha}$$
where $C'$ does not depend on $\alpha \in (0,1)$. To see such a constant exists, note

$$\lim_{\alpha \to 1} \left( \frac{\alpha C}{1 - \alpha} \right)^{1-\alpha} = 1$$

and

$$\lim_{\alpha \to 1} \left( \frac{\alpha C}{1 - \alpha} \right)^{-\alpha} = 0$$

while

$$\lim_{\alpha \to 0} \left( \frac{\alpha C}{1 - \alpha} \right)^{1-\alpha} = 0, \quad \lim_{\alpha \to 0} \left( \frac{\alpha C}{1 - \alpha} \right)^{-\alpha} = 1$$

Of course $C'$ depends on $C$ but as shown above, this did not depend on $\alpha \in (0,1)$. This proves 52.2.16.

The following corollary follows from the proof of the above theorem.

**Corollary 52.2.19** Let $\alpha \in (0,1)$. Then for all $\varepsilon > 0$, there exists a constant $C(\alpha, \varepsilon)$ such that

$$\left\| (-A)^{-\alpha} x \right\| \leq \varepsilon \|x\| + C(\varepsilon, \alpha) \left\| (-A)^{-1} x \right\|$$

Also if $A^{-1}$ is compact, then so is $(-A)^{-\alpha}$ for all $\alpha \in (0,1)$.

**Proof:** The first part is done in the above theorem. Let $S$ be a bounded set and let $\eta > 0$. Then let $\varepsilon > 0$ be small enough that for all $x \in S, \varepsilon \|x\| < \eta/4$. Let $\{(-A)^{-1} x_n\}$ be a $\eta/(2 + 2C(\varepsilon, \alpha))$ net for $(-A)^{-1}(S)$. Then if $(-A)^{-\alpha} x \in (-A)^{-\alpha} S$, there exists $x_n$ such that

$$\left\| (-A)^{-1} x_n - (-A)^{-1} x \right\| < \frac{\eta}{2 + 2C(\varepsilon, \alpha)}.$$ 

Then

$$\left\| (-A)^{-\alpha} x_n - (-A)^{-\alpha} x \right\| \leq \varepsilon \|x_n - x\| + C(\varepsilon, \alpha) \left\| (-A)^{-1} x_n - (-A)^{-1} x \right\|$$

$$< \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

showing $(-A)^{-\alpha}(S)$ has a $\eta$ net. Thus $(-A)^{-\alpha}$ is compact. This proves the corollary.

The next proposition gives a general interpolation inequality.

**Proposition 52.2.20** Let $0 < \alpha < \beta$ and let

$$\gamma = \theta \beta + (1 - \theta) \alpha, \quad \theta \in (0,1).$$

Then there exists a constant, $C$ such that for all $x \in D\left((-A)^{\beta}\right)$,

$$\|(-A)^{\gamma} x\| \leq C \left\| (-A)^{\beta} x \right\|^\theta \left\| (-A)^{\alpha} x \right\|^{1-\theta}.$$
Proof: This is an exercise in using \(52.2.13\). Letting \(x \in D((-A)^{\theta})\),

\[(-A)^{\gamma} x = (-A)^{\theta} (-A)^{-\theta} (-A)^{\gamma} x\]

Therefore, letting \(C\) denote a generic constant, it follows since \((-A)^{\theta}\) is closed,

\[
\Gamma (\theta) \|(A)^{\gamma} x\| = \left\| \int_0^\infty t^{\theta-1} (-A)^{\theta} S(t) (-A)^{\gamma} x dt \right\|
\]

\[
\leq \int_0^\eta t^{\theta-1} \left\| (-A)^{\gamma} S(t) (-A)^{\theta} x \right\| dt + \int_\eta^\infty t^{\theta-1} \left\| (-A)^{\gamma} S(t) (-A)^{\alpha} x \right\| dt
\]

\[
\leq C \int_0^\eta t^{\theta-1} t^{-\theta} t^{\beta-\gamma} dt \left\| (-A)^{\beta} x \right\| + C \int_\eta^\infty t^{\theta-1} t^{-\theta} t^{\alpha-\gamma} dt \left\| (-A)^{\alpha} x \right\|
\]

and now writing in what \(\gamma\) is in terms of \(\theta\) yields

\[
\Gamma (\theta) \|(A)^{\gamma} x\| \leq C \left( \frac{1}{\beta-\alpha} \right) \left( \frac{\eta^{\beta-\alpha}}{(1-\theta)} \right) \left\| (-A)^{\beta} x \right\| + \frac{\eta^{(\gamma-\alpha)}}{\gamma-\alpha} \left\| (-A)^{\alpha} x \right\|
\]

Letting \(\lambda = \frac{\eta^{\beta-\alpha}}{\beta-\alpha}\), it follows

\[
\Gamma (\theta) \|(A)^{\gamma} x\| \leq C \left( \frac{1}{\beta-\alpha} \right) \left( \frac{\lambda^{1-\theta}}{(1-\theta)} \right) \left\| (-A)^{\beta} x \right\| + \frac{\lambda^{-\theta}}{\theta} \left\| (-A)^{\alpha} x \right\|
\]

then let

\[
\lambda = \left\| (-A)^{\alpha} x \right\| \left\| (-A)^{\beta} x \right\|
\]

which is obtained from minimizing the expression on the right in the above. then placing this in the inequality yields

\[
\Gamma (\theta) \|(A)^{\gamma} x\| \leq C \left( \frac{1}{\beta-\alpha} \right) \left( \frac{\left\| (-A)^{\alpha} x \right\|}{\left\| (-A)^{\beta} x \right\|} \right)^{1-\theta} \left\| (-A)^{\beta} x \right\|
\]

\[
+ \frac{\left\| (-A)^{\alpha} x \right\|}{\left\| (-A)^{\beta} x \right\|} \left\| (-A)^{\alpha} x \right\|
\]
and this proves the proposition.

Note that the constant is not bounded as \( \theta \to 1 \).

Here is another interesting result about compactness.

**Proposition 52.2.21**

Let \( A \) be sectorial for \( S_{-a,\phi} \) where \( -a < 0 \). Then the following are equivalent.

1. \( (−A)^{−α} \) is compact for all \( α > 0 \).
2. \( S(t) \) is compact for each \( t > 0 \).

**Proof:** First suppose \( (−A)^{−α} \) is compact for all \( α > 0 \). Then

\[
\Gamma (α) (−A)^{−α} = \int_0^t s^{α−1} S(s) \, ds + \int_t^∞ s^{α−1} S(s) \, ds
\]

\[
= \frac{t^α}{α} S(t) - \int_0^t \frac{s^α}{α} AS(s) \, ds + s^{α−1} S(s) A−1 \bigg|_t^∞
\]

\[- (α - 1) \int_t^∞ s^{α−2} S(s) A−1 \, ds
\]

Now

\[
\left\| \frac{s^α}{α} AS(s) \right\| ≤ C \frac{s^{α−1}}{α}
\]

and so the second integral satisfies

\[
\left\| \int_0^t \frac{s^α}{α} AS(s) \, ds \right\| ≤ C \frac{t^α}{α^2}
\]

\[
\Gamma (α) (−A)^{−α} = O \left( \frac{t^α}{α^2} \right) + \frac{t^α}{α} S(t)
\]

\[- t^{α−1} A−1 S(t) - (α - 1) \int_t^∞ s^{α−2} S(s) \, ds A−1
\]

It follows that for \( t > 0 \), and \( ε > 0 \) given,

\[
S(t) = \left( \frac{t^α}{α} - t^{α−1} \right)^{−1} \left( \Gamma (α) (−A)^{−α}
\right.
\]

\[+ (α - 1) \int_t^∞ s^{α−2} S(s) \, ds A−1 + O \left( \frac{t^α}{α^2} \right)
\]

\[= \left( \frac{t^α}{α} - t^{α−1} \right)^{−1} \left( \Gamma (α) (−A)^{−α}
\right.
\]

\[+ (α - 1) \int_t^∞ s^{α−2} S(s) \, ds A−1 + O \left( \frac{1}{α} \right)
\]

\[= N_α + O \left( \frac{1}{α} \right).
\]
where $N_\alpha$ is a compact operator. Now let $B$ be a bounded set in $H$, $\|x\| \leq M$ for all $x \in B$ and let $\eta > 0$ be given. Then choose $\alpha$ large enough that $\|O\left(\frac{1}{\alpha}\right)\| < \frac{\eta}{4 + 4M}$. Then there exists a $\eta/2$ net, $\{N_\alpha x_n\}_{n=1}^N$ for $N_\alpha(B)$. Then consider $\{S(t)x_n\}_{n=1}^N$.

For $x \in B$, there exists $x_n$ such that $\|N_\alpha x_n - N_\alpha x\| < \eta/2$. Then

\[
\|S(t)x - S(t)x_n\| \leq \|S(t)x - N_\alpha x\| + \|N_\alpha x - N_\alpha x_n\| + \|N_\alpha x_n - S(t)x_n\| \\
\leq \frac{\eta}{4 + 4M}M + \frac{\eta}{2} + \frac{\eta}{4 + 4M}M < \eta
\]

Thus $S(t)(B)$ has an $\eta$ net for every $\eta > 0$ and so $S(t)$ is compact.

Next suppose $S(t)$ is compact for all $t > 0$. Then

\[
(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt
\]

and the integral is a limit in norm of Riemann sums of the form

\[
\sum_{k=1}^m t_k^{\alpha-1} S(t_k) \Delta t_k
\]

and each of these operators is compact. Since $(-A)^{-\alpha}$ is the limit in norm of compact operators, it must also be compact. This proves the proposition.

Here are some observations which are listed in the book by Henry [57]. Like the above proposition, these are exercises in this book.

**Observation 52.2.22** For each $x \in H$, $t \to tAS(t)$ is continuous and $\lim_{t \to 0^+} tAS(t)x = 0$.

The reason for this is that if $x \in D(A)$, then

\[
tAS(t)x = |tS(t)Ax| \to 0
\]

as $t \to 0$. Now suppose $y \in H$ is arbitrary. Then letting $x \in D(A)$,

\[
|tAS(t)y| \leq |tAS(t)(y-x)| + |tAS(t)x| \\
\leq \varepsilon + |tAS(t)x|
\]

provided $x$ is close enough to $y$. The last term converges to 0 and so

\[
\lim_{t \to 0^+} \sup_{y \in H} |tAS(t)y| \leq \varepsilon
\]

where $\varepsilon > 0$ is arbitrary. Thus

\[
\lim_{t \to 0^+} |tAS(t)y| = 0.
\]
Why is \( t \to tAS(t)x \) continuous on \([0,T]\)? This is true if \( x \in D(A) \) because \( t \to tS(t)Ax \) is continuous. If \( y \in H \) is arbitrary, let \( x_n \) converge to \( y \) in \( H \) where \( x_n \in D(A) \). Then
\[
|tAS(t)y - tAS(t)x_n| \leq C|y - x_n|
\]
and so the convergence is uniform. Thus \( t \to tAS(t)y \) is continuous because it is the uniform limit of a sequence of continuous functions.

**Observation 52.2.23** If \( x \in H \) and \( A \) is sectorial for \( S_{-a,\phi}, -a < 0 \), then for any \( \alpha \in [0,1] \),
\[
\lim_{t \to 0^+} t^\alpha \|(−A)^\alpha S(t)x\| = 0.
\]

This follows as above because you can verify this is true for \( x \in D(A) \) and then use the fact shown above that
\[
t^\alpha \|(−A)^\alpha S(t)\| \leq C
\]
to extend it to \( x \) arbitrary.

### 52.2.5 A Scale Of Banach Spaces

Next I will present an important and interesting theorem which can be used to prove equivalence of certain norms.

**Theorem 52.2.24** Let \( A, B \) be sectorial for \( S_{-a,\phi} \) where \( -a < 0 \) and suppose \( D(A) = D(B) \). Also suppose
\[
(A - B)(−A)^{-\alpha}, (A - B)(−B)^{-\alpha}
\]
are both bounded on \( D(A) \) for some \( \alpha \in (0,1) \). Then for all \( \beta \in [0,1] \),
\[
(−A)^\beta (−B)^{-\beta}, (−B)^\beta (−A)^{-\beta}
\]
are both bounded on \( D(A) = D(B) \). Also \( D((-A)^\beta) = D((-B)^\beta) \).

**Proof:** First of all it is a good idea to verify \( (A - B)(−A)^{-\alpha}, (A - B)(−B)^{-\alpha} \) make sense on \( D(A) \). If \( x \in D(A) \), then why is \( (−A)^{-\alpha}x \in D(A) \)? Here is why. Since \( x \in D(A) \),
\[x = (−A)^{-1} y\]
for some \( y \in H \). Then
\[
(−A)^{-\alpha}x = (−A)^{-\alpha}(−A)^{-1} y = (−A)^{-1}(−A)^{-\alpha} y \in D(A).
\]
The case of \( (A - B)(−B)^{-\alpha} \) is similar.
Next for $\beta \in (0, 1)$ and $\lambda > 0$, use (52.2.16) to write
\[
\left\| (A)^{\beta} (\lambda I - A)^{-1} x \right\| 
\leq C \left\| (A) (\lambda I - A)^{-1} x \right\|^{\beta} \left\| (\lambda I - A)^{-1} x \right\|^{1-\beta}
\]
\[
\leq C \left\| (\lambda I - A)^{-1} \right\|^{\beta} \left\| (\lambda I - A)^{-1} x \right\|^{1-\beta} \left\| x \right\|
\leq C \left( 1 + \frac{\lambda}{\lambda + \delta} \right)^{\beta} \frac{M}{(\lambda + \delta)^{1-\beta}} \left\| x \right\| \equiv \frac{C}{(\lambda + \delta)^{1-\beta}} \left\| x \right\| \quad (52.2.20)
\]
where $-a < -\delta < 0$ where $C$ denotes a generic constant. Similarly, for all $\beta \in (0, 1)$,
\[
\left\| (B)^{\beta} (\lambda I - B)^{-1} x \right\| \leq \frac{C}{(\lambda + \delta)^{1-\beta}} \left\| x \right\| \quad (52.2.21)
\]
Now from Theorem (52.2.16) and letting $\beta \in (0, 1)$,
\[
(B)^{\beta} - (A)^{\beta} = \frac{\sin (\pi \beta)}{\pi} \int_{0}^{\infty} \lambda^{-\beta} \left( (\lambda I - B)^{-1} - (\lambda I - A)^{-1} \right) d\lambda
\]
\[
= \frac{\sin (\pi \beta)}{\pi} \int_{0}^{\infty} \lambda^{-\beta} (\lambda I - B)^{-1} (A - B) (\lambda I - A)^{-1} d\lambda. \quad (52.2.22)
\]
Therefore, letting $x \in D(A)$ and letting $C$ denote a generic constant which can be changed from line to line and using (52.2.20) and (52.2.21),
\[
\left\| x - (B)^{\beta} (A)^{-\beta} x \right\| 
\leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta}} \left\| (B)^{\beta} (\lambda I - B)^{-1} (A - B) (\lambda I - A)^{-1} x \right\| d\lambda
\]
The reason $(B)^{\beta}$ goes inside the integral is that it is a closed operator. Then the above
\[
\leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta} (\lambda + \delta)^{1-\beta}} \left\| (A - B)^{\alpha} (A)^{-\alpha} (\lambda I - A)^{-1} x \right\| d\lambda
\leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta} (\lambda + \delta)^{1-\beta}} \left\| (A)^{\alpha} (\lambda I - A)^{-1} x \right\| d\lambda
\leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta} (\lambda + \delta)^{1-\beta}} \frac{1}{(\lambda + \delta)^{1-\alpha}} d\lambda \left\| x \right\| = C \left\| x \right\|.
\]
It follows $(B)^{\beta} (A)^{-\beta}$ is bounded on $D(A)$. 

1774CHAPTER 52. SOME IMPORTANT FUNCTIONAL ANALYSIS APPLICATIONS
Next reverse $A$ and $B$ in $\mathbb{D}$. This yields

$$( -A )^{-\beta} - ( -B )^{-\beta} = \frac{\sin ( \pi \beta )}{\pi} \int_{0}^{\infty} \lambda^{-\beta} ( \lambda I - A )^{-1} ( B - A ) ( \lambda I - B )^{-1} d\lambda.$$

Letting $x \in D ( A )$,

$$\left\| x - ( -A )^{\beta} ( -B )^{-\beta} x \right\| \leq C \int_{0}^{\infty} \lambda^{-\beta} \left\| ( -A )^{\beta} ( \lambda I - A )^{-1} ( B - A ) ( \lambda I - B )^{-1} x \right\| d\lambda$$

$$\leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta} ( \lambda + \delta )^{1-\beta}} \left\| ( B - A ) ( -B )^{-\alpha} ( -B )^{\alpha} ( \lambda I - B )^{-1} x \right\| d\lambda \leq C \int_{0}^{\infty} \frac{1}{\lambda^{\beta} ( \lambda + \delta )^{1-\beta} ( \lambda + \delta )^{1-\alpha}} d\lambda \left\| x \right\| = C \left\| x \right\| \quad (52.2.23)$$

This shows $(-A)^\beta (-B)^{-\beta}$ is bounded on $D(A) = D(B)$. Note the assertion these are bounded refers to the norm on $H$.

It remains to verify $D \left( ( -A )^{\beta} \right) = D \left( ( -B )^{\beta} \right)$. Since $D(A)$ is dense in $H$ there exists a unique $L(A, B) \in L(H, H)$ such that $L(A, B) = (-A)^\beta (-B)^{-\beta}$ on $D(A)$. Let $L(B, A)$ be defined similarly as a continuous linear map which equals $(-B)^\beta (-A)^{-\beta}$ on $D(A)$. Then

$$(-A)^{-\beta} L(A, B) = (-B)^{-\beta}$$

$$(-B)^{-\beta} L(B, A) = (-A)^{-\beta}$$

The first of these equations shows $D \left( ( -B )^{\beta} \right) \subseteq D \left( ( -A )^{\beta} \right)$ and the second turns the inclusion around. Thus they are equal as claimed.

Next consider the case where $\beta = 1$. In this case

$$(A - B)B^{-\alpha}$$

is bounded on $D(A)$ and so

$$(A - B)B^{-\alpha}B^{-1+\alpha}$$

is also bounded on $D(A)$. But this equals

$$(A - B)B^{-1}.$$

Thus $AB^{-1}$ is bounded on $D(A)$. Similarly you can show

$$(B - A)A^{-1}$$

is bounded which implies $BA^{-1}$ is bounded on $D(A)$. This proves the theorem.
Definition 52.2.25 Let $A$ be sectorial for the sector $S_{a,\phi}$. Let $b > a$ so that $A - bI$ is sectorial for $S_{\delta,\phi}$ where $\delta = b - a$. Then for each $\alpha \in [0, 1]$, define a norm on $D((bI - A)^\alpha) \equiv H_\alpha$ by

$$||x||_\alpha \equiv ||(bI - A)^\alpha x||$$

The $\{H_\alpha\}_{\alpha \in [0, 1]}$ is called a scale of Banach spaces.

Proposition 52.2.26 The $H_\alpha$ above are Banach spaces and they decrease in $\alpha$. Furthermore, if $b_i > a$ for $i = 1, 2$ then the two norms associated with the $b_i$ are equivalent.

Proof: That the $H_\alpha$ are decreasing was shown above in Theorem 52.2.16. They are Banach spaces because $(bI - A)^\alpha$ is a closed mapping which is also one to one.

It only remains to verify the claim about the equivalence of the norms. Let $b_2 > b_1 > a$. Then if $\alpha \in (0, 1)$,

$$((b_1 I - A) - (b_2 I - A)) (b_2 I - A)^{-\alpha}$$

$$= (b_1 - b_2) (b_2 I - A)^{-\alpha} \in \mathcal{L}(H, H)$$

and so by Theorem 52.2.24, for each $\beta \in [0, 1]$,

$$D((b_1 I - A)^\beta) = D((b_2 I - A)^\beta)$$

so the spaces, $H_\beta$ are the same for either choice of $b > a$. Also from this theorem,

$$(b_1 I - A)^\beta (b_2 I - A)^{-\beta}, (b_2 I - A)^\beta (b_1 I - A)^{-\beta}$$

are both bounded on $D(A)$. Therefore, for $x \in H_\beta$

$$||((b_1 I - A)^\beta x)|| = ||(b_1 I - A)^\beta (b_2 I - A)^{-\beta} (b_2 I - A)^\beta x||$$

$$\leq C ||(b_2 I - A)^\beta x||$$

Similarly using the boundedness of $(b_2 I - A)^\beta (b_1 I - A)^{-\beta}$, it follows

$$||((b_2 I - A)^\beta x)|| \leq C' ||(b_1 I - A)^\beta x||$$

Thus showing the two norms are equivalent. This proves the proposition.
Chapter 53

Complex Mappings

53.1 Conformal Maps

If \( \gamma(t) = x(t) + iy(t) \) is a \( C^1 \) curve having values in \( U \), an open set of \( \mathbb{C} \), and if \( f : U \to \mathbb{C} \) is analytic, consider \( f \circ \gamma \), another \( C^1 \) curve having values in \( \mathbb{C} \). Also, \( \gamma'(t) \) and \( (f \circ \gamma)'(t) \) are complex numbers so these can be considered as vectors in \( \mathbb{R}^2 \) as follows. The complex number, \( x + iy \) corresponds to the vector, \( (x, y) \). Suppose that \( \gamma \) and \( \eta \) are two such \( C^1 \) curves having values in \( U \) and that \( \gamma(t_0) = \eta(s_0) = z \) and suppose that \( f : U \to \mathbb{C} \) is analytic. What can be said about the angle between \( (f \circ \gamma)'(t_0) \) and \( (f \circ \eta)'(s_0) \)? It turns out this angle is the same as the angle between \( \gamma'(t_0) \) and \( \eta'(s_0) \) assuming that \( f'(z) \neq 0 \). To see this, note \( (x, y) \cdot (a, b) = \frac{1}{2} (zw + \overline{zw}) \) where \( z = x + iy \) and \( w = a + ib \). Therefore, letting \( \theta \) be the cosine between the two vectors, \( (f \circ \gamma)'(t_0) \) and \( (f \circ \eta)'(s_0) \), it follows from calculus that

\[
\cos \theta = \frac{(f \circ \gamma)'(t_0) \cdot (f \circ \eta)'(s_0)}{|(f \circ \gamma)'(t_0)|| (f \circ \eta)'(s_0)|}
\]

\[
= \frac{1}{2} \frac{f'(z) \gamma'(t_0) \eta'(s_0) + \overline{f'(z)} \gamma'(t_0) \eta'(s_0)}{|f'(z)||f'(\eta(s_0))|}
\]

which equals the angle between the vectors, \( \gamma'(t_0) \) and \( \eta'(s_0) \). Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal.

Actually, they also preserve orientations. If \( z = x + iy \) and \( w = a + ib \) are two complex numbers, then \( (x, y, 0) \) and \( (a, b, 0) \) are two vectors in \( \mathbb{R}^3 \). Recall that the
cross product, \((x, y, 0) \times (a, b, 0)\), yields a vector normal to the two given vectors such that the triple, \((x, y, 0), (a, b, 0)\), and \((x, y, 0) \times (a, b, 0)\) satisfies the right hand rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product will produce a vector which is a multiple of \(k\), the unit vector in the direction of the \(z\) axis. In fact, you can verify by computing both sides that, letting \(z = x + iy\) and \(w = a + ib\),
\[
(x, y, 0) \times (a, b, 0) = \text{Re}(ziw)k.
\]
Therefore, in the above situation,
\[
(f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) = \text{Re} \left( f'(\gamma(t_0)) \gamma'(t_0) i\overline{f'(\eta(s_0))\eta'(s_0)} \right) k
\]  
\[
= |f'(z)|^2 \text{Re} \left( \gamma'(t_0) \overline{\eta'(s_0)} \right) k
\]
which shows that the orientation of \(\gamma'(t_0)\), \(\eta'(s_0)\) is the same as the orientation of \((f \circ \gamma)'(t_0), (f \circ \eta)'(s_0)\). Mappings which preserve both orientation and angles are called conformal mappings and this has shown that analytic functions are conformal mappings if the derivative does not vanish.

### 53.2 Fractional Linear Transformations

#### 53.2.1 Circles And Lines

These mappings map lines and circles to either lines or circles.

**Definition 53.2.1** A fractional linear transformation is a function of the form
\[
f(z) = \frac{az + b}{cz + d} \tag{53.2.1}
\]
where \(ad - bc \neq 0\).

Note that if \(c = 0\), this reduces to a linear transformation \((a/d)z + (b/d)\). Special cases of these are defined as follows.

- **dilations**: \(z \rightarrow \delta z, \delta \neq 0\),
- **inversions**: \(z \rightarrow \frac{1}{z}\),
- **translations**: \(z \rightarrow z + \rho\).

The next lemma is the key to understanding fractional linear transformations.

**Lemma 53.2.2** The fractional linear transformation, \(f(z)\), can be written as a finite composition of dilations, inversions, and translations.
53.2. FRACTIONAL LINEAR TRANSFORMATIONS

Proof: Let

\[ S_1(z) = z + \frac{d}{c}, S_2(z) = \frac{1}{z}, S_3(z) = \frac{(bc - ad)}{c^2}z \]

and

\[ S_4(z) = z + \frac{a}{c} \]

in the case where \( c \neq 0 \). Then \( f(z) \) given in (53.2.1) is of the form

\[ f(z) = S_4 \circ S_3 \circ S_2 \circ S_1. \]

Here is why.

\[ S_2(S_1(z)) = S_2\left(z + \frac{d}{c}\right) = \frac{1}{z + \frac{d}{c}} = \frac{c}{zc + d}. \]

Now consider

\[ S_3\left(\frac{c}{zc + d}\right) = \frac{(bc - ad)}{c^2} \left(\frac{c}{zc + d}\right) = \frac{bc - ad}{c(zc + d)}. \]

Finally, consider

\[ S_4\left(\frac{bc - ad}{c(zc + d)}\right) = \frac{bc - ad}{c(zc + d)} + \frac{a}{c} = \frac{b + az}{zc + d}. \]

In case that \( c = 0 \), \( f(z) = \frac{a}{2}z + \frac{b}{3} \) which is a translation composed with a dilation. Because of the assumption that \( ad - bc \neq 0 \), it follows that since \( c = 0 \), both \( a \) and \( d \neq 0 \). This proves the lemma.

This lemma implies the following corollary.

Corollary 53.2.3 Fractional linear transformations map circles and lines to circles or lines.

Proof: It is obvious that dilations and translations map circles to circles and lines to lines. What of inversions? If inversions have this property, the above lemma implies a general fractional linear transformation has this property as well.

Note that all circles and lines may be put in the form

\[ \alpha(x^2 + y^2) - 2ax - 2by = r^2 - (a^2 + b^2) \]

where \( \alpha = 1 \) gives a circle centered at \((a, b)\) with radius \( r \) and \( \alpha = 0 \) gives a line. In terms of complex variables you may therefore consider all possible circles and lines in the form

\[ \alpha z\overline{z} + \beta z + \overline{\beta}z + \gamma = 0, \quad (53.2.2) \]

To see this let \( \beta = \beta_1 + i\beta_2 \) where \( \beta_1 \equiv -a \) and \( \beta_2 \equiv b \). Note that even if \( \alpha \) is not 0 or 1 the expression still corresponds to either a circle or a line because you can
divide by $\alpha$ if $\alpha \neq 0$. Now I verify that replacing $z$ with $\frac{1}{z}$ results in an expression of the form in \(53.2.2\). Thus, let $w = \frac{1}{z}$ where $z$ satisfies \(53.2.2\). Then
\[
(\alpha + \beta w + \gamma w + \eta w) = \frac{1}{z} (\alpha z + \beta z + \gamma z + \eta z) = 0
\]
and so $w$ also satisfies a relation like \(53.2.2\). One simply switches $\alpha$ with $\gamma$ and $\beta$ with $\eta$. Note the situation is slightly different than with dilations and translations.

In the case of an inversion, a circle becomes either a line or a circle and similarly, a line becomes either a circle or a line. This proves the corollary.

The next example is quite important.

**Example 53.2.4** Consider the fractional linear transformation, $w = \frac{z-i}{z+i}$.

First consider what this mapping does to the points of the form $z = x + i0$. Substituting into the expression for $w$,
\[
w = \frac{x - i}{x + i} = \frac{x^2 - 1 - 2xi}{x^2 + 1},
\]
a point on the unit circle. Thus this transformation maps the real axis to the unit circle.

The upper half plane is composed of points of the form $x + iy$ where $y > 0$. Substituting in to the transformation,
\[
w = \frac{x + i(y - 1)}{x + i(y + 1)},
\]
which is seen to be a point on the interior of the unit disk because $|y - 1| < |y + 1|$ which implies $|x + i(y + 1)| > |x + i(y - 1)|$. Therefore, this transformation maps the upper half plane to the interior of the unit disk.

One might wonder whether the mapping is one to one and onto. The mapping is clearly one to one because it has an inverse, $z = -i \frac{w+1}{w-1}$ for all $w$ in the interior of the unit disk. Also, a short computation verifies that $z$ so defined is in the upper half plane. Therefore, this transformation maps \(\{z \in \mathbb{C} \text{ such that } \Im z > 0\}\) one to one and onto the unit disk \(\{z \in \mathbb{C} \text{ such that } |z| < 1\}\).

A fancy way to do part of this is to use Theorem 50.3.5: \(\limsup_{z \to a} \left| \frac{z-i}{z+i} \right| \leq 1\) whenever $a$ is the real axis or $\infty$. Therefore, \(\left| \frac{z-i}{z+i} \right| \leq 1\). This is a little shorter.

### 53.2.2 Three Points To Three Points
There is a simple procedure for determining fractional linear transformations which map a given set of three points to another set of three points. The problem is as follows: There are three distinct points in the extended complex plane, $z_1, z_2,$ and $z_3$ and it is desired to find a fractional linear transformation such that $z_i \rightarrow w_i$ for $i = 1, 2, 3$ where here $w_1, w_2,$ and $w_3$ are three distinct points in the extended
To find the desired fractional linear transformation, solve the following equation for $w$:

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

The result will be a fractional linear transformation with the desired properties. If any of the points equals $\infty$, then the quotient containing this point should be adjusted.

Why should this procedure work? Here is a heuristic argument to indicate why you would expect this to happen rather than a rigorous proof. The reader may want to tighten the argument to give a proof. First suppose $z = z_1$. Then the right side equals zero and so the left side also must equal zero. However, this requires $w = w_1$. Next suppose $z = z_2$. Then the right side equals 1. To get a 1 on the left, you need $w = w_2$. Finally suppose $z = z_3$. Then the right side involves division by 0. To get the same bad behavior, on the left, you need $w = w_3$.

**Example 53.2.5** Let $\mathrm{Im} \xi > 0$ and consider the fractional linear transformation which takes $\xi$ to 0, $\bar{\xi}$ to $\infty$ and 0 to $\xi/\bar{\xi}$.

The equation for $w$ is

$$\frac{w - 0}{w - (\xi/\bar{\xi})} = \frac{z - \xi}{z - 0} \cdot \frac{\bar{\xi} - 0}{\bar{\xi} - \xi}$$

After some computations,

$$w = \frac{z - \xi}{z - \bar{\xi}}.$$

Note that this has the property that $\frac{z - \xi}{z - \bar{\xi}}$ is always a point on the unit circle because it is a complex number divided by its conjugate. Therefore, this fractional linear transformation maps the real line to the unit circle. It also takes the point, $\xi$ to 0 and so it must map the upper half plane to the unit disk. You can verify the mapping is onto as well.

**Example 53.2.6** Let $z_1 = 0, z_2 = 1$, and $z_3 = 2$ and let $w_1 = 0, w_2 = i$, and $w_3 = 2i$.

Then the equation to solve is

$$\frac{w}{w - 2i} \cdot \frac{-i}{i} = \frac{z}{z - 2} \cdot \frac{-1}{1}$$

Solving this yields $w = iz$ which clearly works.
53.3 Riemann Mapping Theorem

From the open mapping theorem analytic functions map regions to other regions or else to single points. The Riemann mapping theorem states that for every simply connected region, \( \Omega \) which is not equal to all of \( \mathbb{C} \) there exists an analytic function, \( f \) such that \( f(\Omega) = B(0,1) \) and in addition to this, \( f \) is one to one. The proof involves several ideas which have been developed up to now. The proof is based on the following important theorem, a case of Montel’s theorem. Before, beginning, note that the Riemann mapping theorem is a classic example of a major existence theorem. In mathematics there are two sorts of questions, those related to whether something exists and those involving methods for finding it. The real questions are often related to questions of existence. There is a long and involved history for proofs of this theorem. The first proofs were based on the Dirichlet principle and turned out to be incorrect, thanks to Weierstrass who pointed out the errors. For more on the history of this theorem, see Hille [59].

The following theorem is really wonderful. It is about the existence of a subsequence having certain salubrious properties. It is this wonderful result which will give the existence of the mapping desired. The other parts of the argument are technical details to set things up and use this theorem.

53.3.1 Montel’s Theorem

**Theorem 53.3.1** Let \( \Omega \) be an open set in \( \mathbb{C} \) and let \( F \) denote a set of analytic functions mapping \( \Omega \) to \( B(0,M) \subseteq \mathbb{C} \). Then there exists a sequence of functions from \( F \), \( \{ f_n \}_{n=1}^{\infty} \) and an analytic function, \( f \) such that \( f_n^{(k)} \) converges uniformly to \( f^{(k)} \) on every compact subset of \( \Omega \).

**Proof:** First note there exists a sequence of compact sets, \( K_n \) such that \( K_n \subseteq \text{int} K_{n+1} \subseteq \Omega \) for all \( n \) where here \( \text{int} \) denotes the interior of the set \( K \), the union of all open sets contained in \( K \) and \( \bigcup_{n=1}^{\infty} K_n = \Omega \). In fact, you can verify that \( B(0,n) \cap \{ z \in \Omega : \text{dist}(z,\Omega^C) \leq \frac{1}{n} \} \) works for \( K_n \). Then there exist positive numbers, \( \delta_n \) such that if \( z \in K_n \), then \( B(z,\delta_n) \subseteq \text{int} K_{n+1} \). Now denote by \( F_n \) the set of restrictions of functions of \( F \) to \( K_n \). Then let \( z \in K_n \) and let \( \gamma(t) \equiv z + \delta_n e^{it}, t \in [0,2\pi] \). It follows that for \( z_1 \in B(z,\delta_n) \), and \( f \in F \),

\[
|f(z) - f(z_1)| = \left| \frac{1}{2\pi i} \int_{\gamma} f(w) \left( \frac{1}{w-z} - \frac{1}{w-z_1} \right) dw \right| \\
\leq \frac{1}{2\pi} \left| \int_{\gamma} f(w) \left( \frac{z-z_1}{(w-z)(w-z_1)} \right) dw \right|
\]

Letting \( |z_1 - z| < \frac{\delta_n}{2} \),

\[
|f(z) - f(z_1)| \leq \frac{M}{2\pi} 2\pi \delta_n \frac{|z-z_1|}{\delta_n^2/2} \\
\leq 2M \frac{|z-z_1|}{\delta_n}.
\]
It follows that $\mathcal{F}_n$ is equicontinuous and uniformly bounded so by the Arzela Ascoli theorem there exists a sequence, $\{f_{nk}\}_{k=1}^{\infty} \subseteq \mathcal{F}$ which converges uniformly on $K_n$. Let $\{f_{1k}\}_{k=1}^{\infty}$ converge uniformly on $K_1$. Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\{f_{2k}\}_{k=1}^{\infty}$ which also converges uniformly on $K_2$. Continue in this way to obtain $\{f_{mn}\}_{m=1}^{\infty}$ is a subsequence of $\{f_{mk}\}_{k=1}^{\infty}$ and so it converges uniformly on $K_m$ for all $m$. Denoting $f_{mn}$ by $f_n$ for short, this is the sequence of functions promised by the theorem. It is clear $\{f_n\}_{n=1}^{\infty}$ converges uniformly on every compact subset of $\Omega$ because every such set is contained in $K_m$ for all $m$ large enough. Let $f(z)$ be the point to which $f_n(z)$ converges. Then $f$ is a continuous function defined on $\Omega$. Is $f$ analytic? Yes it is by Lemma 49.3.13. Alternatively, you could let $T \subseteq \Omega$ be a triangle. Then
\[
\int_{\partial T} f(z) \, dz = \lim_{n \to \infty} \int_{\partial T} f_n(z) \, dz = 0.
\]
Therefore, by Morera’s theorem, $f$ is analytic.

As for the uniform convergence of the derivatives of $f$, recall Theorem 50.4.7 about the existence of a cycle. Let $K$ be a compact subset of $\text{int}(K_n)$ and let $\{\gamma_k\}_{k=1}^{m}$ be closed oriented curves contained in $\text{int}(K_n) \setminus K$ such that $\sum_{j=1}^{m} n(\gamma_k, z) = 1$ for every $z \in K$. Also let $\eta$ denote the distance between $\cup_j \gamma_j$ and $K$. Then for $z \in K$,
\[
|f^{(k)}(z) - f_n^{(k)}(z)| = \left\| \frac{k!}{2\pi i} \sum_{j=1}^{m} \int_{\gamma_j} \frac{f(w) - f_n(w)}{(w-z)^{k+1}} \, dw \right\|
\leq \frac{k!}{2\pi} \|f_k - f\|_{K_n} \sum_{j=1}^{m} \text{(length of } \gamma_j) \frac{1}{\eta^{k+1}}.
\]
where here $\|f_k - f\|_{K_n} \equiv \sup \{|f_k(z) - f(z) : z \in K_n\}$. Thus you get uniform convergence of the derivatives.

Since the family, $\mathcal{F}$ satisfies the conclusion of Theorem 50.4.7 it is known as a normal family of functions. More generally,

**Definition 53.3.2** Let $\mathcal{F}$ denote a collection of functions which are analytic on $\Omega$, a region. Then $\mathcal{F}$ is normal if every sequence contained in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of $\Omega$.

The following result is about a certain class of fractional linear transformations. Recall Lemma 50.4.7 which is listed here for convenience.

**Lemma 53.3.3** For $\alpha \in B(0,1)$, let
\[
\phi_{\alpha}(z) \equiv \frac{z - \alpha}{1 - \overline{\alpha}z}.
\]
Then $\phi_\alpha$ maps $B(0,1)$ one to one and onto $B(0,1)$, $\phi_\alpha^{-1} = \phi_{-\alpha}$, and
\[
\phi'_\alpha(\alpha) = \frac{1}{1 - |\alpha|^2}.
\]

The next lemma, known as Schwarz’s lemma is interesting for its own sake but will also be an important part of the proof of the Riemann mapping theorem. It was stated and proved earlier but for convenience it is given again here.

**Lemma 53.3.4** Suppose $F : B(0,1) \to B(0,1)$, $F$ is analytic, and $F(0) = 0$. Then for all $z \in B(0,1)$,
\[
|F(z)| \leq |z|,
\]
and
\[
|F'(0)| \leq 1.
\]

If equality holds in (53.3.4) then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and
\[
F(z) = \lambda z.
\]

**Proof:** First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,
\[
\left| \frac{F(z)}{z} \right| \leq \max_{t \in [0,2\pi]} \left| \frac{F(re^{it})}{r} \right| \leq \frac{1}{r}.
\]

Then letting $r \to 1$,
\[
\left| \frac{F(z)}{z} \right| \leq 1
\]
this shows (53.3.4) and it also verifies (53.3.3) on taking the limit as $z \to 0$. If equality holds in (53.3.4) then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, $\lambda$ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$. This proves the lemma.

**Definition 53.3.5** A region, $\Omega$ has the square root property if whenever $f, \frac{1}{f} : \Omega \to \mathbb{C}$ are both analytic, it follows there exists $\phi : \Omega \to \mathbb{C}$ such that $\phi$ is analytic and $f(z) = \phi^2(z)$.

The next theorem will turn out to be equivalent to the Riemann mapping theorem.

1This implies $f$ has no zero on $\Omega$. 
53.3. RIEMANN MAPPING THEOREM

53.3.2 Regions With Square Root Property

Theorem 53.3.6 Let $\Omega \neq \mathbb{C}$ for $\Omega$ a region and suppose $\Omega$ has the square root property. Then for $z_0 \in \Omega$ there exists $h : \Omega \rightarrow B(0,1)$ such that $h$ is one to one, onto, analytic, and $h(z_0) = 0$.

Proof: Define $\mathcal{F}$ to be the set of functions, $f$ such that $f : \Omega \rightarrow B(0,1)$ is one to one and analytic. The first task is to show $\mathcal{F}$ is nonempty. Then, using Montel’s theorem it will be shown there is a function in $\mathcal{F}$, $h$, such that $|h'(z_0)| \geq |\psi'(z_0)|$ for all $\psi \in \mathcal{F}$. When this has been done it will be shown that $h$ is actually onto. This will prove the theorem.

Claim 1: $\mathcal{F}$ is nonempty.

Proof of Claim 1: Since $\Omega \neq \mathbb{C}$ it follows there exists $\xi \notin \Omega$. Then it follows $z - \xi$ and $\frac{1}{z - \xi}$ are both analytic on $\Omega$. Since $\Omega$ has the square root property, there exists an analytic function, $\phi : \Omega \rightarrow \mathbb{C}$ such that $\phi^2(z) = z - \xi$ for all $z \in \Omega$, $\phi(z) = \sqrt{z - \xi}$. Since $z - \xi$ is not constant, neither is $\phi$ and it follows from the open mapping theorem that $\phi(\Omega)$ is a region. Note also that $\phi$ is one to one because if $\phi(z_1) = \phi(z_2)$, then you can square both sides and conclude $z_1 - \xi = z_2 - \xi$ implying $z_1 = z_2$.

Now pick $a \in \phi(\Omega)$. Thus $\sqrt{a - \xi} = a$. I claim there exists a positive lower bound to $|\sqrt{z - \xi} + a|$ for $z \in \Omega$. If not, there exists a sequence, ${\{z_n\} \subseteq \Omega}$ such that

$$\sqrt{z_n - \xi} + a = \sqrt{z_n - \xi} + \sqrt{z_a - \xi} \equiv \varepsilon_n \rightarrow 0.$$  

Then

$$\sqrt{z_n - \xi} = (\varepsilon_n - \sqrt{z_a - \xi})$$  \hspace{1cm} (53.3.6)

and squaring both sides,

$$z_n - \xi = \varepsilon_n^2 + z_a - \xi - 2\varepsilon_n\sqrt{z_a - \xi}.$$  

Consequently, $(z_n - z_a) = \varepsilon_n^2 - 2\varepsilon_n\sqrt{z_a - \xi}$ which converges to 0. Taking the limit in (53.3.6), it follows $2\sqrt{z_a - \xi} = 0$ and so $\xi = z_a$, a contradiction to $\xi \notin \Omega$. Choose $r > 0$ such that for all $z \in \Omega$, $|\sqrt{z - \xi} + a| > r > 0$. Then consider

$$\psi(z) \equiv \frac{r}{\sqrt{z - \xi} + a}.$$  \hspace{1cm} (53.3.7)

This is one to one, analytic, and maps $\Omega$ into $B(0,1)$ ($|\sqrt{z - \xi} + a| > r$). Thus $\mathcal{F}$ is not empty and this proves the claim.

Claim 2: Let $z_0 \in \Omega$. There exists a finite positive real number, $\eta$, defined by

$$\eta \equiv \sup \{|\psi'(z_0)| : \psi \in \mathcal{F}\}$$  \hspace{1cm} (53.3.8)

and an analytic function, $h \in \mathcal{F}$ such that $|h'(z_0)| = \eta$. Furthermore, $h(z_0) = 0$.

Proof of Claim 2: First you show $\eta < \infty$. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and $r$ is small enough that $B(z_0, r) \subseteq \Omega$. Then for $\psi \in \mathcal{F}$, the Cauchy integral
formula for the derivative implies

\[ \psi'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{\psi(w)}{(w - z_0)^2} dw \]

and so \( |\psi'(z_0)| \leq (1/2\pi)2\pi r (1/r^2) = 1/r. \) Therefore, \( \eta < \infty \) as desired. For \( \psi \) defined above in \( 53.3.7 \)

\[ \psi'(z_0) = \frac{-r \phi'(z_0)}{(\phi(z_0) + a)^2} = \frac{-r (1/2) (\sqrt{z_0 - \xi})^{-1}}{(\phi(z_0) + a)^2} \neq 0. \]

Therefore, \( \eta > 0. \) It remains to verify the existence of the function, \( h. \)

By Theorem \( 53.3.1 \), there exists a sequence, \( \{\psi_n\} \), of functions in \( F \) and an analytic function, \( h \), such that

\[ |\psi'_n(z_0)| \to \eta \] (53.3.9)

and

\[ \psi_n \to h, \psi'_n \to h', \] (53.3.10)

uniformly on all compact subsets of \( \Omega \). It follows

\[ |h'(z_0)| = \lim_{n \to \infty} |\psi'_n(z_0)| = \eta \] (53.3.11)

and for all \( z \in \Omega, \)

\[ |h(z)| = \lim_{n \to \infty} |\psi_n(z)| \leq 1. \] (53.3.12)

By \( 53.3.7 \), \( h \) is not a constant. Therefore, in fact, \( |h(z)| < 1 \) for all \( z \in \Omega \) in \( 53.3.12 \) by the open mapping theorem.

Next it must be shown that \( h \) is one to one in order to conclude \( h \in F \). Pick \( z_1 \in \Omega \) and suppose \( z_2 \) is another point of \( \Omega \). Since the zeros of \( h - h(z_1) \) have no limit point, there exists a circular contour bounding a circle which contains \( z_2 \) but not \( z_1 \) such that \( \gamma^* \) contains no zeros of \( h - h(z_1) \).

Using the theorem on counting zeros, Theorem \( 50.6.1 \) and the fact that \( \psi_n \) is one to one,

\[ 0 = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{\psi'_n(w)}{\psi_n(w) - \psi_n(z_1)} dw = \frac{1}{2\pi i} \int_{\gamma} \frac{h'(w)}{h(w) - h(z_1)} dw, \]
which shows that \( h - h ( z_1 ) \) has no zeros in \( B ( z_2, r ) \). In particular \( z_2 \) is not a zero of \( h - h ( z_1 ) \). This shows that \( h \) is one to one since \( z_2 \neq z_1 \) was arbitrary. Therefore, \( h \in \mathcal{F} \). It only remains to verify that \( h ( z_0 ) = 0 \).

If \( h ( z_0 ) \neq 0 \), consider \( \phi_{h(z_0)} \circ h \) where \( \phi_\alpha \) is the fractional linear transformation defined in Lemma 53.3.3. By this lemma it follows \( \phi_{h(z_0)} \circ h \in \mathcal{F} \). Now using the chain rule,

\[
\left| \left( \phi_{h(z_0)} \circ h \right)' (z_0) \right| = \left| \phi_{h(z_0)}' (h (z_0)) \right| \left| h' (z_0) \right|
\]

\[
= \left| \frac{1}{1 - |h (z_0)|^2} \right| \left| h' (z_0) \right|
\]

\[
= \left| \frac{1}{1 - |h (z_0)|^2} \right| \eta > \eta
\]

Contradicting the definition of \( \eta \). This proves Claim 2.

**Claim 3:** The function, \( h \) just obtained maps \( \Omega \) onto \( B (0, 1) \).

**Proof of Claim 3:** To show \( h \) is onto, use the fractional linear transformation of Lemma 53.3.3. Suppose \( h \) is not onto. Then there exists \( \alpha \in B (0, 1) \setminus h (\Omega) \). Then \( 0 \neq \phi_\alpha \circ h (z) \) for all \( z \in \Omega \) because

\[
\phi_\alpha \circ h (z) = \frac{h (z) - \alpha}{1 - \alpha h (z)}
\]

and it is assumed \( \alpha \notin h (\Omega) \). Therefore, since \( \Omega \) has the square root property, you can consider an analytic function \( z \to \sqrt{\phi_\alpha \circ h (z)} \). This function is one to one because both \( \phi_\alpha \) and \( h \) are. Also, the values of this function are in \( B (0, 1) \) by Lemma 53.3.3 so it is in \( \mathcal{F} \).

Now let

\[
\psi \equiv \phi_{\phi_\alpha \circ h (z_0)} \circ \sqrt{\phi_\alpha \circ h}.
\]

Thus

\[
\psi (z_0) = \phi_{\phi_\alpha \circ h (z_0)} \circ \sqrt{\phi_\alpha \circ h (z_0)} = 0
\]

and \( \psi \) is a one to one mapping of \( \Omega \) into \( B (0, 1) \) so \( \psi \) is also in \( \mathcal{F} \). Therefore,

\[
|\psi' (z_0)| \leq \eta, \quad \left| \left( \sqrt{\phi_\alpha \circ h} \right)' (z_0) \right| \leq \eta.
\]

(53.3.14)

Define \( s (w) \equiv w^2 \). Then using Lemma 53.3.3, in particular, the description of \( \phi^{-1}_\alpha = \phi_{-\alpha} \), you can solve \( \phi^{-1}_\alpha \) for \( h \) to obtain

\[
h (z) = \phi_{-\alpha} \circ s \circ \phi_{\phi_\alpha \circ h (z_0)} \circ \psi
\]

\[
= \left( \phi_{-\alpha} \circ s \circ \phi_{\phi_\alpha \circ h (z_0)} \circ \psi \right) \equiv F \circ \psi (z)
\]

(53.3.15)
Now
\[ F(0) = \phi_{-\alpha} \circ s \circ \sqrt{\phi_{\alpha} \circ h}(z_0) = \phi_{\alpha}^{-1}(\phi_{\alpha} \circ h(z_0)) = h(z_0) = 0 \]
and \( F \) maps \( B(0,1) \) into \( B(0,1) \). Also, \( F \) is not one to one because it maps \( B(0,1) \) to \( B(0,1) \) and has \( s \) in its definition. Thus there exists \( z_1 \in B(0,1) \) such that
\[ \phi_{-\sqrt{\phi_{\alpha} \circ h}(z_0)}(z_1) = -\frac{1}{2} \]
and another point \( z_2 \in B(0,1) \) such that
\[ \phi_{-\sqrt{\phi_{\alpha} \circ h}(z_0)}(z_2) = \frac{1}{2}. \]
However, thanks to \( s \), \( F(z_1) = F(z_2) \).

Since \( F(0) = h(z_0) = 0 \), you can apply the Schwarz lemma to \( F \). Since \( F \) is not one to one, it can’t be true that \( F(z) = \lambda z \) for \( |\lambda| = 1 \) and so by the Schwarz lemma it must be the case that \( |F'(0)| < 1 \). But this implies from 53.3.15 and 53.3.14 that
\[ \eta = |h'(z_0)| = |F'(\psi(z_0))| |\psi'(z_0)| = |F'(0)||\psi'(z_0)| < |\psi'(z_0)| \leq \eta, \]
a contradiction. This proves the theorem.

The following lemma yields the usual form of the Riemann mapping theorem.

**Lemma 53.3.7** Let \( \Omega \) be a simply connected region with \( \Omega \neq \mathbb{C} \). Then \( \Omega \) has the square root property.

**Proof:** Let \( f \) and \( \frac{1}{f} \) both be analytic on \( \Omega \). Then \( \frac{L'}{L} \) is analytic on \( \Omega \) so by Corollary 53.7.23, there exists \( F \), analytic on \( \Omega \) such that \( F' = \frac{L'}{L} \) on \( \Omega \). Then \( \left( f e^{-\tilde{F}} \right)' = 0 \) and so \( f(z) = C e^{\tilde{F}} = e^{a+ib} e^{\tilde{F}}. \) Now let \( F = \tilde{F} + a + ib \). Then \( F \) is still a primitive of \( f'/f \) and \( f(z) = e^{F(z)} \). Now let \( \phi(z) \equiv e^{\frac{1}{2}F(z)}. \) Then \( \phi \) is the desired square root and so \( \Omega \) has the square root property.

**Corollary 53.3.8** (Riemann mapping theorem) Let \( \Omega \) be a simply connected region with \( \Omega \neq \mathbb{C} \) and let \( z_0 \in \Omega \). Then there exists a function, \( f : \Omega \to B(0,1) \) such that \( f \) is one to one, analytic, and onto with \( f(z_0) = 0 \). Furthermore, \( f^{-1} \) is also analytic.

**Proof:** From Theorem 53.3.17 and Lemma 53.3.18 there exists a function, \( f : \Omega \to B(0,1) \) which is one to one, onto, and analytic such that \( f(z_0) = 0 \). The assertion that \( f^{-1} \) is analytic follows from the open mapping theorem.

## 53.4 Analytic Continuation

### 53.4.1 Regular And Singular Points

Given a function which is analytic on some set, can you extend it to an analytic function defined on a larger set? Sometimes you can do this. It was done in the proof of the Cauchy integral formula. There are also reflection theorems like those
discussed in the exercises starting with Problem 10 on Page 1685. Here I will give a systematic way of extending an analytic function to a larger set. I will emphasize simply connected regions. The subject of analytic continuation is much larger than the introduction given here. A good source for much more on this is found in Alfors [2]. The approach given here is suggested by Rudin [102] and avoids many of the standard technicalities.

**Definition 53.4.1** Let $f$ be analytic on $B(a, r)$ and let $\beta \in \partial B(a, r)$. Then $\beta$ is called a regular point of $f$ if there exists some $\delta > 0$ and a function, $g$ analytic on $B(\beta, \delta)$ such that $g = f$ on $B(\beta, \delta) \cap B(a, r)$. Those points of $\partial B(a, r)$ which are not regular are called singular.

**Theorem 53.4.2** Suppose $f$ is analytic on $B(a, r)$ and the power series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

has radius of convergence $r$. Then there exists a singular point on $\partial B(a, r)$.

**Proof:** If not, then for every $z \in \partial B(a, r)$ there exists $\delta_z > 0$ and $g_z$ analytic on $B(z, \delta_z)$ such that $g_z = f$ on $B(z, \delta_z) \cap B(a, r)$. Since $\partial B(a, r)$ is compact, there exist $z_1, \ldots, z_n$, points in $\partial B(a, r)$ such that $\{B(z_k, \delta_{z_k})\}_{k=1}^{n}$ covers $\partial B(a, r)$. Now define

$$g(z) = \begin{cases} f(z) & \text{if } z \in B(a, r) \\ g_{z_k}(z) & \text{if } z \in B(z_k, \delta_{z_k}) \end{cases}$$

Is this well defined? If $z \in B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j})$, is $g_{z_i}(z) = g_{z_j}(z)$? Consider the following picture representing this situation.
You see that if \( z \in B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j}) \) then \( I \equiv B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j}) \) \( \cap B(a, r) \) is a nonempty open set. Both \( g_{z_i} \) and \( g_{z_j} \) equal \( f \) on \( I \). Therefore, they must be equal on \( B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j}) \) because \( I \) has a limit point. Therefore, \( g \) is well defined and analytic on an open set containing \( \overline{B(a, r)} \) contrary to the assumption that the radius of convergence of the above power series equals \( r \). This proves the theorem.

### 53.4.2 Continuation Along A Curve

Next I will describe what is meant by continuation along a curve. The following definition is standard and is found in Rudin [Rud].

**Definition 53.4.3** A function element is an ordered pair, \((f, D)\) where \( D \) is an open ball and \( f \) is analytic on \( D \). \((f_0, D_0)\) and \((f_1, D_1)\) are direct continuations of each other if \( D_1 \cap D_0 \neq \emptyset \) and \( f_0 = f_1 \) on \( D_1 \cap D_0 \). In this case I will write \((f_0, D_0) \sim (f_1, D_1)\). A chain is a finite sequence, of disks, \( \{D_0, \cdots, D_n\} \) such that \( D_{i-1} \cap D_i \neq \emptyset \). If \((f_0, D_0)\) is a given function element and there exist function elements, \((f_i, D_i)\) such that \( \{D_0, \cdots, D_n\} \) is a chain and \((f_{j-1}, D_{j-1}) \sim (f_j, D_j)\) then \((f_n, D_n)\) is called the analytic continuation of \((f_0, D_0)\) along the chain \( \{D_0, \cdots, D_n\} \). Now suppose \( \gamma \) is an oriented curve with parameter interval \([a, b]\) and there exists a chain, \( \{D_0, \cdots, D_n\} \) such that \( \gamma^* \subseteq \cup_{k=1}^n D_k \), \( \gamma(a) \) is the center of \( D_0 \), \( \gamma(b) \) is the center of \( D_n \), and there is an increasing list of numbers in \([a, b]\), \( a = s_0 < s_1 \cdots < s_n = b \) such that \( \gamma([s_i, s_{i+1}]) \subseteq D_i \) and \((f_n, D_n)\) is an analytic continuation of \((f_0, D_0)\) along the chain. Then \((f_n, D_n)\) is called an analytic continuation of \((f_0, D_0)\) along the curve \( \gamma \). \( \gamma \) will always be a continuous curve. Nothing more is needed.

In the above situation it does not follow that if \( D_n \cap D_0 \neq \emptyset \), that \( f_n = f_0 \)! However, there are some cases where this will happen. This is the monodromy theorem which follows. This is as far as I will go on the subject of analytic continuation. For more on this subject including a development of the concept of Riemann surfaces, see Alfors [Alf].

**Lemma 53.4.4** Suppose \((f, B(0, r))\) for \( r < 1 \) is a function element and \((f, B(0, r))\) can be analytically continued along every curve in \( B(0, 1) \) that starts at \( 0 \). Then there exists an analytic function, \( g \) defined on \( B(0, 1) \) such that \( g = f \) on \( B(0, r) \).

**Proof:** Let

\[
R = \sup\{r_1 \geq r \text{ such that there exists } g_{r_1} \text{ analytic on } B(0, r_1) \text{ which agrees with } f \text{ on } B(0, r)\}.
\]

Define \( g_R(z) \equiv g_{r_1}(z) \) where \(|z| < r_1\). This is well defined because if you use \( r_1 \) and \( r_2 \), both \( g_{r_1} \) and \( g_{r_2} \) agree with \( f \) on \( B(0, r) \), a set with a limit point and so the two functions agree at every point in both \( B(0, r_1) \) and \( B(0, r_2) \). Thus \( g_R \) is analytic on \( B(0, R) \). If \( R < 1 \), then by the assumption there are no singular points
53.4. ANALYTIC CONTINUATION

on \( B(0, R) \) and so Theorem \ref{analytic-continuation} implies the radius of convergence of the power series for \( g_R \) is larger than \( R \) contradicting the choice of \( R \). Therefore, \( R = 1 \) and this proves the lemma. Let \( g = g_R \).

The following theorem is the main result in this subject, the monodromy theorem.

**Theorem 53.4.5** Let \( \Omega \) be a simply connected proper subset of \( \mathbb{C} \) and suppose \((f, B(a, r))\) is a function element with \( B(a, r) \subseteq \Omega \). Suppose also that this function element can be analytically continued along every curve through \( a \). Then there exists \( G \) analytic on \( \Omega \) such that \( G \) agrees with \( f \) on \( B(a, r) \).

**Proof:** By the Riemann mapping theorem, there exists \( h : \Omega \to B(0, 1) \) which is analytic, one to one and onto such that \( f(a) = 0 \). Since \( h \) is an open map, there exists \( \delta > 0 \) such that \( B(0, \delta) \subseteq h(B(a, r)) \).

It follows \( f \circ h^{-1} \) can be analytically continued along every curve through 0. By Lemma \ref{analytic-continuation} there exists \( g \) analytic on \( B(0, 1) \) which agrees with \( f \circ h^{-1} \) on \( B(0, \delta) \). Define \( G(z) \equiv g(h(z)) \). For \( z = h^{-1}(w) \), it follows \( G(h^{-1}(w)) = g(w) \). If \( w \in B(0, \delta) \), then \( G(h^{-1}(w)) = f \circ h^{-1}(w) \) and so \( G = f \) on \( h^{-1}(B(0, \delta)) \), an open set contained in \( B(a, r) \). Therefore, \( G = f \) on \( B(a, r) \) because \( h^{-1}(B(0, \delta)) \) has a limit point. This proves the theorem.

Actually, you sometimes want to consider the case where \( \Omega = \mathbb{C} \). This requires a small modification to obtain from the above theorem.

**Corollary 53.4.6** Suppose \((f, B(a, r))\) is a function element with \( B(a, r) \subseteq \mathbb{C} \). Suppose also that this function element can be analytically continued along every curve through \( a \). Then there exists \( G \) analytic on \( \mathbb{C} \) such that \( G \) agrees with \( f \) on \( B(a, r) \).

**Proof:** Let \( \Omega_1 \equiv \{ z \in \mathbb{C} : a + it : t > a \} \) and \( \Omega_2 \equiv \{ z \in \mathbb{C} : a - it : t > a \} \). Here is a picture of \( \Omega_1 \).

A picture of \( \Omega_2 \) is similar except the line extends down from the boundary of \( B(a, r) \). Thus \( B(a, r) \subseteq \Omega_1 \) and \( \Omega_1 \) is simply connected and proper. By Theorem \ref{analytic-continuation} there exist analytic functions, \( G_i \) analytic on \( \Omega_1 \) such that \( G_i = f \) on \( B(a, r) \). Thus \( G_1 = G_2 \) on \( B(a, r) \), a set with a limit point. Therefore, \( G_1 = G_2 \) on \( \Omega_1 \cap \Omega_2 \). Now let \( G(z) = G_i(z) \) where \( z \in \Omega_i \). This is well defined and analytic on \( \mathbb{C} \). This proves the corollary.
The Picard theorem says that if $f$ is an entire function and there are two complex numbers not contained in $f(\mathbb{C})$, then $f$ is constant. This is certainly one of the most amazing things which could be imagined. However, this is only the little Picard theorem. The big Picard theorem is even more incredible. This one asserts that to be non constant the entire function must take every value of $\mathbb{C}$ but two infinitely many times! I will begin with the little Picard theorem. The method of proof I will use is the one found in Saks and Zygmund [104], Conway [31] and Hille [59]. This is not the way Picard did it in 1879. That approach is very different and is presented at the end of the material on elliptic functions. This approach is much more recent dating it appears from around 1924.

**Lemma 53.5.1** Let $f$ be analytic on a region containing $B(0,r)$ and suppose

$$|f'(0)| = b > 0, f(0) = 0,$$

and $|f(z)| \leq M$ for all $z \in B(0,r)$. Then $f(B(0,r)) \supseteq B\left(0, \frac{r^2b^2}{6M}\right)$.

**Proof:** By assumption,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, |z| \leq r. \quad (53.5.16)$$

Then by the Cauchy integral formula for the derivative,

$$a_k = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{f(w)}{w^{k+1}} \, dw$$

where the integral is in the counter clockwise direction. Therefore,

$$|a_k| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(re^{i\theta})}{r^k} \right| d\theta \leq \frac{M}{r^k}.$$

In particular, $br \leq M$. Therefore, from \[\text{Lemma 53.5.1}\]

$$|f(z)| \geq b|z| - \sum_{k=2}^{\infty} \frac{M}{r^k} |z|^k = b|z| - \frac{M \left(\frac{|z|}{r}\right)^2}{1 - \frac{|z|}{r}} = b|z| - \frac{M |z|^2}{r^2 - r|z|}$$

Suppose $|z| = \frac{r^2b^2}{4M} < r$. Then this is no larger than

$$\frac{1}{4} \frac{b^2r^2}{M} \frac{3M - br}{M(4M - br)} \geq \frac{1}{4} \frac{b^2r^2}{M} \frac{3M - M}{M(4M - M)} = \frac{r^2b^2}{6M}.$$
53.5. THE PICARD THEOREMS

Let \(|w| < \frac{r^2b}{4M}\). Then for \(|z| = \frac{r^2b}{4M}\) and the above,

\[|w| = |(f(z) - w) - f(z)| < \frac{r^2b}{4M} \leq |f(z)|\]

and so by Rouche’s theorem, \(z \rightarrow f(z) - w\) and \(z \rightarrow f(z)\) have the same number of zeros in \(B \left(0, \frac{r^2b}{4M}\right)\). But \(f\) has at least one zero in this ball and so this shows there exists at least one \(z \in B \left(0, \frac{r^2b}{4M}\right)\) such that \(f(z) - w = 0\). This proves the lemma.

53.5.1 Two Competing Lemmas

Lemma 53.5.1 is a really nice lemma but there is something even better, Bloch’s lemma. This lemma does not depend on the bound of \(f\). Like the above two lemmas it is interesting for its own sake and in addition is the key to a fairly short proof of Picard’s theorem. It features the number \(\frac{1}{24}\). The best constant is not currently known.

Lemma 53.5.2 Let \(f\) be analytic on an open set containing \(B(0, R)\) and suppose \(|f'(0)| > 0\). Then there exists \(a \in B(0, R)\) such that

\[f(B(0, R)) \supseteq B \left(f(a), \frac{|f'(0)| R}{24}\right).\]

Proof: Let \(K(\rho) \equiv \max \{|f'(z)| : |z| = \rho\}\). For simplicity, let \(C_\rho \equiv \{z : |z| = \rho\}\).

Claim: \(K\) is continuous from the left.

Proof of claim: Let \(z_\rho \in C_\rho\) such that \(|f'(z_\rho)| = K(\rho)\). Then by the maximum modulus theorem, if \(\lambda \in (0, 1)\),

\[|f'(\lambda z_\rho)| \leq K(\lambda \rho) \leq K(\rho) = |f'(z_\rho)|.\]

Letting \(\lambda \rightarrow 1\) yields the claim.

Let \(\rho_0\) be the largest such that \((R - \rho_0) K(\rho_0) = R |f'(0)|\). (Note \((R - 0) K(0) = R |f'(0)|\).) Thus \(\rho_0 < R\) because \((R - R) K(R) = 0\). Let \(|a| = \rho_0\) such that \(|f'(a)| = K(\rho_0)\). Thus

\[|f'(a)(R - \rho_0) = |f'(0)| R\] (53.5.17)

Now let \(r = \frac{R - \rho_0}{2}\). From 53.5.14,

\[|f'(a)| r = \frac{1}{2} |f'(0)| R, B(a, r) \subseteq B(0, \rho_0 + r) \subseteq B(0, R)\.\] (53.5.18)
Therefore, if \( z \in B(a, r) \), it follows from the maximum modulus theorem and the definition of \( \rho_0 \) that

\[
|f'(z)| \leq K(\rho_0 + r) < \frac{R|f'(0)|}{R - \rho_0 - r} = \frac{2R|f'(0)|}{R - \rho_0} = \frac{R|f'(0)|}{2r}
\]

(53.5.19)

Let \( g(z) = f(a + z) - f(a) \) where \( z \in B(0, r) \). Then \( |g'(0)| = |f'(a)| > 0 \) and for \( z \in B(0, r) \),

\[
|g(z)| \leq \left| \int_{\gamma(a, z)} g'(w) \, dw \right| \leq |z - a| \frac{R|f'(0)|}{r} = R|f'(0)|.
\]

By Lemma 53.5.1 and 53.5.18,

\[
g(B(0, r)) \supseteq B \left( 0, \frac{r^2|f'(a)|^2}{6R|f'(0)|} \right) = B \left( 0, \frac{r^2 \left( \frac{1}{2r} \right) |f'(0)| R^2}{6R|f'(0)|} \right) = B \left( 0, \frac{|f'(0)| R^2}{24} \right)
\]

Now \( g(B(0, r)) = f(B(a, r)) - f(a) \) and so this implies

\[
f(B(0, R)) \supseteq f(B(a, r)) \supseteq B \left( f(a), \frac{|f'(0)| R}{24} \right).
\]

This proves the lemma.

Here is a slightly more general version which allows the center of the open set to be arbitrary.

**Lemma 53.5.3** Let \( f \) be analytic on an open set containing \( B(z_0, R) \) and suppose \( |f'(z_0)| > 0 \). Then there exists \( a \in B(z_0, R) \) such that

\[
f(B(z_0, R)) \supseteq B \left( f(a), \frac{|f'(z_0)| R}{24} \right).
\]
53.5. THE PICARD THEOREMS

Proof: You look at \( g(z) \equiv f(z_0 + z) \) for \( z \in B(0, R) \). Then \( g'(0) = f'(z_0) \) and so by Lemma 53.5.2 there exists \( a_1 \in B(0, R) \) such that

\[
g(B(0, R)) \supseteq B_g(a_1, \frac{|f'(z_0)| R}{24})
\]

Now \( g(B(0, R)) = f(B(z_0, R)) - f(z_0) \) and \( g(a_1) = f(a) - f(z_0) \) for some \( a \in B(z_0, R) \) and so

\[
f(B(z_0, R)) - f(z_0) \geq B_g(a_1, \frac{|f'(z_0)| R}{24}) = B_{f(a) - f(z_0), \frac{|f'(z_0)| R}{24}}
\]

which implies

\[
f(B(z_0, R)) \supseteq B_{f(a), \frac{|f'(z_0)| R}{24}}
\]

as claimed. This proves the lemma.

No attempt was made to find the best number to multiply by \( R|f'(z_0)| \). A discussion of this is given in Conway [31]. See also [59]. Much larger numbers than 1/24 are available and there is a conjecture due to Alfors about the best value. The conjecture is that 1/24 can be replaced with

\[
\frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{11}{12}\right)}{(1 + \sqrt{3})^{1/2} \Gamma\left(\frac{1}{4}\right)} \approx 0.47186
\]

You can see there is quite a gap between the constant for which this lemma is proved above and what is thought to be the best constant.

Bloch’s lemma above gives the existence of a ball of a certain size inside the image of a ball. By contrast the next lemma leads to conditions under which the values of a function do not contain a ball of certain radius. It concerns analytic functions which do not achieve the values 0 and 1.

**Lemma 53.5.4** Let \( \mathcal{F} \) denote the set of functions, \( f \) defined on \( \Omega \), a simply connected region which do not achieve the values 0 and 1. Then for each such function, it is possible to define a function analytic on \( \Omega \), \( H(z) \) by the formula

\[
H(z) \equiv \log \left[ \sqrt{\frac{\log (f(z))}{2\pi i}} - \sqrt{\frac{\log (f(z))}{2\pi i}} - 1 \right].
\]

There exists a constant \( C \) independent of \( f \in \mathcal{F} \) such that \( H(\Omega) \) does not contain any ball of radius \( C \).

Proof: Let \( f \in \mathcal{F} \). Then since \( f \) does not take the value 0, there exists \( g_1 \) a primitive of \( f'/f \). Thus

\[
\frac{d}{dz} (e^{-g_1} f) = 0
\]
so there exists $a, b$ such that $f(z) e^{-g_1(z)} = e^{a+bi}$. Letting $g(z) = g_1(z) + a + ib$, it follows $e^{g(z)} = f(z)$. Let $\log(f(z)) = g(z)$. Then for $n \in \mathbb{Z}$, the integers,

$$\frac{\log(f(z))}{2\pi i}, \frac{\log(f(z))}{2\pi i} - 1 \neq n$$

because if equality held, then $f(z) = 1$ which does not happen. It follows $\frac{\log(f(z))}{2\pi i} \quad \text{and} \quad \frac{\log(f(z))}{2\pi i} - 1$ are never equal to zero. Therefore, using the same reasoning, you can define a logarithm of these two quantities and therefore, a square root. Hence there exists a function analytic on $\Omega$,

$$\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \quad \text{(53.5.20)}$$

For $n$ a positive integer, this function cannot equal $\sqrt{n} \pm \sqrt{n-1}$ because if it did, then

$$\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right) = \sqrt{n} \pm \sqrt{n-1} \quad \text{(53.5.21)}$$

and you could take reciprocals of both sides to obtain

$$\left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right) = \sqrt{n} \mp \sqrt{n-1}. \quad \text{(53.5.22)}$$

Then adding (53.5.21) and (53.5.22)

$$2\sqrt{\frac{\log(f(z))}{2\pi i}} = 2\sqrt{n}$$

which contradicts the above observation that $\frac{\log(f(z))}{2\pi i}$ is not equal to an integer.

Also, the function of (53.5.20) is never equal to zero. Therefore, you can define the logarithm of this function also. It follows

$$H(z) = \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right) \neq \ln \left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$$

where $m$ is an arbitrary integer and $n$ is a positive integer. Now

$$\lim_{n \to \infty} \ln \left(\sqrt{n} + \sqrt{n-1}\right) = \infty$$

and $\lim_{n \to \infty} \ln \left(\sqrt{n} - \sqrt{n-1}\right) = -\infty$ and so $\mathbb{C}$ is covered by rectangles having vertices at points $\ln \left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ as described above. Each of these rectangles has height equal to $2\pi$ and a short computation shows their widths are bounded. Therefore, there exists $C$ independent of $f \in F$ such that $C$ is larger than the diameter of all these rectangles. Hence $H(\Omega)$ cannot contain any ball of radius larger than $C$. 


Now here is the little Picard theorem. It is easy to prove from the above.

**Theorem 53.5.5** If \( h \) is an entire function which omits two values then \( h \) is a constant.

**Proof:** Suppose the two values omitted are \( a \) and \( b \) and that \( h \) is not constant. Let \( f(z) = (h(z) - a) / (b - a) \). Then \( f \) omits the two values 0 and 1. Let \( H \) be defined in Lemma 53.5.4. Then \( H(z) \) is clearly not of the form \( az + b \) because then it would have values equal to the vertices \( \ln(\sqrt{n} ± \sqrt{n-1}) + 2m\pi i \) or else be constant neither of which happen if \( h \) is not constant. Therefore, by Liouville's theorem, \( H' \) must be unbounded. Pick \( \xi \) such that \( |H'(\xi)| > 24C \) where \( C \) is such that \( H(\xi) \) contains no balls of radius larger than \( C \). But by Lemma 53.5.3 \( H(B(\xi,1)) \) must contain a ball of radius \( \frac{|H'(\xi)|}{24} > \frac{24C}{24} = C \), a contradiction. This proves Picard's theorem.

The following is another formulation of this theorem.

**Corollary 53.5.6** If \( f \) is a meromorphic function defined on \( \mathbb{C} \) which omits three distinct values, \( a, b, c, \) then \( f \) is a constant.

**Proof:** Let \( \phi(z) \equiv \frac{z-a}{z-b} \). Then \( \phi(c) = \infty, \phi(a) = 0, \) and \( \phi(b) = 1. \) Now consider the function, \( h = \phi \circ f. \) Then \( h \) misses the three points \( \infty, 0, \) and 1. Since \( h \) is meromorphic and does not have \( \infty \) in its values, it must actually be analytic. Thus \( h \) is an entire function which misses the two values 0 and 1. Therefore, \( h \) is constant by Theorem 53.5.5.

### 53.5.3 Schottky's Theorem

**Lemma 53.5.7** Let \( f \) be analytic on an open set containing \( \overline{B(0,R)} \) and suppose that \( f \) does not take on either of the two values 0 or 1. Also suppose \( |f(0)| \leq \beta. \) Then letting \( \theta \in (0,1) \), it follows

\[
|f(z)| \leq M(\beta, \theta)
\]

for all \( z \in B(0,\theta R) \), where \( M(\beta, \theta) \) is a function of only the two variables \( \beta, \theta. \) (In particular, there is no dependence on \( R. \))

**Proof:** Consider the function, \( H(z) \) used in Lemma 53.5.4 given by

\[
H(z) = \log \left( \sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right). \tag{53.5.23}
\]

You notice there are two explicit uses of logarithms. Consider first the logarithm inside the radicals. Choose this logarithm such that

\[
\log(f(0)) = \ln|f(0)| + i \arg(f(0)), \quad \arg(f(0)) \in (-\pi, \pi]. \tag{53.5.24}
\]
You can do this because
\[ e^{\log(f(0))} = f(0) = e^{\ln|f(0)|}e^{i\alpha} = e^{\ln|f(0)|+i\alpha} \]
and by replacing \( \alpha \) with \( \alpha + 2m\pi \) for a suitable integer, \( m \) it follows the above equation still holds. Therefore, you can assume \( \text{53.5.24} \). Similar reasoning applies to the logarithm on the outside of the parenthesis. It can be assumed \( H(0) \) equals
\[
\ln \left| \frac{\log \left( \sqrt{\log (f(0))-1} \right)}{2\pi i} \right| + i \arg \left( \frac{\log \left( \sqrt{\log (f(0))-1} \right)}{2\pi i} \right) - \left( \frac{i}{2\pi i} \right)
\]
where the imaginary part is no larger than \( \pi \) in absolute value.

Now if \( \xi \in B(0,R) \) is a point where \( H'(\xi) \neq 0 \), then by Lemma \( \text{53.5.2} \)
\[
H(B(\xi,R-|\xi|)) \supseteq B\left( H(a), \frac{|H'(\xi)|(R-|\xi|)}{24} \right)
\]
where \( a \) is some point in \( B(\xi,R-|\xi|) \). But by Lemma \( \text{53.5.4} \) \( H(B(\xi,R-|\xi|)) \) contains no balls of radius \( C \) where \( C \) depended only on the maximum diameters of those rectangles having vertices \( \ln \left( \sqrt{n} \pm \sqrt{n-1} \right) + 2m\pi i \) for \( n \) a positive integer and \( m \) an integer. Therefore,
\[
\frac{|H'(\xi)|(R-|\xi|)}{24} < C
\]
and consequently
\[
|H'(\xi)| < \frac{24C}{R-|\xi|}.
\]
Even if \( H'(\xi) = 0 \), this inequality still holds. Therefore, if \( z \in B(0,R) \) and \( \gamma(0,z) \) is the straight segment from 0 to \( z \),
\[
|H(z) - H(0)| = \left| \int_{\gamma(0,z)} H'(w) \, dw \right| = \left| \int_{0}^{1} H'(tz) \, z \, dt \right|
\leq \int_{0}^{1} |H'(tz)z| \, dt \leq \int_{0}^{1} \frac{24C}{R-t|z|} |z| \
= 24C \ln \left( \frac{R}{R-|z|} \right).
\]
Therefore, for \( z \in \partial B(0,\theta R) \),
\[
|H(z)| \leq |H(0)| + 24C \ln \left( \frac{1}{1-\theta} \right).
\]
By the maximum modulus theorem, the above inequality holds for all \( |z| < \theta R \) also.
Next I will use \(53.5.23\) to get an inequality for \(|f(z)|\) in terms of \(|H(z)|\). From \(53.5.23\),

\[
H(z) = \log \left( \sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i}} - 1 \right)
\]

and so

\[
2H(z) = \log \left( \sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i}} - 1 \right)^2
\]
\[
-2H(z) = \log \left( \sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i}} - 1 \right)^{-2}
\]
\[
= \log \left( \sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i}} - 1 \right)^2
\]

Therefore,

\[
\left( \sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i}} - 1 \right)^2
\]
\[
+ \left( \sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i}} - 1 \right)^2
\]
\[
= \exp(2H(z)) + \exp(-2H(z))
\]

and

\[
\left( \frac{\log(f(z))}{\pi i} - 1 \right) = \frac{1}{2} \left( \exp(2H(z)) + \exp(-2H(z)) \right).
\]

Thus

\[
\log(f(z)) = \pi i + \frac{\pi i}{2} \left( \exp(2H(z)) + \exp(-2H(z)) \right)
\]

which shows

\[
|f(z)| = \left| \exp \left[ \frac{\pi i}{2} \left( \exp(2H(z)) + \exp(-2H(z)) \right) \right] \right|
\]
\[
\leq \exp \left[ \frac{\pi i}{2} \left( \exp(2H(z)) + \exp(-2H(z)) \right) \right]
\]
\[
\leq \exp \left[ \frac{\pi}{2} \left( \exp(2|H(z)|) + \exp(-2|H(z)|) \right) \right]
\]
\[
\leq \exp \left[ \frac{\pi}{2} \left( \exp(2|H(z)|) + \exp(-2|H(z)|) \right) \right]
\]
\[
= \exp(\pi \exp 2|H(z)|).
\]
CHAPTER 53. COMPLEX MAPPINGS

Now from this is dominated by

$$
\exp \left( \pi \exp 2 \left| H(0) \right| + 24C \ln \left( \frac{1}{1 - \theta} \right) \right)
$$

$$
= \exp \left( \pi \exp (2 \left| H(0) \right|) \exp \left( 48C \ln \left( \frac{1}{1 - \theta} \right) \right) \right)
$$

(53.5.27)

Consider $\exp (2 \left| H(0) \right|)$. I want to obtain an inequality for this which involves $\beta$. This is where I will use the convention about the logarithms discussed above.

From

$$
2 \left| H(0) \right| = 2 \left| \ln \left( \sqrt{\frac{\log (f(0))}{2\pi i}} - \sqrt{\frac{\log (f(0))}{2\pi i} - 1} \right) \right|
$$

$$
\leq 2 \left( \left( \ln \left( \sqrt{\frac{\log (f(0))}{2\pi i}} - \sqrt{\frac{\log (f(0))}{2\pi i} - 1} \right) \right)^2 + \pi^2 \right)^{1/2}
$$

$$
\leq 2 \left( \ln \left( \sqrt{\frac{\log (f(0))}{2\pi i}} + \sqrt{\frac{\log (f(0))}{2\pi i} - 1} \right) \right)^2 + \pi^2 \right)^{1/2}
$$

$$
\leq 2 \ln \left( \sqrt{\frac{\log (f(0))}{2\pi i}} + \sqrt{\frac{\log (f(0))}{2\pi i} - 1} \right) + 2\pi
$$

$$
= \ln \left( \left( \frac{\log (f(0))}{\pi i} \right) + \left| \frac{\log (f(0))}{\pi i} - 2 \right| \right) + 2\pi
$$

(53.5.28)

Consider $\left| \frac{\log (f(0))}{\pi i} \right|$,

$$
\frac{\log (f(0))}{\pi i} = - \ln \left| f(0) \right| i + \frac{\arg (f(0))}{\pi}
$$

and so

$$
\left| \frac{\log (f(0))}{\pi i} \right| = \left( \left| \ln \left| f(0) \right| \right|^2 + \left( \frac{\arg (f(0))}{\pi} \right)^2 \right)^{1/2}
$$

$$
\leq \left( \left| \ln \beta \right|^2 + \left( \pi \right)^2 \right)^{1/2}
$$

$$
= \left( \left| \ln \beta \right|^2 + 1 \right)^{1/2}.
$$
Similarly,
\[
\left| \frac{\log(f(0))}{\pi i} - 2 \right| \leq \left( \frac{\ln \beta}{\pi} \right)^2 + (2 + 1)^2 \left( \frac{\ln \beta}{\pi} \right)^2 + 9 \left( 2 + \frac{1}{\theta} \right) \exp \left( 48C \ln \left( \frac{1}{1 - \theta} \right) \right)
\]

It follows from Lemma that
\[
2 |H(0)| \leq \ln \left( 2 \left( \frac{\ln \beta}{\pi} \right)^2 + 9 \right) + 2\pi.
\]

Hence from Lemma
\[
\exp \left( \pi \exp \left( \ln \left( 2 \left( \frac{\ln \beta}{\pi} \right)^2 + 9 \right) + 2\pi \right) \exp \left( 48C \ln \left( \frac{1}{1 - \theta} \right) \right) \right)
\]

and so, letting \( M(\beta, \theta) \) be given by the above expression on the right, the lemma is proved.

The following theorem will be referred to as Schottky’s theorem. It looks just like the above lemma except it is only assumed that \( f \) is analytic on \( B(0, R) \) rather than on an open set containing \( B(0, R) \). Also, the case of an arbitrary center is included along with arbitrary points which are not attained as values of the function.

**Theorem 53.5.8** Let \( f \) be analytic on \( B(z_0, R) \) and suppose that \( f \) does not take on either of the two distinct values \( a \) or \( b \). Also suppose \( |f(z_0)| \leq \beta \). Then letting \( \theta \in (0, 1) \), it follows
\[
|f(z)| \leq M(a, b, \beta, \theta)
\]
for all \( z \in B(z_0, \theta R) \), where \( M(a, b, \beta, \theta) \) is a function of only the variables \( \beta, \theta, a, b \). (In particular, there is no dependence on \( R \).)

**Proof:** First you can reduce to the case where the two values are 0 and 1 by considering
\[
h(z) = f(z) - a \left( b - a \right).
\]

If there exists an estimate of the desired sort for \( h \), then there exists such an estimate for \( f \). Of course here the function, \( M \) would depend on \( a \) and \( b \). Therefore, there is no loss of generality in assuming the points which are missed are 0 and 1.

Apply Lemma to \( B(0, R_1) \) for the function, \( g(z) = f(z_0 + z) \) and \( R_1 < R \). Then if \( \beta \geq |f(z_0)| = |g(0)| \), it follows \( |g(z)| = |f(z_0 + z)| \leq M(\beta, \theta) \) for every \( z \in B(0, \theta R_1) \). Now let \( \theta \in (0, 1) \) and choose \( R_1 < R \) large enough that \( \theta R = \theta_1 R_1 \) where \( \theta_1 \in (0, 1) \). Then if \( |z - z_0| < \theta R \), it follows
\[
|f(z)| \leq M(\beta, \theta_1).
\]

Now let \( R_1 \to R \) so \( \theta_1 \to \theta \).
53.5.4 A Brief Review

First recall the definition of the metric on \( \mathbb{C} \). For convenience it is listed here again. Consider the unit sphere, \( S^2 \) given by \((z - 1)^2 + y^2 + x^2 = 1\). Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, \( p \) in the complex plane to the point \((0, 0, 2)\) on the top of this sphere and let \( \theta (p) \) denote the point of this sphere which the line intersects. Define \( \theta (\infty) \equiv (0, 0, 2) \).

Then \( \theta^{-1} \) is sometimes called stereographic projection. The mapping \( \theta \) is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that \( \theta^{-1} \) is also continuous. In terms of the extended complex plane, \( \mathbb{C} \), a sequence, \( z_n \) converges to \( \infty \) if and only if \( \theta z_n \) converges to \((0, 0, 2) \) and a sequence, \( z_n \) converges to \( z \in \mathbb{C} \) if and only if \( \theta (z_n) \to \theta (z) \).

In fact this makes it easy to define a metric on \( \mathbb{C} \).

**Definition 53.5.9** Let \( z, w \in \mathbb{C} \). Then let \( d(x, y) \equiv |\theta (z) - \theta (w)| \) where this last distance is the usual distance measured in \( \mathbb{R}^3 \).

**Theorem 53.5.10** \((\mathbb{C}, d)\) is a compact, hence complete metric space.

**Proof:** Suppose \( \{z_n\} \) is a sequence in \( \mathbb{C} \). This means \( \{\theta (z_n)\} \) is a sequence in \( S^2 \) which is compact. Therefore, there exists a subsequence, \( \{\theta z_{n_k}\} \) and a point, \( z \in S^2 \) such that \( \theta z_{n_k} \to \theta z \) in \( S^2 \) which implies immediately that \( d(z_{n_k}, z) \to 0 \). A compact metric space must be complete.

Also recall the interesting fact that meromorphic functions are continuous with values in \( \mathbb{C} \) which is reviewed here for convenience. It came from the theory of classification of isolated singularities.

**Theorem 53.5.11** Let \( \Omega \) be an open subset of \( \mathbb{C} \) and let \( f : \Omega \to \mathbb{C} \) be meromorphic. Then \( f \) is continuous with respect to the metric, \( d \) on \( \mathbb{C} \).

**Proof:** Let \( z_n \to z \) where \( z \in \Omega \). Then if \( z \) is a pole, it follows from Theorem 49.7.11 that
\[
d(f(z_n), \infty) \equiv d(f(z_n), f(z)) \to 0.
\]
If \( z \) is not a pole, then \( f(z_n) \to f(z) \) in \( \mathbb{C} \) which implies \( |\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \to 0 \). Recall that \( \theta \) is continuous on \( \mathbb{C} \).

The fundamental result behind all the theory about to be presented is the Ascoli Arzela theorem also listed here for convenience.
53.5. THE PICARD THEOREMS

**Definition 53.5.12** Let \((X, d)\) be a complete metric space. Then it is said to be locally compact if \(B(x, r)\) is compact for each \(r > 0\).

Thus if you have a locally compact metric space, then if \(\{a_n\}\) is a bounded sequence, it must have a convergent subsequence.

Let \(K\) be a compact subset of \(\mathbb{R}^n\) and consider the continuous functions which have values in a locally compact metric space, \((X, d)\) where \(d\) denotes the metric on \(X\). Denote this space as \(C(K, X)\).

**Definition 53.5.13** For \(f, g \in C(K, X)\), where \(K\) is a compact subset of \(\mathbb{R}^n\) and \(X\) is a locally compact complete metric space define

\[
\rho_K(f, g) \equiv \sup \{d(f(x), g(x)) : x \in K\}.
\]

The Ascoli Arzela theorem, Theorem 6.4.4 is a major result which tells which subsets of \(C(K, X)\) are sequentially compact.

**Definition 53.5.14** Let \(A \subseteq C(K, X)\) for \(K\) a compact subset of \(\mathbb{R}^n\). Then \(A\) is said to be uniformly equicontinuous if for every \(\varepsilon > 0\) there exists a \(\delta > 0\) such that whenever \(x, y \in K\) with \(|x - y| < \delta\) and \(f \in A\),

\[
d(f(x), f(y)) < \varepsilon.
\]

The set, \(A\) is said to be uniformly bounded if for some \(M < \infty\), and \(a \in X\),

\[
f(x) \in B(a, M)
\]

for all \(f \in A\) and \(x \in K\).

The Ascoli Arzela theorem follows.

**Theorem 53.5.15** Suppose \(K\) is a nonempty compact subset of \(\mathbb{R}^n\) and \(A \subseteq C(K, X)\), is uniformly bounded and uniformly equicontinuous where \(X\) is a locally compact complete metric space. Then if \(\{f_k\} \subseteq A\), there exists a function, \(f \in C(K, X)\) and a subsequence, \(f_{k_l}\) such that

\[
\lim_{l \to \infty} \rho_K(f_{k_l}, f) = 0.
\]

In the cases of interest here, \(X = \widehat{\mathbb{C}}\) with the metric defined above.

### 53.5.5 Montel’s Theorem

The following lemma is another version of Montel’s theorem. It is this which will make possible a proof of the big Picard theorem.
Lemma 53.5.16 Let \( \Omega \) be a region and let \( \mathcal{F} \) be a set of functions analytic on \( \Omega \) none of which achieve the two distinct values, \( a \) and \( b \). If \( \{ f_n \} \subseteq \mathcal{F} \) then one of the following hold: Either there exists a function, \( f \) analytic on \( \Omega \) and a subsequence, \( \{ f_{n_k} \} \) such that for any compact subset, \( K \) of \( \Omega \),
\[
\lim_{k \to \infty} \| f_{n_k} - f \|_{K, \infty} = 0. \tag{53.5.29}
\]
or there exists a subsequence \( \{ f_{n_k} \} \) such that for all compact subsets \( K \),
\[
\lim_{k \to \infty} \rho_K (f_{n_k}, \infty) = 0. \tag{53.5.30}
\]

Proof: Let \( B(z_0, 2R) \subseteq \Omega \). There are two cases to consider. The first case is that there exists a subsequence, \( \{ f_{n_k} \} \) such that \( \{ f_{n_k}(z_0) \} \) is bounded. The second case is that \( \lim_{n \to \infty} |f_{n_k}(z_0)| = \infty \).

Consider the first case. By Theorem 53.5.16 \( \{ f_{n_k}(z) \} \) is uniformly bounded on \( B(z_0, R) \) because by this theorem, and letting \( \theta = 1/2 \) applied to \( B(z_0, 2R) \), it follows \( |f_{n_k}(z)| \leq M(a, b, \frac{1}{2}, \beta) \) where \( \beta \) is an upper bound to the numbers, \( |f_{n_k}(z_0)| \).

The Cauchy integral formula implies the existence of a uniform bound on the \( \{ f'_{n_k} \} \) which implies the functions are equicontinuous and uniformly bounded. Therefore, by the Ascoli Arzelà theorem there exists a further subsequence which converges uniformly on \( B(z_0, R) \) to a function, \( f \) analytic on \( B(z_0, R) \). Thus denoting this subsequence by \( \{ f_{n_k} \} \) to save on notation,
\[
\lim_{k \to \infty} \| f_{n_k} - f \|_{B(z_0, R), \infty} = 0. \tag{53.5.31}
\]

Consider the second case. In this case, it follows \( \{ 1/f_n(z_0) \} \) is bounded on \( B(z_0, R) \) and so by the same argument just given \( \{ 1/f_n(z) \} \) is uniformly bounded on \( B(z_0, R) \). Therefore, a subsequence converges uniformly on \( B(z_0, R) \). But \( \{ 1/f_n(z) \} \) converges to 0 and so this requires that \( \{ 1/f_n(z) \} \) must converge uniformly to 0. Therefore,
\[
\lim_{k \to \infty} \rho_{B(z_0, R)} (f_{n_k}, \infty) = 0. \tag{53.5.32}
\]

Now let \( \{ D_k \} \) denote a countable set of closed balls, \( D_k = B(z_k, R_k) \) such that \( B(z_k, 2R_k) \subseteq \Omega \) and \( \bigcup_{k=1}^{\infty} \text{int}(D_k) = \Omega \). Using a Cantor diagonal process, there exists a subsequence, \( \{ f_{n_k} \} \) of \( \{ f_n \} \) such that for each \( D_j \), one of the above two alternatives holds. That is, either
\[
\lim_{k \to \infty} \| f_{n_k} - g_j \|_{D_j, \infty} = 0 \tag{53.5.33}
\]
or,
\[
\lim_{k \to \infty} \rho_{D_j} (f_{n_k}, \infty). \tag{53.5.34}
\]

Let \( A = \{ \cup \text{int}(D_j) : \text{alternatives holds} \} \), \( B = \{ \cup \text{int}(D_j) : \text{alternatives holds} \} \). Note that the balls whose union is \( A \) cannot intersect any of the balls whose union is \( B \). Therefore, one of \( A \) or \( B \) must be empty since otherwise, \( \Omega \) would not be connected.
If $K$ is any compact subset of $\Omega$, it follows $K$ must be a subset of some finite collection of the $D_j$. Therefore, one of the alternatives in the lemma must hold. That the limit function, $f$ must be analytic follows easily in the same way as the proof in Theorem 53.5.14 on Page 1752. You could also use Morera’s theorem. This proves the lemma.

53.5.6 The Great Big Picard Theorem

The next theorem is the main result which the above lemmas lead to. It is the Big Picard theorem, also called the Great Picard theorem. Recall $B'(a, r)$ is the deleted ball consisting of all the points of the ball except the center.

**Theorem 53.5.17** Suppose $f$ has an isolated essential singularity at 0. Then for every $R > 0$, and $\beta \in \mathbb{C}$, $f^{-1}(\beta) \cap B'(0, R)$ is an infinite set except for one possible exceptional $\beta$.

**Proof:** Suppose this is not true. Then there exists $R_1 > 0$ and two points, $\alpha$ and $\beta$ such that $f^{-1}(\beta) \cap B'(0, R_1)$ and $f^{-1}(\alpha) \cap B'(0, R_1)$ are both finite sets. Then shrinking $R_1$ and calling the result $R$, there exists $B(0, R)$ such that

$$f^{-1}(\beta) \cap B'(0, R) = \emptyset, \quad f^{-1}(\alpha) \cap B'(0, R) = \emptyset.$$

Now let $A_0$ denote the annulus $\{z \in \mathbb{C} : \frac{R}{2^n} < |z| < \frac{3R}{2^n}\}$ and let $A_n$ denote the annulus $\{z \in \mathbb{C} : \frac{R}{2^n} < |z| < \frac{3R}{2^n}\}$. The reason for the $3$ is to insure that $A_n \cap A_{n+1} = \emptyset$. This follows from the observation that $3R/2^{2+n+1} > R/2^{2+n}$. Now define a set of functions on $A_0$ as follows:

$$f_n(z) = f\left(\frac{z}{2^n}\right).$$

By the choice of $R$, this set of functions missed the two points $\alpha$ and $\beta$. Therefore, by Lemma 53.5.10 there exists a subsequence such that one of the two options presented there holds.

First suppose $\lim_{k \to \infty} \|f_{n_k} - f\|_{K, \infty} = 0$ for all $K$ a compact subset of $A_0$ and $f$ is analytic on $A_0$. In particular, this happens for $\gamma_0$ the circular contour having radius $R/2$. Thus $f_{n_k}$ must be bounded on this contour. But this says the same thing as $f(z/2^{n_k})$ is bounded for $|z| = R/2$, this holding for each $k = 1, 2, \ldots$. Thus there exists a constant, $M$ such that on each of a shrinking sequence of concentric circles whose radii converge to 0, $|f(z)| \leq M$. By the maximum modulus theorem, $|f(z)| \leq M$ at every point between successive circles in this sequence. Therefore, $|f(z)| \leq M$ in $B'(0, R)$ contradicting the Weierstrass Casorati theorem.

The other option which might hold from Lemma 53.5.10 is that $\lim_{k \to \infty} \rho_K(f_{n_k}, \infty) = 0$ for all $K$ compact subset of $A_0$. Since $f$ has an essential singularity at 0 the zeros of $f$ in $B(0, R)$ are isolated. Therefore, for all $k$ large enough, $f_{n_k}$ has no zeros for $|z| < 3R/2^2$. This is because the values of $f_{n_k}$ are the values of $f$ on $A_{n_k}$, a small annulus which avoids all the zeros of $f$ whenever $k$ is large enough. Only consider $k$ such large. Then use the above argument on the analytic functions $1/f_{n_k}$. By the assumption that $\lim_{k \to \infty} \rho_K(f_{n_k}, \infty) = 0$, it follows $\lim_{k \to \infty} \|1/f_{n_k} - 0\|_{K, \infty} = 0$ and
so as above, there exists a shrinking sequence of concentric circles whose radii converge to 0 and a constant, $M$ such that for $z$ on any of these circles, $|1/f(z)| \leq M$. This implies that on some deleted ball, $B'(0,r)$ where $r \leq R$, $|f(z)| \geq 1/M$ which again violates the Weierstrass Casorati theorem. This proves the theorem.

As a simple corollary, here is what this remarkable theorem says about entire functions.

**Corollary 53.5.18** Suppose $f$ is entire and nonconstant and not a polynomial. Then $f$ assumes every complex value infinitely many times with the possible exception of one.

**Proof:** Since $f$ is entire, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Define for $z \neq 0$,

$$g(z) := f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n.$$ 

Thus 0 is an isolated essential singular point of $g$. By the big Picard theorem, it follows $g$ takes every complex number but possibly one an infinite number of times. This proves the corollary.

Note the difference between this and the little Picard theorem which says that an entire function which is not constant must achieve every value but two.

### 53.6 Exercises

1. Prove that in Theorem 53.3.1 it suffices to assume $F$ is uniformly bounded on each compact subset of $\Omega$.

2. Find conditions on $a,b,c,d$ such that the fractional linear transformation, $rac{az+b}{cz+d}$ maps the upper half plane onto the upper half plane.

3. Let $D$ be a simply connected region which is a proper subset of $\mathbb{C}$. Does there exist an entire function, $f$ which maps $\mathbb{C}$ onto $D$? Why or why not?

4. Verify the conclusion of Theorem 53.3.1 involving the higher order derivatives.

5. What if $\Omega = \mathbb{C}$? Does there exist an analytic function, $f$ mapping $\Omega$ one to one and onto $B(0,1)$? Explain why or why not. Was $\Omega \neq \mathbb{C}$ used in the proof of the Riemann mapping theorem?

6. Verify that $|\phi_\alpha(z)| = 1$ if $|z| = 1$. Apply the maximum modulus theorem to conclude that $|\phi_\alpha(z)| \leq 1$ for all $|z| < 1$.

7. Suppose that $|f(z)| \leq 1$ for $|z| = 1$ and $f(\alpha) = 0$ for $|\alpha| < 1$. Show that $|f(z)| \leq |\phi_\alpha(z)|$ for all $z \in B(0,1)$. **Hint:** Consider $\frac{f(z)}{z-\alpha}$ which has a removable singularity at $\alpha$. Show the modulus of this function is bounded by 1 on $|z| = 1$. Then apply the maximum modulus theorem.
8. Let $U$ and $V$ be open subsets of $\mathbb{C}$ and suppose $u : U \to \mathbb{R}$ is harmonic while $h$ is an analytic map which takes $V$ one to one onto $U$. Show that $u \circ h$ is harmonic on $V$.

9. Show that for a harmonic function, $u$ defined on $B(0,R)$, there exists an analytic function, $h = u + iv$ where

$$v(x,y) = \int_0^y u_x(x,t) \, dt - \int_0^x u_y(t,0) \, dt.$$ 

10. Suppose $\Omega$ is a simply connected region and $u$ is a real valued function defined on $\Omega$ such that $u$ is harmonic. Show there exists an analytic function, $f$ such that $u = \text{Re } f$. Show this is not true if $\Omega$ is not a simply connected region. **Hint**: You might use the Riemann mapping theorem and Problems 8 and 9. For the second part it might be good to try something like $u(x,y) = \ln(x^2 + y^2)$ on the annulus $1 < |z| < 2$.

11. Show that $w = \frac{1+z}{1-z}$ maps \{ $z \in \mathbb{C} : \text{Im } z > 0$ and $|z| < 1$ \} to the first quadrant, \{ $z = x + iy : x, y > 0$ \}.

12. Let $f(z) = \frac{az+b}{cz+d}$ and let $g(z) = \frac{a_1z+b_1}{c_1z+d_1}$. Show that $f \circ g (z)$ equals the quotient of two expressions, the numerator being the top entry in the vector

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) \left( \begin{array}{c} z \\ 1 \end{array} \right)$$

and the denominator being the bottom entry. Show that if you define

$$\phi \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \right) \equiv \frac{az+b}{cz+d},$$

then $\phi(AB) = \phi(A) \circ \phi(B)$. Find an easy way to find the inverse of $f(z) = \frac{az+b}{cz+d}$ and give a condition on the $a,b,c,d$ which insures this function has an inverse.

13. The modular group\footnote{This is the terminology used in Rudin’s book Real and Complex Analysis.} is the set of fractional linear transformations, $\frac{az+b}{cz+d}$ such that $a,b,c,d$ are integers and $ad - bc = 1$. Using Problem 12 or brute force show this modular group is really a group with the group operation being composition. Also show the inverse of $\frac{az+b}{cz+d}$ is $\frac{dz-b}{-cz+a}$.

14. Let $\Omega$ be a region and suppose $f$ is analytic on $\Omega$ and that the functions $f_n$ are also analytic on $\Omega$ and converge to $f$ uniformly on compact subsets of $\Omega$. Suppose $f$ is one to one. Can it be concluded that for an arbitrary compact set, $K \subseteq \Omega$ that $f_n$ is one to one for all $n$ large enough?
15. The Vitali theorem says that if $\Omega$ is a region and $\{f_n\}$ is a uniformly bounded sequence of functions which converges pointwise on a set, $S \subseteq \Omega$ which has a limit point in $\Omega$, then in fact, $\{f_n\}$ must converge uniformly on compact subsets of $\Omega$ to an analytic function. Prove this theorem. **Hint:** If the sequence fails to converge, show you can get two different subsequences converging uniformly on compact sets to different functions. Then argue these two functions coincide on $S$.

16. Does there exist a function analytic on $B(0,1)$ which maps $B(0,1)$ onto $B'(0,1)$, the open unit ball in which 0 has been deleted?
Chapter 54

Approximation By Rational Functions

54.1 Runge’s Theorem

Consider the function, \( \frac{1}{z} = f(z) \) for \( z \) defined on \( \Omega \equiv B(0,1) \setminus \{0\} = B'(0,1) \). Clearly \( f \) is analytic on \( \Omega \). Suppose you could approximate \( f \) uniformly by polynomials on \( \text{ann}(0,\frac{1}{2},\frac{3}{4}) \), a compact subset of \( \Omega \). Then, there would exist a suitable polynomial \( p(z) \), such that \( \left| \frac{1}{2\pi i} \int_{\gamma} f(z) - p(z) \, dz \right| < \frac{1}{10} \) where here \( \gamma \) is a circle of radius \( \frac{2}{3} \). However, this is impossible because \( \frac{1}{2\pi i} \int_{\gamma} f(z) \, dz = 1 \) while \( \frac{1}{2\pi i} \int_{\gamma} p(z) \, dz = 0 \). This shows you can’t expect to be able to uniformly approximate analytic functions on compact sets using polynomials. This is just horrible!

In real variables, you can approximate any continuous function on a compact set with a polynomial. However, that is just the way it is. It turns out that the ability to approximate an analytic function on \( \Omega \) with polynomials is dependent on \( \Omega \) being simply connected.

All these theorems work for \( f \) having values in a complex Banach space. However, I will present them in the context of functions which have values in \( \mathbb{C} \). The changes necessary to obtain the extra generality are very minor.

Definition 54.1.1 Approximation will be taken with respect to the following norm.

\[
\|f - g\|_{K,\infty} \equiv \sup \{ |f(z) - g(z)| : z \in K \}
\]

54.1.1 Approximation With Rational Functions

It turns out you can approximate analytic functions by rational functions, quotients of polynomials. The resulting theorem is one of the most profound theorems in complex analysis. The basic idea is simple. The Riemann sums for the Cauchy integral formula are rational functions. The idea used to implement this observation
is that if you have a compact subset, \( K \) of an open set, \( \Omega \) there exists a cycle composed of closed oriented curves \( \{ \gamma_j \}_{j=1}^n \) which are contained in \( \Omega \setminus K \) such that for every \( z \in K \), \( \sum_{k=1}^n n(\gamma_k, z) = 1 \). One more ingredient is needed and this is a theorem which lets you keep the approximation but move the poles.

To begin with, consider the part about the cycle of closed oriented curves. Recall Theorem 49.7.25 which is stated for convenience.

**Theorem 54.1.2** Let \( K \) be a compact subset of an open set, \( \Omega \). Then there exist continuous, closed, bounded variation oriented curves \( \{ \gamma_j \}_{j=1}^m \) for which \( \gamma_j^* \cap K = \emptyset \) for each \( j \), \( \gamma_j^* \subseteq \Omega \), and for all \( p \in K \),

\[
\sum_{k=1}^m n(p, \gamma_k) = 1.
\]

and

\[
\sum_{k=1}^m n(z, \gamma_k) = 0
\]

for all \( z \notin \Omega \).

This theorem implies the following.

**Theorem 54.1.3** Let \( K \subseteq \Omega \) where \( K \) is compact and \( \Omega \) is open. Then there exist oriented closed curves, \( \gamma_k \) such that \( \gamma_k^* \cap K = \emptyset \) but \( \gamma_k^* \subseteq \Omega \), such that for all \( z \in K \),

\[
f(z) = \frac{1}{2\pi i} \sum_{k=1}^p \int_{\gamma_k} \frac{f(w)}{w-z} \, dw.
\] (54.1.1)

**Proof:** This follows from Theorem 49.7.25 and the Cauchy integral formula. As shown in the proof, you can assume the \( \gamma_k \) are linear mappings but this is not important.

Next I will show how the Cauchy integral formula leads to approximation by rational functions, quotients of polynomials.

**Lemma 54.1.4** Let \( K \) be a compact subset of an open set, \( \Omega \) and let \( f \) be analytic on \( \Omega \). Then there exists a rational function, \( Q \) whose poles are not in \( K \) such that

\[
\|Q - f\|_{K, \infty} < \varepsilon.
\]

**Proof:** By Theorem 49.7.25 there are oriented curves, \( \gamma_k \) described there such that for all \( z \in K \),

\[
f(z) = \frac{1}{2\pi i} \sum_{k=1}^p \int_{\gamma_k} \frac{f(w)}{w-z} \, dw.
\] (54.1.2)
Defining \( g(w, z) \equiv \frac{f(w)}{w-z} \) for \((w, z) \in \bigcup_{k=1}^{p} \gamma_k^* \times K\), it follows since the distance between \( K \) and \( \bigcup_{k=1}^{p} \gamma_k^* \) is positive that \( g \) is uniformly continuous and so there exists a \( \delta > 0 \) such that if \( ||P|| < \delta \), then for all \( z \in K \),

\[
\left| f(z) - \frac{1}{2\pi i} \sum_{k=1}^{p} \sum_{j=1}^{n} \frac{f(\gamma_k(t_j))(\gamma_k(t_j) - \gamma_k(t_{j-1}))}{\gamma_k(t_j) - z} \right| < \frac{\varepsilon}{2}.
\]

The complicated expression is obtained by replacing each integral in 54.1.2 with a Riemann sum. Simplifying the appearance of this, it follows there exists a rational function of the form

\[
R(z) = \sum_{k=1}^{M} \frac{A_k}{w_k - z}
\]

where the \( w_k \) are elements of components of \( \mathbb{C} \setminus K \) and \( A_k \) are complex numbers or in the case where \( f \) has values in \( X \), these would be elements of \( X \) such that

\[
||R - f||_{K, \infty} < \frac{\varepsilon}{2}.
\]

This proves the lemma.

### 54.1.2 Moving The Poles And Keeping The Approximation

Lemma 54.1.3 is a nice lemma but needs refining. In this lemma, the Riemann sum handed you the poles. It is much better if you can pick the poles. The following theorem from advanced calculus, called Merten’s theorem, will be used.

#### 54.1.3 Merten’s Theorem.

**Theorem 54.1.5** Suppose \( \sum_{i=r}^{\infty} a_i \) and \( \sum_{j=r}^{\infty} b_j \) both converge absolutely\(^1\). Then

\[
\left( \sum_{i=r}^{\infty} a_i \right) \left( \sum_{j=r}^{\infty} b_j \right) = \sum_{n=r}^{\infty} c_n
\]

where

\[
c_n = \sum_{k=r}^{n} a_k b_{n-k+r}.
\]

**Proof:** Let \( p_{nk} = 1 \) if \( r \leq k \leq n \) and \( p_{nk} = 0 \) if \( k > n \). Then

\[
c_n = \sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r}.
\]

\(^1\)Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten’s theorem and may be read in the 1974 book by Apostol listed in the bibliography.
Also,
\[ \sum_{k=r}^{\infty} \sum_{n=r}^{\infty} p_{nk} |a_k| |b_{n-k+r}| = \sum_{k=r}^{\infty} |a_k| \sum_{n=r}^{\infty} p_{nk} |b_{n-k+r}| = \sum_{k=r}^{\infty} |a_k| \sum_{n=k}^{\infty} |b_{n-k+r}| = \sum_{k=r}^{\infty} |a_k| \sum_{m=r}^{\infty} |b_m| < \infty. \]

Therefore,
\[ \sum_{n=r}^{\infty} c_n = \sum_{n=r}^{\infty} \sum_{k=r}^{n} a_k b_{n-k+r} = \sum_{n=r}^{\infty} \sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r} \]
\[ = \sum_{k=r}^{\infty} a_k \sum_{n=r}^{\infty} p_{nk} b_{n-k+r} = \sum_{k=r}^{\infty} a_k \sum_{n=k}^{\infty} b_{n-k+r} \]
\[ = \sum_{k=r}^{\infty} a_k \sum_{m=r}^{\infty} b_m \]
and this proves the theorem.

It follows that \( \sum_{n=r}^{\infty} c_n \) converges absolutely. Also, you can see by induction that you can multiply any number of absolutely convergent series together and obtain a series which is absolutely convergent. Next, here are some similar results related to Merten’s theorem.

**Lemma 54.1.6** Let \( \sum_{n=0}^{\infty} a_n(z) \) and \( \sum_{n=0}^{\infty} b_n(z) \) be two convergent series for \( z \in K \) which satisfy the conditions of the Weierstrass \( M \) test. Thus there exist positive constants, \( A_n \) and \( B_n \) such that \( |a_n(z)| \leq A_n, |b_n(z)| \leq B_n \) for all \( z \in K \) and \( \sum_{n=0}^{\infty} A_n < \infty, \sum_{n=0}^{\infty} B_n < \infty. \) Then defining the Cauchy product,
\[ c_n(z) \equiv \sum_{k=0}^{n} a_{n-k}(z) b_k(z), \]
it follows \( \sum_{n=0}^{\infty} c_n(z) \) also converges absolutely and uniformly on \( K \) because \( c_n(z) \) satisfies the conditions of the Weierstrass \( M \) test. Therefore,
\[ \sum_{n=0}^{\infty} c_n(z) = \left( \sum_{k=0}^{\infty} a_k(z) \right) \left( \sum_{n=0}^{\infty} b_n(z) \right). \] (54.1.3)

**Proof:**
\[ |c_n(z)| \leq \sum_{k=0}^{n} |a_{n-k}(z)| |b_k(z)| \leq \sum_{k=0}^{n} A_{n-k} B_k. \]
Also,
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k} B_k = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A_{n-k} B_k = \sum_{k=0}^{\infty} B_k \sum_{n=0}^{\infty} A_n < \infty.
\]

The claim of 54.1.3 follows from Merten's theorem. This proves the lemma.

**Corollary 54.1.7** Let \( P \) be a polynomial and let \( \sum_{n=0}^{\infty} a_n(z) \) converge uniformly and absolutely on \( K \) such that the \( a_n \) satisfy the conditions of the Weierstrass \( M \) test. Then there exists a series for \( P(\sum_{n=0}^{\infty} a_n(z)) \), \( \sum_{n=0}^{\infty} c_n(z) \), which also converges absolutely and uniformly for \( z \in K \) because \( c_n(z) \) also satisfies the conditions of the Weierstrass \( M \) test.

The following picture is descriptive of the following lemma. This lemma says that if you have a rational function with one pole off a compact set, then you can approximate on the compact set with another rational function which has a different pole.

**Lemma 54.1.8** Let \( R \) be a rational function which has a pole only at \( a \in V \), a component of \( \mathbb{C} \setminus K \) where \( K \) is a compact set. Suppose \( b \in V \). Then for \( \varepsilon > 0 \) given, there exists a rational function, \( Q \), having a pole only at \( b \) such that
\[
||R - Q||_{K, \infty} < \varepsilon. \tag{54.1.4}
\]

If it happens that \( V \) is unbounded, then there exists a polynomial, \( P \) such that
\[
||R - P||_{K, \infty} < \varepsilon. \tag{54.1.5}
\]

**Proof:** Say that \( b \in V \) satisfies \( P \) if for all \( \varepsilon > 0 \) there exists a rational function, \( Q_b \), having a pole only at \( b \) such that
\[
||R - Q_b||_{K, \infty} < \varepsilon
\]

Now define a set,
\[
S = \{ b \in V : b \text{ satisfies } P \}.
\]
Observe that $S \neq \emptyset$ because $a \in S$.

I claim $S$ is open. Suppose $b_1 \in S$. Then there exists a $\delta > 0$ such that

$$\frac{|b_1 - b|}{|z - b|} < \frac{1}{2} \quad (54.1.6)$$

for all $z \in K$ whenever $b \in B(b_1, \delta)$. In fact, it suffices to take $|b - b_1| < \text{dist}(b_1, K)/4$ because then

$$\frac{|b_1 - b|}{|z - b|} \leq \frac{\text{dist}(b_1, K)/4}{\text{dist}(b_1, K) - \text{dist}(b_1, K)/4} \leq \frac{1}{3} < \frac{1}{2}.$$

Since $b_1$ satisfies $\mathcal{P}$, there exists a rational function $Q_{b_1}$ with the desired properties. It is shown next that you can approximate $Q_{b_1}$ with $Q_b$ thus yielding an approximation to $R$ by the use of the triangle inequality,

$$||R - Q_{b_1}||_{K, \infty} + ||Q_{b_1} - Q_b||_{K, \infty} \geq ||R - Q_b||_{K, \infty}.$$

Since $Q_{b_1}$ has poles only at $b_1$, it follows it is a sum of functions of the form $\frac{1}{(z - b_1)^n}$. Therefore, it suffices to consider the terms of $Q_{b_1}$ or that $Q_{b_1}$ is of the special form

$$Q_{b_1}(z) = \frac{1}{(z - b_1)^n}.$$

However,

$$\frac{1}{(z - b_1)^n} = \frac{1}{(z - b)^n} \left(1 - \frac{b_1 - b}{z - b}\right)^n.$$

Now from the choice of $b_1$, the series

$$\sum_{k=0}^{\infty} \left(\frac{b_1 - b}{z - b}\right)^k = \frac{1}{(1 - \frac{b_1 - b}{z - b})}$$

converges absolutely independent of the choice of $z \in K$ because

$$\left|\left(\frac{b_1 - b}{z - b}\right)^k\right| < \frac{1}{2^k}.$$

By Corollary 54.1.7, the same is true of the series for $\frac{1}{(1 - \frac{b_1 - b}{z - b})}$. Thus a suitable partial sum can be made uniformly on $K$ as close as desired to $\frac{1}{(z - b_1)^n}$. This shows that $b$ satisfies $\mathcal{P}$ whenever $b$ is close enough to $b_1$ verifying that $S$ is open.

Next it is shown $S$ is closed in $V$. Let $b_n \in S$ and suppose $b_n \to b \in V$. Then since $b_n \in S$, there exists a rational function, $Q_{b_n}$, such that

$$||Q_{b_n} - R||_{K, \infty} < \frac{\varepsilon}{2}.$$
Then for all \(n\) large enough,
\[
\frac{1}{2} \text{dist}(b, K) \geq |b_n - b|
\]
and so for all \(n\) large enough,
\[
\left| \frac{b - b_n}{z - b_n} \right| < \frac{1}{2},
\]
for all \(z \in K\). Pick such a \(b_n\). As before, it suffices to assume \(Q_{b_n}\), is of the form
\[
\frac{1}{(z - b_n)^\eta - \sum_{k=0}^{M} a_k \left( \frac{b_n - b}{z - b} \right)^k}.
\]
and because of the estimate, there exists \(M\) such that for all \(z \in K\)
\[
\left| \frac{1}{(z - b_n)^\eta - \sum_{k=0}^{M} a_k \left( \frac{b_n - b}{z - b} \right)^k} \right| < \varepsilon \left( \text{dist}(b, K) \right)^n. \tag{54.1.7}
\]
Therefore, for all \(z \in K\)
\[
\left| Q_{b_n}(z) - \frac{1}{(z - b_n)^\eta - \sum_{k=0}^{M} a_k \left( \frac{b_n - b}{z - b} \right)^k} \right| = 
\left| \frac{1}{(z - b_n)^\eta - \sum_{k=0}^{M} a_k \left( \frac{b_n - b}{z - b} \right)^k} \right| 
\leq \frac{\varepsilon \left( \text{dist}(b, K) \right)^n}{2}.
\]
and so, letting \(Q_b(z) = \frac{1}{(z - b)^\eta - \sum_{k=0}^{M} a_k \left( \frac{b_n - b}{z - b} \right)^k}\),
\[
||R - Q_b||_{K, \infty} \leq ||R - Q_{b_n}||_{K, \infty} + ||Q_{b_n} - Q_b||_{K, \infty} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
showing that \(b \in S\). Since \(S\) is both open and closed in \(V\) it follows that, since \(S \neq \emptyset, S = V\). Otherwise \(V\) would fail to be connected.

It remains to consider the case where \(V\) is unbounded. Pick \(b \in V\) large enough that
\[
\left| \frac{\varepsilon}{b_n} \right| < \frac{1}{2} \tag{54.1.8}
\]
for all \(z \in K\). From what was just shown, there exists a rational function, \(Q_b\) having a pole only at \(b\) such that \(||Q_b - R||_{K, \infty} < \frac{\varepsilon}{2}\). It suffices to assume that \(Q_b\) is of the
CHAPTER 54. APPROXIMATION BY RATIONAL FUNCTIONS

form

\[ Q_b(z) = \frac{p(z)}{(z-b)^n} = p(z)(-1)^n \frac{1}{b^n} \left(1 - \frac{z}{b}\right)^n \]

Then by an application of Corollary 54.1.7 there exists a partial sum of the power series for \( Q_b \) which is uniformly close to \( Q_b \) on \( K \). Therefore, you can approximate \( Q_b \) and therefore also \( R \) uniformly on \( K \) by a polynomial consisting of a partial sum of the above infinite sum. This proves the theorem.

If \( f \) is a polynomial, then \( f \) has a pole at \( \infty \). This will be discussed more later.

54.1.4 Runge’s Theorem

Now what follows is the first form of Runge’s theorem.

**Theorem 54.1.9** Let \( K \) be a compact subset of an open set, \( \Omega \) and let \( \{b_j\} \) be a set which consists of one point from each component of \( \mathbb{C} \setminus K \). Let \( f \) be analytic on \( \Omega \). Then for each \( \varepsilon > 0 \), there exists a rational function, \( Q \) whose poles are all contained in the set, \( \{b_j\} \) such that

\[ ||Q - f||_{K, \infty} < \varepsilon. \]  

(54.1.9)

If \( \mathbb{C} \setminus K \) has only one component, then \( Q \) may be taken to be a polynomial.

**Proof:** By Lemma 54.1.8 there exists a rational function of the form

\[ R(z) = \sum_{k=1}^{M} \frac{A_k}{w_k - z} \]

where the \( w_k \) are elements of components of \( \mathbb{C} \setminus K \) and \( A_k \) are complex numbers such that

\[ ||R - f||_{K, \infty} < \frac{\varepsilon}{2}. \]

Consider the rational function, \( R_k(z) \equiv \frac{A_k}{w_k - z} \) where \( w_k \in V_j \), one of the components of \( \mathbb{C} \setminus K \), the given point of \( V_j \) being \( b_j \). By Lemma 54.1.8 there exists a function, \( Q_k \) which is either a rational function having its only pole at \( b_j \) or a polynomial, depending on whether \( V_j \) is bounded such that

\[ ||R_k - Q_k||_{K, \infty} < \frac{\varepsilon}{2M}. \]

Letting \( Q(z) \equiv \sum_{k=1}^{M} Q_k(z) \),

\[ ||R - Q||_{K, \infty} < \frac{\varepsilon}{2}. \]
It follows
\[ ||f - Q||_{K, \infty} \leq ||f - R||_{K, \infty} + ||R - Q||_{K, \infty} < \varepsilon. \]

In the case of only one component of \( \mathbb{C} \setminus K \), this component is the unbounded component and so you can take \( Q \) to be a polynomial. This proves the theorem.

The next version of Runge’s theorem concerns the case where the given points are contained in \( \mathbb{C} \setminus \Omega \) for \( \Omega \) an open set rather than a compact set. Note that here there could be uncountably many components of \( \mathbb{C} \setminus \Omega \) because the components are no longer open sets. An easy example of this phenomenon in one dimension is where \( \Omega = [0, 1] \setminus P \) for \( P \) the Cantor set. Then you can show that \( \mathbb{R} \setminus \Omega \) has uncountably many components. Nevertheless, Runge’s theorem will follow from Theorem 54.1.9 with the aid of the following interesting lemma.

**Lemma 54.1.10** Let \( \Omega \) be an open set in \( \mathbb{C} \). Then there exists a sequence of compact sets, \( \{K_n\} \) such that
\[ \Omega = \bigcup_{n=1}^{\infty} K_n, \cdots, K_n \subseteq \text{int} \, K_{n+1} \cdots, \tag{54.1.10} \]
and for any \( K \subseteq \Omega \),
\[ K \subseteq K_n, \tag{54.1.11} \]
for all \( n \) sufficiently large, and every component of \( \mathbb{C} \setminus K_n \) contains a component of \( \mathbb{C} \setminus \Omega \).

**Proof:** Let
\[ V_n \equiv \{ z : |z| > n \} \cup \bigcup_{z \in \Omega} B \left( z, \frac{1}{n} \right). \]
Thus \( \{ z : |z| > n \} \) contains the point, \( \infty. \) Now let
\[ K_n \equiv \mathbb{C} \setminus V_n = \mathbb{C} \setminus \mathbb{C} \setminus V_n \subseteq \Omega. \]
You should verify that \( \text{54.1.10} \) and \( \text{54.1.11} \) hold. It remains to show that every component of \( \mathbb{C} \setminus K_n \) contains a component of \( \mathbb{C} \setminus \Omega \). Let \( D \) be a component of \( \mathbb{C} \setminus K_n \).

If \( \infty \notin D \), then \( D \) contains no point of \( \{ z : |z| > n \} \) because this set is connected and \( D \) is a component. (If it did contain a point of this set, it would have to contain the whole set.) Therefore, \( D \subseteq \bigcup_{z \notin \Omega} B \left( z, \frac{1}{n} \right) \) and so \( D \) contains some point of \( B \left( z, \frac{1}{n} \right) \) for some \( z \notin \Omega \). Therefore, since this ball is connected, it follows \( D \) must contain the whole ball and consequently \( D \) contains some point of \( \Omega^c \). (The point \( z \) at the center of the ball will do.) Since \( D \) contains \( z \notin \Omega \), it must contain the component, \( H_z \), determined by this point. The reason for this is that
\[ H_z \subseteq \mathbb{C} \setminus \Omega \subseteq \mathbb{C} \setminus K_n \]
and \( H_z \) is connected. Therefore, \( H_z \) can only have points in one component of \( \mathbb{C} \setminus K_n \). Since it has a point in \( D \), it must therefore, be totally contained in \( D \). This verifies the desired condition in the case where \( \infty \notin D \).
Now suppose that $\infty \in D$. $\infty \notin \Omega$ because $\Omega$ is given to be a set in $\mathbb{C}$. Letting $H_\infty$ denote the component of $\mathbb{C} \setminus \Omega$ determined by $\infty$, it follows both $D$ and $H_\infty$ contain $\infty$. Therefore, the connected set, $H_\infty$ cannot have any points in another component of $\mathbb{C} \setminus K_n$ and it is a set which is contained in $\mathbb{C} \setminus K_n$ so it must be contained in $D$. This proves the lemma.

The following picture is a very simple example of the sort of thing considered by Runge’s theorem. The picture is of a region which has a couple of holes.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example_region.png}
\caption{Example region for Runge’s theorem.}
\end{figure}

However, there could be many more holes than two. In fact, there could be infinitely many. Nor does it follow that the components of the complement of $\Omega$ need to have any interior points. Therefore, the picture is certainly not representative.

**Theorem 54.1.11** *(Runge)* Let $\Omega$ be an open set, and let $A$ be a set which has one point in each component of $\mathbb{C} \setminus \Omega$ and let $f$ be analytic on $\Omega$. Then there exists a sequence of rational functions, $\{R_n\}$ having poles only in $A$ such that $R_n \to f$ uniformly on compact subsets of $\Omega$.

**Proof:** Let $K_n$ be the compact sets of Lemma 54.1.10 where each component of $\mathbb{C} \setminus K_n$ contains a component of $\mathbb{C} \setminus \Omega$. It follows each component of $\mathbb{C} \setminus K_n$ contains a point of $A$. Therefore, by Theorem 54.1.9 there exists $R_n$ a rational function with poles only in $A$ such that

$$||R_n - f||_{K_n, \infty} < \frac{1}{n}.$$ 

It follows, since a given compact set, $K$ is a subset of $K_n$ for all $n$ large enough, that $R_n \to f$ uniformly on $K$. This proves the theorem.

**Corollary 54.1.12** Let $\Omega$ be simply connected and $f$ analytic on $\Omega$. Then there exists a sequence of polynomials, $\{p_n\}$ such that $p_n \to f$ uniformly on compact sets of $\Omega$.

**Proof:** By definition of what is meant by simply connected, $\mathbb{C} \setminus \Omega$ is connected and so there are no bounded components of $\mathbb{C} \setminus \Omega$. Therefore, in the proof of Theorem 54.1.11 when you use Theorem 54.1.9, you can always have $R_n$ be a polynomial by Lemma 54.1.8.

### 54.2 The Mittag-Leffler Theorem

#### 54.2.1 A Proof From Runge’s Theorem

This theorem is fairly easy to prove once you have Theorem 54.1.11. Given a set of complex numbers, does there exist a meromorphic function having its poles equal
54.2. THE MITTAG-LEFFLER THEOREM

The Mittag-Leffler theorem provides a very satisfactory answer to this question. Actually, it says somewhat more. You can specify, not just the location of the pole but also the kind of singularity the meromorphic function is to have at that pole.

**Theorem 54.2.1** Let \( P \equiv \{z_k\}_{k=1}^\infty \) be a set of points in an open subset of \( \mathbb{C} \), \( \Omega \). Suppose also that \( P \subseteq \Omega \subseteq \mathbb{C} \). For each \( z_k \), denote by \( S_k(z) \) a function of the form

\[
S_k(z) = \sum_{j=1}^{m_k} \frac{a_{kj}}{z - z_k^j}.
\]

Then there exists a meromorphic function, \( Q \) defined on \( \Omega \) such that the poles of \( Q \) are the points, \( \{z_k\}_{k=1}^\infty \) and the singular part of the Laurent expansion of \( Q \) at \( z_k \) equals \( S_k(z) \). In other words, for \( z \) near \( z_k \), \( Q(z) = g_k(z) + S_k(z) \) for some function, \( g_k \) analytic near \( z_k \).

**Proof:** Let \( \{K_n\} \) denote the sequence of compact sets described in Lemma \( \text{54.1.10} \). Thus \( \cup_{n=1}^\infty K_n = \Omega \), \( K_n \subseteq \text{int} (K_{n+1}) \subseteq K_{n+1} \cdots \), and the components of \( \mathbb{C} \setminus K_n \) contain the components of \( \mathbb{C} \setminus \Omega \). Renumbering if necessary, you can assume each \( K_n \neq \emptyset \). Also let \( K_0 = \emptyset \). Let \( P_m \equiv P \cap (K_m \setminus K_{m-1}) \) and consider the rational function, \( R_m \) defined by

\[
R_m(z) = \sum_{z_k \in K_m \setminus K_{m-1}} S_k(z).
\]

Since each \( K_m \) is compact, it follows \( P_m \) is finite and so the above really is a rational function. Now for \( m > 1 \), this rational function is analytic on some open set containing \( K_{m-1} \). There exists a set of points, \( A \) one point in each component of \( \mathbb{C} \setminus \Omega \). Consider \( \mathbb{C} \setminus K_{m-1} \). Each of its components contains a component of \( \mathbb{C} \setminus \Omega \) and so for each of these components of \( \mathbb{C} \setminus K_{m-1} \), there exists a point of \( A \) which is contained in it. Denote the resulting set of points by \( A' \). By Theorem \( \text{54.1.3} \) there exists a rational function, \( Q_m \) whose poles are all contained in the set, \( A' \subseteq \Omega' \) such that

\[
||R_m - Q_m||_{K_{m-1},\infty} < \frac{1}{2^n}.
\]

The meromorphic function is

\[
Q(z) = R_1(z) + \sum_{k=2}^\infty (R_k(z) - Q_k(z)).
\]

It remains to verify this function works. First consider \( K_1 \). Then on \( K_1 \), the above sum converges uniformly. Furthermore, the terms of the sum are analytic in some open set containing \( K_1 \). Therefore, the infinite sum is analytic on this open set and so for \( z \in K_1 \) The function, \( f \) is the sum of a rational function, \( R_1 \), having poles at
CHAPTER 54. APPROXIMATION BY RATIONAL FUNCTIONS

$P_1$ with the specified singular terms and an analytic function. Therefore, $Q$ works on $K_1$. Now consider $K_m$ for $m > 1$. Then

$$Q(z) = R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z)) + \sum_{k=m+2}^{\infty} (R_k(z) - Q_k(z)).$$

As before, the infinite sum converges uniformly on $K_{m+1}$ and hence on some open set, $O$ containing $K_m$. Therefore, this infinite sum equals a function which is analytic on $O$. Also,

$$R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z))$$

is a rational function having poles at $\bigcup_{k=1}^m P_k$ with the specified singularities because the poles of each $Q_k$ are not in $\Omega$. It follows this function is meromorphic because it is analytic except for the points in $P$. It also has the property of retaining the specified singular behavior.

54.2.2 A Direct Proof Without Runge’s Theorem

There is a direct proof of this important theorem which is not dependent on Runge’s theorem in the case where $\Omega = \mathbb{C}$. I think it is arguably easier to understand and the Mittag-Leffler theorem is very important so I will give this proof here.

**Theorem 54.2.2** Let $P \equiv \{z_k\}_{k=1}^\infty$ be a set of points in $\mathbb{C}$ which satisfies $\lim_{n \to \infty} |z_n| = \infty$. For each $z_k$, denote by $S_k(z)$ a polynomial in $1/z - z_k$ which is of the form

$$S_k(z) = \sum_{j=1}^{m_k} a_j^k (z - z_k)^j.$$

Then there exists a meromorphic function, $Q$ defined on $\mathbb{C}$ such that the poles of $Q$ are the points, $\{z_k\}_{k=1}^\infty$ and the singular part of the Laurent expansion of $Q$ at $z_k$ equals $S_k(z)$. In other words, for $z$ near $z_k$,

$$Q(z) = g_k(z) + S_k(z)$$

for some function, $g_k$ analytic in some open set containing $z_k$.

**Proof:** First consider the case where none of the $z_k = 0$. Letting

$$K_k \equiv \{z : |z| \leq |z_k|/2\},$$

there exists a power series for $1/z - z_k$ which converges uniformly and absolutely on this set. Here is why:

$$\frac{1}{z - z_k} = \left( -1 \frac{1}{1 - \frac{z}{z_k}} \right) \frac{1}{z_k} = - \frac{1}{z_k} \sum_{l=0}^{\infty} \left( \frac{z}{z_k} \right)^l.$$
and the Weierstrass $M$ test can be applied because
\[
\left| \frac{z}{z_k} \right| < \frac{1}{2}
\]
on this set. Therefore, by Corollary 54.1.7, $S_k(z)$, being a polynomial in $\frac{1}{z-z_k}$, has a power series which converges uniformly to $S_k(z)$ on $K_k$. Therefore, there exists a polynomial, $P_k(z)$ such that
\[
||P_k - S_k||_{B(0,|z_k|/2),\infty} < \frac{1}{2^k}.
\]
Let
\[
Q(z) \equiv \sum_{k=1}^{\infty} (S_k(z) - P_k(z)) .
\]
(54.2.12)
Consider $z \in K_m$ and let $N$ be large enough that if $k > N$, then $|z_k| > 2|z|$.
\[
Q(z) = \sum_{k=1}^{N} (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).
\]
On $K_m$, the second sum converges uniformly to a function analytic on int($K_m$) (interior of $K_m$) while the first is a rational function having poles at $z_1, \cdots , z_N$. Since any compact set is contained in $K_m$ for large enough $m$, this shows $Q(z)$ is meromorphic as claimed and has poles with the given singularities.
Now consider the case where the poles are at $\{z_k\}_{k=0}^{\infty}$ with $z_0 = 0$. Everything is similar in this case. Let
\[
Q(z) \equiv S_0(z) + \sum_{k=1}^{\infty} (S_k(z) - P_k(z)).
\]
The series converges uniformly on every compact set because of the assumption that $\lim_{n \to \infty} |z_n| = \infty$ which implies that any compact set is contained in $K_k$ for $k$ large enough. Choose $N$ such that $z \in \text{int}(K_N)$ and $z_n \notin K_N$ for all $n \geq N + 1$. Then
\[
Q(z) = S_0(z) + \sum_{k=1}^{N} (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).
\]
The last sum is analytic on int($K_N$) because each function in the sum is analytic due to the fact that none of its poles are in $K_N$. Also, $S_0(z) + \sum_{k=1}^{N} (S_k(z) - P_k(z))$ is a finite sum of rational functions so it is a rational function and $P_k$ is a polynomial so $z_m$ is a pole of this function with the correct singularity whenever $z_m \in \text{int}(K_N)$. 

54.2. THE MITTAG-LEFFLER THEOREM
Sometimes it is useful to think of isolated singular points at $\infty$.

**Definition 54.2.3** Suppose $f$ is analytic on $\{z \in \mathbb{C} : |z| > r\}$. Then $f$ is said to have a removable singularity at $\infty$ if the function, $g(z) \equiv f\left(\frac{1}{z}\right)$ has a removable singularity at 0. $f$ is said to have a pole at $\infty$ if the function, $g(z) = f\left(\frac{1}{z}\right)$ has a pole at 0. Then $f$ is said to be meromorphic on $\hat{\mathbb{C}}$ if all its singularities are isolated and either poles or removable.

So what is $f$ like for these cases? First suppose $f$ has a removable singularity at $\infty$. Then $zg(z)$ converges to 0 as $z \to 0$. It follows $g(z)$ must be analytic near 0 and so can be given as a power series. Thus $f(z)$ is of the form $f(z) = g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$. Next suppose $f$ has a pole at $\infty$. This means $g(z)$ has a pole at 0 so $g(z)$ is of the form $g(z) = \sum_{k=1}^{m} \frac{b_k}{z^k} + h(z)$ where $h(z)$ is analytic near 0. Thus in the case of a pole at $\infty$, $f(z)$ is of the form $f(z) = g\left(\frac{1}{z}\right) = \sum_{k=1}^{m} b_k z^k + \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$.

It turns out that the functions which are meromorphic on $\hat{\mathbb{C}}$ are all rational functions. To see this suppose $f$ is meromorphic on $\hat{\mathbb{C}}$ and note that there exists $r > 0$ such that $f(z)$ is analytic for $|z| > r$. This is required if $\infty$ is to be isolated. Therefore, there are only finitely many poles of $f$ for $|z| \leq r$, $\{a_1, \ldots, a_m\}$, because by assumption, these poles are isolated and this is a compact set. Let the singular part of $f$ at $a_k$ be denoted by $S_k(z)$. Then $f(z) - \sum_{k=1}^{m} S_k(z)$ is analytic on all of $\mathbb{C}$. Therefore, it is bounded on $|z| \leq r$. In one case, $f$ has a removable singularity at $\infty$. In this case, $f$ is bounded as $z \to \infty$ and $\sum_{k=1}^{m} S_k$ also converges to 0 as $z \to \infty$. Therefore, by Liouville’s theorem, $f(z) - \sum_{k=1}^{m} S_k(z)$ equals a constant and so $f - \sum_{k} S_k$ is a constant. Thus $f$ is a rational function. In the other case that $f$ has a pole at $\infty$, $f(z) - \sum_{k=1}^{m} S_k(z) = \sum_{k=1}^{m} b_k z^k = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n - \sum_{k=1}^{m} S_k(z)$. Now $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k$ is analytic on $\mathbb{C}$ and so is bounded on $|z| \leq r$. But now $\sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n - \sum_{k=1}^{m} S_k(z)$ converges to 0 as $z \to \infty$ and so by Liouville’s theorem, $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k$ must equal a constant and again, $f(z)$ equals a rational function.

**54.2.4 Great And Glorious Theorem, Simply Connected Regions**

Here is a laundry list of properties which are equivalent to an open set being simply connected. Recall Definition [397.2.41] on Page [1050] which said that an open set, $\Omega$ is simply connected means $\hat{\mathbb{C}} \setminus \Omega$ is connected. Recall also that this is not the same thing at all as saying $\mathbb{C} \setminus \Omega$ is connected. Consider the outside of a disk for example. I will continue to use this definition for simply connected because it is the most convenient one for complex analysis. However, there are many other equivalent conditions. First here is an interesting lemma which is interesting for its own sake. Recall $n(p, \gamma)$ means the winding number of $\gamma$ about $p$. Now recall Theorem [397.2.39] implies the following lemma in which $B^C$ is playing the role of $\Omega$ in Theorem [397.2.39].
Lemma 54.2.4 Let $K$ be a compact subset of $B^c$, the complement of a closed set. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j \cap K = \emptyset$ for each $j$, $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,

$$\sum_{k=1}^m n(\Gamma_k, p) = 1.$$ 

while for all $z \in B$

$$\sum_{k=1}^m n(\Gamma_k, z) = 0.$$

Definition 54.2.5 Let $\gamma$ be a closed curve in an open set, $\Omega$, $\gamma : [a, b] \to \Omega$. Then $\gamma$ is said to be homotopic to a point, $p$ in $\Omega$ if there exists a continuous function, $H : [0, 1] \times [a, b] \to \Omega$ such that $H(0, t) = p, H(\alpha, a) = H(\alpha, b)$, and $H(1, t) = \gamma(t)$. This function, $H$ is called a homotopy.

Lemma 54.2.6 Suppose $\gamma$ is a closed continuous bounded variation curve in an open set, $\Omega$ which is homotopic to a point. Then if $a \not\in \Omega$, it follows $n(a, \gamma) = 0$.

Proof: Let $H$ be the homotopy described above. The problem with this is that it is not known that $H(\alpha, \cdot)$ is of bounded variation. There is no reason it should be. Therefore, it might not make sense to take the integral which defines the winding number. There are various ways around this. Extend $H$ as follows. $H(\alpha, t) = H(\alpha, a)$ for $t < a$, $H(\alpha, t) = H(\alpha, b)$ for $t > b$. Let $\varepsilon > 0$.

$$\frac{1}{2\varepsilon} \int_{t+\varepsilon}^{t+2\varepsilon} H(\alpha, s) \, ds, \quad H_\varepsilon(0, t) = p.$$ 

Thus $H_\varepsilon(\alpha, \cdot)$ is a closed curve which has bounded variation and when $\alpha = 1$, this converges to $\gamma$ uniformly on $[a, b]$. Therefore, for $\varepsilon$ small enough, $n(a, H_\varepsilon(1, \cdot)) = n(a, \gamma)$ because they are both integers and as $\varepsilon \to 0$, $n(a, H_\varepsilon(1, \cdot)) \to n(a, \gamma)$. Also, $H_\varepsilon(\alpha, t) \to H(\alpha, t)$ uniformly on $[0, 1] \times [a, b]$ because of uniform continuity of $H$. Therefore, for small enough $\varepsilon$, you can also assume $H_\varepsilon(\alpha, t) \in \Omega$ for all $\alpha, t$. Now $\alpha \to n(a, H_\varepsilon(\alpha, \cdot))$ is continuous. Hence it must be constant because the winding number is integer valued. But

$$\lim_{\alpha \to 0} \frac{1}{2\pi i} \int_{H_\varepsilon(\alpha, \cdot)} \frac{1}{z-a} \, dz = 0$$

because the length of $H_\varepsilon(\alpha, \cdot)$ converges to 0 and the integrand is bounded because $a \not\in \Omega$. Therefore, the constant can only equal 0. This proves the lemma.

Now it is time for the great and glorious theorem on simply connected regions. The following equivalence of properties is taken from Rudin. There is a slightly different list in Conway and a shorter list in Ash.

Theorem 54.2.7 The following are equivalent for an open set, $\Omega$. 

54.2. THE MITTAG-LEFFLER THEOREM
1. $\Omega$ is homeomorphic to the unit disk, $B(0,1)$.

2. Every closed curve contained in $\Omega$ is homotopic to a point in $\Omega$.

3. If $z \notin \Omega$, and if $\gamma$ is a closed bounded variation continuous curve in $\Omega$, then $n(\gamma,z) = 0$.

4. $\Omega$ is simply connected, $(\mathbb{C} \setminus \Omega$ is connected and $\Omega$ is connected).

5. Every function analytic on $\Omega$ can be uniformly approximated by polynomials on compact subsets.

6. For every $f$ analytic on $\Omega$ and every closed continuous bounded variation curve, $\gamma$,
$$\int_\gamma f(z) \, dz = 0.$$ 

7. Every function analytic on $\Omega$ has a primitive on $\Omega$.

8. If $f, 1/f$ are both analytic on $\Omega$, then there exists an analytic, $g$ on $\Omega$ such that $f = \exp(g)$.

9. If $f, 1/f$ are both analytic on $\Omega$, then there exists $\phi$ analytic on $\Omega$ such that $f = \phi^2$.

**Proof:** $1 \Rightarrow 2$. Assume $1$ and let $\gamma$ be a closed curve in $\Omega$. Let $h$ be the homeomorphism, $h : B(0,1) \to \Omega$. Let $H(\alpha,t) = h(\alpha(h^{-1}(\gamma(t))))$. This works.

$2 \Rightarrow 3$. Suppose $2$ but $3$ fails to hold. Then if $\mathbb{C} \setminus \Omega$ is not connected, there exist disjoint nonempty sets, $A$ and $B$ such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. It follows each of these sets must be closed because neither can have a limit point in $\Omega$ nor in the other. Also, one and only one of them contains $\infty$. Let this set be $B$. Thus $A$ is a closed set which must also be bounded. Otherwise, there would exist a sequence of points in $A$, $\{a_n\}$ such that $\lim_{n \to \infty} a_n = \infty$ which would contradict the requirement that no limit points of $A$ can be in $B$. Therefore, $A$ is a compact set contained in the open set, $B^C \equiv \{ z \in \mathbb{C} : z \notin B \}$. Pick $p \in A$. By Lemma $1 \Rightarrow 2$, there exist continuous bounded variation closed curves $\{\Gamma_k\}_{k=1}^m$ which are contained in $B^C$, do not intersect $A$ and such that
$$1 = \sum_{k=1}^m n(p, \Gamma_k).$$

However, if these curves do not intersect $A$ and they also do not intersect $B$ then they must be all contained in $\Omega$. Since $p \notin \Omega$, it follows by $3$ that for each $k$, $n(p, \Gamma_k) = 0$, a contradiction.

$5 \Rightarrow 6$. This is Corollary $54.2.12$ on Page 1823.
Every polynomial has a primitive and so the integral over any closed bounded variation curve of a polynomial equals 0. Let \( f \) be analytic on \( \Omega \). Then let \( \{f_n\} \) be a sequence of polynomials converging uniformly to \( f \) on \( \gamma^* \). Then
\[
0 = \lim_{n \to \infty} \int_{\gamma} f_n(z) \, dz = \int_{\gamma} f(z) \, dz.
\]

Pick \( z_0 \in \Omega \). Letting \( \gamma(z_0,z) \) be a bounded variation continuous curve joining \( z_0 \) to \( z \) in \( \Omega \), you define a primitive for \( f \) as follows.
\[
F(z) = \int_{\gamma(z_0,z)} f(w) \, dw.
\]
This is well defined by \( \mathfrak{R} \) and is easily seen to be a primitive. You just write the difference quotient and take a limit using \( \mathfrak{I} \)
\[
\lim_{w \to 0} \frac{F(z+w) - F(z)}{w} = \lim_{w \to 0} \frac{1}{w} \left( \int_{\gamma(z_0,z+w)} f(u) \, du - \int_{\gamma(z_0,z)} f(u) \, du \right)
\]
\[
= \lim_{w \to 0} \frac{1}{w} \int_{\gamma(z,z+w)} f(u) \, du
\]
\[
= \lim_{w \to 0} \frac{1}{w} \int_{0}^{1} f(z + tw) \, w \, dt = f(z).
\]

Suppose then that \( f,1/f \) are both analytic. Then \( f'/f \) is analytic and so it has a primitive by \( \mathfrak{I} \). Let this primitive be \( g_1 \). Then
\[
\left( e^{-g_1} f \right)' = e^{-g_1} (-g'_1) f + e^{-g_1} f' = e^{-g_1} \left( f' - f \right) = 0.
\]

Therefore, since \( \Omega \) is connected, it follows \( e^{-g_1} f \) must equal a constant. (Why?) Let the constant be \( e^{a+ib} \). Then \( f(z) = e^{g_1(z)} e^{a+ib} \). Therefore, you let \( g(z) = g_1(z) + a + ib \).

Suppose then that \( f,1/f \) are both analytic on \( \Omega \). Then by \( \mathfrak{S} \) \( f(z) = e^{g(z)} \). Let \( \phi(z) = e^{g(z)/2} \).

There are two cases. First suppose \( \Omega = \mathbb{C} \). This satisfies condition \( \mathfrak{S} \) because if \( f,1/f \) are both analytic, then the same argument involved in \( \mathfrak{S} \Rightarrow \mathfrak{I} \) gives the existence of a square root. A homeomorphism is \( h(z) = z / \sqrt{1+|z|^2} \). It obviously maps onto \( B(0,1) \) and is continuous. To see it is 1 - 1 consider the case of \( z_1 \) and \( z_2 \) having different arguments. Then \( h(z_1) \neq h(z_2) \). If \( z_2 = t z_1 \) for a positive \( t \neq 1 \), then it is also clear \( h(z_1) \neq h(z_2) \). To show \( h^{-1} \) is continuous, note that if you have an open set in \( \mathbb{C} \) and a point in this open set, you can get a small open set containing this point by allowing the modulus and the argument to lie in some open interval. Reasoning this way, you can verify \( h \) maps open sets to open sets. In the case where \( \Omega \neq \mathbb{C} \), there exists a one to one analytic map which maps \( \Omega \) onto \( B(0,1) \) by the Riemann mapping theorem. This proves the theorem.
CHAPTER 54. APPROXIMATION BY RATIONAL FUNCTIONS

54.3 Exercises

1. Let \( a \in \mathbb{C} \). Show there exists a sequence of polynomials, \( \{ p_n \} \) such that \( p_n(a) = 1 \) but \( p_n(z) \to 0 \) for all \( z \neq a \).

2. Let \( l \) be a line in \( \mathbb{C} \). Show there exists a sequence of polynomials \( \{ p_n \} \) such that \( p_n(z) \to 1 \) on one side of this line and \( p_n(z) \to -1 \) on the other side of the line. **Hint:** The complement of this line is simply connected.

3. Suppose \( \Omega \) is a simply connected region, \( f \) is analytic on \( \Omega \), \( f \neq 0 \) on \( \Omega \), and \( n \in \mathbb{N} \). Show that there exists an analytic function, \( g \) such that \( g(z)^n = f(z) \) for all \( z \in \Omega \). That is, you can take the \( n \)th root of \( f(z) \). If \( \Omega \) is a region which contains 0, is it possible to find \( g(z) \) such that \( g \) is analytic on \( \Omega \) and \( g(z)^2 = z \)?

4. Suppose \( \Omega \) is a region (connected open set) and \( f \) is an analytic function defined on \( \Omega \) such that \( f(z) \neq 0 \) for any \( z \in \Omega \). Suppose also that for every positive integer, \( n \) there exists an analytic function, \( g_n \) defined on \( \Omega \) such that \( g_n^n(z) = f(z) \). Show that then it is possible to define an analytic function, \( L \) on \( f(\Omega) \) such that \( e^{L(f(z))} = f(z) \) for all \( z \in \Omega \).

5. You know that \( \phi(z) \equiv \frac{z + i}{z - i} \) maps the upper half plane onto the unit ball. Its inverse, \( \psi(z) = \frac{i + z}{1 - z} \) maps the unit ball onto the upper half plane. Also for \( z \) in the upper half plane, you can define a square root as follows. If \( z = |z| e^{i\theta} \) where \( \theta \in (0, \pi) \), let \( z^{1/2} = |z|^{1/2} e^{i\theta/2} \) so the square root maps the upper half plane to the first quadrant. Now consider

\[
z \to \exp \left( -i \log \left[ i \left( \frac{1 + z}{1 - z} \right)^{1/2} \right] \right) . \tag{54.3.13}
\]

Show this is an analytic function which maps the unit ball onto an annulus. Is it possible to find a one to one analytic map which does this?
Chapter 55

Infinite Products

The Mittag-Leffler theorem gives existence of a meromorphic function which has specified singular part at various poles. It would be interesting to do something similar to zeros of an analytic function. That is, given the order of the zero at various points, does there exist an analytic function which has these points as zeros with the specified orders? You know that if you have the zeros of the polynomial, you can factor it. Can you do something similar with analytic functions which are just limits of polynomials? These questions involve the concept of an infinite product.

**Definition 55.0.1** \[ \prod_{n=1}^{\infty} (1 + u_n) \equiv \lim_{n \to \infty} \prod_{k=1}^{n} (1 + u_k) \] whenever this limit exists. If \( u_n = u_n(z) \) for \( z \in H \), we say the infinite product converges uniformly on \( H \) if the partial products, \( \prod_{k=1}^{n} (1 + u_k(z)) \) converge uniformly on \( H \).

The main theorem is the following.

**Theorem 55.0.2** Let \( H \subseteq C \) and suppose that \( \sum_{n=1}^{\infty} |u_n(z)| \) converges uniformly on \( H \) where \( u_n(z) \) bounded on \( H \). Then

\[ P(z) \equiv \prod_{n=1}^{\infty} (1 + u_n(z)) \]

converges uniformly on \( H \). If \( (n_1, n_2, \cdots) \) is any permutation of \( (1, 2, \cdots) \), then for all \( z \in H \),

\[ P(z) = \prod_{k=1}^{\infty} (1 + u_{n_k}(z)) \]

and \( P \) has a zero at \( z_0 \) if and only if \( u_n(z_0) = -1 \) for some \( n \).
**Proof:** First a simple estimate:

\[
\prod_{k=m}^{n} (1 + |u_k(z)|) = \exp\left(\ln\left(\prod_{k=m}^{n} (1 + |u_k(z)|)\right)\right) = \exp\left(\sum_{k=m}^{n} \ln (1 + |u_k(z)|)\right) \\
\leq \exp\left(\sum_{k=m}^{\infty} |u_k(z)|\right) < e
\]

for all \(z \in H\) provided \(m\) is large enough. Since \(\sum_{k=1}^{\infty} |u_k(z)|\) converges uniformly on \(H\), \(|u_k(z)| < \frac{1}{2}\) for all \(z \in H\) provided \(k\) is large enough. Thus you can take \(\log(1 + u_k(z))\). Pick \(N_0\) such that for \(n > m \geq N_0\),

\[
|u_m(z)| < \frac{1}{2} \prod_{k=m}^{n} (1 + |u_k(z)|) < e. \tag{55.0.1}
\]

Now having picked \(N_0\), the assumption the \(u_n\) are bounded on \(H\) implies there exists a constant, \(C\), independent of \(z \in H\) such that for all \(z \in H\),

\[
\prod_{k=1}^{N_0} (1 + |u_k(z)|) < C. \tag{55.0.2}
\]

Let \(N_0 < M < N\). Then

\[
\left|\prod_{k=1}^{N} (1 + u_k(z)) - \prod_{k=1}^{M} (1 + u_k(z))\right| \\
\leq \prod_{k=1}^{N_0} (1 + u_k(z)) \left|\prod_{k=N_0+1}^{N} (1 + u_k(z)) - \prod_{k=N_0+1}^{M} (1 + u_k(z))\right| \\
\leq C \left|\prod_{k=N_0+1}^{N} (1 + u_k(z)) - \prod_{k=N_0+1}^{M} (1 + u_k(z))\right| \\
\leq C \left(\prod_{k=N_0+1}^{M} (1 + |u_k(z)|)\right) \left|\prod_{k=M+1}^{N} (1 + u_k(z)) - 1\right| \\
\leq C e \left|\prod_{k=M+1}^{N} (1 + |u_k(z)|) - 1\right|.
\]
Since \( 1 \leq \prod_{k=M+1}^{N} (1 + |u_k(z)|) \leq e \), it follows the term on the far right is dominated by

\[
Ce^2 \left| \ln \left( \prod_{k=M+1}^{N} (1 + |u_k(z)|) \right) - \ln 1 \right| 
\]

\[
\leq Ce^2 \sum_{k=M+1}^{N} \ln (1 + |u_k(z)|) 
\]

\[
\leq Ce^2 \sum_{k=M+1}^{N} |u_k(z)| < \varepsilon 
\]

uniformly in \( z \in H \) provided \( M \) is large enough. This follows from the simple observation that if \( 1 < x < e \), then \( x - 1 \leq e (\ln x - \ln 1) \). Therefore, \( \{ \prod_{k=1}^{m} (1 + u_k(z)) \}_{m=1}^{\infty} \) is uniformly Cauchy on \( H \) and therefore, converges uniformly on \( H \). Let \( P(z) \) denote the function it converges to.

What about the permutations? Let \( \{ n_1, n_2, \cdots \} \) be a permutation of the indices. Let \( \varepsilon > 0 \) be given and let \( N_0 \) be such that if \( n > N_0 \),

\[
\left| \prod_{k=1}^{n} (1 + u_k(z)) - P(z) \right| < \varepsilon 
\]

for all \( z \in H \). Let \( \{ 1, 2, \cdots, n \} \subseteq \{ n_1, n_2, \cdots, n_{p(n)} \} \) where \( p(n) \) is an increasing sequence. Then from \( 55.0.1 \) and \( 55.0.2 \),

\[
\left| P(z) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| 
\]

\[
\leq \left| P(z) - \prod_{k=1}^{n} (1 + u_k(z)) \right| \left| \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) 
\]

\[
\leq \varepsilon + \prod_{k=1}^{n} (1 + |u_k(z)|) \left| 1 - \prod_{n_k > n} (1 + u_{n_k}(z)) \right| 
\]

\[
\leq \varepsilon + \prod_{k=1}^{N_0} (1 + |u_k(z)|) \left| \prod_{k=N_0+1}^{n} (1 + |u_k(z)|) \right| \left| 1 - \prod_{n_k > n} (1 + u_{n_k}(z)) \right| 
\]

\[
\leq \varepsilon + Ce \left| \prod_{n_k > n} (1 + |u_{n_k}(z)|) - 1 \right| \leq \varepsilon + Ce \left| \prod_{k=n+1}^{M(p(n))} (1 + |u_{n_k}(z)|) - 1 \right| 
\]
where \(M(p(n))\) is the largest index in the permuted list, \(\{n_1, n_2, \ldots, n_{p(n)}\}\), then from \(55.0.1\), this last term is dominated by

\[
\varepsilon + Ce^2 \left| \ln \left( \prod_{k=n+1}^{M(p(n))} (1 + |u_{n_k}(z)|) \right) - \ln 1 \right| \leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} \ln (1 + |u_{n_k}|) \leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} |u_{n_k}| < 2\varepsilon
\]

for all \(n\) large enough uniformly in \(z \in H\). Therefore, \(\left| P(z) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| < 2\varepsilon\) whenever \(n\) is large enough. This proves the part about the permutation.

It remains to verify the assertion about the points, \(z_0\), where \(P(z_0) = 0\). Obviously, if \(u_n(z_0) = -1\), then \(P(z_0) = 0\). Suppose then that \(P(z_0) = 0\) and \(M > N_0\).

Then

\[
\left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| = \prod_{k=1}^{M} (1 + u_k(z_0)) - \prod_{k=1}^{\infty} (1 + u_k(z_0))
\]

\[
\leq \prod_{k=1}^{M} (1 + u_k(z_0)) \left| 1 - \prod_{k=M+1}^{\infty} (1 + u_k(z_0)) \right|
\]

\[
\leq \prod_{k=1}^{M} (1 + u_k(z_0)) \left| \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - 1 \right|
\]

\[
\leq e^{\left( \sum_{k=M+1}^{\infty} \ln (1 + |u_k(z)|) \right)} \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|
\]

\[
\leq e^{\sum_{k=M+1}^{\infty} |u_k(z)|} \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|
\]

\[
\leq \frac{1}{2} \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|
\]

whenever \(M\) is large enough. Therefore, for such \(M\),

\[\prod_{k=1}^{M} (1 + u_k(z_0)) = 0\]

and so \(u_k(z_0) = -1\) for some \(k \leq M\). This proves the theorem.
55.1 Analytic Function With Prescribed Zeros

Suppose you are given complex numbers, \( \{z_n\} \) and you want to find an analytic function, \( f \) such that these numbers are the zeros of \( f \). How can you do it? The problem is easy if there are only finitely many of these zeros, \( \{z_1, z_2, \cdots, z_m\} \). You just write \((z - z_1)(z - z_2)\cdots(z - z_m)\). Now if none of the \( z_k = 0 \) you could also write it at \( \prod_{k=1}^{m} \left(1 - \frac{z}{z_k}\right) \) and this might have a better chance of success in the case of infinitely many prescribed zeros. However, you would need to verify something like \( \sum_{n=1}^{\infty} \left| \frac{z}{z_n} \right| < \infty \) which might not be so. The way around this is to adjust the product, making it \( \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{g_k(z)} \) where \( g_k(z) \) is some analytic function. Recall also that for \( |x| < 1 \), \( \ln \left((1 - x)^{-1}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n} \). If you had \( x/x_n \) small and real, then \( 1 = (1 - x/x_n) \exp \left(\ln \left((1 - x/x_n)^{-1}\right)\right) \) and \( \prod_{k=1}^{\infty} 1 \) of course converges but loses all the information about zeros. However, this is why it is not too unreasonable to consider factors of the form

\[
\left(1 - \frac{z}{z_k}\right) e^{\sum_{k=1}^{p_k} \left(\frac{z}{z_k}\right)^k}
\]

where \( p_k \) is suitably chosen.

First here are some estimates.

**Lemma 55.1.1** For \( z \in \mathbb{C} \),

\[
|e^z - 1| \leq |z| e^{|z|},
\]

(55.1.3)

and if \( |z| \leq 1/2 \),

\[
\left| \sum_{k=m}^{\infty} \frac{z^k}{k!} \right| \leq \frac{1}{m} \frac{|z|^m}{1 - |z|} \leq \frac{2}{m} |z|^m \leq \frac{1}{m} \frac{1}{2^{m-1}}.
\]

(55.1.4)

**Proof:** Consider (55.1.3)

\[
|e^z - 1| = \left| \sum_{k=1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=1}^{\infty} \left| \frac{z^k}{k!} \right| = e^{|z|} - 1 \leq |z| e^{|z|}
\]

the last inequality holding by the mean value theorem. Now consider (55.1.4)

\[
\left| \sum_{k=m}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=m}^{\infty} \frac{|z|^k}{k} \leq \frac{1}{m} \sum_{k=m}^{\infty} |z|^k = \frac{1}{m} \frac{|z|^m}{1 - |z|} \leq \frac{2}{m} |z|^m \leq \frac{1}{m} \frac{1}{2^{m-1}}.
\]

This proves the lemma.

The functions, \( E_p \) in the next definition are called the elementary factors.
Definition 55.1.2 Let $E_0(z) \equiv 1 - z$ and for $p \geq 1$,

$$E_p(z) \equiv (1 - z) \exp \left( z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$$

In terms of this new symbol, here is another estimate. A sharper inequality is available in Rudin [102] but it is more difficult to obtain.

Corollary 55.1.3 For $E_p$ defined above and $|z| \leq 1/2$,

$$|E_p(z) - 1| \leq 3|z|^{p+1}.$$ 

Proof: From elementary calculus, $\log(1 - x) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ for all $|x| < 1$. Therefore, for $|z| < 1$,

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \log((1 - z)^{-1}) = \sum_{n=1}^{\infty} \frac{z^n}{n},$$

because the function $\log(1 - z)$ and the analytic function, $-\sum_{n=1}^{\infty} \frac{z^n}{n}$ both are equal to $\log(1 - x)$ on the real line segment $(-1, 1)$, a set which has a limit point. Therefore, using Lemma 55.1.1,

$$|E_p(z) - 1| = \left| (1 - z) \exp \left( z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right) - 1 \right| = \left| (1 - z) \exp \left( \log((1 - z)^{-1}) - \sum_{n=p+1}^{\infty} \frac{z^n}{n} \right) - 1 \right| = \exp \left( -\sum_{n=p+1}^{\infty} \frac{z^n}{n} \right) - 1 \leq -\sum_{n=p+1}^{\infty} \frac{z^n}{n} e^{\sum_{n=p+1}^{\infty} \frac{z^n}{n}} \leq \frac{1}{p+1} \cdot 2 \cdot e^{1/(p+1)} |z|^{p+1} \leq 3|z|^{p+1}.$$ 

This proves the corollary.

With this estimate, it is easy to prove the Weierstrass product formula.

Theorem 55.1.4 Let $\{z_n\}$ be a sequence of nonzero complex numbers which have no limit point in $\mathbb{C}$ and let $\{p_n\}$ be a sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty} \left( \frac{R}{|z_n|} \right)^{p_n+1} < \infty$$ (55.1.5)
for all \( R \in \mathbb{R} \). Then
\[
P(z) \equiv \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{z_n} \right)
\]
is analytic on \( \mathbb{C} \) and has a zero at each point, \( z_n \), and at no others. If \( w \) occurs \( m \) times in \( \{ z_n \} \), then \( P \) has a zero of order \( m \) at \( w \).

**Proof:** Since \( \{ z_n \} \) has no limit point, it follows \( \lim_{n \to \infty} |z_n| = \infty \). Therefore, if \( p_n = n - 1 \) the condition, holds for this choice of \( p_n \). Now by Theorem 55.0.2, the infinite product in this theorem will converge uniformly on \( |z| \leq R \) if the same is true of the sum,
\[
\sum_{n=1}^{\infty} \left| E_{p_n} \left( \frac{z}{z_n} \right) - 1 \right| = 1.
\]
But by Corollary 55.1.3, the \( n \)th term of this sum satisfies
\[
\left| E_{p_n} \left( \frac{z}{z_n} \right) - 1 \right| \leq 3 \left( \frac{z}{z_n} \right)^{p_n + 1}.
\]
Since \( |z_n| \to \infty \), there exists \( N \) such that for \( n > N, |z_n| > 2R \). Therefore, for \( |z| < R \) and letting \( 0 < a = \min \{|z_n| : n \leq N\} \),
\[
\sum_{n=1}^{\infty} \left| E_{p_n} \left( \frac{z}{z_n} \right) - 1 \right| \leq 3 \sum_{n=1}^{N} \left( \frac{R}{a} \right)^{p_n + 1} + 3 \sum_{n=N}^{\infty} \left( \frac{R}{2R} \right)^{p_n + 1} < \infty.
\]
By the Weierstrass M test, the series in 55.1.6 converges uniformly for \( |z| < R \) and so the same is true of the infinite product. It follows from Lemma 49.3.13 on Page 1659 that \( P(z) \) is analytic on \( |z| < R \) because it is a uniform limit of analytic functions.

Also by Theorem 55.0.2, the zeros of the analytic \( P(z) \) are exactly the points, \( \{ z_n \} \), listed according to multiplicity. That is, if \( z_n \) is a zero of order \( m \), then if it is listed \( m \) times in the formula for \( P(z) \), then it is a zero of order \( m \) for \( P \). This proves the theorem.

The following corollary is an easy consequence and includes the case where there is a zero at 0.

**Corollary 55.1.5** Let \( \{ z_n \} \) be a sequence of nonzero complex numbers which have no limit point and let \( \{ p_n \} \) be a sequence of nonnegative integers such that
\[
\sum_{n=1}^{\infty} \left( \frac{r}{|z_n|} \right)^{1+p_n} < \infty
\]
for all $r \in \mathbb{R}$. Then
\[ P(z) \equiv z^m \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{z_n} \right) \]
is analytic $\Omega$ and has a zero at each point, $z_n$ and at no others along with a zero of order $m$ at 0. If $w$ occurs $m$ times in $\{z_n\}$, then $P$ has a zero of order $m$ at $w$.

The above theory can be generalized to include the case of an arbitrary open set. First, here is a lemma.

**Lemma 55.1.6** Let $\Omega$ be an open set. Also let $\{z_n\}$ be a sequence of points in $\Omega$ which is bounded and which has no point repeated more than finitely many times such that $\{z_n\}$ has no limit point in $\Omega$. Then there exist $\{w_n\} \subseteq \partial \Omega$ such that $\lim_{n \to \infty} |z_n - w_n| = 0$.

**Proof:** Since $\partial \Omega$ is closed, there exists $w_n \in \partial \Omega$ such that $\text{dist} (z_n, \partial \Omega) = |z_n - w_n|$. Now if there is a subsequence, $\{z_{n_k}\}$ such that $|z_{n_k} - w_{n_k}| \geq \varepsilon$ for all $k$, then $\{z_{n_k}\}$ must possess a limit point because it is a bounded infinite set of points. However, this limit point can only be in $\Omega$ because $\{z_{n_k}\}$ is a bounded away from $\partial \Omega$. This is a contradiction. Therefore, $\lim_{n \to \infty} |z_n - w_n| = 0$. This proves the lemma.

**Corollary 55.1.7** Let $\{z_n\}$ be a sequence of complex numbers contained in $\Omega$, an open subset of $\mathbb{C}$ which has no limit point in $\Omega$. Suppose each $z_n$ is repeated no more than finitely many times. Then there exists a function $f$ which is analytic on $\Omega$ whose zeros are exactly $\{z_n\}$. If $w \in \{z_n\}$ and $w$ is listed $m$ times, then $w$ is a zero of order $m$ of $f$.

**Proof:** There is nothing to prove if $\{z_n\}$ is finite. You just let $f(z) = \prod_{j=1}^{m} (z - z_j)$ where $\{z_n\} = \{z_1, \cdots, z_m\}$.

Pick $w \in \Omega \setminus \{z_n\}_{n=1}^{\infty}$ and let $h(z) \equiv \frac{1}{z-w}$. Since $w$ is not a limit point of $\{z_n\}$, there exists $r > 0$ such that $B(w, r)$ contains no points of $\{z_n\}$. Let $\Omega_1 \equiv \Omega \setminus \{w\}$. Now $h$ is not constant and so $h(\Omega_1)$ is an open set by the open mapping theorem. In fact, $h$ maps each component of $\Omega$ to a region. $|z_n - w| > r$ for all $z_n$ and so $|h(z_n)| < r^{-1}$. Thus the sequence, $\{h(z_n)\}$ is a bounded sequence in the open set $h(\Omega_1)$. It has no limit point in $h(\Omega_1)$ because this is true of $\{z_n\}$ and $\Omega_1$.

By Lemma 55.1.6 there exist $w_n \in \partial (h(\Omega_1))$ such that $\lim_{n \to \infty} |w_n - h(z_n)| = 0$. Consider for $z \in \Omega_1$
\[ f(z) \equiv \prod_{n=1}^{\infty} E_n \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right). \] (55.1.8)

Letting $K$ be a compact subset of $\Omega_1$, $h(K)$ is a compact subset of $h(\Omega_1)$ and so if $z \in K$, then $|h(z) - w_n|$ is bounded below by a positive constant. Therefore, there exists $N$ large enough that for all $z \in K$ and $n \geq N$,
\[ \frac{|h(z_n) - w_n|}{|h(z) - w_n|} < \frac{1}{2} \]
and so by Corollary 55.1.3, for all $z \in K$ and $n \geq N$,

$$
|E_n \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right) - 1| \leq 3 \left( \frac{1}{2} \right)^n. \tag{55.1.9}
$$

Therefore,

$$
\sum_{n=1}^{\infty} \left| E_n \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right) - 1 \right|
$$

converges uniformly for $z \in K$. This implies $\prod_{n=1}^{\infty} E_n \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right)$ also converges uniformly for $z \in K$ by Theorem 55.1.4. Since $K$ is arbitrary, this shows $f$ defined in 55.1.8 is analytic on $\Omega$.

Also if $z_n$ is listed $m$ times so it is a zero of multiplicity $m$ and $w_n$ is the point from $\partial(h(\Omega))$ closest to $h(z_n)$, then there are $m$ factors in 55.1.8 which are of the form

$$
E_n \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right) = \left( 1 - \frac{h(z_n) - w_n}{h(z) - w_n} \right) e^{g_n(z)}
$$

$$
= \left( \frac{h(z) - h(z_n)}{h(z) - w_n} \right) e^{g_n(z)}
$$

$$
= \frac{z_n - z}{(z - w)(z_n - w)} \left( \frac{1}{h(z) - w_n} \right) e^{g_n(z)}
$$

$$
= (z - z_n) G_n(z) \tag{55.1.10}
$$

where $G_n$ is an analytic function which is not zero at and near $z_n$. Therefore, $f$ has a zero of order $m$ at $z_n$. This proves the theorem except for the point, $w$ which has been left out of $\Omega$. It is necessary to show $f$ is analytic at this point also and right now, $f$ is not even defined at $w$.

The $\{w_n\}$ are bounded because $\{h(z_n)\}$ is bounded and $\lim_{n \to \infty} |w_n - h(z_n)| = 0$ which implies $|w_n - h(z_n)| \leq C$ for some constant, $C$. Therefore, there exists $\delta > 0$ such that if $z \in B'(w, \delta)$, then for all $n$,

$$
\frac{h(z_n) - w}{1 - w_n} - w_n = \frac{h(z_n) - w_n}{h(z) - w_n} < \frac{1}{2}.
$$

Thus 55.1.4 holds for all $z \in B'(w, \delta)$ and $n$ so by Theorem 55.1.8, the infinite product in 55.1.8 converges uniformly on $B'(w, \delta)$. This implies $f$ is bounded in $B'(w, \delta)$ and so $w$ is a removable singularity and $f$ can be extended to $w$ such that the result is analytic. It only remains to verify $f(w) \neq 0$. After all, this would not do because it would be another zero other than those in the given list. By 55.1.10, a partial product is of the form

$$
\prod_{n=1}^{N} \left( \frac{h(z) - h(z_n)}{h(z) - w_n} \right) e^{g_n(z)} \tag{55.1.11}
$$
where

\[ g_n(z) \equiv \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right) + \frac{1}{2} \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right)^2 + \cdots + \frac{1}{n} \left( \frac{h(z_n) - w_n}{h(z) - w_n} \right)^n \]

Each of the quotients in the definition of \( g_n(z) \) converges to 0 as \( z \to w \) and so the partial product of \( g_n(z) \) converges to 1 as \( z \to w \) because \( \frac{h(z) - h(z_n)}{h(z) - w_n} \to 1 \) as \( z \to w \).

If \( f(w) = 0 \), then if \( z \) is close enough to \( w \), it follows \( |f(z)| < \frac{1}{2} \). Also, by the uniform convergence on \( B'(w, \delta) \), it follows that for some \( N \), the partial product up to \( N \) must also be less than \( 1/2 \) in absolute value for all \( z \) close enough to \( w \) and as noted above, this does not occur because such partial products converge to 1 as \( z \to w \). Hence \( f(w) \neq 0 \). This proves the corollary.

Recall the definition of a meromorphic function on Page 1673. It was a function which is analytic everywhere except at a countable set of isolated points at which the function has a pole. It is clear that the quotient of two analytic functions yields a meromorphic function but is this the only way it can happen?

**Theorem 55.1.8** Suppose \( Q \) is a meromorphic function on an open set, \( \Omega \). Then there exist analytic functions on \( \Omega \), \( f(z) \) and \( g(z) \) such that \( Q(z) = f(z)/g(z) \) for all \( z \) not in the set of poles of \( Q \).

**Proof:** Let \( Q \) have a pole of order \( m(z) \) at \( z \). Then by Corollary 55.1.7 there exists an analytic function, \( g \) which has a zero of order \( m(z) \) at every \( z \in \Omega \). It follows \( gQ \) has a removable singularity at the poles of \( Q \). Therefore, there is an analytic function, \( f \) such that \( f(z) = g(z)Q(z) \). This proves the theorem.

**Corollary 55.1.9** Suppose \( \Omega \) is a region and \( Q \) is a meromorphic function defined on \( \Omega \) such that the set, \( \{ z \in \Omega : Q(z) = c \} \) has a limit point in \( \Omega \). Then \( Q(z) = c \) for all \( z \in \Omega \).

**Proof:** From Theorem 55.1.8 there are analytic functions, \( f, g \) such that \( Q = \frac{f}{g} \). Therefore, the zero set of the function, \( f(z) - cg(z) \) has a limit point in \( \Omega \) and so \( f(z) - cg(z) = 0 \) for all \( z \in \Omega \). This proves the corollary.

### 55.2 Factoring A Given Analytic Function

The next theorem is the Weierstrass factorization theorem which can be used to factor a given analytic function \( f \). If \( f \) has a zero of order \( m \) when \( z = 0 \), then you could factor out a \( z^m \) and from there consider the factorization of what remains when you have factored out the \( z^m \). Therefore, the following is the main thing of interest.
Theorem 55.2.1 Let $f$ be analytic on $\mathbb{C}$, $f(0) \neq 0$, and let the zeros of $f$, be $\{z_k\}$, listed according to order. (Thus if $z$ is a zero of order $m$, it will be listed $m$ times in the list, $\{z_k\}$.) Choosing nonnegative integers, $p_n$ such that for all $r > 0$, 

$$
\sum_{n=1}^{\infty} \left( \frac{r}{|z_n|} \right)^{p_n+1} < \infty,
$$

There exists an entire function, $g$ such that 

$$
f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{z_n} \right). \tag{55.2.12}
$$

Note that $e^{g(z)} \neq 0$ for any $z$ and this is the interesting thing about this function.

**Proof:** \( \{z_n\} \) cannot have a limit point because if there were a limit point of this sequence, it would follow from Theorem 49.5.3 that $f(z) = 0$ for all $z$, contradicting the hypothesis that $f(0) \neq 0$. Hence $\lim_{n \to \infty} |z_n| = \infty$ and so 

$$
\sum_{n=1}^{\infty} \left( \frac{r}{|z_n|} \right)^{1+n-1} = \sum_{n=1}^{\infty} \left( \frac{r}{|z_n|} \right)^n < \infty
$$

by the root test. Therefore, by Theorem 55.1.4 $P(z) = \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{z_n} \right)$ is analytic on $\mathbb{C}$ by picking $p_n = n - 1$ or perhaps some other choice. ( $p_n = n - 1$ works but there might be another choice that would work.) Then $f/P$ has only removable singularities in $\mathbb{C}$ and no zeros thanks to Theorem 55.1.4. Thus, letting $h(z) = f(z)/P(z)$, Corollary 49.7.23 implies that $h'/h$ has a primitive, $\tilde{g}$. Then 

$$
(h e^{-\tilde{g}})' = 0
$$

and so 

$$
h(z) = e^{a+ib} e^{\tilde{g}(z)}
$$

for some constants, $a, b$. Therefore, letting $g(z) = \tilde{g}(z) + a + ib$, $h(z) = e^{g(z)}$ and thus 55.2.12 holds. This proves the theorem.

**Corollary 55.2.2** Let $f$ be analytic on $\mathbb{C}$, $f$ has a zero of order $m$ at 0, and let the other zeros of $f$ be $\{z_k\}$, listed according to order. (Thus if $z$ is a zero of order $l$, it will be listed $l$ times in the list, $\{z_k\}$.) Also let 

$$
\sum_{n=1}^{\infty} \left( \frac{r}{|z_n|} \right)^{1+p_n} < \infty \tag{55.2.13}
$$

for any choice of $r > 0$. Then there exists an entire function, $g$ such that 

$$
f(z) = z^{m} e^{g(z)} \prod_{n=1}^{\infty} E_{p_n} \left( \frac{z}{z_n} \right). \tag{55.2.14}
$$
Proof: Since $f$ has a zero of order $m$ at $0$, it follows from Theorem 49.5.3 that \( \{z_k\} \) cannot have a limit point in $\mathbb{C}$ and so you can apply Theorem 55.2.1 to the function, $f(z)/z^m$ which has a removable singularity at $0$. This proves the corollary.

### 55.2.1 Factoring Some Special Analytic Functions

Factoring a polynomial is in general a hard task. It is true it is easy to prove the factors exist but finding them is another matter. Corollary 55.2.2 gives the existence of factors of a certain form but it does not tell how to find them. This should not be surprising. You can’t expect things to get easier when you go from polynomials to analytic functions. Nevertheless, it is possible to factor some popular analytic functions. These factorizations are based on the following Mitag-Leffler expansions.

By an auspicious choice of the contour and the method of residues it is possible to obtain a very interesting formula for $\cot(\pi z)$.

#### Example 55.2.3

Let $\gamma_N$ be the contour which goes from $-N - \frac{1}{2} - Ni$ horizontally to $N + \frac{1}{2} - Ni$ and from there, vertically to $N + \frac{1}{2} + Ni$ and then horizontally to $-N - \frac{1}{2} + Ni$ and finally vertically to $-N - \frac{1}{2} - Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the integral

\[
I_N = \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z (\alpha^2 - z^2)} \, dz
\]

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag-Leffler,

\[
\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot(\pi \alpha). \quad (55.2.15)
\]

First you show that $\cot(\pi z)$ is bounded on this contour. This is easy using the formula for $\cot(z) = \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}$. Therefore, $I_N \to 0$ as $N \to \infty$ because the integrand is of order $1/N^2$ while the diameter of $\gamma_N$ is of order $N$. Next you compute the residues of the integrand at $\pm \alpha$ and at $n$ where $|n| < N + \frac{1}{2}$ for $n$ an integer. These are the only singularities of the integrand in this contour and therefore, using the residue theorem, you can evaluate $I_N$ by using these. You can calculate these residues and find that the residue at $\pm \alpha$ is

\[
-\frac{\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}
\]

while the residue at $n$ is

\[
\frac{1}{\alpha^2 - n^2}.
\]

Therefore

\[
0 = \lim_{N \to \infty} I_N = 2\pi i \left[ \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot(\pi \alpha)}{\alpha} \right]
\]

This proves the corollary.
which establishes the following formula of Mittag-Leffler:

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$  

Writing this in a slightly nicer form, you obtain

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

This is a very interesting formula. This will be used to factor \( \sin(\pi z) \). The zeros of this function are at the integers. Therefore, considering

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

you can pick \( p_n = 1 \) in the Weierstrass factorization formula. Therefore, by Corollary

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

there exists an analytic function \( g(z) \) such that

$$\sin(\pi z) = ze^{g(z)} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right) e^{z/n}$$

where the \( z_n \) are the nonzero integers. Remember you can permute the factors in these products. Therefore, this can be written more conveniently as

$$\sin(\pi z) = ze^{g(z)} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right)^2$$

and it is necessary to find \( g(z) \). Differentiating both sides of

$$\sin(\pi z) = ze^{g(z)} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right) e^{z/n}$$

Now divide both sides by \( \sin(\pi z) \) to obtain

$$\frac{\pi \cot (\pi z)}{\sin(\pi z)} = \frac{1}{z} + g'(z) - \sum_{n=1}^{\infty} \frac{2z/n^2}{1 - z^2/n^2}$$

By

$$\frac{\pi \cot (\pi z)}{\sin(\pi z)} = \frac{1}{z} + g'(z) - \sum_{n=1}^{\infty} \frac{2z/n^2}{1 - z^2/n^2}.$$  

this yields \( g'(z) = 0 \) for \( z \) not an integer and so \( g(z) = c \), a constant. So far this yields

$$\sin(\pi z) = ze^{c} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{n} \right)^2$$

and it only remains to find \( c \). Divide both sides by \( \pi z \) and take a limit as \( z \to 0 \). Using the power series of \( \sin(\pi z) \), this yields

$$1 = \frac{e^c}{\pi}.$$
and so \( c = \ln \pi \). Therefore,

\[
\sin(\pi z) = z\pi \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right).
\]

(55.2.17)

**Example 55.2.4** Find an interesting formula for \( \tan(\pi z) \).

This is easy to obtain from the formula for \( \cot(\pi z) \).

\[
\cot \left( \pi \left( z + \frac{1}{2} \right) \right) = -\tan \pi z
\]

for \( z \) real and therefore, this formula holds for \( z \) complex also. Therefore, for \( z + \frac{1}{2} \)
not an integer

\[
\pi \cot \left( \pi \left( z + \frac{1}{2} \right) \right) = \frac{2}{2z+1} + \sum_{n=1}^{\infty} \frac{2z+1}{(2z+1)^2 - n^2}
\]

55.3 The Existence Of An Analytic Function With Given Values

The Weierstrass product formula, Theorem 55.1.4, along with the Mittag-Leffler theorem, Theorem 54.2.1, can be used to obtain an analytic function which has given values on a countable set of points, having no limit point. This is clearly an amazing result and indicates how potent these theorems are. In fact, you can show that it isn’t just the values of the function which may be specified at the points in this countable set of points but the derivatives up to any finite order.

**Theorem 55.3.1** Let \( P = \{ z_k \}_{k=1}^{\infty} \) be a set of points in \( \mathbb{C} \), which has no limit point. For each \( z_k \), consider

\[
\sum_{j=0}^{m_k} a_j^k (z - z_k)^j.
\]

(55.3.18)

Then there exists an analytic function defined on \( \mathbb{C} \) such that the Taylor series of \( f \) at \( z_k \) has the first \( m_k \) terms given by \( \{ a_j^k \}_{k=1}^{\infty} \).

**Proof:** By the Weierstrass product theorem, Theorem 55.1.4, there exists an analytic function, \( f \) defined on all of \( \Omega \) such that \( f \) has a zero of order \( m_k + 1 \) at \( z_k \). Consider this \( z_k \) Thus for \( z \) near \( z_k \),

\[
f(z) = \sum_{j=m_k+1}^{\infty} c_j (z - z_k)^j
\]

\( ^1 \)This says you can specify the first \( m_k \) derivatives of the function at the point \( z_k \).
where \( c_{m+1} \neq 0 \). You choose \( b_1, b_2, \ldots, b_{m+1} \) such that

\[
f(z) = \sum_{j=0}^{m} a_j^k (z - z_k)^j + \sum_{k=m+1}^{\infty} c_j^k (z - z_k)^j.
\]

Thus you need

\[
\sum_{l=1}^{m+1} \sum_{j=m+1}^{\infty} c_j b_l (z - z_k)^{j-l} = \sum_{r=0}^{m} a_r^k (z - z_k)^r + \text{Higher order terms}.
\]

It follows you need to solve the following system of equations for \( b_1, \ldots, b_{m+1} \).

\[
c_{m+1} b_{m+1} = a_0^k
\]

\[
c_{m+2} b_{m+1} + c_{m+1} b_m = a_1^k
\]

\[
c_{m+3} b_{m+1} + c_{m+2} b_m + c_{m+1} b_{m-1} = a_2^k
\]

\[\vdots\]

\[
c_{m+m} b_{m+1} + c_{m+m-1} b_m + \cdots + c_{m+1} b_1 = a_{m+k}^k
\]

Since \( c_{m+1} \neq 0 \), it follows there exists a unique solution to the above system. You first solve for \( b_{m+1} \) in the top. Then, having found it, you go to the next and use \( c_{m+1} \neq 0 \) again to find \( b_{m} \) and continue in this manner. Let \( S_k(z) \) be determined in this manner for each \( z_k \). By the Mittag-Leffler theorem, there exists a Meromorphic function, \( g \) such that \( g \) has exactly the singularities, \( S_k(z) \). Therefore, \( f(z)g(z) \) has removable singularities at each \( z_k \) and for \( z \) near \( z_k \), the first \( m_k \) terms of \( fg \) are as prescribed. This proves the theorem.

**Corollary 55.3.2** Let \( P = \{z_k\}_{k=1}^{\infty} \) be a set of points in \( \Omega \), an open set such that \( P \) has no limit points in \( \Omega \). For each \( z_k \), consider

\[
\sum_{j=0}^{m} a_j^k (z - z_k)^j.
\]  

(55.3.19)

Then there exists an analytic function defined on \( \Omega \) such that the Taylor series of \( f \) at \( z_k \) has the first \( m_k \) terms given by (55.3.19).

**Proof:** The proof is identical to the above except you use the versions of the Mittag-Leffler theorem and Weierstrass product which pertain to open sets.

**Definition 55.3.3** Denote by \( H(\Omega) \) the analytic functions defined on \( \Omega \), an open subset of \( \mathbb{C} \). Then \( H(\Omega) \) is a commutative ring\(^2\) with the usual operations of addition and multiplication. A set, \( I \subseteq H(\Omega) \) is called a finitely generated ideal of the ring if \( I \) is of the form

\[
\left\{ \sum_{k=1}^{n} g_k f_k : f_k \in H(\Omega) \text{ for } k = 1, 2, \ldots, n \right\}
\]

\(^2\)It is not a field because you can’t divide two analytic functions and get another one.
where \(g_1, \cdots, g_n\) are given functions in \(H(\Omega)\). This ideal is also denoted as \([g_1, \cdots, g_n]\) and is called the ideal generated by the functions, \([g_1, \cdots, g_n]\). Since there are finitely many of these functions it is called a finitely generated ideal. A principal ideal is one which is generated by a single function. An example of such a thing is \([1] = H(\Omega)\).

Then there is the following interesting theorem.

**Theorem 55.3.4** Every finitely generated ideal in \(H(\Omega)\) for \(\Omega\) a connected open set (region) is a principal ideal.

**Proof:** Let \(I = [g_1, \cdots, g_n]\) be a finitely generated ideal as described above. Then if any of the functions has no zeros, this ideal would consist of \(H(\Omega)\) because then \(g_i^{-1} \in H(\Omega)\) and so \(1 \in I\). It follows all the functions have zeros. If any of the functions has a zero of infinite order, then the function equals zero on \(\Omega\) because \(\Omega\) is connected and can be deleted from the list. Similarly, if the zeros of any of these functions have a limit point in \(\Omega\), then the function equals zero and can be deleted from the list. Thus, without loss of generality, all zeros are of finite order and there are no limit points of the zeros in \(\Omega\). Let \(m(g_i, z)\) denote the order of the zero of \(g_i\) at \(z\). If \(g_i\) has no zero at \(z\), then \(m(g_i, z) = 0\).

I claim that if no point of \(\Omega\) is a zero of all the \(g_i\), then the conclusion of the theorem is true and in fact \([g_1, \cdots, g_n] = [1] = H(\Omega)\). The claim is obvious if \(n = 1\) because this assumption that no point is a zero of all the functions implies \(g \neq 0\) and so \(g^{-1}\) is analytic. Hence \(1 \in [g_1]\). Suppose it is true for \(n - 1\) and consider \([g_1, \cdots, g_n]\) where no point of \(\Omega\) is a zero of all the \(g_i\). Even though this may be true of \([g_1, \cdots, g_n]\), it may not be true of \([g_1, \cdots, g_{n-1}]\). By Corollary 55.4 there exists \(\phi\), a function analytic on \(\Omega\) such that \(m(\phi, z) = \min\{m(g_i, z), i = 1, 2, \cdots, n - 1\}\). Thus the functions \([g_1/\phi, \cdots, g_{n-1}/\phi]\) are all analytic. Could they all equal zero at some point, \(z\)? If so, pick \(i\) where \(m(\phi, z) = m(g_i, z)\). Thus \(g_i/\phi\) is not equal to zero at \(z\) after all and so these functions are analytic there is no point of \(\Omega\) which is a zero of all of them. By induction, \([g_1/\phi, \cdots, g_{n-1}/\phi] = H(\Omega)\). (Also there are no new zeros obtained in this way.)

Now this means there exist functions \(f_i \in H(\Omega)\) such that

\[
\sum_{i=1}^{n} f_i \left( \frac{g_i}{\phi} \right) = 1
\]

and so \(\phi = \sum_{i=1}^{n} f_i g_i\). Therefore, \([\phi] \subseteq [g_1, \cdots, g_{n-1}]\). On the other hand, if \(\sum_{k=1}^{n} h_k g_k \in [g_1, \cdots, g_{n-1}]\) you could define \(h \equiv \sum_{k=1}^{n} h_k (g_k/\phi)\), an analytic function with the property that \(h\phi = \sum_{k=1}^{n-1} h_k g_k\) which shows \([\phi] = [g_1, \cdots, g_{n-1}]\).

Therefore,

\([g_1, \cdots, g_n] = [\phi, g_n]\)

Now \(\phi\) has no zeros in common with \(g_n\) because the zeros of \(\phi\) are contained in the set of zeros for \(g_1, \cdots, g_{n-1}\). Now consider a zero, \(\alpha\) of \(\phi\). It is not a zero of \(g_n\) and...
55.3. THE EXISTENCE OF AN ANALYTIC FUNCTION WITH GIVEN VALUES

so near $\alpha$, these functions have the form

$$
\phi(z) = \sum_{k=m}^{\infty} a_k (z - \alpha)^k, \quad g_n(z) = \sum_{k=0}^{\infty} b_k (z - \alpha)^k, \quad b_0 \neq 0.
$$

I want to determine coefficients for an analytic function, $h$ such that

$$
m(1 - hg_n, \alpha) \geq m(\phi, \alpha).
$$

Let

$$
h(z) = \sum_{k=0}^{\infty} c_k (z - \alpha)^k
$$

and the $c_k$ must be determined. Using Merten’s theorem, the power series for $1 - hg_n$ is of the form

$$
1 - b_0 c_0 - \sum_{j=1}^{\infty} \left( \sum_{r=0}^{j} b_{j-r} c_r \right) (z - \alpha)^j.
$$

First determine $c_0$ such that $1 - c_0 b_0 = 0$. This is no problem because $b_0 \neq 0$. Next you need to get the coefficients of $(z - \alpha)$ to equal zero. This requires

$$
b_1 c_0 + b_0 c_1 = 0.
$$

Again, there is no problem because $b_0 \neq 0$. In fact, $c_1 = (-b_1 c_0 / b_0)$. Next consider the second order terms if $m \geq 2$.

$$
b_2 c_0 + b_1 c_1 + b_0 c_2 = 0
$$

Again there is no problem in solving, this time for $c_2$ because $b_0 \neq 0$. Continuing this way, you see that in every step, the $c_k$ which needs to be solved for is multiplied by $b_0 \neq 0$. Therefore, by Corollary 55.1.7 there exists an analytic function, $h$ satisfying (55.3.20). Therefore, $(1 - hg_n) / \phi$ has a removable singularity at every zero of $\phi$ and so may be considered an analytic function. Therefore,

$$
1 = \frac{1 - hg_n}{\phi} \phi + hg_n \in [\phi, g_n] = [g_1 \cdots g_n]
$$

which shows $[g_1 \cdots g_n] = H(\Omega) = [1]$. It follows the claim is established.

Now suppose $\{g_1, \cdots g_n\}$ are just elements of $H(\Omega)$. As explained above, it can be assumed they all have zeros of finite order and the zeros have no limit point in $\Omega$ since if these occur, you can delete the function from the list. By Corollary 55.1.4 there exists $\phi \in H(\Omega)$ such that $m(\phi, z) \leq \min \{m(g_i, z) : i = 1, \cdots, n\}$. Then $g_k / \phi$ has a removable singularity at each zero of $g_k$ and so can be regarded as an analytic function. Also, as before, there is no point which is a zero of each $g_k / \phi$ and so by the first part of this argument, $[g_1 / \phi \cdots g_n / \phi] = H(\Omega)$. As in the first part of the argument, this implies $[g_1 \cdots g_n] = [\phi]$ which proves the theorem. $[g_1 \cdots g_n]$ is a principal ideal as claimed.

The following corollary follows from the above theorem. You don’t need to assume $\Omega$ is connected.
Corollary 55.3.5 Every finitely generated ideal in $H(\Omega)$ for $\Omega$ an open set is a principal ideal.

**Proof:** Let $[g_1, \cdots, g_n]$ be a finitely generated ideal in $H(\Omega)$. Let $\{U_k\}$ be the components of $\Omega$. Then applying the above to each component, there exists $h_k \in H(U_k)$ such that restricting each $g_i$ to $U_k, [g_1, \cdots, g_n] = [h_k]$. Then let $h(z) = h_k(z)$ for $z \in U_k$. This is an analytic function which works.

### 55.4 Jensen’s Formula

This interesting formula relates the zeros of an analytic function to an integral. The proof given here follows Alfors, [2]. First, here is a technical lemma.

**Lemma 55.4.1**

$$\int_{-\pi}^{\pi} \ln |1 - e^{i\theta}| d\theta = 0.$$  

**Proof:** First note that the only problem with the integrand occurs when $\theta = 0$. However, this is an integrable singularity so the integral will end up making sense. Letting $z = e^{i\theta}$, you could get the above integral as a limit as $\varepsilon \to 0$ of the following contour integral where $\gamma_\varepsilon$ is the contour shown in the following picture with the radius of the big circle equal to 1 and the radius of the little circle equal to $\varepsilon$.

$$\int_{\gamma_\varepsilon} \frac{\ln|1 - z|}{iz} \, dz.$$  

On the indicated contour, $1 - z$ lies in the half plane $\text{Re} \, z > 0$ and so $\log(1 - z) = \ln|1 - z| + i \arg(1 - z)$. The above integral equals

$$\int_{\gamma_\varepsilon} \frac{\log(1 - z)}{iz} \, dz - \int_{\gamma_\varepsilon} \frac{\arg(1 - z)}{z} \, dz.$$  

The first of these integrals equals zero because the integrand has a removable singularity at 0. The second equals

$$i \int_{-\pi}^{-\eta_\varepsilon} \arg(1 - e^{i\theta}) \, d\theta + i \int_{\eta_\varepsilon}^{\pi} \arg(1 - e^{i\theta}) \, d\theta$$

$$+ \varepsilon i \int_{-\pi}^{-\pi - \lambda_\varepsilon} \theta d\theta + \varepsilon i \int_{\pi}^{\pi + \lambda_\varepsilon} \theta d\theta$$
where $\eta, \lambda \to 0$ as $\varepsilon \to 0$. The last two terms converge to 0 as $\varepsilon \to 0$ while the first two add to zero. To see this, change the variable in the first integral and then recall that when you multiply complex numbers you add the arguments. Thus you end up integrating $\text{arg}$ (real valued function) which equals zero.

In this material on Jensen’s equation, $\varepsilon$ will denote a small positive number. Its value is not important as long as it is positive. Therefore, it may change from place to place. Now suppose $f$ is analytic on $B(0, r + \varepsilon)$, and $f$ has no zeros on $\overline{B}(0, r)$. Then you can define a branch of the logarithm which makes sense for complex numbers near $f(z)$. Thus $z \to \log(f(z))$ is analytic on $B(0, r + \varepsilon)$. Therefore, its real part, $u(x, y) \equiv \ln|f(x + iy)|$ must be harmonic. Consider the following lemma.

**Lemma 55.4.2** Let $u$ be harmonic on $B(0, r + \varepsilon)$. Then

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) \, d\theta.$$ 

**Proof:** For a harmonic function, $u$ defined on $B(0, r + \varepsilon)$, there exists an analytic function, $h = u + iv$ where

$$v(x, y) \equiv \int_{0}^{y} u_x(x, t) \, dt - \int_{0}^{x} u_y(t, 0) \, dt.$$ 

By the Cauchy integral theorem,

$$h(0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{h(z)}{z} \, dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(re^{i\theta}) \, d\theta.$$ 

Therefore, considering the real part of $h$,

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) \, d\theta.$$ 

This proves the lemma.

Now this shows the following corollary.

**Corollary 55.4.3** Suppose $f$ is analytic on $B(0, r + \varepsilon)$ and has no zeros on $\overline{B}(0, r)$. Then

$$\ln |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})|$$ 

(55.4.21)

What if $f$ has some zeros on $|z| = r$ but none on $B(0, r)$? It turns out 55.4.21 is still valid. Suppose the zeros are at $\{re^{i\theta_k}\}_{k=1}^{m}$, listed according to multiplicity. Then let

$$g(z) = \frac{f(z)}{\prod_{k=1}^{m} (z - re^{i\theta_k})}.$$
It follows $g$ is analytic on $B(0, r + \varepsilon)$ but has no zeros in $\overline{B(0, r)}$. Then \[55.4.21\] holds for $g$ in place of $f$. Thus

$$
\ln |f(0)| - \sum_{k=1}^{m} \ln |r|
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln |re^{i\theta} - re^{i\theta_k}| \, d\theta
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln |e^{i\theta} - e^{i\theta_k}| \, d\theta - \sum_{k=1}^{m} \ln |r|
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln |e^{i\theta} - 1| \, d\theta - \sum_{k=1}^{m} \ln |r|
$$

Therefore, \[55.4.21\] will continue to hold exactly when \[\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{m} \ln |e^{i\theta} - 1| \, d\theta = 0.\] But this is the content of Lemma \[55.4.1\]. This proves the following lemma.

**Lemma 55.4.4** Suppose $f$ is analytic on $B(0, r + \varepsilon)$ and has no zeros on $B(0, r)$. Then

$$
\ln |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| \quad (55.4.22)
$$

With this preparation, it is now not too hard to prove Jensen’s formula. Suppose there are $n$ zeros of $f$ in $B(0, r), \{a_k\}_{k=1}^{n},$ listed according to multiplicity, none equal to zero. Let

$$
F(z) \equiv f(z) \prod_{i=1}^{n} \frac{r^2 - \overline{a_i}z}{r(z - a_i)}.
$$

Then $F$ is analytic on $B(0, r + \varepsilon)$ and has no zeros in $B(0, r)$. The reason for this is that $f(z) / \prod_{i=1}^{n} r(z - a_i)$ has no zeros there and $r^2 - \overline{a_i}z$ cannot equal zero if $|z| < r$ because if this expression equals zero, then

$$
|z| = \frac{r^2}{|a_i|} > r.
$$

The other interesting thing about $F(z)$ is that when $z = re^{i\theta},$

$$
F(re^{i\theta}) = f(re^{i\theta}) \prod_{i=1}^{n} \frac{r^2 - \overline{a_i}re^{i\theta}}{r(re^{i\theta} - a_i)}
= f(re^{i\theta}) \prod_{i=1}^{n} \frac{r - \overline{a_i}e^{i\theta}}{re^{i\theta} - a_i} = f(re^{i\theta}) e^{i\theta} \prod_{i=1}^{n} \frac{re^{-i\theta} - \overline{a_i}}{re^{i\theta} - a_i}
$$

so $|F(re^{i\theta})| = |f(re^{i\theta})|.$
**Theorem 55.4.5** Let $f$ be analytic on $B(0, r + \varepsilon)$ and suppose $f(0) \neq 0$. If the zeros of $f$ in $B(0, r)$ are $\{a_k\}_{k=1}^n$, listed according to multiplicity, then

$$\ln |f(0)| = -\sum_{i=1}^{n} \ln \left(\frac{r}{|a_i|}\right) + \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| \, d\theta.$$ 

**Proof:** From the above discussion and Lemma [55.4.4](#),

$$\ln |F(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| \, d\theta$$

But $F(0) = f(0) \prod_{i=1}^{n} \frac{z}{a_i}$ and so $\ln |F(0)| = \ln |f(0)| + \sum_{i=1}^{n} \ln \left|\frac{z}{a_i}\right|$. Therefore,

$$\ln |f(0)| = -\sum_{i=1}^{n} \ln \left|\frac{r}{a_i}\right| + \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| \, d\theta$$

as claimed.

Written in terms of exponentials this is

$$|f(0)| \prod_{k=1}^{n} \left|\frac{r}{a_k}\right| = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| \, d\theta\right).$$

### 55.5 Blaschke Products

The Blaschke product is a way to produce a function which is bounded and analytic on $B(0, 1)$ which also has given zeros in $B(0, 1)$. The interesting thing here is that there may be infinitely many of these zeros. Thus, unlike the above case of Jensen’s inequality, the function is not analytic on $B(0, 1)$. Recall for purposes of comparison, Liouville’s theorem which says bounded entire functions are constant. The Blaschke product gives examples of bounded functions on $B(0, 1)$ which are definitely not constant.

**Theorem 55.5.1** Let $\{\alpha_n\}$ be a sequence of nonzero points in $B(0, 1)$ with the property that

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$ 

Then for $k \geq 0$, an integer

$$B(z) \equiv z^k \prod_{k=1}^{\infty} \frac{\alpha_n - z}{1 - \alpha_n \bar{z}} \frac{|\alpha_n|}{\alpha_n}$$

is a bounded function which is analytic on $B(0, 1)$ which has zeros only at $0$ if $k > 0$ and at the $\alpha_n$.

\(^3\text{Wilhelm Blaschke, 1915}\)
Proof: From Theorem 55.0.2 the above product will converge uniformly on
$B(0, r)$ for $r < 1$ to an analytic function if

$$\sum_{k=1}^{\infty} \left| \frac{\alpha_n - z}{1 - \alpha_n z} \frac{|\alpha_n|}{\alpha_n} - 1 \right|$$

converges uniformly on $B(0, r)$. But for $|z| < r$,

$$\left| \frac{\alpha_n - z}{1 - \alpha_n z} \frac{|\alpha_n|}{\alpha_n} - 1 \right| = \left| \frac{\alpha_n - z}{1 - \alpha_n z} \frac{|\alpha_n|}{\alpha_n} - \frac{\alpha_n (1 - \alpha_n z)}{\alpha_n (1 - \alpha_n z)} \right|$$

$$= \left| \frac{|\alpha_n| \alpha_n - |\alpha_n| z - \alpha_n + |\alpha_n|^2 z}{(1 - \alpha_n z) \alpha_n} \right|$$

$$= \left| \frac{|\alpha_n| \alpha_n - |\alpha_n| z + |\alpha_n|^2 z}{(1 - \alpha_n z) \alpha_n} \right|$$

$$= ||\alpha_n| - 1| \frac{\alpha_n + z |\alpha_n|}{(1 - \alpha_n z) \alpha_n}$$

$$= ||\alpha_n| - 1| \frac{1 + z (|\alpha_n| / \alpha_n)}{(1 - \alpha_n z)}$$

$$\leq ||\alpha_n| - 1| \frac{1 + |z|}{1 - |z|} \leq ||\alpha_n| - 1| \frac{1 + r}{1 - r}$$

and so the assumption on the sum gives uniform convergence of the product on
$B(0, r)$ to an analytic function. Since $r < 1$ is arbitrary, this shows $B(z)$ is analytic
on $B(0, 1)$ and has the specified zeros because the only place the factors equal zero
are at the $\alpha_n$ or 0.

Now consider the factors in the product. The claim is that they are all no larger
in absolute value than 1. This is very easy to see from the maximum modulus
theorem. Let $|\alpha| < 1$ and $\phi(z) = \frac{\alpha - z}{1 - \alpha z}$. Then $\phi$ is analytic near $B(0, 1)$ because its
only pole is $1/\alpha$. Consider $z = e^{i\theta}$. Then

$$|\phi(e^{i\theta})| = \left| \frac{\alpha - e^{i\theta}}{1 - \alpha e^{i\theta}} \right| = \left| \frac{1 - \alpha e^{-i\theta}}{1 - \alpha e^{i\theta}} \right| = 1.$$ 

Thus the modulus of $\phi(z)$ equals 1 on $\partial B(0, 1)$. Therefore, by the maximum modu-
lus theorem, $|\phi(z)| < 1$ if $|z| < 1$. This proves the claim that the terms in the
product are no larger than 1 and shows the function determined by the Blaschke
product is bounded. This proves the theorem.

Note in the conditions for this theorem the one for the sum, $\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty$. The Blaschke product gives an analytic function, whose absolute value is bounded
by 1 and which has the $\alpha_n$ as zeros. What if you had a bounded function, analytic
on $B(0, 1)$ which had zeros at $\{\alpha_k\}$? Could you conclude the condition on the sum?
The answer is yes. In fact, you can get by with less than the assumption that \( f \) is bounded but this will not be presented here. See Rudin [102]. This theorem is an exciting use of Jensen’s equation.

**Theorem 55.5.2** Suppose \( f \) is an analytic function on \( B(0,1) \), \( f(0) \neq 0 \), and \( |f(z)| \leq M \) for all \( z \in B(0,1) \). Suppose also that the zeros of \( f \) are \( \{\alpha_k\}_{k=1}^{\infty} \), listed according to multiplicity. Then \( \sum_{k=1}^{\infty} (1 - |\alpha_k|) < \infty \).

**Proof:** If there are only finitely many zeros, there is nothing to prove so assume there are infinitely many. Also let the zeros be listed such that \( |\alpha_n| \leq |\alpha_{n+1}| \cdots \). Let \( n(r) \) denote the number of zeros in \( B(0,r) \). By Jensen’s formula,

\[
\ln |f(0)| + \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| \, d\theta \leq \ln (M).
\]

Therefore, by the mean value theorem,

\[
\sum_{i=1}^{n(r)} \frac{1}{r} (r - |\alpha_i|) \leq \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i| \leq \ln (M) - \ln |f(0)|
\]

As \( r \to 1^- \), \( n(r) \to \infty \), and so an application of Fatou’s lemma yields

\[
\sum_{i=1}^{\infty} (1 - |\alpha_i|) \leq \lim \inf_{r \to 1^-} \sum_{i=1}^{n(r)} \frac{1}{r} (r - |\alpha_i|) \leq \ln (M) - \ln |f(0)|.
\]

This proves the theorem.

You don’t need the assumption that \( f(0) \neq 0 \).

**Corollary 55.5.3** Suppose \( f \) is an analytic function on \( B(0,1) \) and \( |f(z)| \leq M \) for all \( z \in B(0,1) \). Suppose also that the nonzero zeros of \( f \) are \( \{\alpha_k\}_{k=1}^{\infty} \), listed according to multiplicity. Then \( \sum_{k=1}^{\infty} (1 - |\alpha_k|) < \infty \).

**Proof:** Suppose \( f \) has a zero of order \( m \) at 0. Then consider the analytic function, \( g(z) \equiv f(z)/z^m \) which has the same zeros except for 0. The argument goes the same way except here you use \( g \) instead of \( f \) and only consider \( r > r_0 > 0 \).

---

\(^4\) This is a fun thing to say: nonzero zeros.
Thus from Jensen’s equation,

\[
\ln |g(0)| + \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i|
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \ln |g(re^{i\theta})| \, d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \ln |f(re^{i\theta})| \, d\theta - \frac{1}{2\pi} \int_{0}^{2\pi} m \ln (r)
\]

\[
\leq M + \frac{1}{2\pi} \int_{0}^{2\pi} m \ln (r^{-1})
\]

\[
\leq M + m \ln \left( \frac{1}{r_0} \right).
\]

Now the rest of the argument is the same.

An interesting restatement yields the following amazing result.

**Corollary 55.5.4** Suppose \( f \) is analytic and bounded on \( B(0,1) \) having zeros \( \{\alpha_n\} \). Then if \( \sum_{k=1}^{\infty} (1 - |\alpha_n|) = \infty \), it follows \( f \) is identically equal to zero.

### 55.5.1 The Müntz-Szasz Theorem Again

Corollary makes possible an easy proof of a remarkable theorem named above which yields a wonderful generalization of the Weierstrass approximation theorem. In what follows \( b > 0 \). The Weierstrass approximation theorem states that linear combinations of \( 1, t, t^2, t^3, \ldots \) (polynomials) are dense in \( C([0,b]) \). Let \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) be an increasing list of positive real numbers. This theorem tells when linear combinations of \( 1, t^{\lambda_1}, t^{\lambda_2}, \ldots \) are dense in \( C([0,b]) \). The proof which follows is like the one given in Rudin [102]. There is a much longer one in Cheney [32] which discusses more aspects of the subject. See also Page 522 where the version given in Cheney is presented. This other approach is much more elementary and does not depend in any way on the theory of functions of a complex variable. There are those of us who automatically prefer real variable techniques. Nevertheless, this proof by Rudin is a very nice and insightful application of the preceding material. Cheney refers to the theorem as the second Müntz theorem. I guess Szasz must also have been involved.

**Theorem 55.5.5** Let \( \lambda_1 < \lambda_2 < \lambda_3 < \cdots \) be an increasing list of positive real numbers and let \( a > 0 \). If

\[
\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty,
\]

then linear combinations of \( 1, t^{\lambda_1}, t^{\lambda_2}, \ldots \) are dense in \( C([0,b]) \).
**Proof:** Let $X$ denote the closure of linear combinations of \( \{1, t^{\lambda_1}, t^{\lambda_2}, \ldots \} \) in $C([0, b])$. If $X \neq C([0, b])$, then letting $f \in C([0, b]) \setminus X$, define $\Lambda \in C([0, b])'$ as follows. First let $\Lambda_0 : X + \mathbb{C}f$ be given by $\Lambda_0 (g + \alpha f) = \alpha \|f\|_\infty$. Then

\[
\sup_{\|g + \alpha f\| \leq 1} |\Lambda_0 (g + \alpha f)| = \sup_{\|g + \alpha f\| \leq 1} |\alpha| \|f\|_\infty
\]

\[
= \sup_{\|g/\alpha + f\| \leq \frac{1}{|\alpha|}} |\alpha| \|f\|_\infty
\]

\[
= \sup_{\|g + f\| \leq \frac{1}{|\alpha|}} |\alpha| \|f\|_\infty
\]

Now $\operatorname{dist} (f, X) > 0$ because $X$ is closed. Therefore, there exists a lower bound, $\eta > 0$ to $\|g + f\|$ for $g \in X$. Therefore, the above is no larger than

\[
\sup_{|\alpha| \leq \frac{1}{\eta}} |\alpha| \|f\|_\infty = \left( \frac{1}{\eta} \right) \|f\|_\infty
\]

which shows that $\|\Lambda_0\| \leq \left( \frac{1}{\eta} \right) \|f\|_\infty$. By the Hahn Banach theorem $\Lambda_0$ can be extended to $\Lambda \in C([0, b])'$ which has the property that $\Lambda(X) = 0$ but $\Lambda(f) = \|f\| \neq 0$. By the Weierstrass approximation theorem, Theorem \ref{thm:weierstrass} or one of its cases, there exists a polynomial, $p$ such that $\Lambda(p) \neq 0$. Therefore, if it can be shown that whenever $\Lambda(X) = 0$, it is the case that $\Lambda(p) = 0$ for all polynomials, it must be the case that $X$ is dense in $C([0, b])$.

By the Riesz representation theorem the elements of $C([0, b])'$ are complex measures. Suppose then that for $\mu$ a complex measure it follows that for all $t^{\lambda_k}$,

\[
\int_{[0, b]} t^{\lambda_k} d\mu = 0.
\]

I want to show that then

\[
\int_{[0, b]} t^{k} d\mu = 0
\]

for all positive integers. It suffices to modify $\mu$ is necessary to have $\mu \{0\} = 0$ since this will not change any of the above integrals. Let $\mu_1 (E) = \mu (E \cap (0, b))$ and use $\mu_1$. I will continue using the symbol, $\mu$.

For $\operatorname{Re} z > 0$, define

\[
F(z) = \int_{[0, b]} t^{z} d\mu = \int_{(0, b]} t^{z} d\mu
\]

The function $t^z = \exp (z \ln (t))$ is analytic. I claim that $F(z)$ is also analytic for $\operatorname{Re} z > 0$. Apply Morera’s theorem. Let $T$ be a triangle in $\operatorname{Re} z > 0$. Then

\[
\int_{\partial T} F(z) dz = \int_{\partial T} \int_{(0, b]} e^{(z \ln (t))} |\xi| |\mu| dz
\]
Now \( \int_{\partial T} \) can be split into three integrals over intervals of \( \mathbb{R} \) and so this integral is essentially a Lebesgue integral taken with respect to Lebesgue measure. Furthermore, \( e^{(z \ln(t))} \) is a continuous function of the two variables and \( \xi \) is a function of only the one variable, \( t \). Thus the integrand is product measurable. The iterated integral is also absolutely integrable because \( |e^{(z \ln(t))}| \leq e^{x \ln t} \leq e^{x \ln b} \) where \( x + iy = z \) and \( x \) is given to be positive. Thus the integrand is actually bounded. Therefore, you can apply Fubini’s theorem and write

\[
\int_{\partial T} F(z) \, dz = \int_{\partial T} \int_{(0,b]} e^{(z \ln(t))} \xi \, |\mu| \, dz = \int_{(0,b]} \xi \int_{\partial T} e^{(z \ln(t))} \, dz \, |\mu| = 0.
\]

By Morera’s theorem, \( F \) is analytic on \( \text{Re } z > 0 \) which is given to have zeros at the \( \lambda_k \).

Now let \( \phi(z) = \frac{1+z}{1-z} \). Then \( \phi \) maps \( B(0,1) \) one to one onto \( \text{Re } z > 0 \). To see this let \( 0 < r < 1 \).

\[
\phi(re^{i\theta}) = \frac{1 + re^{i\theta}}{1-re^{i\theta}} = \frac{1 - r^2 + i2r \sin \theta}{1 + r^2 - 2r \cos \theta}
\]

and so \( \text{Re } \phi(re^{i\theta}) > 0 \). Now the inverse of \( \phi \) is \( \phi^{-1}(z) = \frac{z-1}{z+1} \). For \( \text{Re } z > 0 \),

\[
|\phi^{-1}(z)|^2 = \frac{z-1}{z+1} \cdot \frac{\overline{z}-1}{\overline{z}+1} = \frac{|z|^2 - 2\text{Re } z + 1}{|z|^2 + 2\text{Re } z + 1} < 1.
\]

Consider \( F \circ \phi \), an analytic function defined on \( B(0,1) \). This function is given to have zeros at \( z_n \) where \( \phi(z_n) = \frac{1+z_n}{1-z_n} = \lambda_n \). This reduces to \( z_n = \frac{-1+\lambda_n}{1+\lambda_n} \). Now

\[
1 - |z_n| \geq \frac{c}{1+\lambda_n}
\]

for a positive constant, \( c \). It is given that \( \sum \frac{1}{\lambda_n} = \infty \), so it follows \( \sum (1 - |z_n|) = \infty \) also. Therefore, by Corollary 55.5.4, \( F \circ \phi = 0 \). It follows \( F = 0 \) also. In particular, \( F(k) \) for \( k \) a positive integer equals zero. This has shown that if \( \Lambda \in C((0,b))^\prime \) and \( \Lambda \) sends 1 and all the \( t^\lambda \) to 0, then \( \Lambda \) sends 1 and all \( t^k \) for \( k \) a positive integer to zero. As explained above, \( X \) is dense in \( C((0,b)) \).

The converse of this theorem is also true and is proved in Rudin [102].

### 55.6 Exercises

1. Suppose \( f \) is an entire function with \( f(0) = 1 \). Let

\[
M(r) = \max \{|f(z)| : |z| = r\}.
\]

Use Jensen’s equation to establish the following inequality.

\[
M(2r) \geq 2^n(r)
\]

where \( n(r) \) is the number of zeros of \( f \) in \( B(0,r) \).
2. The version of the Blaschke product presented above is that found in most complex variable texts. However, there is another one in Section 80. Instead of \( \frac{\alpha_n - z}{1 - \alpha_n z} \frac{|\alpha_n|}{\alpha_n} \) you use

\[ \frac{\alpha_n - z}{\frac{1}{\alpha_n} - z} \]

Prove a version of Theorem 55.5.1 using this modification.

3. The Weierstrass approximation theorem holds for polynomials of \( n \) variables on any compact subset of \( \mathbb{R}^n \). Give a multidimensional version of the Müntz-Szasz theorem which will generalize the Weierstrass approximation theorem for \( n \) dimensions. You might just pick a compact subset of \( \mathbb{R}^n \) in which all components are positive. You have to do something like this because otherwise, \( t^\lambda \) might not be defined.

4. Show \( \cos(\pi z) = \prod_{k=1}^{\infty} \left( 1 - \frac{4z^2}{(2k-1)^2} \right) \).

5. Recall \( \sin(\pi z) = \frac{z\pi}{2} \prod_{n=1}^{\infty} \left( 1 - \left( \frac{z}{n} \right)^2 \right) \). Use this to derive Wallis product, \( \frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{4k^2}{(2k-1)(2k+1)} \).

6. The order of an entire function, \( f \) is defined as

\[ \inf \left\{ a \geq 0 : |f(z)| \leq e^{|z|^a} \text{ for all large enough } |z| \right\} \]

If no such \( a \) exists, the function is said to be of infinite order. Show the order of an entire function is also equal to \( \limsup_{r \to \infty} \frac{\ln(\ln(M(r)))}{\ln(r)} \) where \( M(r) \equiv \max \{|f(z)| : |z| = r\} \).

7. Suppose \( \Omega \) is a simply connected region and let \( f \) be meromorphic on \( \Omega \). Suppose also that the set, \( S \equiv \{z \in \Omega : f(z) = c\} \) has a limit point in \( \Omega \). Can you conclude \( f(z) = c \) for all \( z \in \Omega \)?

8. This and the next collection of problems are dealing with the gamma function. Show that

\[ \left| \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} - 1 \right| \leq \frac{C(z)}{n^2} \]

and therefore,

\[ \sum_{n=1}^{\infty} \left| \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} - 1 \right| < \infty \]

with the convergence uniform on compact sets.

9. Show \( \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right) e^{-\frac{z}{n}} \) converges to an analytic function on \( \mathbb{C} \) which has zeros only at the negative integers and that therefore,

\[ \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{-1} e^{\frac{z}{n}} \]
is a meromorphic function (Analytic except for poles) having simple poles at the negative integers.

10. Show there exists $\gamma$ such that if

$$\Gamma (z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{-1} e^{\frac{z}{n}},$$

then $\Gamma (1) = 1$. Thus $\Gamma$ is a meromorphic function having simple poles at the negative integers. **Hint:** $\prod_{n=1}^{\infty} (1 + n) e^{-1/n} = c = e^{\gamma}$.

11. Now show that $\gamma = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} \frac{1}{k} - \ln n \right]$.

12. Justify the following argument leading to Gauss’s formula

$$\Gamma (z) = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \left( \frac{k}{k+z} \right) e^{\frac{z}{k}} \right) \frac{e^{-\gamma z}}{z}$$

$$= \lim_{n \to \infty} \left( \frac{n!}{(1+z)(2+z)\cdots(n+z)} e^{z(\sum_{k=1}^{n} \frac{1}{k})} \right) \frac{e^{-\gamma z}}{z}$$

$$= \lim_{n \to \infty} \frac{n!}{(1+z)(2+z)\cdots(n+z)} e^{z[\sum_{k=1}^{n} \frac{1}{k} - \ln n]}$$

$$= \lim_{n \to \infty} \frac{n! z^{x} (x+1) \cdots (x+n)}{n! n^{x}}.$$}

13. Verify from the Gauss formula above that $\Gamma (z+1) = \Gamma (z) z$ and that for $n$ a nonnegative integer, $\Gamma (n+1) = n!$.

14. The usual definition of the gamma function for positive $x$ is

$$\Gamma (x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt.$$ 

Show $(1 - \frac{t}{n})^{n} \leq e^{-t}$ for $t \in [0, n]$. Then show

$$\int_{0}^{n} \left( 1 - \frac{t}{n} \right)^{n} t^{x-1} dt = \frac{n! n^{x}}{x(x+1)\cdots(x+n)}.$$ 

Use the first part to conclude that

$$\Gamma (x) = \lim_{n \to \infty} \frac{n! n^{x}}{x(x+1)\cdots(x+n)} = \Gamma (x).$$ 

**Hint:** To show $(1 - \frac{t}{n})^{n} \leq e^{-t}$ for $t \in [0, n]$, verify this is equivalent to showing $(1 - u)^{n} \leq e^{-nu}$ for $u \in [0, 1]$. 

55.6. EXERCISES

15. Show \( \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \) whenever \( \text{Re} \ z > 0 \). \textbf{Hint:} You have already shown that this is true for positive real numbers. Verify this formula for \( \text{Re} \ z > 0 \) yields an analytic function.

16. Show \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \). Then find \( \Gamma\left(\frac{5}{2}\right) \).

17. Show that \( \int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi} \). \textbf{Hint:} Denote this integral by \( I \) and observe that \( I^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dxdy \). Then change variables to polar coordinates, \( x = r \cos(\theta) \), \( y = r \sin \theta \).

18. Now that you know what the gamma function is, consider in the formula for \( \Gamma(\alpha+1) \) the following change of variables. \( t = \alpha + \alpha^{1/2}s \). Then in terms of the new variable, \( s \), the formula for \( \Gamma(\alpha+1) \) is

\[
\begin{align*}
&\quad e^{-\alpha} \alpha^{\alpha+\frac{1}{2}} \int_{-\sqrt{\alpha}}^{\infty} e^{-\sqrt{\alpha} s} \left(1 + \frac{s}{\sqrt{\alpha}}\right)^\alpha ds \\
&= e^{-\alpha} \alpha^{\alpha+\frac{1}{2}} \int_{-\sqrt{\alpha}}^{\infty} e^\alpha \left[\ln\left(1 + \frac{s}{\sqrt{\alpha}}\right) - \frac{s}{\sqrt{\alpha}}\right] ds
\end{align*}
\]

Show the integrand converges to \( e^{-\frac{1}{2} s^2} \). Show that then

\[
\lim_{\alpha \to \infty} \frac{\Gamma(\alpha+1)}{e^{-\alpha} \alpha^{\alpha+1/2}} = \int_{-\infty}^{\infty} e^{-\frac{1}{2} s^2} ds = \sqrt{2\pi}.
\]

\textbf{Hint:} You will need to obtain a dominating function for the integral so that you can use the dominated convergence theorem. You might try considering \( s \in (-\sqrt{\alpha}, \sqrt{\alpha}) \) first and consider something like \( e^{1 - (s^2/4)} \) on this interval. Then look for another function for \( s > \sqrt{\alpha} \). This formula is known as Stirling’s formula.

19. This and the next several problems develop the zeta function and give a relation between the zeta and the gamma function. Define for \( 0 < r < 2\pi \)

\[
I_r(z) = \int_0^{2\pi} e^{(z-1)(\ln r + i\theta)} ire^{i\theta} d\theta + \int_r^{\infty} e^{(z-1)(\ln t + 2\pi i)} e^t - 1 dt \quad (55.6.24)
\]

Show that \( I_r \) is an entire function. The reason \( 0 < r < 2\pi \) is that this prevents \( e^{re^{i\theta}} - 1 \) from equaling zero. The above is just a precise description of the contour integral, \( \int_\gamma \frac{e^{w^{z-1}}}{w^{z-1}} dw \) where \( \gamma \) is the contour shown below.
in which on the integrals along the real line, the argument is different in going from $r$ to $\infty$ than it is in going from $\infty$ to $r$. Now I have not defined such contour integrals over contours which have infinite length and so have chosen to simply write out explicitly what is involved. You have to work with these integrals given above anyway but the contour integral just mentioned is the motivation for them. **Hint:** You may want to use convergence theorems from real analysis if it makes this more convenient but you might not have to.

20. ↑ In the context of Problem 19 define for small $\delta > 0$

\[ I_{r\delta}(z) \equiv \int_{\gamma_{r,\delta}} \frac{w^{z-1}}{e^{w} - 1} \, dw \]

where $\gamma_{r\delta}$ is shown below.

Show that $\lim_{\delta \to 0} I_{r\delta}(z) = I_r(z)$. **Hint:** Use the dominated convergence theorem if it makes this go easier. This is not a hard problem if you use these theorems but you can probably do it without them with more work.

21. ↑ In the context of Problem 20 show that for $r_1 < r$, $I_{r\delta}(z) - I_{r_1\delta}(z)$ is a contour integral,

\[ \int_{\gamma_{r,r_1,\delta}} \frac{w^{z-1}}{e^{w} - 1} \, dw \]

where the oriented contour is shown below.
In this contour integral, \( w^{z-1} \) denotes \( e^{(z-1) \log(w)} \) where \( \log(w) = \ln|w| + i \arg(w) \) for \( \arg(w) \in (0, 2\pi) \). Explain why this integral equals zero. From Problem 21 it follows that \( I_r = I_{r_1} \). Therefore, you can define an entire function, \( I(z) \equiv I_r(z) \) for all \( r \) positive but sufficiently small. \textbf{Hint:} Remember the Cauchy integral formula for analytic functions defined on simply connected regions. You could argue there is a simply connected region containing \( \gamma_{r,r_1,\delta} \).

22. ↑ In case \( \Re z > 1 \), you can get an interesting formula for \( I(z) \) by taking the limit as \( r \to 0 \). Recall that

\[
I_r(z) = \int_0^{2\pi} e^{(z-1)(\ln r + i\theta)} \frac{ire^{i\theta}}{e^{re^{i\theta}} - 1} d\theta + \int_r^\infty e^{(z-1)(\ln t + 2\pi i)} \frac{e^{(z-1)\ln t}}{e^t - 1} dt \tag{55.6.25}
\]

and now it is desired to take a limit in the case where \( \Re z > 1 \). Show the first integral above converges to 0 as \( r \to 0 \). Next argue the sum of the two last integrals converges to

\[
\left( e^{(z-1)2\pi i} - 1 \right) \int_0^\infty e^{(z-1)\ln t} \frac{e^{(z-1)\ln t}}{e^t - 1} dt.
\]

Thus

\[
I(z) = \left( e^{z\pi i} - 1 \right) \int_0^\infty e^{(z-1)\ln t} \frac{e^{(z-1)\ln t}}{e^t - 1} dt \tag{55.6.26}
\]

when \( \Re z > 1 \).

23. ↑ So what does all this have to do with the zeta function and the gamma function? The zeta function is defined for \( \Re z > 1 \) by

\[
\sum_{n=1}^{\infty} \frac{1}{n^z} \equiv \zeta(z).
\]

By Problem 11, whenever \( \Re z > 0 \),

\[
\Gamma(z) = \int_0^\infty e^{-t}t^{z-1} dt.
\]
Change the variable and conclude

\[ \Gamma(z) \frac{1}{n^z} = \int_0^\infty e^{-ns} s^{z-1} ds. \]

Therefore, for \( \text{Re} \ z > 1 \),

\[ \zeta(z) \Gamma(z) = \sum_{n=1}^{\infty} \int_0^\infty e^{-ns} s^{z-1} ds. \]

Now show that you can interchange the order of the sum and the integral. This is possibly most easily done by using Fubini’s theorem. Show that \( \sum_{n=1}^{\infty} \int_0^\infty |e^{-ns}s^{z-1}| ds < \infty \) and then use Fubini’s theorem. I think you could do it other ways though. It is possible to do it without any reference to Lebesgue integration. Thus

\[
\zeta(z) \Gamma(z) = \int_0^\infty s^{z-1} \sum_{n=1}^{\infty} e^{-ns} ds \\
= \int_0^\infty s^{z-1} e^{-s} ds = \int_0^\infty s^{z-1} \frac{1}{e^s - 1} ds 
\]

By 55.6.26,

\[
I(z) = (e^{2\pi iz} - 1) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt \\
= (e^{2\pi iz} - 1) \zeta(z) \Gamma(z) \\
= (e^{2\pi iz} - 1) \zeta(z) \Gamma(z)
\]

whenever \( \text{Re} \ z > 1 \).

24. ↑ Now show there exists an entire function, \( h(z) \) such that

\[ \zeta(z) = \frac{1}{z - 1} + h(z) \]

for \( \text{Re} \ z > 1 \). Conclude \( \zeta(z) \) extends to a meromorphic function defined on all of \( \mathbb{C} \) which has a simple pole at \( z = 1 \), namely, the right side of the above formula. **Hint**: Use Problem 10 to observe that \( \Gamma(z) \) is never equal to zero but has simple poles at every nonnegative integer. Then for \( \text{Re} \ z > 1 \),

\[ \zeta(z) \equiv \frac{I(z)}{(e^{2\pi iz} - 1) \Gamma(z)}. \]

By 55.6.26 \( \zeta \) has no poles for \( \text{Re} \ z > 1 \). The right side of the above equation is defined for all \( z \). There are no poles except possibly when \( z \) is a nonnegative integer. However, these points are not poles either because of Problem II which states that \( \Gamma \) has simple poles at these points thus cancelling the simple
zeros of \((e^{2\pi iz} - 1)\). The only remaining possibility for a pole for \(\zeta\) is at \(z = 1\). Show it has a simple pole at this point. You can use the formula for \(I(z)\)

\[
I(z) \equiv \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^r - 1} i r e^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt
\]

Thus \(I(1)\) is given by

\[
I(1) \equiv \int_0^{2\pi} \frac{1}{e^r - 1} i r e^{i\theta} d\theta + \int_r^\infty \frac{1}{e^t - 1} dt + \int_\infty^r \frac{1}{e^t - 1} dt = \int_{\gamma_r} \frac{dw}{e^w - 1}\]

where \(\gamma_r\) is the circle of radius \(r\). This contour integral equals \(2\pi i\) by the residue theorem. Therefore,

\[
\frac{I(z)}{(e^{2\pi iz} - 1) \Gamma(z)} = \frac{1}{z - 1} + h(z)
\]

where \(h(z)\) is an entire function. People worry a lot about where the zeros of \(\zeta\) are located. In particular, the zeros for \(\text{Re } z \in (0, 1)\) are of special interest. The Riemann hypothesis says they are all on the line \(\text{Re } z = 1/2\). This is a good problem for you to do next.

25. There is an important relation between prime numbers and the zeta function due to Euler. Let \(\{p_n\}_{n=1}^\infty\) be the prime numbers. Then for \(\text{Re } z > 1\),

\[
\prod_{n=1}^\infty \frac{1}{1 - p_n^{-z}} = \zeta(z).
\]

To see this, consider a partial product.

\[
\prod_{n=1}^N \frac{1}{1 - p_n^{-z}} = \prod_{n=1}^N \sum_{j_n=1}^\infty \left( \frac{1}{p_n^{j_n}} \right)^z.
\]

Let \(S_N\) denote all positive integers which use only \(p_1, \cdots, p_N\) in their prime factorization. Then the above equals \(\sum_{\sigma \in S_N} \frac{1}{n^z}\). Letting \(N \to \infty\) and using the fact that \(\text{Re } z > 1\) so that the order in which you sum is not important (See Theorem 56.0.1 or recall advanced calculus. ) you obtain the desired equation. Show \(\sum_{n=1}^\infty \frac{1}{p_n} = \infty\).
Chapter 56

Elliptic Functions

This chapter is to give a short introduction to elliptic functions. There is much more available. There are books written on elliptic functions. What I am presenting here follows Alfors [2] although the material is found in many books on complex analysis. Hille, [59] has a much more extensive treatment than what I will attempt here. There are also many references and historical notes available in the book by Hille. Another good source for more having much the same emphasis as what is presented here is in the book by Saks and Zygmund [104]. This is a very interesting subject because it has considerable overlap with algebra.

Before beginning, recall that an absolutely convergent series can be summed in any order and you always get the same answer. The easy way to see this is to think of the series as a Lebesgue integral with respect to counting measure and apply convergence theorems as needed. The following theorem provides the necessary results.

Theorem 56.0.1 Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$ and let $\theta, \phi : \mathbb{N} \rightarrow \mathbb{N}$ be one to one and onto mappings. Then $\sum_{n=1}^{\infty} a_{\phi(n)}$ and $\sum_{n=1}^{\infty} a_{\theta(n)}$ both converge and the two sums are equal.

Proof: By the monotone convergence theorem,

$$\sum_{n=1}^{\infty} |a_n| = \lim_{n \to \infty} \sum_{k=1}^{n} |a_{\phi(k)}| = \lim_{n \to \infty} \sum_{k=1}^{n} |a_{\theta(k)}|$$

but these last two equal $\sum_{k=1}^{\infty} |a_{\phi(k)}|$ and $\sum_{k=1}^{\infty} |a_{\theta(k)}|$ respectively. Therefore, $\sum_{k=1}^{\infty} a_{\phi(k)}$ and $\sum_{k=1}^{\infty} a_{\theta(k)}$ exist ($n \rightarrow a_{\theta(n)}$ is in $L^1$ with respect to counting measure.) It remains to show the two are equal. There exists $M$ such that if $n > M$ then

$$\sum_{k=n+1}^{\infty} |a_{\theta(k)}| < \varepsilon, \quad \sum_{k=n+1}^{\infty} |a_{\phi(k)}| < \varepsilon$$

$$\sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n} a_{\phi(k)} < \varepsilon, \quad \sum_{k=1}^{\infty} a_{\theta(k)} - \sum_{k=1}^{n} a_{\theta(k)} < \varepsilon$$
CHAPTER 56. ELLIPTIC FUNCTIONS

Pick such an \( n \) denoted by \( n_1 \). Then pick \( n_2 > n_1 > M \) such that

\[
\{ \theta(1), \ldots, \theta(n_1) \} \subseteq \{ \phi(1), \ldots, \phi(n_2) \}.
\]

Then

\[
\sum_{k=1}^{n_2} a_{\phi(k)} = \sum_{k=1}^{n_1} a_{\theta(k)} + \sum_{\phi(k) \notin \{\theta(1), \ldots, \theta(n_1)\}} a_{\phi(k)}.
\]

Therefore,

\[
\left| \sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)} \right| = \left| \sum_{\phi(k) \notin \{\theta(1), \ldots, \theta(n_1)\}, k \leq n_2} a_{\phi(k)} \right|.
\]

Now all of these \( \phi(k) \) in the last sum are contained in \( \{ \theta(n_1 + 1), \ldots \} \) and so the last sum above is dominated by

\[
\leq \sum_{k=n_1+1}^{\infty} |a_{\theta(k)}| < \varepsilon.
\]

Therefore,

\[
\left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{\infty} a_{\theta(k)} \right| \leq \left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n_2} a_{\phi(k)} \right| + \left| \sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\phi(k)} \right| + \left| \sum_{k=1}^{n_1} a_{\theta(k)} - \sum_{k=1}^{\infty} a_{\theta(k)} \right| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\]

and since \( \varepsilon \) is arbitrary, it follows \( \sum_{k=1}^{\infty} a_{\phi(k)} = \sum_{k=1}^{\infty} a_{\theta(k)} \) as claimed. This proves the theorem.

56.1 Periodic Functions

Definition 56.1.1 A function defined on \( \mathbb{C} \) is said to be periodic if there exists \( w \) such that \( f(z + w) = f(z) \) for all \( z \in \mathbb{C} \). Denote by \( M \) the set of all periods. Thus if \( w_1, w_2 \in M \) and \( a, b \in \mathbb{Z} \), then \( aw_1 + bw_2 \in M \). For this reason \( M \) is called the module of periods.\(^1\)

Theorem 56.1.2 Let \( f \) be a meromorphic function and let \( M \) be the module of periods. Then if \( M \) has a limit point, then \( f \) equals a constant. If this does not happen then either there exists \( w_1 \in M \) such that \( \mathbb{Z}w_1 = M \) or there exist \( w_1, w_2 \in M \) such that \( M = \{ aw_1 + bw_2 : a, b \in \mathbb{Z} \} \) and \( w_1/w_2 \) is not real. Also if \( \tau = w_2/w_1 \),

\[
|\tau| \geq 1, \quad -\frac{1}{2} \leq \text{Re}\, \tau \leq \frac{1}{2}
\]

\(^1\)A module is like a vector space except instead of a field of scalars, you have a ring of scalars.
56.1. PERIODIC FUNCTIONS

Proof: Suppose $f$ is meromorphic and $M$ has a limit point, $w_0$. By Theorem on Page there exist analytic functions, $p, q$ such that $f(z) = \frac{p(z)}{q(z)}$. Now pick $z_0$ such that $z_0$ is not a pole of $f$. Then letting $w_n \rightarrow w_0$ where $\{w_n\} \subseteq M$, $f(z_0 + w_n) = f(z_0)$. Therefore, $p(z_0 + w_n) = q(z_0 + w_n)$ and so the analytic function, $p(z) - f(z_0)q(z)$ has a zero set which has a limit point. Therefore, this function is identically equal to zero because of Theorem on Page. Thus $f$ equals a constant as claimed.

This has shown that if $f$ is not constant, then $M$ is discrete. Therefore, there exists $w_1 \in M$ such that $|w_1| = \min \{|w| : w \in M\}$. Suppose first that every element of $M$ is a real multiple of $w_1$. Thus, if $w \in M$, it follows there exists a real number, $x$ such that $w = xw_1$. Then there exist positive integers, $k, k + 1$ such that $k < x < k + 1$. If $x > k$, then $w - kw_1 = (x - k)w_1$ is a period having smaller absolute value than $|w_1|$ which would be a contradiction. Hence, $x = k$ and so $M = Zw_1$.

Now suppose there exists $w_2 \in M$ which is not a real multiple of $w_1$. You can let $w_2$ be the element of $M$ having this property which has smallest absolute value. Now let $w \in M$. Since $w_1$ and $w_2$ point in different directions, it follows $w = xw_1 + yw_2$ for some real numbers, $x, y$. Let $|m - x| \leq \frac{1}{2}$ and $|n - y| \leq \frac{1}{2}$ where $m, n$ are integers. Therefore,

$$w = mw_1 + nw_2 + (x - m)w_1 + (y - n)w_2$$

and so

$$w - mw_1 - nw_2 = (x - m)w_1 + (y - n)w_2$$

(56.1.1)

Now since $w_2/w_1 \not\in \mathbb{R}$,

$$|(x - m)w_1 + (y - n)w_2| < |(x - m)w_1| + |(y - n)w_2|$$

$$= \frac{1}{2}|w_1| + \frac{1}{2}|w_2|.$$ 

Therefore, from this,

$$|w - mw_1 - nw_2| = |(x - m)w_1 + (y - n)w_2|$$

$$< \frac{1}{2}|w_1| + \frac{1}{2}|w_2| \leq |w_2|$$

and so the period, $w - mw_1 - nw_2$ cannot be a non real multiple of $w_1$ because $w_2$ is the one which has smallest absolute value and this period has smaller absolute value than $w_2$. Therefore, the ratio $w - mw_1 - nw_2/w_1$ must be a real number, $x$. Thus

$$w - mw_1 - nw_2 = xw_1$$

Since $w_1$ has minimal absolute value of all periods, it follows $|x| \geq 1$. Let $k \leq x < k + 1$ for some integer, $k$. If $x > k$, then

$$w - mw_1 - nw_2 - kw_1 = (x - k)w_1$$

which would contradict the choice of $w_1$ as being the period having minimal absolute value because the expression on the left in the above is a period and it equals
something which has absolute value less than $|w_1|$. Therefore, $x = k$ and $w$ is an integer linear combination of $w_1$ and $w_2$. It only remains to verify the claim about $\tau$.

From the construction, $|w_1| \leq |w_2|$ and $|w_2| \leq |w_1 - w_2|, |w_2| \leq |w_1 + w_2|$. Therefore,

$$|\tau| \geq 1, |\tau| \leq |1 - \tau|, |\tau| \leq |1 + \tau|.$$  

The last two of these inequalities imply $-1/2 \leq \Re \tau \leq 1/2$.

This proves the theorem.

**Definition 56.1.3** For $f$ a meromorphic function which has the last of the above alternatives holding in which $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$, the function, $f$ is called elliptic. This is also called doubly periodic.

**Theorem 56.1.4** Suppose $f$ is an elliptic function which has no poles. Then $f$ is constant.

**Proof:** Since $f$ has no poles it is analytic. Now consider the parallelograms determined by the vertices, $mw_1 + nw_2$ for $m, n \in \mathbb{Z}$. By periodicity of $f$ it must be bounded because its values are identical on each of these parallelograms. Therefore, it equals a constant by Liouville’s theorem.

**Definition 56.1.5** Define $P_a$ to be the parallelogram determined by the points

$$a + mw_1 + nw_2, a + (m + 1)w_1 + nw_2, a + mw_1 + (n + 1)w_2,$$

$$a + (m + 1)w_1 + (n + 1)w_2$$

Such $P_a$ will be referred to as a period parallelogram. The sum of the orders of the poles in a period parallelogram which contains no poles or zeros of $f$ on its boundary is called the order of the function. This is well defined because of the periodic property of $f$.

**Theorem 56.1.6** The sum of the residues of any elliptic function, $f$ equals zero on every $P_a$ if $a$ is chosen so that there are no poles on $\partial P_a$.

**Proof:** Choose $a$ such that there are no poles of $f$ on the boundary of $P_a$. By periodicity,

$$\int_{\partial P_a} f(z) \, dz = 0$$

because the integrals over opposite sides of the parallelogram cancel out because the values of $f$ are the same on these sides and the orientations are opposite. It follows from the residue theorem that the sum of the residues in $P_a$ equals 0.

**Theorem 56.1.7** Let $P_a$ be a period parallelogram for a nonconstant elliptic function, $f$ which has order equal to $m$. Then $f$ assumes every value in $f(P_a)$ exactly $m$ times.
**Proof:** Let \( c \in f(P_a) \) and consider \( P_{a'} \) such that \( f^{-1}(c) \cap P_{a'} = f^{-1}(c) \cap P_a \) and \( P_{a'} \) contains the same poles and zeros of \( f - c \) as \( P_a \) but \( P_{a'} \) has no zeros of \( f(z) - c \) or poles of \( f \) on its boundary. Thus \( f'(z) / (f(z) - c) \) is also an elliptic function and so Theorem 56.1.6 applies. Consider

\[
\frac{1}{2\pi i} \int_{\partial P_a} \frac{f'(z)}{f(z) - c} \, dz.
\]

By the argument principle, this equals \( N_z - N_p \) where \( N_z \) equals the number of zeros of \( f(z) - c \) and \( N_p \) equals the number of the poles of \( f(z) \). From Theorem 56.1.6 this must equal zero because it is the sum of the residues of \( f'/f - c \) and so \( N_z = N_p \). Now \( N_p \) equals the number of poles in \( P_a \) counted according to multiplicity.

There is an even better theorem than this one.

**Theorem 56.1.8** Let \( f \) be a non constant elliptic function and suppose it has poles \( p_1, \ldots, p_m \) and zeros, \( z_1, \ldots, z_m \) in \( P_a \), listed according to multiplicity where \( \partial P_a \) contains no poles or zeros of \( f \). Then \( \sum_{k=1}^m z_k - \sum_{k=1}^m p_k \in M \), the module of periods.

**Proof:** You can assume \( \partial P_a \) contains no poles or zeros of \( f \) because if it did, then you could consider a slightly shifted period parallelogram, \( P_{a'} \) which contains no new zeros and poles but which has all the old ones but no poles or zeros on its boundary. By Theorem 56.1.8 on Page 1865

\[
\frac{1}{2\pi i} \int_{\partial P_a} z f'(z) f(z) \, dz = \sum_{k=1}^m z_k - \sum_{k=1}^m p_k. \tag{56.1.2}
\]

Denoting by \( \gamma(z,w) \) the straight oriented line segment from \( z \) to \( w \),

\[
\int_{\partial P_a} z f'(z) f(z) \, dz = \int_{\gamma(a,a+w_1)} z f'(z) f(z) \, dz + \int_{\gamma(a+a+w_1, a+w_2)} z f'(z) f(z) \, dz + \int_{\gamma(a+w_1, a+w_2+w_1)} z f'(z) f(z) \, dz + \int_{\gamma(a+w_2, a)} z f'(z) f(z) \, dz
\]

\[
= \int_{\gamma(a,a+w_1)} (z - (z + w_2)) f'(z) f(z) \, dz + \int_{\gamma(a,a+w_2)} (z - (z + w_1)) f'(z) f(z) \, dz
\]

Now near these line segments \( f'(z) f(z) \) is analytic and so there exists a primitive, \( g_{w_1}(z) \) on \( \gamma(a,a + w_1) \) by Corollary 56.1.7 on Page 1865 which satisfies \( e^{g_{w_1}(z)} = f(z) \). Therefore,

\[
= -w_2 (g_{w_1} (a + w_1) - g_{w_1} (a)) - w_1 (g_{w_2} (a + w_2) - g_{w_2} (a)).
\]
Now by periodicity of $f$ it follows $f(a + w) = f(a) = f(a + w_2)$. Hence

$$g_{w_1}(a + w) - g_{w_1}(a) = 2m\pi i$$

for some integer, $m$ because

$$e^{g_{w_1}(a+w)} - e^{g_{w_1}(a)} = f(a + w) - f(a) = 0.$$

Therefore, from 56.1.2, there exist integers, $k, l$ such that

$$\frac{1}{2\pi i} \int_{\partial P_{\alpha}} z \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} [w_2 (g_{w_1}(a + w_1) - g_{w_1}(a)) - w_1 (g_{w_2}(a + w_2) - g_{w_2}(a))]
= \frac{1}{2\pi i} [w_2 (2k\pi i) - w_1 (2l\pi i)]
= -w_2 k - w_1 l \in M.$$

From 56.1.2 it follows

$$\sum_{k=1}^{m} z_k - \sum_{k=1}^{m} p_k \in M.$$

This proves the theorem.

Hille says this relation is due to Liouville. There is also a simple corollary which follows from the above theorem applied to the elliptic function, $f(z) - c$.

**Corollary 56.1.9** Let $f$ be a non constant elliptic function and suppose the function, $f(z) - c$ has poles $p_1, \ldots, p_m$ and zeros, $z_1, \ldots, z_m$ on $P_{\alpha}$, listed according to multiplicity where $\partial P_{\alpha}$ contains no poles or zeros of $f(z) - c$. Then $\sum_{k=1}^{m} z_k - \sum_{k=1}^{m} p_k \in M$, the module of periods.

### 56.1.1 The Unimodular Transformations

**Definition 56.1.10** Suppose $f$ is a nonconstant elliptic function and the module of periods is of the form $\{aw_1 + bw_2\}$ where $a, b$ are integers and $w_1/w_2$ is not real. Then by analogy with linear algebra, $\{w_1, w_2\}$ is referred to as a basis. The unimodular transformations will refer to matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where all entries are integers and

$$ad - bc = \pm 1.$$ 

These linear transformations are also called the modular group.

The following is an interesting lemma which ties matrices with the fractional linear transformations.
Lemma 56.1.11 Define

\[ \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \frac{az + b}{cz + d}. \]

Then

\[ \phi(AB) = \phi(A) \circ \phi(B), \quad (56.1.3) \]

\[ \phi(A)(z) = z \text{ if and only if } A = cI \]

where I is the identity matrix and c ≠ 0. Also if \( f(z) = \frac{az + b}{cz + d} \), then \( f^{-1}(z) \) exists if and only if \( ad - cb \neq 0 \). Furthermore it is easy to find \( f^{-1} \).

**Proof:** The equation in (56.1.3) is just a simple computation. Now suppose \( \phi(A)(z) = z \). Then letting \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), this requires

\[ az + b = z(cz + d) \]

and so \( az + b = cz^2 + dz \). Since this is to hold for all \( z \) it follows \( c = 0 = b \) and \( a = d \). The other direction is obvious.

Consider the claim about the existence of an inverse. Let \( ad - cb \neq 0 \) for \( f(z) = \frac{az + b}{cz + d} \). Then

\[ f(z) = \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \]

It follows \( \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \) exists and equals \( \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). Therefore,

\[ z = \phi(I)(z) = \phi \left( \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \cdot \left( \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right) \right)(z) \]

\[ = \phi \left( \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \right) \circ \phi \left( \phi \left( \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right) \right)(z) \]

\[ = f \circ f^{-1}(z) \]

which shows \( f^{-1} \) exists and it is easy to find.

Next suppose \( f^{-1} \) exists. I need to verify the condition \( ad - cb \neq 0 \). If \( f^{-1} \) exists, then from the process used to find it, you see that it must be a fractional linear transformation. Letting \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) so \( \phi(A) = f \), it follows there exists a matrix \( B \) such that

\[ \phi(BA)(z) = \phi(B) \circ \phi(A)(z) = z. \]

However, it was shown that this implies \( BA \) is a nonzero multiple of I which requires that \( A^{-1} \) must exist. Hence the condition must hold.
Theorem 56.1.12 If $f$ is a nonconstant elliptic function with a basis $\{w_1, w_2\}$ for the module of periods, then $\{w'_1, w'_2\}$ is another basis, if and only if there exists a unimodular transformation, $
abla = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$
\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
$$

(56.1.4)

Proof: Since $\{w_1, w_2\}$ is a basis, there exist integers, $a, b, c, d$ such that holds. It remains to show the transformation determined by the matrix is unimodular. Taking conjugates,

$$
\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
$$

Therefore,

$$
\begin{pmatrix} w'_1 & w'_1 \\ w'_2 & w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & w_1 \\ w_2 & w_2 \end{pmatrix}.
$$

Now since $\{w'_1, w'_2\}$ is also given to be a basis, there exits another matrix having all integer entries, $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that

$$
\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix}.
$$

and

$$
\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix}.
$$

Therefore,

$$
\begin{pmatrix} w'_1 & w_1 \\ w'_2 & w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w'_1 & w'_1 \\ w'_2 & w'_2 \end{pmatrix}.
$$

However, since $w'_1/w'_2$ is not real, it is routine to verify that

$$
\text{det} \begin{pmatrix} w'_1 & w'_1 \\ w'_2 & w'_2 \end{pmatrix} \neq 0.
$$

Therefore,

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}
$$

and so $\text{det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{det} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = 1$. But the two matrices have all integer entries and so both determinants must equal either 1 or −1.

Next suppose

$$
\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
$$

(56.1.5)
where \((\begin{array}{cc} a & b \\ c & d \end{array})\) is unimodular. I need to verify that \(\{w'_1, w'_2\}\) is a basis. If \(w \in M\), there exist integers, \(m, n\) such that

\[
 w = mw_1 + nw_2 = (m \ n) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
\]

From 56.1.5

\[
 \pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}
\]

and so

\[
 w = \pm \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix}
\]

which is an integer linear combination of \(\{w'_1, w'_2\}\). It only remains to verify that \(w'_1/w'_2\) is not real.

**Claim:** Let \(w_1\) and \(w_2\) be nonzero complex numbers. Then \(w_2/w_1\) is not real if and only if

\[
\det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0
\]

**Proof of the claim:** Let \(\lambda = w_2/w_1\). Then

\[
 w_1 \overline{w_2} - \overline{w_1} w_2 = \overline{\lambda} w_1 \overline{w_1} - \overline{\lambda} w_1 = (\overline{\lambda} - \lambda) |w_1|^2
\]

Thus the ratio is not real if and only if \((\overline{\lambda} - \lambda) \neq 0\) if and only if \(w_1 \overline{w_2} - \overline{w_1} w_2 \neq 0\).

Now to verify \(w'_2/w'_1\) is not real,

\[
 \det \begin{pmatrix} w'_1 & \overline{w'_1} \\ w'_2 & \overline{w'_2} \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} = \pm \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0
\]

This proves the theorem.

### 56.1.2 The Search For An Elliptic Function

By Theorem 56.1.4 and 56.1.6 if you want to find a nonconstant elliptic function it must fail to be analytic and also have either no terms in its Laurent expansion which are of the form \(b_1 (z - a)^{-1}\) or else these terms must cancel out. It is simplest to look for a function which simply does not have them. Weierstrass looked for a function of the form

\[
 \wp(z) = \frac{1}{z^2} + \sum_{w \neq 0} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right)
\]  

(56.1.6)
where \( w \) consists of all numbers of the form \( aw_1 + bw_2 \) for \( a, b \) integers. Sometimes people write this as \( \wp(z, w_1, w_2) \) to emphasize its dependence on the periods, \( w_1 \) and \( w_2 \) but I won’t do so. It is understood there exist these periods, which are given. This is a reasonable thing to try. Suppose you formally differentiate the right side. Never mind whether this is justified for now. This yields

\[
\wp'(z) = -\frac{2}{z^3} - \sum_{w \neq 0} \frac{-2}{(z-w)^3} = \sum_w \frac{-2}{(z - w)^3}
\]

which is clearly periodic having both periods \( w_1 \) and \( w_2 \). Therefore, \( \wp(z + w_1) - \wp(z) \) and \( \wp(z + w_2) - \wp(z) \) are both constants, \( c_1 \) and \( c_2 \) respectively. The reason for this is that since \( \wp' \) is periodic with periods \( w_1 \) and \( w_2 \), it follows \( \wp'(z + w_i) - \wp'(z) = 0 \) as long as \( z \) is not a period. From 56.1.6 you can see right away that

\[
\wp(z) = \wp(-z)
\]

Indeed

\[
\wp(-z) = \frac{1}{z^2} + \sum_{w \neq 0} \left( \frac{1}{(-z-w)^2} - \frac{1}{w^2} \right) = \wp(z).
\]

and so

\[
c_1 = \wp\left(\frac{w_1}{2} + w_1\right) - \wp\left(-\frac{w_1}{2}\right) = \wp\left(\frac{w_1}{2}\right) - \wp\left(-\frac{w_1}{2}\right) = 0
\]

which shows the constant for \( \wp(z + w_1) - \wp(z) \) must equal zero. Similarly the constant for \( \wp(z + w_2) - \wp(z) \) also equals zero. Thus \( \wp \) is periodic having the two periods \( w_1, w_2 \).

Of course to justify this, you need to consider whether the series of 56.1.6 converges. Consider the terms of the series.

\[
\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = |z| \left| \frac{2w - z}{(z-w)^2 w^2} \right|
\]

If \( |w| > 2|z| \), this can be estimated more. For such \( w \),

\[
\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = |z| \left| \frac{2w - z}{(z-w)^2 w^2} \right| \leq |z| \frac{(5/2)|w|}{|w|^2 (|w| - |z|)^2}
\]

\[
\leq |z| \frac{(5/2)|w|}{|w|^2 ((1/2)|w|)^2} = |z| \frac{10}{|w|^3}.
\]
It follows the series in $56.1.6$ converges uniformly and absolutely on every compact set, $K$ provided $\sum_{w \neq 0} \frac{1}{|w|^3}$ converges. This question is considered next.

**Claim:** There exists a positive number, $k$ such that for all pairs of integers, $m, n$, not both equal to zero,

$$\frac{|mw_1 + nw_2|}{|m| + |n|} \geq k > 0.$$  

**Proof of claim:** If not, there exists $m_k$ and $n_k$ such that

$$\lim_{k \to \infty} \frac{m_k}{|m_k| + |n_k|} w_1 + \frac{n_k}{|m_k| + |n_k|} w_2 = 0.$$

However, $\left( \frac{m_k}{|m_k| + |n_k|}, \frac{n_k}{|m_k| + |n_k|} \right)$ is a bounded sequence in $\mathbb{R}^2$ and so, taking a subsequence, still denoted by $k$, you can have

$$\left( \frac{m_k}{|m_k| + |n_k|}, \frac{n_k}{|m_k| + |n_k|} \right) \to (x, y) \in \mathbb{R}^2$$

and so there are real numbers, $x, y$ such that $xw_1 + yw_2 = 0$ contrary to the assumption that $w_2/w_1$ is not equal to a real number. This proves the claim.

Now from the claim,

$$\sum_{w \neq 0} \frac{1}{|w|^3} = \sum_{(m,n) \neq (0,0)} \frac{1}{|mw_1 + nw_2|^3} \leq \sum_{(m,n) \neq (0,0)} \frac{1}{k^3(|m| + |n|)^3} $$

$$= \frac{1}{k^3} \sum_{j=1}^{\infty} \sum_{|m| + |n| = j} \frac{1}{(|m| + |n|)^3} = \frac{1}{k^3} \sum_{j=1}^{\infty} \frac{4j}{j^3} < \infty.$$

Now consider the series in $56.1.6$. Letting $z \in B(0, R),$

$$\wp(z) = \frac{1}{z^2} + \sum_{w \neq 0, |w| \leq R} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

$$+ \sum_{w \neq 0, |w| > R} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

and the last series converges uniformly on $B(0, R)$ to an analytic function. Thus $\wp$ is a meromorphic function and also the argument given above involving differentiation of the series termwise is valid. Thus $\wp$ is an elliptic function as claimed. This is called the Weierstrass $\wp$ function. This has proved the following theorem.

**Theorem 56.1.13** The function $\wp$ defined above is an example of an elliptic function. On any compact set, $\wp$ equals a rational function added to a series which is uniformly and absolutely convergent on the compact set.
56.1.3 The Differential Equation Satisfied By $\wp$

For $z$ not a pole,

$$ \wp' (z) = \frac{-2}{z^3} - \sum_{w \neq 0} \frac{2}{(z - w)^3} $$

Also since there are no poles of order 1 you can obtain a primitive for $\wp$, $-\zeta$.

To do so, recall

$$ \wp (z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right) $$

where for $|z| < R$ this is the sum of a rational function with a uniformly convergent series. Therefore, you can take the integral along any path, $\gamma (0, z)$ from 0 to $z$ which misses the poles of $\wp$. By the uniform convergence of the above integral, you can interchange the sum with the integral and obtain

$$ \zeta (z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z - w} + \frac{z}{w^2} + \frac{1}{w} \quad (56.1.7) $$

This function is odd. Here is why.

$$ \zeta (-z) = \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{-z - w} - \frac{z}{w^2} + \frac{1}{w} $$

while

$$ -\zeta (z) = \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{z - w} - \frac{z}{w^2} - \frac{1}{w} $$

$$ = \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{z + w} - \frac{z}{w^2} + \frac{1}{w}. $$

Now consider $56.1.7$. It will be used to find the Laurent expansion about the origin for $\zeta$ which will then be differentiated to obtain the Laurent expansion for $\wp$ at the origin. Since $w \neq 0$ and the interest is for $z$ near 0 so $|z| < |w|$,

$$ \frac{1}{z - w} + \frac{z}{w^2} + \frac{1}{w} = \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w} \frac{1 - z}{w} $$

$$ = \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w} \sum_{k=0}^{\infty} \left( \frac{z}{w} \right)^k $$

$$ = -\frac{1}{w} \sum_{k=2}^{\infty} \left( \frac{z}{w} \right)^k $$

---

2I don’t know why it is traditional to refer to this antiderivative as $-\zeta$ rather than $\zeta$ but I am following the convention. I think it is to minimize the number of minus signs in the next expression.
From \[ \zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \left( -\sum_{k=2}^{\infty} \frac{z^k}{w^{k+1}} \right) \] \[ = \frac{1}{z} - \sum_{k=2}^{\infty} \sum_{w \neq 0} \frac{z^k}{w^{k+1}} = \frac{1}{z} - \sum_{k=2}^{\infty} \sum_{w \neq 0} \frac{z^{2k-1}}{w^{2k}} \]

because the sum over odd powers must be zero because for each \( w \neq 0 \), there exists \( -w \neq 0 \) such that the two terms \( \frac{z^{2k}}{w^{2k}} \) and \( \frac{z^{2k}}{(-w)^{2k}} \) cancel each other. Hence

\[ \zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} G_k z^{2k-1} \]

where \( G_k = \sum_{w \neq 0} \frac{1}{w^{2k}} \). Now with this,

\[ -\zeta'(z) = \wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} G_k (2k - 1) z^{2k-2} \]

\[ = \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \cdots \]

Therefore,

\[ \wp'(z) = -\frac{2}{z^3} + 6G_2 z + 20G_3 z^3 + \cdots \]

\[ \wp'(z)^2 = \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 + \cdots \]

\[ 4\wp(z)^3 = \left( \frac{1}{z^6} + 3G_2 z^2 + 5G_3 z^4 + \cdots \right)^3 \]

\[ = \frac{4}{z^6} + 36 \frac{z^2}{z^2}G_2 + 60G_3 + \cdots \]

and finally

\[ 60G_2\wp(z) = \frac{60G_2}{z^2} + 0 + \cdots \]

where in the above, the positive powers of \( z \) are not listed explicitly. Therefore,

\[ \wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3 = \sum_{n=1}^{\infty} a_n z^n \]

In deriving the equation it was assumed \( |z| < |w| \) for all \( w = aw_1 + bw_2 \) where \( a, b \) are integers not both zero. The left side of the above equation is periodic with respect to \( w_1 \) and \( w_2 \) where \( w_2/w_1 \) is not a real number. The only possible poles of the left side are at 0, \( w_1 \), \( w_2 \), and \( w_1 + w_2 \), the vertices of the parallelogram determined by \( w_1 \) and \( w_2 \). This follows from the original formula for \( \wp(z) \). However, the above
equation shows the left side has no pole at 0. Since the left side is periodic with
periods $w_1$ and $w_2$, it follows it has no pole at the other vertices of this parallelogram
either. Therefore, the left side is periodic and has no poles. Consequently, it equals
a constant by Theorem 56.1.4. But the right side of the above equation shows this
constant is 0 because this side equals zero when $z = 0$. Therefore, $\wp$ satisfies the
differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3 = 0.$$  

It is traditional to define $60G_2 \equiv g_2$ and $140G_3 \equiv g_3$. Then in terms of these new
quantities the differential equation is

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$  

Suppose $e_1, e_2$ and $e_3$ are zeros of the polynomial $4w^3 - g_2w - g_3 = 0$. Then the
above equation can be written in the form

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \quad (56.1.8)$$  

56.1.4 A Modular Function

The next task is to find the $e_i$ in 56.1.8. First recall that $\wp$ is an even function. That
is $\wp(-z) = \wp(z)$. This follows from 56.1.10 which is listed here for convenience.

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left( \frac{1}{(z - w)^2} - \frac{1}{w^2} \right) \quad (56.1.9)$$  

Thus

$$\wp(-z) = \frac{1}{z^2} + \sum_{w \neq 0} \left( \frac{1}{(-z - w)^2} - \frac{1}{w^2} \right)$$

$$= \frac{1}{z^2} + \sum_{w \neq 0} \left( \frac{1}{(-z + w)^2} - \frac{1}{w^2} \right) = \wp(z).$$

Therefore, $\wp(w_1 - z) = \wp(z - w_1) = \wp(z)$ and so $-\wp'(w_1 - z) = \wp'(z)$. Letting
$z = w_1/2$, it follows $\wp'(w_1/2) = 0$. Similarly, $\wp'(w_2/2) = 0$ and $\wp'((w_1 + w_2)/2) = 0$. Therefore, from

$$0 = 4(\wp(w_1/2) - e_1)(\wp(w_1/2) - e_2)(\wp(w_1/2) - e_3).$$

It follows one of the $e_i$ must equal $\wp(w_1/2)$. Similarly, one of the $e_i$ must equal
$\wp(w_2/2)$ and one must equal $\wp((w_1 + w_2)/2$).

**Lemma 56.1.14** The numbers, $\wp(w_1/2), \wp(w_2/2), \text{ and } \wp((w_1 + w_2)/2)$ are dist-

**IC-56**
56.1. PERIODIC FUNCTIONS

Proof: Choose $P_a$, a period parallelogram which contains the pole 0, and the points $w_1/2, w_2/2$, and $(w_1 + w_2)/2$ but no other pole of $\varphi(z)$. Also $\partial P_a^*$ does not contain any zeros of the elliptic function, $z \to \varphi(z) - \varphi(w_1/2)$. This can be done by shifting $P_0$ slightly because the poles are only at the points $aw_1 + bw_2$ for $a, b$ integers and the zeros of $\varphi(z) - \varphi(w_1/2)$ are discrete.

If $\varphi(w_2/2) = \varphi(w_1/2)$, then $\varphi(z) - \varphi(w_1/2)$ has two zeros, $w_2/2$ and $w_1/2$ and since the pole at 0 is of order 2, this is the order of $\varphi(z) - \varphi(w_1/2)$ on $P_a$ hence by Theorem 56.1.7 on Page 1864 these are the only zeros of this function on $P_a$. It follows by Corollary 56.1.9 on Page 1866 which says the sum of the zeros minus the sum of the poles is in $M$, $w_1^2 + w_2^2 \in M$. Thus there exist integers, $a, b$ such that

$$\frac{w_1 + w_2}{2} = aw_1 + bw_2$$

which implies $(2a - 1)w_1 + (2b - 1)w_2 = 0$ contradicting $w_2/w_1$ not being real. Similar reasoning applies to the other pairs of points in $\{w_1/2, w_2/2, (w_1 + w_2)/2\}$. For example, consider $(w_1 + w_2)/2$ and choose $P_a$ such that its boundary contains no zeros of the elliptic function, $z \to \varphi(z) - \varphi((w_1 + w_2)/2)$ and $P_a$ contains no poles of $\varphi$ on its interior other than 0. Then if $\varphi(w_2/2) = \varphi((w_1 + w_2)/2)$, it follows from Theorem 56.1.7 on Page 1864 $w_2/2$ and $(w_1 + w_2)/2$ are the only two zeros of $\varphi(z) - \varphi((w_1 + w_2)/2)$ on $P_a$ and by Corollary 56.1.9 on Page 1864

$$\frac{w_1 + w_1 + w_2}{2} = aw_1 + bw_2 \in M$$

for some integers $a, b$ which leads to the same contradiction as before about $w_1/w_2$ not being real. The other cases are similar. This proves the lemma.

Lemma 56.1.14 proves the $e_i$ are distinct. Number the $e_i$ such that

$$e_1 = \varphi(w_1/2), e_2 = \varphi(w_2/2)$$

and

$$e_3 = \varphi((w_1 + w_2)/2).$$

To summarize, it has been shown that for complex numbers, $w_1$ and $w_2$ with $w_2/w_1$ not real, an elliptic function, $\varphi$ has been defined. Denote this function as
\( \wp(z) = \wp(z, w_1, w_2) \). This in turn determined numbers, \( e_1 \) as described above. Thus these numbers depend on \( w_1 \) and \( w_2 \) and as described above,

\[
e_1(w_1, w_2) = \wp \left( \frac{w_1}{2}, w_1, w_2 \right), \quad e_2(w_1, w_2) = \wp \left( \frac{w_2}{2}, w_1, w_2 \right), \quad e_3(w_1, w_2) = \wp \left( \frac{w_1 + w_2}{2}, w_1, w_2 \right).
\]

Therefore, using the formula for \( \wp \),

\[
\wp(z) = 1 + \sum_{w \neq 0} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)
\]
you see that if the two periods \( w_1 \) and \( w_2 \) are replaced with \( tw_1 \) and \( tw_2 \) respectively, then

\[
e_i(tw_1, tw_2) = t^{-2}e_i(w_1, w_2).
\]

Let \( \tau \) denote the complex number which equals the ratio, \( w_2/w_1 \) which was assumed in all this to not be real. Then

\[
e_i(w_1, w_2) = w_1^{-2}e_i(1, \tau)
\]

Now define the function, \( \lambda(\tau) \)

\[
\lambda(\tau) = \frac{e_3(1, \tau) - e_2(1, \tau)}{e_1(1, \tau) - e_2(1, \tau)} = \left( \frac{e_3(w_1, w_2) - e_2(w_1, w_2)}{e_1(w_1, w_2) - e_2(w_1, w_2)} \right).
\]

This function is meromorphic for \( \text{Im } \tau > 0 \) or for \( \text{Im } \tau < 0 \). However, since the denominator is never equal to zero the function must actually be analytic on both the upper half plane and the lower half plane. It never is equal to 0 because \( e_3 \neq e_2 \) and it never equals 1 because \( e_3 \neq e_1 \). This is stated as an observation.

**Observation 56.1.15** The function, \( \lambda(\tau) \) is analytic for \( \tau \) in the upper half plane and never assumes the values 0 and 1.

This is a very interesting function. Consider what happens when

\[
\begin{pmatrix}
w'_1 \\
w'_2
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
\]

and the matrix is unimodular. By Theorem 56.1.13 on Page 1869 \( \{w'_1, w'_2\} \) is just another basis for the same module of periods. Therefore, \( \wp(z, w_1, w_2) = \wp(z, w'_1, w'_2) \) because both are defined as sums over the same values of \( w \), just in different order which does not matter because of the absolute convergence of the sums on compact subsets of \( \mathbb{C} \). Since \( \wp \) is unchanged, it follows \( \wp'(z) \) is also unchanged and so the numbers, \( e_i \) are also the same. However, they might be permuted in which case the function \( \lambda(\tau) \) defined above would change. What would it take for \( \lambda(\tau) \) to not change? In other words, for which unimodular transformations will \( \lambda \) be left...
unchanged? This happens if and only if no permuting takes place for the \(e_i\). This occurs if \(\varphi \left( \frac{w_1}{2} \right) = \varphi \left( \frac{w'_1}{2} \right)\) and \(\varphi \left( \frac{w_2}{2} \right) = \varphi \left( \frac{w'_2}{2} \right)\). If

\[
\frac{w'_1}{2} - \frac{w_1}{2} \in M, \quad \frac{w'_2}{2} - \frac{w_2}{2} \in M
\]

then \(\varphi \left( \frac{w_1}{2} \right) = \varphi \left( \frac{w'_1}{2} \right)\) and so \(e_1\) will be unchanged and similarly for \(e_2\) and \(e_3\).

This occurs exactly when

\[
\frac{1}{2} \left( (a - 1) w_1 + bw_2 \right) \in M, \quad \frac{1}{2} \left( cw_1 + (d - 1) w_2 \right) \in M.
\]

This happens if \(a\) and \(d\) are odd and if \(b\) and \(c\) are even. Of course the stylish way to say this is

\[
a \equiv 1 \mod 2, \quad d \equiv 1 \mod 2, \quad b \equiv 0 \mod 2, \quad c \equiv 0 \mod 2.
\]

(56.1.11)

This has shown that for unimodular transformations satisfying 56.1.11 \(\lambda\) is unchanged. Letting \(\tau\) be defined as above,

\[
\tau' = \frac{w'_2}{w'_1} = \frac{cw_1 + dw_2}{aw_1 + bw_2} = \frac{c + d\tau}{a + b\tau}.
\]

Thus for unimodular transformations, \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) satisfying 56.1.11, or more succinctly,

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2
\]

it follows that

\[
\lambda \left( \frac{c + d\tau}{a + b\tau} \right) = \lambda (\tau).
\]

(56.1.13)

Furthermore, this is the only way this can happen.

**Lemma 56.1.16** \(\lambda (\tau) = \lambda (\tau')\) if and only if

\[
\tau' = \frac{a\tau + b}{c\tau + d}
\]

where 56.1.11 holds.

**Proof:** It only remains to verify that if \(\varphi \left( \frac{w'_1}{2} \right) = \varphi \left( \frac{w_1}{2} \right)\) then it is necessary that

\[
\frac{w'_1}{2} - \frac{w_1}{2} \in M
\]

with a similar requirement for \(w_2\) and \(w'_2\). If \(\frac{w'_1}{2} - \frac{w_1}{2} \notin M\), then there exist integers, \(m, n\) such that

\[
\frac{-w'_1}{2} + mw_1 + nw_2
\]
is in the interior of \( P_0 \), the period parallelogram whose vertices are 0, \( w_1 \), \( w_1 + w_2 \), and \( w_2 \). Therefore, it is possible to choose small \( a \) such that \( P_a \) contains the pole, 0, \( w_1 \), and \( -\frac{w_1'}{2} + mw_1 + nw_2 \) but no other poles of \( \wp \) and in addition, \( \partial P^*_a \) contains no zeros of \( z \to \wp(z) - \wp\left(\frac{w_1}{2}\right) \). Then the order of this elliptic function is 2. By assumption, and the fact that \( \wp \) is even,

\[
\wp\left(\frac{-w_1'}{2} + mw_1 + nw_2\right) = \wp\left(\frac{-w_1'}{2}\right) = \wp\left(\frac{w_1'}{2}\right) = \wp\left(\frac{w_1}{2}\right).
\]

It follows both \( -\frac{w_1'}{2} + mw_1 + nw_2 \) and \( \frac{w_1}{2} \) are zeros of \( \wp(z) - \wp\left(\frac{w_1}{2}\right) \) and so by Theorem 56.1.13 on Page 1864 these are the only two zeros of this function in \( P_a \). Therefore, from Corollary 56.1.9 on Page 1866

\[
\frac{w_1}{2} - \frac{w_1'}{2} + mw_1 + nw_2 \in M
\]

which shows \( \frac{w_1}{2} - \frac{w_1'}{2} \in M \). This completes the proof of the lemma.

Note the condition in the lemma is equivalent to the condition 56.1.13 because you can relabel the coefficients. The message of either version is that the coefficient of \( \tau \) in the numerator and denominator is odd while the constant in the numerator and denominator is even.

Next, \( \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \) and therefore,

\[
\lambda\left(\frac{2+\tau}{1}\right) = \lambda(\tau + 2) = \lambda(\tau).
\]

(56.1.14)

Thus \( \lambda \) is periodic of period 2.

Thus \( \lambda \) leaves invariant a certain subgroup of the unimodular group. According to the next definition, \( \lambda \) is an example of something called a modular function.

**Definition 56.1.17** When an analytic or meromorphic function is invariant under a group of linear transformations, it is called an automorphic function. A function which is automorphic with respect to a subgroup of the modular group is called a modular function or an elliptic modular function.

Now consider what happens for some other unimodular matrices which are not congruent to the identity mod 2. This will yield other functional equations for \( \lambda \) in addition to the fact that \( \lambda \) is periodic of period 2. As before, these functional equations come about because \( \wp \) is unchanged when you change the basis for \( M \), the module of periods. In particular, consider the unimodular matrices

\[
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

(56.1.15)

Consider the first of these. Thus

\[
\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 + w_2 \end{pmatrix}
\]
Hence \( \tau' = w_2'/w_1' = (w_1 + w_2)/w_1 = 1 + \tau \). Then from the definition of \( \lambda \),

\[
\lambda (\tau') = \lambda (1 + \tau)
\]

\[
= \varphi \left( \frac{w_1' + w_2'}{2} \right) - \varphi \left( \frac{w_1'}{2} \right)
\]

\[
= \frac{\varphi \left( \frac{w_1 + w_2 + w_1}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)}{\varphi \left( \frac{w_1}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)}
\]

\[
= \frac{\varphi \left( \frac{w_1'}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)}{\varphi \left( \frac{w_1}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)}
\]

\[
= -\frac{\varphi \left( \frac{w_1'}{2} \right) - \varphi \left( \frac{w_2}{2} \right)}{\varphi \left( \frac{w_1}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)}
\]

\[
= -\frac{\varphi \left( \frac{w_1 + w_2}{2} \right) - \varphi \left( \frac{w_1'}{2} \right)}{\varphi \left( \frac{w_1}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)}
\]

\[
= \frac{\varphi \left( \frac{w_1 + w_2}{2} \right) - \varphi \left( \frac{w_1'}{2} \right)}{\varphi \left( \frac{w_1}{2} \right) - \varphi \left( \frac{w_1 + w_2}{2} \right)} - 1
\]

\[
= \frac{\lambda (\tau)}{\lambda (\tau) - 1}.
\]

(56.1.16)

Summarizing the important feature of the above,

\[
\lambda (1 + \tau) = \frac{\lambda (\tau)}{\lambda (\tau) - 1}.
\]

(56.1.17)
Next consider the other unimodular matrix in 56.1.13. In this case \( w'_1 = w_2 \) and \( w'_2 = w_1 \). Therefore, \( \tau' = w'_2/w'_1 = w_1/w_2 = 1/\tau \). Then

\[
\lambda (\tau') = \lambda (1/\tau) = \frac{\wp \left(\frac{w'_1 + w'_2}{2}\right) - \wp \left(\frac{w'_2}{2}\right)}{\wp \left(\frac{w'_2}{2}\right) - \wp \left(\frac{w'_1}{2}\right)}
\]

\[
= \frac{\wp \left(\frac{w_1 + w_2}{2}\right) - \wp \left(\frac{w_1}{2}\right)}{\wp \left(\frac{w_2}{2}\right) - \wp \left(\frac{w_1}{2}\right)}
\]

\[
= \frac{e_3 - e_1}{e_2 - e_1} = \frac{-e_3 - e_2 + e_2 - e_1}{e_1 - e_2} = -\left(\lambda (\tau) - 1\right) = -\lambda (\tau) + 1.
\]

(56.1.18)

You could try other unimodular matrices and attempt to find other functional equations if you like but this much will suffice here.

56.1.5 A Formula For \( \lambda \)

Recall the formula of Mittag-Leffler for \( \cot (\pi \alpha) \) given in 55.2.15. For convenience, here it is.

\[
\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi \alpha.
\]

As explained in the derivation of this formula it can also be written as

\[
\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2 - n^2} = \pi \cot \pi \alpha.
\]

Differentiating both sides yields

\[
\pi^2 \csc^2 (\pi \alpha) = \sum_{n=-\infty}^{\infty} \frac{\alpha^2 + n^2}{(\alpha^2 - n^2)^2}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{(\alpha + n)^2 - 2\alpha n}{(\alpha + n)^2 (\alpha - n)^2}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{(\alpha + n)^2}{(\alpha + n)^2 (\alpha - n)^2} - \sum_{n=-\infty}^{\infty} \frac{2\alpha n}{(\alpha^2 - n^2)^2}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{1}{(\alpha - n)^2}.
\]

(56.1.19)

Now this formula can be used to obtain a formula for \( \lambda (\tau) \). As pointed out above, \( \lambda \) depends only on the ratio \( w_2/w_1 \) and so it suffices to take \( w_1 = 1 \) and
56.1. PERIODIC FUNCTIONS

\( w_2 = \tau \). Thus

\[
\lambda (\tau) = \frac{\wp \left( \frac{1+\tau}{2} \right) - \wp \left( \frac{\tau}{2} \right)}{\wp \left( \frac{1}{2} \right) - \wp \left( \frac{\tau}{2} \right)}. \tag{56.1.20}
\]

From the original formula for \( \wp \),

\[
\wp \left( 1 + \frac{\tau}{2} \right) - \wp \left( \frac{\tau}{2} \right) = \frac{1}{(1+\tau)^2} - \frac{1}{(\tau/2)^2} + \sum_{(k,m) \neq (0,0)} \frac{1}{(k-\frac{1}{2} + (m-\frac{1}{2}) \tau)^2} - \frac{1}{(k+(m-\frac{1}{2}) \tau)^2}
\]

\[
= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{(k-\frac{1}{2} + (m-\frac{1}{2}) \tau)^2} - \frac{1}{(k+(m-\frac{1}{2}) \tau)^2}
\]

\[
= \sum_{(k,m) \in \mathbb{Z}^2} \left[ \frac{1}{(k-\frac{1}{2} + (m-\frac{1}{2}) \tau)^2} - \frac{1}{(k+(m-\frac{1}{2}) \tau)^2} \right]
\]

\[
= \sum_{(k,m) \in \mathbb{Z}^2} \left[ \frac{1}{(1/2 + (m+\frac{1}{2}) \tau - k)^2} - \frac{1}{((m+\frac{1}{2}) \tau - k)^2} \right]. \tag{56.1.21}
\]

Similarly,

\[
\wp \left( \frac{1}{2} \right) - \wp \left( \frac{\tau}{2} \right) = \frac{1}{(1/2)^2} - \frac{1}{(\tau/2)^2} + \sum_{(k,m) \neq (0,0)} \frac{1}{(k-\frac{1}{2} + m\tau)^2} - \frac{1}{(k+(m-\frac{1}{2}) \tau)^2}
\]

\[
= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{(k-\frac{1}{2} + m\tau)^2} - \frac{1}{(k+(m-\frac{1}{2}) \tau)^2}
\]

\[
= \sum_{(k,m) \in \mathbb{Z}^2} \left[ \frac{1}{(1/2 + m\tau - k)^2} - \frac{1}{((m+\frac{1}{2}) \tau - k)^2} \right]. \tag{56.1.22}
\]

Now use \( \varphi \) to sum these over \( k \). This yields,

\[
\wp \left( \frac{1+\tau}{2} \right) - \wp \left( \frac{\tau}{2} \right)
\]

\[
= \sum_m \sin^2 \left( \pi \left( \frac{1}{2} + (m+\frac{1}{2}) \tau \right) \right) - \sin^2 \left( \pi \left( m+\frac{1}{2} \tau \right) \right)
\]

\[
= \sum_m \cos^2 \left( \pi \left( m+\frac{1}{2} \tau \right) \right) - \sin^2 \left( \pi \left( m+\frac{1}{2} \tau \right) \right)
\]
and
\[
\wp \left( \frac{1}{2} \right) - \wp \left( \frac{\tau}{2} \right) = \sum_m \frac{\pi^2}{\sin^2 \left( \pi \left( \frac{1}{2} + m\tau \right) \right)} - \frac{\pi^2}{\sin^2 \left( \pi \left( m + \frac{1}{2} \right) \tau \right)}
\]
\[
= \sum_m \frac{\pi^2}{\cos^2 (\pi m \tau)} - \frac{\pi^2}{\sin^2 \left( \pi \left( m + \frac{1}{2} \right) \tau \right)}.
\]

The following interesting formula for \( \lambda \) results.
\[
\lambda (\tau) = \frac{\sum_m \cos^2 (\pi (m+\frac{1}{2}) \tau) - \sin^2 (\pi (m+\frac{1}{2}) \tau)}{\sum_m \cos^2 (\pi m \tau) - \sin^2 (\pi (m+\frac{1}{2}) \tau)}.
\]

(56.1.23)

From this it is obvious \( \lambda (-\tau) = \lambda (\tau) \). Therefore, from (56.1.18),
\[
-\lambda (\tau) + 1 = \lambda \left( \frac{1}{\tau} \right) = \lambda \left( -\frac{1}{\tau} \right)
\]

(56.1.24)

(It is good to recall that \( \lambda \) has been defined for \( \tau \not\in \mathbb{R} \).)

56.1.6 Mapping Properties Of \( \lambda \)

The two functional equations, (56.1.24) and (56.1.17), along with some other properties presented above are of fundamental importance. For convenience, they are summarized here in the following lemma.

**Lemma 56.1.18** *The following functional equations hold for \( \lambda \).*

\[
\lambda (1 + \tau) = \frac{\lambda (\tau)}{\lambda (\tau) - 1}, \quad 1 = \lambda (\tau) + \lambda \left( \frac{-1}{\tau} \right)
\]

(56.1.25)

\[
\lambda (\tau + 2) = \lambda (\tau)
\]

(56.1.26)

\[\lambda (z) = \lambda (w) \text{ if and only if there exists a unimodular matrix,} \]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2
\]

such that
\[
w = \frac{az + b}{cz + d}
\]

(56.1.27)

Consider the following picture.
In this picture, \( l_1 \) is the y axis and \( l_2 \) is the line, \( x = 1 \) while \( C \) is the top half of the circle centered at \( \left( \frac{1}{2}, 0 \right) \) which has radius 1/2. Note the above formula implies \( \lambda \) has real values on \( l_1 \) which are between 0 and 1. This is because \( 56.1.27 \) implies

\[
\lambda (ib) = \frac{\sum_m \cos^2(\pi (m + \frac{1}{2})b) - \sin^2(\pi (m + \frac{1}{2})b)}{\sum_m \cos(\pi m b) - \sin(\pi (m + \frac{1}{2})b)} = \frac{\sum_m \cos^2(\pi (m + \frac{1}{2})b) + \sin^2(\pi (m + \frac{1}{2})b)}{\sum_m \cos(\pi m b) + \sin(\pi (m + \frac{1}{2})b)} \in (0, 1). \tag{56.1.28}
\]

This follows from the observation that

\[
\cos (ix) = \cosh (x), \sin (ix) = i \sinh (x).
\]

Thus it is clear from \( 56.1.28 \) that \( \lim_{b \to 0^+} \lambda (ib) = 1. \)

Next I need to consider the behavior of \( \lambda (\tau) \) as \( \text{Im} (\tau) \to \infty. \) From \( 56.1.28 \) listed here for convenience,

\[
\lambda (\tau) = \frac{\sum_m \cos^2(\pi (m + \frac{1}{2})\tau) - \sin^2(\pi (m + \frac{1}{2})\tau)}{\sum_m \cos(\pi m \tau) - \sin(\pi (m + \frac{1}{2})\tau)}, \tag{56.1.29}
\]

it follows

\[
\lambda (\tau) = \frac{\sum_m \cos(\pi (m + \frac{1}{2})\tau) - \sin(\pi (m + \frac{1}{2})\tau) + A (\tau)}{1 + B (\tau)}
\]

\[
= \frac{2 \cos(\pi (\frac{1}{2})\tau) - \sin(\pi (\frac{1}{2})\tau) + A (\tau)}{1 + B (\tau)} \tag{56.1.30}
\]

Where \( A (\tau), B (\tau) \to 0 \) as \( \text{Im} (\tau) \to \infty. \) I took out the \( m = 0 \) term involving \( \frac{1}{\cos^2(\pi m \tau)} \) in the denominator and the \( m = -1 \) and \( m = 0 \) terms in the numerator of \( 56.1.29. \) In fact, \( e^{-i\pi (a + ib)} A (a + ib), e^{-i\pi (a + ib)} B (a + ib) \) converge to zero uniformly in \( a \) as \( b \to \infty. \)

**Lemma 56.1.19** For \( A, B \) defined in \( 56.1.28, e^{-\pi (a + ib)} C (a + ib) \to 0 \) uniformly in \( a \) for \( C = A, B. \)

**Proof:** From \( 56.1.28 \)

\[
e^{-i\pi \tau} A (\tau) = \sum_{m \neq 0, m \neq -1} e^{-i\pi \tau} \frac{\cos^2 \left( \pi (m + \frac{1}{2}) \tau \right)}{\cos^2 \left( \pi (m + \frac{1}{2}) \tau \right)} - \frac{e^{-i\pi \tau}}{\sin^2 \left( \pi (m + \frac{1}{2}) \tau \right)}
\]

Now let \( \tau = a + ib. \) Then letting \( \alpha_m = \pi \left( m + \frac{1}{2} \right), \)

\[
\cos (\alpha_m a + i\alpha_m b) = \cos (\alpha_m a) \cosh (\alpha_m b) - i \sinh (\alpha_m b) \sin (\alpha_m a)
\]

\[
\sin (\alpha_m a + i\alpha_m b) = \sin (\alpha_m a) \cosh (\alpha_m b) + i \cos (\alpha_m a) \sinh (\alpha_m b)
\]
Therefore,
\[
\cos^2(\alpha_m a + i\alpha_m b) = \cos^2(\alpha_m a) \cosh^2(\alpha_m b) + \sinh^2(\alpha_m b) \sin^2(\alpha_m a) \\
\geq \sinh^2(\alpha_m b).
\]

Similarly,
\[
\sin^2(\alpha_m a + i\alpha_m b) = \sin^2(\alpha_m a) \cosh^2(\alpha_m b) + \cos^2(\alpha_m a) \sinh^2(\alpha_m b) \\
\geq \sinh^2(\alpha_m b).
\]

It follows that for \(\tau = a + ib\) and \(b\) large
\[
|e^{-i\pi \tau} A(\tau)| \leq 2e^{\pi b} \\
\sum_{m \neq 0} \frac{2e^{\pi b}}{\sinh^2(\pi (m + \frac{1}{2}) b)} \\
\leq \sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2(\pi (m + \frac{1}{2}) b)} + \sum_{m=-\infty}^{-2} \frac{2e^{\pi b}}{\sinh^2(\pi (m + \frac{1}{2}) b)} \\
= 2 \sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2(\pi (m + \frac{1}{2}) b)} = 4 \sum_{m=1}^{\infty} \frac{e^{\pi b}}{\sinh^2(\pi (m + \frac{1}{2}) b)}
\]

Now a short computation shows
\[
\frac{e^{\pi b}}{\sinh^2(\pi (m + \frac{1}{2}) b)} = \frac{\sinh^2(\pi (m + \frac{1}{2}) b)}{\sinh^2(\pi (m + \frac{3}{2}) b)} \leq \frac{1}{e^{3\pi b}}.
\]

Therefore, for \(\tau = a + ib\),
\[
|e^{-i\pi \tau} A(\tau)| \leq 4 \frac{e^{\pi b}}{\sinh^2(\frac{3\pi b}{2})} \sum_{m=1}^{\infty} \left(\frac{1}{e^{3\pi b}}\right)^m \\
\leq 4 \frac{e^{\pi b}}{\sinh^2(\frac{3\pi b}{2})} \frac{1}{1 - \left(\frac{1}{e^{3\pi b}}\right)}
\]
which converges to zero as \(b \to \infty\). Similar reasoning will establish the claim about \(B(\tau)\). This proves the lemma.

**Lemma 56.1.20** \(\lim_{b \to \infty} \lambda(a + ib) e^{-i\pi(a + ib)} = 16 \) uniformly in \(a \in \mathbb{R}\).

**Proof:** From **6b.1.21** and Lemma **56.1.19**, this lemma will be proved if it is shown
\[
\lim_{b \to \infty} \left(\frac{2}{\cos^2(\pi (\frac{1}{2}) (a + ib))} - \frac{2}{\sin^2(\pi (\frac{1}{2}) (a + ib))}\right) e^{-i\pi(a + ib)} = 16
\]
uniformly in \( a \in \mathbb{R} \). Let \( \tau = a + ib \) to simplify the notation. Then the above expression equals

\[
\left( \frac{8}{(e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{2}})^2} + \frac{8}{(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}})^2} \right) e^{-i\pi \tau}
\]

\[
= \left( \frac{8e^{i\pi \tau}}{(e^{i\pi \tau} + 1)^2} + \frac{8e^{i\pi \tau}}{(e^{i\pi \tau} - 1)^2} \right) e^{-i\pi \tau}
\]

\[
= \frac{8}{(e^{i\pi \tau} + 1)^2} + \frac{8}{(e^{i\pi \tau} - 1)^2}
\]

\[
= 16 \frac{1 + e^{2i\pi \tau}}{(1 - e^{2i\pi \tau})^2}.
\]

Now

\[
\left| \frac{1 + e^{2i\pi \tau}}{(1 - e^{2i\pi \tau})^2} - 1 \right| = \left| \frac{1 + e^{2i\pi \tau}}{(1 - e^{2i\pi \tau})^2} - \frac{(1 - e^{2i\pi \tau})^2}{(1 - e^{2i\pi \tau})^2} \right|
\]

\[
\leq \left| \frac{3e^{2i\pi \tau} - e^{4i\pi \tau}}{(1 - e^{-2\pi \tau})^2} \right| \leq \frac{3e^{-2\pi \tau} + e^{-4\pi \tau}}{(1 - e^{-2\pi \tau})^2}
\]

and this estimate proves the lemma.

**Corollary 56.1.21** \( \lim_{b \to \infty} \lambda(a + ib) = 0 \) uniformly in \( a \in \mathbb{R} \). Also \( \lambda(ib) \) for \( b > 0 \) is real and is between 0 and 1, \( \lambda \) is real on the line, \( l_2 \) and on the curve, \( C \) and \( \lim_{b \to 0^+} \lambda(1 + ib) = -\infty \).

**Proof:** From Lemma 56.1.20

\[
\left| \lambda(a + ib) e^{-i\pi(a + ib)} - 16 \right| < 1
\]

for all \( a \) provided \( b \) is large enough. Therefore, for such \( b \),

\[
|\lambda(a + ib)| \leq 17e^{-\pi b}.
\]

proves the assertion about \( \lambda(-ib) \) real.

By the first part, \( \lim_{b \to \infty} |\lambda(ib)| = 0 \). Now from 56.1.21

\[
\lim_{b \to 0^+} \lambda(ib) = \lim_{b \to 0^+} \left( 1 - \lambda \left( \frac{-1}{ib} \right) \right) = \lim_{b \to 0^+} \left( 1 - \lambda \left( \frac{i}{b} \right) \right) = 1. \quad (56.1.31)
\]

by Corollary 56.1.21.

Next consider the behavior of \( \lambda \) on line \( l_2 \) in the above picture. From 56.1.17 and 56.1.23,

\[
\lambda(1 + ib) = \frac{\lambda(ib)}{\lambda(ib) - 1} < 0
\]
and so as $b \to 0+$ in the above, $\lambda(1 + ib) \to -\infty$.

It is left as an exercise to show that the map $\tau \to 1 - \frac{1}{\tau}$ maps $l_2$ onto the curve, $C$. Therefore, by (56.1.25), for $\tau \in l_2$,

$$\lambda \left(1 - \frac{1}{\tau}\right) = \frac{\lambda \left(\frac{-1}{\tau}\right)}{\lambda \left(\frac{-1}{\tau}\right) - 1} = \frac{1 - \lambda(\tau)}{(1 - \lambda(\tau)) - 1} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \in \mathbb{R} \quad (56.1.32)$$

It follows $\lambda$ is real on the boundary of $\Omega$ in the above picture. This proves the corollary.

Now, following Alfors [2], cut off $\Omega$ by considering the horizontal line segment, $z = a + ib_0$ where $b_0$ is very large and positive and $a \in [0, 1]$. Also cut $\Omega$ off by the images of this horizontal line, under the transformations $z = \frac{1}{\tau}$ and $z = 1 - \frac{1}{\tau}$. These are arcs of circles because the two transformations are fractional linear transformations. It is left as an exercise for you to verify these arcs are situated as shown in the following picture. The important thing to notice is that for $b_0$ large the points of these circles are close to the origin and $(1, 0)$ respectively. The following picture is a summary of what has been obtained so far on the mapping by $\lambda$.

In the picture, the descriptions are of $\lambda$ acting on points of the indicated boundary of $\Omega$. Consider the oriented contour which results from $\lambda(z)$ as $z$ moves first up $l_2$ as indicated, then along the line $z = a + ib$ and then down $l_1$ and then along $C_1$ to $C$ and along $C$ till $C_2$ and then along $C_2$ to $l_2$. As indicated in the picture, this involves going from a large negative real number to a small negative real number and then over a smooth curve which stays small to a real positive number and from there to a real number near $1$. $\lambda(z)$ stays fairly near $1$ on $C_1$ provided $b_0$ is large so that the circle, $C_1$ has very small radius. Then along $C$, $\lambda(z)$ is real until it hits $C_2$. What about the behavior of $\lambda$ on $C_2$? For $z \in C_2$, it follows from the definition of $C_2$ that $z = 1 - \frac{1}{\tau}$ where $\tau$ is on the line, $a + ib_0$. Therefore, by Lemma (56.1.20).
56.1. PERIODIC FUNCTIONS

\[ \lambda(z) = \lambda \left( 1 - \frac{1}{\tau} \right) = \frac{\lambda \left( \frac{1}{\tau} \right)}{\lambda \left( \frac{1}{\tau} \right)} - 1 = 1 - \frac{1}{\lambda(\tau)} \]

which is approximately equal to

\[ 1 - \frac{1}{16e^{\pi(a+ib_0)}} = 1 - e^{\pi b_0} e^{-ia\pi} \frac{1}{16} \]

These points are essentially on a large half circle in the upper half plane which has radius approximately \( e^{\pi b_0} \).

Now let \( w \in \mathbb{C} \) with \( \text{Im}(w) \neq 0 \). Then for \( b_0 \) large enough, the motion over the boundary of the truncated region indicated in the above picture results in \( \lambda \) tracing out a large simple closed curve oriented in the counter clockwise direction which includes \( w \) on its interior if \( \text{Im}(w) > 0 \) but which excludes \( w \) if \( \text{Im}(w) < 0 \).

**Theorem 56.1.22** Let \( \Omega \) be the domain described above. Then \( \lambda \) maps \( \Omega \) one to one and onto the upper half plane of \( \mathbb{C} \), \( \{ z \in \mathbb{C} \text{ such that } \text{Im}(z) > 0 \} \). Also, the line \( \lambda(l_1) = (0,1), \lambda(l_2) = (-\infty,0) \), and \( \lambda(C) = (1,\infty) \).

**Proof:** Let \( \text{Im}(w) > 0 \) and denote by \( \gamma \) the oriented contour described above and illustrated in the above picture. Then the winding number of \( \lambda \circ \gamma \) about \( w \) equals 1. Thus

\[ \frac{1}{2\pi i} \int_{\lambda \circ \gamma} \frac{1}{z - w} dz = 1. \]

But, splitting the contour integrals into \( l_2, \) the top line, \( l_1, C_1, C \), and \( C_2 \) and changing variables on each of these, yields

\[ 1 = \frac{1}{2\pi i} \int_{\gamma} \frac{\lambda'(z)}{\lambda(z) - w} dz \]

and by the theorem on counting zeros, Theorem 50.6.1 on Page 1701, the function, \( z \to \lambda(z) - w \) has exactly one zero inside the truncated \( \Omega \). However, this shows this function has exactly one zero inside \( \Omega \) because \( b_0 \) was arbitrary as long as it is sufficiently large. Since \( w \) was an arbitrary element of the upper half plane, this verifies the first assertion of the theorem. The remaining claims follow from the above description of \( \lambda \), in particular the estimate for \( \lambda \) on \( C_2 \). This proves the theorem.

Note also that the argument in the above proof shows that if \( \text{Im}(w) < 0 \), then \( w \) is not in \( \lambda(\Omega) \). However, if you consider the reflection of \( \Omega \) about the \( y \) axis, then it will follow that \( \lambda \) maps this set one to one onto the lower half plane. The argument will make significant use of Theorem 50.6.3 on Page 1703 which is stated here for convenience.
Theorem 56.1.23 Let \( f : B(a, R) \to \mathbb{C} \) be analytic and let\[
f(z) - \alpha = (z - a)^m g(z), \quad \infty > m \geq 1\]
where \( g(z) \neq 0 \) in \( B(a, R) \). \((f(z) - \alpha \) has a zero of order \( m \) at \( z = a \). Then there exist \( \varepsilon, \delta > 0 \) with the property that for each \( z \) satisfying \( 0 < |z - \alpha| < \delta \), there exist points,
\[
\{a_1, \ldots, a_m\} \subseteq B(a, \varepsilon),
\]
such that\[
f^{-1}(z) \cap B(a, \varepsilon) = \{a_1, \ldots, a_m\}
\]
and each \( a_k \) is a zero of order 1 for the function \( f(z) - \alpha \).

Corollary 56.1.24 Let \( \Omega \) be the region above. Consider the set of points, \( Q = \overline{\Omega} \cup \Omega' \setminus \{0, 1\} \) described by the following picture.

Then \( \lambda(Q) = \mathbb{C} \setminus \{0, 1\} \). Also \( \lambda'(z) \neq 0 \) for every \( z \) in \( \bigcup_{k=-\infty}^{\infty} (Q + 2k) \equiv H \).

Proof: By Theorem 56.1.22 this will be proved if it can be shown that \( \lambda(\Omega') = \{z \in \mathbb{C} : \text{Im}(z) < 0\} \). Consider \( \lambda_1 \) defined on \( \Omega' \) by
\[
\lambda_1(x + iy) \equiv \overline{\lambda(-x + iy)}.
\]

Claim: \( \lambda_1 \) is analytic.

Proof of the claim: You just verify the Cauchy Riemann equations. Letting \( \lambda(x + iy) = u(x, y) + iv(x, y) \),
\[
\lambda_1(x + iy) = u(-x, y) - iv(-x, y) \equiv u_1(x, y) + iv(x, y).
\]

Then \( u_{1x}(x, y) = -u_x(-x, y) \) and \( v_{1y}(x, y) = -v_y(-x, y) = -u_x(-x, y) \) since \( \lambda \) is analytic. Thus \( u_{1x} = v_{1y} \). Next, \( u_{1y}(x, y) = u_y(-x, y) \) and \( v_{1x}(x, y) = v_x(-x, y) = -u_y(-x, y) \) and so \( u_{1y} = -v_x \).

Now recall that on \( l_1 \), \( \lambda \) takes real values. Therefore, \( \lambda_1 = \lambda \) on \( l_1 \), a set with a limit point. It follows \( \lambda = \lambda_1 \) on \( \Omega' \cup \Omega \). By Theorem 56.1.22 \( \lambda \) maps \( \Omega \) one to one onto the upper half plane. Therefore, from the definition of \( \lambda_1 = \lambda \), it follows \( \lambda \) maps \( \Omega' \) one to one onto the lower half plane as claimed. This has shown that \( \lambda \)
is one to one on $\Omega \cup \Omega'$. This also verifies from Theorem 56.1.3 on Page 1889 that $\lambda' \neq 0$ on $\Omega \cup \Omega'$.

Now consider the lines $l_2$ and $C$. If $\lambda'(z) = 0$ for $z \in l_2$, a contradiction can be obtained. Pick such a point. If $\lambda'(z) = 0$, then $z$ is a zero of order $m \geq 2$ of the function, $\lambda - \lambda(z)$. Then by Theorem 56.1.3 there exist $\delta, \varepsilon > 0$ such that if $w \in B(\lambda(z), \delta)$, then $\lambda^{-1}(w) \cap B(z, \varepsilon)$ contains at least $m$ points.

In particular, for $z_1 \in \Omega \cap B(z, \varepsilon)$ sufficiently close to $z, \lambda(z_1) \in B(\lambda(z), \delta)$ and so the function $\lambda - \lambda(z_1)$ has at least two distinct zeros. These zeros must be in $B(z, \varepsilon) \cap \Omega$ because $\lambda(z_1)$ has positive imaginary part and the points on $l_2$ are mapped by $\lambda$ to a real number while the points of $B(z, \varepsilon) \setminus \overline{\Omega}$ are mapped by $\lambda$ to the lower half plane thanks to the relation, $\lambda(z + 2) = \lambda(z)$. This contradicts $\lambda$ one to one on $\Omega$. Therefore, $\lambda' \neq 0$ on $l_2$. Consider $C$. Points on $C$ are of the form $1 - \frac{1}{\tau}$ where $\tau \in l_2$. Therefore, using 56.1.33,

$$
\lambda \left(1 - \frac{1}{\tau}\right) = \frac{\lambda(\tau) - 1}{\lambda(\tau)}.
$$

Taking the derivative of both sides,

$$
\lambda' \left(1 - \frac{1}{\tau}\right) \left(\frac{1}{\tau^2}\right) = \frac{\lambda'(\tau)}{\lambda(\tau)} \neq 0.
$$

Since $\lambda$ is periodic of period 2 it follows $\lambda'(z) \neq 0$ for all $z \in \cup_{k=-\infty}^{\infty} (Q + 2k)$.

**Lemma 56.1.25** If $\text{Im} (\tau) > 0$ then there exists a unimodular $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$
\frac{c + d\tau}{a + b\tau}.
$$
is contained in the interior of $Q$. In fact, $\left| \frac{c+d\tau}{a+b\tau} \right| \geq 1$ and

$$-1/2 \leq \text{Re} \left( \frac{c+d\tau}{a+b\tau} \right) \leq 1/2.$$ 

**Proof:** Letting a basis for the module of periods of $\wp$ be $\{1, \tau\}$, it follows from Theorem 56.1.4 on Page 1862 that there exists a basis for the same module of periods, $\{w'_1, w'_2\}$ with the property that for $\tau' = w'_2/w'_1$

$$|\tau'| \geq 1, \quad -1/2 \leq \text{Re} \tau' \leq 1/2.$$ 

Since this is a basis for the same module of periods, there exists a unimodular matrix, \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) such that

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}.$$ 

Hence,

$$\tau' = \frac{w'_2}{w'_1} = \frac{c+d\tau}{a+b\tau}.$$ 

Thus $\tau'$ is in the interior of $H$. In fact, it is on the interior of $\Omega' \cup \Omega \equiv Q$.

**56.1.7 A Short Review And Summary**

With this lemma, it is easy to extend Corollary 56.1.24. First, a simple observation and review is a good idea. Recall that when you change the basis for the module of periods, the Weierstrass $\wp$ function does not change and so the set of $e_i$ used in defining $\lambda$ also do not change. Letting the new basis be $\{w'_1, w'_2\}$, it was shown that

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$
56.1. PERIODIC FUNCTIONS

for some unimodular transformation, \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Letting \( \tau = w_2/w_1 \) and \( \tau' = w'_2/w'_1 \)

\[
\tau' = \frac{c + d\tau}{a + b\tau} \equiv \phi(\tau)
\]

Now as discussed earlier

\[
\lambda(\tau') = \lambda(\phi(\tau)) = \frac{\varphi \left( \frac{w_1' + w_2'}{2} \right) - \varphi \left( \frac{w_2'}{2} \right)}{\varphi \left( \frac{w_1}{2} \right) - \varphi \left( \frac{w'_2}{2} \right)}
\]

These numbers in the above fraction must be the same as \( \varphi \left( \frac{1+\tau'}{2} \right) \), \( \varphi \left( \frac{\tau'}{2} \right) \), and \( \varphi \left( \frac{1}{2} \right) \) but they might occur differently. This is because \( \varphi \) does not change and these numbers are the zeros of a polynomial having coefficients involving only numbers and \( \varphi(z) \). It could happen for example that \( \varphi \left( \frac{1+\tau'}{2} \right) = \varphi \left( \frac{\tau'}{2} \right) \) in which case this would change the value of \( \lambda \). In effect, you can keep track of all possibilities by simply permuting the \( e_i \) in the formula for \( \lambda(\tau) \) given by \( \frac{e_3 - e_2}{e_1 - e_2} \). Thus consider the following permutation table.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Corresponding to this list of 6 permutations, all possible formulas for \( \lambda(\phi(\tau)) \) can be obtained as follows. Letting \( \tau' = \phi(\tau) \) where \( \phi \) is a unimodular matrix corresponding to a change of basis, \( \lambda(\tau') = \frac{e_3 - e_2}{e_1 - e_2} = \lambda(\tau) \) \( (56.1.34) \)

\[
\lambda(\tau') = \frac{e_1 - e_3}{e_2 - e_3} = \frac{e_3 - e_2 + e_2 - e_1}{e_3 - e_2} = 1 - \frac{1}{\lambda(\tau)} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \quad (56.1.35)
\]

\[
\lambda(\tau') = e_2 - e_1 e_3 - e_1 = -\left[ \frac{e_3 - e_2 - (e_1 - e_2)}{e_1 - e_2} \right]^{-1} = -[\lambda(\tau) - 1]^{-1} = \frac{1}{1 - \lambda(\tau)} \quad (56.1.36)
\]

\[
\lambda(\tau') = \frac{e_3 - e_1}{e_2 - e_1} = -\left[ \frac{e_3 - e_2 - (e_1 - e_2)}{e_1 - e_2} \right] = -[\lambda(\tau) - 1] = 1 - \lambda(\tau) \quad (56.1.37)
\]
Corollary 56.1.26 \( \lambda'(\tau) \neq 0 \) for all \( \tau \) in the upper half plane, denoted by \( P_+ \).

Proof: Let \( \tau \in P_+ \). By Lemma 56.1.39 there exists \( \phi \) a unimodular transformation and \( \tau' \) in the interior of \( Q \) such that \( \tau' = \phi(\tau) \). Now from the definition of \( \lambda \) in terms of the \( e_i \), there is at worst a permutation of the \( e_i \) and so it might be the case that \( \lambda(\phi(\tau)) \neq \lambda(\tau) \) but it is the case that \( \lambda(\phi(\tau)) = \xi(\lambda(\tau)) \) where \( \xi'(z) \neq 0 \). Here \( \xi \) is one of the functions determined by 56.1.28 and its corollary on Page 56.1.34. \( \xi \) and these are the same as the possibilities for \( \xi^{-1} \). Therefore, \( \lambda'(\phi(\tau))\phi'(\tau) = \xi'(\lambda(\tau))\lambda'(\tau) \) and so \( \lambda'(\tau) \neq 0 \) as claimed.

Now I will present a lemma which is of major significance. It depends on the remarkable mapping properties of the modular function and the monodromy theorem from analytic continuation. A review of the monodromy theorem will be listed here for convenience. First recall the definition of the concept of function elements and analytic continuation.

Definition 56.1.27 A function element is an ordered pair, \((f, D)\) where \( D \) is an open ball and \( f \) is analytic on \( D \). \((f_0, D_0)\) and \((f_1, D_1)\) are direct continuations of each other if \( D_1 \cap D_0 \neq \emptyset \) and \( f_0 = f_1 \) on \( D_1 \cap D_0 \). In this case I will write \((f_0, D_0) \sim (f_1, D_1)\). A chain is a finite sequence, of disks, \( \{D_0, \ldots , D_n\} \) such that \( D_{i-1} \cap D_i \neq \emptyset \). If \((f_0, D_0)\) is a given function element and there exist function elements, \((f_i, D_i)\) such that \( \{D_0, \ldots , D_n\} \) is a chain and \((f_{j-1}, D_{j-1}) \sim (f_j, D_j)\) then \((f_n, D_n)\) is called the analytic continuation of \((f_0, D_0)\) along the chain \( \{D_0, \ldots , D_n\} \). Now suppose \( \gamma \) is an oriented curve with parameter interval \([a, b]\) and there exists a chain, \( \{D_0, \ldots , D_n\} \) such that \( \gamma^* \subseteq \bigcup_{k=1}^n D_k \), \( \gamma(a) \) is the center of \( D_0 \), \( \gamma(b) \) is the center of \( D_n \), and there is an increasing list of numbers in \([a, b]\), \( a = s_0 < s_1 \cdots < s_n = b \) such that \( \gamma([s_i, s_{i+1}]) \subseteq D_i \) and \((f_n, D_n)\) is an analytic continuation of \((f_0, D_0)\) along \( D_n \). Then \((f_n, D_n)\) is called an analytic continuation of \((f_0, D_0)\) along the curve \( \gamma \). (\( \gamma \) will always be a continuous curve. Nothing more is needed.)

Then the main theorem is the monodromy theorem listed next, Theorem 56.1.28 and its corollary on Page 56.1.29.
56.1. PERIODIC FUNCTIONS

Here is the lemma.

**Lemma 56.1.29** Let $\lambda$ be the modular function defined on $P_+$, the upper half plane. Let $V$ be a simply connected region in $\mathbb{C}$ and let $f : V \to \mathbb{C} \setminus \{0,1\}$ be analytic and nonconstant. Then there exists an analytic function, $g : V \to P_+$ such that $\lambda \circ g = f$.

**Proof:** Let $a \in V$ and choose $r_0$ small enough that $f(B(a,r_0))$ contains neither 0 nor 1. You need only let $B(a,r_0) \subseteq V$. Now there exists a unique point in $Q, \lambda_0$ such that $\lambda(\lambda_0) = f(a)$. By Corollary 56.1.28 $\lambda'(r_0) \neq 0$ and so by the open mapping theorem, Theorem 56.1.24 on Page 1882. There exists $B(\tau_0, R_0) \subseteq P_+$ such that $\lambda$ is one to one on $B(\tau_0, R_0)$ and has a continuous inverse. Then picking $r_0$ still smaller, it can be assumed $f(B(a,r_0)) \subseteq \lambda(B(\tau_0, R_0))$. Thus there exists a local inverse for $\lambda$, $\lambda_0^{-1}$ defined on $f(B(a,r_0))$ having values in $B(\tau_0, R_0) \cap \lambda^{-1}(f(B(a,r_0)))$. Then defining $g_0 \equiv \lambda_0^{-1} \circ f$, $(g_0, B(a,r_0))$ is a function element. I need to show this can be continued along every curve starting at $a$ in such a way that each function in each function element has values in $P_+$.

Let $\gamma : [\alpha, \beta] \to V$ be a continuous curve starting at $a, (\gamma(\alpha) = a)$ and suppose that if $t < T$ there exists a nonnegative integer $m$ and a function element $(g_m, B(\gamma (t), r_m))$ which is an analytic continuation of $(g_0, B(a,r_0))$ along $\gamma$ where $g_m(\gamma(t)) \in P_+$ and each function in every function element for $j \leq m$ has values in $P_+$. Thus for some small $T > 0$ this has been achieved.

Then consider $f(\gamma(T)) \in \mathbb{C} \setminus \{0,1\}$. As in the first part of the argument, there exists a unique $\tau_T \in Q$ such that $\lambda(\tau_T) = f(\gamma(T))$ and for $r$ small enough there is an analytic local inverse, $\lambda_T^{-1}$ between $f(B(\gamma(T),r))$ and $\lambda^{-1}(f(B(\gamma(T),r))) \cap B(\tau_T, R_T) \subseteq P_+$ for some $R_T > 0$. By the assumption that the analytic continuation can be carried out for $t < T$, there exists $\{t_0, \cdots, t_m = t\}$ and function elements $(g_j, B(\gamma (t_j), r_j)), j = 0, \cdots, m$ as just described with $g_j(\gamma(t_j)) \in P_+, \lambda \circ g_j = f$ on $B(\gamma (t_j), r_j)$ such that for $t \in [t_m, T], \gamma(t) \in B(\gamma(T), r)$. Let

$$I = B(\gamma(t_m), r_m) \cap B(\gamma(T), r).$$

Then since $\lambda_T^{-1}$ is a local inverse, it follows for all $z \in I$

$$\lambda(g_m(z)) = f(z) = \lambda(\lambda_T^{-1} \circ f(z)).$$

Pick $z_0 \in I$. Then by Lemma 56.1.18 on Page 1884 there exists a unimodular mapping of the form

$$\phi(z) = \frac{az + b}{cz + d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2$$

such that

$$g_m(z_0) = \phi(\lambda_T^{-1} \circ f(z_0)).$$

Since both $g_m(z_0)$ and $\phi(\lambda_T^{-1} \circ f(z_0))$ are in the upper half plane, it follows $ad - cb = 1$ and $\phi$ maps the upper half plane to the upper half plane. Note the pole of
\( \phi \) is real and all the sets being considered are contained in the upper half plane so \( \phi \) is analytic where it needs to be.

**Claim:** For all \( z \in I \),

\[
g_m(z) = \phi \circ \lambda^{-1} \circ f(z). \tag{56.1.40}
\]

**Proof:** For \( z = z_0 \) the equation holds. Let

\[
A = \{ z \in I : g_m(z) = \phi (\lambda^{-1} \circ f(z)) \}.
\]

Thus \( z_0 \in I \). If \( z \in I \) and if \( w \) is close enough to \( z \), then \( w \in I \) also and so both sides of (56.1.40) with \( w \) in place of \( z \) are in \( \lambda^{-1}_m(f(I)) \). But by construction, \( \lambda \) is one to one on this set and since \( \lambda \) is invariant with respect to \( \phi \),

\[
\lambda(g_m(w)) = \lambda(\lambda^{-1}_T \circ f(w)) = \lambda(\phi \circ \lambda^{-1}_T \circ f(w))
\]

and consequently, \( w \in A \). This shows \( A \) is open. But \( A \) is also closed in \( I \) because the functions are continuous. Therefore, \( A = I \) and so 56.1.40 is obtained.

Letting \( f(z) \in f(B(\gamma(T)), r) \),

\[
\lambda(\phi(\lambda^{-1}_T(f(z)))) = \lambda(\lambda^{-1}_T(f(z))) = f(z)
\]

and so \( \phi \circ \lambda^{-1}_T \) is a local inverse for \( \lambda \) on \( f(B(\gamma(T)), r) \). Let the new function be \( \phi \circ \lambda^{-1}_T \circ f, B(\gamma(T), r) \). This has shown the initial function element can be continued along every curve through \( a \).

By the monodromy theorem, there exists \( g \) analytic on \( V \) such that \( g \) has values in \( P_r \) and \( g = g_0 \) on \( B(a, r_0) \). By the construction, it also follows \( \lambda \circ g = f \). This last claim is easy to see because \( \lambda \circ g = f \) on \( B(a, r_0) \), a set with a limit point so the equation holds for all \( z \in V \). This proves the lemma.

### 56.2 The Picard Theorem Again

Having done all this work on the modular function which is important for its own sake, there is an easy proof of the Picard theorem. In fact, this is the way Picard did it in 1879. I will state it slightly differently since it is no trouble to do so, [56].

**Theorem 56.2.1** Let \( f \) be meromorphic on \( \mathbb{C} \) and suppose \( f \) misses three distinct points, \( a, b, c \). Then \( f \) is a constant function.

**Proof:** Let \( \phi(z) = \frac{z-a}{z-b} \). Then \( \phi(c) = \infty, \phi(a) = 0 \), and \( \phi(b) = 1 \). Now consider the function, \( h = \phi \circ f \). Then \( h \) misses the three points \( \infty, 0, \) and \( 1 \). Since \( h \) is meromorphic and does not have \( \infty \) in its values, it must actually be analytic. Thus \( h \) is an entire function which misses the two values \( 0 \) and \( 1 \). If \( h \) is not constant, then by Lemma 56.1.29 there exists a function, \( g \) analytic on \( \mathbb{C} \) which has values
56.3. **EXERCISES**

in the upper half plane, \( P_+ \) such that \( \lambda \circ g = h \). However, \( g \) must be a constant because there exists \( \psi \) an analytic map on the upper half plane which maps the upper half plane to \( B(0, 1) \). You can use the Riemann mapping theorem or more simply, \( \psi(z) = \frac{z-i}{z+i} \). Thus \( \psi \circ g \) equals a constant by Liouville’s theorem. Hence \( g \) is a constant and so \( h \) must also be a constant because \( \lambda(g(z)) = h(z) \). This proves \( f \) is a constant also. This proves the theorem.

56.3 **Exercises**

1. Show the set of modular transformations is a group. Also show those modular transformations which are congruent mod 2 to the identity as described above is a subgroup.

2. Suppose \( f \) is an elliptic function with period module \( M \). If \( \{w_1, w_2\} \) and \( \{w'_1, w'_2\} \) are two bases, show that the resulting period parallelograms resulting from the two bases have the same area.

3. Given a module of periods with basis \( \{w_1, w_2\} \) and letting a typical element of this module be denoted by \( w \) as described above, consider the product

\[
\sigma(z) = z \prod_{w \neq 0} \left( 1 - \frac{z}{w} \right) e^{(z/w)+\frac{1}{2}(z/w)^2}.
\]

Show this product converges uniformly on compact sets, is an entire function, and satisfies

\[
\frac{\sigma'(z)}{\sigma(z)} = \zeta(z)
\]

where \( \zeta(z) \) was defined above as a primitive of \( \wp(z) \) and is given by

\[
\zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w}.
\]

4. Show \( \zeta(z + w_i) = \zeta(z) + \eta_i \) where \( \eta_i \) is a constant.

5. Let \( P_a \) be the parallelogram shown in the following picture.
Show that \( \frac{1}{2\pi i} \int_{\partial P} \zeta(z) \, dz = 1 \) where the contour is taken once around the parallelogram in the clockwise direction. Next evaluate this contour integral directly to obtain Legendre’s relation,
\[ \eta_1 w_2 - \eta_2 w_1 = 2\pi i. \]

6. For \( \sigma \) defined in Problem 3\# explain the following steps. For \( j = 1, 2 \)
\[ \frac{\sigma'(z + w_j)}{\sigma(z + w_j)} = \zeta(z + w_j) = \zeta(z) + \eta_j = \frac{\sigma'(z)}{\sigma(z)} + \eta_j \]
Therefore, there exists a constant, \( C_j \) such that
\[ \sigma(z + w_j) = C_j \sigma(z) e^{\eta_j z}. \]
Next show \( \sigma \) is an odd function, \( (\sigma(-z) = -\sigma(z)) \) and then let \( z = -w_j/2 \) to find \( C_j = -e^{\eta_j w_j/2} \) and so
\[ \sigma(z + w_j) = -\sigma(z) e^{\alpha_j (z + w_j/2)}. \] (56.3.41)

7. Show any even elliptic function, \( f \) with periods \( w_1 \) and \( w_2 \) for which 0 is neither a pole nor a zero can be expressed in the form
\[ f(0) \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)} \]
where \( C \) is some constant. Here \( \wp \) is the Weierstrass function which comes from the two periods, \( w_1 \) and \( w_2 \). **Hint:** You might consider the above function in terms of the poles and zeros on a period parallelogram and recall that an entire function which is elliptic is a constant.

8. Suppose \( f \) is any elliptic function with \( \{w_1, w_2\} \) a basis for the module of periods. Using Theorem 56.1.8 and 56.3.11 show that there exists constants \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) such that for some constant \( C \),
\[ f(z) = C \prod_{k=1}^{n} \frac{\sigma(z - a_k)}{\sigma(z - b_k)}. \]
**Hint:** You might try something like this: By Theorem 56.1.8, it follows that if \( \{\alpha_k\} \) are the zeros and \( \{b_k\} \) the poles in an appropriate period parallelogram, \( \sum \alpha_k - \sum b_k \) equals a period. Replace \( \alpha_k \) with \( a_k \) such that \( \sum a_k - \sum b_k = 0 \). Then use 56.3.11 to show that the given formula for \( f \) is bi periodic. Anyway, you try to arrange things such that the given formula has the same poles as \( f \). Remember an entire elliptic function equals a constant.

9. Show that the map \( \tau \to 1 - \frac{1}{\tau} \) maps \( l_2 \) onto the curve, \( C \) in the above picture on the mapping properties of \( \lambda \).

10. Modify the proof of Theorem 56.1.22 to show that \( \lambda(\Omega) \cap \{z \in \mathbb{C} : \text{Im}(z) < 0\} = \emptyset \).
Part VI

Topics In Probability
Chapter 57

Basic Probability

Caution: This material on probability and stochastic processes may be half baked in places. I have not yet rewritten it several times. This is not to say that nothing else is half baked. However, the probability is higher here.

57.1 Random Variables And Independence

Recall Lemma 12.2.3 on Page 367 which is stated here for convenience.

Lemma 57.1.1 Let $M$ be a metric space with the closed balls compact and suppose $\lambda$ is a measure defined on the Borel sets of $M$ which is finite on compact sets. Then there exists a unique Radon measure, $\bar{\lambda}$ which equals $\lambda$ on the Borel sets. In particular $\lambda$ must be both inner and outer regular on all Borel sets.

Also important is the following fundamental result which is called the Borel Cantelli lemma.

Lemma 57.1.2 Let $(\Omega, F, \lambda)$ be a measure space and let $\{A_i\}$ be a sequence of measurable sets satisfying

$$\sum_{i=1}^{\infty} \lambda(A_i) < \infty.$$ 

Then letting $S$ denote the set of $\omega \in \Omega$ which are in infinitely many $A_i$, it follows $S$ is a measurable set and $\lambda(S) = 0$.

Proof: $S = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_m$. Therefore, $S$ is measurable and also

$$\lambda(S) \leq \lambda(\bigcup_{m=k}^{\infty} A_m) \leq \sum_{m=k}^{\infty} \lambda(A_m)$$

and this converges to 0 as $k \to \infty$ because of the convergence of the series. ■

Here is another nice observation.
Proposition 57.1.3 Suppose $E_i$ is a separable Banach space. Then if $B_i$ is a Borel set of $E_i$, it follows $\prod_{i=1}^n B_i$ is a Borel set in $\prod_{i=1}^n E_i$.

Proof: An easy way to do this is to consider the projection maps.

$$\pi_i x \equiv x_i$$

Then these projection maps are continuous. Hence for $U$ an open set,

$$\pi_i^{-1}(U) \equiv \prod_{j=1}^n A_j, \ A_j = E_j \text{ if } j \neq i \text{ and } A_i = U.$$ 

Thus $\pi_i^{-1}(\text{open})$ equals an open set. Let

$$S \equiv \{ V \subseteq \mathbb{R} : \pi_i^{-1}(V) \text{ is Borel} \}$$

Then $S$ contains all the open sets and is clearly a $\sigma$ algebra. Therefore, $S$ contains the Borel sets. Let $B_i$ be a Borel set in $E_i$. Then

$$\prod_{i=1}^n B_i = \cap_{i=1}^n \pi_i^{-1}(B_i),$$

a finite intersection of Borel sets. \qed

Definition 57.1.4 A probability space is a measure space, $(\Omega, \mathcal{F}, P)$ where $P$ is a measure satisfying $P(\Omega) = 1$. A random vector (variable) is a measurable function, $X : \Omega \to Z$ where $Z$ is some topological space. It is often the case that $Z$ will equal $\mathbb{R}^p$. Assume $Z$ is a separable Banach space. Define the following $\sigma$ algebra.

$$\sigma(X) \equiv \{ X^{-1}(E) : E \text{ is Borel in } Z \}$$

Thus $\sigma(X) \subseteq \mathcal{F}$. For $E$ a Borel set in $Z$ define

$$\lambda_X(E) \equiv P(X^{-1}(E)).$$

This is called the distribution of the random variable, $X$. If

$$\int_{\Omega} |X(\omega)| \, dP < \infty$$

then define

$$E(X) \equiv \int_{\Omega} X \, dP$$

where the integral is defined as the Bochner integral.

Recall the following fundamental result which was proved earlier but which I will give a short proof of now.
**Proposition 57.1.5** Let \((\Omega, S, \mu)\) be a measure space and let \(X : \Omega \to Z\) where \(Z\) is a separable Banach space. Then \(X\) is strongly measurable if and only if \(X^{-1}(U) \in S\) for all \(U\) open in \(Z\).

**Proof:** To begin with, let \(D(a, r)\) be the closure of the open ball \(B(a, r)\). By Lemma 57.1.6, there exists \(\{f_i\} \subseteq B'\), the unit ball in \(Z'\) such that
\[
||z||_Z = \sup_i |f_i(z)|
\]
Then
\[
D(a, r) = \{z : ||a - z|| \leq r\} = \bigcap_i \{z : |f_i(z) - f_i(a)| \leq r\}
= \bigcap_i f_i^{-1}(B(f_i(a), r))
\]
It follows that
\[
X^{-1}(D(a, r)) = \bigcap_i X^{-1}(f_i^{-1}(B(f_i(a), r)))
= \bigcap_i (f_i \circ X)^{-1}(B(f_i(a), r))
\]
If \(X\) is strongly measurable, then it is weakly measurable and so each \(f_i \circ X\) is a real (complex) valued measurable function. Hence the expression on the right in the above is measurable. Now if \(U\) is any open set in \(Z\), then it is the countable union of such closed disks \(U = \bigcup_i D_i\). Therefore, \(X^{-1}(U) = \bigcap_i X^{-1}(D_i) \in S\). It follows that strongly measurable implies inverse images of open sets are in \(S\).

Conversely, suppose \(X^{-1}(U) \in S\) for every open \(U\). Then for \(f \in Z', f \circ X\) is real valued and measurable. Therefore, \(X\) is weakly measurable. By the Pettis theorem, it follows that \(f \circ X\) is strongly measurable. \(\blacksquare\)

**Proposition 57.1.6** If \(X : \Omega \to Z\) is measurable, then \(\sigma(X)\) equals the smallest \(\sigma\) algebra such that \(X\) is measurable with respect to it. Also if \(X_i\) are random variables having values in separable Banach spaces \(Z_i\), then \(\sigma(X) = \sigma(X_1, \cdots, X_n)\) where \(X\) is the vector mapping \(\Omega\) to \(\prod_{i=1}^n Z_i\) and \(\sigma(X_1, \cdots, X_n)\) is the smallest \(\sigma\) algebra such that each \(X_i\) is measurable with respect to it.

**Proof:** Let \(\mathcal{G}\) denote the smallest \(\sigma\) algebra such that \(X\) is measurable with respect to this \(\sigma\) algebra. By definition \(X^{-1}(open) \in \mathcal{G}\). Furthermore, the set of all \(E\) such that \(X^{-1}(E) \in \mathcal{G}\) is a \(\sigma\) algebra. Hence it includes all the Borel sets. Hence \(X^{-1}(Borel) \in \mathcal{G}\) and so \(\mathcal{G} \supseteq \sigma(X)\). However, \(\sigma(X)\) defined above is a \(\sigma\) algebra such that \(X\) is measurable with respect to \(\sigma(X)\). Therefore, \(\mathcal{G} = \sigma(X)\).

Letting \(B_i\) be a Borel set in \(Z_i\), \(\prod_{i=1}^n B_i\) is a Borel set by Proposition 57.1.5 and so
\[
X^{-1}\left(\prod_{i=1}^n B_i\right) = \bigcap_{i=1}^n X_i^{-1}(B_i) \in \sigma(X_1, \cdots, X_n)
\]
If \(\mathcal{G}\) denotes the Borel sets \(F \subseteq \prod_{i=1}^n Z_i\) such that \(X^{-1}(F) \in \sigma(X_1, \cdots, X_n)\), then \(\mathcal{G}\) is clearly a \(\sigma\) algebra which contains the open sets. Hence \(\mathcal{G} = \mathcal{B}\) the Borel sets of...
\[ \prod_{i=1}^{n} Z_i. \] This shows that \( \sigma(X) \subseteq \sigma(X_1, \cdots, X_n) \). Next we observe that \( \sigma(X) \) is a \( \sigma \)-algebra with the property that each \( X_i \) is measurable with respect to \( \sigma(X) \). This follows from \( X_i^{-1}(B_i) = X^{-1}\left(\bigcap_{j=1}^{n} A_j\right) \in \sigma(X) \), where each \( A_j = Z_j \) except for \( A_i = B_i \). Since \( \sigma(X_1, \cdots, X_n) \) is defined as the smallest such \( \sigma \)-algebra, it follows that \( \sigma(X) \supseteq \sigma(X_1, \cdots, X_n) \).

For random variables having values in a separable Banach space or even more generally for a separable metric space, much can be said about regularity of \( \lambda_X \).

**Definition 57.1.7** A measure, \( \mu \) defined on \( \mathcal{B}(E) \) will be called **inner regular** if for all \( F \in \mathcal{B}(E) \),

\[
\mu(F) = \sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ is closed} \}
\]

A measure, \( \mu \) defined on \( \mathcal{B}(E) \) will be called **outer regular** if for all \( F \in \mathcal{B}(E) \),

\[
\mu(F) = \inf \{ \mu(V) : V \supseteq F \text{ and } V \text{ is open} \}
\]

When a measure is both inner and outer regular, it is called **regular**.

For probability measures, the above definition of regularity tends to come free. Note it is a little weaker than the usual definition of regularity because \( K \) is only assumed to be closed, not compact.

**Lemma 57.1.8** Let \( \mu \) be a finite measure defined on \( \mathcal{B}(E) \) where \( E \) is a metric space. Then \( \mu \) is regular.

**Proof**: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let \( V \) be an open set and let

\[ K_k \equiv \{ x \in V : \text{dist}(x, V^C) \geq 1/k \}. \]

Then clearly the union of the \( K_k \) equals \( V \). Next, for \( K \) closed let

\[ V_k \equiv \{ x \in E : \text{dist}(x, K) < 1/k \}. \]

Clearly the intersection of the \( V_k \) equals \( K \). Therefore, letting \( V \) denote an open set and \( K \) a closed set,

\[
\mu(V) = \sup \{ \mu(K) : K \subseteq V \text{ and } K \text{ is closed} \}
\]

\[
\mu(K) = \inf \{ \mu(V) : V \supseteq K \text{ and } V \text{ is open} \}.
\]

Also since \( V \) is open and \( K \) is closed,

\[
\mu(V) = \inf \{ \mu(U) : U \supseteq V \text{ and } V \text{ is open} \}
\]

\[
\mu(K) = \sup \{ \mu(L) : L \subseteq K \text{ and } L \text{ is closed} \}.
\]

In words, \( \mu \) is regular on open and closed sets. Let

\[ \mathcal{F} \equiv \{ F \in \mathcal{B}(E) \text{ such that } \mu \text{ is regular on } F \}. \]
57.1. RANDOM VARIABLES AND INDEPENDENCE

Then $\mathcal{F}$ contains the open sets and the closed sets.

Suppose $F \in \mathcal{F}$. Then there exists $V \supseteq F$ with $\mu(V \setminus F) < \varepsilon$. It follows $V^c \subseteq F^c$ and

$$\mu(F^c \setminus V^c) = \mu(V \setminus F) < \varepsilon.$$ 

Thus $F^c$ is inner regular. Since $F \in \mathcal{F}$, there exists $K \subseteq F$ where $K$ is closed and $\mu(F \setminus K) < \varepsilon$. Then also $K^c \supseteq F^c$ and

$$\mu(K^c \setminus F^c) = \mu(F \setminus K) < \varepsilon.$$ 

Thus if $F \in \mathcal{F}$ so is $F^c$.

Suppose now that \{\(F_i\)\} \(\subseteq \mathcal{F}\), the $F_i$ being disjoint. Is $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$? There exists $K_i \subseteq F_i$ such that $\mu(K_i) + \varepsilon/2^i > \mu(F_i)$. Then

$$\mu(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i) \leq \varepsilon + \sum_{i=1}^{\infty} \mu(K_i)$$

$$< 2\varepsilon + \sum_{i=1}^{N} \mu(K_i) = 2\varepsilon + \mu(\bigcup_{i=1}^{N} K_i)$$

provided $N$ is large enough. Thus it follows $\bigcup_{i=1}^{\infty} F_i$ is inner regular. Why is it outer regular? Let $V_i \supseteq F_i$ such that $\mu(F_i) + \varepsilon/2^i > \mu(V_i)$ and

$$\mu(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i) > -\varepsilon + \sum_{i=1}^{\infty} \mu(V_i) \geq -\varepsilon + \mu(\bigcup_{i=1}^{N} V_i)$$

which shows $\bigcup_{i=1}^{\infty} F_i$ is outer regular. It follows $\mathcal{F}$ contains the $\pi$ system consisting of open sets and so by the Lemma on $\pi$ systems, Lemma 57.1.9, $\mathcal{F}$ contains $\sigma(\tau)$ where $\tau$ is the set of open sets. Hence $\mathcal{F}$ contains the Borel sets and is itself a subset of the Borel sets by definition. Therefore, $\mathcal{F} = B(E)$. \[\square\]

One can say more if the metric space is complete and separable. In fact in this case the above definition of inner regularity can be shown to imply the usual one.

**Lemma 57.1.9** Let $\mu$ be a finite measure on a $\sigma$ algebra containing $B(E)$, the Borel sets of $E$, a separable complete metric space. Then if $C$ is a closed set,

$$\mu(C) = \sup \{ \mu(K) : K \subseteq C \text{ and } K \text{ is compact} \}.$$

**Proof:** Let \(\{a_k\}\) be a countable dense subset of $C$. Thus $\bigcup_{k=1}^{\infty} B(a_k, \frac{1}{n}) \supseteq C$. Therefore, there exists $m_n$ such that

$$\mu \left( C \setminus \bigcup_{k=1}^{m_n} B \left( a_k, \frac{1}{n} \right) \right) = \mu(C \setminus C_n) < \frac{\varepsilon}{2^n}.$$ 

Now let $K = C \cap (\cap_{n=1}^{\infty} C_n)$. Then $K$ is a subset of $C_n$ for each $n$ and so for each $\varepsilon > 0$ there exists an $\varepsilon$ net for $K$ since $C_n$ has a $1/n$ net, namely $a_1, \ldots, a_{m_n}$. Since
Chapter 57. Basic Probability

K is closed, it is complete and so it is also compact since it is complete and totally bounded. Now

\[ \mu (C \setminus K) = \mu (\bigcup_{n=1}^{\infty} (C \setminus C_n)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon. \]

Thus \( \mu (C) \) can be approximated by \( \mu (K) \) for \( K \) a compact subset of \( C \). This proves the lemma.

**Definition 57.1.10** A measurable function \( X : (\Omega, \mathcal{F}, \mu) \to Z \) a topological space is called a random variable when \( \mu (\Omega) = 1 \). For such a random variable, one can define a distribution measure \( \lambda_X \) on the Borel sets of \( Z \) as follows.

\[ \lambda_X (G) \equiv \mu (X^{-1} (G)) \]

This is a well defined measure on the Borel sets of \( Z \) because it makes sense for every \( G \) open and \( \mathcal{G} \equiv \{ G \subseteq Z : X^{-1} (G) \in \mathcal{F} \} \) is a \( \sigma \) algebra which contains the open sets, hence the Borel sets. Such a measurable function is also called a random vector.

**Corollary 57.1.11** Let \( X \) be a random variable (random vector) with values in a complete metric space, \( Z \). Then \( \lambda_X \) is an inner and outer regular measure defined on \( \mathcal{B}(Z) \).

**Proposition 57.1.12** For \( X \) a random vector defined above, \( X \) having values in a complete separable metric space \( Z \), then \( \lambda_X \) is inner and outer regular and Borel.

\[ (\Omega, P) \xrightarrow{X} (Z, \lambda_X) \xrightarrow{h} E \]

If \( h \) is Borel measurable and \( h \in L^1 (Z, \lambda_X; E) \) for \( E \) a Banach space, then

\[ \int_{\Omega} h (X (\omega)) \, dP = \int_{Z} h (x) \, d\lambda_X. \]  

(57.1.1)

In the case where \( Z = E \), a separable Banach space, if \( X \) is measurable then \( X \in L^1 (\Omega; E) \) if and only if the identity map on \( E \) is in \( L^1 (E; \lambda_X) \) and

\[ \int_{\Omega} X (\omega) \, dP = \int_{E} x \, d\lambda_X (x) \]  

(57.1.2)

**Proof:** The regularity claims are established above. It remains to verify \( 57.1.3 \).

Since \( h \in L^1 (Z, E) \), it follows there exists a sequence of simple functions \( \{ h_n \} \) such that

\[ h_n (x) \to h (x), \quad \int_{Z} ||h_m - h_n|| \, d\lambda_X \to 0 \text{ as } m, n \to \infty. \]

The first convergence above implies

\[ h_n \circ X \to h \circ X \text{ pointwise on } \Omega \]  

(57.1.3)
Then letting \( h_n(x) = \sum_{k=1}^m x_k x_{E_k}(x) \), where the \( E_k \) are disjoint and Borel, it follows easily that \( h_n \circ X \) is also a simple function of the form \( h_n \circ X(\omega) = \sum_{k=1}^m x_k x_{X^{-1}(E_k)}(\omega) \) and by assumption \( X^{-1}(E_k) \in \mathcal{F} \). From the definition of the integral, it is easily seen

\[
\int h_n \circ X d\mu = \int h_n d\lambda_X, \quad \int ||h_n|| \circ X d\mu = \int ||h_n|| d\lambda_X
\]

Also, \( h_n \circ X - h_m \circ X \) is a simple function and so

\[
\int ||h_n \circ X - h_m \circ X|| d\mu = \int ||h_n - h_m|| d\lambda_X \quad (57.1.4)
\]

It follows from the definition of the Bochner integral and 57.1.3 and 57.1.4 that \( h \circ X \) is in \( L^1(\Omega; E) \) and

\[
\int h \circ X d\mu = \lim_{n \to \infty} \int h_n \circ X d\mu = \lim_{n \to \infty} \int h_n d\lambda_X = \int h d\lambda_X.
\]

Finally consider the case that \( E = \mathbb{Z} \) and suppose \( X \in L^1(\Omega; E) \). Then letting \( h \) be the identity map on \( E \), it follows \( h \) is obviously separably valued and \( h^{-1}(U) \in \mathcal{B}(E) \) for all \( U \) open and so \( h \) is measurable. Why is it in \( L^1(\Omega; E) \)?

\[
\int_E ||h(\omega)|| d\lambda_X = \int_0^\infty \lambda_X(||h|| > t) dt = \int_0^\infty P(||X|| > t) dt = \int_\Omega ||X|| d\mu < \infty
\]

Thus the identity map on \( E \) is in \( L^1(\Omega; E) \). Next suppose the identity map \( h \) is in \( L^1(\Omega; \lambda_X) \). Then \( X(\omega) = h \circ X(\omega) \) and so from the first part, \( X \in L^1(\Omega; E) \) and from 57.1.3 57.1.2 follows. 

### 57.2 Kolmogorov Extension Theorem For Polish Spaces

Let \( M_t \) be a complete separable metric space. This is called a Polish space. \( I \) will denote a totally ordered index set, (Like \( \mathbb{R} \)) and the interest will be in building a measure on the product space, \( \prod_{t \in I} M_t \). By the well ordering principle, you can always put an order on any index set so this order is no restriction, but we do not insist on a well order and in fact, index sets of great interest are \( \mathbb{R} \) or \([0, \infty)\). Also for \( X \) a topological space, \( \mathcal{B}(X) \) will denote the Borel sets.

**Notation 57.2.1** The symbol \( J \) will denote a finite subset of \( I \), \( J = (t_1, \ldots, t_n) \), the \( t_i \) taken in order. \( E_J \) will denote a set which has a set \( E_t \) of \( \mathcal{B}(M_t) \) in the \( t \text{th} \) position for \( t \in J \) and for \( t \notin J \), the set in the \( t \text{th} \) position will be \( M_t \). \( K_J \) will denote a set which has a compact set in the \( t \text{th} \) position for \( t \in J \) and for \( t \notin J \), the
set in the $t^{th}$ position will be $M_t$. Also denote by $\mathcal{R}_J$ the sets $E_J$ and $\mathcal{R}$ the union of the $\mathcal{R}_J$. Let $\mathcal{E}_J$ denote finite disjoint unions of sets of $\mathcal{R}_J$ and let $\mathcal{E}$ denote finite disjoint unions of sets of $\mathcal{R}$. Thus if $F$ is a set of $\mathcal{E}$, there exists $J$ such that $F$ is a finite disjoint union of sets of $\mathcal{R}_J$. For $F \in \Omega$, denote by $\pi_J(F)$ the set $\prod_{t \in I} F_t$ where $F = \prod_{t \in I} F_t$.

**Lemma 57.2.2** The sets, $\mathcal{E}$, $\mathcal{E}_J$ defined above form an algebra of sets of $\prod_{t \in I} M_t$.

**Proof:** First consider $\mathcal{R}_J$. If $A, B \in \mathcal{R}_J$, then $A \cap B \in \mathcal{R}_J$ also. Is $A \setminus B$ a finite disjoint union of sets of $\mathcal{R}_J$? It suffices to verify that $\pi_J(A \setminus B)$ is a finite disjoint union of $\pi_J(\mathcal{R}_J)$. Let $|J|$ denote the number of indices in $J$. If $|J| = 1$, then it is obvious that $\pi_J(A \setminus B)$ is a finite disjoint union of sets of $\pi_J(\mathcal{R}_J)$. In fact, letting $J = (t)$ and the $t^{th}$ entry of $A$ is $A$ and the $t^{th}$ entry of $B$ is $B$, then the $t^{th}$ entry of $A \setminus B$ is $A \setminus B$, a Borel set of $M_t$, a finite disjoint union of Borel sets of $M_t$.

Suppose then that for $A, B$ sets of $\mathcal{R}_J$, $\pi_J(A \setminus B)$ is a finite disjoint union of sets of $\pi_J(\mathcal{R}_J)$ for $|J| = n$, and consider $J = (t_1, \ldots, t_n, t_{n+1})$. Let the $t_i^{th}$ entry of $A$ and $B$ be respectively $A_i$ and $B_i$. It follows that $\pi_J(A \setminus B)$ has the following in the entries for $J$

$$(A_1 \times A_2 \times \cdots \times A_n \times A_{n+1}) \setminus (B_1 \times B_2 \times \cdots \times B_n \times B_{n+1})$$

Letting $A$ represent $A_1 \times A_2 \times \cdots \times A_n$ and $B$ represent $B_1 \times B_2 \times \cdots \times B_n$, this is of the form

$$A \times (A_{n+1} \setminus B_{n+1}) \cup (A \setminus B) \times (A_{n+1} \cap B_{n+1})$$

By induction, $(A \setminus B)$ is the finite disjoint union of sets of $\mathcal{R}_{(t_1, \ldots, t_n)}$. Therefore, the above is the finite disjoint union of sets of $\mathcal{R}_J$. It follows that $\mathcal{E}_J$ is an algebra.

Now suppose $A, B \in \mathcal{R}$. Then for some finite set $J$, both are in $\mathcal{R}_J$. Then from what was just shown,

$$A \setminus B \in \mathcal{E}_J \subseteq \mathcal{E}, A \cap B \in \mathcal{R}.$$ 

By Lemma [57.2.2](#) on Page [57.2.2](#) this shows $\mathcal{E}$ is an algebra. ■

With this preparation, here is the Kolmogorov extension theorem. In the statement and proof of the theorem, $F_i, G_i$, and $E_i$ will denote Borel sets. Any list of indices from $I$ will always be assumed to be taken in order. Thus, if $J \subseteq I$ and $J = (t_1, \ldots, t_n)$, it will always be assumed $t_1 < t_2 < \cdots < t_n$.

**Theorem 57.2.3** For each finite set $J = (t_1, \cdots, t_n) \subseteq I$,

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1, \cdots, t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ such that the following consistency condition holds. If

$$(t_1, \cdots, t_n) \subseteq (s_1, \cdots, s_p),$$

...
57.2. KOLMOGOROV EXTENSION THEOREM FOR POLISH SPACES

then

\[ \nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p} (G_{s_1} \times \cdots \times G_{s_p}) \] (57.2.5)

where if \( s_i = t_j \), then \( G_{s_i} = F_{t_j} \) and if \( s_i \) is not equal to any of the indices, \( t_k \), then \( G_{s_i} = M_{s_i} \). Then for \( \mathcal{E} \) defined in Definition [57.2.1], there exists a probability measure, \( P \) and a \( \sigma \)-algebra \( \mathcal{F} = \sigma(\mathcal{E}) \) such that

\[
\left( \prod_{t \in I} M_t, P, \mathcal{F} \right)
\]

is a probability space. Also there exist measurable functions, \( X_s : \prod_{t \in I} M_t \to M_s \) defined as

\[
X_s \equiv x_s
\]

for each \( s \in I \) such that for each \( (t_1 \cdots t_n) \subseteq I \),

\[
\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = P ([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}])
\]

\[ = P \left( (X_{t_1}, \cdots, X_{t_n}) \in \prod_{j=1}^n F_{t_j} \right) = P \left( \prod_{t \in I} F_t \right) \] (57.2.6)

where \( F_t = M_t \) for every \( t \notin \{t_1 \cdots t_n\} \) and \( F_t \) is a Borel set. Also if \( f \) is a non-negative function of finitely many variables, \( x_{t_1}, \cdots, x_{t_n} \), measurable with respect to \( \mathcal{B} \left( \prod_{j=1}^n M_{t_j} \right) \), then \( f \) is also measurable with respect to \( \mathcal{F} \) and

\[
\int_{M_{t_1} \times \cdots \times M_{t_n}} f(x_{t_1}, \cdots, x_{t_n}) \, d\nu_{t_1 \cdots t_n} = \int_{\prod_{t \in I} M_t} f(x_{t_1}, \cdots, x_{t_n}) \, dP \] (57.2.7)

**Proof:** Let \( \mathcal{E} \) be the algebra of sets defined in Definition [57.2.1]. I want to define a measure on \( \mathcal{E} \). For \( F \in \mathcal{E} \), there exists \( J \) such that \( F \) is the finite disjoint unions of sets of \( \mathcal{R}_J \). Define

\[
P_0(F) \equiv \nu_J (\pi_J(F))
\]

Then \( P_0 \) is well defined because of the consistency condition on the measures \( \nu_J \). \( P_0 \) is clearly finitely additive because the \( \nu_J \) are measures and one can pick \( J \) as large as desired to include all \( t \) where there may be something other than \( M_t \). Also, from the definition,

\[
P_0(\Omega) \equiv P_0 \left( \prod_{t \in I} M_t \right) = \nu_{t_1} (M_{t_1}) = 1.
\]

Next I will show \( P_0 \) is a finite measure on \( \mathcal{E} \). After this it is only a matter of using the Caratheodory extension theorem to get the existence of the desired probability measure \( P \).
**Claim:** Suppose \( E^n \) is in \( \mathcal{E} \) and suppose \( E^n \downarrow \emptyset \). Then \( P_0 (E^n) \downarrow 0 \).

**Proof of the claim:** If not, there exists a sequence such that although \( E^n \downarrow \emptyset \), \( P_0 (E^n) \downarrow \varepsilon > 0 \). Let \( E^n \in \mathcal{E}_{J_n} \). Thus it is a finite disjoint union of sets of \( \mathcal{R}_{J_n} \).

By regularity of the measures \( \nu_J \), which follows from Lemmas 57.1.8 and 57.1.9, there exists a compact set \( K_{J_n} \subseteq E^n \) such that

\[
\nu_{J_n} (\pi_{J_n} (K_{J_n})) + \frac{\varepsilon}{2^{n+2}} > \nu_{J_n} (\pi_{J_n} (E^n))
\]

Thus

\[
P_0 (K_{J_n}) + \frac{\varepsilon}{2^{n+2}} = \nu_{J_n} (\pi_{J_n} (K_{J_n})) + \frac{\varepsilon}{2^{n+2}} > \nu_{J_n} (\pi_{J_n} (E^n)) \equiv P_0 (E^n)
\]

The interesting thing about these \( K_{J_n} \) is: they have the finite intersection property. Here is why.

\[
\varepsilon \leq P_0 (\bigcap_{k=1}^m K_{J_k}) + P_0 (E^n \setminus \bigcap_{k=1}^m K_{J_k}) \\
\leq P_0 (\bigcap_{k=1}^m K_{J_k}) + P_0 (\bigcup_{k=1}^m E^k \setminus K_{J_k}) \\
< P_0 (\bigcap_{k=1}^m K_{J_k}) + \sum_{k=1}^\infty \frac{\varepsilon}{2^{k+2}} < P_0 (\bigcap_{k=1}^m K_{J_k}) + \varepsilon/2,
\]

and so \( P_0 (\bigcap_{k=1}^m K_{J_k}) > \varepsilon/2 \). In considering all the \( E^n \), there are countably many entries in the product space which have something other than \( M_t \) in them. Say these are \( \{t_1, t_2, \ldots \} \).

Let \( p_{t_i} \) be a point which is in the intersection of the \( t_i \) components of the sets \( K_{J_n} \). The compact sets in the \( t_i \) position must have the finite intersection property also because if not, the sets \( K_{J_n} \) can’t have it. Thus there is such a point. As to the other positions, use the axiom of choice to pick something in each of these.

Thus the intersection of these \( K_{J_n} \) contains a point which is contrary to \( E^n \downarrow \emptyset \) because these sets are contained in the \( E^n \).

With the claim, it follows \( P_0 \) is a measure on \( \mathcal{E} \). Here is why: If \( E = \bigcup_{k=1}^\infty E^k \) where \( E, E^k \in \mathcal{E} \), then \( (E \setminus \bigcup_{k=1}^n E^k) \downarrow \emptyset \) and so

\[
P_0 (\bigcup_{k=1}^n E^k) \rightarrow P_0 (E).
\]

Hence if the \( E_k \) are disjoint, \( P_0 (\bigcup_{k=1}^n E^k) = \sum_{k=1}^n P_0 (E_k) \rightarrow P_0 (E) \). Thus for disjoint \( E_k \) having \( \bigcup_{k=1} E^k = E \in \mathcal{E} \),

\[
P_0 (\bigcup_{k=1}^\infty E_k) = \sum_{k=1}^\infty P_0 (E_k).
\]

Now to conclude the proof, apply the Caratheodory extension theorem to obtain \( P \) a probability measure which extends \( P_0 \) to a \( \sigma \) algebra which contains \( \sigma (\mathcal{E}) \) the sigma algebra generated by \( \mathcal{E} \) with \( P = P_0 \) on \( \mathcal{E} \). Thus for \( E_J \in \mathcal{E} \), \( P(E_J) = P_0 (E_J) = \nu_J (P_0 E_J) \).
Next, let \((\prod_{t \in I} M_t, \mathcal{F}, P)\) be the probability space and for \(x \in \prod_{t \in I} M_t\) let \(X_t(x) = x_t\), the \(t^{th}\) entry of \(x\). It follows \(X_t\) is measurable (also continuous) because if \(U\) is open in \(M_t\), then \(X_t^{-1}(U)\) has a \(U\) in the \(t^{th}\) slot and \(M_s\) everywhere else for \(s \neq t\). Thus inverse images of open sets are measurable. Also, letting \(J\) be a finite subset of \(I\) and for \(J = (t_1, \cdots, t_n)\), and \(F_{t_1}, \cdots, F_{t_n}\) Borel sets in \(M_{t_1} \cdots M_{t_n}\) respectively, it follows \(F_J\), where \(F_J\) has \(F_{t_i}\) in the \(t_i^{th}\) entry, is in \(\mathcal{E}\) and therefore,

\[
P([X_{t_1} \in F_{t_1}] \cap [X_{t_2} \in F_{t_2}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]) =
P([X_{t_1}, X_{t_2}, \cdots, X_{t_n}] \in F_{t_1} \times \cdots \times F_{t_n}) = P(F_J) = P_0(F_J)
\]

Finally consider the claim about the integrals. Suppose \(f(x_{t_1}, \cdots, x_{t_n}) = \chi_F\) where \(F\) is a Borel set of \(\prod_{t \in J} M_t\) where \(J = (t_1, \cdots, t_n)\). To begin with suppose \(F = F_{t_1} \times \cdots \times F_{t_n}\) \(\quad (57.2.8)\)

where each \(F_{t_j}\) is in \(\mathcal{B}(M_{t_j})\). Then

\[
\int_{M_{t_1} \times \cdots \times M_{t_n}} \chi_F(x_{t_1}, \cdots, x_{t_n}) \, d\nu_{t_1 \cdots t_n} = \nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n})
\]

\[
= P\left(\prod_{t \in I} F_t\right) = \int_{\Omega} \chi_{\prod_{t \in I} F_t}(x) \, dP
\]

\[
= \int_{\Omega} \chi_F(x_{t_1}, \cdots, x_{t_n}) \, dP \quad (57.2.9)
\]

where \(F_t = M_t\) if \(t \notin J\). Let \(\mathcal{K}\) denote sets, \(F\) of the sort in \(\prod_{t \in J} M_t\). It is clearly a \(\pi\) system. Now let \(\mathcal{G}\) denote those sets \(F\) in \(\mathcal{B}(\prod_{t \in J} M_t)\) such that \(\prod_{t \in J} M_t\) holds. Thus \(\mathcal{G} \supseteq \mathcal{K}\). It is clear that \(\mathcal{G}\) is closed with respect to countable disjoint unions and complements. Hence \(\mathcal{G} \supseteq \sigma(\mathcal{K})\) but \(\sigma(\mathcal{K}) = \mathcal{B}(\prod_{t \in J} M_t)\) because every open set in \(\prod_{t \in J} M_t\) is the countable union of rectangles like \(\prod_{t \in J} M_t\) in which each \(F_{t_i}\) is open. Therefore, \(\prod_{t \in J} M_t\) holds for every \(F \in \mathcal{B}(\prod_{t \in J} M_t)\).

Passing to simple functions and then using the monotone convergence theorem yields the final claim of the theorem. \(\Box\)

### 57.3 Independence

The concept of independence is probably the main idea which separates probability from analysis and causes some of us to struggle to understand what is going on.

**Definition 57.3.1** Let \((\Omega, \mathcal{F}, P)\) be a probability space. The sets in \(\mathcal{F}\) are called events. A set of events, \(\{A_i\}_{i \in I}\) is called independent if whenever \(\{A_{i_k}\}_{k=1}^m\) is a finite subset

\[
P(\bigcap_{k=1}^m A_{i_k}) = \prod_{k=1}^m P(A_{i_k}) .
\]
Each of these events defines a rather simple \( \sigma \) algebra, \((A_i, A_i^C, \emptyset, \Omega)\) denoted by \( \mathcal{F}_i \). Now the following lemma is interesting because it motivates a more general notion of independent \( \sigma \) algebras.

**Lemma 57.3.2** Suppose \( B_i \in \mathcal{F}_i \) for \( i \in I \). Then for any \( m \in \mathbb{N} \)

\[
P(\cap_{k=1}^{m} B_{i_k}) = \prod_{k=1}^{m} P(B_{i_k}).
\]

**Proof:** The proof is by induction on the number \( l \) of the \( B_{i_k} \) which are not equal to \( A_{i_k} \). First suppose \( l = 0 \). Then the above assertion is true by assumption. Suppose it is so for some \( l \) and there are \( l + 1 \) sets not equal to \( A_{i_k} \). If any equals \( \emptyset \) there is nothing to show. Both sides equal 0. If any equals \( \Omega \), there is also nothing to show. You can ignore that set in both sides and then you have by induction the two sides are equal because you have no more than \( l \) sets different than \( A_{i_k} \). The only remaining case is where some \( B_{i_k} = A_{i_k}^C \). Say \( B_{i_{m+1}} = A_{i_{m+1}}^C \) for simplicity.

\[
P(\cap_{k=1}^{m+1} B_{i_k}) = P\left(A_{i_{m+1}}^C \cap \cap_{k=1}^{m} B_{i_k}\right)
\]

\[
= P\left(\cap_{k=1}^{m} B_{i_k}\right) - P\left(A_{i_{m+1}} \cap \cap_{k=1}^{m} B_{i_k}\right)
\]

Then by induction,

\[
= \prod_{k=1}^{m} P(B_{i_k}) - P\left(A_{i_{m+1}}\right) \prod_{k=1}^{m} P(B_{i_k}) = \prod_{k=1}^{m} P(B_{i_k}) \left(1 - P\left(A_{i_{m+1}}\right)\right)
\]

\[
= \prod_{k=1}^{m} P(B_{i_k}) \left(A_{i_{m+1}}^C\right) \prod_{k=1}^{m} P(B_{i_k}) = \prod_{k=1}^{m+1} P(B_{i_k})
\]

thus proving it for \( l + 1 \). \( \blacksquare \)

This motivates a more general notion of independence in terms of \( \sigma \) algebras.

**Definition 57.3.3** If \( \{\mathcal{F}_i\}_{i \in I} \) is any set of \( \sigma \) algebras contained in \( \mathcal{F} \), they are said to be independent if whenever \( A_{i_k} \in \mathcal{F}_{i_k} \) for \( k = 1, 2, \ldots, m \), then

\[
P(\cap_{k=1}^{m} A_{i_k}) = \prod_{k=1}^{m} P(A_{i_k}).
\]

A set of random variables \( \{X_i\}_{i \in I} \) is independent if the \( \sigma \) algebras \( \{\sigma(X_i)\}_{i \in I} \) are independent \( \sigma \) algebras. Here \( \sigma(X) \) denotes the smallest \( \sigma \) algebra such that \( X \) is measurable. Thus \( \sigma(X) = \{X^{-1}(U) : U \text{ is a Borel set}\} \). More generally, \( \sigma(X_i : i \in I) \) is the smallest \( \sigma \) algebra such that each \( X_i \) is measurable.

Note that by Lemma 57.3.2 you can consider independent events in terms of independent \( \sigma \) algebras. That is, a set of independent events can always be considered as events taken from a set of independent \( \sigma \) algebras. This is a more general notion because here the \( \sigma \) algebras might have infinitely many sets in them.
Lemma 57.3.4 Suppose the set of random variables, \( \{X_i\}_{i \in I} \) is independent. Also suppose \( I_1 \subseteq I \) and \( j \notin I_1 \). Then the \( \sigma \) algebras \( \sigma(X_i : i \in I_1), \sigma(X_j) \) are independent \( \sigma \) algebras.

Proof: Let \( B \in \sigma(X_j) \). I want to show that for any \( A \in \sigma(X_i : i \in I_1) \), it follows that \( P(A \cap B) = P(A) P(B) \). Let \( \mathcal{K} \) consist of finite intersections of sets of the form \( X_k^{-1}(B_k) \) where \( B_k \) is a Borel set and \( k \in I_1 \). Thus \( \mathcal{K} \) is a \( \pi \) system and \( \sigma(\mathcal{K}) = \sigma(X_i : i \in I_1) \). Now if you have one of these sets of the form \( A = \cap_{k=1}^m X_k^{-1}(B_k) \) where without loss of generality, it can be assumed the \( k \) are distinct since \( X_k^{-1}(B_k) \cap X_k^{-1}(B'_k) = X_k^{-1}(B_k \cap B'_k) \), then

\[
P(A \cap B) = P(\cap_{k=1}^m X_k^{-1}(B_k) \cap B) = P(B) \prod_{k=1}^m P(X_k^{-1}(B_k))
\]

Thus \( \mathcal{K} \) is contained in

\[
\mathcal{G} \equiv \{ A \in \sigma(X_i : i \in I_1) : P(A \cap B) = P(A) P(B) \}.
\]

Now \( \mathcal{G} \) is closed with respect to complements and countable disjoint unions. Here is why: If each \( A_i \in \mathcal{G} \) and the \( A_i \) are disjoint,

\[
P(\cup_{i=1}^\infty A_i \cap B) = P(\cup_{i=1}^\infty (A_i \cap B)) = \sum_{i=1}^{\infty} P(A_i \cap B) = \sum_{i=1}^{\infty} P(A_i) P(B) = P(B) \sum_{i}^{\infty} P(A_i) = P(B) P(\cup_{i=1}^\infty A_i)
\]

If \( A \in \mathcal{G} \),

\[
P(A^C \cap B) + P(A \cap B) = P(B)
\]

and so

\[
P(A^C \cap B) = P(B) - P(A \cap B)
\]

\[
= P(B) - P(A) P(B)
\]

\[
= P(B)(1 - P(A)) = P(B) P(A^C).
\]

Therefore, from the lemma on \( \pi \) systems, Lemma 57.2.4 on Page 191, it follows \( \mathcal{G} \supseteq \sigma(\mathcal{K}) = \sigma(X_i : i \in I_1) \).

Lemma 57.3.5 If \( \{X_k\}_{k=1}^r \) are independent random variables having values in \( \mathbb{Z} \) a separable metric space, and if \( g_k \) is a Borel measurable function, then \( \{g_k(X_k)\}_{k=1}^r \) is also independent. Furthermore, if the random variables have values in \( \mathbb{R} \), and they are all bounded, then

\[
E\left(\prod_{i=1}^r X_i\right) = \prod_{i=1}^r E(X_i).
\]
More generally, the above formula holds if it is only known that each $X_i \in L^1(\Omega; \mathbb{R})$ and 

$$
\prod_{i=1}^{r} X_i \in L^1(\Omega; \mathbb{R}).
$$

**Proof:** First consider the claim about $\{g_k(X_k)\}_{k=1}^{r}$. Letting $O$ be an open set in $Z$,

$$(g_k \circ X_k)^{-1}(O) = X_k^{-1}(g_k^{-1}(O)) = X_k^{-1} (\text{Borel set}) \in \sigma(X_k).$$

It follows $(g_k \circ X_k)^{-1}(E)$ is in $\sigma(X_k)$ whenever $E$ is Borel because the sets whose inverse images are measurable includes the Borel sets. Thus $\sigma(g_k \circ X_k) \subseteq \sigma(X_k)$ and this proves the first part of the lemma.

Let $X_1 = \sum_{i=1}^{m} c_i X_{E_i}, X_2 = \sum_{j=1}^{m} d_j X_{F_j}$, where $P(E_i F_j) = P(E_i) P(F_j)$. Then

$$\int X_1 X_2 dP = \sum_{i,j} d_j c_i P(E_i) P(F_j) = \left( \int X_1 dP \right) \left( \int X_2 dP \right).$$

In general for $X_1, X_2$ independent, there exist sequences of bounded simple functions $\{s_n\}, \{t_n\}$ measurable with respect to $\sigma(X_1)$ and $\sigma(X_2)$ respectively such that $s_n \to X_1$ pointwise and $t_n \to X_2$ pointwise. Then from the above and the dominated convergence theorem,

$$\int X_1 X_2 dP = \lim_{n \to \infty} \int s_n t_n dP = \lim_{n \to \infty} \left( \int s_n dP \right) \left( \int t_n dP \right)$$

$$= \left( \int X_1 dP \right) \left( \int X_2 dP \right).$$

Next suppose there are $m$ of these independent bounded random variables. Then $\prod_{i=2}^{m} X_i \in \sigma(X_2, \cdots, X_m)$ and by Lemma 57.3.4 the two random variables $X_1$ and $\prod_{i=2}^{m} X_i$ are independent. Hence from the above and induction,

$$\int \prod_{i=1}^{m} X_i dP = \int X_1 \prod_{i=2}^{m} X_i dP = \int X_1 dP \int \prod_{i=2}^{m} X_i dP = \prod_{i=1}^{m} \int X_i dP$$

Now consider the last claim. Replace each $X_i$ with $X_i^n$ where this is just a truncation of the form

$$X_i^n = \begin{cases} 
X_i & \text{if } |X_i| \leq n \\
n & \text{if } X_i > n \\
-n & \text{if } X_i < n
\end{cases}$$

Then by the first part

$$E \left( \prod_{i=1}^{r} X_i^n \right) = \prod_{i=1}^{r} E(X_i^n)$$

Now $|\prod_{i=1}^{r} X_i^n| \leq |\prod_{i=1}^{r} X_i| \in L^1$ and so by the dominated convergence theorem, you can pass to the limit in both sides to get the desired result. \[\square\]
57.3. INDEPENDENCE

Maybe this would be a good place to put a really interesting result known as the Doob Dynkin lemma. This amazing result is illustrated with the following diagram in which $\mathbf{X} = (X_1, \ldots, X_m)$. By Proposition 57.1.6 $\sigma(\mathbf{X}) = \sigma(X_1, \ldots, X_m)$.

\[ (\Omega, \sigma(\mathbf{X})) \xrightarrow{X} F \]
\[ (\prod_{i=1}^m E_i, B(\prod_{i=1}^m E_i)) \]

You start with $\mathbf{X}$ and can write it as the composition $g \circ \mathbf{X}$ provided $\mathbf{X}$ is measurable.

**Lemma 57.3.6** Let $(\Omega, \mathcal{F})$ be a measure space and let $X_i : \Omega \to E_i$ where $E_i$ is a separable Banach space. Suppose also that $X : \Omega \to F$ where $F$ is a separable Banach space. Then $X$ is $\sigma(X_1, \ldots, X_m)$ measurable if and only if there exists a Borel measurable function $g : \prod_{i=1}^m E_i \to F$ such that $X = g(X_1, \ldots, X_m)$.

**Proof:** First suppose $X(\omega) = fX_W(\omega)$ where $f \in F$ and $W \in \sigma(X_1, \ldots, X_m)$. Then by Proposition 57.1.6, $W$ is of the form $(X_1, \ldots, X_m)^{-1}(B) \equiv X^{-1}(B)$ where $B$ is Borel in $\prod_{i=1}^m E_i$. Therefore,

$X(\omega) = fX_{X^{-1}(B)}(\omega) = fX_B(\mathbf{X}(\omega))$.

Now suppose $X$ is measurable with respect to $\sigma(X_1, \ldots, X_m)$. Then there exist simple functions

$X_n(\omega) = \sum_{k=1}^{m_n} f_k X_{B_k}(\mathbf{X}(\omega)) \equiv g_n(\mathbf{X}(\omega))$

where the $B_k$ are Borel sets in $\prod_{i=1}^m E_i$, such that $X_n(\omega) \to X(\omega)$, each $g_n$ being Borel. Thus $g_n$ converges on $\mathbf{X}(\Omega)$. Furthermore, the set on which $g_n$ does converge is a Borel set equal to

$\bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \bigcap_{p,q \geq m} \left[ \||g_p - g_q\| < \frac{1}{n} \right]$}

which contains $\mathbf{X}(\Omega)$. Therefore, modifying $g_n$ by multiplying it by the indicator function of this Borel set containing $\mathbf{X}(\Omega)$, we can conclude that $g_n$ converges to a Borel function $g$ and, passing to a limit in the above,

$X(\omega) = g(\mathbf{X}(\omega))$

Conversely, suppose $X(\omega) = g(\mathbf{X}(\omega))$. Why is $X \sigma(\mathbf{X})$ measurable?

$X^{-1}(\text{open}) = X^{-1}(g^{-1}(\text{open})) = X^{-1}(\text{Borel}) \in \sigma(\mathbf{X})$
57.4 Independence For Banach Space Valued Random Variables

Recall that for $X$ a random variable, $\sigma(X)$ is the smallest $\sigma$ algebra containing all the sets of the form $X^{-1}(F)$ where $F$ is Borel. Since such sets, $X^{-1}(F)$ for $F$ Borel, form a $\sigma$ algebra it follows $\sigma(X) = \{ X^{-1}(F) : F \text{ is Borel} \}$.

Next consider the case where you have a set of $\sigma$ algebras. The following lemma is helpful when you try to verify such a set of $\sigma$ algebras is independent. It says you only need to check things on $\pi$ systems contained in the $\sigma$ algebras. This is really nice because it is much easier to consider the smaller $\pi$ systems than the whole $\sigma$ algebra.

**Lemma 57.4.1** Suppose $\{F_i\}_{i \in I}$ is a set of $\sigma$ algebras contained in $F$ where $F$ is a $\sigma$ algebra of sets of $\Omega$. Suppose that $K_j \subseteq F_i$ is a $\pi$ system and $F_i = \sigma(K_i)$. Suppose also that whenever $J$ is a finite subset of $I$ and $A_j \in K_j$ for $j \in J$, it follows

$$P(\cap_{j \in J} A_j) = \prod_{j \in J} P(A_j).$$

Then $\{F_i\}_{i \in I}$ is independent.

**Proof:** I need to verify that under the given conditions, if $\{j_1, j_2, \ldots, j_n\} \subseteq I$ and $A_{j_k} \subseteq F_{j_k}$, then

$$P(\cap_{k=1}^n A_{j_k}) = \prod_{k=1}^n P(A_{j_k}).$$

By hypothesis, this is true if each $A_{j_k} \in K_{j_k}$. Suppose it is true whenever there are at most $r - 1 \geq 0$ of the $A_{j_k}$ which are not in $K_{j_k}$. Consider

$$\cap_{k=1}^n A_{j_k}$$

where there are $r$ sets which are not in the corresponding $K_{j_k}$. Without loss of generality, say there are at most $r - 1$ sets in the first $n - 1$ which are not in the corresponding $K_{j_k}$.

Pick $(A_{j_1}, \ldots, A_{j_{n-1}})$ let

$$\mathcal{G}_{(A_{j_1}, \ldots, A_{j_{n-1}})} = \left\{ B \in F_{j_n} : P(\cap_{k=1}^{n-1} A_{j_k} \cap B) = \prod_{k=1}^{n-1} P(A_{j_k}) P(B) \right\}$$

I am going to show $\mathcal{G}_{(A_{j_1}, \ldots, A_{j_{n-1}})}$ is closed with respect to complements and countable disjoint unions and then apply the Lemma on $\pi$ systems. By the induction
hypothesis, \( \mathcal{K}_{jn} \subseteq \mathcal{G}_{(A_{j_1} \cdots A_{j_{n-1}})} \). If \( B \in \mathcal{G}_{(A_{j_1} \cdots A_{j_{n-1}})} \),

\[
\prod_{k=1}^{n-1} P(A_{j_k}) = P(\cap_{k=1}^{n-1} A_{j_k})
\]

\[
= P((\cap_{k=1}^{n-1} A_{j_k} \cap B^C) \cup (\cap_{k=1}^{n-1} A_{j_k} \cap B))
\]

\[
= P(\cap_{k=1}^{n-1} A_{j_k} \cap B^C) + P(\cap_{k=1}^{n-1} A_{j_k} \cap B)
\]

\[
= P(\cap_{k=1}^{n-1} A_{j_k} \cap B^C) + \prod_{k=1}^{n-1} P(A_{j_k}) P(B)
\]

and so

\[
P(\cap_{k=1}^{n-1} A_{j_k} \cap B^C) = \prod_{k=1}^{n-1} P(A_{j_k}) (1 - P(B))
\]

\[
= \prod_{k=1}^{n-1} P(A_{j_k}) P(B^C)
\]

showing if \( B \in \mathcal{G}_{(A_{j_1} \cdots A_{j_{n-1}})} \), then so is \( B^C \). It is clear that \( \mathcal{G}_{(A_{j_1} \cdots A_{j_{n-1}})} \) is closed with respect to disjoint unions also. Here is why. If \( \{B_j\}_{j=1}^{\infty} \) are disjoint sets in \( \mathcal{G}_{(A_{j_1} \cdots A_{j_{n-1}})} \),

\[
P(\bigcup_{i=1}^{\infty} B_i \cap \cap_{k=1}^{n-1} A_{j_k}) = \sum_{i=1}^{\infty} P(B_i \cap \cap_{k=1}^{n-1} A_{j_k})
\]

\[
= \sum_{i=1}^{\infty} P(B_i) \prod_{k=1}^{n-1} P(A_{j_k})
\]

\[
= \prod_{k=1}^{n-1} P(A_{j_k}) \sum_{i=1}^{\infty} P(B_i)
\]

\[
= \prod_{k=1}^{n-1} P(A_{j_k}) P(\bigcup_{i=1}^{\infty} B_i)
\]

Therefore, by the \( \pi \) system lemma, Lemma 57.4.2, \( \mathcal{G}_{(A_{j_1} \cdots A_{j_{n-1}})} = \mathcal{F}_{j_n} \). This proves the induction step in going from \( r-1 \) to \( r \). □

What is a useful \( \pi \) system for \( \mathcal{B}(E) \), the Borel sets of \( E \) where \( E \) is a Banach space?

Recall the fundamental lemma used to prove the Pettis theorem. It was proved on Page 629.

**Lemma 57.4.2** Let \( E \) be a separable real Banach space. Sets of the form

\[
\{x \in E : x_i^*(x) \leq \alpha_i, i = 1, 2, \cdots, m\}
\]
where \( x^*_i \in D' \), a dense subspace of the unit ball of \( E' \) and \( \alpha_i \in [-\infty, \infty) \) are a \( \pi \) system, and denoting this \( \pi \) system by \( K \), it follows \( \sigma(K) = \mathcal{B}(E) \). The sets of \( K \) are examples of cylindrical sets. The \( D' \) is that set for the proof of the Pettis theorem.

**Proof:** The sets described are obviously a \( \pi \) system. I want to show \( \sigma(K) \) contains the closed balls because then \( \sigma(K) \) contains the open balls and hence the open sets and the result will follow. Let \( D' \) be described in Lemma [19.1.3]. As pointed out earlier it can be any dense subset of \( B' \). Then

\[
\{ x \in E : ||x - a|| \leq r \} = \{ x \in E : \sup_{f \in D'} |f(x) - f(a)| \leq r \} = \bigcap_{f \in D'} \{ x \in E : f(a) - r \leq f(x) \leq f(a) + r \} = \bigcap_{f \in D'} \{ x \in E : f(x) \leq f(a) + r \text{ and } (-f)(x) \leq r - f(a) \}
\]

which equals a countable intersection of sets of the given \( \pi \) system. Therefore, every closed ball is contained in \( \sigma(K) \). It follows easily that every open ball is also contained in \( \sigma(K) \) because

\[
B(a, r) = \bigcup_{n=1}^{\infty} B(a, r - \frac{1}{n}).
\]

Since the Banach space is separable, it is completely separable and so every open set is the countable union of balls. This shows the open sets are in \( \sigma(K) \) and so \( \sigma(K) \supseteq \mathcal{B}(E) \). However, all the sets in the \( \pi \) system are closed hence Borel because they are inverse images of closed sets. Therefore, \( \sigma(K) \subseteq \mathcal{B}(E) \) and so \( \sigma(K) = \mathcal{B}(E) \). ■

As mentioned above, we can replace \( D' \) in the above with \( M \), any dense subset of \( E' \).

**Observation 57.4.3** Denote by \( C_{\alpha, n} \) the set \( \{ \beta \in \mathbb{R}^n : \beta_i \leq \alpha_i \} \). Also denote by \( g_n \) an element of \( M^n \) with the understanding that \( g_n : E \to \mathbb{R}^n \) according to the rule

\[
g_n(x) \equiv (g_1(x), \ldots, g_n(x)).
\]

Then the sets in the above lemma can be written as \( g_n^{-1}(C_{\alpha, n}) \). In other words, sets of the form \( g_n^{-1}(C_{\alpha, n}) \) form a \( \pi \) system for \( \mathcal{B}(E) \).

Next suppose you have some random variables having values in a separable Banach space, \( E, \{ X_i \}_{i \in I} \). How can you tell if they are independent? To show they are independent, you need to verify that

\[
P \left( \bigcap_{k=1}^{n} X_k^{-1}(F_i) \right) = \prod_{k=1}^{n} P \left( X_k^{-1}(F_i) \right)
\]
whenever the $F_i_k$ are Borel sets in $E$. It is desirable to find a way to do this easily.

**Lemma 57.4.4** Let $\mathcal{K}$ be a $\pi$ system of sets of $E$, a separable real Banach space and let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \rightarrow E$ be a random variable. Then

$$X^{-1}(\sigma(\mathcal{K})) = \sigma(X^{-1}(\mathcal{K}))$$

**Proof:** First note that $X^{-1}(\sigma(\mathcal{K}))$ is a $\sigma$ algebra which contains $X^{-1}(K)$ and so it contains $\sigma(X^{-1}(K))$. Thus

$$X^{-1}(\sigma(\mathcal{K})) \supseteq \sigma(X^{-1}(K))$$

Now let

$$\mathcal{G} \equiv \{ A \in \sigma(K) : X^{-1}(A) \in \sigma(X^{-1}(K)) \}$$

Then $\mathcal{G} \supseteq K$. If $A \in \mathcal{G}$, then $X^{-1}(A) \in \sigma(X^{-1}(K))$ and so

$$X^{-1}(A) \subseteq \sigma(X^{-1}(K))$$

and so

$$X^{-1}(A)^C = X^{-1}(A^C) \in \sigma(X^{-1}(K))$$

because $\sigma(X^{-1}(K))$ is a $\sigma$ algebra. Hence $A^C \in \mathcal{G}$. Finally suppose $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$. Then

$$X^{-1}(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} X^{-1}(A_i) \in \sigma(X^{-1}(K))$$

again because $\sigma(X^{-1}(K))$ is a $\sigma$ algebra. It follows from Lemma 10.12 on Page 312 that $\mathcal{G} \supseteq \sigma(K)$ and this shows that whenever

$$A \in \sigma(K), X^{-1}(A) \in \sigma(X^{-1}(K)).$$

Thus $X^{-1}(\sigma(\mathcal{K})) \subseteq \sigma(X^{-1}(K))$. \[ ]

With this lemma, here is the desired result about independent random variables. Essentially, you can reduce to the case of random vectors having values in $\mathbb{R}^n$.

### 57.5 Reduction To Finite Dimensions

Let $E$ be a Banach space and let $g \in (E')^n$. Then for $x \in E$, $g \circ x$ is the vector in $F^n$ which equals $(g_1(x), g_2(x), \ldots, g_n(x))$.

**Theorem 57.5.1** Let $X_i$ be a random variable having values in $E$ a real separable Banach space. The random variables $\{X_i\}_{i \in I}$ are independent if whenever

$$\{i_1, \ldots, i_n\} \subseteq I,$$

$m_{i_1}, \ldots, m_{i_n}$ are positive integers, and $g_{m_{i_1}}, \ldots, g_{m_{i_n}}$ are respectively in

$$(M)^{m_{i_1}}, \ldots, (M)^{m_{i_n}}$$

for $M$ a dense subspace of $E'$, $\left\{ g_{m_{i_j}} \circ X_{i_j} \right\}_{j=1}^{n}$ are independent random vectors having values in $\mathbb{R}^{m_{i_1}}, \ldots, \mathbb{R}^{m_{i_n}}$ respectively.
Proof: It is necessary to show that the events $X_i^{-1}(B_{ij})$ are independent events whenever $B_{ij}$ are Borel sets. By Lemma 57.4.1 and the above Lemma 57.4.2, it suffices to verify that the events

$$X_i^{-1}\left(g_{m_{ij}}^{-1}(C_{\tilde{\alpha},m_{ij}})\right) = \left(g_{m_{ij}} \circ X_i\right)^{-1}(C_{\tilde{\alpha},m_{ij}})$$

are independent where $C_{\tilde{\alpha},m_{ij}}$ are the cones described in Lemma 57.4.2. Thus

$$\tilde{\alpha} = (\alpha_{k_1}, \cdots, \alpha_{k_m})$$

$$C_{\tilde{\alpha},m_{ij}} = \prod_{i=1}^{m_{ij}}(-\infty, \alpha_{k_i}]$$

But this condition is implied when the finite dimensional valued random vectors $g_{m_{ij}} \circ X_i$ are independent. ■

The above assertion also goes the other way as you may want to show.

57.6 0, 1 Laws

I am following [108] for the proof of many of the following theorems. Recall the set of $\omega$ which are in infinitely many of the sets $\{A_n\}$ is

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$ 

This is because $\omega$ is in the above set if and only if for every $n$ there exists $m \geq n$ such that it is in $A_m$.

Theorem 57.6.1 Suppose $A_n \in \mathcal{F}_n$ where the $\sigma$ algebras $\{\mathcal{F}_n\}_{n=1}^{\infty}$ are independent. Suppose also that

$$\sum_{k=1}^{\infty} P(A_k) = \infty.$$ 

Then

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = 1.$$ 

Proof: It suffices to verify that

$$P\left(\bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^C\right) = 0$$

which can be accomplished by showing

$$P\left(\bigcap_{m=n}^{\infty} A_m^C\right) = 0$$
for each \( n \). The sets \( \{ A^C_k \} \) satisfy \( A^C_k \in \mathcal{F}_k \). Therefore, noting that \( e^{-x} \geq 1 - x \),

\[
P \left( \bigcap_{m=n}^{\infty} A^C_m \right) = \lim_{N \to \infty} P \left( \bigcap_{m=n}^{N} A^C_m \right) = \lim_{N \to \infty} \prod_{m=n}^{N} P \left( A^C_m \right)
\]

\[
= \lim_{N \to \infty} \prod_{m=n}^{N} (1 - P(A_m)) \leq \lim_{N \to \infty} \prod_{m=n}^{N} e^{-P(A_m)}
\]

\[
= \lim_{N \to \infty} \exp \left( - \sum_{m=n}^{N} P(A_m) \right) = 0. \quad \blacksquare
\]

The Kolmogorov zero one law follows next. It has to do with something called a tail event.

**Definition 57.6.2** Let \( \{ \mathcal{F}_n \} \) be a sequence of \( \sigma \) algebras. Then \( \mathcal{T}_n \equiv \sigma \left( \bigcup_{k=n}^{\infty} \mathcal{F}_k \right) \) where this means the smallest \( \sigma \) algebra which contains each \( \mathcal{F}_k \) for \( k \geq n \). Then a tail event is a set which is in the \( \sigma \) algebra, \( \mathcal{T} \equiv \bigcap_{n=1}^{\infty} \mathcal{T}_n \).

As usual, \( (\Omega, \mathcal{F}, P) \) is the underlying probability space such that all \( \sigma \) algebras are contained in \( \mathcal{F} \).

**Lemma 57.6.3** Suppose \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) are independent \( \sigma \) algebras and suppose \( A \) is a tail event and \( A_k \in \mathcal{F}_k, \ i = 1, \ldots, m \) are given sets. Then

\[
P \left( A_{k_1} \cap \cdots \cap A_{k_m} \cap A \right) = P \left( A_{k_1} \cap \cdots \cap A_{k_m} \right) P \left( A \right)
\]

**Proof:** Let \( \mathcal{K} \) be the \( \pi \) system consisting of finite intersections of the form

\[
B_{m_1} \cap B_{m_2} \cap \cdots \cap B_{m_j}
\]

where \( m_i \in \mathcal{F}_{k_i} \) for \( k_i > \max \{ k_1, \ldots, k_m \} \equiv N \). Thus \( \sigma \left( \mathcal{K} \right) = \sigma \left( \bigcup_{i=N+1}^{\infty} \mathcal{F}_i \right) \). Now let

\[
\mathcal{G} \equiv \{ B \in \sigma \left( \mathcal{K} \right) : P \left( A_{k_1} \cap \cdots \cap A_{k_m} \cap B \right) = P \left( A_{k_1} \cap \cdots \cap A_{k_m} \right) P \left( B \right) \}
\]

Then clearly \( \mathcal{K} \subseteq \mathcal{G} \). It is also true that \( \mathcal{G} \) is closed with respect to complements and countable disjoint unions. By the lemma on \( \pi \) systems, \( \mathcal{G} = \sigma \left( \mathcal{K} \right) = \sigma \left( \bigcup_{i=N+1}^{\infty} \mathcal{F}_i \right) \).

Since \( A \) is in \( \sigma \left( \bigcup_{i=N+1}^{\infty} \mathcal{F}_i \right) \) due to the assumption that it is a tail event, it follows that

\[
P \left( A_{k_1} \cap \cdots \cap A_{k_m} \cap A \right) = P \left( A_{k_1} \cap \cdots \cap A_{k_m} \right) P \left( A \right) \quad \blacksquare
\]

**Theorem 57.6.4** Suppose the \( \sigma \) algebras, \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) are independent and suppose \( A \) is a tail event. Then \( P \left( A \right) \) either equals 0 or 1.

**Proof:** Let \( A \in \mathcal{T} \). I want to show that \( P \left( A \right) = P \left( A \right)^2 \). Let \( \mathcal{K} \) denote sets of the form \( A_{k_1} \cap \cdots \cap A_{k_m} \) for some \( m, A_{k_j} \in \mathcal{F}_{k_j} \) where each \( k_j > n \). Thus \( \mathcal{K} \) is a \( \pi \) system and

\[
\sigma \left( \mathcal{K} \right) = \sigma \left( \bigcup_{k=n+1}^{\infty} \mathcal{F}_k \right) \equiv \mathcal{T}_{n+1}
\]
Let \( G \equiv \{ B \in \mathcal{T}_{n+1} \equiv \sigma \left( \bigcup_{k=n+1}^{\infty} \mathcal{F}_k \right) : P (A \cap B) = P (A) P (B) \} \)

Thus \( \mathcal{K} \subseteq G \) because

\[
P (A_{k_1} \cap \cdots \cap A_{k_m} \cap A) = P (A_{k_1} \cap \cdots \cap A_{k_m}) P (A)
\]

by Lemma 57.6.3. However, \( G \) is closed with respect to countable disjoint unions and complements. Here is why. If \( B \in G \),

\[
P \left( A \cap B^C \right) + P (A \cap B) = P (A)
\]

and so

\[
P (A \cap B^C) = P (A) - P (A \cap B) = P (A) (1 - P (B)) = P (A) P (B^C).
\]

and so \( B^C \in G \). If \( \{ B_i \}_{i=1}^{\infty} \) are disjoint sets in \( G \),

\[
P (A \cap \bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} P (A \cap B_k) = P (A) \sum_{k=1}^{\infty} P (B_k)
\]

and so \( \bigcup_{k=1}^{\infty} B_k \in G \). Therefore by the Lemma on \( \pi \) systems Lemma 10.12.3 on Page 312, it follows \( G = \sigma (\mathcal{K}) = \sigma \left( \bigcup_{k=n+1}^{\infty} \mathcal{F}_k \right) \).

Thus for any \( B \in \sigma \left( \bigcup_{k=n+1}^{\infty} \mathcal{F}_k \right) = \mathcal{T}_{n+1}, P (A \cap B) = P (A) P (B) \). However, \( A \) is in all of these \( \mathcal{T}_{n+1} \) and so \( P (A \cap A) = P (A) = P (A)^2 \) so \( P (A) \) equals either 0 or 1.

What sorts of things are tail events of independent \( \sigma \) algebras?

**Theorem 57.6.5** Let \( \{ X_k \} \) be a sequence of independent random variables having values in \( Z \) a Banach space. Then

\[ A \equiv \{ \omega : \{ X_k (\omega) \} \text{ converges} \} \]

is a tail event of the independent \( \sigma \) algebras \( \{ \sigma (X_k) \} \). So is

\[ B \equiv \left\{ \omega : \left\{ \sum_{k=1}^{\infty} X_k (\omega) \right\} \text{ converges} \right\}. \]

**Proof:** Since \( Z \) is complete, \( A \) is the same as the set where \( \{ X_k (\omega) \} \) is a Cauchy sequence. This set is

\[
\cap_{p=1}^{\infty} \bigcap_{m=p}^{\infty} \bigcup_{l,k \geq m} \{ \omega : ||X_k (\omega) - X_l (\omega)|| < 1/n \}
\]

Note that

\[
\bigcup_{m=p}^{\infty} \bigcap_{l,k \geq m} \{ \omega : ||X_k (\omega) - X_l (\omega)|| < 1/n \} \in \sigma \left( \bigcup_{j=p}^{\infty} \sigma (X_j) \right)
\]
for every $p$ is the set where ultimately any pair of $X_k, X_l$ are closer together than $1/n$,

$$\cap_{p=1}^\infty \cup_{m=p}^\infty \cap_{l,k\geq m} \{ \omega : ||X_k(\omega) - X_l(\omega)|| < 1/n \}$$

is a tail event. The set where $\{X_k(\omega)\}$ is a Cauchy sequence is the intersection of all these and is therefore, also a tail event.

Now consider $B$. This set is the same as the set where the partial sums are Cauchy sequences. Let $S_n \equiv \sum_{k=1}^n X_k$. The set where the sum converges is then

$$\cap_{n=1}^\infty \cap_{p=2}^\infty \cup_{m=p}^\infty \cap_{l,k\geq m} \{ \omega : ||S_k(\omega) - S_l(\omega)|| < 1/n \}$$

Say $k < l$ and consider for $m \geq p$

$$\{ \omega : ||S_k(\omega) - S_l(\omega)|| < 1/n, k \geq m \}$$

This is the same as

$$\left\{ \omega : \left\| \sum_{j=k-1}^{l} X_j(\omega) \right\| < 1/n, k \geq m \right\} \in \sigma \left( \cup_{j=p-1}^\infty \sigma (X_j) \right)$$

Thus

$$\cup_{m=p}^\infty \cap_{l,k\geq m} \{ \omega : ||S_k(\omega) - S_l(\omega)|| < 1/n \} \in \sigma \left( \cup_{j=p-1}^\infty \sigma (X_j) \right)$$

and so the intersection for all $p$ of these is a tail event. Then the intersection over all $n$ of these tail events is a tail event. ■

From this it can be concluded that if you have a sequence of independent random variables, $\{X_k\}$ the set where it converges is either of probability 1 or probability 0. A similar conclusion holds for the set where the infinite sum of these random variables converges. This is stated in the next corollary. This incredible assertion is the next corollary.

**Corollary 57.6.6** Let $\{X_k\}$ be a sequence of random variables having values in a Banach space. Then

$$\lim_{n \to \infty} X_n(\omega)$$

either exists for a.e. $\omega$ or the convergence fails to take place for a.e. $\omega$. Also if

$$A \equiv \left\{ \omega : \sum_{k=1}^\infty X_k(\omega) \text{ converges} \right\},$$

then $P(A) = 0$ or 1.
57.7 Kolmogorov’s Inequality, Strong Law Of Large Numbers

Kolmogorov’s inequality is a very interesting inequality which depends on independence of a set of random vectors. The random vectors have values in \( \mathbb{R}^n \) or more generally some real separable Hilbert space.

**Lemma 57.7.1** If \( Y, X \) are independent random variables having values in a real separable Hilbert space, \( H \) with \( E \left( |X|^2 \right), E \left( |Y|^2 \right) < \infty \), then

\[
\int_{\Omega} (X, Y) dP = \left( \int_{\Omega} X dP, \int_{\Omega} Y dP \right).
\]

**Proof:** Let \( \{e_k\} \) be a complete orthonormal basis. Thus

\[
\int_{\Omega} (X, Y) dP = \int_{\Omega} \sum_{k=1}^{\infty} (X, e_k) (Y, e_k) dP
\]

Now

\[
\int_{\Omega} \sum_{k=1}^{\infty} |(X, e_k) (Y, e_k)| dP \leq \int_{\Omega} \left( \sum_{k} |(X, e_k)|^2 \right)^{1/2} \left( \sum_{k} |(Y, e_k)|^2 \right)^{1/2} dP
\]

\[
= \int_{\Omega} |X| |Y| dP \leq \left( \int_{\Omega} |X|^2 dP \right)^{1/2} \left( \int_{\Omega} |Y|^2 dP \right)^{1/2} < \infty
\]

and so by Fubini’s theorem,

\[
\int_{\Omega} (X, Y) dP = \int_{\Omega} \sum_{k=1}^{\infty} (X, e_k) (Y, e_k) dP = \sum_{k=1}^{\infty} \int_{\Omega} (X, e_k) (Y, e_k) dP
\]

\[
= \sum_{k=1}^{\infty} \int_{\Omega} (X, e_k) dP \int_{\Omega} (Y, e_k) dP = \sum_{k=1}^{\infty} \left( \int_{\Omega} X dP, e_k \right) \left( \int_{\Omega} Y dP, e_k \right) dP
\]

\[
= \left( \int_{\Omega} X dP, \int_{\Omega} Y dP \right) \quad \blacksquare
\]

Now here is Kolmogorov’s inequality.

**Theorem 57.7.2** Suppose \( \{X_k\}_{k=1}^{n} \) are independent with \( E \left( |X_k| \right) < \infty, E (X_k) = 0 \). Then for any \( \varepsilon > 0 \),

\[
P \left( \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} X_j \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^{n} E \left( |X_k|^2 \right).
\]
57.7. KOLMOGOROV’S INEQUALITY, STRONG LAW OF LARGE NUMBERS

Proof: Let

\[ A = \left[ \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} X_j \right| \geq \varepsilon \right] \]

Now let \( A_1 \equiv [|X_1| \geq \varepsilon] \) and if \( A_1, \ldots, A_m \) have been chosen,

\[ A_{m+1} \equiv \left[ \sum_{j=1}^{m+1} X_j \geq \varepsilon \right] \cap \bigcap_{r=1}^{m} \left[ \sum_{j=1}^{r} X_j < \varepsilon \right] \]

Thus the \( A_k \) partition \( A \) and \( \omega \in A_k \) means

\[ \sum_{j=1}^{k} X_j \geq \varepsilon \]

but this did not happen for \( \sum_{r=1}^{r} X_j \) for any \( r < k \). Note also that \( A_k \in \sigma(X_1, \ldots, X_k) \). Then from algebra,

\[
\left| \sum_{j=1}^{n} X_j \right|^2 = \left( \sum_{i=1}^{k} X_i + \sum_{j=k+1}^{n} X_j \sum_{i=1}^{k} X_i + \sum_{j=k+1}^{n} X_j \right)
\]

\[
= \sum_{j=1}^{k} X_j^2 + \sum_{i \leq k, j > k} (X_i, X_j) + \sum_{i \leq k, j > k} (X_j, X_i) + \sum_{i > k, j > k} (X_j, X_i)
\]

Written more succinctly,

\[
\left| \sum_{j=1}^{n} X_j \right|^2 = \sum_{j=1}^{k} X_j^2 + \sum_{j > k \text{ or } i > k} (X_i, X_j)
\]

Now multiply both sides by \( \mathcal{X}_{A_k} \) and integrate. Suppose \( i \leq k \) for one of the terms in the second sum. Then by Lemma 57.7.4 and \( A_k \in \sigma(X_1, \ldots, X_k) \), the two random vectors \( \mathcal{X}_{A_k} X_i, X_j \) are independent,

\[
\int_{\Omega} \mathcal{X}_{A_k} (X_i, X_j) \, dP = \left( \int_{\Omega} \mathcal{X}_{A_k} X_i \, dP \right) \left( \int_{\Omega} X_j \, dP \right) = 0
\]

the last equality holding because by assumption \( E(X_j) = 0 \). Therefore, it can be assumed both \( i, j \) are larger than \( k \) and

\[
\int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^{n} X_j \right|^2 \, dP = \int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^{k} X_j \right|^2 \, dP
\]

\[ + \sum_{j > k, i > k} \int_{\Omega} \mathcal{X}_{A_k} (X_i, X_j) \, dP \quad (57.7.10) \]
The last term on the right is interesting. Suppose \( i > j \). The integral inside the sum is of the form
\[
\int_{\Omega} (\mathbf{X}_i, \mathcal{X}_k \mathbf{X}_j) \, dP
\] (57.7.11)

The second factor in the inner product is in \( \sigma (\mathbf{X}_1, \cdots, \mathbf{X}_k, \mathbf{X}_j) \) and \( \mathbf{X}_i \) is not included in the list of random vectors. Thus by Lemma 57.3.4, the two random vectors \( \mathbf{X}_i, \mathcal{X}_k \mathbf{X}_j \) are independent and so (57.7.11) reduces to
\[
\left( \int_{\Omega} \mathbf{X}_i \, dP, \int_{\Omega} \mathcal{X}_k \mathbf{X}_j \, dP \right) = (0, \int_{\Omega} \mathcal{X}_k \mathbf{X}_j \, dP) = 0.
\]

A similar result holds if \( j > i \). Thus the mixed terms in the last term of (57.7.10) are all equal to 0. Hence (57.7.10) reduces to
\[
\int_{\Omega} \mathcal{X}_k \left| \sum_{j=1}^{n} \mathbf{X}_j \right|^2 \, dP = \int_{\Omega} \mathcal{X}_k \left| \sum_{j=1}^{k} \mathbf{X}_j \right|^2 \, dP + \sum_{i>k} \int_{\Omega} \mathcal{X}_k |\mathbf{X}_i|^2 \, dP
\]

and so
\[
\int_{\Omega} \mathcal{X}_k \left| \sum_{j=1}^{n} \mathbf{X}_j \right|^2 \, dP \geq \int_{\Omega} \mathcal{X}_k \left| \sum_{j=1}^{k} \mathbf{X}_j \right|^2 \, dP \geq \varepsilon^2 P (A_k).
\]

Now, summing these yields
\[
\varepsilon^2 P (A) \leq \int_{\Omega} \mathcal{X}_k \left| \sum_{j=1}^{n} \mathbf{X}_j \right|^2 \, dP \leq \int_{\Omega} \left| \sum_{j=1}^{n} \mathbf{X}_j \right|^2 \, dP
\]
\[
= \sum_{i,j} \int_{\Omega} (\mathbf{X}_i, \mathbf{X}_j) \, dP
\]

By independence of the random vectors the mixed terms of the above sum equal zero and so it reduces to
\[
\sum_{i=1}^{n} \int_{\Omega} |\mathbf{X}_i|^2 \, dP
\]

This theorem implies the following amazing result.

**Theorem 57.7.3** Let \( \{\mathbf{X}_k\}_{k=1}^{\infty} \) be independent random vectors having values in a separable real Hilbert space and suppose \( E (|\mathbf{X}_k|) < \infty \) for each \( k \) and \( E (\mathbf{X}_k) = 0 \). Suppose also that
\[
\sum_{j=1}^{\infty} E (|\mathbf{X}_j|^2) < \infty.
\]
Then
\[ \sum_{j=1}^{\infty} X_j \]
converges a.e.

**Proof:** Let \( \varepsilon > 0 \) be given. By Kolmogorov’s inequality, Theorem \( 57.7.2 \), it follows that for \( p \leq m < n \)
\[ P \left( \left[ \max_{m \leq k \leq n} \left| \sum_{j=m}^{k} X_j \right| \geq \varepsilon \right] \right) \leq \frac{1}{\varepsilon^2} \sum_{j=p}^{n} E \left( |X_j|^2 \right) \]
\[ \leq \frac{1}{\varepsilon^2} \sum_{j=p}^{n} E \left( |X_j|^2 \right). \]
Therefore, letting \( n \to \infty \) it follows that for all \( m, n \) such that \( p \leq m \leq n \)
\[ P \left( \left[ \max_{p \leq m \leq n} \left| \sum_{j=m}^{n} X_j \right| \geq \varepsilon \right] \right) \leq \frac{1}{\varepsilon^2} \sum_{j=p}^{\infty} E \left( |X_j|^2 \right). \]
It follows from the assumption
\[ \sum_{j=1}^{\infty} E \left( |X_j|^2 \right) < \infty \]
there exists a sequence, \( \{p_n\} \) such that if \( m \geq p_n \)
\[ P \left( \left[ \max_{k \geq m \geq p_n} \left| \sum_{j=m}^{k} X_j \right| \geq 2^{-n} \right] \right) \leq 2^{-n}. \]
By the Borel Cantelli lemma, Lemma \( 57.1.2 \), there is a set of measure 0, \( N \) such that for \( \omega \notin N \), \( \omega \) is in only finitely many of the sets,
\[ \left[ \max_{k \geq m \geq p_n} \left| \sum_{j=m}^{k} X_j \right| \geq 2^{-n} \right] \]
and so for \( \omega \notin N \), it follows that for large enough \( n \),
\[ \left[ \max_{k \geq m \geq p_n} \left| \sum_{j=m}^{k} X_j(\omega) \right| < 2^{-n} \right] \]
However, this says the partial sums \( \left\{ \sum_{j=1}^{k} X_j(\omega) \right\}_{k=1}^{\infty} \) are a Cauchy sequence. Therefore, they converge. ■

With this amazing result, there is a simple proof of the strong law of large numbers. In the following lemma, \( s_k \) and \( a_j \) could have values in any normed linear space.
Lemma 57.7.4 Suppose \( s_k \to s \). Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} s_k = s.
\]

Also if

\[
\sum_{j=1}^{\infty} \frac{a_j}{j}
\]

converges, then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j = 0.
\]

**Proof:** Consider the first part. Since \( s_k \to s \), it follows there is some constant, \( C \) such that \( |s_k| < C \) for all \( k \) and \( |s| < C \) also. Choose \( K \) so large that if \( k \geq K \), then for \( n > K \),

\[
|s - s_k| < \varepsilon/2.
\]

\[
|s - \frac{1}{n} \sum_{k=1}^{n} s_k| \leq \frac{1}{n} \sum_{k=1}^{n} |s_k - s|
\]

\[
= \frac{1}{n} \sum_{k=1}^{K} |s_k - s| + \frac{1}{n} \sum_{k=K}^{n} |s_k - s|
\]

\[
\leq \frac{2CK}{n} + \frac{\varepsilon}{2} \frac{n-K}{n} < \frac{2CK}{n} + \frac{\varepsilon}{2}
\]

Therefore, whenever \( n \) is large enough,

\[
|s - \frac{1}{n} \sum_{k=1}^{n} s_k| < \varepsilon.
\]

Now consider the second claim. Let

\[
s_k = \sum_{j=1}^{k} \frac{a_j}{j}
\]

and \( s = \lim_{k \to \infty} s_k \) Then by the first part,

\[
s = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} s_k = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} \frac{a_j}{j}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j \sum_{k=j}^{n} 1 = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j (n-j)
\]

\[
= \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{a_j}{j} - \frac{1}{n} \sum_{j=1}^{n} a_j \right) = s - \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j \quad \blacksquare
\]

Now here is the strong law of large numbers.
Theorem 57.7.5 Suppose \( \{X_k\} \) are independent random variables and \( E(|X_k|) < \infty \) for each \( k \) and \( E(X_k) = m_k \). Suppose also

\[
\sum_{j=1}^{\infty} \frac{1}{j^2} E \left( |X_j - m_j|^2 \right) < \infty. \tag{57.7.12}
\]

Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (X_j - m_j) = 0
\]

Proof: Consider the sum

\[
\sum_{j=1}^{\infty} \frac{X_j - m_j}{j}.
\]

This sum converges a.e. because of \(\text{[\textit{Weaker}] and Theorem 57.7.3 applied to the random vectors } \left\{ \frac{X_j - m_j}{j} \right\}. \) Therefore, from Lemma 57.7.4 it follows that for a.e. \( \omega \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (X_j(\omega) - m_j) = 0 \quad \blacksquare
\]

The next corollary is often called the strong law of large numbers. It follows immediately from the above theorem.

Corollary 57.7.6 Suppose \( \{X_j\}_{j=1}^{\infty} \) are independent having mean \( m \) and variance equal to

\[
\sigma^2 \equiv \int_{\Omega} |X_j - m|^2 dP < \infty.
\]

Then for a.e. \( \omega \in \Omega \)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} X_j(\omega) = m
\]

57.8 The Characteristic Function

One of the most important tools in probability is the characteristic function. To begin with, assume the random variables have values in \( \mathbb{R}^p \).

Definition 57.8.1 Let \( X \) be a random variable as above. The characteristic function is

\[
\phi_X(t) \equiv E(e^{it \cdot X}) \equiv \int_{\Omega} e^{it \cdot X(\omega)} dP = \int_{\mathbb{R}^p} e^{it \cdot x} d\lambda_X
\]

the last equation holding by Proposition 57.1.12.

Recall the following fundamental lemma and definition, Lemma 57.1.12 on Page 1059.
Definition 57.8.2 For $T \in G^*$, define $F T, F T^{-1} T \in G^*$ by

$$F T (\phi) \equiv T (F \phi), \quad F T^{-1} (\phi) \equiv T (F^{-1} \phi)$$

Lemma 57.8.3 $F$ and $F^{-1}$ are both one to one, onto, and are inverses of each other.

The main result on characteristic functions is the following.

Theorem 57.8.4 Let $X$ and $Y$ be random vectors with values in $\mathbb{R}^p$ and suppose $E (e^{it \cdot X}) = E (e^{it \cdot Y})$ for all $t \in \mathbb{R}^p$. Then $\lambda_X = \lambda_Y$.

Proof: For $\psi \in G$, let $\lambda_X (\psi) \equiv \int_{\mathbb{R}^p} \psi d\lambda_X$ and $\lambda_Y (\psi) \equiv \int_{\mathbb{R}^p} \psi d\lambda_Y$. Thus both $\lambda_X$ and $\lambda_Y$ are in $G^*$. Then letting $\psi \in G$ and using Fubini’s theorem,

$$\int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot y} \psi (t) dt d\lambda_Y = \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot y} d\lambda_Y \psi (t) dt$$

$$= \int_{\mathbb{R}^p} E (e^{it \cdot Y}) \psi (t) dt$$

$$= \int_{\mathbb{R}^p} E (e^{it \cdot X}) \psi (t) dt$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot x} d\lambda_X \psi (t) dt$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot x} \psi (t) dt d\lambda_X.$$

Thus $\lambda_Y (F^{-1} \psi) = \lambda_X (F^{-1} \psi)$. Since $\psi \in G$ is arbitrary and $F^{-1}$ is onto, this implies $\lambda_X = \lambda_Y$ in $G^*$. But $G$ is dense in $C_0 (\mathbb{R}^p)$ from the Stone Weierstrass theorem and so $\lambda_X = \lambda_Y$ as measures. Recall from real analysis the dual space of $C_0 (\mathbb{R}^p)$ is the space of complex measures.

Alternatively, the above shows that since $F^{-1}$ is onto, for all $\psi \in G$,

$$\int_{\mathbb{R}^p} \psi d\lambda_Y = \int_{\mathbb{R}^p} \psi d\lambda_X$$

and then, by a use of the Stone Weierstrass theorem, the above will hold for all $\psi \in C_0 (\mathbb{R}^p)$ and now, by the Riesz representation theorem for positive linear functionals, the two measures are equal.

You can also give a version of this theorem in which reference is made only to the probability distribution measures.

Definition 57.8.5 For $\mu$ a probability measure on the Borel sets of $\mathbb{R}^n$,

$$\phi_\mu (t) \equiv \int_{\mathbb{R}^n} e^{it \cdot x} d\mu.$$

Theorem 57.8.6 Let $\mu$ and $\nu$ be probability measures on the Borel sets of $\mathbb{R}^p$ and suppose $\phi_\mu (t) = \phi_\nu (t)$. Then $\mu = \nu$.

Proof: The proof is identical to the above. Just replace $\lambda_X$ with $\mu$ and $\lambda_Y$ with $\nu$. ■
57.9 Conditional Probability

Here I will consider the concept of conditional probability depending on the theory of differentiation of general Radon measures. This leads to a different way of thinking about independence.

If \( X, Y \) are two random vectors defined on a probability space having values in \( \mathbb{R}^{p_1} \) and \( \mathbb{R}^{p_2} \) respectively, and if \( E \) is a Borel set in the appropriate space, then \((X, Y)\) is a random vector with values in \( \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \) and \( \lambda_{(X,Y)}(E \times \mathbb{R}^{p_2}) = \lambda_{X}(E) \), \( \lambda_{(X,Y)}(\mathbb{R}^{p_1} \times E) = \lambda_{Y}(E) \). Thus, by Theorem 28.3.3 on Page 1071, there exist probability measures, denoted here by \( \lambda_{X|Y} \), such that whenever \( E \) is a Borel set in \( \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \),

\[
\int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} X_E d\lambda_{X}(X, Y) = \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} X_E d\lambda_{Y|X} d\lambda_{X},
\]

and

\[
\int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} X_E d\lambda_{Y}(X, Y) = \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} X_E d\lambda_{X|Y} d\lambda_{Y}.
\]

**Definition 57.9.1** Let \( X \) and \( Y \) be two random vectors defined on a probability space. The conditional probability measure of \( Y \) given \( X \) is the measure \( \lambda_{Y|X} \) in the above. Similarly the conditional probability measure of \( X \) given \( Y \) is the measure \( \lambda_{X|Y} \).

More generally, one can use the theory of slicing measures to consider any finite list of random vectors, \( \{X_i\} \), defined on a probability space with \( X_i \in \mathbb{R}^{p_i} \), and write the following for \( E \) a Borel set in \( \prod_{i=1}^{n} \mathbb{R}^{p_i} \).

\[
\int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} X_E d\lambda_{(x_1, \cdots, x_n)} = \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_{n-1}}} \int_{\mathbb{R}^{p_n}} X_E d\lambda_{X_n|(x_1, \cdots, x_{n-1})} d\lambda_{(x_1, \cdots, x_{n-1})}
\]

\[
= \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_{n-2}}} \int_{\mathbb{R}^{p_{n-1}}} \int_{\mathbb{R}^{p_n}} X_E d\lambda_{X_n|(x_1, \cdots, x_{n-1})} d\lambda_{X_{n-1}|(x_1, \cdots, x_{n-2})} d\lambda_{(x_1, \cdots, x_{n-2})}
\]

\[
\vdots
\]

\[
\int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_n}} X_E d\lambda_{X_n|(x_1, \cdots, x_{n-1})} d\lambda_{X_{n-1}|(x_1, \cdots, x_{n-2})} \cdots d\lambda_{X_2|x_1} d\lambda_{X_1}. \quad (57.9.13)
\]

Obviously, this could have been done in any order in the iterated integrals by simply modifying the “given” variables, those occurring after the symbol |, to be those which have been integrated in an outer level of the iterated integral. For simplicity, write

\[
\lambda_{X_n|(x_1, \cdots, x_{n-1})} = \lambda_{X_n|x_1, \cdots, x_{n-1}}
\]

**Definition 57.9.2** Let \( \{X_1, \cdots, X_n\} \) be random vectors defined on a probability space having values in \( \mathbb{R}^{p_1}, \cdots, \mathbb{R}^{p_n} \) respectively. The random vectors are independent if for every \( E \) a Borel set in \( \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n} \),

\[
\int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} X_E d\lambda_{(x_1, \cdots, x_n)}
\]
\[ = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \lambda_E d\lambda_{X_n} d\lambda_{X_{n-1}} \cdots d\lambda_{X_2} d\lambda_{X_1} \tag{57.9.14} \]

and the iterated integration may be taken in any order. If \( A \) is any set of random vectors defined on a probability space, \( A \) is independent if any finite set of random vectors from \( A \) is independent.

Thus, the random vectors are independent exactly when the dependence on the givens in 57.9.13 can be dropped.

Does this amount to the same thing as discussed earlier? Suppose you have three random variables \( X, Y, Z \). Let \( A = X^{-1}(E), B = Y^{-1}(F), C = Z^{-1}(G) \) where \( E, F, G \) are Borel sets. Thus these inverse images are typical sets in \( \sigma(X), \sigma(Y), \sigma(Z) \) respectively. First suppose that the random variables are independent in the earlier sense. Then

\[ P(A \cap B \cap C) = P(A) P(B) P(C) \]

\[ = \int_{\mathbb{R}^n} \lambda_E(x) d\lambda_X \int_{\mathbb{R}^n} \lambda_F(y) d\lambda_Y \int_{\mathbb{R}^n} \lambda_G(z) d\lambda_Z \]

\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda_E(x) \lambda_F(y) \lambda_G(z) d\lambda_Z d\lambda_Y d\lambda_X \]

Also

\[ P(A \cap B \cap C) = \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} \lambda_E(x) \lambda_F(y) \lambda_G(z) d\lambda(x,y,z) \]

\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda_E(x) \lambda_F(y) \lambda_G(z) d\lambda_Z d\lambda_Y d\lambda_X |x| \]

Thus

\[ = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda_E(x) \lambda_F(y) \lambda_G(z) d\lambda_Z d\lambda_Y d\lambda_X |x| \]

Now letting \( G = \mathbb{R}^p \), it follows that

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda_E(x) \lambda_F(y) d\lambda_Y d\lambda_X = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \lambda_E(x) \lambda_F(y) d\lambda_Y |x| d\lambda_X \]

By uniqueness of the slicing measures or an application of the Besikovitch differentiation theorem, it follows that for \( \lambda_X \) a.e. \( x \),

\[ \lambda_Y = \lambda_Y|x| \]
Thus, using this in the above,

\[
\int_{\mathbb{R}^1} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{X}_E (x) \mathcal{X}_F (y) \mathcal{X}_G (z) \, d\lambda_Z d\lambda_Y d\lambda_X
\]

and also it reduces to

\[
\int_{\mathbb{R}^1 \times \mathbb{R}^2} \int_{\mathbb{R}^3} \mathcal{X}_E (x) \mathcal{X}_F (y) \mathcal{X}_G (z) \, d\lambda_Z d\lambda_{(X,Y)}
\]

Now by uniqueness of the slicing measures again, for \(\lambda_{(X,Y)}\) a.e. \((x, y)\), it follows that

\[
\lambda_Z = \lambda_{Z|XY}
\]

Similar conclusions hold for \(\lambda_X, \lambda_Y\). In each case, off a set of measure zero the distribution measures equal the slicing measures.

Conversely, if the distribution measures equal the slicing measures off sets of measure zero as described above, then it is obvious that the random variables are independent. The same reasoning applies for any number of random variables.

Thus this gives a different and more analytical way to think of independence of finitely many random variables. Clearly, the argument given above will apply to any finite set of random variables.

**Proposition 57.9.3** Equations [57.9.14] and [57.9.13] hold with \(\mathcal{X}_E\) replaced by any nonnegative Borel measurable function and for any bounded continuous function or for any function in \(L^1\).

**Proof:** The two equations hold for simple functions in place of \(\mathcal{X}_E\) and so an application of the monotone convergence theorem applied to an increasing sequence of simple functions converging pointwise to a given nonnegative Borel measurable function yields the conclusion of the proposition in the case of the nonnegative Borel function. For a bounded continuous function or one in \(L^1\), one can apply the result just established to the positive and negative parts of the real and imaginary parts of the function.

**Lemma 57.9.4** Let \(X_1, \ldots, X_n\) be random vectors with values in \(\mathbb{R}^{p_1}, \ldots, \mathbb{R}^{p_n}\) respectively and let \(g : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n} \to \mathbb{R}^k\) be Borel measurable. Then \(g(X_1, \ldots, X_n)\) is a random vector with values in \(\mathbb{R}^k\) and if \(h : \mathbb{R}^k \to [0, \infty)\), then

\[
\int_{\mathbb{R}^k} h(y) \, d\lambda_{g(X_1, \ldots, X_n)} (y) = \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} h(g(X_1, \ldots, X_n)) \, d\lambda_{(X_1, \ldots, X_n)}. \quad (57.9.15)
\]
If \( X_i \) is a random vector with values in \( \mathbb{R}^{p_i}, i = 1, 2, \cdots \) and if \( g_i : \mathbb{R}^{p_i} \to \mathbb{R}^{k_i} \), where \( g_i \) is Borel measurable, then the random vectors \( g_i(X_i) \) are also independent whenever the \( X_i \) are independent.

**Proof:** First let \( E \) be a Borel set in \( \mathbb{R}^k \). From the definition,

\[
\lambda_{g(X_1, \cdots, X_n)}(E) = P(g(X_1, \cdots, X_n) \in E) = P((X_1, \cdots, X_n) \in g^{-1}(E)) = \lambda(X_1, \cdots, X_n)(g^{-1}(E))
\]

\[
\int_{\mathbb{R}^k} \mathcal{L}_E d\lambda_{g(X_1, \cdots, X_n)} = \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{L}_{g^{-1}(E)} d\lambda(X_1, \cdots, X_n)
\]

This proves 57.9.15 in the case when \( h \) is \( \mathcal{L}_E \). To prove it in the general case, approximate the nonnegative Borel measurable function with simple functions for which the formula is true, and use the monotone convergence theorem.

It remains to prove the last assertion that functions of independent random vectors are also independent random vectors. Let \( E \) be a Borel set in \( \mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n} \). Then for

\[
\pi_i (x_1, \cdots, x_n) \equiv x_i,
\]

\[
\int_{\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}} \mathcal{L}_E d\lambda_{(g_1, \cdots, g_n)(X_1, \cdots, X_n)}
\]

\[
= \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{L}_E (g_1 \circ \pi_1, \cdots, g_n \circ \pi_n) d\lambda(X_1, \cdots, X_n)
\]

\[
= \int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_n}} \mathcal{L}_E (g_1 \circ \pi_1, \cdots, g_n \circ \pi_n) d\lambda_{X_1} \cdots d\lambda_{X_n}
\]

and this proves the last assertion.

**Proposition 57.9.5** Let \( \nu_1, \cdots, \nu_n \) be Radon probability measures defined on \( \mathbb{R}^p \). Then there exists a probability space and independent random vectors \( \{X_1, \cdots, X_n\} \) defined on this probability space such that \( \lambda_{X_i} = \nu_i \).

**Proof:** Let \( (\Omega, \mathcal{S}, P) \equiv ((\mathbb{R}^p)^n, \mathcal{S}_1 \times \cdots \times \mathcal{S}_n, \nu_1 \times \cdots \times \nu_n) \) where this is just the product \( \sigma \) algebra and product measure which satisfies the following for measurable rectangles.

\[
(\nu_1 \times \cdots \times \nu_n)\left(\prod_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} \nu_i(E_i).
\]
Now let $X_i(x_1, \ldots, x_i, \ldots, x_n) = x_i$. Then from the definition, if $E$ is a Borel set in $\mathbb{R}^p$, 
$$
\lambda_{X_i}(E) \equiv P\{X_i \in E\} = (\nu_1 \times \cdots \times \nu_n)(\mathbb{R}^p \times \cdots \times E \times \cdots \times \mathbb{R}^p) = \nu_i(E).
$$
Let $\mathcal{M}$ consist of all Borel sets of $(\mathbb{R}^p)^n$ such that 
$$
\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} X_E(x_1, \ldots, x_n) \ d\lambda_{X_1} \cdots d\lambda_{X_n} = \int_{(\mathbb{R}^p)^n} X_E d\lambda_{(X_1, \ldots, X_n)}.
$$
From what was just shown and the definition of $(\nu_1 \times \cdots \times \nu_n)$ that $\mathcal{M}$ contains all sets of the form $\prod_{i=1}^n E_i$ where each $E_i \in$ Borel sets of $\mathbb{R}^p$. Therefore, $\mathcal{M}$ contains the algebra of all finite disjoint unions of such sets. It is also clear that $\mathcal{M}$ is a monotone class and so by the theorem on monotone classes, $\mathcal{M}$ equals the Borel sets. You could also note that $\mathcal{M}$ is closed with respect to complements and countable disjoint unions and apply Lemma 10.12.3. Therefore, the given random vectors are independent and this proves the proposition.

The following Lemma was proved earlier in a different way.

**Lemma 57.9.6** If $\{X_i\}_{i=1}^n$ are independent random variables having values in $\mathbb{R}$, 
$$
E\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n E(X_i).
$$

**Proof:** By Lemma 57.4.3 and denoting by $P$ the product, $\prod_{i=1}^n X_i$, 
$$
E\left(\prod_{i=1}^n X_i\right) = \int_{\mathbb{R}} z d\lambda_P(z) = \int_{\mathbb{R}^n} \prod_{i=1}^n x_i d\lambda_{(X_1, \ldots, X_n)} = \int_{\mathbb{R}^n} \prod_{i=1}^n x_i d\lambda_{X_1} \cdots d\lambda_{X_n} = \prod_{i=1}^n E(X_i).
$$

### 57.10 Conditional Expectation

**Definition 57.10.1** Let $X$ and $Y$ be random vectors having values in $\mathbb{F}^{p_1}$ and $\mathbb{F}^{p_2}$ respectively. Then if 
$$
\int |x| d\lambda_{X|Y}(x) < \infty,
$$
we define 
$$
E(X|Y) \equiv \int x d\lambda_{X|Y}(x).
$$

**Proposition 57.10.2** Suppose $\int_{\mathbb{F}^{p_1} \times \mathbb{F}^{p_2}} |x| d\lambda_{(X,Y)}(x) < \infty$. Then $E(X|Y)$ exists for $\lambda_Y$ a.e. $y$ and 
$$
\int_{\mathbb{F}^{p_2}} E(X|Y) d\lambda_Y = \int_{\mathbb{F}^{p_1}} x d\lambda_X(x) = E(X).$$
Proof: $\infty > \int_{\mathcal{F}_1 \times \mathcal{F}_2} |x| \, d\lambda(x, y) = \int_{\mathcal{F}_2} \int_{\mathcal{F}_1} |x| \, d\lambda_{X|y} (x) \, d\lambda_Y (y)$ and so

$$\int_{\mathcal{F}_1} |x| \, d\lambda_{X|y} (x) < \infty$$

$\lambda_Y$ a.e. Now

$$\int_{\mathcal{F}_2} E(X|y) \, d\lambda_Y$$

$$= \int_{\mathcal{F}_2} \int_{\mathcal{F}_1} xd\lambda_{X|y} (x) \, d\lambda_Y (y) = \int_{\mathcal{F}_1 \times \mathcal{F}_2} xd\lambda_{(X,Y)}$$

$$= \int_{\mathcal{F}_1} \int_{\mathcal{F}_2} xd\lambda_{X|y} (y) \, d\lambda_X (x) = \int_{\mathcal{F}_2} xd\lambda_X (x) = E(X).$$

**Definition 57.10.3** Let $\{X_n\}$ be any sequence, finite or infinite, of random variables with values in $\mathbb{R}$ which are defined on some probability space, $(\Omega, \mathcal{S}, P)$. We say $\{X_n\}$ is a Martingale if

$$E(X_n|x_{n-1}, \ldots, x_1) = x_{n-1}$$

and we say $\{X_n\}$ is a submartingale if

$$E(X_n|x_{n-1}, \ldots, x_1) \geq x_{n-1}.$$

Next we define what is meant by an upcrossing.

**Definition 57.10.4** Let $\{x_i\}_{i=1}^I$ be any sequence of real numbers, $I \leq \infty$. Define an increasing sequence of integers $\{m_k\}$ as follows. $m_1$ is the first integer $\geq 1$ such that $x_{m_1} \leq a, m_2$ is the first integer larger than $m_1$ such that $x_{m_2} \geq b, m_3$ is the first integer larger than $m_2$ such that $x_{m_3} \leq a$, etc. Then each sequence, $\{x_{m_{2k-1}}, \ldots, x_{m_{2k}}\}$, is called an upcrossing of $[a,b]$.

**Proposition 57.10.5** Let $\{X_i\}_{i=1}^\infty$ be a finite sequence of real random variables defined on $\Omega$ where $(\Omega, \mathcal{S}, P)$ is a probability space. Let $U_{[a,b]} (\omega)$ denote the number of upcrossings of $X_i (\omega)$ of the interval $[a,b]$. Then $U_{[a,b]}$ is a random variable.

**Proof:** Let $X_0 (\omega) \equiv a+1, let Y_0 (\omega) \equiv 0, and let Y_k (\omega) remain 0 for $k = 0, \ldots, l$ until $X_{l} (\omega) \leq a$. When this happens (if ever), $Y_{l+1} (\omega) \equiv 1$. Then let $Y_i (\omega) remain 1$ for $i = l + 1, \ldots, r$ until $X_r (\omega) \geq b$ when $Y_{r+1} (\omega) \equiv 0$. Let $Y_k (\omega) remain 0 for k \geq r + 1 until X_k (\omega) \leq a$ when $Y_k (\omega) \equiv 1 and continue in this way. Thus the upcrossings of $X_i (\omega)$ are identified as unbroken strings of ones with a zero at each end, with the possible exception of the last string of ones which may be missing the zero at the upper end and may or may not be an upcrossing.

Note also that $Y_0$ is measurable because it is identically equal to 0 and that if $Y_k$ is measurable, then $Y_{k+1}$ is measurable because the only change in going from
57.10. CONDITIONAL EXPECTATION

$k$ to $k + 1$ is a change from 0 to 1 or from 1 to 0 on a measurable set determined by $X_k$. Now let

$$Z_k(\omega) = \begin{cases} 
1 & \text{if } Y_k(\omega) = 1 \text{ and } Y_{k+1}(\omega) = 0, \\
0 & \text{otherwise},
\end{cases}$$

if $k < n$ and

$$Z_n(\omega) = \begin{cases} 
1 & \text{if } Y_n(\omega) = 1 \text{ and } X_n(\omega) \geq b, \\
0 & \text{otherwise}.
\end{cases}$$

Thus $Z_k(\omega) = 1$ exactly when an upcrossing has been completed and each $Z_i$ is a random variable.

$$U_{[a,b]}(\omega) = \sum_{k=1}^{n} Z_k(\omega)$$

so $U_{[a,b]}$ is a random variable as claimed.

The following corollary collects some key observations found in the above construction.

**Corollary 57.10.6** $U_{[a,b]}(\omega) \leq$ the number of unbroken strings of ones in the sequence, $\{Y_k(\omega)\}$ there being at most one unbroken string of ones which produces no upcrossing. Also

$$Y_i(\omega) = \psi_i \left( \left\{ X_j(\omega) \right\}_{j=1}^{i-1} \right), \quad (57.10.16)$$

where $\psi_i$ is some function of the past values of $X_j(\omega)$.

**Lemma 57.10.7** (upcrossing lemma) Let $\{X_i\}_{i=1}^{n}$ be a submartingale and suppose $E(|X_n|) < \infty$.

Then

$$E(U_{[a,b]}) \leq \frac{E(|X_n| + |a|)}{b - a}.$$

**Proof:** Let $\phi(x) \equiv a + (x - a)^+$. Thus $\phi$ is a convex and increasing function.

$$\phi(X_{k+r}) - \phi(X_k) = \sum_{i=k+1}^{k+r} \phi(X_i) - \phi(X_{i-1})$$

$$= \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) Y_i + \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1}))(1 - Y_i).$$

The upcrossings of $\phi(X_i)$ are exactly the same as the upcrossings of $X_i$ and from Formula 57.10.16

$$E \left( \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1}))(1 - Y_i) \right)$$
By Jensen’s inequality, Problem III of Chapter 55.

$$\geq \sum_{i=k+1}^{k+r} \int_{\mathbb{R}^{i-1}} \left( 1 - \psi_i \left( \{x_j\}_{j=1}^{i-1} \right) \right) d\lambda_{X_i | x_{i-1}, \ldots, x_{i-1}}$$

Because of the assumption that our sequence of random variables is a submartingale and the observation that \( \phi \) is both convex and increasing.

Now let the unbroken strings of ones for \( \{Y_i(\omega)\} \) be

$$\{k_1, \ldots, k_1 + r_1\}, \{k_2, \ldots, k_2 + r_2\}, \ldots, \{k_m, \ldots, k_m + r_m\} \quad (57.10.17)$$

where \( m = V(\omega) \equiv \) the number of unbroken strings of ones in the sequence \( \{Y_i(\omega)\} \).

By Corollary 57.10.6 \( V(\omega) \geq U_{[a,b]}(\omega) \).

$$\phi(\omega) \leq \sum_{k=1}^{n} (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) Y_k(\omega)$$

Summing the first sum over the unbroken strings of ones (the terms in which \( Y_i(\omega) = 0 \) contribute nothing), implies

$$\phi(\omega) \leq \sum_{k=1}^{n} (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) (1 - Y_k(\omega)).$$
\[ \phi (X_k (\omega)) - \phi (X_{k-1} (\omega)) (1 - Y_k (\omega)) \]

where the zero on the right side results from a string of ones which does not produce an upcrossing. It is here that we use \( \phi (x) \geq a \). Such a string begins with \( \phi (X_k (\omega)) = a \) and results in an expression of the form \( \phi (X_{k+m} (\omega)) - \phi (X_k (\omega)) \geq 0 \) since \( \phi (X_{k+m} (\omega)) \geq a \). If we had not replaced \( X_k \) with \( \phi (X_k) \), it would have been possible for \( \phi (X_{k+m} (\omega)) \) to be less than \( a \) and the zero in the above could have been a negative number.

Therefore from Formula 57.10.18,
\[
(b - a) E (U_{[a,b]} (\omega)) \leq E (\phi (X_n) - \phi (X_1)) \leq E (\phi (X_n) - a) \\
= E \left( (X_n - a)^+ \right) \leq |a| + E (|X_n|)
\]

and this proves the lemma.

**Theorem 57.10.8 (submartingale convergence theorem)** Let \( \{X_i\}_{i=1}^\infty \) be a submartingale with \( K \equiv \sup \{E (|X_n|) : n \geq 1\} < \infty \). Then there exists a random variable, \( X_\infty \), such that \( E (|X_\infty|) \leq K \) and \( \lim_{n \to \infty} X_n (\omega) = X_\infty (\omega) \) a.e.

**Proof:** Let \( a, b \in \mathbb{Q} \) and let \( a < b \). Let \( U_{[a,b]}^n (\omega) \) be the number of upcrossings of \( \{X_i (\omega)\}_{i=1}^n \). Then let
\[ U_{[a,b]} (\omega) \equiv \lim_{n \to \infty} U_{[a,b]}^n (\omega) = \text{number of upcrossings of } \{X_i (\omega)\} \]

By the upcrossing lemma,
\[ E \left( U_{[a,b]}^n \right) \leq \frac{E (|X_n|) + |a|}{b - a} \leq \frac{K + |a|}{b - a} \]

and so by the monotone convergence theorem,
\[ E (U_{[a,b]}) \leq \frac{K + |a|}{b - a} < \infty \]

which shows \( U_{[a,b]} (\omega) \) is finite a.e., for all \( \omega \notin S_{[a,b]} \) where \( P (S_{[a,b]}) = 0 \). Define
\[ S \equiv \cup \{S_{[a,b]} : a, b \in \mathbb{Q}, a < b\} \]

Then \( P (S) = 0 \) and if \( \omega \notin S \), \( \{X_k\}_{k=1}^\infty \) has only finitely many upcrossings of every interval having rational endpoints. Thus, for \( \omega \notin S \), \( \limsup_{k \to \infty} X_k (\omega) = \liminf_{k \to \infty} X_k (\omega) = \lim_{k \to \infty} X_k (\omega) \equiv X_\infty (\omega) \). Letting \( X_\infty (\omega) = 0 \) for \( \omega \in S \), Fatou’s lemma implies
\[
\int_\Omega |X_\infty| dP = \int_\Omega \lim_{n \to \infty} \inf |X_n| dP \leq \liminf_{n \to \infty} \int_\Omega |X_n| dP \leq K
\]

and so this proves the theorem.
CHAPTER 57. BASIC PROBABILITY

57.11 Characteristic Functions And Independence

There is a way to tell if random vectors are independent by using their characteristic functions.

**Proposition 57.11.1** If $X_i$ is a random vector having values in $\mathbb{R}^p$, then the random vectors are independent if and only if

$$E(e^{iP}) = \prod_{j=1}^n E(e^{it_jX_j})$$

where $P \equiv \sum_{j=1}^n t_j \cdot X_j$ for $t_j \in \mathbb{R}^p$.

The proof of this proposition will depend on the following lemma.

**Lemma 57.11.2** Let $Y$ be a random vector with values in $\mathbb{R}^p$ and let $f$ be bounded and measurable with respect to the Radon measure $\lambda_Y$, and satisfy

$$\int f(y) e^{it \cdot y} d\lambda_Y = 0$$

for all $t \in \mathbb{R}^p$. Then $f(y) = 0$ for $\lambda_Y$ a.e. $y$.

**Proof:** You could write the following for $\phi \in \mathcal{G}$

$$\int \phi(t) \int f(y) e^{it \cdot y} d\lambda_Y dt = 0 = \int f(y) \left( \int \phi(t) e^{it \cdot y} dt \right) d\lambda_Y$$

and now recall that the inverse Fourier transform maps $\mathcal{G}$ onto $\mathcal{G}$. Hence

$$\int f(y) \psi(y) d\lambda_Y = 0$$

for all $\psi \in \mathcal{G}$. Thus this is also so for every $\psi \in C_0^\infty(\mathbb{R}^p)$ by an obvious application of the Stone Weierstrass theorem. Let $\{\phi_k\}$ be a sequence of functions in $C_0^\infty(\mathbb{R}^p)$ which converges to

$$\text{sgn}(f) \equiv \begin{cases} \tilde{f}/|f| & \text{if } f \neq 0 \\ 0 & \text{if } f = 0 \end{cases}$$

pointwise and in $L^1(\mathbb{R}^p, \lambda_Y)$, each $|\phi_k| \leq 2$. Then for any $\psi \in C_0^\infty(\mathbb{R}^p)$,

$$0 = \int f(y) \phi_n(y) \psi(y) d\lambda_Y \rightarrow \int |f(y)| \psi(y) d\lambda_Y$$

Also, the above holds for any $\psi \in C_0(\mathbb{R}^p)$ as can be seen by taking such a $\psi$ and convolving with a mollifier. By the Riesz representation theorem, $f(y) = 0$ $\lambda_Y$ a.e. (The measure $\mu(E) \equiv \int_E |f(y)| d\lambda_Y$ equals 0.)

**Proof of the proposition:** If the $X_j$ are independent, the formula follows from Lemma 57.9.4 and Lemma 57.9.3.
57.11. CHARACTERISTIC FUNCTIONS AND INDEPENDENCE

Now suppose the formula holds. Thus

\[
\prod_{j=1}^{n} E(e^{it_j X_j}) = \\
= \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} e^{i t_1 x_1} e^{i t_2 x_2} \cdots e^{i t_n x_n} d\lambda x_1, d\lambda x_2, \cdots, d\lambda x_n = E(e^{iP})
\]

Then from the above Lemma 57.11.2, the following equals 0 for \(\lambda x_n\) a.e. \(x_n\).

\[
\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} e^{i t_1 x_1} e^{i t_2 x_2} \cdots e^{i t_{n-1} x_{n-1}} d\lambda x_1 d\lambda x_2 \cdots d\lambda x_{n-1} = 0.
\]

Let \(t_i = 0\) for \(i = 1, 2, \cdots, n - 2\). Then this implies

\[
\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} e^{i t_{n-1} x_{n-1}} d\lambda x_{n-1} = \int_{\mathbb{R}^p} e^{i t_{n-1} x_{n-1}} d\lambda x_{n-1} x_n
\]

By the fact that the characteristic function determines the distribution measure, Theorem 57.8.3, it follows that for these \(x_n\) off a set of \(\lambda x_n\) measure zero, \(\lambda x_{n-1} x_n = \lambda x_{n-1} x_{n}x_n\). Returning to 57.11.2, one can replace \(\lambda x_{n-1} x_n\) with \(\lambda x_{n-1}\) to obtain

\[
\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} e^{i t_1 x_1} e^{i t_2 x_2} \cdots e^{i t_n x_n} d\lambda x_1 d\lambda x_2 \cdots d\lambda x_{n-1} d\lambda x_n
\]

Next let \(t_n = 0\) and applying the above Lemma 57.11.2 again, this implies that for \(\lambda x_{n-1}\) a.e. \(x_{n-1}\), the following equals 0.

\[
\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} e^{i t_1 x_1} e^{i t_2 x_2} \cdots e^{i t_{n-2} x_{n-2}} d\lambda x_1 d\lambda x_2 \cdots d\lambda x_{n-2} d\lambda x_{n-1} x_n
\]

Let \(t_i = 0\) for \(i = 1, 2, \cdots, n - 3\). Then you obtain

\[
\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} e^{i t_{n-2} x_{n-2}} d\lambda x_{n-2} = \int_{\mathbb{R}^p} e^{i t_{n-2} x_{n-2}} d\lambda x_{n-2} x_{n-1}
\]

and so \(\lambda x_{n-2} = \lambda x_{n-2} x_{n-1}\) for \(x_{n-1}\) off a set of \(\lambda x_{n-1}\) measure zero. Continuing this way, it follows that

\[
\lambda x_{n-k} = \lambda x_{n-k} x_{n-k-1} \cdots x_{n-k+1}
\]
for $x_{n-k+1}$ off a set of $\lambda x_{n-k+1}$ measure zero. Thus if $E$ is Borel in $\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_1}$,

$$\int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_1}} E_X d\lambda(x_1, \ldots, x_n) =$$

$$\int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_1}} E_X d\lambda x_1 | x_2 \ldots x_n d\lambda x_2 | x_3 \ldots x_n d\lambda x_3 | x_4 \ldots x_n d\lambda x_n \cdots d\lambda x_{n-1} | x_n d\lambda x_n$$

$$\vdots$$

$$= \int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_1}} E_X d\lambda x_1 d\lambda x_2 \cdots d\lambda x_n$$

One could achieve this iterated integral in any order by similar arguments to the above. By Definition 57.9.2 and the discussion which follows, this implies that the random variables $X_i$ are independent. 

Here is another proof of the Doob Dynkin lemma based on differentiation theory.

**Lemma 57.11.3** Suppose $X, Y_1, Y_2, \ldots, Y_k$ are random vectors $X$ having values in $\mathbb{R}^n$ and $Y_j$ having values in $\mathbb{R}^{p_j}$ and $X, Y_j \in L^1(\Omega)$.

Suppose $X$ is $\sigma(Y_1, \ldots, Y_k)$ measurable. Thus

$$\{X^{-1}(E) : E \text{ Borel} \} \subseteq \left\{(Y_1, \ldots, Y_k)^{-1}(F) : F \text{ Borel in } \prod_{j=1}^k \mathbb{R}^{p_j} \right\}$$

Then there exists a Borel function, $g : \prod_{j=1}^k \mathbb{R}^{p_j} \rightarrow \mathbb{R}^n$ such that

$$X = g(Y_1, Y_2, \ldots, Y_k).$$

**Proof:** For the sake of brevity, denote by $Y$ the vector $(Y_1, \ldots, Y_k)$ and by $y$ the vector $(y_1, \ldots, y_k)$ and let $\prod_{j=1}^k \mathbb{R}^{p_j} \equiv \mathbb{R}^P$. For $E$ a Borel set of $\mathbb{R}^n$,

$$\int_{Y^{-1}(E)} XdP = \int_{\mathbb{R}^n} X_{\mathbb{R}^n \times E}(x, y) x d\lambda(x, y)$$

$$= \int_E \int_{\mathbb{R}^n} x d\lambda x | y d\lambda y. \quad (57.11.20)$$

Consider the function

$$y \rightarrow \int_{\mathbb{R}^n} x d\lambda x | y.$$
Since $d\lambda_Y$ is a Radon measure having inner and outer regularity, it follows the above function is equal to a Borel function for $\lambda_Y$ a.e. $y$. This function will be denoted by $g$. Then from \textbf{57.11.20} 
\[
\int_{Y^{-1}(E)} XdP = \int_E g(y) d\lambda_Y = \int_{\mathbb{R}^p} \chi_E(y) g(y) d\lambda_Y \\
= \int_{\Omega} \chi_E(Y(\omega)) g(Y(\omega)) dP \\
= \int_{Y^{-1}(E)} g(Y(\omega)) dP
\]
and since $Y^{-1}(E)$ is an arbitrary element of $\sigma(Y)$, this shows that since $X$ is $\sigma(Y)$ measurable, 
\[X = g(Y) \text{ P a.e.} \]
What about the case where $X$ is not necessarily measurable in $\sigma(Y_1, \cdots, Y_k)$?

**Lemma 57.11.4** There exists a unique function $Z(\omega)$ which satisfies 
\[
\int_F X(\omega) dP = \int_F Z(\omega) dP
\]
for all $F \in \sigma(Y_1, \cdots, Y_k)$ such that $Z$ is $\sigma(Y_1, \cdots, Y_k)$ measurable. It is denoted by 
\[E(X|\sigma(Y_1, \cdots, Y_k))\]

**Proof:** It is like the above. Letting $E$ be a Borel set in $\mathbb{R}^p$, 
\[
\int_{Y^{-1}(E)} XdP = \int_{\mathbb{R}^n \times \mathbb{R}^p} \chi_{\mathbb{R}^n \times E}(x,y) xd\lambda_{(X,Y)} \\
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^p} xd\lambda_{X|Y} d\lambda_Y.
\]
Now let $g(y) \equiv E(X|y_1, \cdots, y_k)$ be a Borel representative of 
\[
\int_{\mathbb{R}^n} xd\lambda_{X|Y}
\]
It follows $\omega \to g(Y(\omega)) = E(X|Y_1(\omega), \cdots, Y_k(\omega))$ is $\sigma(Y_1, \cdots, Y_k)$ measurable because by definition $\omega \to Y(\omega)$ is $\sigma(Y_1, \cdots, Y_k)$ measurable and a Borel measurable function composed with a measurable one is still measurable. It follows that for all $E$ Borel in $\mathbb{R}^p$, 
\[
\int_{Y^{-1}(E)} XdP = \int_E E(X|y_1, \cdots, y_k) d\lambda_Y \\
= \int_{Y^{-1}(E)} E(X|Y_1(\omega), \cdots, Y_k(\omega)) dP
\]
and so \( Z(\omega) = E(X|Y_1(\omega), \cdots, Y_k(\omega)) \) works because a generic set of \( \sigma(Y_1, \cdots, Y_k) \)
is \( Y^{-1}(E) \) for \( E \) a Borel set in \( \mathbb{R}^p \). If both \( Z, Z_1 \) work, then for all \( F \in \sigma(Y_1, \cdots, Y_k) \),
\[
\int_F (Z - Z_1) \, dP = 0
\]
Since \( F \) is arbitrary, some routine computations show \( Z = Z_1 \) a.e. □

**Observation 57.11.5** Note that a.e.
\[
E(X|Y_1(\omega), \cdots, Y_k(\omega)) = E(X|\sigma(Y_1, \cdots, Y_k))
\]
where the one on the left is the expected value of \( X \) given values of \( Y_j(\omega) \). This one corresponds to the sort of thing we say in words. The one on the right is an abstract concept which is usually obtained using the Radon Nikodym theorem and its description is given in the lemma. This lemma shows that its meaning is really to take the expected value of \( X \) given values for the \( Y_k \).

### 57.12 Characteristic Functions For Measures

Recall the characteristic function for a random variable having values in \( \mathbb{R}^n \). I will give a review of this to begin with. Then the concept will be generalized to random variables (vectors) which have values in a real separable Banach space.

**Definition 57.12.1** Let \( X \) be a random variable. The characteristic function is
\[
\phi_X(t) \equiv E(e^{it \cdot X}) = \int_{\Omega} e^{it \cdot X(\omega)} \, dP = \int_{\mathbb{R}^p} e^{it \cdot x} \, d\lambda_X
\]
the last equation holding by Proposition 57.1.12 on Page 1904.

Recall the following fundamental lemma and definition, Lemma 29.3.4 on Page 1089.

**Definition 57.12.2** For \( T \in G^* \), define \( FT, F^{-1}T \in G^* \) by
\[
FT(\phi) \equiv T(F\phi), \quad F^{-1}T(\phi) \equiv T(F^{-1}\phi)
\]

**Lemma 57.12.3** \( F \) and \( F^{-1} \) are both one to one, onto, and are inverses of each other.

The main result on characteristic functions is the following in Theorem 57.8.4 on Page 1928 which is stated here for convenience.

**Theorem 57.12.4** Let \( X \) and \( Y \) be random vectors with values in \( \mathbb{R}^p \) and suppose \( E(e^{it \cdot X}) = E(e^{it \cdot Y}) \) for all \( t \in \mathbb{R}^p \). Then \( \lambda_X = \lambda_Y \).

I want to do something similar for random variables which have values in a separable real Banach space, \( E \) instead of \( \mathbb{R}^p \).
Corollary 57.12.5 Let \( K \) be a \( \pi \) system of subsets of \( \Omega \) and suppose two probability measures, \( \mu \) and \( \nu \) defined on \( \sigma (K) \) are equal on \( K \). Then \( \mu = \nu \).

Proof: This follows from the Lemma 10.12.3 on Page 312. Let
\[
G \equiv \{ E \in \sigma (K) : \mu (E) = \nu (E) \}
\]
Then \( K \subseteq G \), since \( \mu \) and \( \nu \) are both probability measures, it follows that if \( E \in G \), then so is \( E^C \). Since these are measures, if \( \{ A_i \} \) is a sequence of disjoint sets from \( G \) then
\[
\mu (\bigcup_{i=1}^{\infty} A_i) = \sum_{i} \mu (A_i) = \sum_{i} \nu (A_i) = \nu (\bigcup_{i=1}^{\infty} A_i)
\]
and so from Lemma 10.12.3, \( G = \sigma (K) \).

Next recall the following fundamental lemma used to prove Pettis' theorem. It is proved on Page 629 but is stated here for convenience.

Lemma 57.12.6 If \( E \) is a separable Banach space with \( B' \) the closed unit ball in \( E' \), then there exists a sequence \( \{ f_n \}_{n=1}^{\infty} \equiv D' \subseteq B' \) with the property that for every \( x \in E \),
\[
||x|| = \sup_{f \in D'} |f(x)|
\]

Definition 57.12.7 Let \( E \) be a separable real Banach space. A cylindrical set is one which is of the form
\[
\{ x \in E : x_i^* (x) \in \Gamma_i, i = 1, 2, \ldots, m \}
\]
where here \( x_i^* \in E' \) and \( \Gamma_i \) is a Borel set in \( \mathbb{R} \).

It is obvious that \( \emptyset \) is a cylindrical set and that the intersection of two cylindrical sets is another cylindrical set. Thus the cylindrical sets form a \( \pi \) system. What is the smallest \( \sigma \) algebra containing the cylindrical sets? It is the Borel sets of \( E \). This is a special case of Lemma 57.4.2. Recall why this was. Letting \( \{ f_n \}_{n=1}^{\infty} = D' \) be the sequence of Lemma 57.12.6 it follows that
\[
\{ x \in E : ||x-A|| \leq \delta \}
\]
\[
= \left\{ x \in E : \sup_{f \in D'} |f(x-A)| \leq \delta \right\}
\]
\[
= \left\{ x \in E : \sup_{f \in D'} |f(x) - f(a)| \leq \delta \right\}
\]
\[
= \bigcap_{n=1}^{\infty} \left\{ x \in E : f_n(x) \in B(f_n(a), \delta) \right\}
\]
which yields a countable intersection of cylindrical sets. It follows the smallest \( \sigma \) algebra containing the cylindrical sets contains the closed balls and hence the open balls and consequently the open sets and so it contains the Borel sets. However, each cylindrical set is a Borel set and so in fact this \( \sigma \) algebra equals \( B(E) \).

From Corollary 57.12.5 it follows that two probability measures which are equal on the cylindrical sets are equal on the Borel sets \( B(E) \).
Definition 57.12.8 Let $\mu$ be a probability measure on $\mathcal{B}(E)$ where $E$ is a real separable Banach space. Then for $x^* \in E'$,

$$\phi_\mu(x^*) \equiv \int_E e^{ix^*(x)} d\mu(x).$$

$\phi_\mu$ is called the characteristic function for the measure $\mu$.

Note this is a little different than earlier when the symbol $\phi_X(t)$ was used and $X$ was a random variable. Here the focus is more on the measure than a random variable, $X$ such that $\mathcal{L}(X) = \mu$. It might appear this is a more general concept but in fact this is not the case. You could just consider the separable Banach space or Polish space with the Borel $\sigma$-algebra as your probability space and then consider the identity map as a random variable having the given measure as a distribution measure. Of course a major result is the one which says that the characteristic function determines the measures.

Theorem 57.12.9 Let $\mu$ and $\nu$ be two probability measures on $\mathcal{B}(E)$ where $E$ is a separable real Banach space. Suppose

$$\phi_\mu(x^*) = \phi_\nu(x^*)$$

for all $x^* \in E'$. Then $\mu = \nu$.

Proof: It suffices to verify that $\mu(A) = \nu(A)$ for all $A \in \mathcal{K}$ where $\mathcal{K}$ is the set of cylindrical sets. Fix $g_n \in (E')^n$. Thus the two measures are equal if for all such $g_n$, $n \in \mathbb{N}$,

$$\mu(g_n^{-1}(B)) = \nu(g_n^{-1}(B))$$

for $B$ a Borel set in $\mathbb{R}^n$. Of course, for such a choice of $g_n \in (E')^n$, there are measures defined on the Borel sets of $\mathbb{R}^n$ $\mu_n$ and $\nu_n$ which are given by

$$\mu_n(B) \equiv \mu(g_n^{-1}(B)), \nu_n(B) \equiv \nu(g_n^{-1}(B))$$

and so it suffices to verify that these two measures are equal. So what are their characteristic functions? Note that $g_n$ is a random variable taking $E$ to $\mathbb{R}^n$ and $\mu_n$, $\nu_n$ are just the probability distribution measures of this random variable. Therefore,

$$\phi_{\mu_n}(t) \equiv \int_{\mathbb{R}^n} e^{it \cdot s} d\mu_n = \int_E e^{it \cdot g_n(x)} d\mu$$

Similarly,

$$\phi_{\nu_n}(t) \equiv \int_{\mathbb{R}^n} e^{it \cdot s} d\nu_n = \int_E e^{it \cdot g_n(x)} d\nu$$

Now $t \cdot g_n \in E'$ and so by assumption, the two ends of the above are equal. Hence $\phi_{\mu_n}(t) = \phi_{\nu_n}(t)$ and so by Theorem 57.12.8, $\mu_n = \nu_n$ which, as shown above, implies $\mu = \nu$. ■
57.13 Characteristic Functions And Independence In Banach Space

I will consider the relation between the characteristic function and independence of random variables having values in a Banach space. Recall an earlier proposition which relates independence of random vectors with characteristic functions. It is proved starting on Page 1938.

Proposition 57.13.1 Let \( \{X_k\}_{k=1}^{n} \) be random vectors such that \( X_k \) has values in \( \mathbb{R}^{p_k} \). Then the random vectors are independent if and only if

\[
E(e^{iP}) = \prod_{j=1}^{n} E(e^{it_j \cdot X_j})
\]

where \( P \equiv \sum_{j=1}^{n} t_j \cdot X_j \) for \( t_j \in \mathbb{R}^{p_j} \).

It turns out there is a generalization of the above proposition to the case where the random variables have values in a real separable Banach space. Before proving this recall an earlier theorem which had to do with reducing to the case where the random variables had values in \( \mathbb{R}^n \), Theorem 57.5.1. It is restated here for convenience.

Theorem 57.13.2 The random variables \( \{X_i\}_{i \in I} \) are independent if whenever

\[
\{i_1, \cdots, i_n\} \subseteq I,
\]

\( m_{i_1}, \cdots, m_{i_n} \) are positive integers, and \( g_{m_{i_1}}, \cdots, g_{m_{i_n}} \) are in

\[
(E')^{m_{i_1}}, \cdots, (E')^{m_{i_n}}
\]

respectively, \( \{g_{m_{i_j}} \circ X_{i_j}\}_{j=1}^{n} \) are independent random vectors having values in

\[
\mathbb{R}^{m_{i_1}}, \cdots, \mathbb{R}^{m_{i_n}}
\]

respectively.

Now here is the theorem about independence and the characteristic functions.

Theorem 57.13.3 Let \( \{X_k\}_{k=1}^{n} \) be random variables such that \( X_k \) has values in \( E_k \), a real separable Banach space. Then the random variables are independent if and only if

\[
E(e^{iP}) = \prod_{j=1}^{n} E(e^{it_{j}^*(X_j)})
\]

where \( P \equiv \sum_{j=1}^{n} t_{j}^*(X_j) \) for \( t_{j}^* \in E_j' \).
CHAPTER 57. BASIC PROBABILITY

**Proof:** If the random variables are independent, then so are the random variables, \( t_j^* (X_j) \) and so the equation follows.

The interesting case is when the equation holds.

It suffices to consider only the case where each \( E_k = E \). This is because you can consider each \( X_j \) to have values in \( \prod_{k=1}^n E_k \) by letting \( X_j \) take its values in the \( j \)th component of the product and 0 in the other components. Can you draw the conclusion the random variables are independent? By Theorem 57.5.1, it suffices to show the random variables \( \{ g_{mk} \circ X_k \}^n_{k=1} \) are independent for \( g_{mk} = (x_1^*, \ldots, x_{mk}^*) \in (E')^{mk} \). This happens if whenever \( t_{mk} \in \mathbb{R}^{mk} \)

\[
P = \sum_{k=1}^n t_{mk} \cdot (g_{mk} \circ X_k),
\]

it follows

\[
E (e^{itP}) = \prod_{k=1}^n E \left( e^{it_{mk} \cdot (g_{mk} \circ X_k)} \right),
\]

(57.13.21)

However, the expression on the right in (57.13.21) equals

\[
\prod_{k=1}^n E \left( e^{it_{mk} \cdot (g_{mk} \circ X_k)} \right)
\]

and \( t_{mk} \cdot g_{mk} = \sum_{j=1}^{mk} t_j x_j^* \in E' \). Also the expression on the left equals \( E (e^{i \sum_{k=1}^n t_{mk} \cdot g_{mk} \circ X_k}) \).

Therefore, by assumption, (57.13.21) holds. □

There is an obvious corollary which is useful.

**Corollary 57.13.4** Let \( \{ X_k \}^n_{k=1} \) be random variables such that \( X_k \) has values in \( E_k \), a real separable Banach space. Then the random variables are independent if and only if

\[
E (e^{iP}) = \prod_{j=1}^n E \left( e^{it_j^* (X_j)} \right)
\]

where \( P = \sum_{j=1}^n t_j^* (X_j) \) for \( t_j^* \in M_j \) where \( M_j \) is a dense subset of \( E_j' \).

**Proof:** The easy direction follows from Theorem 57.13.3. Suppose then the above equation holds for all \( t_j^* \in M_j \). Then let \( t_j^* \in E' \) and let \( \{ t_{nj}^* \} \) be a sequence in \( M_j \) such that

\[
\lim_{n \to \infty} t_{nj}^* = t_j^* \text{ in } E'
\]

Then define

\[
P = \sum_{j=1}^n t_j^* X_j, \quad P_n = \sum_{j=1}^n t_{nj}^* X_j.
\]
It follows
\[
E(e^{it}) = \lim_{n \to \infty} E(e^{it_n}) = \lim_{n \to \infty} \prod_{j=1}^{n} E(e^{it_j(X_j)}) = \prod_{j=1}^{n} E(e^{it_j(X_j)})
\]

57.14 Convolution And Sums

Lemma 57.1.9 on Page 1903 makes possible a definition of convolution of two probability measures defined on \(B(E)\) where \(E\) is a separable Banach space as well as some other interesting theorems which held earlier in the context of locally compact spaces. I will first show a little theorem about density of continuous functions in \(L^p(E)\) and then define the convolution of two finite measures. First here is a simple technical lemma.

Lemma 57.14.1 Suppose \(K\) is a compact subset of \(U\) an open set in \(E\) a metric space. Then there exists \(\delta > 0\) such that \(\text{dist}(x, K) + \text{dist}(x, U^c) \geq \delta\) for all \(x \in E\).

Proof: For each \(x \in K\), there exists a ball, \(B(x, \delta_x)\) such that \(B(x, 3\delta_x) \subseteq U\). Finitely many of these balls cover \(K\) because \(K\) is compact, say \(\{B(x_i, \delta_x)\}_{i=1}^{m}\). Let
\[
0 < \delta < \min(\delta_x : i = 1, 2, \cdots, m).
\]
Now pick any \(x \in K\). Then \(x \in B(x_i, \delta_x)\) for some \(x_i\) and so \(B(x, \delta) \subseteq B(x_i, 2\delta_x) \subseteq U\). Therefore, for any \(x \in K\), \(\text{dist}(x, U^c) \geq \delta\). If \(x \in B(x_i, 2\delta_x)\) for some \(x_i\), it follows \(\text{dist}(x, U^c) \geq \delta\) because then \(B(x, \delta) \subseteq B(x_i, 3\delta_x) \subseteq U\). If \(x \notin B(x_i, 2\delta_x)\) for any of the \(x_i\), then \(x \notin B(y, \delta)\) for any \(y \in K\) because all these sets are contained in some \(B(x_i, 2\delta_x)\). Consequently \(\text{dist}(x, K) \geq \delta\). This proves the lemma.

From this lemma, there is an easy corollary.

Corollary 57.14.2 Suppose \(K\) is a compact subset of \(U\), an open set in \(E\) a metric space. Then there exists a uniformly continuous function \(f\) defined on all of \(E\), having values in \([0, 1]\) such that \(f(x) = 0\) if \(x \notin U\) and \(f(x) = 1\) if \(x \in K\).

Proof: Consider
\[
f(x) = \frac{\text{dist}(x, U^c)}{\text{dist}(x, U^c) + \text{dist}(x, K)}.
\]
Then some algebra yields
\[
|f(x) - f(x')| \leq ...
\]
where $\delta$ is the constant of Lemma 57.14.1. Now it is a general fact that

$$|\text{dist} (x, S) - \text{dist} (x', S)| \leq d(x,x').$$

Therefore,

$$|f(x) - f(x')| \leq \frac{2}{\delta} d(x,x')$$

and this proves the corollary.

Now suppose $\mu$ is a finite measure defined on the Borel sets of a separable Banach space, $E$. It was shown above that $\mu$ is inner and outer regular. Lemma 57.1.9 on Page 1903 shows that $\mu$ is inner regular in the usual sense with respect to compact sets. This makes possible the following theorem.

**Theorem 57.14.3** Let $\mu$ be a finite measure on $\mathcal{B}(E)$ where $E$ is a separable Banach space and let $f \in L^p(E; \mu)$. Then for any $\varepsilon > 0$, there exists a uniformly continuous, bounded $g$ defined on $E$ such that

$$||f - g||_{L^p(E)} < \varepsilon.$$

**Proof:** As usual in such situations, it suffices to consider only $f \geq 0$. Then by Theorem 9.3.9 on Page 227 and an application of the monotone convergence theorem, there exists a simple measurable function,

$$s(x) \equiv \sum_{k=1}^{m} c_k \chi_{A_k} (x)$$

such that $||f - s||_{L^p(E)} < \varepsilon/2$. Now by regularity of $\mu$ there exist compact sets, $K_k$ and open sets, $V_k$ such that $2 \sum_{k=1}^{m} |c_k| \mu (V_k \setminus K_k)^{1/p} < \varepsilon/2$ and by Corollary 57.14.2 there exist uniformly continuous functions $g_k$ having values in $[0,1]$ such that $g_k = 1$ on $K_k$ and 0 on $V_k^C$. Then consider

$$g(x) = \sum_{k=1}^{m} c_k g_k (x).$$
This function is bounded and uniformly continuous. Furthermore,
\[
\|s - g\|_{L^p(E)} \leq \left( \int_E \left| \sum_{k=1}^{m} c_k \mathcal{X}_{A_k}(x) - \sum_{k=1}^{m} c_k g_k(x) \right|^p d\mu \right)^{1/p} 
\]
\[
\leq \left( \int_E \left| \sum_{k=1}^{m} |c_k| \left| \mathcal{X}_{A_k}(x) - g_k(x) \right| \right|^p d\mu \right)^{1/p} 
\]
\[
\leq \sum_{k=1}^{m} |c_k| \left( \int_E \left| \mathcal{X}_{A_k}(x) - g_k(x) \right|^p d\mu \right)^{1/p} 
\]
\[
\leq \sum_{k=1}^{m} |c_k| \left( \int_{V_k \setminus K_k} 2^p d\mu \right)^{1/p} 
\]
\[
= \sum_{k=1}^{m} |c_k| \mu(V_k \setminus K)^{1/p} < \varepsilon/2. 
\]

Therefore,
\[
\|f - g\|_{L^p} \leq \|f - s\|_{L^p} + \|s - g\|_{L^p} < \varepsilon/2 + \varepsilon/2. 
\]

This proves the theorem.

**Lemma 57.14.4** Let \( A \in B(E) \) where \( \mu \) is a finite measure on \( B(E) \) for \( E \) a separable Banach space. Also let \( x_i \in E \) for \( i = 1, 2, \cdots, m \). Then for \( x \in E^m \),
\[
x \rightarrow \mu \left( A + \sum_{i=1}^{m} x_i \right), \quad x \rightarrow \mu \left( A - \sum_{i=1}^{m} x_i \right) 
\]
are Borel measurable functions. Furthermore, the above functions are
\[
B(E) \times \cdots \times B(E) 
\]
measurable where the above denotes the product measurable sets as described in Theorem 10.12.6 on Page 316.

**Proof:** First consider the case where \( A = U \), an open set. Let
\[
y \in \left\{ x \in E^m : \mu \left( U + \sum_{i=1}^{m} x_i \right) > \alpha \right\} \tag{57.14.22} 
\]
Then from Lemma 57.14.3 on Page 1903 there exists a compact set, \( K \subseteq U + \sum_{i=1}^{m} y_i \) such that \( \mu(K) > \alpha \). Then if \( y' \) is close enough to \( y \), it follows \( K \subseteq U + \sum_{i=1}^{m} y'_i \) also. Therefore, for all \( y' \) close enough to \( y \),
\[
\mu \left( U + \sum_{i=1}^{m} y'_i \right) \geq \mu(K) > \alpha. 
\]
In other words the set described in 57.14.22 is an open set and so $y \rightarrow \mu(U + \sum_{i=1}^{m} y_i)$ is Borel measurable whenever $U$ is an open set in $E$.

Define a $\pi$ system, $K$ to consist of all open sets in $E$. Then define $\mathcal{G}$ as

$$\left\{ A \in \sigma(K) = \mathcal{B}(E) : y \rightarrow \mu(A + \sum_{i=1}^{m} y_i) \text{ is Borel measurable} \right\}$$

I just showed $\mathcal{G} \supseteq K$. Now suppose $A \in \mathcal{G}$. Then

$$\mu \left( A^C + \sum_{i=1}^{m} y_i \right) = \mu(E) - \mu \left( A + \sum_{i=1}^{m} y_i \right)$$

and so $A^C \in \mathcal{G}$ whenever $A \in \mathcal{G}$. Next suppose $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$. Then

$$\mu \left( \bigcup_{i=1}^{\infty} A_i + \sum_{j=1}^{m} y_j \right) = \mu \left( \bigcup_{i=1}^{\infty} \left( A_i + \sum_{j=1}^{m} y_j \right) \right)$$

$$= \sum_{i=1}^{\infty} \mu \left( A_i + \sum_{j=1}^{m} y_j \right)$$

and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$ because the above is the sum of Borel measurable functions. By the lemma on $\pi$ systems, Lemma 10.12.3 on Page 312, it follows $\mathcal{G} = \sigma(K) = \mathcal{B}(E)$.

Similarly, $x \rightarrow \mu \left( A - \sum_{j=1}^{m} x_j \right)$ is also Borel measurable whenever $A \in \mathcal{B}(E)$. Finally note that

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

contains the open sets of $E^m$ because the separability of $E$ implies the existence of a countable basis for the topology of $E^m$ consisting of sets of the form $\prod_{i=1}^{m} U_i$ where the $U_i$ come from a countable basis for $E$. Since every open set is the countable union of sets like the above, each being a measurable box, the open sets are contained in

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

which implies $\mathcal{B}(E^m) \subseteq \mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$ also. This proves the lemma.

With this lemma, it is possible to define the convolution of two finite measures.

**Definition 57.14.5** Let $\mu$ and $\nu$ be two finite measures on $\mathcal{B}(E)$, for $E$ a separable Banach space. Then define a new measure, $\mu \ast \nu$ on $\mathcal{B}(E)$ as follows

$$\mu \ast \nu (A) \equiv \int_{E} \nu (A - x) \, d\mu (x).$$

This is well defined because of Lemma 57.14.4 which says that $x \rightarrow \nu (A - x)$ is Borel measurable.
Here is an interesting theorem about convolutions. However, first here is a little lemma. The following picture is descriptive of the set described in the following lemma.

Lemma 57.14.6 For $A$ a Borel set in $E$, a separable Banach space, define

$$S_A \equiv \{(x, y) \in E \times E : x + y \in A\}$$

Then $S_A \in \mathcal{B}(E) \times \mathcal{B}(E)$, the $\sigma$ algebra of product measurable sets, the smallest $\sigma$ algebra which contains all the sets of the form $A \times B$ where $A$ and $B$ are Borel.

Proof: Let $K$ denote the open sets in $E$. Then $K$ is a $\pi$ system. Let

$$G \equiv \{A \in \sigma(K) = \mathcal{B}(E) : S_A \in \mathcal{B}(E) \times \mathcal{B}(E)\}.$$

Then $K \subseteq G$ because if $U \in K$ then $S_U$ is an open set in $E \times E$ and all open sets are in $\mathcal{B}(E) \times \mathcal{B}(E)$ because a countable basis for the topology of $E \times E$ are sets of the form $B \times C$ where $B$ and $C$ come from a countable basis for $E$. Therefore, $K \subseteq G$. Now let $A \in G$. For $(x, y) \in E \times E$, either $x + y \in A$ or $x + y \notin A$. Hence $E \times E = S_A \cup S_A^C$ which shows that if $A \in G$ then so is $A^C$. Finally if $\{A_i\}$ is a sequence of disjoint sets of $G$

$$S_{\bigcup_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} S_{A_i},$$

and this shows that $G$ is also closed with respect to countable unions of disjoint sets. Therefore, by the lemma on $\pi$ systems, Lemma 10.12.3 on Page 612 it follows $G = \sigma(K) = \mathcal{B}(E)$. This proves the lemma.

Theorem 57.14.7 Let $\mu$, $\nu$, and $\lambda$ be finite measures on $\mathcal{B}(E)$ for $E$ a separable Banach space. Then

$$\mu * \nu = \nu * \mu$$

(57.14.23)

$$(\mu * \nu) * \lambda = \mu * (\nu * \lambda)$$

(57.14.24)

If $\mu$ is the distribution for an $E$ valued random variable, $X$ and if $\nu$ is the distribution for an $E$ valued random variable, $Y$, and $X$ and $Y$ are independent, then $\mu * \nu$ is the distribution for the random variable, $X + Y$. Also the characteristic function of a convolution equals the product of the characteristic functions.
**Proof:** First consider 57.14.23. Letting \( A \in \mathcal{B}(E) \), the following computation holds from Fubini’s theorem and Lemma 57.14.6:

\[
\mu \ast \nu(A) \equiv \int_E \nu(A-x)\,d\mu(x) = \int_E \int_E \chi_{S_A}(x,y)\,d\nu(y)\,d\mu(x) = \int_E \int_E \chi_{S_A}(x,y)\,d\mu(x)\,d\nu(y) = \nu \ast \mu(A).
\]

Next consider 57.14.24. Using 57.14.23 whenever convenient,

\[
(\mu \ast \nu) \ast \lambda(A) \equiv \int_E (\mu \ast \nu)(A-x)\,d\lambda(x) = \int_E \nu(A-x-y)\,d\mu(y)\,d\lambda(x)
\]
while

\[
\mu \ast (\nu \ast \lambda)(A) = \int_E (\nu \ast \lambda)(A-y)\,d\mu(y) = \int_E \nu(A-y-x)\,d\lambda(x)\,d\mu(y) = \int_E \nu(A-y-x)\,d\mu(y)\,d\lambda(x).
\]

The necessary product measurability comes from Lemma 57.14.4.

Recall

\[
(\mu \ast \nu)(A) \equiv \int_E \nu(A-x)\,d\mu(x).
\]

Therefore, if \( s \) is a simple function, \( s(x) = \sum_{k=1}^n c_k \chi_{A_k}(x) \),

\[
\int_E sd(\mu \ast \nu) = \sum_{k=1}^n c_k \int_E \nu(A_k-x)\,d\mu(x) = \int_E \sum_{k=1}^n c_k \nu(A_k-x)\,d\mu(x) = \int_E \sum_{k=1}^n c_k \chi_{A_k-x}(y)\,d\nu(y)\,d\mu(x) = \int_E \int_E s(x+y)\,d\nu(y)\,d\mu(x)
\]

Approximating with simple functions it follows that whenever \( f \) is bounded and measurable or nonnegative and measurable,

\[
\int_E f d(\mu \ast \nu) = \int_E \int_E f(x+y)\,d\nu(y)\,d\mu(x) \quad (57.14.25)
\]
Therefore, letting $Z = X + Y$, and $\lambda$ the distribution of $Z$, it follows from independence of $X$ and $Y$ that for $t^* \in E'$,

$$\phi_{\lambda}(t^*) = E \left( e^{it^*(Z)} \right) = E \left( e^{it^*(X+Y)} \right) = E \left( e^{it^*(X)} \right) E \left( e^{it^*(Y)} \right)$$

But also, it follows from

$$\phi_{(\mu \ast \nu)}(t^*) = \int_E e^{it^*_x} d(\mu \ast \nu)(z) = \int_E \int_E e^{it^*_x(y)} d\nu(y) d\mu(x) = \int_E \int_E e^{it^*_x(z)} e^{it^*_y(y)} d\nu(y) d\mu(x) = \left( \int_E e^{it^*_y(y)} d\nu(y) \right) \left( \int_E e^{it^*_x(x)} d\mu(x) \right) = E \left( e^{it^*_x(X)} \right) E \left( e^{it^*_y(Y)} \right)$$

Since $\phi_{\lambda}(t^*) = \phi_{(\mu \ast \nu)}(t^*)$, it follows $\lambda = \mu \ast \nu$.

Note the last part of this argument shows the characteristic function of a convolution equals the product of the characteristic functions. This proves the theorem.

### 57.15 The Convergence Of Sums Of Symmetric Random Variables

It turns out that when random variables have symmetric distributions, some remarkable things can be said about infinite sums of these random variables. Conditions are given here that enable one to conclude the convergence of the sequence of partial sums from the convergence of some subsequence of partial sums.

The following lemma is like an earlier result but is proved here for convenience.

**Definition 57.15.1** Let $X$ be a random variable. $L(X) = \mu$ means $\lambda_X = \mu$. This is called the law of $X$. It is the same as saying the distribution measure of $X$ is $\mu$.

**Lemma 57.15.2** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to E$ be a random variable, where $E$ is a real separable Banach space. Also let $L(X) = \mu$, a probability measure defined on $\mathcal{B}(E)$, the Borel sets of $E$. Suppose $h : E \to \mathbb{R}$ is in $L^1(E; \mu)$ or is nonnegative and Borel measurable. Then

$$\int_{\Omega} (h \circ X) dP = \int_{E} h(x) d\mu.$$ 

**Proof:** First suppose $A$ is a Borel set in $E$. Then

$$\int_{E} \chi_{A} (x) d\mu = \mu(A) \equiv P([X \in A])$$

$$\int_{\Omega} (\chi_{A} \circ X) dP = \int_{\Omega} \chi_{X^{-1}(A)} (\omega) dP \equiv P(X^{-1}(A)) \equiv P([X \in A])$$
Thus for nonnegative simple Borel measurable functions $s$, it follows
\[
\int_{\Omega} (s \circ X) \, dP = \int_{E} s(x) \, d\mu
\]
Now approximating with an increasing sequence of nonnegative simple functions and using the monotone convergence theorem, the desired formula holds for nonnegative Borel measurable functions $h$.

If $h$ is Borel measurable and in $L^1(E;\mu)$, then you can consider the formula for the positive and negative parts and get the result in this case also. This proves the lemma.

Here is a simple definition and lemma about random variables whose distribution is symmetric.

**Definition 57.15.3** Let $X$ be a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ having values in a Banach space, $E$. Then it has a symmetric distribution if whenever $A$ is a Borel set,
\[
P([X \in A]) = P([X \in -A])
\]
In terms of the distribution,
\[
\lambda_X = \lambda_{-X}.
\]

It is good to observe that if $X, Y$ are independent random variables defined on a probability space, $(\Omega, \mathcal{F}, P)$ such that each has symmetric distribution, then $X + Y$ also has symmetric distribution. Here is why. Let $A$ be a Borel set in $E$. Then by Theorem 57.14.7 on Page 1951,
\[
\lambda_{X+Y}(A) = \int_{E} \lambda_X(A - z) \, d\lambda_Y(z)
\]
\[
= \int_{E} \lambda_X(A - z) \, d\lambda_{-Y}(z)
\]
\[
= \lambda_{-(X+Y)}(A) = \lambda_{X+Y}(-A)
\]
By induction, it follows that if you have $n$ independent random variables each having symmetric distribution, then their sum has symmetric distribution.

Here is a simple lemma about random variables having symmetric distributions. It will depend on Lemma 57.15.2 on Page 1953.

**Lemma 57.15.4** Let $X \equiv (X_1, \cdots, X_n)$ and $Y$ be random variables defined on a probability space, $(\Omega, \mathcal{F}, P)$ such that $X_i, i = 1, 2, \cdots, n$ and $Y$ have values in $E$ a separable Banach space. Thus $X$ has values in $E^n$. Suppose also that $\{X_1, \cdots, X_n, Y\}$ are independent and that $Y$ has symmetric distribution. Then if $A \in \mathcal{B}(E^n)$, it follows
\[
P\left([X \in A] \cap \left\{ \left\| \sum_{i=1}^{n} X_i + Y \right\| < r \right\} \right)
\]
\[
= P\left([X \in A] \cap \left\{ \left\| \sum_{i=1}^{n} X_i - Y \right\| < r \right\} \right)
\]
You can also change the inequalities in the obvious way, < to ≤, > or ≥.

**Proof:** Denote by \( \lambda_X \) and \( \lambda_Y \) the distribution measures for \( X \) and \( Y \) respectively. Since the random variables are independent, the distribution for the random variable, \((X, Y)\) mapping into \( E^{n+1} \) is \( \lambda_X \times \lambda_Y \) where this denotes product measure. Since the Banach space is separable, the Borel sets are contained in the product measurable sets. Then by symmetry of the distribution of \( Y \)

\[
P \left( [X \in A] \cap \left[ \left\| \sum_{i=1}^{n} X_i + Y \right\| < r \right] \right)
= \int_{E^n \times E} X_A(x) X_{B(0,r)} \left( \sum_{i=1}^{n} x_i + y \right) d(\lambda_X \times \lambda_Y) (x, y)
= \int_{E} \int_{E^n} X_A(x) X_{B(0,r)} \left( \sum_{i=1}^{n} x_i + y \right) d\lambda_X d\lambda_Y
= \int_{E} \int_{E^n} X_A(x) X_{B(0,r)} \left( \sum_{i=1}^{n} x_i + y \right) d\lambda_X d\lambda_{-Y}
= \int_{E^n \times E} X_A(x) X_{B(0,r)} \left( \sum_{i=1}^{n} x_i + y \right) d(\lambda_X \times \lambda_{-Y}) (x, y)
= P \left( [X \in A] \cap \left[ \left\| \sum_{i=1}^{n} X_i + (-Y) \right\| < r \right] \right)
\]

This proves the lemma. Other cases are similar.

Now here is a really interesting lemma.

**Lemma 57.15.5** Let \( E \) be a real separable Banach space. Assume \( \xi_1, \ldots, \xi_N \) are independent random variables having values in \( E \), a separable Banach space which have symmetric distributions. Also let \( S_k = \sum_{i=1}^{k} \xi_i. \) Then for any \( r > 0 \),

\[
P \left( \sup_{k \leq N} \|S_k\| > r \right) \leq 2P \left( \|S_N\| > r \right).
\]

**Proof:** First of all,

\[
P \left( \sup_{k \leq N} \|S_k\| > r \right)
= P \left( \sup_{k \leq N} \|S_k\| > r \text{ and } \|S_N\| > r \right)
\]
I need to estimate the second of these terms. Let \( A_1 \equiv [||S_1|| > r], \ldots, A_k \equiv [||S_k|| > r, \, ||S_j|| \leq r \text{ for } j < k] \).

Thus \( A_k \) consists of those \( \omega \) where \( ||S_k(\omega)|| > r \) for the first time at \( k \). Thus

\[
\bigg[ \sup_{k \leq N-1} ||S_k|| > r \text{ and } ||S_N|| \leq r \bigg] = \bigcup_{j=1}^{N-1} A_j \cap [||S_N|| \leq r]
\]

and the sets in the above union are disjoint. Consider \( A_j \cap [||S_N|| \leq r] \). For \( \omega \) in this set,

\[
||S_j(\omega)|| > r, \, ||S_i(\omega)|| \leq r \text{ if } i < j.
\]

Since \( ||S_N(\omega)|| \leq r \) in this set, it follows

\[
||S_N(\omega)|| = \left| S_j(\omega) + \sum_{i=j+1}^{N} \xi_i(\omega) \right| \leq r
\]

Thus

\[
P(A_j \cap [||S_N|| \leq r]) = P \left( \bigcap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \cap \left[ \left| S_j + \sum_{i=j+1}^{N} \xi_i \right| \leq r \right] \right)
\]

(57.15.27)

Now \( \bigcap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \) is of the form

\[
[ (\xi_1, \ldots, \xi_j) \in A]
\]

for some Borel set, \( A \). Then letting \( Y = \sum_{i=j+1}^{N} \xi_i \) in Lemma 57.15.4 and \( X_i = \xi_i \),

\[
P \left( \bigcap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \cap \left[ \left| S_j - \sum_{i=j+1}^{N} \xi_i \right| \leq r \right] \right)
\]

Now since \( ||S_j(\omega)|| > r \),

\[
||2S_j - S_N|| \leq r \subseteq [2||S_j|| - ||S_N|| \leq r]
\]

\[
\subseteq [2r - ||S_N|| < r]
\]

\[
= [||S_N|| > r]
\]
and so, referring to 57.15.27, this has shown

\[ P (A_j \cap \|S_N\| \leq r) \]

\[ = P \left( \bigcap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap [\|2S_j - S_N\| \leq r] \right) \]

\[ \leq P \left( \bigcap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap [\|S_N\| > r] \right) \]

\[ = P (A_j \cap \|S_N\| > r). \]

It follows that

\[ P \left( \sup_{k \leq N-1} \|S_k\| > r \text{ and } \|S_N\| \leq r \right) = \sum_{i=1}^{N-1} P (A_j \cap \|S_N\| \leq r) \]

\[ \leq \sum_{i=1}^{N-1} P (A_j \cap \|S_N\| > r) \leq P (\|S_N\| > r) \]

and using 57.15.26, this proves the lemma.

This interesting lemma will now be used to prove the following which concludes a sequence of partial sums converges given a subsequence of the sequence of partial sums converges.

**Lemma 57.15.6** Let \( \{\zeta_k\} \) be a sequence of independent random variables having values in a separable real Banach space, \( E \) whose distributions are symmetric. Letting \( S_k \equiv \sum_{i=1}^{k} \zeta_i \), suppose \( \{S_{n_k}\} \) converges a.e. Also suppose that for every \( m > n_k \),

\[ P \left( \|S_m - S_{n_k}\|_E > 2^{-k} \right) < 2^{-k}. \] (57.15.29)

Then in fact,

\[ S_k (\omega) \rightarrow S (\omega) \text{ a.e.} \omega \] (57.15.30)

and off a set of measure zero, the convergence of \( S_k \) to \( S \) is uniform.

**Proof:** Let \( n_k \leq l \leq m \). Then by Lemma 57.15.6

\[ P \left( \left[ \sup_{n_k \leq l \leq m} \|S_l - S_{n_k}\| > 2^{-k} \right] \right) \leq 2P \left( \|S_m - S_{n_k}\| > 2^{-k} \right) \]

In using this lemma, you could renumber the \( \zeta_i \) so that the sum

\[ \sum_{j=n_k+1}^{l} \zeta_j \]

corresponds to

\[ \sum_{j=1}^{l-n_k} \zeta_j \]
where $\xi_j = \zeta_j + n_k$.

Then using (57.15.29),

$$P \left( \left\{ \sup_{n_k < l \leq m} ||S_l - S_{n_k}|| > 2^{-k} \right\} \right) \leq 2P \left( ||S_m - S_{n_k}|| > 2^{-k} \right) < 2^{-(k-1)}$$

If $S_l(\omega)$ fails to converge then $\omega$ must be in infinitely many of the sets,

$$\left[ \sup_{n_k < l} ||S_l - S_{n_k}|| > 2^{-k} \right]$$

each of which has measure no more than $2^{-(k-1)}$. Thus $\omega$ must be in a set of measure zero. This proves the lemma.

### 57.16 The Multivariate Normal Distribution

**Definition 57.16.1** A random vector, $X$, with values in $\mathbb{R}^p$ has a multivariate normal distribution written as $X \sim N_p (m, \Sigma)$ if for all Borel $E \subseteq \mathbb{R}^p$,

$$\lambda_X (E) = \int_{\mathbb{R}^p} \mathcal{N}_E (x) \frac{1}{(2\pi)^{p/2} \det (\Sigma)^{1/2}} e^{-\frac{1}{2}(x-m)^* \Sigma^{-1} (x-m)} dx$$

for $\mu$ a given vector and $\Sigma$ a given positive definite symmetric matrix.

**Theorem 57.16.2** For $X \sim N_p (m, \Sigma)$, $m = E (X)$ and

$$\Sigma = E ((X - m)(X - m)^*)$$

**Proof:** Let $R$ be an orthogonal transformation such that

$$R \Sigma R^* = D = \text{diag} (\sigma_1^2, \cdots, \sigma_p^2).$$

Changing the variable by $x - m = R^* y$,

$$E (X) = \int_{\mathbb{R}^p} x e^{-\frac{1}{2}(x-m)^* \Sigma^{-1} (x-m)} dx \left( \frac{1}{(2\pi)^{p/2} \det (\Sigma)^{1/2}} \right)$$

$$= \int_{\mathbb{R}^p} (R^* y + m) e^{-\frac{1}{2}y^* D^{-1} y} dy \left( \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right)$$

$$= \int_{\mathbb{R}^p} e^{-\frac{1}{2}y^* D^{-1} y} dy \left( \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) = m$$

by Fubini’s theorem and the easy to establish formula

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma}} dy = 1.$$
Next let \( M \equiv E \left( (X - m)(X - m)' \right) \). Thus, changing the variable as above by \( x - m = R^* y \)

\[
M = \int_{\mathbb{R}^p} (x - m)(x - m)' e^{-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)} dx \left( \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} \right)
\]

\[
= R^* \int_{\mathbb{R}^p} yy' e^{-\frac{1}{2}y'D^{-1}y} dy \left( \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) R
\]

Therefore,

\[
(RMR^*)_{ij} = \int_{\mathbb{R}^p} y_i y_j e^{-\frac{1}{2}y'D^{-1}y} dy \left( \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) = 0,
\]

so; \( RMR^* \) is a diagonal matrix.

\[
(RMR^*)_{ii} = \int_{\mathbb{R}^p} y_i^2 e^{-\frac{1}{2}y'D^{-1}y} dy \left( \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right).
\]

Using Fubini’s theorem and the easy to establish equations,

\[
\frac{1}{\sqrt{2\pi \sigma}} \int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy = 1, \quad \frac{1}{\sqrt{2\pi \sigma}} \int_{\mathbb{R}} y^2 e^{-\frac{y^2}{2\sigma^2}} dy = \sigma^2,
\]

it follows \((RMR^*)_{ii} = \sigma_i^2\). Hence \( RMR^* = D \) and so \( M = R^* DR = \Sigma \). \( \blacksquare \)

**Theorem 57.16.3** Suppose \( X_1 \sim N_p(m_1, \Sigma_1) \), \( X_2 \sim N_p(m_2, \Sigma_2) \) and the two random vectors are independent. Then

\( X_1 + X_2 \sim N_p(m_1 + m_2, \Sigma_1 + \Sigma_2). \) (57.16.31)

Also, if \( X \sim N_p(m, \Sigma) \) then \(-X \sim N_p(-m, \Sigma)\). Furthermore, if \( X \sim N_p(m, \Sigma) \) then

\[
E(e^{it'X}) = e^{it'm} e^{-\frac{1}{2}t'\Sigma t} \quad (57.16.32)
\]

Also if \( a \) is a constant and \( X \sim N_p(m, \Sigma) \) then \( aX \sim N_p(am, a^2\Sigma) \).

**Proof:** Consider \( E(e^{it'X}) \) for \( X \sim N_p(m, \Sigma) \).

\[
E(e^{it'X}) = \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{it'x} e^{-\frac{1}{2}(x-m)'\Sigma^{-1}(x-m)} dx.
\]

Let \( R \) be an orthogonal transformation such that

\[
R\Sigma R^* = D = \text{diag}(\sigma_1^2, \ldots, \sigma_p^2).
\]

Then let \( R(x - m) = y \). Then

\[
E(e^{it'X}) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \int_{\mathbb{R}^p} e^{it'(R'y + m)} e^{-\frac{1}{2}y'D^{-1}y} dy.
\]
Therefore
\[ E(e^{itX}) = \frac{1}{(2\pi)^{p/2}} \prod_{i=1}^{p} \int_{\mathbb{R}} e^{is(y+Rtm)} e^{-\frac{1}{2}y^2 dy} \]
where \( s = Rt \). This equals
\[ e^{itm} \prod_{i=1}^{p} \left( \int_{\mathbb{R}} e^{is_i y} e^{-\frac{1}{2}y^2} dy \right) \frac{1}{\sqrt{2\pi} \sigma_i} \]
\[ = e^{itm} \prod_{i=1}^{p} \left( \int_{\mathbb{R}} e^{is_i y} e^{-\frac{1}{2}y^2} dy \right) \frac{1}{\sqrt{2\pi}} \]
\[ = e^{itm} \prod_{i=1}^{p} e^{-\frac{1}{2} \sigma_i^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} dy \]
\[ = e^{itm} e^{-\frac{1}{2} \sum_{i=1}^{p} \sigma_i^2} = e^{itm} e^{-\frac{1}{2}t^* \Sigma t} \]
This proves the assertion about \(-X\).

Since \( X_1 \) and \( X_2 \) are independent, \( e^{itX_1} \) and \( e^{itX_2} \) are also independent. Hence
\[ E(e^{it(X_1+X_2)}) = E(e^{itX_1}) E(e^{itX_2}) \]
Thus,
\[ E(e^{itX_1+X_2}) = E(e^{itX_1}) E(e^{itX_2}) \]
\[ = e^{itm_1} e^{-\frac{1}{2} t^* \Sigma_1 t} e^{itm_2} e^{-\frac{1}{2} t^* \Sigma_2 t} \]
\[ = e^{it(m_1+m_2)} e^{-\frac{1}{2} t^* (\Sigma_1+\Sigma_2) t} \]
which is the characteristic function of a random vector distributed as \( N_p (m_1 + m_2, \Sigma_1 + \Sigma_2) \).

Now it follows that \( X_1 + X_2 \sim N_p (m_1 + m_2, \Sigma_1 + \Sigma_2) \) by Theorem 57.3.3. This proves the assertion about \(-X\).

The assertion about \(-X\) is also easy to see because
\[ E(e^{it(-X)}) = E(e^{i(-t)X}) \]
\[ = \frac{1}{(2\pi)^{p/2} (\det \Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{i(-t)x} e^{-\frac{1}{2}(x-m)^* \Sigma^{-1}(x-m)} dx \]
\[ = \frac{1}{(2\pi)^{p/2} (\det \Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{itx} e^{-\frac{1}{2}(x+m)^* \Sigma^{-1}(x+m)} dx \]
which is the characteristic function of a random variable which is \( N (-m, \Sigma) \). Theorem 57.3.3 again implies \(-X \sim N (-m, \Sigma) \). Finally consider the last claim. You
apply what is known about $X$ with $t$ replaced with $at$ and then massage things. This gives the characteristic function for $aX$ is given by
\[
E \left( \exp \left( it \cdot aX \right) \right) = \exp \left( it \cdot am \right) \exp \left( - \frac{1}{2} t^* \Sigma a^2 t \right)
\]
which is the characteristic function of a normal random vector having mean $am$ and covariance $a^2 \Sigma$. This proves the theorem.

Following [94], a random vector has a generalized normal distribution if its characteristic function is given as
\[
e^{it \cdot m} e^{-\frac{1}{2} t^* \Sigma t}
\] (57.16.33)
where $\Sigma$ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $m$ is scalar and so is $\Sigma$ so the characteristic function of such a generalized normally distributed random variable is
\[
e^{it \mu} e^{-\frac{1}{2} t^2 \sigma^2}
\] (57.16.34)
These generalized normal distributions do not require $\Sigma$ to be invertible, only that the eigenvalues be nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at $\mu$. In higher dimensions, it could be a mixture of such things with more familiar things. I won’t try very hard to distinguish between generalized normal distributions and normal distributions in which the covariance matrix has all positive eigenvalues.

Here are some other interesting results about normal distributions found in [94]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense.

**Theorem 57.16.4** Let $X = (X_1, \cdots, X_p)$ where each $X_i$ is a real valued random variable. Then $X$ is normally distributed in the above generalized sense if and only if every linear combination, $\sum_{j=1}^{p} a_i X_i$ is normally distributed. In this case the mean of $X$ is
\[
m = (E(X_1), \cdots, E(X_p))
\] and the covariance matrix for $X$ is
\[
\Sigma_{jk} = E \left( (X_j - m_j) (X_k - m_k)^* \right).
\]

**Proof:** Suppose first $X$ is normally distributed. Then its characteristic function is of the form
\[
\phi_X (t) = E \left( e^{it \cdot X} \right) = e^{it \cdot m} e^{-\frac{1}{2} t^* \Sigma t}.
\]
Then letting $a = (a_1, \cdots, a_p)$
\[
E \left( e^{it \sum_{j=1}^{p} a_j X_i} \right) = E \left( e^{ita \cdot X} \right) = e^{ita \cdot m} e^{-\frac{1}{2} a^* \Sigma a t^2}
\]
which is the characteristic function of a normally distributed random variable with mean $a \cdot m$ and variance $\sigma^2 = a^* \Sigma a$. This proves half of the theorem. If $X$ is normally distributed, then every linear combination is normally distributed.
Next suppose $\sum_{j=1}^{p} a_j X_j = a \cdot X$ is normally distributed with mean $\mu$ and variance $\sigma^2$ so that its characteristic function is given in (57.16.34). I will now relate $\mu$ and $\sigma^2$ to various quantities involving the $X_j$. Letting $m_j = E(X_j), m = (m_1, \cdots, m_p)^*$

$$
\begin{align*}
\mu &= \sum_{j=1}^{p} a_j E(X_j) = \sum_{j=1}^{p} a_j m_j, \\
\sigma^2 &= E \left( \left\| \sum_{j=1}^{p} a_j X_j - \sum_{j=1}^{p} a_j m_j \right\|^2 \right) \\
&= E \left( \left( \sum_{j=1}^{p} a_j (X_j - m_j) \right)^2 \right) = \sum_{j,k} a_j a_k E((X_j - m_j)(X_k - m_k))
\end{align*}
$$

It follows the mean of the normally distributed random variable, $a \cdot X$ is

$$
\mu = \sum_{j} a_j m_j = a \cdot m
$$

and its variance is

$$
\sigma^2 = a^* E((X - m)(X - m)^*) a
$$

Therefore,

$$
E \left( e^{ita} X \right) = e^{ita} e^{-\frac{1}{2}t^2 \sigma^2} = e^{i a \cdot m} e^{-\frac{1}{2}t^2 a^* E((X-m)(X-m)^*) a}.
$$

Then letting $s = ta$ this shows

$$
E \left( e^{isX} \right) = e^{is \cdot m} e^{-\frac{1}{2}s^2 E((X-m)(X-m)^*) s}
$$

which is the characteristic function of a normally distributed random variable with $m$ given above and $\Sigma$ given by

$$
\Sigma_{jk} = E((X_j - m_j)(X_k - m_k)).
$$

By assumption, $a$ is completely arbitrary and so it follows that $s$ is also. Hence, $X$ is normally distributed as claimed.

**Corollary 57.16.5** Let $X = (X_1, \cdots, X_p), Y = (Y_1, \cdots, Y_p)$ where each $X_i, Y_i$ is a real valued random variable. Suppose also that for every $a \in \mathbb{R}^p$, $a \cdot X$ and $a \cdot Y$ are both normally distributed with the same mean and variance. Then $X$ and $Y$ are both multivariate normal random vectors with the same mean and variance.

**Proof:** In the Proof of Theorem 57.16.4 the proof implies that the characteristic functions of $a \cdot X$ and $a \cdot Y$ are both of the form

$$
e^{im \cdot X} e^{-\frac{1}{2} \sigma^2 t^2}.
$$
Then as in the proof of that theorem, it must be the case that

\[ m = \sum_{j=1}^{p} a_j m_j \]

where \( E(X_i) = m_i = E(Y_i) \) and

\[ \sigma^2 = a^* E((X - m)(X - m)^*) a = a^* E((Y - m)(Y - m)^*) a \]

and this last equation must hold for every \( a \). Therefore,

\[ E((X - m)(X - m)^*) = E((Y - m)(Y - m)^*) \equiv \Sigma \]

and so the characteristic function of both \( X \) and \( Y \) is

\[ e^{is \cdot m - \frac{1}{2} s^* \Sigma s} \]

as in the proof of Theorem 57.16.4.

**Theorem 57.16.6** Suppose \( X = (X_1, \cdots, X_p) \) is normally distributed with mean \( m \) and covariance \( \Sigma \). Then if \( X_1 \) is uncorrelated with any of the \( X_i \), meaning

\[ E((X_1 - m_1)(X_j - m_j)) = 0 \quad \text{for } j > 1, \]

then \( X_1 \) and \( (X_2, \cdots, X_p) \) are both normally distributed and the two random vectors are independent. Here \( m_j \equiv E(X_j) \). More generally, if the covariance matrix is a diagonal matrix, the random variables, \( \{X_1, \cdots, X_p\} \) are independent.

**Proof:** From Theorem 57.16.4,

\[ \Sigma = E((X - m)(X - m)^*). \]

Then by assumption,

\[ \Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \Sigma_{p-1} \end{pmatrix}. \quad (57.16.35) \]

I need to verify that if \( E \in \sigma(X_1) \) and \( F \in \sigma(X_2, \cdots, X_p) \), then

\[ P(E \cap F) = P(E) P(F). \]

Let \( E = X_1^{-1}(A) \) and

\[ F = (X_2, \cdots, X_p)^{-1}(B) \]

where \( A \) and \( B \) are Borel sets in \( \mathbb{R} \) and \( \mathbb{R}^{p-1} \) respectively. Thus I need to verify that

\[ P([(X_1, X_2, \cdots, X_p)) \in (A, B)]) = \mu_{(X_1, X_2, \cdots, X_p)}(A \times B) = \mu_{X_1}(A) \mu_{(X_2, \cdots, X_p)}(B). \quad (57.16.36) \]

Using Fubini’s theorem, and definitions,

\[ \mu_{(X_1, X_2, \cdots, X_p)}(A \times B) = \]
\[ \int_{\mathbb{R}^p} X_{A \times B}(x) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(x-m)^\top \Sigma^{-1} (x-m)} \, dx \]

\[ = \int_{\mathbb{R}} X_A(x_1) \int_{\mathbb{R}^{p-1}} X_B(x_2, \cdots, x_p) \cdot \frac{1}{(2\pi)^{(p-1)/2} \sqrt{2\pi} \sigma_{p-1}^{1/2} \det(\Sigma_{p-1})^{1/2}} e^{-\frac{1}{2\sigma_{p-1}^2} (x_1-m_1)^2} \cdot e^{-\frac{1}{2}(x'-m')^\top \Sigma_{p-1}^{-1} (x'-m')} \, dx' \, dx_1 \]

where \( x' = (x_2, \cdots, x_p) \) and \( m' = (m_2, \cdots, m_p) \). Now this equals

\[ \int_{\mathbb{R}} X_A(x_1) \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{1}{2\sigma_1^2} (x_1-m_1)^2} \int_{B} \frac{1}{(2\pi)^{(p-1)/2} \sqrt{2\pi} \sigma_{p-1}^{1/2} \det(\Sigma_{p-1})^{1/2}} e^{-\frac{1}{2\sigma_{p-1}^2} (x'-m')^\top \Sigma_{p-1}^{-1} (x'-m')} \, dx' \, dx_1 \]

where \( X_1 \) is normally distributed as claimed. Similarly, letting \( A = \mathbb{R} \), \( B = \mathbb{R}^{p-1} \), the inside integral equals 1 and

\[ \mu_{X_1}(A) = \mu_{(X_1, (X_2, \cdots, X_p))}(A \times \mathbb{R}^{p-1}) \]

which shows \( X_1 \) is normally distributed as claimed. Similarly, letting \( A = \mathbb{R} \), \( B = \mathbb{R}^{p-1} \), the inside integral equals 1 and

\[ \mu_{(X_2, \cdots, X_p)}(B) = \mu_{(X_1, (X_2, \cdots, X_p))}(\mathbb{R} \times B) \]

and \( (X_2, \cdots, X_p) \) is also normally distributed with mean \( m' \) and covariance \( \Sigma_{p-1} \). Now from (57.16.37), 57.16.36 follows. In case the covariance matrix is diagonal, the above reasoning extends in an obvious way to prove the random variables, \( \{X_1, \cdots, X_p\} \) are independent.

However, another way to prove this is to use Proposition 57.11.1 on Page 1938 and consider the characteristic function. Let \( E(X_j) = m_j \) and

\[ P = \sum_{j=1}^p t_j X_j. \]

Then since \( X \) is normally distributed and the covariance is a diagonal,

\[ D = \begin{pmatrix} \sigma_1^2 & 0 \\ & \ddots \\ 0 & \sigma_p^2 \end{pmatrix} \]
57.17. USE OF CHARACTERISTIC FUNCTIONS TO FIND MOMENTS

Let $X$ be a random variable with characteristic function

$$
\phi_X(t) \equiv E(\exp(itX))
$$

Then this can be used to find moments of the random variable assuming they exist. The $k^{th}$ moment is defined as

$$
E(X^k).
$$

This can be done by using the dominated convergence theorem to differentiate the characteristic function with respect to $t$ and then plugging in $t = 0$. For example,

$$
\phi'_X(t) = E(iX \exp(itX))
$$

and now plugging in $t = 0$ you get $iE(X)$. Doing another differentiation you obtain

$$
\phi''_X(t) = E(-X^2 \exp(itX))
$$
and plugging in \( t = 0 \) you get \( -E(X^2) \) and so forth.

An important case is where \( X \) is normally distributed with mean 0 and variance \( \sigma^2 \). In this case, as shown above, the characteristic function is

\[
e^{-\frac{1}{2}t^2\sigma^2}
\]

Also all moments exist when \( X \) is normally distributed. So what are these moments?

\[
D_t \left( e^{-\frac{1}{2}t^2\sigma^2} \right) = -t\sigma^2 e^{-\frac{1}{2}t^2\sigma^2}
\]

and plugging in \( t = 0 \) you find the mean equals 0 as expected.

\[
D_t \left( -\sigma^2 e^{-\frac{1}{2}t^2\sigma^2} \right) = -\sigma^2 e^{-\frac{1}{2}t^2\sigma^2} + t^2\sigma^4 e^{-\frac{1}{2}t^2\sigma^2}
\]

and plugging in \( t = 0 \) you find the second moment is \( \sigma^2 \). Then do it again.

\[
D_t \left( -\sigma^2 e^{-\frac{1}{2}t^2\sigma^2} + t^2\sigma^4 e^{-\frac{1}{2}t^2\sigma^2} \right) = 3\sigma^4 t e^{-\frac{1}{2}t^2\sigma^2} - t^3\sigma^6 e^{-\frac{1}{2}t^2\sigma^2}
\]

Then \( E(X^3) = 0 \).

\[
D_t \left( 3\sigma^4 t e^{-\frac{1}{2}t^2\sigma^2} - t^3\sigma^6 e^{-\frac{1}{2}t^2\sigma^2} \right)
= 3\sigma^4 e^{-\frac{1}{2}t^2\sigma^2} - 6\sigma^6 t^2 e^{-\frac{1}{2}t^2\sigma^2} + t^4\sigma^8 e^{-\frac{1}{2}t^2\sigma^2}
\]

and so \( E(X^4) = 3\sigma^4 \). By now you can see the pattern. If you continue this way, you find the odd moments are all 0 and

\[
E(X^{2m}) = C_m (\sigma^2)^m.
\]  

(57.17.40)

This is an important observation.

**57.18 The Central Limit Theorem**

The central limit theorem is one of the most marvelous theorems in mathematics. It can be proved through the use of characteristic functions. Recall for \( x \in \mathbb{R}^p \),

\[
||x||_\infty \equiv \max \{|x_j|, j = 1, \cdots, p\}.
\]

Also recall the definition of the distribution function for a random vector, \( X \).

\[
F_X(x) \equiv P(X_j \leq x_j, j = 1, \cdots, p).
\]

**Definition 57.18.1** Let \( \{X_n\} \) be random vectors with values in \( \mathbb{R}^p \). Then \( \{\lambda_{X_n}\}_{n=1}^\infty \) is called “tight” if for all \( \varepsilon > 0 \) there exists a compact set, \( K_\varepsilon \) such that

\[
\lambda_{X_n}(|x| \notin K_\varepsilon) < \varepsilon
\]
for all \( \lambda x_n \). Similarly, if \( \{\mu_n\} \) is a sequence of probability measures defined on the Borel sets of \( \mathbb{R}^p \), then this sequence is “tight” if for each \( \varepsilon > 0 \) there exists a compact set, \( K_\varepsilon \) such that

\[
\mu_n ([x \notin K_\varepsilon]) < \varepsilon
\]

for all \( \mu_n \).

**Lemma 57.18.2** If \( \{X_n\} \) is a sequence of random vectors with values in \( \mathbb{R}^p \) such that

\[
\lim_{n \to \infty} \phi_{X_n}(t) = \psi(t)
\]

for all \( t \), where \( \psi(0) = 1 \) and \( \psi \) is continuous at \( 0 \), then \( \{\lambda x_n\}_{n=1}^\infty \) is tight.

**Proof:** Let \( e_j \) be the \( j \)th standard unit basis vector.

\[
\begin{align*}
\left| \frac{1}{u} \int_{-u}^{u} \left( 1 - \phi_{X_n}(te_j) \right) dt \right| &= \left| \frac{1}{u} \int_{-u}^{u} \left( 1 - \int_{\mathbb{R}^p} e^{itx} d\lambda_{X_n} \right) dt \right| \\
&= \left| \frac{1}{u} \int_{-u}^{u} \left( \int_{\mathbb{R}^p} \left( 1 - e^{itx} \right) d\lambda_{X_n} \right) dt \right| \\
&= \left| \int_{\mathbb{R}^p} \frac{1}{u} \int_{-u}^{u} \left( 1 - e^{itx} \right) dt d\lambda_{X_n}(x) \right| \\
&= \left| 2 \int_{\mathbb{R}^p} \left( 1 - \frac{\sin (ux_j)}{ux_j} \right) d\lambda_{X_n}(x) \right| \\
&\geq 2 \int_{|x_j| \geq \frac{u}{2}} \left( 1 - \frac{1}{ux_j} \right) d\lambda_{X_n}(x) \\
&\geq 2 \int_{|x_j| \geq \frac{u}{2}} \left( 1 - \frac{1}{|u|^2} \right) d\lambda_{X_n}(x) \\
&= \int_{|x_j| \geq \frac{u}{2}} 1 d\lambda_{X_n}(x) \\
&= \lambda_{X_n} \left( \left\{ x : |x_j| \geq \frac{2}{u} \right\} \right).
\end{align*}
\]

If \( \varepsilon > 0 \) is given, there exists \( r > 0 \) such that if \( u \leq r \),

\[
\frac{1}{u} \int_{-u}^{u} (1 - \psi(t e_j)) dt < \varepsilon / p
\]
for all \( j = 1, \cdots, p \) and so, by the dominated convergence theorem, the same is true with \( \phi_{X_n} \) in place of \( \psi \) provided \( n \) is large enough, say \( n \geq N(u) \). Thus, if \( u \leq r \), and \( n \geq N(u) \),

\[
\lambda_{X_n} \left( \left[ x : |x_j| \geq \frac{2}{u} \right] \right) < \varepsilon/p
\]

for all \( j \in \{1, \cdots, p\} \). It follows that for \( u \leq r \) and \( n \geq N(u) \),

\[
\lambda_{X_n} \left( \left[ x : ||x||_\infty \geq \frac{2}{u} \right] \right) < \varepsilon.
\]

because

\[
\left[ x : ||x||_\infty \geq \frac{2}{u} \right] \subseteq \bigcup_{j=1}^{p} \left[ x : |x_j| \geq \frac{2}{u} \right]
\]

This proves the lemma because there are only finitely many measures, \( \lambda_{X_n} \) for \( n < N(u) \) and the compact set can be enlarged finitely many times to obtain a single compact set, \( K_\varepsilon \) such that for all \( n, \lambda_{X_n} ([x \notin K_\varepsilon]) < \varepsilon \). This proves the lemma.

**Lemma 57.18.3** If \( \phi_{X_n}(t) \to \phi_X(t) \) for all \( t \), then whenever \( \psi \in \mathcal{S} \),

\[
\lambda_{X_n}(\psi) = \int_{\mathbb{R}^p} \psi(y) d\lambda_{X_n}(y) \to \int_{\mathbb{R}^p} \psi(y) d\lambda_X(y) = \lambda_X(\psi)
\]

as \( n \to \infty \).

**Proof:** Recall that if \( X \) is any random vector, its characteristic function is given by

\[
\phi_X(y) = \int_{\mathbb{R}^p} e^{iy \cdot x} d\lambda_X(x).
\]

Also remember the inverse Fourier transform. Letting \( \psi \in \mathcal{S} \), the Schwartz class,

\[
F^{-1}(\lambda_X)(\psi) = \lambda_X(F^{-1}\psi) = \int_{\mathbb{R}^p} F^{-1}\psi d\lambda_X
\]

\[
= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{iy \cdot x} \psi(x) dx d\lambda_X(y)
\]

\[
= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(x) \int_{\mathbb{R}^p} e^{iy \cdot x} d\lambda_X(y) dx
\]

\[
= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(x) \phi_X(x) dx
\]

and so, considered as elements of \( \mathcal{S}^* \),

\[
F^{-1}(\lambda_X) = \phi_X(\cdot) (2\pi)^{-(p/2)} \in L^\infty.
\]
By the dominated convergence theorem

\[
(2\pi)^{p/2} F^{-1}(\lambda X_n) (\psi) \equiv \int_{\mathbb{R}^p} \phi_{X_n} (t) \psi (t) dt \\
\to \int_{\mathbb{R}^p} \phi_X (t) \psi (t) dt \\
= (2\pi)^{p/2} F^{-1}(\lambda X) (\psi)
\]

whenever \( \psi \in \mathcal{S} \). Thus

\[
\lambda_{X_n} (\psi) = FF^{-1}\lambda_{X_n} (\psi) \equiv F^{-1}\lambda_{X_n} (F\psi) \to F^{-1}\lambda_X (F\psi) \\
\equiv F^{-1}F\lambda_X (\psi) = \lambda_X (\psi).
\]

This proves the lemma.

**Lemma 57.18.4** If \( \phi_{X_n} (t) \to \phi_X (t) \), then if \( \psi \) is any bounded uniformly continuous function,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^p} \psi d\lambda_{X_n} = \int_{\mathbb{R}^p} \psi d\lambda_X.
\]

**Proof:** Let \( \varepsilon > 0 \) be given, let \( \psi \) be a bounded function in \( C^\infty (\mathbb{R}^p) \). Now let \( \eta \in C^\infty_c (Q_r) \) where \( Q_r \equiv [-r, r]^p \) satisfy the additional requirement that \( \eta = 1 \) on \( Q_{r/2} \) and \( \eta (x) \in [0, 1] \) for all \( x \). By Lemma 57.18.2 the set, \( \{\lambda_{X_n}\}_{n=1}^\infty \), is tight and so if \( \varepsilon > 0 \) is given, there exists \( r \) sufficiently large such that for all \( n \),

\[
\int_{[x \not\in Q_{r/2}]} |1 - \eta| |\psi| d\lambda_{X_n} < \frac{\varepsilon}{3},
\]

and

\[
\int_{[x \not\in Q_{r/2}]} |1 - \eta| |\psi| d\lambda_X < \frac{\varepsilon}{3}.
\]

Thus,

\[
\left|\int_{\mathbb{R}^p} \psi d\lambda_{X_n} - \int_{\mathbb{R}^p} \psi d\lambda_X\right| \leq \left|\int_{\mathbb{R}^p} \psi d\lambda_{X_n} - \int_{\mathbb{R}^p} \psi \eta d\lambda_{X_n}\right| + \\
\left|\int_{\mathbb{R}^p} \psi \eta d\lambda_{X_n} - \int_{\mathbb{R}^p} \psi \eta d\lambda_X\right| + \left|\int_{\mathbb{R}^p} \psi \eta d\lambda_X - \int_{\mathbb{R}^p} \psi d\lambda_X\right| \\
\leq \frac{2\varepsilon}{3} + \left|\int_{\mathbb{R}^p} \psi \eta d\lambda_{X_n} - \int_{\mathbb{R}^p} \psi \eta d\lambda_X\right| < \varepsilon
\]

whenever \( n \) is large enough by Lemma 57.18.3 because \( \psi \eta \in \mathcal{S} \). This establishes the conclusion of the lemma in the case where \( \psi \) is also infinitely differentiable. To consider the general case, let \( \psi \) only be uniformly continuous and let \( \psi_k = \psi \ast \phi_k \) where \( \phi_k \) is a mollifier whose support is in \((- (1/k), (1/k))^p \). Then \( \psi_k \) converges uniformly to \( \psi \) and so the desired conclusion follows for \( \psi \) after a routine estimate. This proves the lemma.
Definition 57.18.5 Let $\mu$ be a Radon measure on $\mathbb{R}^p$. A Borel set, $A$, is a $\mu$-continuity set if $\mu(\partial A) = 0$ where $\partial A \equiv \overline{A} \setminus \text{int}(A)$ and $\text{int}$ denotes the interior.

The main result is the following continuity theorem. More can be said about the equivalence of various criteria [15].

Theorem 57.18.6 If $\phi_{X_n}(t) \to \phi_X(t)$ then $\lambda_{X_n}(A) \to \lambda_X(A)$ whenever $A$ is a $\lambda_X$-continuity set.

Proof: First suppose $K$ is a closed set and let

$$
\psi_k(x) \equiv (1 - k \text{dist}(x,K))^+.
$$

Thus, since $K$ is closed $\lim_{k \to \infty} \psi_k(x) = \chi_K(x)$. Choose $k$ large enough that

$$
\int_{\mathbb{R}^p} \psi_k d\lambda_X \leq \lambda_X(K) + \varepsilon.
$$

Then by Lemma 57.18.4, applied to the bounded uniformly continuous function $\psi_k$,

$$
\limsup_{n \to \infty} \lambda_{X_n}(K) \leq \limsup_{n \to \infty} \int \psi_k d\lambda_{X_n} = \int \psi_k d\lambda_X \leq \lambda_X(K) + \varepsilon.
$$

Since $\varepsilon$ is arbitrary, this shows

$$
\limsup_{n \to \infty} \lambda_{X_n}(K) \leq \lambda_X(K)
$$

for all $K$ closed.

Next suppose $V$ is open and let

$$
\psi_k(x) = 1 - \left(1 - k \text{dist}(x,V^C)\right)^+.
$$

Thus $\psi_k(x) \in [0,1], \psi_k = 1$ if $\text{dist}(x,V^C) \geq 1/k$, and $\psi_k = 0$ on $V^C$. Since $V$ is open, it follows

$$
\lim_{k \to \infty} \psi_k(x) = \chi_V(x).
$$

Choose $k$ large enough that

$$
\int \psi_k d\lambda_X \geq \lambda_X(V) - \varepsilon.
$$

Then by Lemma 57.18.3,

$$
\liminf_{n \to \infty} \lambda_{X_n}(V) \geq \liminf_{n \to \infty} \int \psi_k(x) d\lambda_{X_n} = \int \psi_k(x) d\lambda_X \geq \lambda_X(V) - \varepsilon.
$$
57.18. THE CENTRAL LIMIT THEOREM

and since \(\varepsilon\) is arbitrary,

\[
\lim \inf_{n \to \infty} \lambda_{X_n}(V) \geq \lambda_X(V).
\]

Now let \(\lambda_X(\partial A) = 0\) for \(A\) a Borel set.

\[
\lambda_X(\text{int}(A)) \leq \lim \inf_{n \to \infty} \lambda_{X_n}(\text{int}(A)) \leq \lim \inf_{n \to \infty} \lambda_{X_n}(A) \leq \lim \sup_{n \to \infty} \lambda_{X_n}(A) \leq \lambda_X(A).
\]

But \(\lambda_X(\text{int}(A)) = \lambda_X(A)\) by assumption and so \(\lim_{n \to \infty} \lambda_{X_n}(A) = \lambda_X(A)\) as claimed. This proves the theorem.

As an application of this theorem the following is a version of the central limit theorem in the situation in which the limit distribution is multivariate normal. It concerns a sequence of random vectors, \(\{X_k\}_{k=1}^\infty\), which are identically distributed, have finite mean \(m\), and satisfy

\[E\left(\left|X_k\right|^2\right) < \infty.\] (57.18.41)

**Theorem 57.18.7** Let \(\{X_k\}_{k=1}^\infty\) be random vectors satisfying (57.18.41) which are independent and identically distributed with mean \(m\) and positive definite covariance \(\Sigma = E((X - m)(X - m)^\ast)\). Let

\[Z_n \equiv \sum_{j=1}^n \frac{X_j - m}{\sqrt{n}}.\] (57.18.42)

Then for \(Z \sim N_p(0, \Sigma)\),

\[\lim_{n \to \infty} F_{Z_n}(x) = F_Z(x)\] (57.18.43)

for all \(x\).

**Proof:** The characteristic function of \(Z_n\) is given by

\[
\phi_{Z_n}(t) = E\left(e^{it \cdot \sum_{j=1}^n \frac{X_j - m}{\sqrt{n}}}\right) = \prod_{j=1}^n E\left(e^{it \cdot \frac{X_j - m}{\sqrt{n}}}\right).
\]

By Taylor’s theorem applied to real and imaginary parts of \(e^{ix}\), it follows

\[e^{ix} = 1 + ix - f(x)\frac{x^2}{2}\]

where \(|f(x)| < 2\) and

\[\lim_{x \to 0} f(x) = 1.\]

Denoting \(X_j\) as \(X\), this implies

\[e^{it \cdot \frac{X - m}{\sqrt{n}}} = 1 + it \cdot \frac{X - m}{\sqrt{n}} - f \left( t \cdot \frac{X - m}{\sqrt{n}} \right) \left( t \cdot \frac{(X - m)}{2n} \right)^2 \]
Thus
\[ e^{it\left(\frac{X - m}{\sqrt{n}}\right)} = 1 + it\cdot\frac{X - m}{\sqrt{n}} - \left(\frac{t \cdot (X - m)}{2n}\right)^2 + \left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2. \]

Thus
\[ \phi_{Z_n}(t) = \prod_{j=1}^{n} \left[ 1 - E\left(\frac{\left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2}{2n}\right) \right] + E\left(\left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2\right) \]
\[ = \prod_{j=1}^{n} \left[ 1 - \frac{1}{2n} t^* \Sigma t + \frac{1}{2n} E\left(\left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2\right)\right]. \]

(Note \((t \cdot (X - m))^2 = t^* (X - m) (X - m)^* t\). Now here is a simple inequality for complex numbers whose moduli are no larger than one. I will give a proof of this at the end. It follows easily by induction.}

\[ |z_1 \cdots z_n - w_1 \cdots w_n| \leq \sum_{k=1}^{n} |z_k - w_k|. \]

Also for each \(t\), and all \(n\) large enough,
\[ \left| \frac{1}{2n} E\left(\left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2\right) \right| < 1. \]

Applying (57.18.44) to (57.18.45),
\[ \phi_{Z_n}(t) = \left( \prod_{j=1}^{n} \left( 1 - \frac{1}{2n} t^* \Sigma t \right) + e_n \right) \]
\[ = \left( 1 - \frac{1}{2n} t^* \Sigma t \right)^n + e_n \]
where
\[ |e_n| \leq \sum_{j=1}^{n} \frac{1}{2n} E\left(\left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2\right) \]
\[ = \frac{1}{2} \left| E\left(\left(1 - f\left(\frac{t \cdot (X - m)}{\sqrt{n}}\right)\right) \left(\frac{t \cdot (X - m)}{2n}\right)^2\right) \right| \]
which converges to 0 as \( n \to \infty \) by the Dominated Convergence theorem. Therefore,

\[
\lim_{n \to \infty} \left| \phi_{Z_n}(t) - \left( 1 - \frac{t^* \Sigma t}{2n} \right)^n \right| = 0
\]

and so

\[
\lim_{n \to \infty} \phi_{Z_n}(t) = e^{-\frac{1}{2} t^* \Sigma t} = \phi_Z(t)
\]

where \( Z \sim N_p(0, \Sigma) \). Therefore, \( F_{Z_n}(x) \to F_Z(x) \) for all \( x \) because \( R_x \equiv \prod_{k=1}^{p} (-\infty, x_k] \) is a set of \( \lambda_Z \) continuity due to the assumption that \( \lambda_Z \ll m_p \) which is implied by \( Z \sim N_p(0, \Sigma) \). This proves the theorem.

Here is the proof of the little inequality used above. The inequality is obviously true if \( n = 1 \). Assume it is true for \( n \). Then since all the numbers have absolute value no larger than one,

\[
\left| \prod_{i=1}^{n+1} z_i - \prod_{i=1}^{n+1} w_i \right| \leq \left| \prod_{i=1}^{n+1} z_i - \prod_{i=1}^{n} w_i \right| + \left| \prod_{i=1}^{n} w_i - \prod_{i=1}^{n+1} w_i \right|
\]

\[
\leq \left| \prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i \right| + |z_{n+1} - w_{n+1}|
\]

\[
\leq \sum_{k=1}^{n+1} |z_k - w_k|
\]

by induction.

Suppose \( X \) is a random vector with covariance \( \Sigma \) and mean \( m \), and suppose also that \( \Sigma^{-1} \) exists. Consider \( \Sigma^{-(1/2)} (X - m) \equiv Y \). Then \( E(Y) = 0 \) and

\[
E(YY^*) = E \left( \Sigma^{-(1/2)} (X - m) (X^* - m) \Sigma^{-(1/2)} \right)
= \Sigma^{-(1/2)} E \left( (X - m) (X^* - m) \right) \Sigma^{-(1/2)} = I.
\]

Thus \( Y \) has zero mean and covariance \( I \). This implies the following corollary to Theorem 57.18.7.

**Corollary 57.18.8** Let independent identically distributed random variables,

\[
\{X_j\}_{j=1}^{\infty}
\]

have mean \( m \) and positive definite covariance \( \Sigma \) where \( \Sigma^{-1} \) exists. Then if

\[
Z_n = \sum_{j=1}^{n} \Sigma^{-(1/2)} \frac{(X_j - m)}{\sqrt{n}},
\]
it follows that for \( Z \sim N_p (0, I) \),

\[
F_{Z_n} (x) \to F_Z (x)
\]

for all \( x \).

### 57.19 Characteristic Functions Of Probability Measures, Prokhorov Theorem

Recall one can define the characteristic function of a probability measure. In a sense it is more natural.

**Definition 57.19.1** Let \( \mu \) be a probability measure defined on the Borel sets of \( \mathbb{R}^p \). Then

\[
\phi_\mu (t) \equiv \int_{\mathbb{R}^p} e^{itx} d\mu.
\]

Also \( \{ \mu_n \}_{n=1}^\infty \) is called “tight” if for all \( \varepsilon > 0 \) there exists a compact set, \( K_\varepsilon \) such that

\[
\mu_n ([x \notin K_\varepsilon]) < \varepsilon
\]

for all \( \mu_n \).

Then there is a version of Lemma 57.18.2 whose proof is identical to the proof of that lemma.

**Lemma 57.19.2** If \( \{ \mu_n \} \) is a sequence of Borel probability measures defined on the Borel sets of \( \mathbb{R}^p \) such that

\[
\lim_{n \to \infty} \phi_{\mu_n} (t) = \psi (t)
\]

for all \( t \), where \( \psi (0) = 1 \) and \( \psi \) is continuous at \( 0 \), then \( \{ \mu_n \}_{n=1}^\infty \) is tight.

**Proof:** Let \( e_j \) be the \( j^{th} \) standard unit basis vector. Letting \( t = te_j \) in the definition,

\[
\left| \frac{1}{u} \int_{-u}^u \left( 1 - \phi_{\mu_n}(te_j) \right) dt \right| = \left| \frac{1}{u} \int_{-u}^u \left( 1 - \int_{\mathbb{R}^p} e^{itx} d\mu_n (x) \right) dt \right| \tag{57.19.46}
\]

\[
= \left| \frac{1}{u} \int_{-u}^u \left( \int_{\mathbb{R}^p} (1 - e^{itx}) d\mu_n (x) \right) dt \right|
\]

\[
= \left| \int_{\mathbb{R}^p} \frac{1}{u} \int_{-u}^u (1 - e^{itx}) dt d\mu_n (x) \right|
\]
57.19. CHARACTERISTIC FUNCTIONS OF PROBABILITY MEASURES, PROKHOROV THEOREM

\[
\begin{align*}
\int_{\mathbb{R}^p} & \left| 2 \int \frac{1 - \sin \left( \frac{ux_j}{ux_j} \right)}{ux_j} \, d\mu_n(x) \right| \\
\geq & \quad 2 \int_{\{ |x_j| \geq \frac{u}{2} \}} \left( 1 - \frac{1}{|ux_j|} \right) \, d\mu_n(x) \\
\geq & \quad 2 \int_{\{ |x_j| \geq \frac{u}{2} \}} \left( 1 - \frac{1}{|u|(2/u)} \right) \, d\mu_n(x) \\
= & \quad \int_{\{ |x_j| \geq \frac{u}{2} \}} 1 \, d\mu_n(x) \\
= & \quad \mu_n \left( \left\{ x : |x_j| \geq \frac{2}{u} \right\} \right).
\end{align*}
\]

If \( \varepsilon > 0 \) is given, there exists \( r > 0 \) such that if \( u \leq r \),

\[
\frac{1}{u} \int_{-u}^{u} (1 - \psi(t e_j)) \, dt < \varepsilon/p
\]

for all \( j = 1, \cdots, p \) and so, by the dominated convergence theorem, the same is true with \( \phi_{\mu_n} \) in place of \( \psi \) provided \( n \) is large enough, say \( n \geq N(u) \). Thus, from (57.19.30), if \( u \leq r \), and \( n \geq N(u) \),

\[
\mu_n \left( \left\{ x : |x_j| \geq \frac{2}{u} \right\} \right) < \varepsilon/p
\]

for all \( j \in \{ 1, \cdots, p \} \). It follows that for \( u \leq r \) and \( n \geq N(u) \),

\[
\mu_n \left( \left\{ x : \| x \|_\infty \geq \frac{2}{u} \right\} \right) < \varepsilon.
\]

because

\[
\left\{ x : \| x \|_\infty \geq \frac{2}{u} \right\} \subseteq \bigcup_{j=1}^{p} \left\{ x : |x_j| \geq \frac{2}{u} \right\}
\]

This proves the lemma because there are only finitely many measures, \( \mu_n \) for \( n < N(u) \) and the compact set can be enlarged finitely many times to obtain a single compact set, \( K_\varepsilon \) such that for all \( n, \mu_n \left( \{ x \notin K_\varepsilon \} \right) < \varepsilon. \]

As before, there are simple modifications of Lemmas (57.18.3) and (57.18.4). The first of these is as follows.

**Lemma 57.19.3** If \( \phi_{\mu_n} (t) \to \phi_\mu (t) \) for all \( t \), then whenever \( \psi \in S \), the Schwartz class,

\[
\mu_n (\psi) \equiv \int_{\mathbb{R}^p} \psi (y) \, d\mu_n(y) \to \int_{\mathbb{R}^p} \psi (y) \, d\mu (y) \equiv \mu (\psi)
\]

as \( n \to \infty \).
Proof: By definition,
\[ \phi_{\mu}(y) \equiv \int_{\mathbb{R}^p} e^{iy \cdot x} d\mu(x). \]
Also remember the inverse Fourier transform. Letting \( \psi \in \mathcal{S} \), the Schwartz class,
\[ F^{-1}(\mu)(\psi) \equiv \mu(F^{-1}\psi) \equiv \int_{\mathbb{R}^p} F^{-1}\psi d\mu \]
\[ = \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{iy \cdot x} \psi(x) dx d\mu(y) \]
\[ = \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(x) \int_{\mathbb{R}^p} e^{iy \cdot x} d\mu(y) dx \]
\[ = \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(x) \phi_{\mu}(x) dx \]
and so, considered as elements of \( \mathcal{S}^* \),
\[ F^{-1}(\mu) = \phi_{\mu}(\cdot) (2\pi)^{-p/2} \in L^{\infty}. \]
By the dominated convergence theorem
\[ (2\pi)^{p/2} F^{-1}(\mu_n)(\psi) = \int_{\mathbb{R}^p} \phi_{\mu_n}(t) \psi(t) dt \]
\[ \rightarrow \int_{\mathbb{R}^p} \phi_{\mu}(t) \psi(t) dt \]
\[ = (2\pi)^{p/2} F^{-1}(\mu)(\psi) \]
whenever \( \psi \in \mathcal{S} \). Thus
\[ \mu_n(\psi) = FF^{-1}\mu_n(\psi) \equiv F^{-1}(\mu_n(F\psi) \rightarrow F^{-1}\mu(F\psi) \]
\[ \equiv F^{-1}F\mu(\psi) = \mu(\psi). \]

The version of Lemma 57.19.2 is the following.

Lemma 57.19.4 If \( \phi_{\mu_n}(t) \rightarrow \phi_{\mu}(t) \) where \( \{\mu_n\} \) and \( \mu \) are probability measures defined on the Borel sets of \( \mathbb{R}^p \), then if \( \psi \) is any bounded uniformly continuous function,
\[ \lim_{n \to \infty} \int_{\mathbb{R}^p} \psi d\mu_n = \int_{\mathbb{R}^p} \psi d\mu. \]

Proof: Let \( \varepsilon > 0 \) be given, let \( \psi \) be a bounded function in \( C^{\infty}(\mathbb{R}^p) \). Now let \( \eta \in C^{\infty}(Q_r) \) where \( Q_r \equiv [-r, r]^p \) satisfy the additional requirement that \( \eta = 1 \) on \( Q_{r/2} \) and \( \eta(x) \in [0, 1] \) for all \( x \). By Lemma 57.19.2 the set, \( \{\mu_n\}_{n=1}^{\infty} \), is tight and so if \( \varepsilon > 0 \) is given, there exists \( r \) sufficiently large such that for all \( n \),
\[ \int_{[x \notin Q_{r/2}]} |1 - \eta| |\psi| d\mu_n < \frac{\varepsilon}{3}. \]
and
\[ \int_{|x| \leq Q_r/2} |1 - \eta| |\psi| d\mu < \frac{\varepsilon}{3}. \]
Thus,
\[ \left| \int_{\mathbb{R}^p} \psi d\mu_n - \int_{\mathbb{R}^p} \psi d\mu \right| \leq \int_{\mathbb{R}^p} \psi d\mu_n - \int_{\mathbb{R}^p} \psi \eta d\mu_n + \int_{\mathbb{R}^p} \psi \eta d\mu - \int_{\mathbb{R}^p} \psi d\mu \]
\[ \leq \frac{2\varepsilon}{3} + \left| \int_{\mathbb{R}^p} \psi \eta d\mu_n - \int_{\mathbb{R}^p} \psi \eta d\mu \right| < \varepsilon \]
whenever \( n \) is large enough by Lemma 57.19.3 because \( \psi \eta \in \mathcal{S} \). This establishes the conclusion of the lemma in the case where \( \psi \) is also infinitely differentiable. To consider the general case, let \( \psi \) only be uniformly continuous and let \( \psi_k = \psi * \phi_k \)
where \( \phi_k \) is a mollifier whose support is in \((- (1/k) , (1/k))^p \). Then \( \psi_k \) converges uniformly to \( \psi \) and so the desired conclusion follows for \( \psi \) after a routine estimate.

The next theorem is really important. It gives the existence of a measure based on the assumption that a set of measures is tight. The next theorem is Prokhorov’s theorem about a tight set of measures. Recall that \( \Lambda \) is tight means that for every \( \varepsilon > 0 \) there exists \( K \) compact such that \( \mu(K^c) < \varepsilon \) for all \( \mu \in \Lambda \).

**Theorem 57.19.5** Let \( \Lambda = \{ \mu_n \}_{n=1}^{\infty} \) be a sequence of probability measures defined on the Borel sets of \( \mathbb{R}^p \). If \( \Lambda \) is tight then there exists a probability measure, \( \lambda \) and a subsequence of \( \{ \mu_n \}_{n=1}^{\infty} \), still denoted by \( \{ \mu_n \}_{n=1}^{\infty} \) such that whenever \( \phi \) is a continuous bounded complex valued function defined on \( E \),

\[ \lim_{n \to \infty} \int \phi d\mu_n = \int \phi d\lambda. \]

**Proof:** By tightness, there exists an increasing sequence of compact sets, \( \{ K_n \} \) such that
\[ \mu(K_n) > 1 - \frac{1}{n} \]
for all \( \mu \in \Lambda \). Now letting \( \mu \in \Lambda \) and \( \phi \in C(K_n) \) such that \( ||\phi||_{\infty} \leq 1 \), it follows
\[ \left| \int_{K_n} \phi d\mu \right| \leq \mu(K_n) \leq 1 \]
and so the restrictions of the measures of \( \Lambda \) to \( K_n \) are contained in the unit ball of \( C(K_n)' \). Recall from the Riesz representation theorem, the dual space of \( C(K_n) \) is a space of complex Borel measures. Theorem 15.5.5 on Page 447 implies the unit ball of \( C(K_n)' \) is weak * sequentially compact. This follows from the observation that \( C(K_n) \) is separable which follows easily from the Weierstrass approximation theorem. Thus the unit ball in \( C(K_n)' \) is actually metrizable by Theorem 15.5.5.
Therefore, there exists a subsequence of \( \Lambda \), \( \{\mu_{1k}\} \) such that their restrictions to \( K_1 \) converge weak \( * \) to a measure, \( \lambda_1 \in C(K_1)' \). That is, for every \( \phi \in C(K_1) \),

\[
\lim_{k \to \infty} \int_{K_1} \phi d\mu_{1k} = \int_{K_1} \phi d\lambda_1
\]

By the same reasoning, there exists a further subsequence \( \{\mu_{2k}\} \) such that the restrictions of these measures to \( K_2 \) converge weak \( * \) to a measure \( \lambda_2 \in C(K_2)' \) etc. Continuing this way,

\[
\mu_{11}, \mu_{12}, \mu_{13}, \ldots \to \text{Weak } * \text{ in } C(K_1)'
\]

\[
\mu_{21}, \mu_{22}, \mu_{23}, \ldots \to \text{Weak } * \text{ in } C(K_2)'
\]

\[
\mu_{31}, \mu_{32}, \mu_{33}, \ldots \to \text{Weak } * \text{ in } C(K_3)'
\]

... 

Here the \( j \)th sequence is a subsequence of the \((j-1)\)th. Let \( \lambda_n \) denote the measure in \( C(K_n)' \) to which the sequence \( \{\mu_{nk}\}_{k=1}^{\infty} \) converges weak *. Let \( \{\mu_n\} \equiv \{\mu_{nn}\} \), the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so \( \mu_n \) converges weak * in \( C(K_m)' \) to \( \lambda_m \) for each \( m \).

**Claim:** For \( p > n \), the restriction of \( \lambda_p \) to the Borel sets of \( K_n \) equals \( \lambda_n \).

**Proof of claim:** Let \( H \) be a compact subset of \( K_n \). Then there are sets, \( V_l \) open in \( K_n \) which are decreasing and whose intersection equals \( H \). This follows because this is a metric space. Then let \( H \prec \phi_l \prec V_l \). It follows

\[
\lambda_n(V_l) \geq \int_{K_n} \phi_l d\lambda_n = \lim_{k \to \infty} \int_{K_n} \phi_l d\mu_k
\]

\[
= \lim_{k \to \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \geq \lambda_p(H).
\]

Now considering the ends of this inequality, let \( l \to \infty \) and pass to the limit to conclude

\[
\lambda_n(H) \geq \lambda_p(H).
\]

Similarly,

\[
\lambda_n(H) \leq \int_{K_n} \phi_l d\lambda_n = \lim_{k \to \infty} \int_{K_n} \phi_l d\mu_k
\]

\[
= \lim_{k \to \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \leq \lambda_p(V_l).
\]

Then passing to the limit as \( l \to \infty \), it follows

\[
\lambda_n(H) \leq \lambda_p(H).
\]

Thus the restriction of \( \lambda_p, \lambda_p|_{K_n} \) to the compact sets of \( K_n \) equals \( \lambda_n \). Then by inner regularity it follows the two measures, \( \lambda_p|_{K_n} \), and \( \lambda_n \) are equal on all Borel
57.19. CHARACTERISTIC FUNCTIONS OF PROBABILITY MEASURES, PROKHOROV THEOREM

sets of $K_n$. Recall that for finite measures on the Borel sets of separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that $\lambda_m (F) \geq 0$ for all $F$ a Borel subset of $K_m$. (Whenever $\phi \geq 0$, $\int_{K_m} \phi d\lambda_m \geq 0$ because $\int_{K_m} \phi d\mu_k \geq 0$. Now you can approximate $\chi_F$ with a suitable nonnegative $\phi$ using regularity of the measure.) Also, letting $\phi \equiv 1$,

$$1 \geq \lambda_m (K_m) \geq 1 - \frac{1}{m}, \quad (57.19.47)$$

Define for $F$ a Borel set,

$$\lambda (F) \equiv \lim_{n \to \infty} \lambda_n (F \cap K_n).$$

The limit exists because the sequence on the right is increasing due to the above observation that $\lambda_n = \lambda_m$ on the Borel subsets of $K_m$ whenever $n > m$. Thus for $n > m$

$$\lambda_n (F \cap K_n) \geq \lambda_n (F \cap K_m) = \lambda_m (F \cap K_m).$$

Now let $\{F_k\}$ be a sequence of disjoint Borel sets. Then

$$\lambda (\bigcup_{k=1}^{\infty} F_k) = \lim_{n \to \infty} \lambda_n (\bigcup_{k=1}^{\infty} F_k \cap K_n) = \lim_{n \to \infty} \lambda_n (\bigcup_{k=1}^{\infty} (F_k \cap K_n)) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \lambda_n (F_k \cap K_n) = \sum_{k=1}^{\infty} \lambda (F_k)$$

the last equation holding by the monotone convergence theorem.

It remains to verify

$$\lim_{k \to \infty} \int \phi d\mu_k = \int \phi d\lambda$$

for every $\phi$ bounded and continuous. This is where tightness is used again. Suppose $\|\phi\|_{\infty} < M$. Then as noted above,

$$\lambda_n (K_n) = \lambda (K_n)$$

because for $p > n$, $\lambda_p (K_n) = \lambda_n (K_n)$ and so letting $p \to \infty$, the above is obtained.

Also, from

$$\lambda (K_n^C) = \lim_{p \to \infty} \lambda_p (K_n^C \cap K_p) \leq \lim \sup_{p \to \infty} (\lambda_p (K_p) - \lambda_p (K_n)) \leq \lim \sup_{p \to \infty} (\lambda_p (K_p) - \lambda_n (K_n)) \leq \lim \sup_{p \to \infty} \left(1 - \left(1 - \frac{1}{n}\right)\right) = \frac{1}{n}$$
Consequently,
\[
\left| \int \phi d\mu_k - \int \phi d\lambda \right| \leq \left| \int_{K^C_n} \phi d\mu_k - \int_{K^C_n} \phi d\mu - \left( \int_{K_n} \phi d\lambda + \int_{K^C_n} \phi d\lambda \right) \right| \\
\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\mu_n \right| + \left| \int_{K^C_n} \phi d\mu_k - \int_{K^C_n} \phi d\mu \right| \\
\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\mu_n \right| + \left| \int_{K^C_n} \phi d\mu_k \right| + \left| \int_{K^C_n} \phi d\mu \right| \\
\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\mu_n \right| + \frac{M}{n} + \frac{M}{n}
\]

First let \( n \) be so large that \( 2M/n < \varepsilon/2 \) and then pick \( k \) large enough that the above expression is less than \( \varepsilon \).

**Definition 57.19.6** Let \( \mu, \{\mu_n\} \) be probability measures defined on the Borel sets of \( \mathbb{R}^p \) and let the sequence of probability measures, \( \{\mu_n\} \) satisfy
\[
\lim_{n \to \infty} \int \phi d\mu_n = \int \phi d\mu.
\]
for every \( \phi \) a bounded continuous function. Then \( \mu_n \) is said to converge weakly to \( \mu \).

With the above, it is possible to prove the following amazing theorem of Levy.

**Theorem 57.19.7** Suppose \( \{\mu_n\} \) is a sequence of probability measures defined on the Borel sets of \( \mathbb{R}^p \) and let \( \{\phi_{\mu_n}\} \) denote the corresponding sequence of characteristic functions. If there exists \( \psi \) which is continuous at \( 0 \), \( \psi(0) = 1 \), and for all \( t \),
\[
\phi_{\mu_n}(t) \to \psi(t),
\]
then there exists a probability measure, \( \lambda \) defined on the Borel sets of \( \mathbb{R}^p \) and
\[
\phi_{\lambda}(t) = \psi(t).
\]
That is, \( \psi \) is a characteristic function of a probability measure. Also, \( \{\mu_n\} \) converges weakly to \( \lambda \).

**Proof:** By Lemma 57.19.2 \( \{\mu_n\} \) is tight. Therefore, there exists a subsequence \( \{\mu_{n_k}\} \) converging weakly to a probability measure, \( \lambda \). In particular,
\[
\phi_{\lambda}(t) = \int e^{it \cdot x} d\lambda(x) = \lim_{n \to \infty} \int e^{it \cdot x} d\mu_{n_k}(x) = \lim_{n \to \infty} \phi_{\mu_{n_k}}(t) = \psi(t)
\]
The last claim follows from this and Lemma 57.19.4. Note how it was only necessary to assume \( \psi(0) = 1 \) and \( \psi \) is continuous at \( 0 \) in order to conclude that \( \psi \) is a characteristic function. Thus you find that \( |\psi(t)| \leq 1 \) for free. This helps to see why Prokhorov’s and Levy’s theorems are so amazing.
57.20 Generalized Multivariate Normal

In this section is a further explanation of generalized multivariable normal random variables. Recall that these have characteristic function equal to \( e^{it^t \mathbf{m} - \frac{1}{2} t^t \Sigma t} \) where \( \Sigma \geq 0, \Sigma = \Sigma^* \). The new detail is the case that \( \det(\Sigma) = 0 \).

**Definition 57.20.1** A random vector, \( \mathbf{X} \), with values in \( \mathbb{R}^p \) has a multivariate normal distribution written as

\[
\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)
\]

if for all Borel \( E \subseteq \mathbb{R}^p \), the distribution measure is given by

\[
\lambda_{\mathbf{X}}(E) = \int_{\mathbb{R}^p} \lambda_{E}(x) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(x^t - \mathbf{m})^t \Sigma^{-1}(x^t - \mathbf{m})} dx
\]

for \( \mathbf{m} \) a given vector and \( \Sigma \) a given positive definite symmetric matrix. Recall also that the characteristic function of this random variable is

\[
E(e^{it^t \mathbf{X}}) = e^{it^t \mathbf{m} - \frac{1}{2} t^t \Sigma t} \tag{57.20.48}
\]

So what if \( \det(\Sigma) = 0 \)? Is there a probability measure having characteristic function \( e^{it^t \mathbf{X}} \)? Let \( \Sigma_n \rightarrow \Sigma \) in the Frobenius norm, \( \det(\Sigma_n) > 0 \). That is the \( ij^{th} \) components converge. Let \( \mathbf{X}_n \) be the random variable which is associated with \( \mathbf{m} \) and \( \Sigma_n \). Thus for \( \phi \in C_0(\mathbb{R}^p) \),

\[
|\lambda_{\mathbf{X}_n}(\phi)| \equiv \left| \int_{\mathbb{R}^p} \phi(x) \frac{1}{(2\pi)^{p/2} \det(\Sigma_n)^{1/2}} e^{-\frac{1}{2}(x^t - \mathbf{m})^t \Sigma_n^{-1}(x^t - \mathbf{m})} dx \right| \leq \|\phi\|_{C_0(\mathbb{R}^p)}
\]

Thus these \( \lambda_{\mathbf{X}_n} \) are bounded in the weak * topology of \( C_0(\mathbb{R}^p)' \) which is the space of signed measures. By the separability of \( C_0(\mathbb{R}^p) \) and the Banach Alaoglu theorem and the Riesz representation theorem for \( C_0(\mathbb{R}^p)' \), there is a subsequence still denoted as \( \lambda_{\mathbf{X}_n} \) which converges weak * to a finite measure \( \mu \). Is \( \mu \) a probability measure? Is the characteristic function of this measure \( e^{it^t \mathbf{X}} \)?

Note that \( E(e^{it^t \mathbf{X}_n}) = e^{it^t \mathbf{m} - \frac{1}{2} t^t \Sigma_n t} \rightarrow e^{it^t \mathbf{m} - \frac{1}{2} t^t \Sigma t} \) and this last function of \( t \) is continuous at 0. Therefore, by Lemma 57.18.2, these measures \( \lambda_{\mathbf{X}_n} \) are also tight. Let \( \varepsilon > 0 \) be given. Then there is a compact set \( K_\varepsilon \) such that \( \lambda_{\mathbf{X}_n}(\mathbf{x} \notin K_\varepsilon) < \varepsilon \). Now let \( \phi = 1 \) on \( K_\varepsilon \) and \( \phi \in C_c(\mathbb{R}^p) \), \( \phi \geq 0 \), \( \phi(\mathbf{x}) \in [0, 1] \). Then

\[
(1 - \varepsilon) \leq \int_{\mathbb{R}^p} \phi d\lambda_{\mathbf{X}_n} \rightarrow \int_{\mathbb{R}^p} \phi d\mu \leq \mu(\mathbb{R}^p)
\]

and so, since \( \varepsilon \) is arbitrary, this shows that \( \mu(\mathbb{R}^p) \geq 1 \). However, \( \mu(\mathbb{R}^p) \leq 1 \) because

\[
\mu(\mathbb{R}^p) \leq \int_{\mathbb{R}^p} \psi d\mu + \varepsilon \leq \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} + 2\varepsilon \leq 1 + 2\varepsilon
\]
for suitable $\psi \in C_c(\mathbb{R}^p)$ having values in $[0, 1]$ and $n$. Thus $\mu$ is indeed a probability measure.

Now what of its characteristic function?

$$e^{it \cdot m} e^{-\frac{1}{2} t^* \Sigma t} = \lim_{n \to \infty} e^{it \cdot m} e^{-\frac{1}{2} t^* \Sigma_n t} = \lim_{n \to \infty} \int_{\mathbb{R}^p} e^{it \cdot x} d\lambda_{X_n}(x) \quad (57.20.49)$$

Is this equal to

$$\int_{\mathbb{R}^p} e^{it \cdot x} d\mu(x)?$$

Using tightness again,

$$\left| \int_{\mathbb{R}^p} e^{it \cdot x} d\mu(x) - \int_{\mathbb{R}^p} e^{it \cdot x} d\lambda_{X_n}(x) \right| \leq \left| \int_{\mathbb{R}^p} e^{it \cdot x} d\mu(x) - \int_{\mathbb{R}^p} \psi e^{it \cdot x} d\mu(x) \right|$$

$$+ \left| \int_{\mathbb{R}^p} \psi e^{it \cdot x} d\mu(x) - \int_{\mathbb{R}^p} \psi e^{it \cdot x} d\lambda_{X_n}(x) \right| \leq \varepsilon + \left| \int_{\mathbb{R}^p} \psi e^{it \cdot x} d\mu(x) - \int_{\mathbb{R}^p} \psi e^{it \cdot x} d\lambda_{X_n}(x) \right|$$

for a suitable choice of $\psi \in C_c(\mathbb{R}^p)$ having values in $[0, 1]$. The middle term is less than $\varepsilon$ if $n$ large enough thanks to the weak $^*$ convergence of $\lambda_{X_n}$ to $\mu$. Hence the last limit in $57.20.49$ equals $\int_{\mathbb{R}^p} e^{it \cdot x} d\mu(x)$ as hoped. Letting $X$ be a random variable having $\mu$ as its distribution measure, (You could take $\Omega = \mathbb{R}^p$ and the measurable sets the Borel sets.) what about $E((X \cdot m) (X \cdot m)^* )$? Is it equal to $\Sigma$? What about the question whether $X \in L^q(\Omega; \mathbb{R}^p)$ for all $q > 1$? This is clearly true for the case where $\Sigma^{-1}$ exists, but what of the case where $\det(\Sigma) = 0$?

For simplicity, say $m = 0$.

$$\int_{\Omega} |X|^q dP = \int_0^\infty P(|X|^q > \lambda) d\lambda = \int_0^\infty \mu(|X|^q > \lambda) d\lambda$$

$$\leq \int_0^\infty \mu(|X|^q > \lambda) d\lambda \leq \int_0^\infty (1 - \psi_\lambda) d\mu d\lambda$$

where $\psi_\lambda = 1$ on $B\left(0, \frac{1}{2} \lambda^{1/q}\right)$ is nonnegative, and is in $C_c\left(B\left(0, \lambda^{1/q}\right)\right)$. Now from the above, $\mu(\mathbb{R}^p) = \lambda_{X_n}(\mathbb{R}^p) = 1$ and so the inside integral satisfies

$$\int_{\mathbb{R}^p} (1 - \psi_\lambda) d\mu = \lim_{n \to \infty} \int_{\mathbb{R}^p} (1 - \psi_\lambda) d\lambda_{X_n} \quad (57.20.50)$$

because

$$\int_{\mathbb{R}^p} d\mu = \int_{\mathbb{R}^p} d\lambda_{X_n} = 1$$

and as to the other terms, the weak $^*$ convergence gives

$$\int_{\mathbb{R}^p} \psi_\lambda d\mu = \lim_{n \to \infty} \int_{\mathbb{R}^p} \psi_\lambda d\lambda_{X_n}$$
Each of these integrals in $\mathbb{R}^p$ is no larger than 1. Hence from Fatou’s lemma,

$$\int_\Omega |X|^q dP \leq \int_0^\infty \int_{\mathbb{R}^p} (1 - \psi_\lambda) d\mu d\lambda \leq \lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^p} (1 - \psi_\lambda) d\lambda_X d\lambda$$

Is this on the right finite? It is dominated by

$$\lim_{n \to \infty} \int_0^\infty \lambda_X \left(|x|^q > \frac{1}{2^q} \lambda \right) d\lambda = \lim_{n \to \infty} 2^q \int_0^\infty \lambda_X \left(|x|^q > \delta \right) d\delta$$

So is a subsequence of $\{E(|X_n|^q)\}$ bounded? It equals

$$\int_{\mathbb{R}^p} \frac{1}{(2\pi)^{p/2}} \frac{1}{\det(\Sigma_n)^{1/2}} e^{-\frac{1}{2}((x-m)^* \Sigma_n^{-1} (x-m))} d\lambda$$

and for $q$ an even integer, this moment can be computed using the characteristic function.

$$e^{-\frac{1}{2} t^* \Sigma_n t} = \int_{\mathbb{R}^p} e^{i t^* x} d\lambda_X$$

Also, it suffices to consider $E(X^q_k)$. Differentiate both sides. Using the repeated index summation convention,

$$e^{-\frac{1}{2} t^* \Sigma_n t} (\sum_{n}{t}^j t_j) = \int_{\mathbb{R}^p} i x_k e^{i t^* x} d\mu$$

Now differentiate again.

$$e^{-\frac{1}{2} t^* \Sigma_n t} (\sum_{n}{t}^j t_j) (\sum_{n}{t}^j t_j) + (\sum_{n}{t}^k t_k) = \int_{\mathbb{R}^p} x_k^2 e^{i t^* x} d\lambda_X$$

Next let $t = 0$ to conclude that $E(X^2_{nk}) = \Sigma_{nk}$. Of course you can continue differentiating as long as desired and obtain $E(X^2_{nk})$ is equal to some polynomial formula involving $\Sigma_{nk}$ and these are given to converge to $\Sigma_{kk}$. Therefore, for any $q > 1$, $\{E(|X_n|^q)\}$ is bounded and so from the above,

$$\int_\Omega |X|^q dP \leq \lim_{n \to \infty} 2^q E(|X_n|^q) < \infty$$

So yes, $X$ is indeed in $L^q(\Omega, \mathbb{R}^p)$ for every $q$. What about the covariance?

From the definition of the characteristic function,

$$e^{-\frac{1}{2} t^* \Sigma t} = \int_{\mathbb{R}^p} e^{i t^* x} d\mu$$

and so taking the derivative with respect to $t_k$ of both sides,

$$e^{-\frac{1}{2} t^* \Sigma t} (\sum_{n}{t}^j t_j) = \int_{\mathbb{R}^p} i x_k e^{i t^* x} d\mu$$
Now differentiate with respect to $t_i$ on both sides.

\[
e^{-\frac{1}{2}t^*\Sigma t} (-\Sigma_{il}t_i) (-\Sigma_{kj}t_j) + e^{-\frac{1}{2}t^*\Sigma t} (-\Sigma_{kl}) = \int_{\mathbb{R}^p} ix_k (ix_l) e^{it\cdot x} d\mu = -\int_{\mathbb{R}^p} x_kx_le^{it\cdot x} d\mu
\]

Now let $t = 0$ to obtain

\[
\Sigma_{kl} = \int_{\mathbb{R}^p} x_kx_le^{it\cdot x} d\mu = E (X_kX_l)
\]

If $m \neq 0$, the same kind of argument holds with a little more details. This proves the following theorem.

**Theorem 57.20.2** Let $\Sigma$ be nonnegative and self adjoint $p \times p$ matrix. Then there exists a random variable $X$ whose distribution measure $\lambda_X$ has characteristic function

\[
e^{it\cdot m} e^{-\frac{1}{2}t^*\Sigma t}
\]

Also

\[
E \left( (X - m)^2 \right) = \Sigma
\]

that is

\[
E \left( (X - m)_i (X - m)_j \right) = \Sigma_{ij}
\]

This is generalized normally distributed random variable.

There is an interesting corollary to this theorem.

**Corollary 57.20.3** Let $H$ be a real Hilbert space. Then there exist random variables $W(h)$ for $h \in H$ such that each is normally distributed with mean 0 and for every $h, g, (W(h), W(g))$ is normally distributed and

\[
E (W(h)W(g)) = (h, g)_H
\]

Furthermore, if $\{e_i\}$ is an orthogonal set of vectors of $H$, then $\{W(e_i)\}$ are independent random variables. Also for any finite set $\{f_1, f_2, \cdots, f_n\}$,

\[
(W(f_1), W(f_2), \cdots, W(f_n))
\]

is normally distributed.

**Proof:** Let $\mu_{h_1 \cdots h_m}$ be a multivariate normal distribution with covariance $\Sigma_{ij} = (h_i, h_j)$ and mean 0. Thus the characteristic function of this measure is

\[
e^{-\frac{1}{2}t^*\Sigma t}
\]

Now suppose $\mu_{k_1 \cdots k_n}$ is another such measure where for simplicity,

\[
\{h_1 \cdots h_m, k_{m+1} \cdots k_n\} = \{k_1 \cdots k_n\}
\]
Let $\nu$ be a measure on $B(\mathbb{R}^m)$ which is given by
\[
\nu(E) = \mu_{k_1 \cdots k_n}(E \times \mathbb{R}^{n-m})
\]
Then does it follow that $\nu = \mu_{h_1 \cdots h_n}$? If so, then the Kolmogorov consistency condition will hold for these measures $\mu_{h_1 \cdots h_n}$. To determine whether this is so, take the characteristic function of $\nu$. Let $\Sigma_1$ be the $n \times n$ matrix which comes from the $\{k_1 \cdots k_n\}$ and let $\Sigma_2$ be the one which comes from the $\{h_1 \cdots h_m\}$.
\[
\int_{\mathbb{R}^m} e^{it \cdot x} d\nu(x) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} e^{i(x,x')} d\mu_{k_1 \cdots k_n}(x,y)
\]
which is the characteristic function for $\mu_{h_1 \cdots h_n}$. Therefore, these two measures are the same and the Kolmogorov consistency condition holds. It follows that there exists a measure $\mu$ defined on the Borel sets of $\prod_{h \in H} \mathbb{R}$ which extends all of these measures. This argument also shows that if a random vector $X$ has characteristic function $e^{-\frac{1}{2}(x',\Sigma x)}$ then if $X_k$ is one of its components, then the characteristic function of $X_k$ is $e^{-\frac{1}{2}x_k^2}$ so this scalar valued random variable has mean zero and variance $|h_k|^2$. Then if $\omega \in \prod_{h \in H} \mathbb{R}$
\[
W(h)(\omega) \equiv \pi_h(\omega)
\]
where $\pi_h$ denotes the projection onto position $h$ in this product. Also define
\[
(W(f_1), W(f_2), \cdots , W(f_n)) \equiv \pi_{f_1 \cdots f_n}(\omega)
\]
Then this is a random variable whose covariance matrix is just $\Sigma_{ij} = (f_i, f_j)_H$ and whose characteristic equation is $e^{-\frac{1}{2}(x',\Sigma x)}$ so this verifies that
\[
(W(f_1), W(f_2), \cdots , W(f_n))
\]
is normally distributed with covariance $\Sigma$. If you have two of them, $W(g), W(h)$, then
\[
E(W(h)W(g)) = (h,g)_H. \quad \text{This follows from what was just shown that (W(f), W(g)) is normally distributed and so the covariance will be}
\]
\[
\begin{pmatrix}
|f|^2 & (f,g) \\
(f,g) & |g|^2
\end{pmatrix} = \begin{pmatrix}
E(W(f)^2) & E(W(f)W(g)) \\
E(W(f)W(g)) & E(W(g)^2)
\end{pmatrix}
\]
Finally consider the claim about independence. Any finite subset of $\{W(e_i)\}$ is generalized normal with the covariance matrix being a diagonal. Therefore, writing in terms of the distribution measures, this diagonal matrix allows the iterated integrals to be split apart and it follows that
\[
E\left(\exp\left(\sum_{k=1}^{m} t_k W(e_k)\right)\right) = \prod_{k=1}^{m} \exp(it_k W(e_k))
\]
and so this follows from Proposition 57.20. Note that in this case, the covariance matrix will not have zero determinant.
57.21 Positive Definite Functions, Bochner’s Theorem

First here is a nice little lemma about matrices.

**Lemma 57.21.1** Suppose \( M \) is an \( n \times n \) matrix. Suppose also that
\[
\alpha^* M \alpha = 0
\]
for all \( \alpha \in \mathbb{C}^n \). Then \( M = 0 \).

**Proof:** Suppose \( \lambda \) is an eigenvalue for \( M \) and let \( \alpha \) be an associated eigenvector.
\[
0 = \alpha^* M \alpha = \alpha^* \lambda \alpha = \lambda \alpha^* \alpha = \lambda |\alpha|^2
\]
and so all the eigenvalues of \( M \) equal zero. By Schur’s theorem there is a unitary matrix \( U \) such that
\[
M = U \begin{pmatrix}
0 & *_1 \\
. & . \\
0 & 0
\end{pmatrix} U^* \tag{57.21.51}
\]
where the matrix in the middle has zeros down the main diagonal and zeros below the main diagonal. Thus
\[
M^* = U \begin{pmatrix}
0 & 0 \\
. & . \\
*_2 & 0
\end{pmatrix} U^*
\]
where \( M^* \) has zeros down the main diagonal and zeros above the main diagonal.
Also taking the adjoint of the given equation for \( M \), it follows that for all \( \alpha \),
\[
\alpha^* M^* \alpha = 0
\]
Therefore, \( M + M^* \) is Hermitian and has the property that
\[
\alpha^* (M + M^*) \alpha = 0.
\]
Thus \( M + M^* = 0 \) because it is unitarily similar to a diagonal matrix and the above equation can only hold for all \( \alpha \) if \( M + M^* \) has all zero eigenvalues which implies the diagonal matrix has zeros down the main diagonal. Therefore, from the formulas for \( M, M^* \),
\[
0 = U \begin{pmatrix}
0 & 0 \\
. & . \\
*_2 & 0
\end{pmatrix} + \begin{pmatrix}
0 & *_1 \\
. & . \\
0 & 0
\end{pmatrix} U^*
\]
and so the sum of the two matrices in the middle must also equal 0. Hence the entries of the matrix in the middle in [57.21.51] are all equal to zero. Thus \( M = 0 \) as claimed.
Definition 57.21.2 A Borel measurable function, \( f : \mathbb{R}^n \to \mathbb{C} \) is called positive definite if whenever \( \{ t_k \}_{k=1}^p \subseteq \mathbb{R}^n, \alpha \in \mathbb{C}^p \)

\[
\sum_{k,j} f(t_j - t_k) \alpha_j \alpha_k \geq 0 \tag{57.21.52}
\]

The first thing to notice about a positive definite function is the following which implies these functions are automatically bounded.

Lemma 57.21.3 If \( f \) is positive definite then whenever \( \{ t_k \}_{k=1}^p \) are \( p \) points in \( \mathbb{R}^n \), \( |f(t_j - t_k)| \leq f(0) \). In particular, for all \( t \), \( |f(t)| \leq f(0) \).

Proof: Let \( F \) be the \( p \times p \) matrix such that

\[
F_{kj} = f(t_j - t_k).
\]

Then is of the form

\[
\alpha^* F \alpha = (F \alpha, \alpha) \geq 0 \tag{57.21.53}
\]

where this is the inner product in \( \mathbb{C}^p \). Letting \( [\alpha, \beta] \equiv (F \alpha, \beta) \equiv \beta^* F \alpha \), it is obvious that \( [\alpha, \beta] \) satisfies

\[
[a \alpha + b \beta, \gamma] = a [\alpha, \gamma] + b [\beta, \gamma].
\]

I claim it also satisfies

\[
[\alpha, \beta] = [\beta, \alpha].
\]

To verify this last claim, note that since \( \alpha^* F \alpha \) is real,

\[
\alpha^* F^* \alpha = \alpha^* F \alpha \geq 0
\]

and so for all \( \alpha \in \mathbb{C}^p \),

\[
\alpha^* (F^* - F) \alpha = 0
\]

which from Lemma 57.21.1 implies \( F^* = F \). Hence \( F \) is self-adjoint and it follows

\[
[\alpha, \beta] \equiv \beta^* F \alpha = \beta^* F^* \alpha = \alpha^T F^* T \beta = \alpha^* F \beta = [\beta, \alpha].
\]

Therefore, the Cauchy Schwarz inequality holds for \( [,] \) and it follows

\[
||[\alpha, \beta]|| = ||(F \alpha, \beta)|| \leq (F \alpha, \alpha)^{1/2} (F \beta, \beta)^{1/2}.
\]

Letting \( \alpha = e_k \) and \( \beta = e_j \), it follows \( F_{ss} \geq 0 \) for all \( s \) and

\[
|F_{kj}| \leq F_{kk}^{1/2} F_{jj}^{1/2}
\]

which says nothing more than

\[
|f(t_j - t_k)| \leq f(0)^{1/2} f(0)^{1/2} = f(0).
\]

This proves the lemma.

With this information, here is another useful lemma involving positive definite functions. It is interesting because it looks like the formula which defines what it means for the function to be positive definite.
Lemma 57.21.4 Let $f$ be a positive definite function as defined above and let $\mu$ be a finite Borel measure. Then
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y) \, d\mu(x) \, d\mu(y) \geq 0. \tag{57.21.54}
\]
If $\mu$ also has the property that it is symmetric, $\mu(F) = \mu(-F)$ for all $F$ Borel, then
\[
\int_{\mathbb{R}^n} f(x) \, d(\mu * \mu)(x) \geq 0. \tag{57.21.55}
\]

Proof: By definition if \( \{t_j\}_{j=1}^p \subseteq \mathbb{R}^n \), and letting $\alpha = (1, \ldots, 1)^T \in \mathbb{R}^n$,
\[
\sum_{j,k} f(t_j - t_k) \geq 0.
\]
Therefore, integrating over each of the variables,
\[
0 \leq \sum_{j=1}^p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t_j - t_j) \, d\mu(t_j) \, d\mu(t_j) + \sum_{j \neq k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t_j - t_k) \, d\mu(t_j) \, d\mu(t_k)
\]
and so
\[
0 \leq f(0) \mu(\mathbb{R}^n)^2 + p(p-1) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y) \, d\mu(x) \, d\mu(y).
\]
Dividing both sides by $p(p-1)$ and letting $p \to \infty$, it follows
\[
0 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y) \, d\mu(x) \, d\mu(y)
\]
which shows $f(0)$.

To verify $\mu_t * \mu_t = \mu_{2t}$, use $\mu_t$.\]
\[
\int_{\mathbb{R}^n} f d(\mu * \mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x + y) \, d\mu(x) \, d\mu(y)
\]
and since $\mu$ is symmetric, this equals
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x - y) \, d\mu(x) \, d\mu(y) \geq 0
\]
by the first part of the lemma. This proves the lemma.

Lemma 57.21.5 Let $\mu_t$ be the measure defined on $\mathcal{B}(\mathbb{R}^n)$ by
\[
\mu_t(F) = \int_F \frac{1}{(\sqrt{2\pi t})^n} e^{-\frac{1}{2} ||x||^2} \, dx
\]
for $t > 0$. Then $\mu_t * \mu_t = \mu_{2t}$ and each $\mu_t$ is a probability measure.
Proof: By Theorem 57.14.7

\[ \phi_{\mu_t*\mu_t}(s) = \phi_{\mu_t}(s) \phi_{\mu_t}(s) = \left( e^{-\frac{1}{2}t|s|^2} \right)^2 = e^{-\frac{1}{2}(2t)|s|^2} = \phi_{\mu_{2t}}(s). \]

Each \( \mu_t \) is a probability measure because it is the distribution of a normally distributed random variable of mean \( 0 \) and covariance \( tI. \)

Now let \( \mu \) be a probability measure on \( B(\mathbb{R}^n) \).

\[ \phi_{\mu}(t) \equiv \int e^{it\cdot y} d\mu(y) \]

and so by the dominated convergence theorem, \( \phi_{\mu} \) is continuous and also \( \phi_{\mu}(0) = 1. \)

I claim \( \phi_{\mu} \) is also positive definite. Let \( \alpha \in \mathbb{C}^p \) and \( \{t_k\}_{k=1}^p \) a sequence of points of \( \mathbb{R}^n. \) Then

\[ \sum_{k,j} \phi_{\mu}(t_k - t_j) \alpha_k \overline{\alpha_j} = \sum_{k,j} \int e^{it_k \cdot y} \alpha_k e^{-it_j \cdot y} \overline{\alpha_j} d\mu(y) \]

\[ = \int \sum_{k,j} e^{it_k \cdot y} \alpha_k e^{it_j \cdot y} \overline{\alpha_j} d\mu(y). \]

Now let \( \beta(y) \equiv (e^{it_1 \cdot y} \alpha_1, \ldots, e^{it_p \cdot y} \alpha_p)^T. \) Then the above equals

\[ \int (1, \ldots, 1) \beta(y) \beta^*(y) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} d\mu \]

The integrand is of the form

\[ \beta^* \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \beta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \geq 0 \]

because it is just a complex number times its conjugate.

Thus every characteristic function is continuous, equals 1 at 0, and is positive definite. Bochner’s theorem goes the other direction.

To begin with, suppose \( \mu \) is a finite measure on \( B(\mathbb{R}^n). \) Then for \( \mathcal{S} \) the Schwartz class, \( \mu \) can be considered to be in the space of linear transformations defined on \( \mathcal{S}, \mathcal{S}^* \) as follows.

\[ \mu(f) \equiv \int f d\mu. \]

Recall \( F^{-1}(\mu) \) is defined as

\[ F^{-1}(\mu)(f) \equiv \mu(F^{-1}f) = \int_{\mathbb{R}^n} F^{-1}f d\mu \]
\[
\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{ix \cdot y} f(y) \, dy \, d\mu = \int_{\mathbb{R}^n} \left( \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} d\mu \right) f(y) \, dy
\]

and so \( F^{-1}(\mu) \) is the bounded continuous function \( y \to \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix \cdot y} d\mu \).

Now the following lemma has the main ideas for Bochner’s theorem.

**Lemma 57.21.6** Suppose \( \psi(t) \) is positive definite, \( t \to \psi(t) \) is in \( L^1(\mathbb{R}^n, m_n) \) where \( m_n \) is Lebesgue measure, \( \psi(0) = 1 \), and \( \psi \) is continuous at \( 0 \). Then there exists a unique probability measure, \( \mu \) defined on the Borel sets of \( \mathbb{R}^n \) such that

\[
\phi_\mu(t) = \psi(t).
\]

**Proof:** If the conclusion is true, then

\[
\psi(t) = \int_{\mathbb{R}^n} e^{it \cdot x} d\mu(x) = (2\pi)^{n/2} F^{-1}(\mu)(t).
\]

Recall that \( \mu \in \mathcal{S}^\ast \), the algebraic dual of \( \mathcal{S} \). Therefore, in \( \mathcal{S}^\ast \),

\[
\frac{1}{(2\pi)^{n/2}} F(\psi) = \mu.
\]

That is, for all \( f \in \mathcal{S} \),

\[
\int_{\mathbb{R}^n} f(y) \, d\mu(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(\psi)(y) f(y) \, dy = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} e^{-iy \cdot x} \psi(x) \, dx \right) \, dy. \tag{57.21.56}
\]

I will show

\[
f \to \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} e^{-iy \cdot x} \psi(x) \, dx \right) \, dy
\]

is a positive linear functional and then it will follow from \( \text{57.21.56} \) that \( \mu \) is unique. Thus it is needed to show the inside integral in \( \text{57.21.56} \) is nonnegative. First note that the integrand is a positive definite function of \( x \) for each fixed \( y \). This follows from

\[
\sum_{k,j} e^{-iy \cdot (x_k - x_j)} \psi(x_k - x_j) \alpha_k \bar{\alpha}_j
\]

\[
= \sum_{k,j} \psi(x_k - x_j) \left( e^{-iy \cdot x_k} \alpha_k \right) \overline{e^{-iy \cdot x_j} \alpha_j} \geq 0.
\]
57.21. POSITIVE DEFINITE FUNCTIONS, BOCHNER’S THEOREM

Let \( t > 0 \) and
\[
 h_{2t}(x) \equiv \frac{1}{(4\pi t)^{n/2}} e^{-\frac{1}{4} |x|^2}.
\]

Then by dominated convergence theorem,
\[
 \int_{\mathbb{R}^n} e^{-i\mathbf{y} \cdot \mathbf{x}} \psi(x) \, dx = \lim_{t \to \infty} \int_{\mathbb{R}^n} e^{-i\mathbf{y} \cdot \mathbf{x}} \psi(x) h_{2t}(x) \, dx
\]

Letting \( d\eta_{2t} = h_{2t}(x) \, dx \), it follows from Lemma 57.21.5 \( \eta_{2t} = \eta_t \ast \eta_t \) and since these are symmetric measures, it follows from Lemma 57.21.4 the above equals
\[
\lim_{t \to \infty} \int_{\mathbb{R}^n} e^{-i\mathbf{y} \cdot \mathbf{x}} \psi(x) \, d(\eta_t \ast \eta_t) \geq 0
\]

Thus the above functional is a positive linear functional and so there exists a unique Radon measure, \( \mu \) satisfying
\[
\int_{\mathbb{R}^n} f(y) \, d\mu(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(\psi)(y) f(y) \, dy
\]
\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y) \left( \int_{\mathbb{R}^n} e^{-i\mathbf{y} \cdot \mathbf{x}} \psi(x) \, dx \right) \, dy
\]
\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \psi(x) \left( \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-i\mathbf{y} \cdot \mathbf{x}} \, dy \right) \, dx
\]

for all \( f \in C_c(\mathbb{R}^n) \). Thus from the dominated convergence theorem, the above holds for all \( f \in \mathcal{S} \) also. Hence for all \( f \in \mathcal{S} \) and considering \( \mu \) as an element of \( \mathcal{S}^* \),
\[
F^{-1} \mu(Ff) = \mu(f) = \int_{\mathbb{R}^n} f(y) \, d\mu(y)
\]
\[
= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \psi(x) F(f)(x) \, dx
\]
\[
= \frac{1}{(2\pi)^{n/2}} F(\psi)(f) \equiv \frac{1}{(2\pi)^{n/2}} \psi(Ff).
\]

It follows that in \( \mathcal{S}^* \),
\[
\psi = (2\pi)^{n/2} F^{-1} \mu
\]

Thus
\[
\psi(t) = \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} \, d\mu
\]
in \( L^1 \). Since the right side is continuous and the left is given continuous at \( t = 0 \) and equal to 1 there, it follows
\[
1 = \psi(0) = \int_{\mathbb{R}^n} e^{i0 \cdot \mathbf{x}} \, d\mu = \mu(\mathbb{R}^n)
\]

and so \( \mu \) is a probability measure as claimed. This proves the lemma.

The following is Bochner’s theorem.
Theorem 57.21.7 Let $\psi$ be positive definite, continuous at $0$, and $\psi(0) = 1$. Then there exists a unique Radon probability measure $\mu$ such that $\psi = \phi_\mu$.

**Proof:** If $\psi \in L^1(\mathbb{R}^n, m_n)$, then the result follows from Lemma 57.21.6. By Lemma 57.21.3 $\psi$ is bounded. Consider

$$\psi_t(x) \equiv \psi(x) \frac{1}{(2\pi t)^{n/2}} e^{-\frac{1}{2}t|x|^2}.$$ 

Then $\psi_t(0) = 1$, $x \to \psi_t(x)$ is continuous at $0$, and $\psi_t \in L^1(\mathbb{R}^n, m_n)$. Therefore, by Lemma 57.21.6 there exists a unique Radon probability measure $\mu_t$ such that

$$\psi_t(x) = \int_{\mathbb{R}^n} e^{ix \cdot y} d\mu_t(y) = \phi_{\mu_t}(x).$$

Now letting $t \to \infty$,

$$\lim_{t \to \infty} \psi_t(x) = \lim_{t \to \infty} \phi_{\mu_t}(x) = \psi(x).$$

By Levy’s theorem, Theorem 57.19.7 it follows there exists $\mu$, a probability measure on $\mathcal{B}(\mathbb{R}^n)$ such that $\psi(x) = \phi_{\mu}(x)$. The measure is unique because the characteristic functions are uniquely determined by the measure. This proves the theorem.
Chapter 58

Conditional Expectation And Martingales

58.1 Conditional Expectation

From Observation 57.11.5 on Page 1942, it was shown that the conditional expectation of a random variable $X$ given some others really is just what the words suggest. Given $\omega \in \Omega$, it results in a value for the “other” random variables and then you essentially take the expectation of $X$ given this information which yields the value of the conditional expectation of $X$ given the other random variables. It was also shown in Lemma 57.11.4 that this gives the same result as finding a $\sigma(X_1, \cdots, X_n)$ measurable function $Z$ such that for all $F \in \sigma(X_1, \cdots, X_n)$,

$$\int_F X dP = \int_F Z dP$$

This was done for a particular type of $\sigma$ algebra but there is no need to be this specialized. The following is the general version of conditional expectation given a $\sigma$ algebra and this is what will be done from now on.

Definition 58.1.1 Let $(\Omega, \mathcal{M}, P)$ be a probability space and let $S \subseteq \mathcal{F}$ be two $\sigma$ algebras contained in $\mathcal{M}$. Let $f$ be $\mathcal{F}$ measurable and in $L^1(\Omega)$. Then $E(f|S)$, called the conditional expectation of $f$ with respect to $S$ is defined as follows:

$E(f|S)$ is $S$ measurable

For all $E \in S$,

$$\int_E E(f|S) dP = \int_E f dP$$
Lemma 58.1.2 The above is well defined. Also, if $S \subseteq F$ then
\[ E(X|S) = E(E(X|F)|S). \] (58.1.1)

If $Z$ is bounded and measurable in $S$ then
\[ ZE(X|S) = E(ZX|S). \] (58.1.2)

Proof: Let a finite measure on $S$, $\mu$ be given by
\[ \mu(E) \equiv \int_E f dP. \]
Then $\mu \ll P$ and so by the Radon Nikodym theorem, there exists a unique $S$ measurable function, $E(f|S)$ such that
\[ \int_E f dP \equiv \mu(E) = \int_E E(f|S) dP \]
for all $E \in S$.
Let $F \in S$. Then
\[ \int_F E(E(X|F)|S) dP = \int_F E(X|F) dP \]
\[ = \int_F X dP = \int_F E(X|S) dP \]
and so, by uniqueness, $E(E(X|F)|S) = E(X|S)$. This shows 58.1.1.

To establish 58.1.2, note that if $Z = \chi_F$ where $F \in S$,
\[ \int \chi_F E(X|S) dP = \int \chi_F X dP = \int E(\chi_F X|S) dP \]
which shows 58.1.2 in the case where $Z$ is the indicator function of a set in $S$.
It follows this also holds for simple functions. Let $\{s_n\}$ be a sequence of simple functions which converges uniformly to $Z$ and let $F \in S$. Then by what was just shown,
\[ \int_F s_n E(X|S) dP = \int_F s_n X dP = \int_F E(s_n X|S) dP \]
Now
\[ \left| \int_F E(s_n X|S) dP - \int_F E(ZX|S) dP \right| \]
\[ \leq \int_F |s_n X - ZX| dP = \int_F |s_n - Z| |X| dP \]
which converges to 0 by the dominated convergence theorem. Then passing to the limit using the dominated convergence theorem, yields
\[ \int_F ZE(X|S) dP = \int_F ZX dP \equiv \int_F E(ZX|S) dP. \]
58.1. CONDITIONAL EXPECTATION

Since this holds for every $F \in \mathcal{S}$, this shows □

The next major result is a generalization of Jensen’s inequality whose proof depends on the following lemma about convex functions.

**Lemma 58.1.3** Let $\phi$ be a convex real valued function defined on an interval $I$. Then for each $x \in I$, there exists $a_x$ such that for all $t \in I$, $\phi(t) \geq a_x(t - x) + \phi(x)$.

Also $\phi$ is continuous on $I$.

**Proof:** Let $x \in I$ and let $t > x$. Then by convexity of $\phi$, 
\[
\frac{\phi(x + \lambda(t - x)) - \phi(x)}{\lambda(t - x)} \leq \frac{\phi(x)(1 - \lambda) + \lambda \phi(t) - \phi(x)}{\lambda(t - x)}
\]
\[
= \frac{\phi(t) - \phi(x)}{t - x}.
\]

Therefore $t \to \frac{\phi(t) - \phi(x)}{t - x}$ is increasing if $t > x$. If $t < x$
\[
\frac{\phi(x + \lambda(t - x)) - \phi(x)}{\lambda(t - x)} \geq \frac{\phi(x)(1 - \lambda) + \lambda \phi(t) - \phi(x)}{\lambda(t - x)}
\]
\[
= \frac{\phi(t) - \phi(x)}{t - x}
\]
and so $t \to \frac{\phi(t) - \phi(x)}{t - x}$ is increasing for $t \neq x$. Let 
\[
a_x \equiv \inf \left\{ \frac{\phi(t) - \phi(x)}{t - x} : t > x \right\}.
\]

Then if $t_1 < x$, and $t > x$,
\[
\frac{\phi(t_1) - \phi(x)}{t_1 - x} \leq a_x \leq \frac{\phi(t) - \phi(x)}{t - x}.
\]

Thus for all $t \in I$, 
\[
\phi(t) \geq a_x(t - x) + \phi(x). 
\]

(58.1.3)

The continuity of $\phi$ follows easily from this and the observation that convexity simply says that the graph of $\phi$ lies below the line segment joining two points on its graph. Thus, we have the following picture which clearly implies continuity. □
Lemma 58.1.4 Let $I$ be an open interval on $\mathbb{R}$ and let $\phi$ be a convex function defined on $I$. Then there exists a sequence $\{(a_n, b_n)\}$ such that

$$
\phi(t) = \sup \{a_n t + b_n, n = 1, \cdots\}.
$$

Proof: Let $a_x$ be as defined in the above lemma. Let

$$
\psi(x) \equiv \sup \{a_r (x - r) + \phi(r) : r \in \mathbb{Q} \cap I\}.
$$

Thus if $r_1 \in \mathbb{Q}$,

$$\psi(r_1) \equiv \sup \{a_r (r_1 - r) + \phi(r) : r \in \mathbb{Q} \cap I\} \geq \phi(r_1)
$$

Then $\psi$ is convex on $I$ so $\psi$ is continuous. Therefore, $\psi(t) \geq \phi(t)$ for all $t \in I$. By 

$$
\psi(t) \geq \phi(t) \geq \sup \{a_r (t - r) + \phi(r), r \in \mathbb{Q} \cap I\} \equiv \psi(t).
$$

Thus $\psi(t) = \phi(t)$. Let $\mathbb{Q} \cap I = \{r_n\}$, $a_n = a_{r_n}$ and $b_n = -a_{r_n} r_n + \phi(r_n)$.

Lemma 58.1.5 If $X \leq Y$, then $E(X|S) \leq E(Y|S)$ a.e. Also

$$
X \to E(X|S)
$$

is linear.

Proof: Let $A \in S$,

$$
\int_A E(X|S) dP = \int_A X dP
$$

$$
\leq \int_A Y dP = \int_A E(Y|S) dP.
$$

Hence $E(X|S) \leq E(Y|S)$ a.e. as claimed. It is obvious $X \to E(X|S)$ is linear.

Theorem 58.1.6 (Jensen’s inequality) Let $X(\omega) \in I$ and let $\phi : I \to \mathbb{R}$ be convex. Suppose

$$
E(\|X\|), E(\|\phi(X)\|) < \infty.
$$

Then

$$
\phi(E(X|S)) \leq E(\phi(X)|S).
$$

Proof: Let $\phi(x) = \sup \{a_n x + b_n\}$. Letting $A \in S$,

$$
\frac{1}{P(A)} \int_A E(X|S) dP =\frac{1}{P(A)} \int_A X dP \in I \text{ a.e.}
$$

whenever $P(A) \neq 0$. Hence $E(X|S)(\omega) \in I$ a.e. and so it makes sense to consider $\phi(E(X|S))$. Now

$$
a_n E(X|S) + b_n = E(a_n X + b_n|S) \leq E(\phi(X)|S).
$$

Thus

$$
\sup \{a_n E(X|S) + b_n\}
$$

$$
= \phi(E(X|S)) \leq E(\phi(X)|S) \text{ a.e.} \quad \blacksquare
$$
58.2 Discrete Martingales

Definition 58.2.1 Let $S_k$ be an increasing sequence of $\sigma$-algebras which are subsets of $S$ and $X_k$ be a sequence of real-valued random variables with $E(|X_k|) < \infty$ such that $X_k$ is $S_k$ measurable. Then this sequence is called a martingale if

$$E(X_{k+1}|S_k) = X_k,$$

a submartingale if

$$E(X_{k+1}|S_k) \geq X_k,$$

and a supermartingale if

$$E(X_{k+1}|S_k) \leq X_k.$$

Saying that $X_k$ is $S_k$ measurable is referred to by saying $\{X_k\}$ is adapted to $S_k$.

Note that if $\{X_k\}$ is a martingale, then $\{|X_k|\}$ is a submartingale and that if $\{X_k\}$ is a submartingale and $\phi$ is convex and increasing, then $\{\phi(X_k)\}$ is a submartingale.

An upcrossing occurs when a sequence goes from $a$ up to $b$. Thus it crosses the interval, $[a, b]$ in the up direction, hence upcrossing. More precisely,

Definition 58.2.2 Let $\{x_i\}_{i=1}^I$ be any sequence of real numbers, $I \leq \infty$. Define an increasing sequence of integers $\{m_k\}$ as follows. $m_1$ is the first integer $\geq 1$ such that $x_{m_1} \leq a$, $m_2$ is the first integer larger than $m_1$ such that $x_{m_2} \geq b$, $m_3$ is the first integer larger than $m_2$ such that $x_{m_3} \leq a$, etc. Then each sequence, $\{x_{m_{2k-1}}, \ldots, x_{m_{2k}}\}$, is called an upcrossing of $[a, b]$.

Here is a picture of an upcrossing.

![Upcrossing Diagram]

Proposition 58.2.3 Let $\{X_i\}_{i=1}^n$ be a finite sequence of real random variables defined on $\Omega$ where $(\Omega, S, P)$ is a probability space. Let $U_{[a,b]}(\omega)$ denote the number of upcrossings of $X_i(\omega)$ of the interval $[a, b]$. Then $U_{[a,b]}$ is a random variable.

Proof: Let $X_0(\omega) \equiv a+1$, let $Y_0(\omega) \equiv 0$, and let $Y_k(\omega)$ remain 0 for $k = 0, \ldots, l$ until $X_l(\omega) \leq a$. When this happens (if ever), $Y_{l+1}(\omega) \equiv 1$. Then let $Y_i(\omega)$ remain 1 for $i = l+1, \ldots, r$ until $X_r(\omega) \geq b$ when $Y_{r+1}(\omega) \equiv 0$. Let $Y_k(\omega)$ remain 0 for $k \geq r+1$ until $X_k(\omega) \leq a$ when $Y_k(\omega) \equiv 1$ and continue in this way. Thus the upcrossings of $X_i(\omega)$ are identified as unbroken strings of ones for $Y_k$ with a zero
at each end, with the possible exception of the last string of ones which may be missing the zero at the upper end and may or may not be an upcrossing.

Note also that $Y_0$ is measurable because it is identically equal to 0 and that if $Y_k$ is measurable, then $Y_{k+1}$ is measurable because the only change in going from $k$ to $k+1$ is a change from 0 to 1 or from 1 to 0 on a measurable set determined by $X_k$. In particular,

$$Y_{k+1}^{-1}(1) = ([Y_k = 1] \cap [X_k < b]) \cup ([Y_k = 0] \cap [X_k \leq a])$$

This set is in $\mathcal{S}$ by induction. Of course, $Y_{k+1}^{-1}(0)$ is just the complement of this set. Thus $Y_{k+1}$ is $\mathcal{S}$ measurable since 0, 1 are the only two values possible. Now let

$$Z_k(\omega) = \begin{cases} 1 & \text{if } Y_k(\omega) = 1 \text{ and } Y_{k+1}(\omega) = 0, \\ 0 & \text{otherwise}, \end{cases}$$

if $k < n$ and

$$Z_n(\omega) = \begin{cases} 1 & \text{if } Y_n(\omega) = 1 \text{ and } X_n(\omega) \geq b, \\ 0 & \text{otherwise}. \end{cases}$$

Thus $Z_k(\omega) = 1$ exactly when an upcrossing has been completed and each $Z_i$ is a random variable.

$$U_{[a,b]}(\omega) = \sum_{k=1}^n Z_k(\omega)$$

so $U_{[a,b]}$ is a random variable as claimed.

The following corollary collects some key observations found in the above construction.

**Corollary 58.2.4** $U_{[a,b]}(\omega) \leq$ the number of unbroken strings of ones in the sequence, $\{Y_k(\omega)\}$ there being at most one unbroken string of ones which produces no upcrossing. Also

$$Y_i(\omega) = \psi_i \left( \{X_j(\omega)\}_{j=1}^{i-1} \right), \quad (58.2.4)$$

where $\psi_i$ is some function of the past values of $X_j(\omega)$.

**Lemma 58.2.5** Let $\phi$ be a convex and increasing function and suppose

$$\{(X_n, S_n)\}$$

is a submartingale. Then if $E(\phi(X_n)) < \infty$, it follows

$$\{\phi(X_n), S_n\}$$

is also a submartingale.

**Proof:** It is given that $E(X_{n+1}, S_n) \geq X_n$ and so

$$\phi(X_n) \leq \phi(E(X_{n+1}|S_n)) \leq E(\phi(X_{n+1})|S_n)$$

by Jensen’s inequality.

The following is called the upcrossing lemma.
58.2. DISCRETE MARTINGALES

58.2.1 Upcrossings

Lemma 58.2.6 (upcrossing lemma) Let \( \{ (X_i, S_i) \}_{i=1}^n \) be a submartingale and let \( U_{[a,b]} (\omega) \) be the number of upcrossings of \([a,b]\). Then

\[
E \left( U_{[a,b]} \right) \leq \frac{E( |X_n| ) + |a|}{b - a}.
\]

Proof: Let \( \phi(x) \equiv a + (x - a)^+ \) so that \( \phi \) is an increasing convex function always at least as large as \( a \). By Lemma 58.2.5 it follows that \( \{ (\phi(X_k), S_k) \} \) is also a submartingale.

\[
\phi(X_{k+r}) - \phi(X_k) = \sum_{i=k+1}^{k+r} \phi(X_i) - \phi(X_{i-1})
\]

\[
= \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) Y_i + \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) (1 - Y_i).
\]

Observe that \( Y_i \) is \( S_{i-1} \) measurable from its construction in Proposition 58.2.3, \( Y_i \) depending only on \( X_j \) for \( j < i \).

Now let the unbroken strings of ones for \( \{ Y_i(\omega) \} \) be

\[
\{ k_1, \ldots, k_1 + r_1 \}, \{ k_2, \ldots, k_2 + r_2 \}, \ldots, \{ k_m, \ldots, k_m + r_m \}
\]

(58.2.5)

where \( m = V(\omega) \equiv \) the number of unbroken strings of ones in the sequence \( \{ Y_i(\omega) \} \).

By Corollary 58.2.7 \( V(\omega) \geq U_{[a,b]} (\omega) \).

\[
\phi(X_n(\omega)) - \phi(X_1(\omega))
\]

\[
= \sum_{k=1}^n (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) Y_k(\omega)
\]

\[
+ \sum_{k=1}^n (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) (1 - Y_k(\omega)).
\]

The first sum in the above reduces to summing over the unbroken strings of ones because the terms in which \( Y_i(\omega) = 0 \) contribute nothing. Therefore,

\[
\phi(X_n(\omega)) - \phi(X_1(\omega)) \geq U_{[a,b]} (\omega) (b - a) + 0 + \sum_{k=1}^n (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) (1 - Y_k(\omega))
\]

(58.2.6)

where the zero on the right side results from a string of ones which does not produce an upcrossing. It is here that it is important that \( \phi(x) \geq a \). Such
a string begins with \( \phi(X_k(\omega)) = a \) and results in an expression of the form 
\( \phi(X_{k+m}(\omega)) - \phi(X_k(\omega)) \geq 0 \) since \( \phi(X_{k+m}(\omega)) \geq a \). If \( X_k \) had not been replaced with \( \phi(X_k) \), it would have been possible for \( \phi(X_{k+m}(\omega)) \) to be less than \( a \) and the zero in the above could have been a negative number This would have been inconvenient.

Next take the expected value of both sides in \( \text{58.2.6} \). This results in
\[
E(\phi(X_n) - \phi(X_1)) \geq (b - a) E(U_{[a,b]})
\]
\[
+ E \left( \sum_{k=1}^{n} (\phi(X_k) - \phi(X_{k-1}))(1 - Y_k) \right)
\]
\[
\geq (b - a) E(U_{[a,b]})
\]

The reason for the last inequality where the term at the end was dropped is
\[
E((\phi(X_k) - \phi(X_{k-1}))(1 - Y_k))
\]
\[
= E(E((\phi(X_k) - \phi(X_{k-1}))(1 - Y_k) | \mathcal{F}_{k-1}))
\]
\[
= E((1 - Y_k) E(\phi(X_k) | \mathcal{F}_{k-1}) - (1 - Y_k) E(\phi(X_{k-1}) | \mathcal{F}_{k-1}))
\]
\[
\geq E((1 - Y_k) (\phi(X_{k-1}) - \phi(X_{k-1}))) = 0.
\]

Recall that \( Y_k \) is \( S_{k-1} \) measurable and that \((\phi(X_k), S_k)\) is a submartingale.

The reason for this lemma is to prove the amazing submartingale convergence theorem.

58.2.2 The Submartingale Convergence Theorem

**Theorem 58.2.7 (submartingale convergence theorem)** Let
\[
\{(X_i, S_i)\}_{i=1}^{\infty}
\]
be a submartingale with \( K \equiv \sup E(|X_n|) < \infty \). Then there exists a random variable, \( X \), such that \( E(|X|) \leq K \) and
\[
\lim_{n \to \infty} X_n(\omega) = X(\omega) \quad a.e.
\]

**Proof:** Let \( a, b \in \mathbb{Q} \) and let \( a < b \). Let \( U_{[a,b]}^n(\omega) \) be the number of upcrossings of \( \{X_i(\omega)\}_{i=1}^{n} \). Then let
\[
U_{[a,b]}(\omega) \equiv \lim_{n \to \infty} U_{[a,b]}^n(\omega) = \text{number of upcrossings of } \{X_i\}.
\]

By the upcrossing lemma,
\[
E\left(U_{[a,b]}^n\right) \leq \frac{E(|X_n|) + |a|}{b - a} \leq \frac{K + |a|}{b - a}
\]
and so by the monotone convergence theorem,

\[ E(U_{[a,b]}) \leq \frac{K + |a|}{b-a} < \infty \]

which shows \( U_{[a,b]}(\omega) \) is finite a.e., for all \( \omega \notin S_{[a,b]} \) where \( P(S_{[a,b]}) = 0 \). Define

\[ S \equiv \bigcup \{ S_{[a,b]} : a, b \in \mathbb{Q}, a < b \}. \]

Then \( P(S) = 0 \) and if \( \omega \notin S \), \( \{X_k\}_{k=1}^{\infty} \) has only finitely many upcrossings of every interval having rational endpoints. For such \( \omega \) it cannot be the case that

\[ \lim \sup_{k \to \infty} X_k(\omega) > \lim \inf_{k \to \infty} X_k(\omega) \]

because then you could pick rational \( a, b \) such that \([a, b]\) is between the limsup and the liminf and there would be infinitely many upcrossings of \([a, b]\). Thus, for \( \omega \notin S \),

\[ \lim \sup_{k \to \infty} X_k(\omega) = \lim \inf_{k \to \infty} X_k(\omega) = \lim_{k \to \infty} X_k(\omega) \equiv X_{\infty}(\omega). \]

Letting \( X_{\infty}(\omega) \equiv 0 \) for \( \omega \in S \), Fatou’s lemma implies

\[ \int_{\Omega} |X_{\infty}| \, dP = \int_{\Omega} \lim \inf_{n \to \infty} |X_n| \, dP \leq \lim \inf_{n \to \infty} \int_{\Omega} |X_n| \, dP \leq K \quad \square \]

### 58.2.3 Doob Submartingale Estimate

Another very interesting result about submartingales is the Doob submartingale estimate.

**Theorem 58.2.8** Let \( \{(X_i, S_i)\}_{i=1}^{\infty} \) be a submartingale. Then for \( \lambda > 0 \),

\[ P\left( \max_{1 \leq k \leq n} X_k \geq \lambda \right) \leq \frac{1}{\lambda} \int_{\Omega} X_n^+ \, dP \]

**Proof:** Let

\[ A_1 \equiv [X_1 \geq \lambda], A_2 \equiv [X_2 \geq \lambda] \setminus A_1, \]

\[ \cdots, A_k \equiv [X_k \geq \lambda] \setminus \left( \bigcup_{i=1}^{k-1} A_i \right), \cdots \]
Thus each \( A_k \) is \( S_k \)-measurable, the \( A_k \) are disjoint, and their union equals \([\max_{1 \leq k \leq n} X_k \geq \lambda]\).

Therefore from the definition of a submartingale and Jensen’s inequality,

\[
P\left(\max_{1 \leq k \leq n} X_k \geq \lambda\right) = \sum_{k=1}^{n} P(A_k) \leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_k} X_k \, dP
\]

\[
\leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_k} E(X_n|S_k) \, dP
\]

\[
\leq \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_k} E(X^+_n|S_k) \, dP
\]

\[
= \frac{1}{\lambda} \sum_{k=1}^{n} \int_{A_k} X^+_n \, dP \leq \frac{1}{\lambda} \int_{\Omega} X^+_n \, dP. \quad \blacksquare
\]

58.3 Optional Sampling And Stopping Times

58.3.1 Stopping Times And Their Properties

First it is necessary to define the notion of a stopping time.

**Definition 58.3.1** Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(\{\mathcal{F}_n\}_{n=1}^{\infty}\) be an increasing sequence of \(\sigma\) algebras each contained in \(\mathcal{F}\). A stopping time is a measurable function, \(T\) which maps \(\Omega\) to \(\mathbb{N}\),

\[
T^{-1}(A) \in \mathcal{F} \quad \text{for all } A \in \mathcal{P}(\mathbb{N}) ,
\]

such that for all \(n \in \mathbb{N}\),

\[
[T \leq n] \in \mathcal{F}_n.
\]

Note this is equivalent to saying

\[
[T = n] \in \mathcal{F}_n
\]

because

\[
[T = n] = [T \leq n] \setminus [T \leq n - 1].
\]

For \(T\) a stopping time define \(\mathcal{F}_T\) as follows.

\[
\mathcal{F}_T \equiv \{ A \in \mathcal{F} : A \cap [T \leq n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N} \}
\]

These sets in \(\mathcal{F}_T\) are referred to as “prior” to \(T\).

The following lemma is fundamental to understand.

**Lemma 58.3.2** In the situation of Definition 58.3.1, if \(S \leq T\) for two stopping times, \(S\) and \(T\), then \(\mathcal{F}_S \subseteq \mathcal{F}_T\). Also \(\mathcal{F}_T\) is a \(\sigma\) algebra.
58.3. OPTIONAL SAMPLING AND STOPPING TIMES

**Proof:** Let \( A \in \mathcal{F}_S \). Then this means
\[
A \cap [S \leq n] \in \mathcal{F}_n \quad \text{for all } n.
\]
Then
\[
A \cap [T \leq n] = \bigcup_{i=1}^{n} (A \cap [S \leq i]) \cap [T \leq n] \quad (58.3.7)
\]
Suppose \( \omega \) is in the set on the left. Then if \( T(\omega) < n \), it is clearly in the set on the right. If \( T(\omega) = n \), then \( \omega \in [S \leq i] \) for some \( i \leq n \) and it is also in \([T \leq n]\). Thus the set on the left is contained in the set on the right. Next suppose \( \omega \) is in the set on the right. Then \( \omega \in [T \leq n] \) and it only remains to verify \( \omega \in A \). However, \( \omega \in A \cap [S \leq i] \) for some \( i \) and so \( \omega \in A \) also.

Now from 58.3.7 it follows \( A \cap [T \leq n] \in \mathcal{F}_n \) because
\[
A \cap [S \leq i] \in \mathcal{F}_i \subseteq \mathcal{F}_n
\]
and \([T \leq n] \in \mathcal{F}_n \) because \( T \) is a stopping time. Since \( n \) is arbitrary, this shows \( \mathcal{F}_S \subseteq \mathcal{F}_T \).

It remains to verify \( \mathcal{F}_T \) is a \( \sigma \) algebra. Suppose \( \{A_i\} \) is a sequence of sets in \( \mathcal{F}_T \). Then I need to show that \((\bigcup_{i=1}^{\infty} A_i) \cap [T \leq j] \in \mathcal{F}_j \) for all \( j \).
\[
\bigcup_{i=1}^{\infty} A_i \cap [T \leq j] = \bigcup_{i=1}^{\infty} (A_i \cap [T \leq j])
\]
Now each \((A_i \cap [T \leq j])\) is in \( \mathcal{F}_j \) and so the countable union of these sets is also in \( \mathcal{F}_j \). Next suppose \( A \in \mathcal{F}_T \). I need to verify \( A^C \cap [T \leq j] \in \mathcal{F}_j \) for all \( j \). However, \([T \leq j] \in \mathcal{F}_j \) and \( \Omega \in \mathcal{F}_j \) so \( \Omega \in \mathcal{F}_T \). Thus
\[
\Omega \cap [T \leq j] = (A \cap [T \leq j]) \cup (A^C \cap [T \leq j])
\]
and so
\[
(A^C \cap [T \leq j]) = \Omega \cap [T \leq j] \setminus (A \cap [T \leq j]) \in \mathcal{F}_j.
\]
This proves the lemma.

**Lemma 58.3.3** Let \( T \) be a stopping time and let \( \{X_n\} \) be a sequence of random variables such that \( X_n \) is \( \mathcal{F}_n \) measurable. Then \( X_T(\omega) \equiv X_{T(\omega)}(\omega) \) is also a random variable and it is measurable with respect to \( \mathcal{F}_T \).

**Proof:** I assume the \( X_n \) have values in some topological space and each is measurable because the inverse image of an open set is in \( \mathcal{F}_n \). I need to show \( X_T^{-1}(U) \cap [T \leq n] \in \mathcal{F}_n \) for all \( n \) whenever \( U \) is open.
\[
X_T^{-1}(U) \cap [T < n] = \bigcup_{i=1}^{\infty} X_i^{-1}(U) \cap [T = i] \cap [T < n].
\]
It follows \( X_T^{-1}(U) \in \mathcal{F} \). Furthermore,
\[
X_T^{-1}(U) \cap [T \leq n] = \bigcup_{i=1}^{n} X_i^{-1}(U) \cap [T = i] \cap [T \leq n] = \bigcup_{i=1}^{n} X_i^{-1}(U) \cap [T = i] \cap [T \leq n] \subseteq \mathcal{F}_n
\]
and so \( X_T \) is \( \mathcal{F}_T \) is measurable as claimed. This proves the lemma.
Lemma 58.3.4  Let $S \leq T$ be two stopping times such that $T$ is bounded above and let $\{X_n\}$ be a submartingale (martingale) adapted to the increasing sequence of $\sigma$ algebras, $\{F_n\}$. Then

$$E(X_T|F_S) \geq X_S$$

in the case where $\{X_n\}$ is a submartingale and

$$E(X_T|F_S) = X_S$$

in the case where $\{X_n\}$ is a martingale.

Proof: I will prove the case where $\{X_n\}$ is a submartingale and note the other case will only involve replacing $\geq$ with $\leq$. First recall that from Lemma 58.3.2 $F_S \subseteq F_T$. Also let $m$ be an upper bound for $T$. Then it follows from this that

$$E(|X_T|) = \sum_{i=1}^{m} \int_{T=i} |X_i| \, dP < \infty$$

with a similar formula holding for $E(|X_S|)$. Thus it makes sense to speak of $E(X_T|F_S)$.

I need to show that if $B \in F_S$, so that $B \cap [S \leq n] \in F_n$ for all $n$, then

$$\int_B X_T \, dP \geq \int_B X_S \, dP.$$  \hfill (58.3.8)

It suffices to do this for $B$ of the special form

$$B = A \cap [S = i]$$

because if this is done, then the result follows from summing over all possible values of $S$. Note that if $B = A \cap [S = m]$ then $X_T = X_S = X_m$ and there is nothing to prove in this case so it can be assumed $i \leq m - 1$. Then let $B$ be of this form.

$$\int_{A \cap [S = i]} X_T \, dP = \sum_{j=i}^{m-1} \int_{A \cap [S = i] \cap [T = j]} X_T \, dP$$

$$= \sum_{j=i}^{m-1} \int_{A \cap [S = i] \cap [T = j]} X_T \, dP + \int_{A \cap [S = i] \cap [T \geq m]} X_m \, dP$$

And so

$$\int_{A \cap [S = i]} X_T \, dP = \sum_{j=i}^{m-1} \int_{A \cap [S = i] \cap [T = j]} X_T \, dP + \int_{A \cap [S = i] \cap [T \geq m]} X_m \, dP \quad (58.3.9)$$
The right hand side equals
repeat this process till you get the following inequality
which is exactly the same form as
provided \( m - 1 \geq i \) because \( \{X_n\} \) is a submartingale and
\[
A \cap [S = i] \cap [T \leq m - 1]^{\bar{C}} \in \mathcal{F}_{m-1}
\]
Now combine the top term of the sum with the term on the right to obtain
\[
= \sum_{j=i}^{m-1} \int_{A \cap [S = i] \cap [T = j]} X_T dP + \int_{A \cap [S = i] \cap [T \leq m-1]} X_{m-1} dP
\]
which is exactly the same form as \( \text{Optional Sampling and Stopping Times} \) except \( m \) is replaced with \( m - 1 \). Now repeat this process till you get the following inequality
\[
\int_{A \cap [S = i]} X_T dP \geq \sum_{j=i}^{i+1} \int_{A \cap [S = i] \cap [T = j]} X_T dP + \int_{A \cap [S = i] \cap [T \geq i+2]} X_{i+2} dP
\]
The right hand side equals
\[
\sum_{j=i}^{i+1} \int_{A \cap [S = i] \cap [T = j]} X_T dP + \int_{A \cap [S = i] \cap [T \leq i+1]} X_{i+2} dP
\]
\[
\geq \sum_{j=i}^{i+1} \int_{A \cap [S = i] \cap [T = j]} X_T dP + \int_{A \cap [S = i] \cap [T \leq i+1]} X_{i+1} dP
\]
\[
= \int_{A \cap [S = i] \cap [T = i]} X_T dP + \int_{A \cap [S = i] \cap [T = i+1]} X_T dP + \int_{A \cap [S = i] \cap [T \leq i+1]} X_{i+1} dP
\]
\[
= \int_{A \cap [S = i] \cap [T = i+1]} X_{i+1} dP + \int_{A \cap [S = i] \cap [T \geq i+1]} X_{i+1} dP
\]
\[
= \int_{A \cap [S = i] \cap [T = i]} X_i dP + \int_{A \cap [S = i] \cap [T \geq i+1]} X_{i+1} dP
\]
\[
= \int_{A \cap [S = i] \cap [T = i]} X_i dP + \int_{A \cap [S = i] \cap [T \leq i]} X_{i+1} dP
\]
\[
\int_{A \cap [S = i] \cap [T = i]} X_i \, dP + \int_{A \cap [S = i] \cap [T \leq i]} X_i \, dP = \int_{A \cap [S = i] \cap [T \geq i]} X_i \, dP = \int_{A \cap [S = i]} X_S \, dP
\]

In the case where \( \{X_n\} \) is a martingale, you replace every occurrence of \( \geq \) in the above argument with \( = \). This proves the lemma.

This lemma is called the optional sampling theorem. Another version of this theorem is the case where you have an increasing sequence of stopping times, \( \{T_n\}_{n=1}^\infty \). Thus if \( \{X_n\} \) is a sequence of random variables each \( \mathcal{F}_n \) measurable, the sequence \( \{X_{T_n}\} \) is also a sequence of random variables such that \( X_{T_n} \) is measurable with respect to \( \mathcal{F}_{T_n} \) where \( \mathcal{F}_{T_n} \) is an increasing sequence of \( \sigma \) fields. In the case where \( X_n \) is a submartingale (martingale) it is reasonable to ask whether \( \{X_{T_n}\} \) is also a submartingale (martingale). The optional sampling theorem says this is often the case.

**Theorem 58.3.5** Let \( \{T_n\} \) be an increasing **bounded** sequence of stopping times and let \( \{X_n\} \) be a submartingale (martingale) adapted to the increasing sequence of \( \sigma \) algebras, \( \{\mathcal{F}_n\} \). Then \( \{X_{T_n}\} \) is a submartingale (martingale) adapted to the increasing sequence of \( \sigma \) algebras \( \{\mathcal{F}_{T_n}\} \).

**Proof:** This follows from Lemma 58.3.4.

**Example 58.3.6** Let \( \{X_n\} \) be a sequence of real random variables such that \( X_n \) is \( \mathcal{F}_n \) measurable and let \( A \) be a Borel subset of \( \mathbb{R} \). Let \( T(\omega) \) denote the first time \( X_n(\omega) \) is in \( A \). Then \( T \) is a stopping time. It is called the first hitting time.

To see this is a stopping time,

\[
[T \leq l] = \bigcup_{i=1}^l X_i^{-1}(A) \in \mathcal{F}_i
\]

### 58.4 Optional Stopping Times And Martingales

#### 58.4.1 Stopping Times And Their Properties

The purpose of this section is to consider a special optional sampling theorem for martingales which is superior to the one presented earlier. I have presented a different treatment of the fundamental properties of stopping times also. See Kallenberg [7] for more.

**Definition 58.4.1** Let \( (\Omega, \mathcal{F}, P) \) be a probability space and let \( \{\mathcal{F}_n\}_{n=1}^\infty \) be an increasing sequence of \( \sigma \) algebras each contained in \( \mathcal{F} \). A stopping time is a measurable function, \( \tau \) which maps \( \Omega \) to \( \mathbb{N} \),

\[
\tau^{-1}(A) \in \mathcal{F} \quad \text{for all } A \in \mathcal{P}(\mathbb{N}),
\]
such that for all $n \in \mathbb{N}$,

$$[\tau \leq n] \in \mathcal{F}_n.$$ 

Note this is equivalent to saying

$$[\tau = n] \in \mathcal{F}_n$$

because

$$[\tau = n] = [\tau \leq n] \setminus [\tau \leq n - 1].$$

For $\tau$ a stopping time define $\mathcal{F}_\tau$ as follows.

$$\mathcal{F}_\tau \equiv \{ A \in \mathcal{F} : A \cap [\tau \leq n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N} \}$$

These sets in $\mathcal{F}_\tau$ are referred to as “prior” to $\tau$.

First note that for $\tau$ a stopping time, $\mathcal{F}_\tau$ is a $\sigma$ algebra. This is in the next proposition.

**Proposition 58.4.2** For $\tau$ a stopping time, $\mathcal{F}_\tau$ is a $\sigma$ algebra and if $Y(k)$ is $\mathcal{F}_k$ measurable for all $k$, then

$$\omega \rightarrow Y(\tau(\omega))$$

is $\mathcal{F}_\tau$ measurable.

**Proof:** Let $A_n \in \mathcal{F}_\tau$. I need to show $\cup_n A_n \in \mathcal{F}_\tau$. In other words, I need to show that

$$\cup_n A_n \cap [\tau \leq k] \in \mathcal{F}_k$$

The left side equals

$$\cup_n (A_n \cap [\tau \leq k])$$

which is a countable union of sets of $\mathcal{F}_k$ and so $\mathcal{F}_\tau$ is closed with respect to countable unions. Next suppose $A \in \mathcal{F}_\tau$.

$$(A^C \cap [\tau \leq k]) \cup (A \cap [\tau \leq k]) = \Omega \cap [\tau \leq k]$$

and $\Omega \cap [\tau \leq k] \in \mathcal{F}_k$. Therefore, so is $A^C \cap [\tau \leq k]$. It remains to verify the last claim.

$$[Y(\tau) \leq a] = \cup_k [\tau = k] \cap [Y(k) \leq a]$$

Thus

$$[Y(\tau) \leq a] \cap [\tau \leq l] = \cup_k [\tau = k] \cap [Y(k) \leq a] \cap [\tau \leq l]$$

Consider a term in the union. If $l \geq k$ the term reduces to $[\tau = k] \cap [Y(k) \leq a] \in \mathcal{F}_k$ while if $l < k$, this term reduces to $\emptyset$, also a set of $\mathcal{F}_k$. Therefore, $Y(\tau)$ must be $\mathcal{F}_\tau$ measurable. This proves the proposition.

The following lemma contains the fundamental properties of stopping times.

**Lemma 58.4.3** In the situation of Definition 58.4.1, let $\sigma, \tau$ be two stopping times. Then
CHAPTER 58. CONDITIONAL EXPECTATION AND MARTINGALES

1. \( \tau \) is \( \mathcal{F}_\tau \) measurable
2. \( \mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\sigma \land \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau \)
3. \( \mathcal{F}_\tau = \mathcal{F}_k \) on \([\tau = k]\) for all \( k \). That is if \( A \in \mathcal{F}_k \), then \( A \cap [\tau = k] \in \mathcal{F}_\tau \) and if \( A \in \mathcal{F}_\tau \), then \( A \cap [\tau = k] \in \mathcal{F}_k \). Also if \( A \in \mathcal{F}_\tau \),

\[
\int_{A \cap [\tau = k]} E(Y|\mathcal{F}_\tau) \, dP = \int_{A \cap [\tau = k]} E(Y|\mathcal{F}_k) \, dP
\]

and

\[
E(Y|\mathcal{F}_\tau) = E(Y|\mathcal{F}_k) \quad a.e.
\]
on \([\tau = k]\).

Proof: Consider the first claim. I need to show that \([\tau \leq a] \cap [\tau \leq k] \in \mathcal{F}_k \) for every \( k \). However, this is easy if \( a \geq k \) because the left side is then \([\tau \leq k]\) which is given to be in \( \mathcal{F}_k \) since \( \tau \) is a stopping time. If \( a < k \), it is also easy because then the left side is \([\tau \leq a] \in \mathcal{F}_a \) where \([a]\) is the greatest integer less than or equal to \( a \).

Next consider the second claim. Let \( A \in \mathcal{F}_\sigma \). I want to show first that

\[
A \cap [\sigma \leq \tau] \in \mathcal{F}_\tau
\]

(58.4.10)

In other words, I want to show

\[
A \cap [\sigma \leq \tau] \cap [\tau \leq k] \in \mathcal{F}_k
\]

for all \( k \). This will be done if I can show

\[
A \cap [\sigma \leq j] \cap [\tau \leq k] \in \mathcal{F}_k
\]

for each \( j \leq k \) because

\[
\bigcup_{j \leq k} A \cap [\sigma \leq j] \cap [\tau \leq k] = A \cap [\sigma \leq \tau] \cap [\tau \leq k]
\]

However, since \( \sigma \in \mathcal{F}_\sigma \), it follows \( A \cap [\sigma \leq j] \in \mathcal{F}_j \subseteq \mathcal{F}_k \) for each \( j \leq k \) and \([\tau \leq k] \in \mathcal{F}_k \) and so this has shown what I wanted to show, \( A \cap [\sigma \leq \tau] \in \mathcal{F}_\tau \).

Now replace the stopping time, \( \tau \) with the stopping time \( \tau \land \sigma \) in what was just shown. Note

\[
[\tau \land \sigma \leq n] = [\tau \leq n] \cup [\sigma \leq n] \in \mathcal{F}_n
\]

so \( \tau \land \sigma \) really is a stopping time. This yields

\[
A \cap [\sigma \leq \tau \land \sigma] \in \mathcal{F}_{\tau \land \sigma}
\]

However the left side equals \( A \cap [\sigma \leq \tau] \). Thus

\[
A \cap [\sigma \leq \tau] \in \mathcal{F}_{\tau \land \sigma}
\]
This has shown the first part of 2.), \( \mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\tau \wedge \sigma} \). Now (8.4.10) implies if \( A \in \mathcal{F}_{\sigma \wedge \tau} \),

\[
A = A \cap [\sigma \wedge \tau \leq \tau] \in \mathcal{F}_\tau
\]

and so \( \mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\tau \). Similarly, \( \mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma \) which shows

\[
\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\tau \cap \mathcal{F}_\sigma.
\]

Next let \( A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma \). Then is it in \( \mathcal{F}_{\sigma \wedge \tau} \)? Is \( A \cap [\sigma \wedge \tau \leq k] \in \mathcal{F}_k \)? Of course this is so because

\[
A \cap [\sigma \wedge \tau \leq k] = A \cap ([\sigma \leq k] \cup [\tau \leq k])
\]

\[
= (A \cap [\sigma \leq k]) \cup (A \cap [\tau \leq k]) \in \mathcal{F}_k
\]

since both \( \sigma, \tau \) are stopping times. This proves part 2.).

Now consider part 3.). Note that \( [\tau = k] \) is in both \( \mathcal{F}_k \) and \( \mathcal{F}_\tau \). Let \( A \in \mathcal{F}_k \). I need to show

\[
\mathcal{F}_\tau \cap [\tau = k] = \mathcal{F}_k \cap [\tau = k]
\]

where \( \mathcal{G} \cap [\tau = k] \) means all sets of the form \( A \cap [\tau = k] \) where \( A \in \mathcal{G} \). Let \( A \in \mathcal{F}_\tau \). Then

\[
A \cap [\tau = k] = (A \cap [\tau \leq k]) \setminus (A \cap [\tau \leq k - 1]) \in \mathcal{F}_k
\]

Therefore, there exists \( B \in \mathcal{F}_k \) such that \( B = A \cap [\tau = k] \) and so

\[
B \cap [\tau = k] = A \cap [\tau = k]
\]

which shows \( \mathcal{F}_\tau \cap [\tau = k] \subseteq \mathcal{F}_k \cap [\tau = k] \). Now let \( A \in \mathcal{F}_k \) so that

\[
A \cap [\tau = k] \in \mathcal{F}_k \cap [\tau = k]
\]

Then

\[
A \cap [\tau = k] \cap [\tau \leq j] \in \mathcal{F}_j
\]

because in case \( j < k \), the set on the left is \( \emptyset \) and if \( j \geq k \) it reduces to \( A \cap [\tau = k] \) and both \( A \) and \( [\tau = k] \) are in \( \mathcal{F}_k \subseteq \mathcal{F}_j \). Therefore, the two \( \sigma \) algebras of subsets of \( [\tau = k] \),

\[
\mathcal{F}_\tau \cap [\tau = k] , \mathcal{F}_k \cap [\tau = k]
\]

are equal. Thus for \( A \) in either \( \mathcal{F}_\tau \) or \( \mathcal{F}_k \), \( A \cap [\tau = k] \) is a set of both \( \mathcal{F}_\tau \) and \( \mathcal{F}_k \) because if \( A \in \mathcal{F}_k \), then from the above, there exists \( B \in \mathcal{F}_\tau \) such that

\[
A \cap [\tau = k] = B \cap [\tau = k] \in \mathcal{F}_\tau
\]

with similar reasoning holding if \( A \in \mathcal{F}_\tau \). In other words, if \( g \) is \( \mathcal{F}_\tau \) or \( \mathcal{F}_k \) measurable, then the restriction of \( g \) to \( [\tau = k] \) is measurable with respect to \( \mathcal{F}_\tau \cap [\tau = k] \) and \( \mathcal{F}_k \cap [\tau = k] \). Let \( Y \) be an arbitrary random variable in \( L^1(\Omega, \mathcal{F}) \). It follows

\[
\int_{A \cap [\tau = k]} E(Y | \mathcal{F}_\tau) \, dP = \int_{A \cap [\tau = k]} Y \, dP = \int_{A \cap [\tau = k]} E(Y | \mathcal{F}_k) \, dP
\]
Since this holds for an arbitrary set in \( F \cap [\tau = k] = F_k \cap [\tau = k] \), it follows
\[
E(Y|F_\tau) = E(Y|F_k) \text{ a.e. on } [\tau = k]
\]
This proves the third claim and the Lemma.

With this lemma, here is a major theorem, the optional sampling theorem of Doob. This one is special for martingales.

**Theorem 58.4.4** Let \( \{M(k)\} \) be a real valued martingale with respect to the increasing sequence of \( \sigma \) algebras, \( \{F_k\} \) and let \( \sigma, \tau \) be two stopping times such that \( \tau \) is bounded. Then \( M(\tau) \) defined as
\[
\omega \mapsto M(\tau(\omega))
\]
is integrable and
\[
M(\sigma \land \tau) = E(M(\tau)|F_\sigma).
\]

**Proof:** By Proposition \( \text{Mart.} \) \( M(\tau) \) is \( F_\tau \) measurable. Next note that since \( \tau \) is bounded by some \( l \),
\[
\int_{\Omega} \| M(\tau(\omega)) \| \ dP \leq \sum_{i=1}^{l} \int_{[\tau=i]} \| M(i) \| \ dP < \infty.
\]
This proves the first assertion and makes possible the consideration of conditional expectation.

Let \( l \geq \tau \) as described above. Then for \( k \leq l \), by Lemma \( \text{Mart.} \),
\[
F_k \cap [\tau = k] = F_\tau \cap [\tau = k] \equiv G
\]
implying that if \( g \) is either \( F_k \) measurable or \( F_\tau \) measurable, then its restriction to \( [\tau = k] \) is \( G \) measurable and so if \( A \in F_k \cap [\tau = k] = F_\tau \cap [\tau = k] \),
\[
\int_A E(M(l)|F_\tau) \ dP = \int_A M(l) \ dP = \int_A E(M(l)|F_k) \ dP = \int_A M(k) \ dP = \int_A M(\tau) \ dP \text{ (on } A, \tau = k)\]
Therefore, since \( A \) was arbitrary,
\[
E(M(l)|F_\tau) = M(\tau) \text{ a.e.}
\]
on \( [\tau = k] \) for every \( k \leq l \). It follows
\[
E(M(l)|F_\tau) = M(\tau) \text{ a.e.} \quad (58.4.11)
\]
since it is true on each \([\tau = k]\) for all \(k \leq l\).
Now consider \(E(M(\tau) | F_{\sigma})\) on the set \([\sigma = i] \cap [\tau = j]\). By Lemma 58.4.3, on this set,
\[
E(M(\tau) | F_{\sigma}) = E(M(\tau) | F_i) = E(E(M(l) | F_{\tau}) | F_i) = E(E(M(l) | F_j) | F_i)
\]
If \(j \leq i\), this reduces to
\[
E(M(l) | F_j) = M(j) = M(\sigma \wedge \tau).
\]
If \(j > i\), this reduces to
\[
E(M(l) | F_i) = M(i) = M(\sigma \wedge \tau)
\]
and since this exhausts all possibilities for values of \(\sigma\) and \(\tau\), it follows
\[
E(M(\tau) | F_{\sigma}) = M(\sigma \wedge \tau) \text{ a.e.}
\]
This proves the theorem.

This is a really amazing theorem. Note it says \(M(\sigma \wedge \tau) = E(M(\tau) | F_{\sigma})\). This would not be so surprising if it had said
\[
M(\sigma \wedge \tau) = E(M(\tau) | F_{\sigma \wedge \tau}).
\]

What about submartingales? Recall \(\{X(k)\}_{k=1}^{\infty}\) is a submartingale if
\[
E(X(k+1) | F_k) \geq X(k)
\]
where the \(F_k\) are an increasing sequence of \(\sigma\) algebras in the usual way. The following is a very interesting result.

**Lemma 58.4.5** Let \(\{X(k)\}_{k=0}^{\infty}\) be a submartingale adapted to the increasing sequence of \(\sigma\) algebras, \(\{F_k\}\). Then there exists a unique increasing process \(\{A(k)\}_{k=0}^{\infty}\) such that \(A(0) = 0\) and \(A(k+1)\) is \(F_k\) measurable for all \(k\) and a martingale, \(\{M(k)\}_{k=0}^{\infty}\) such that
\[
X(k) = A(k) + M(k).
\]
Furthermore, for \(\tau\) a stopping time, \(A(\tau)\) is \(F_\tau\) measurable.

**Proof:** Define \(\sum_{k=0}^{n-1} \neq 0\). First consider the uniqueness assertion. Suppose \(A\) is a process which does what is supposed to do.
\[
\sum_{k=0}^{n-1} E(X(k+1) - X(k) | F_k) = \sum_{k=0}^{n-1} E(A(k+1) - A(k) | F_k)
\]
\[
+ \sum_{k=0}^{n-1} E(M(k+1) - M(k) | F_k)
\]
Then since \( \{ M(k) \} \) is a martingale,
\[
\sum_{k=0}^{n-1} E(X(k+1) - X(k) | \mathcal{F}_k) = \sum_{k=0}^{n-1} A(k+1) - A(k) = A(n)
\]
This shows uniqueness and gives a formula for \( A(n) \) assuming it exists. It is only a matter of verifying this does work. Define
\[
A(n) \equiv \sum_{k=0}^{n-1} E(X(k+1) - X(k) | \mathcal{F}_k), \quad A(0) = 0.
\]
Then \( A \) is increasing because from the definition,
\[
A(n+1) - A(n) = E(X(n+1) - X(n) | \mathcal{F}_n) \geq 0.
\]
Also from the definition above, \( A(n) \) is \( \mathcal{F}_{n-1} \) measurable, consider \( \{ X(k) - A(k) \} \).

Why is this a martingale?
\[
E(X(k+1) - A(k+1) | \mathcal{F}_k) = E(X(k+1) | \mathcal{F}_k) - A(k+1)
\]
\[
= E(X(k+1) | \mathcal{F}_k) - \sum_{j=0}^{k} E(X(j+1) - X(j) | \mathcal{F}_j)
\]
\[
= E(X(k+1) | \mathcal{F}_k) - E(X(k+1) - X(k) | \mathcal{F}_k)
\]
\[
- \sum_{j=0}^{k-1} E(X(j+1) - X(j) | \mathcal{F}_j)
\]
\[
= X(k) - \sum_{j=0}^{k-1} E(X(j+1) - X(j) | \mathcal{F}_j) = X(k) - A(k)
\]
Let \( M(k) = X(k) - A(k) \). \( A(\tau) \) is \( \mathcal{F}_\tau \) measurable by Proposition 60.6.3.

Note the nonnegative integers could be replaced with any finite set or ordered countable set of numbers with no change in the conclusions of this lemma or the above optional sampling theorem.

Next consider the case of a submartingale.

**Theorem 58.4.6** Let \( \{ X(k) \} \) be a submartingale with respect to the increasing sequence of \( \sigma \) algebras, \( \{ F_k \} \) and let \( \sigma, \tau \) be two stopping times such that \( \tau \) is bounded. Then \( X(\tau) \) defined as
\[
\omega \rightarrow X(\tau(\omega))
\]
is integrable and
\[
X(\sigma \wedge \tau) \leq E(X(\tau) | \mathcal{F}_\sigma).
\]
**Proof:** The claim about $X(\tau)$ being integrable is the same as in Theorem 58.4.5. If $\tau \leq l$,

$$E (|X(\tau(\omega))|) = \sum_{i=1}^{l} \int_{[\tau=i]} |X(i)| \, dP < \infty$$

By Lemma 58.4.4 there is a martingale, $\{M(k)\}$ and an increasing process $\{A(k)\}$ such that $A(k+1)$ is $\mathcal{F}_k$ measurable such that

$$X(k) = M(k) + A(k).$$

Then using Theorem 58.4.3 on the martingale and the fact $A$ is increasing

$$E(X(\tau) \mid \mathcal{F}_\sigma) = E(M(\tau) + A(\tau) \mid \mathcal{F}_\sigma) = M(\tau \wedge \sigma) + E(A(\tau \wedge \sigma) \mid \mathcal{F}_\sigma) \geq M(\tau \wedge \sigma) + E(A(\tau \wedge \sigma)) \geq M(\tau \wedge \sigma) + A(\tau \wedge \sigma) = X(\tau \wedge \sigma).$$

because in the above, it follows from Lemma 58.4.4, $A(\tau \wedge \sigma)$ is $\mathcal{F}_{\tau \wedge \sigma}$ measurable and from Lemma 58.4.5, $\mathcal{F}_{\tau \wedge \sigma} = \mathcal{F}_\tau \cap \mathcal{F}_\sigma \subseteq \mathcal{F}_\sigma$

and so

$$E(A(\tau \wedge \sigma) \mid \mathcal{F}_\sigma) = A(\tau \wedge \sigma). \blacksquare$$

### 58.5 Submartingale Convergence Theorem

#### 58.5.1 Upcrossings

Let $\{X(k)\}$ be an adapted stochastic process, $k = 0, 1, 2, \ldots, M$ adapted to the increasing $\sigma$ algebras $\mathcal{F}_k$. Also let $[a, b]$ be an interval. An upcrossing occurs when $X(k) < a$ and you have $X(k+1) > b$ while $X(r) < b$ for all $r \in [k, k+1 - 1]$. In order to understand upcrossings, consider the following:

$$\tau_0 = \min(\inf\{k : X(k) \leq a\}, M),$$
$$\tau_1 = \min(\inf\{k : (X(k \lor \tau_0) - X(\tau_0))^+ \geq b - a\}, M),$$
$$\tau_2 = \min(\inf\{k : (X(\tau_1) - X(k \lor \tau_1))^+ \geq b - a\}, M),$$
$$\tau_3 = \min(\inf\{k : (X(k \lor \tau_2) - X(\tau_2))^+ \geq b - a\}, M),$$
$$\tau_4 = \min(\inf\{k : (X(\tau_3) - X(k \lor \tau_3))^+ \geq b - a\}, M),$$

\[ \vdots \]

As usual, $\inf(\emptyset) \equiv \infty$. Are the above stopping times? If $\alpha \geq 0$, and $\tau$ is a stopping time, is $k \rightarrow (X(\tau) - X(k \lor \tau))^+_+ \text{ adapted?}$

$$[(X(\tau) - X(k \lor \tau))^+ > \alpha] = [(X(\tau) - X(k))^+ > \alpha] \cap [\tau \leq k]$$
2014  CHAPTER 58.  CONDITIONAL EXPECTATION AND MARTINGALES

Now
\[(X (\tau) - X (k))_+ > a] \cap [\tau \leq k] = \bigcup_{i=0}^k [(X (i) - X (k))_+ > a] \cap [\tau \leq k] \in \mathcal{F}_k\]

If \( \alpha < 0 \), then \([(X (\tau_1) - X (k \lor \tau_1))_+ > a] = \Omega \) and so \( k \to (X (\tau) - X (k \lor \tau))_+ \) is adapted. Similarly \( k \to (X (k \lor \tau) - X (\tau))_+ \) is adapted. Therefore, all those \( \tau_k \) are stopping times.

Now consider the following random variable for odd \( M, 2n + 1 = M \)

\[U_M^{[a,b]} \equiv \lim_{\varepsilon \to 0} \sum_{k=0}^n \frac{X (\tau_{2k+1}) - X (\tau_{2k})}{\varepsilon + X (\tau_{2k+1}) - X (\tau_{2k})} \leq \frac{1}{b-a} \sum_{k=0}^n X (\tau_{2k+1}) - X (\tau_{2k})\]

Now suppose \( \{X (k)\} \) is a nonnegative submartingale. Then since \( E (X (2\tau) | \mathcal{F}_{2\tau-1}) \geq X (\tau_{2\tau-1}) \)

\[E \left( \sum_{k=1}^n X (\tau_{2k}) - X (\tau_{2k-1}) \right) \geq 0\]

Hence

\[E \left( U_M^{[a,b]} \right) \leq \frac{1}{b-a} \sum_{k=0}^n E (X (\tau_{2k+1}) - X (\tau_{2k})) \]

\[\leq \frac{1}{b-a} \sum_{k=0}^n E (X (\tau_{2k+1}) - X (\tau_{2k})) + \frac{1}{b-a} \sum_{k=1}^n E (X (\tau_{2k}) - X (\tau_{2k-1}))\]

\[= \frac{1}{b-a} \sum_{k=0}^n E (X (\tau_k) - X (\tau_{k-1})) \leq \frac{1}{b-a} E (X (\tau_k))\]

Now by the optional sampling theorem \( X (0), X (\tau_k), X (M) \) is a submartingale. Therefore, the above is no larger than

\[\frac{1}{b-a} E (|X (M)|)\]

Now note that \( U_M^{[a,b]} \) is at least as large as the number of upcrossings of \( \{X (k)\} \) for \( k \leq M \). This is because every time an upcrossing occurs, it will follow that \( X (\tau_{2k+1}) - X (\tau_{2k}) > 0 \) and so a one will occur in the above sum which defines \( U_M^{[a,b]} \). However, this might be larger than the number of upcrossings. The above discussion has proved the following upcrossing lemma.

**Lemma 58.5.1** Let \( \{X (k)\} \) be a nonnegative submartingale. Let

\[U_M^{[a,b]} \equiv \lim_{\varepsilon \to 0} \sum_{k=0}^n \frac{X (\tau_{2k+1}) - X (\tau_{2k})}{\varepsilon + X (\tau_{2k+1}) - X (\tau_{2k})}, 2n + 1 = M\]

Then

\[E \left( U_M^{[a,b]} \right) \leq \frac{1}{b-a} E (X (M))\]
Suppose that there exists a constant $C \geq E(X(M))$ for all $M$. That is, $\{X(k)\}$ is bounded in $L^1(\Omega)$. Then letting

$$U^{[a,b]} = \lim_{M \to \infty} U^{[a,b]}_M,$$

it follows that

$$E\left( U^{[a,b]} \right) \leq C \frac{1}{b-a}$$

The second half follows from the first part and the monotone convergence theorem.

Now with this estimate, it is easy to prove the submartingale convergence theorem.

**Theorem 58.5.2** Let $\{X(k)\}$ be a submartingale which is bounded in $L^1(\Omega)$,

$$\|X(k)\|_{L^1(\Omega)} \leq C$$

Then there is a set of measure zero $N$ such that for $\omega \notin N$, $\lim_{k \to \infty} X(k)(\omega)$ exists. If $X(\omega) = \lim_{k \to \infty} X(k)(\omega)$, then $X \in L^1(\Omega)$.

**Proof:** Let $a < b$ and consider the submartingale $(X(k) - a)_+$. Let $U^{[0,b-a]}$ be the random variable of the above lemma which is associated with this submartingale. Thus

$$E\left( U^{[0,b-a]} \right) \leq \frac{C}{b-a}$$

It follows that $U^{[0,b-a]}$ is finite for a.e. $\omega$. As noted above, $U^{[0,b-a]}$ is an upper bound to the number of upcrossings of $(X(k) - a)_+$ and each of these corresponds to an upcrossing of $[a,b]$ by $X(k)$. Thus for all $\omega \notin N_{a,b}$ where $P(N_{a,b}) = 0$, it follows that

$$U^{[0,b-a]} < \infty.$$ 

If $\lim_{k \to \infty} X(k)(\omega)$ fails to exist, then there exists $a < b$ both rational such that

$$\lim \sup_{k \to \infty} X(k) > b > a > \lim \inf_{k \to \infty} X(k)$$

Thus $\omega \in N_{a,b}$ because there are infinitely many upcrossings of $[a,b]$. Let $N = \bigcup \{N_{a,b} : a, b \in \mathbb{Q}\}$. Then for $\omega \notin N$, the limit just discussed must exist. Letting $X(\omega) = \lim_{k \to \infty} X(k)(\omega)$ for $\omega \notin N$ and letting $X(\omega) = 0$ on $N$, it follows from Fatou’s lemma that $X$ is in $L^1(\Omega)$.

**58.5.2 Maximal Inequalities**

Next I will show that stopping times and the optional sampling theorem, Lemma 58.3.4, can be used to establish maximal inequalities for submartingales very easily.
CHAPTER 58. CONDITIONAL EXPECTATION AND MARTINGALES

Lemma 58.5.3 Let \( \{X (k)\} \) be real valued and adapted to the increasing sequence of \( \sigma \) algebras \( \{F_k\} \). Let

\[
T (\omega) \equiv \inf \{ k : X (k) \geq \lambda \}
\]

Then \( T \) is a stopping time. Similarly,

\[
T (\omega) \equiv \inf \{ k : X (k) \leq \lambda \}
\]

is a stopping time.

Proof: Is \( [T \leq p] \in \mathcal{F}_p \) for all \( p \)?

\[
[T = p] = \bigcap_{i=1}^{p-1} \{ X (i) < \lambda \} \cap \{ X (p) \geq \lambda \}
\]

Therefore,

\[
[T \leq p] = \bigcup_{i=1}^{p} [T = i] \in \mathcal{F}_p
\]

Theorem 58.5.4 Let \( \{X_k\} \) be a real valued submartingale with respect to the \( \sigma \) algebras \( \{F_k\} \). Then for \( \lambda > 0 \)

\[
\lambda P \left( \max_{1 \leq k \leq n} X_k \geq \lambda \right) \leq E \left( X_n^+ \right), \quad (58.5.12)
\]

\[
\lambda P \left( \min_{1 \leq k \leq n} X_k \leq -\lambda \right) \leq E (|X_n| + |X_1|), \quad (58.5.13)
\]

\[
\lambda P \left( \max_{1 \leq k \leq n} |X_k| \geq \lambda \right) \leq 2E (|X_n| + |X_1|). \quad (58.5.14)
\]

Proof: Let \( T (\omega) \) be the first time \( X_k (\omega) \) is \( \geq \lambda \) or if this does not happen for \( k \leq n \), then \( T (\omega) \equiv n \). Thus

\[
T (\omega) \equiv \min \{ \min \{ k : X_k (\omega) \geq \lambda \} \}, n
\]

Note

\[
[T > k] = \bigcap_{i=1}^{k} \{ X_i < \lambda \} \in \mathcal{F}_k
\]

and so the complement, \( [T \leq k] \) is also in \( \mathcal{F}_k \) which shows \( T \) is indeed a stopping time.

Then \( 1, T (\omega), n \) are stopping times, \( 1 \leq T (\omega) \leq n \). Therefore, from the optional sampling theorem, Lemma 58.3.4, \( X_1, X_T, X_n \) is a submartingale. It follows

\[
E (X_n) \geq E (X_T) = \int_{\max_k X_k \geq \lambda} X_T dP + \int_{\max_k X_k < \lambda} X_T dP
\]

\[
= \int_{\max_k X_k \geq \lambda} X_T dP + \int_{\max_k X_k < \lambda} X_n dP
\]
and so, subtracting the last term on the right from both sides,

\[ E( X_n^+) \geq \int_{\max_k X_k \geq \lambda} X_n dP = \int_{\max_k X_k \geq \lambda} X_T dP \geq \lambda P \left( \max_k X_k \geq \lambda \right) \]

because \( X_T(\omega) \geq \lambda \) on \( \max_k X_k \geq \lambda \) from the definition of \( T \). This establishes 58.5.12.

Next let \( T(\omega) \) be the first time \( X_k(\omega) \) is \( \leq -\lambda \) or if this does not happen for \( k \leq n \), then \( T(\omega) \equiv n \). Then this is a stopping time by similar reasoning and \( 1 \leq T(\omega) \leq n \) are stopping times and so by the optional stopping theorem, \( X_1, X_T, X_n \) is a submartingale. Therefore, on \( \min_k X_k \leq -\lambda \), \( X_T(\omega) \leq -\lambda \) and

\[ E(X_1) \leq E( E(X_T|\mathcal{F}_1)) = E(X_T) \]

which implies

\[ E(X_1) \leq E(X_T) = \int_{\min_k X_k \leq -\lambda} X_T dP + \int_{\min_k X_k > -\lambda} X_T dP \]

and so

\[ E(X_1) - \int_{\min_k X_k > -\lambda} X_n dP \leq \int_{\min_k X_k \leq -\lambda} X_T dP \leq -\lambda P \left( \min_k X_k \leq -\lambda \right) \]

which implies

\[ \lambda P \left( \min_k X_k \leq -\lambda \right) \leq \int_{\min_k X_k > -\lambda} X_n dP - E(X_1) \leq \int_{\Omega} (|X_n| + |X_1|) dP \]

and this proves 58.5.13.

The last estimate follows from these. Here is why.

\[ \max_{1 \leq k \leq n} |X_k| \geq \lambda \leq \max_{1 \leq k \leq n} X_k \geq \lambda \cup \min_{1 \leq k \leq n} X_k \leq -\lambda \]

\[ \max_{1 \leq k \leq n} X_k \geq \lambda \leq \max_{1 \leq k \leq n} X_k \geq \lambda \cup \min_{1 \leq k \leq n} X_k \leq -\lambda \]
and so
\[ \lambda P \left( \max_{1 \leq k \leq n} |X_k| \geq \lambda \right) \leq \lambda P \left( \max_{1 \leq k \leq n} X_k \geq \lambda \right) \cup \left( \min_{1 \leq k \leq n} X_k \leq -\lambda \right) \]
\[ \leq \lambda P \left( \max_{1 \leq k \leq n} X_k \geq \lambda \right) + \lambda P \left( \min_{1 \leq k \leq n} X_k \leq -\lambda \right) \]
\[ \leq 2 E (|X_1| + |X_n|) \]
and this proves the last estimate.

58.5.3 The Upcrossing Estimate

A very interesting example of stopping times is next. It has to do with upcrossings. First here is a lemma.

Lemma 58.5.5 Let \( \{F_k\} \) be an increasing sequence of \( \sigma \) algebras and let \( \{X(k)\} \) be adapted to this sequence. Suppose that \( X(k) \) has all values in \([a, b]\) and suppose \( \sigma \) is a stopping time with the property that \( X(\sigma) = a \). Let \( \tau(\omega) \) be the first \( k > \sigma \) such that \( X(k) = b \). If no such \( k \) exists, then \( \tau \equiv \infty \). Then \( \tau \) is a stopping time. Also, you can switch \( a, b \) in the above and obtain the same conclusion that \( \tau \) is a stopping time.

Proof: Let \( I \) be an interval and consider \( X(k \vee \sigma) \). Is \( k \rightarrow X(k \vee \sigma) \) adapted? Let \( I \) be an interval. Is
\[ A \equiv X(k \vee \sigma)^{-1} (I) \in F_k? \]
We know that this set is in \( F_{k \vee \sigma} \).
\[ A = A \cap [\sigma \leq k] \cup (X(k \vee \sigma)^{-1} (I) \cap [\sigma > k]) \]
(♠)
Consider the second set in ♠. There are two cases, \( a \in I \) and \( a \notin I \). First suppose \( a \notin I \). Then if \( \omega \in [\sigma > k] \), it follows that \( X(k \vee \sigma) = X(\sigma) = a \). Therefore, in this case, the set on the right in ♠ is empty and the empty set is in \( F_k \). Next suppose \( a \in I \). Then for \( \omega \in [\sigma > k] \),
\[ X(k \vee \sigma)(\omega) = X(\sigma)(\omega) = a \in I \]
and so each \( \omega \in [\sigma > k] \) is in the set \( X(k \vee \sigma)^{-1} (I) \) and so, in this case, the set on the right equals
\[ [\sigma > k] \in F_k \]
Now consider the first set in ♠,
\[ A \cap [\sigma \leq k] = A \cap [\sigma \vee k \leq k] \in F_k \]
by the definition of what it means for the set \( A \) to be in \( F_{k \vee \sigma} \). The argument proceeds in the same way when you switch \( a, b \).
Let $\{X_k\}$ be a sequence of random variables adapted to the increasing sequence of $\sigma$ algebras, $\{\mathcal{F}_k\}$. Let $[a,b]$ be an interval. An upcrossing is a sequence $X_n(\omega), \cdots, X_{n+p}(\omega)$ such that $X_n(\omega) \leq a, X_{n+i}(\omega) < b$ for $i < p$, and $X_{n+p}(\omega) \geq b$.

**Definition 58.5.6** Let $\{F_n\}$ be an increasing sequence of $\sigma$ algebras contained in $\mathcal{F}$ where $(\Omega, \mathcal{F}, P)$ is a probability space and let $\{X_n\}$ be a sequence of real valued random variables such that $X_n$ is $\mathcal{F}_n$ measurable. Also let $a < b$. Define

$$T_0 \equiv \inf \{n : X(n) \leq a\}$$
$$T_1 \equiv \inf \{n > T_0 : X(n) \geq b\}$$
$$T_2 \equiv \inf \{n > T_1 : X(n) \leq a\}$$
$$\vdots$$
$$T_{2k-1} \equiv \inf \{n > T_{2k-2} : X(n) \geq b\}$$
$$T_{2k} \equiv \inf \{n > T_{2k-1} : X(n) \leq a\}$$

If $X_n(\omega)$ is never in the desired interval for any $n > T_j(\omega)$, then define $T_{j+1}(\omega) \equiv \infty$. Then this is an increasing sequence of stopping times.

It happens that the above gives an increasing sequence of stopping times.

**Lemma 58.5.8** The above example gives an increasing sequence of stopping times.

**Proof:** You could consider the modified random variables

$$Y(k) \equiv (X(k) \lor a) \land b$$

Then these new random variables stay in $[a,b]$ and if you replace $X(n)$ in the above with $Y(n)$, you get the same sequence of stopping times. Now apply Lemma 58.5.6.

Now there is an interesting application of these stopping times to the concept of upcrossings. Let $\{X_n\}$ be a submartingale such that $X_n$ is $\mathcal{F}_n$ measurable and let $a < b$. Assume $X_0(\omega) \leq a$. The function, $x \rightarrow (x - a)^+$ is increasing and convex so $\{(X_n - a)^+\}$ is also a submartingale. Furthermore, $\{X_n\}$ goes from $\leq a$ to $\geq b$ if and only if $\{(X_n - a)^+\}$ goes from $0$ to $\geq b - a$. That is, a subsequence of the form $Y_n(\omega), Y_{n+1}(\omega), \cdots, Y_{n+r}(\omega)$ for $Y$ equal to either $X$ or $(X - a)^+$ starts out below $a$ (0) and ends up above $b$ ($b - a$). Such a sequence is called an upcrossing of $[a,b]$. The idea is to estimate the expected number of upcrossings for $n \leq N$. For the stopping times defined in Example 58.5.7, let $T_k' \equiv \min (T_k, N)$. Thus $T_k'$, a continuous function of the stopping time, is also a stopping time which is bounded. Moreover, $T_k' \leq T_{k+1}'$. Now pick $n$ such that $2n > N$. Then for each $\omega \in \Omega$

$$(X_N(\omega) - a)^+ - (X_0(\omega) - a)^+$$
must equal the sum of all successive terms of the form
\[
\left( X_{T_{k+1}} (\omega) - a \right)^+ - \left( X_{T_k} (\omega) - a \right)^+
\]
for \( k = 1, 2, \cdots, 2n \). This is because \( \{ T'_k (\omega) \} \) is a strictly increasing sequence which starts with 0 due to the assumption \( X_0 (\omega) \leq a \) and ends with \( N < 2n \). Therefore,
\[
(X_N - a)^+ - (X_0 - a)^+ = \sum_{k=1}^{2n} \left( X_{T'_k} - a \right)^+ - \left( X_{T'_{k-1}} - a \right)^+
\]
\[
= \sum_{k=0}^{n-1} \left( X_{T_{2k+1}} - a \right)^+ - \left( X_{T_{2k}} - a \right)^+ + \sum_{k=1}^{n} \left( X_{T'_{2k}} - a \right)^+ - \left( X_{T'_{2k-1}} - a \right)^+.
\]
Now denote by \( U^N_{[a,b]} \) the number of upcrossings. When \( T'_k \) is such that \( k \) is odd, \( (X_{T'_k} - a)^+ \) is above \( b - a \) and when \( k \) is even, it equals 0. Therefore, in the first sum \( X_{T_{2k+1}} = X_{T_{2k}} = b - a \) and there are \( U^N_{[a,b]} \) terms which are nonzero in this sum. (Note this might not be \( n \) because many of the terms in the sum could be 0 due to the definition of \( T'_k \).) Hence
\[
(X_N - a)^+ - (X_0 - a)^+ = (X_N - a)^+
\]
\[
\geq (b - a) U^N_{[a,b]} + \sum_{k=1}^{n} \left( X_{T'_k} - a \right)^+ - \left( X_{T'_{k-1}} - a \right)^+.
\]  
(58.5.15)
Now \( U^N_{[a,b]} \) is a random variable. To see this, let \( Z_k (\omega) = 1 \) if \( T'_{2k+1} > T'_{2k} \) and 0 otherwise. Thus \( U^N_{[a,b]} (\omega) = \sum_{k=0}^{n-1} Z_k (\omega) \). Therefore, it makes sense to take the expected value of both sides of (58.5.15). By the optional sampling theorem, \( \left\{ X_{T'_k} - a \right\} \) is a submartingale and so
\[
E \left( \left( X_{T'_{2k}} - a \right)^+ - \left( X_{T'_{2k-1}} - a \right)^+ \right)
\]
\[
= \int_{\Omega} E \left( \left( X_{T'_{2k}} - a \right)^+ | \mathcal{F}_{T'_{2k-1}} \right) dP - \int_{\Omega} \left( X_{T'_{2k-1}} - a \right)^+ dP \geq 0.
\]
Therefore,
\[
E \left( (X_N - a)^+ \right) \geq (b - a) E \left( U^N_{[a,b]} \right).
\]  
(58.5.16)
This proves most of the following fundamental upcrossing estimate.

**Theorem 58.5.9** Let \( \{X_n\} \) be a real valued submartingale such that \( X_n \) is \( \mathcal{F}_n \) measurable. Then letting \( U^N_{[a,b]} \) denote the upcrossings of \( \{X_n\} \) from \( a \) to \( b \) for \( n \leq N \),
\[
E \left( U^N_{[a,b]} \right) \leq \frac{1}{b - a} E \left( (X_N - a)^+ \right).
\]
58.6. THE SUBMARTINGALE CONVERGENCE THEOREM

Proof: The estimate \(58.5.16\) was based on the assumption that \(X_0(\omega) \leq a\). If this is not so, modify \(X_0\). Change it to \(\min(X_0, a)\) . Then the inequality holds for the modified submartingale which has at least as many upcrossings. Therefore, the inequality remains. ■

Note this theorem holds if the submartingale starts at the index 1 rather than 0. Just adjust the argument.

58.6 The Submartingale Convergence Theorem

With this estimate it is now possible to prove the amazing submartingale convergence theorem.

**Theorem 58.6.1** Let \(\{X_n\}\) be a real valued submartingale such that

\[ E(|X_n|) < M \]

for all \(n\). Then there exists \(X \in L^1(\Omega, \mathcal{F})\) such that \(X_n(\omega)\) converges to \(X(\omega)\) a.e. \(\omega\) and \(X \in L^1(\Omega)\).

Proof: Let \(a < b\) be two rational numbers. From Theorem 58.5.1 it follows that for all \(N\),

\[ \int_{\Omega} U_N^{[a,b]} dP \leq \frac{1}{b-a} E((X_N - a)^+) \]

\[ \leq \frac{1}{b-a} (E(|X_N|) + |a|) \leq \frac{M + |a|}{b-a}. \]

Therefore, letting \(N \to \infty\), it follows that for a.e. \(\omega\), there are only finitely many upcrossings of \([a,b]\). Denote by \(S_{[a,b]}\) the exceptional set. Then letting \(S \equiv \cup_{a,b \in \mathbb{Q}} S_{[a,b]}\), it follows that \(P(S) = 0\) and for \(\omega \notin S\), \(\{X_n(\omega)\}\) is a Cauchy sequence because if

\[ \lim sup_{n \to \infty} X_n(\omega) > \lim inf_{n \to \infty} X_n(\omega) \]

then you can pick \(\lim inf_{n \to \infty} X_n(\omega) < a < b < \lim sup_{n \to \infty} X_n(\omega)\) with \(a, b\) rational and conclude \(\omega \in S_{[a,b]}\).

Let \(X(\omega) = \lim_{n \to \infty} X_n(\omega)\) if \(\omega \notin S\) and let \(X(\omega) = 0\) if \(\omega \in S\). Then it only remains to verify \(X \in L^1(\Omega)\). Since \(X\) is the pointwise limit of measurable functions, it follows \(X\) is measurable. By Fatou’s lemma,

\[ \int_{\Omega} |X(\omega)| dP \leq \lim inf_{n \to \infty} \int_{\Omega} |X_n(\omega)| dP \]

Thus \(X \in L^1(\Omega)\). This proves the theorem.

As a simple application, here is an easy proof of a nice theorem about convergence of sums of independent random variables.
Theorem 58.6.2 Let \( \{X_k\} \) be a sequence of independent real valued random variables such that \( E(|X_k|) < \infty, E(X_k) = 0, \) and
\[
\sum_{k=1}^{\infty} E\left(X_k^2\right) < \infty.
\]
Then \( \sum_{k=1}^{\infty} X_k \) converges a.e.

**Proof:** Let \( F_n \equiv \sigma(X_1, \cdots, X_n) \). Consider \( S_n \equiv \sum_{k=1}^{n} X_k. \)
\[
E(S_{n+1}|F_n) = S_n + E(X_{n+1}|F_n).
\]
Letting \( A \in F_n \) it follows from independence that
\[
\int_A E(X_{n+1}|F_n) \, dP = \int_A X_{n+1} \, dP = \int_{\Omega} X_A X_{n+1} \, dP = P(A) \int_{\Omega} X_{n+1} \, dP = 0
\]
and so \( E(X_{n+1}|F_n) = 0 \). Therefore, \( \{S_n\} \) is a martingale. Now using independence again,
\[
E(|S_n|) \leq E\left(|S_n^2|\right) = \sum_{k=1}^{n} E\left(X_k^2\right) \leq \sum_{k=1}^{\infty} E\left(X_k^2\right) < \infty
\]
and so \( \{S_n\} \) is an \( L^1 \) bounded martingale. Therefore, it converges a.e. and this proves the theorem.

**Corollary 58.6.3** Let \( \{X_k\} \) be a sequence of independent real valued random variables such that \( E(|X_k|) < \infty, E(X_k) = m_k, \) and
\[
\sum_{k=1}^{\infty} E\left(|X_k - m_k|^2\right) < \infty.
\]
Then \( \sum_{k=1}^{\infty} (X_k - m_k) \) converges a.e.

This can be extended to the case where the random variables have values in a separable Hilbert space.

**Theorem 58.6.4** Let \( \{X_k\} \) be a sequence of independent \( H \) valued random variables where \( H \) is a real separable Hilbert space such that \( E(|X_k|_H) < \infty, E(X_k) = 0, \) and
\[
\sum_{k=1}^{\infty} E\left(|X_k|^2_H\right) < \infty.
\]
Then \( \sum_{k=1}^{\infty} X_k \) converges a.e.
58.6. THE SUBMARTINGALE CONVERGENCE THEOREM

Proof: Let \( \{e_k\} \) be an orthonormal basis for \( H \). Then \( \{(X_n, e_k)_H\}_{n=1}^\infty \) are real valued, independent, and their mean equals 0. Also

\[
\sum_{n=1}^\infty E\left( |(X_n, e_k)_H|^2 \right) \leq \sum_{n=1}^\infty E\left( |X_n|_H^2 \right) < \infty
\]

and so from Theorem 58.6.2, the series,

\[
\sum_{n=1}^\infty (X_n, e_k)_H
\]

converges a.e. Therefore, there exists a set of measure zero such that for \( \omega \) not in this set, \( \sum_n (X_n(\omega), e_k)_H \) converges for each \( k \). For \( \omega \) not in this exceptional set, define

\[
Y_k(\omega) \equiv \sum_{n=1}^\infty (X_n(\omega), e_k)_H
\]

Next define

\[
S(\omega) \equiv \sum_{k=1}^\infty Y_k(\omega) e_k. \tag{58.6.17}
\]

Of course it is not clear this even makes sense. I need to show \( \sum_{k=1}^\infty |Y_k(\omega)|^2 < \infty \).

Using the independence of the \( X_n \)

\[
E\left( |Y_k|^2 \right) = E\left( \left( \sum_{n=1}^\infty (X_n, e_k)_H \right)^2 \right)
\]

\[
= E\left( \sum_{n=1}^\infty \sum_{m=1}^\infty (X_n, e_k)_H (X_m, e_k)_H \right)
\]

\[
\leq \lim_{N \to \infty} \inf_{N \to \infty} E\left( \sum_{n=1}^N \sum_{m=1}^N (X_n, e_k)_H (X_m, e_k)_H \right)
\]

\[
= \lim_{N \to \infty} \inf_{N \to \infty} \sum_{n=1}^N (X_n, e_k)_H^2
\]

\[
= \sum_{n=1}^\infty E\left( (X_n, e_k)_H^2 \right)
\]

Hence from the above,

\[
E\left( \sum_{k} |Y_k|^2 \right) = \sum_{k} E\left( |Y_k|^2 \right) \leq \sum_{k} \sum_{n} E\left( (X_n, e_k)_H^2 \right)
\]
and by the monotone convergence theorem or Fubini’s theorem,

\[
E \left( \sum_k \sum_n (X_n, e_k)^2_H \right) = E \left( \sum_n \sum_k (X_n, e_k)^2_H \right) \\
= E \left( \sum_n |X_n|^2_H \right) = \sum_n E \left( |X_n|^2_H \right) < \infty \tag{58.6.18}
\]

Therefore, for \( \omega \) off a set of measure zero, and for

\[
Y_k(\omega) \equiv \sum_{n=1}^{\infty} (X_n(\omega), e_k)_H,
\]

\[
\sum_k |Y_k(\omega)|^2 < \infty
\]

and also for these \( \omega \),

\[
\sum_n \sum_k (X_n(\omega), e_k)^2_H < \infty.
\]

It follows from the estimate 58.6.18 that for \( \omega \) not on a suitable set of measure zero, \( S(\omega) \) defined by 58.6.17,

\[
S(\omega) \equiv \sum_{k=1}^{\infty} Y_k(\omega) e_k
\]

makes sense. Thus for these \( \omega \)

\[
S(\omega) = \sum_l (S(\omega), e_l) e_l = \sum_l Y_l(\omega) e_l = \sum_l \sum_n (X_n(\omega), e_l)_H e_l
\]

\[
= \sum_n \sum_l (X_n(\omega), e_l) e_l = \sum_n X_n(\omega).
\]

This proves the theorem.

Now with this theorem, here is a strong law of large numbers.

**Theorem 58.6.5** Suppose \( \{X_k\} \) are independent random variables and \( E(|X_k|) < \infty \) for each \( k \) and \( E(X_k) = m_k \). Suppose also

\[
\sum_{j=1}^{\infty} \frac{1}{j^2} E \left( |X_j - m_j|^2 \right) < \infty. \tag{58.6.19}
\]

Then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (X_j - m_j) = 0 \text{ a.e.}
\]
Proof: Consider the sum
\[ \sum_{j=1}^{\infty} \frac{X_j - m_j}{j}. \]
This sum converges a.e. because of \texttt{58.6.19} and Theorem \texttt{58.6.4} applied to the random vectors \( \left\{ \frac{X_j - m_j}{j} \right\} \). Therefore, from Lemma \texttt{57.7.4} it follows that for a.e. \( \omega \),
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} (X_j(\omega) - m_j) = 0 \]
This proves the theorem.

The next corollary is often called the strong law of large numbers. It follows immediately from the above theorem.

Corollary 58.6.6 Suppose \( \{X_j\}_{j=1}^\infty \) are independent random vectors, \( \lambda_{X_i} = \lambda_{X_j} \) for all \( i, j \) having mean \( m \) and variance equal to
\[ \sigma^2 \equiv \int_{\Omega} |X_j - m|^2 \, dP < \infty. \]
Then for a.e. \( \omega \in \Omega \)
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} X_j(\omega) = m \]

58.7 A Reverse Submartingale Convergence Theorem

Definition 58.7.1 Let \( \{X_n\}_{n=0}^\infty \) be a sequence of real random variables such that \( E(|X_n|) < \infty \) for all \( n \) and let \( \{F_n\} \) be a sequence of \( \sigma \)-algebras such that \( F_n \supseteq F_{n+1} \) for all \( n \). Then \( \{X_n\} \) is called a reverse submartingale if for all \( n \),
\[ E(X_n | F_{n+1}) \geq X_{n+1}. \]

Note it is just like a submartingale only the indices are going the other way. Here is an interesting lemma. This lemma gives uniform integrability for a reverse submartingale.

Lemma 58.7.2 Suppose \( E(|X_n|) < \infty \) for all \( n \), \( X_n \) is \( F_n \) measurable, \( F_{n+1} \subseteq F_n \) for all \( n \in \mathbb{N} \), and there exist \( X_\infty \) \( F_\infty \) measurable such that \( F_\infty \subseteq F_n \) for all \( n \) and \( X_0 \) \( F_0 \) measurable such that \( F_0 \supseteq F_n \) for all \( n \) such that for all \( n \in \{0, 1, \cdots\} \),
\[ E(X_n | F_{n+1}) \geq X_{n+1}, \quad E(X_n | F_\infty) \geq X_\infty, \]
where \( E(|X_\infty|) < \infty \). Then \( \{X_n : n \in \mathbb{N}\} \) is uniformly integrable.
CHAPTER 58. CONDITIONAL EXPECTATION AND MARTINGALES

Proof: 

\[ E(X_{n+1}) \leq E(E(X_n|\mathcal{F}_{n+1})) = E(X_n) \]

Therefore, the sequence \( \{E(X_n)\} \) is a decreasing sequence bounded below by \( E(X_\infty) \) so it has a limit. I am going to show the functions are equiintegrable. Let \( k \) be large enough that

\[ \left| E(X_k) - \lim_{m \to \infty} E(X_m) \right| < \varepsilon \quad (58.7.20) \]

and suppose \( n > k \). Then if \( \lambda > 0 \),

\[
\int |X_n| \, dP = \int_{[X_n \geq \lambda]} X_n \, dP + \int_{[-X_n < \lambda]} (-X_n) \, dP \\
= \int_{[X_n \geq \lambda]} X_n \, dP + \int_{\Omega} (-X_n) \, dP - \int_{[-X_n < \lambda]} (-X_n) \, dP \\
= \int_{[X_n \geq \lambda]} X_n \, dP - \int_{[-X_n < \lambda]} X_n \, dP
\]

From \( 58.7.20 \)

\[
\leq \int_{[X_n \geq \lambda]} X_n \, dP - \int_{\Omega} X_k \, dP + \varepsilon + \int_{[-X_n < \lambda]} X_n \, dP
\]

By assumption,

\[ E(X_k|\mathcal{F}_n) \geq X_n \]

and so the above

\[
\leq \int_{[X_n \geq \lambda]} E(X_k|\mathcal{F}_n) \, dP - \int_{\Omega} X_k \, dP + \varepsilon + \int_{[-X_n < \lambda]} E(X_k|\mathcal{F}_n) \, dP \\
= \int_{[X_n \geq \lambda]} X_k \, dP - \int_{\Omega} X_k \, dP + \varepsilon + \int_{[-X_n < \lambda]} X_k \, dP \\
= \int_{[X_n \geq \lambda]} X_k \, dP - \int_{\Omega} X_k \, dP + \varepsilon + \int_{[X_n > -\lambda]} X_k \, dP \\
= \int_{[X_n \geq \lambda]} X_k \, dP + \left( \int_{[X_n > -\lambda]} (-X_k) \, dP - \int_{[X_n > -\lambda]} (-X_k) \, dP \right) + \varepsilon \\
= \int_{[X_n \geq \lambda]} X_k \, dP + \int_{[X_n \leq -\lambda]} (-X_k) \, dP + \varepsilon = \int_{[|X_n| \geq \lambda]} |X_k| \, dP + \varepsilon
\]

Applying the maximal inequality for submartingales, Theorem 58.5.3,

\[ P(\max \{|X_j|: j = n, \cdots , 1\} \geq \lambda) \leq \frac{1}{\lambda} (E(|X_0|) + E(|X_\infty|)) \leq \frac{C}{\lambda} \]

and taking sup for all \( n \),

\[ P(\sup \{|X_j| \geq \lambda\}) \leq \frac{C}{\lambda} \]
It follows since the single function, $X_k$ is equiintegrable that for all $\lambda$ large enough,
\[
\int_{|X_n|\geq \lambda} |X_n| dP \leq 2\varepsilon
\]
and since $\varepsilon$ is arbitrary, this shows $\{X_n\}$ for $n > k$ is equiintegrable. Since there are only finitely many $X_j$ for $j \leq k$, this shows $\{X_n\}$ is equiintegrable. Hence $\{X_n\}$ is uniformly integrable. This proves the lemma.

Now with this lemma and the upcrossing lemma it is easy to prove an important convergence theorem.

**Theorem 58.7.3** Let $\{X_n, \mathcal{F}_n\}_{n=0}^{\infty}$ be a backwards submartingale as described above and suppose $\sup_{n \geq 0} E(|X_n|) < \infty$. Then $\{X_n\}$ converges a.e. and in $L^1(\Omega)$ to a function, $X_\infty$.

**Proof:** By the upcrossing lemma applied to the submartingale $\{X_k\}_{k=0}^{N}$, the number of upcrossings (Downcrossings is probably a better term. They are upcrossings as $n$ gets smaller.) of the interval $[a,b]$ satisfies the inequality
\[
E \left( U_{[a,b]}^N \right) \leq \frac{1}{b-a} E \left( (X_0 - a)^+ \right)
\]
Letting $N \to \infty$, it follows the expected number of upcrossings, $E \left( U_{[a,b]} \right)$ is bounded. Therefore, there exists a set of measure 0 $N_{ab}$ such that if $\omega \notin N_{ab}, U_{[a,b]}(\omega) < \infty$. Let $N = \bigcup \{N_{ab} : a, b \in \mathbb{Q}\}$. Then for $\omega \notin N$,
\[
\lim_{n \to \infty} \sup X_n(\omega) = \lim_{n \to \infty} \inf X_n(\omega)
\]
because if inequality holds, then letting
\[
\lim_{n \to \infty} \inf X_n(\omega) < a < b < \lim_{n \to \infty} \sup X_n(\omega)
\]
it would follow $U_{[a,b]}(\omega) = \infty$, contrary to $\omega \notin N_{ab}$.

Let $X_\infty(\omega) \equiv \lim_{n \to \infty} X_n(\omega)$. Then by Fatou’s lemma,
\[
\int_{\Omega} |X_\infty(\omega)| dP \leq \lim_{n \to \infty} \inf \int_{\Omega} |X_n| dP < \infty.
\]
and so $X_\infty$ is in $L^1(\Omega)$. By the Vitali convergence theorem and Lemma which shows $\{|X_n|\}$ is uniformly integrable, it follows
\[
\lim_{n \to \infty} \int_{\Omega} |X_\infty(\omega) - X_n(\omega)| dP = 0.
\]
This proves the theorem.
58.8 Strong Law Of Large Numbers

There is a version of the strong law of large numbers which does not depend on the random variables having finite variance. First are some preparatory lemmas. The approach followed here is from Ash [D].

**Lemma 58.8.1** Let \( \{X_n\} \) be a sequence of independent random variables such that \( E(|X_k|) < \infty \) for all \( k \) and let \( S_n \equiv \sum_{k=1}^{n} X_k \). Then for \( k \leq n \),

\[
E(X_k|\sigma(S_n)) = E(X_k|\sigma(S_n, Y)) \quad \text{a.e.}
\]

where \( Y = (X_{n+1}, X_{n+2}, \cdots) \in \mathbb{R}^N \). Also for \( k \leq n \) as above,

\[
\sigma(S_n, Y) = \sigma(S_n, S_{n+1}, \cdots).
\]

**Proof:** Note that \( \mathbb{R}^N \) with the usual product topology has a countable basis. Here it is. Let \( B_N \) denote sets of the form \( \prod_{i=1}^{\infty} D_i \) where for \( i \leq N, D_i \in B \), a countable basis for \( \mathbb{R} \) and for \( i > N, D_i = \mathbb{R} \). Then \( B_N \) is countable and so is \( D = \bigcup_{N=1}^{\infty} B_N \). From the definition of the product topology, this is a countable basis for the product topology.

Let \( V \in D \) and \( U \) be an open set of \( \mathbb{R} \). Then if \( A \in (S_n, Y)^{-1}(U \times V) \), by independence of the \( \{X_k\} \),

\[
\int_{(S_n, Y)^{-1}(U \times V)} E(X_k|\sigma(S_n, Y)) \, dP = \int_{(S_n, Y)^{-1}(U \times V)} X_k \, dP = \int_{\Omega} X_{S_n^{-1}(U)}(\omega) X_k(\omega) \, dP
\]

\[
= \int_{\Omega} X_{S_n^{-1}(U)}(\omega) X_{Y^{-1}(V)}(\omega) X_k(\omega) \, dP = P(Y^{-1}(V)) \int_{\Omega} X_{S_n^{-1}(U)}(\omega) X_k(\omega) \, dP
\]

\[
= P(Y^{-1}(V)) \int_{S_n^{-1}(U)} E(X_k|\sigma(S_n)) \, dP.
\]

Now by independence again, \( \{S_n, X_{n+1}, X_{n+2}, \cdots\} \) are independent and so the above equals

\[
\int_{S_n^{-1}(U)} X_{Y^{-1}(V)} E(X_k|\sigma(S_n)) \, dP = \int_{(S_n, Y)^{-1}(U \times V)} E(X_k|\sigma(S_n)) \, dP.
\]

Letting

\[
S \equiv \left\{ A \in B\left( \mathbb{R} \times \mathbb{R}^N \right) : \int_{(S_n, Y)^{-1}(A)} E(X_k|\sigma(S_n)) \, dP \right\}
\]

the above has shown this is true for all \( A \) in a countable basis. Therefore, it is true for all \( A \) open in \( \mathbb{R} \times \mathbb{R}^N \). Finally, it is clear that \( S \) is a \( \sigma \) algebra which shows the above holds for all \( A \) Borel in \( \mathbb{R} \times \mathbb{R}^N \). Thus, for all \( B \in \sigma(S_n, Y) \),

\[
\int_B E(X_k|\sigma(S_n)) \, dP = \int_B E(X_k|\sigma(S_n, Y)) \, dP
\]
and thus \( E(X_k|\sigma(S_n)) = E(X_k|\sigma(S_n, Y)) \) a.e.

It only remains to prove the last assertion. For \( k > 0 \),

\[
X_{n+k} = S_{n+k} - S_{n+k-1}
\]

Thus

\[
\sigma(S_n, Y) = \sigma(S_n, X_{n+1}, \ldots) = \sigma(S_n, (S_{n+1} - S_n), (S_{n+2} - S_{n+1}), \ldots)
\subseteq \sigma(S_n, S_{n+1}, \ldots)
\]

On the other hand,

\[
\sigma(S_n, S_{n+1}, \ldots) = \sigma(S_n, X_{n+1} + S_n, X_{n+2} + X_{n+1} + S_n, \ldots)
\subseteq \sigma(S_n, X_{n+1}, X_{n+2}, \ldots)
\]

To see this, note that for an open set, and hence for a Borel set, \( B \),

\[
\left( S_n + \sum_{k=n+1}^m X_k \right)^{-1} (B) = (S_n, X_{n+1}, \ldots, X_m)^{-1} (B')
\]

for some \( B' \in \mathbb{R}^{m+1} \). Thus \( (S_n + \sum_{k=n+1}^m X_k)^{-1} (B) \) for \( B \) a Borel set is contained in \( \sigma(S_n, X_{n+1}, X_{n+2}, \ldots) \). Similar considerations apply to the other inclusion stated earlier. This proves the lemma.

**Lemma 58.8.2** Let \( \{X_k\} \) be a sequence of independent identically distributed random variables such that \( E(|X_k|) < \infty \). Then letting \( S_n = \sum_{k=1}^n X_k \), it follows that for \( k \leq n \)

\[
E(X_k|\sigma(S_n, S_{n+1}, \ldots)) = E(X_k|\sigma(S_n)) = \frac{S_n}{n}.
\]

**Proof:** It was shown in Lemma 58.8.1 the first equality holds. It remains to show the second. Letting \( A = S_n^{-1}(B) \) where \( B \) is Borel, it follows there exists \( B' \subseteq \mathbb{R}^n \) a Borel set such that

\[
S_n^{-1}(B) = (X_1, \ldots, X_n)^{-1}(B').
\]

Then

\[
\int_A E(X_k|\sigma(S_n)) \, dP = \int_{S_n^{-1}(B)} X_k dP
= \int_{(X_1, \ldots, X_n)^{-1}(B')} X_k dP = \int_{(X_1, \ldots, X_n)^{-1}(B')} x_k d\lambda(x_1, \ldots, x_n)
= \int \cdots \int x_k d\lambda x_1 d\lambda x_2 \cdots d\lambda x_n
= \int \cdots \int x_k d\lambda x_1 d\lambda x_2 \cdots d\lambda x_n
\]
\[ = \int_A E(X_l|\sigma(S_n)) \, dP \]
and so since \( A \in \sigma(S_n) \) is arbitrary,
\[ E(X_l|\sigma(S_n)) = E(X_k|\sigma(S_n)) \]
for each \( k, l \leq n \). Therefore,
\[ S_n = E(S_n|\sigma(S_n)) = \sum_{j=1}^{n} E(X_j|\sigma(S_n)) = n E(X_k|\sigma(S_n)) \text{ a.e.} \]
and so
\[ E(X_k|\sigma(S_n)) = \frac{S_n}{n} \text{ a.e.} \]
as claimed. This proves the lemma.

With this preparation, here is the strong law of large numbers for identically distributed random variables.

**Theorem 58.8.3** Let \( \{X_k\} \) be a sequence of independent identically distributed random variables such that \( E(|X_k|) < \infty \) for all \( k \). Letting \( m = E(X_k) \),
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k(\omega) = m \text{ a.e.} \]
and convergence also takes place in \( L^1(\Omega) \).

**Proof:** Consider the reverse submartingale \( \{E(X_1|\sigma(S_n, S_{n+1}, \cdots))\} \). By Theorem 58.7.3 this converges a.e. and in \( L^1(\Omega) \) to a random variable, \( X_\infty \). However, from Lemma 58.8.2, \( E(X_1|\sigma(S_n, S_{n+1}, \cdots)) = S_n/n \). Therefore, \( S_n/n \) converges a.e. and in \( L^1(\Omega) \) to \( X_\infty \). I need to argue that \( X_\infty \) is constant and also that it equals \( m \). For \( a \in \mathbb{R} \) let
\[ E_a \equiv [X_\infty \geq a] \]
For \( a \) small enough, \( P(E_a) \neq 0 \). Then since \( E_a \) is a tail event for the independent random variables, \( \{X_k\} \) it follows from the Kolmogorov zero one law, Theorem 57.6.4, that \( P(E_a) = 1 \). Let \( b = \sup \{a : P(E_a) = 1\} \). The sets, \( E_a \) are decreasing as \( a \) increases. Let \( \{a_n\} \) be a strictly increasing sequence converging to \( b \). Then
\[ [X_\infty \geq b] = \cap_n [X_\infty \geq a_n] \]
and so
\[ 1 = P(E_b) = \lim_{n \to \infty} P(E_{a_n}). \]
On the other hand, if \( c > b \), then \( P(E_c) < 1 \) and so \( P(E_c) = 0 \). Hence \( P([X = b]) = 1 \). It remains to show \( b = m \). This is easy because by the \( L^1 \) convergence,
\[ b = \int_{\Omega} X_\infty dP = \lim_{n \to \infty} \int_{\Omega} \frac{S_n}{n} dP = \lim_{n \to \infty} m = m. \]
This proves the theorem.
Chapter 59

Probability In Infinite Dimensions

59.1 Conditional Expectation In Banach Spaces

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X \in L^1(\Omega; \mathbb{R})$. Also let $\mathcal{G} \subseteq \mathcal{F}$ where $\mathcal{G}$ is also a $\sigma$ algebra. Then the usual conditional expectation is defined by

$$\int_A XdP = \int_A E(X|\mathcal{G})dP$$

where $E(X|\mathcal{G})$ is $\mathcal{G}$ measurable and $A \in \mathcal{G}$ is arbitrary. Recall this is an application of the Radon Nikodym theorem. Also recall $E(X|\mathcal{G})$ is unique up to a set of measure zero.

I want to do something like this here. Denote by $L^1(\Omega; \mathcal{E}, \mathcal{G})$ those functions in $L^1(\Omega; \mathcal{E})$ which are measurable with respect to $\mathcal{G}$.

**Theorem 59.1.1** Let $E$ be a separable Banach space and let $X \in L^1(\Omega; E, \mathcal{F})$ where $X$ is measurable with respect to $\mathcal{F}$ and let $\mathcal{G}$ be a $\sigma$ algebra which is contained in $\mathcal{F}$. Then there exists a unique $Z \in L^1(\Omega; E, \mathcal{G})$ such that for all $A \in \mathcal{G}$,

$$\int_A XdP = \int_A ZdP$$

Denoting this $Z$ as $E(X|\mathcal{G})$, it follows

$$||E(X|\mathcal{G})|| \leq E(||X|||\mathcal{G}) .$$

**Proof:** First consider uniqueness. Suppose $Z'$ is another in $L^1(\Omega; E, \mathcal{G})$ which works. Consider a dense subset of $E \{a_n\}_{n=1}^\infty$. Then the balls $\{B\left(a_n, \frac{||a_n||}{2}\right)\}_{n=1}^\infty$ must cover $E \setminus \{0\}$. Here is why. If $y \neq 0$, pick $a_n \in B\left(y, \frac{||y||}{5}\right)$. 

2031
Then $\|a_n\| \geq 4 \|y\| / 5$ and so $\|a_n - y\| < \|y\| / 5$. Thus

$$y \in B(a_n, \|y\| / 5) \subseteq B\left(a_n, \frac{\|a_n\|}{4}\right)$$

Now suppose $Z$ is $\mathcal{G}$ measurable and

$$\int_A Z dP = 0$$

for all $A \in \mathcal{G}$. The letting $A \equiv Z^{-1}\left(B\left(a_n, \frac{\|a_n\|}{4}\right)\right)$ it follows

$$0 = \int_A Z - a_n + a_n dP$$

and so

$$\|a_n\| P(A) = \left\|\int_A a_n dP\right\| = \left\|\int_A (a_n - Z) dP\right\| \leq \int_{Z^{-1}\left(B\left(a_n, \frac{\|a_n\|}{4}\right)\right)} \|a_n - Z\| dP \leq \frac{\|a_n\|}{4} P(A)$$

which is a contradiction unless $P(A) = 0$. Therefore, letting

$$N \equiv \bigcup_{n=1}^{\infty} Z^{-1}\left(B\left(a_n, \frac{\|a_n\|}{4}\right)\right) = Z^{-1}(E \setminus \{0\})$$

it follows $N$ has measure zero and so $Z = 0$ a.e. This proves uniqueness because if $Z, Z'$ both hold, then from the above argument, $Z - Z' = 0$ a.e.

Next I will show $Z$ exists. To do this recall Theorem [19.2.5] on Page 638 which is stated below for convenience.

**Theorem 59.1.2** An $E$ valued function, $X$, is Bochner integrable if and only if $X$ is strongly measurable and

$$\int_{\Omega} \|X(\omega)\| dP < \infty. \quad (59.1.1)$$

In this case there exists a sequence of simple functions $\{X_n\}$ satisfying

$$\int_{\Omega} \|X_n(\omega) - X_m(\omega)\| dP \to 0 \text{ as } m, n \to \infty. \quad (59.1.2)$$
59.1. CONDITIONAL EXPECTATION IN BANACH SPACES

Let \( X_n(\omega) \) converging pointwise to \( X(\omega) \),

\[
||X_n(\omega)|| \leq 2 ||X(\omega)||
\]

and

\[
\lim_{n \to \infty} \int_{\Omega} ||X(\omega) - X_n(\omega)|| \, dP = 0.
\]

Now let \( \{X_n\} \) be the simple functions just defined and let

\[
X_n(\omega) = \sum_{k=1}^{m} x_k \mathcal{X}_{F_k}(\omega)
\]

where \( F_k \in \mathcal{F} \), the \( F_k \) being disjoint. Then define

\[
Z_n = \sum_{k=1}^{m} x_k E(\mathcal{X}_{F_k} | \mathcal{G})
\]

Thus, if \( A \in \mathcal{G} \),

\[
\int_{A} Z_n \, dP = \sum_{k=1}^{m} x_k \int_{A} E(\mathcal{X}_{F_k} | \mathcal{G}) \, dP
\]

\[
= \sum_{k=1}^{m} x_k \int_{A} \mathcal{X}_{F_k} \, dP
\]

\[
= \sum_{k=1}^{m} x_k P(F_k \cap A) = \int_{A} X_n \, dP
\]

Then since \( E(\mathcal{X}_{F_k} | \mathcal{G}) \geq 0 \),

\[
||Z_n|| \leq \sum_{k=1}^{m} ||x_k|| E(\mathcal{X}_{F_k} | \mathcal{G})
\]

Thus if \( A \in \mathcal{G} \),

\[
E(||Z_n|| \mathcal{X}_A) \leq E\left(\sum_{k=1}^{m} ||x_k|| \mathcal{X}_A E(\mathcal{X}_{F_k} | \mathcal{G})\right) = \sum_{k=1}^{m} ||x_k|| \int_{A} E(\mathcal{X}_{F_k} | \mathcal{G}) \, dP
\]

\[
= \sum_{k=1}^{m} ||x_k|| \int_{A} \mathcal{X}_{F_k} \, dP = E(\mathcal{X}_A ||X_n||).
\]

Note the use of \( \leq \) in the first step in the above. Although the \( F_k \) are disjoint, all that is known about \( E(\mathcal{X}_{F_k} | \mathcal{G}) \) is that it is nonnegative. Similarly,

\[
E(||Z_n - Z_m||) \leq E(||X_n - X_m||)
\]
and this last term converges to 0 as \( n, m \to \infty \) by the properties of the \( X_n \). Therefore, \( \{Z_n\} \) is a Cauchy sequence in \( L^1(\Omega;E;\mathcal{G}) \). It follows it converges to some \( Z \) in \( L^1(\Omega;E;\mathcal{G}) \). Then letting \( A \in \mathcal{G} \), and using \( 59.1.5 \),

\[
\int_A ZdP = \int X_A ZdP = \lim_{n \to \infty} \int X_A Z_n dP = \lim_{n \to \infty} \int_A Z_n dP = \lim_{n \to \infty} \int_A X_n dP = \int_A XdP.
\]

Then define \( Z \equiv E(X|\mathcal{G}) \).

It remains to verify \( \|E(X|\mathcal{G})\| \equiv \|Z\| \leq E(\|X\| |\mathcal{G}) \). This follows because, from the above,

\[
\|Z_n\| \to \|Z\|, \quad \|X_n\| \to \|X\| \quad \text{in} \quad L^1(\Omega)
\]

and so if \( A \in \mathcal{G} \), then from \( 59.1.5 \),

\[
\frac{1}{P(A)} \int_A \|Z_n\| dP \leq \frac{1}{P(A)} \int_A \|X_n\| dP
\]

and so, passing to the limit,

\[
\frac{1}{P(A)} \int_A \|Z\| dP \leq \frac{1}{P(A)} \int_A \|X\| dP = \frac{1}{P(A)} \int_A E(\|X\| |\mathcal{G}) dP
\]

Since \( A \) is arbitrary, this shows that

\[
\|E(X|\mathcal{G})\| \equiv \|Z\| \leq E(\|X\| |\mathcal{G}) . \quad \blacksquare
\]

In the case where \( E \) is reflexive, one could also use Corollary 19.7.5 on Page 669 to get the above result. You would define a vector measure on \( \mathcal{G} \),

\[ \nu(F) \equiv \int_F XdP \]

and then you would use the fact that reflexive separable Banach spaces have the Radon Nikodym property to obtain \( Z \in L^1(\Omega;E,\mathcal{G}) \) such that

\[ \nu(F) = \int_F XdP = \int_F ZdP. \]

The function, \( Z \) whose existence and uniqueness is guaranteed by Theorem 59.1.2, is called \( E(X|\mathcal{G}) \).

### 59.2 Probability Measures And Tightness

Here and in what remains, \( \mathcal{B}(E) \) will denote the Borel sets of \( E \) where \( E \) is a topological space, usually at least a Banach space. Because of the fact that probability measures are finite, you can use a simpler definition of what it means for a measure
to be regular. Recall that there were two ingredients, inner regularity which said that the measure of a set is the supremum of the measures of compact subsets and outer regularity which says that the measure of a set is the infimum of the measures of the open sets which contain the given set. Here the definition will be similar but instead of using compact sets, closed sets are substituted. Thus the following definition is a little different than the earlier one. I will show, however, that in many interesting cases, this definition of regularity is actually the same as the earlier one.

**Definition 59.2.1** A measure, $\mu$ defined on $\mathcal{B}(E)$ will be called inner regular if for all $F \in \mathcal{B}(E)$,

$$\mu(F) = \sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ is closed} \}$$

A measure, $\mu$ defined on $\mathcal{B}(E)$ will be called outer regular if for all $F \in \mathcal{B}(E)$,

$$\mu(F) = \inf \{ \mu(V) : V \supseteq F \text{ and } V \text{ is open} \}$$

When a measure is both inner and outer regular, it is called regular.

For probability measures, regularity tends to come free.

**Lemma 59.2.2** Let $\mu$ be a finite measure defined on $\mathcal{B}(E)$ where $E$ is a metric space. Then $\mu$ is regular.

**Proof:** First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let $V$ be an open set and let

$$K_k \equiv \{ x \in V : \text{dist} (x, V^C) \geq 1/k \}.$$ 

Then clearly the union of the $K_k$ equals $V$. Next, for $K$ closed let

$$V_k \equiv \{ x \in E : \text{dist} (x, K) < 1/k \}.$$ 

Clearly the intersection of the $V_k$ equals $K$. Therefore, letting $V$ denote an open set and $K$ a closed set,

$$\mu(V) = \sup \{ \mu(K) : K \subseteq V \text{ and } K \text{ is closed} \}$$

$$\mu(K) = \inf \{ \mu(V) : V \supseteq K \text{ and } V \text{ is open} \}.$$ 

Also since $V$ is open and $K$ is closed,

$$\mu(V) = \inf \{ \mu(U) : U \supseteq V \text{ and } V \text{ is open} \}$$

$$\mu(K) = \sup \{ \mu(L) : L \subseteq K \text{ and } L \text{ is closed} \}.$$ 

In words, $\mu$ is regular on open and closed sets. Let

$$F \equiv \{ F \in \mathcal{B}(E) \text{ such that } \mu \text{ is regular on } F \}.$$
Then \( F \) contains the open sets. I want to show \( F \) is a \( \sigma \) algebra and then it will follow \( F = \mathcal{B}(E) \).

First I will show \( F \) is closed with respect to complements. Let \( F \in F \). Then since \( \mu \) is finite and \( F \) is inner regular, there exists \( K \subseteq F \) such that \( \mu (F \setminus K) < \varepsilon \).

But \( K^C \setminus F^C = F \setminus K \) and so \( \mu (K^C \setminus F^C) < \varepsilon \) showing that \( F^C \) is outer regular. I have just approximated the measure of \( F^C \) with the measure of \( K^C \), an open set containing \( F^C \). A similar argument works to show \( F^C \) is inner regular. You start with \( V \supseteq F \) such that \( \mu (V \setminus F) < \varepsilon \), note \( F^C \setminus V^C = V \setminus F \), and then conclude \( \mu (F^C \setminus V^C) < \varepsilon \), thus approximating \( F^C \) with the closed subset, \( V^C \).

Next I will show \( F \) is closed with respect to taking countable unions. Let \( \{F_k\} \) be a sequence of sets in \( F \). Then \( \mu \) is inner regular on each of these so there exist \( \{K_k\} \) such that \( K_k \subseteq F_k \) and \( \mu (F_k \setminus K_k) < \varepsilon / 2^{k+1} \). First choose \( m \) large enough that

\[
\mu \left( (\bigcup_{k=1}^{\infty} F_k) \setminus (\bigcup_{k=1}^{m} F_k) \right) < \frac{\varepsilon}{2}.
\]

Then

\[
\mu \left( (\bigcup_{k=1}^{m} F_k) \setminus (\bigcup_{k=1}^{m} K_k) \right) \leq \sum_{k=1}^{m} \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}
\]

and so

\[
\mu \left( (\bigcup_{k=1}^{\infty} F_k) \setminus (\bigcup_{k=1}^{m} K_k) \right) \leq \mu \left( (\bigcup_{k=1}^{\infty} F_k) \setminus (\bigcup_{k=1}^{m} F_k) \right) + \mu \left( (\bigcup_{k=1}^{m} F_k) \setminus (\bigcup_{k=1}^{m} K_k) \right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

showing \( \mu \) is inner regular on \( \bigcup_{k=1}^{\infty} F_k \). Since \( \mu \) is outer regular on \( F_k \), there exists \( V_k \) such that \( \mu (V_k \setminus F_k) < \varepsilon / 2^k \). Then

\[
\mu \left( (\bigcup_{k=1}^{\infty} V_k) \setminus (\bigcup_{k=1}^{\infty} F_k) \right) \leq \sum_{k=1}^{\infty} \mu (V_k \setminus F_k)
\]

\[
< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon
\]

and this shows \( \mu \) is outer regular on \( \bigcup_{k=1}^{\infty} F_k \) and this proves the lemma.

**Lemma 59.2.3** Let \( \mu \) be a finite measure on \( \mathcal{B}(E) \), the Borel sets of \( E \), a separable complete metric space. Then if \( C \) is a closed set,

\[
\mu (C) = \sup \{ \mu (K) : K \subseteq C \text{ and } K \text{ is compact} \}.
\]

**Proof:** Let \( \{a_k\} \) be a countable dense subset of \( C \). Thus \( \bigcup_{k=1}^{\infty} B(a_k, \frac{1}{n}) \supseteq C \). Therefore, there exists \( m_n \) such that

\[
\mu \left( C \setminus \bigcup_{k=1}^{m_n} B \left( a_k, \frac{1}{n} \right) \right) \equiv \mu (C \setminus C_n) < \frac{\varepsilon}{2^n}.
\]
59.3. Tight Measures

Now let $K = C \cap (\bigcap_{n=1}^{\infty} C_n)$. Then $K$ is a subset of $C_n$ for each $n$ and so for each $\varepsilon > 0$ there exists an $\varepsilon$ net for $K$ since $C_n$ has a $1/n$ net, namely $a_1, \ldots, a_m$. Since $K$ is closed, it is complete and so it is also compact. Now

$$
\mu(C \setminus K) = \mu(\bigcup_{n=1}^{\infty} (C_n \setminus C)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.
$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for $K$ a compact subset of $C$. This proves the lemma.

This shows that for a finite measure on the Borel sets of a separable metric space, the above definition of regular coincides with the earlier one.

59.3 Tight Measures

Now here is a definition of what it means for a set of measures to be tight.

**Definition 59.3.1** Let $\Lambda$ be a set of probability measures defined on the Borel sets of a topological space. Then $\Lambda$ is “tight” if for all $\varepsilon > 0$ there exists a compact set, $K_\varepsilon$ such that

$$
\mu([x \notin K_\varepsilon]) < \varepsilon
$$

for all $\mu \in \Lambda$.

Lemma 59.2.3 implies a single probability measure on the Borel sets of a separable metric space is tight. The proof of that lemma generalizes slightly to give a simple criterion for a set of measures to be tight.

**Lemma 59.3.2** Let $E$ be a separable complete metric space and let $\Lambda$ be a set of Borel probability measures. Then $\Lambda$ is tight if and only if for every $\varepsilon > 0$ and $r > 0$ there exists a finite collection of balls, $\{B(a_i, r)\}_{i=1}^{m}$ such that

$$
\mu\left(\bigcup_{i=1}^{m} B(a_i, r)\right) > 1 - \varepsilon
$$

for every $\mu \in \Lambda$.

**Proof:** If $\Lambda$ is tight, then there exists a compact set, $K_\varepsilon$ such that

$$
\mu(K_\varepsilon) > 1 - \varepsilon
$$

for all $\mu \in \Lambda$. Then consider the open cover, $\{B(x, r) : x \in K_\varepsilon\}$. Finitely many of these cover $K_\varepsilon$ and this yields the above condition.

Now suppose the above condition and let

$$
C_n = \bigcup_{i=1}^{\infty} B(a_i^n, 1/n)
$$
satisfy \( \mu(C_n) > 1 - \varepsilon/2^n \) for all \( \mu \in \Lambda \). Then let \( K_\varepsilon \equiv \cap_{n=1}^\infty C_n \). This set \( K_\varepsilon \) is a compact set because it is a closed subset of a complete metric space and is therefore complete, and it is also totally bounded by construction. For \( \mu \in \Lambda \),

\[
\mu(K_\varepsilon^c) = \mu(\cup_n C_n^c) \leq \sum_n \mu(C_n^c) < \sum_n \frac{\varepsilon}{2^n} = \varepsilon
\]

Therefore, \( \Lambda \) is tight. ■

Prokhorov’s theorem is an important result which also involves tightness. In order to give a proof of this important theorem, it is necessary to consider some simple results from topology which are interesting for their own sake.

**Theorem 59.3.3** Let \( H \) be a compact metric space. Then there exists a compact subset of \([0, 1] \times K \) and a continuous function, \( \theta \) which maps \( K \) onto \( H \).

**Proof:** Without loss of generality, it can be assumed \( H \) is an infinite set since otherwise the conclusion is trivial. You could pick finitely many points of \([0, 1] \times K \). Since \( H \) is compact, it is totally bounded. Therefore, \( \mu \) is tight.

Note that \( \theta \equiv \cap_{i=1}^\infty H_i \). Since the diameters of the \( H_i \) converge to 0 as \( i \to \infty \), this function is well defined. It is continuous because if \( x_n \to x \), then ultimately \( x_n \) and \( x \) are both in \( H_j \), the \( k \)th closed interval in the sequence whose intersection is \( x \). Hence, \( d(\theta x_n, \theta x) \leq \text{diameter}(H_j) \leq 1/k \). To see the map is onto, let \( h \in H \). Then from the construction, there exists a sequence \( \{H^k_h\}_{k=1}^\infty \) of the above sets whose intersection equals \( h \). Then \( \theta(\cap_{i=1}^\infty I_k^h) = h \). ■

Note \( \theta \) is maybe not one to one.

As an important corollary, it follows that the continuous functions defined on any compact metric space is separable.

**Corollary 59.3.4** Let \( H \) be a compact metric space and let \( C(H) \) denote the continuous functions defined on \( H \) with the usual norm,

\[
||f||_\infty \equiv \max\{ ||f(x)|| : x \in H \}
\]

Then \( C(H) \) is separable.
**Proof:** The proof is by contradiction. Suppose $C(H)$ is not separable. Let $\mathcal{H}_k$ denote a maximal collection of functions of $C(H)$ with the property that if $f, g \in \mathcal{H}_k$, then $||f - g||_\infty \geq 1/k$. The existence of such a maximal collection of functions is a consequence of a simple use of the Hausdorff maximality theorem. Then $\bigcup_{k=1}^\infty \mathcal{H}_k$ is dense. Therefore, it cannot be countable by the assumption that $C(H)$ is not separable. It follows that for some $k$, $\mathcal{H}_k$ is uncountable. Now by Theorem 59.3.3 there exists a continuous function, $\theta$ defined on a compact subset, $K$ of $[0,1]$ which maps $K$ onto $H$. Now consider the functions defined on $K$

$$G_k \equiv \{f \circ \theta : f \in \mathcal{H}_k\}.$$ 

Then $G_k$ is an uncountable set of continuous functions defined on $K$ with the property that the distance between any two of them is at least as large as $1/k$. This contradicts separability of $C(K)$ which follows from the Weierstrass approximation theorem in which the separable countable set of functions is the restrictions of polynomials that involve only rational coefficients. ■

Now here is Prokhorov’s theorem.

**Theorem 59.3.5** Let $\Lambda = \{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures defined on $\mathcal{B}(E)$ where $E$ is a separable complete metric space. If $\Lambda$ is tight then there exists a probability measure, $\lambda$ and a subsequence of $\{\mu_n\}_{n=1}^\infty$, still denoted by $\{\mu_n\}_{n=1}^\infty$, such that whenever $\phi$ is a continuous bounded complex valued function defined on $E$,

$$\lim_{n \to \infty} \int \phi d\mu_n = \int \phi d\lambda.$$

**Proof:** By tightness, there exists an increasing sequence of compact sets, $\{K_n\}$ such that

$$\mu(K_n) > 1 - \frac{1}{n}$$

for all $\mu \in \Lambda$. Now letting $\mu \in \Lambda$ and $\phi \in C(K_n)$ such that $||\phi||_\infty \leq 1$, it follows

$$\left|\int_{K_n} \phi d\mu\right| \leq \mu(K_n) \leq 1$$

and so the restrictions of the measures of $\Lambda$ to $K_n$ are contained in the unit ball of $C(K_n)'$. Recall from the Riesz representation theorem, the dual space of $C(K_n)$ is a space of complex Borel measures. Theorem 15.3.12 on Page 447 implies the unit ball of $C(K_n)'$ is weak $*$ sequentially compact. This follows from the observation that $C(K_n)$ is separable which is proved in Corollary 59.3.2 and leads to the fact that the unit ball in $C(K_n)'$ is actually metrizable by Theorem 15.3.12 on Page 447. Therefore, there exists a subsequence of $\Lambda$, $\{\mu_{1k}\}$ such that their restrictions to $K_1$ converge weak $*$ to a measure, $\lambda_1 \in C(K_1)'$. That is, for every $\phi \in C(K_1)$,

$$\lim_{k \to \infty} \int_{K_1} \phi d\mu_{1k} = \int_{K_1} \phi d\lambda_1.$$
By the same reasoning, there exists a further subsequence \( \{ \mu_{2k} \} \) such that the restrictions of these measures to \( K_2 \) converge weak * to a measure \( \lambda_2 \in C(K_2)' \) etc. Continuing this way,

\[
\begin{align*}
\mu_{11}, \mu_{12}, \mu_{13}, & \cdots \to \text{ weak * in } C(K_1)' \\
\mu_{21}, \mu_{22}, \mu_{23}, & \cdots \to \text{ weak * in } C(K_2)' \\
\mu_{31}, \mu_{32}, \mu_{33}, & \cdots \to \text{ weak * in } C(K_3)' \\
& \vdots
\end{align*}
\]

Here the \( j \)th sequence is a subsequence of the \((j-1)\)th. Let \( \lambda_n \) denote the measure in \( C(K_n)' \) to which the sequence \( \{ \mu_{nk} \}_{k=1}^{\infty} \) converges weak*. Let \( \{ \mu_n \} \equiv \{ \mu_{nn} \} \), the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so \( \mu_n \) converges weak* in \( C(K_m)' \) to \( \lambda_m \) for each \( m \). Note that this is all happening on different sets so there is no contradiction with something converging to two different things.

**Claim:** For \( p > n \), the restriction of \( \lambda_p \) to the Borel sets of \( K_n \) equals \( \lambda_n \).

**Proof of claim:** Let \( H \) be a compact subset of \( K_n \). Then there are sets, \( V_l \) open in \( K_n \) which are decreasing and whose intersection equals \( H \). This follows because this is a metric space. Then let \( H \prec \phi_l \prec V_l \). It follows

\[
\lambda_n (V_l) \geq \int_{K_n} \phi_l d\lambda_n = \lim_{k \to \infty} \int_{K_n} \phi_l d\mu_k = \lim_{k \to \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \geq \lambda_p (H).
\]

Now considering the ends of this inequality, let \( l \to \infty \) and pass to the limit to conclude

\[
\lambda_n (H) \geq \lambda_p (H).
\]

Similarly,

\[
\lambda_n (H) \leq \int_{K_n} \phi_l d\lambda_n = \lim_{k \to \infty} \int_{K_n} \phi_l d\mu_k = \lim_{k \to \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \leq \lambda_p (V_l).
\]

Then passing to the limit as \( l \to \infty \), it follows

\[
\lambda_n (H) \leq \lambda_p (H).
\]

Thus the restriction of \( \lambda_p, \lambda_p|_{K_n} \) to the compact sets of \( K_n \) equals \( \lambda_n \). Then by inner regularity it follows the two measures, \( \lambda_p|_{K_n} \), and \( \lambda_n \) are equal on all Borel sets of \( K_n \). Recall that for finite measures on separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that \( \lambda_m (F) \geq 0 \) for all \( F \) a Borel subset of \( K_m \). Note that \( \phi \to \int_{K_n} \phi d\lambda_n \) is a positive linear functional
and so \( \lambda_n \geq 0 \). Also, letting \( \phi \equiv 1 \),

\[
1 \geq \lambda_m (K_m) \geq 1 - \frac{1}{m}.
\] (59.3.7)

Define for \( F \) a Borel set,

\[
\lambda (F) \equiv \lim_{n \to \infty} \lambda_n (F \cap K_n).
\]

The limit exists because the sequence on the right is increasing due to the above observation that \( \lambda_n = \lambda_m \) on the Borel subsets of \( K_m \) whenever \( n > m \). Thus for \( n > m \)

\[
\lambda_n (F \cap K_n) \geq \lambda_n (F \cap K_m) = \lambda_m (F \cap K_m).
\]

Now let \( \{ F_k \} \) be a sequence of disjoint Borel sets. Then

\[
\lambda (\cup_{k=1}^{\infty} F_k) \equiv \lim_{n \to \infty} \lambda_n (\cup_{k=1}^{\infty} F_k \cap K_n) = \lim_{n \to \infty} \lambda_n (\cup_{k=1}^{\infty} (F_k \cap K_n)) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \lambda_n (F_k \cap K_n) = \sum_{k=1}^{\infty} \lambda (F_k)
\]

the last equation holding by the monotone convergence theorem.

It remains to verify

\[
\lim_{k \to \infty} \int \phi d\mu_k = \int \phi d\lambda
\]

for every \( \phi \) bounded and continuous. This is where tightness is used again. Then as noted above,

\[
\lambda_n (K_n) = \lambda (K_n)
\]

because for \( p > n \), \( \lambda_p (K_n) = \lambda_n (K_n) \) and so letting \( p \to \infty \), the above is obtained.

Also, from 59.3.7

\[
\lambda (K_n^C) = \lim_{p \to \infty} \lambda_p (K_n^C \cap K_p)
\]

\[
\leq \lim \sup_{p \to \infty} (\lambda_p (K_p) - \lambda_p (K_n))
\]

\[
\leq \lim \sup_{p \to \infty} (\lambda_p (K_p) - \lambda_n (K_n))
\]

\[
\leq \lim \sup_{p \to \infty} \left( 1 - \left( 1 - \frac{1}{n} \right) \right) = \frac{1}{n}
\]

Suppose \( ||\phi||_{\infty} < M \). Then

\[
\left| \int \phi d\mu_k - \int \phi d\lambda \right| \leq \left| \int_{K_n^C} \phi d\mu_k + \int_{K_n} \phi d\mu_k - \left( \int_{K_n} \phi d\lambda + \int_{K_n^C} \phi d\lambda \right) \right|
\]

\[
\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda \right| + \left| \int_{K_n^C} \phi d\mu_k - \int_{K_n^C} \phi d\lambda \right|
\]
First let \( n \) be so large that \( 2M/n < \varepsilon/2 \) and then pick \( k \) large enough that the above expression is less than \( \varepsilon \).

**Definition 59.3.6** Let \( E \) be a complete separable metric space and let \( \mu \) and the sequence of probability measures, \( \{\mu_n\} \), defined on \( \mathcal{B}(E) \) satisfy

\[
\lim_{n \to \infty} \int \phi d\mu_n = \int \phi d\mu.
\]

for every \( \phi \) a bounded continuous function. Then \( \mu_n \) is said to converge weakly to \( \mu \).

### 59.4 A Major Existence And Convergence Theorem

Here is an interesting lemma about weak convergence.

**Lemma 59.4.1** Let \( \mu \) converge weakly to \( \mu \) and let \( U \) be an open set with \( \mu(\partial U) = 0 \). Then

\[
\lim_{n \to \infty} \mu_n(U) = \mu(U).
\]

**Proof:** Let \( \{\psi_k\} \) be a sequence of bounded continuous functions which decrease to \( \chi_U \). Also let \( \{\phi_k\} \) be a sequence of bounded continuous functions which increase to \( \chi_U \). For example, you could let

\[
\psi_k(x) = (1 - k \text{dist}(x, U))^+, \quad \phi_k(x) = 1 - (1 - k \text{dist}(x, U^C))^+.
\]

Let \( \varepsilon > 0 \) be given. Then since \( \mu(\partial U) = 0 \), the dominated convergence theorem implies there exists \( \psi = \psi_k \) and \( \phi = \phi_k \) such that

\[
\varepsilon > \int \psi d\mu - \int \phi d\mu.
\]

Next use the weak convergence to pick \( N \) large enough that if \( n \geq N \),

\[
\int \psi d\mu_n \leq \int \psi d\mu + \varepsilon, \quad \int \phi d\mu_n \geq \int \phi d\mu - \varepsilon.
\]

Therefore, for \( n \) this large,

\[
\mu(U), \mu_n(U) \in \left[ \int \phi d\mu - \varepsilon, \int \psi d\mu + \varepsilon \right]
\]
and so

$$|\mu(U) - \mu_n(U)| < 3\varepsilon.$$  

since $\varepsilon$ is arbitrary, this proves the lemma.

**Definition 59.4.2** Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to E$ be a random variable where here $E$ is some topological space. Then one can define a probability measure, $\lambda_X$ on $\mathcal{B}(E)$ as follows:

$$\lambda_X(F) \equiv P([X \in F]).$$

More generally, if $\mu$ is a probability measure on $\mathcal{B}(E)$, and $X$ is a random variable defined on a probability space, $\mathcal{L}(X) = \mu$ means

$$\mu(F) \equiv P([X \in F]).$$

The following amazing theorem is due to Skorokhod. It starts with a measure, $\mu$ on $\mathcal{B}(E)$ and produces a random variable, $X$ for which $\mathcal{L}(X) = \mu$. It also has something to say about the convergence of a sequence of such random variables.

**Theorem 59.4.3** Let $E$ be a separable complete metric space and let $\{\mu_n\}$ be a sequence of Borel probability measures defined on $\mathcal{B}(E)$ such that $\mu_n$ converges weakly to $\mu$ another probability measure on $\mathcal{B}(E)$. Then there exist random variables, $X_n, X$ defined on the probability space, $([0, 1), \mathcal{B}([0, 1]), m)$ where $m$ is one dimensional Lebesgue measure such that

$$\mathcal{L}(X) = \mu, \mathcal{L}(X_n) = \mu_n,$$

(59.4.8)

each random variable, $X, X_n$ is continuous off a set of measure zero, and

$$X_n(\omega) \to X(\omega) \text{ m a.e.}$$

**Proof:** Let $\{a_k\}$ be a countable dense subset of $E$.

**Construction of sets in $E$**

First I will describe a construction. Letting $C \in \mathcal{B}(E)$ and $r > 0$,

$$C^r_1 \equiv C \cap B(a_1, r), C^r_2 \equiv B(a_2, r) \cap C \setminus C^r_1, \ldots,$$

$$C^r_n \equiv B(a_n, r) \cap C \setminus \left(\bigcup_{k=1}^{n-1} C^r_k\right).$$

Thus the sets, $C^r_k$ for $k = 1, 2, \ldots$ are disjoint Borel sets whose union is all of $C$. Of course many may be empty.

$$C^r_n(\text{size}) \text{, } C^r_n(\text{index of the } \{a_k\} \text{ it is close to})$$

Now let $C = E$, the whole metric space. Also let $\{r_k\}$ be a decreasing sequence of positive numbers which converges to 0. Let

$$A_k \equiv E^r_k, \ k = 1, 2, \ldots$$
Thus \( \{A_k\} \) is a sequence of Borel sets, \( A_k \subseteq B(a_k, r_1) \), and the union of the \( A_k \) equals \( E \). For \( (i_1, \ldots, i_m) \in \mathbb{N}^m \), suppose \( A_{i_1, \ldots, i_m} \) has been defined. Then for \( k \in \mathbb{N} \),
\[
A_{i_1, \ldots, i_m k} \equiv (A_{i_1, \ldots, i_m})^{r_{m+1}}
\]
Thus \( A_{i_1, \ldots, i_m k} \subseteq B(a_k, r_{m+1}) \), is a Borel set, and
\[
\bigcup_{k=1}^{\infty} A_{i_1, \ldots, i_m k} = A_{i_1, \ldots, i_m}.
\] (59.4.9)

Also note that \( A_{i_1, \ldots, i_m} \) could be empty. This is because \( A_{i_1, \ldots, i_m k} \subseteq B(a_k, r_{m+1}) \) but \( A_{i_1, \ldots, i_m} \subseteq B(a_m, r_m) \) which might have empty intersection with \( B(a_k, r_{m+1}) \). However, applying \( 59.4.9 \) repeatedly,
\[
E = \bigcup_{i_1} \cdots \bigcup_{i_m} A_{i_1, \ldots, i_m}
\]
and also, the construction shows the Borel sets, \( A_{i_1, \ldots, i_m} \) are disjoint. Note that to get \( A_{i_1, \ldots, i_m k} \), you do to \( A_{i_1, \ldots, i_m} \) what was done for \( E \) but you consider smaller sized pieces.

Construction of intervals depending on the measure

Next I will construct intervals, \( I'_{i_1, \ldots, i_m} \) in \([0, 1)\) corresponding to these \( A_{i_1, \ldots, i_m} \).
In what follows, \( \nu = \mu_{\alpha} \) or \( \mu \). These intervals will depend on the measure chosen as indicated in the notation.
\[
I'_i = [0, \nu(A_1)], \cdots, I'_j = \left[ \sum_{k=1}^{j-1} \nu(A_k), \sum_{k=1}^{j} \nu(A_k) \right]
\]
for \( j = 1, 2, \cdots \). Note these are disjoint intervals whose union is \([0, 1)\). Also note
\[
m(I'_j) = \nu(A_j).
\]
The endpoints of these intervals as well as their lengths depend on the measures of the sets \( A_k \). Now supposing \( I'_{i_1, \ldots, i_m} = [\alpha, \beta] \) where \( \beta - \alpha = \nu(A_{i_1, \ldots, i_m}) \), define
\[
I'_{i_1, \ldots, i_m, j} = \left[ \alpha + \sum_{k=1}^{j-1} \nu(A_{i_1, \ldots, i_m, k}), \alpha + \sum_{k=1}^{j} \nu(A_{i_1, \ldots, i_m, k}) \right]
\]
Thus \( m(I'_{i_1, \ldots, i_m, j}) = \nu(A_{i_1, \ldots, i_m, j}) \) and
\[

\nu(A_{i_1, \ldots, i_m}) = \sum_{k=1}^{\infty} \nu(A_{i_1, \ldots, i_m, k}) = \sum_{k=1}^{\infty} m(I'_{i_1, \ldots, i_m, k}) = \beta - \alpha,
\]
the intervals, \( I'_{i_1, \ldots, i_m, j} \) being disjoint and
\[
I'_{i_1, \ldots, i_m} = \bigcup_{j=1}^{\infty} I'_{i_1, \ldots, i_m, j}.
\]
These intervals satisfy the same inclusion properties as the sets \( \{A_{i_1, \ldots, i_m}\} \). They are just on \([0, 1)\) rather than on \( E \). The intervals \( I'_{i_1, \ldots, i_m, k} \) correspond to the sets \( A_{i_1, \ldots, i_m, k} \) and in fact the Lebesgue measure of the interval is the same as \( \nu(A_{i_1, \ldots, i_m, k}) \).
Choosing the sequence \( \{r_k\} \) in an auspicious manner

There are at most countably many positive numbers, \( r \) such that for \( \nu = \mu_n \) or \( \mu, \nu (\partial B (a_i, r)) > 0 \). This is because \( \nu \) is a finite measure. Taking the countable union of these countable sets, there are only countably many \( r \) such that \( \nu (\partial B (a_i, r)) > 0 \) for some \( a_i \). Let the sequence avoid all these bad values of \( r \). Thus for

\[
F = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \partial B (a_k, r_m)
\]

and \( \nu = \mu \) or \( \mu_n, \nu (F) = 0 \). Here the \( r_m \) are all good values such that for all \( k, m, \partial B (a_k, r_m) \) has \( \mu \) measure zero and \( \mu_n \) measure zero. The next claim is illustrated in the following picture. In the picture, \( A \) represents one of those \( A_{i_1, \ldots, i_m} \) and \( A_1 \) and \( A_2 \) are two of the sets \( A_{i_1, \ldots, i_m k} \) which partition \( A_{i_1, \ldots, i_m} \).

Claim 1: \( \partial A_{i_1, \ldots, i_k} \subseteq F \). This really follows from the construction. However, the details follow.

Proof of claim: Suppose \( C \) is a Borel set for which \( \partial C \subseteq F \). I need to show \( \partial C_{i_k} \subseteq F \). First consider \( k = 1 \). Then \( C_{i_1} = B (a_1, r_1) \cap C \). If \( x \in \partial C_{i_1} \), then \( B (x, \delta) \) contains points of \( B (a_1, r_1) \cap C \) and points of \( B (a_1, r_1) \cap C \) for every \( \delta > 0 \). First suppose \( x \in B (a_1, r_1) \). Then a small enough neighborhood of \( x \) has no points of \( B (a_1, r_1) \cap C \) and so every \( B (x, \delta) \) has points of \( C \) and points of \( C \) so that \( x \in \partial C \subseteq F \) by assumption. If \( x \in \partial C_{i_1} \), then it can’t happen that \( ||x - a_1|| > r_1 \) because then there would be a neighborhood of \( x \) having no points of \( C_{i_1} \). The only other case to consider is that \( ||x - a_1|| = r_1 \) but this says \( x \in F \). Now assume \( \partial C_{i_1} \subseteq F \) for \( j \leq k - 1 \) and consider \( \partial C_{i_k} \).

\[
C_{i_k} = B (a_k, r_i) \cap C \setminus \bigcup_{j=1}^{k-1} C_{i_j}
= B (a_k, r_i) \cap C \cap \left( \bigcup_{j=1}^{k-1} \left( C_{i_j} \cap C \right) \right)
\]

Consider \( x \in \partial C_{i_k} \). If \( x \in \text{int} (B (a_k, r_i) \cap C) \) (\( \text{int} \equiv \text{interior} \)) then a small enough ball about \( x \) contains no points of \( (B (a_k, r_i) \cap C) \) and so every ball about \( x \) must contain points of

\[
\left( \bigcup_{j=1}^{k-1} \left( C_{i_j} \cap C \right) \right) = \bigcup_{j=1}^{k-1} C_{i_j}
\]

Since there are only finitely many sets in the union, there exists \( s \leq k - 1 \) such that every ball about \( x \) contains points of \( C_{i_s} \) but from the above, every ball about \( x \) contains points of \( (C_{i_s} \cap C) \) which implies \( x \in \partial C_{i_s} \subseteq F \) by induction. It is not

\[
\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \partial B (a_k, r_m)
\]
possible that \( d(x, a_k) > r_i \) and yet have \( x \) in \( \partial C^0_i \). This follows from the description in [10]. If \( d(x, a_k) = r_i \) then by definition, \( x \in F \). The only other case to consider is that \( x \notin \text{int} (B(a_k, r_i) \cap C) \) but \( x \in B(a_k, r_i) \). From [10], every ball about \( x \) contains points of \( C \). However, since \( x \in B(a_k, r_i) \), a small enough ball is contained in \( B(a_k, r_i) \). Therefore, every ball about \( x \) must also contain points of \( C^0 \) since otherwise, \( x \notin \text{int} (B(a_k, r_i) \cap C) \). Thus \( x \in \partial C \subseteq F \) by assumption. Now apply what was just shown to the case where \( C = E \), the whole space. In this case, \( \partial E \subseteq F \) because \( \partial E = \emptyset \). Then keep applying what was just shown to the \( A_{i_1, \ldots, i_n} \). This proves the claim.

From the claim, \( \nu (\text{int} (A_{i_1, \ldots, i_n})) = \nu (A_{i_1, \ldots, i_n}) \) whenever \( \nu = \mu \) or \( \mu_n \). This is because that in \( A_{i_1, \ldots, i_n} \) which is not in \( \text{int} (A_{i_1, \ldots, i_n}) \) is in \( F \) which has measure zero.

**Some functions on \([0, 1]\)**

By the axiom of choice, there exists \( x_{i_1, \ldots, i_m} \in \text{int} (A_{i_1, \ldots, i_m}) \) whenever

\[
\text{int} (A_{i_1, \ldots, i_n}) \neq \emptyset.
\]

For \( \nu = \mu_n \) or \( \mu \), define the following functions. For \( \omega \in I^\nu_{i_1, \ldots, i_m} \)

\[
Z^\nu_m (\omega) \equiv x_{i_1, \ldots, i_m}.
\]

This defines the functions, \( Z^\nu_n \) and \( Z^\mu_n \). Note these functions have the same values but on slightly different intervals. Here is an important claim.

**Claim 2 (Limit on \( \mu_n \))**: For a.e. \( \omega \in [0, 1] \), \( \lim_{n \to \infty} Z^\mu_n (\omega) = Z^\mu_m (\omega) \).

**Proof of the claim**: This follows from the weak convergence of \( \mu_n \) to \( \mu \) and Lemma [10]. This lemma implies \( \mu_n (\text{int} (A_{i_1, \ldots, i_m})) \to \mu (\text{int} (A_{i_1, \ldots, i_m})) \). Thus by the construction described above, \( \mu_n (A_{i_1, \ldots, i_m}) \to \mu (A_{i_1, \ldots, i_m}) \) because of claim 1 and the construction of \( F \) in which it is always a set of measure zero. It follows that if \( \omega \in \text{int} (I^\nu_{i_1, \ldots, i_m}) \), then for all \( n \) large enough, \( \omega \in \text{int} (I^\mu_n_{i_1, \ldots, i_m}) \) and so \( Z^\mu_n (\omega) = Z^\mu_m (\omega) \). Note this convergence is very far from being uniform.

**Claim 3 (Limit on size of sets, fixed measure)**: For \( \nu = \mu_n \) or \( \mu \), \( \{Z^\nu_m\}_{m=1}^{\infty} \) is uniformly Cauchy independent of \( \nu \).

**Proof of the claim**: For \( \omega \in I^\nu_{i_1, \ldots, i_m} \), then by the construction, \( \omega \in I^\nu_{i_1, \ldots, i_m, i_{m+1}, \ldots, i_n} \) for some \( i_{m+1}, \ldots, i_n \). Therefore, \( Z^\nu_m (\omega) \) and \( Z^\nu_n (\omega) \) are both contained in \( A_{i_1, \ldots, i_m} \) which is contained in \( B(a_{i_1, \ldots, i_m}) \). Since \( \omega \in [0, 1] \) was arbitrary, and \( r_m \to 0 \), it follows these functions are uniformly Cauchy as claimed.

Let \( X^\nu (\omega) = \lim_{m \to \infty} Z^\nu_m (\omega) \). Since each \( Z^\nu_m \) is continuous off a set of measure zero, it follows from the uniform convergence that \( X^\nu \) is also continuous off a set of measure zero.

**Claim 4**: For a.e. \( \omega \),

\[
\lim_{n \to \infty} X^{\mu_n} (\omega) = X^\mu (\omega).
\]

**Proof of the claim**: From Claim 3 and letting \( \varepsilon > 0 \) be given, there exists \( m \) large enough that for all \( n \),

\[
\sup_{\omega} d(Z^\nu_m (\omega), X^{\mu_n} (\omega)) < \varepsilon / 3, \ sup_{\omega} d(Z^\nu_m (\omega), X^\mu (\omega)) < \varepsilon / 3.
\]
for \( \omega \) off a set of measure zero. Now pick \( \omega \in [0, 1) \) such that \( \omega \) is not equal to any of the end points of any of the intervals, \( \{I_{i_1, \ldots, i_m}\} \), this countable set of endpoints, a set of Lebesgue measure zero. Then by Claim 2, there exists \( N \) such that if \( n \geq N \), then \( d \left( Z_m^{\mu_n} (\omega), Z_m^{\nu} (\omega) \right) < \varepsilon / 3 \). Therefore, for such \( n \) and this \( \omega \),

\[
d (X^n (\omega), X^\mu (\omega)) \leq d (X^n (\omega), Z_m^{\mu_n} (\omega)) + d (Z_m^{\mu_n} (\omega), Z_m^{\nu} (\omega)) \]

\[
\qquad + d (Z_m^{\nu} (\omega), X^\nu (\omega)) < \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon.
\]

This proves the claim.

**Showing \( \mathcal{L} (X^\nu) = \nu \).**

This has mostly proved the theorem except for the claim that \( \mathcal{L} (X^\nu) = \nu \) for \( \nu = \mu_n \) and \( \mu \). To do this, I will first show \( m \left( (X^\nu)^{-1} (\partial A_{i_1, \ldots, i_m}) \right) = 0 \). By the construction, \( \nu (\partial A_{i_1, \ldots, i_m}) = 0 \). Let \( \varepsilon > 0 \) be given and let \( \delta > 0 \) be small enough that

\[
H_\delta = \{ x \in E : \text{dist} (x, \partial A_{i_1, \ldots, i_m}) \leq \delta \}
\]

is a set of measure less than \( \varepsilon / 2 \). Denote by \( \mathcal{G}_k \) the sets of the form \( A_{i_1, \ldots, i_k} \) where \( (i_1, \ldots, i_k) \in \mathbb{N}^k \). Recall also that corresponding to \( A_{i_1, \ldots, i_k} \) is an interval, \( I^\nu_{i_1, \ldots, i_k} \) having length equal to \( \nu (A_{i_1, \ldots, i_k}) \). Denote by \( \mathcal{B}_k \) those sets of \( \mathcal{G}_k \) which have nonempty intersection with \( H_\delta \) and let the corresponding intervals be denoted by \( I^\nu_k \). If \( \omega \notin \bigcup I^\nu_k \), then from the construction, \( Z^\nu_k (\omega) \) is at a distance of at least \( \delta \) from \( \partial A_{i_1, \ldots, i_m} \) for all \( p \geq k \). (If \( Z^\nu_k (\omega) \) were in some set of \( \mathcal{B}_k \), this would require \( \omega \) to be in the corresponding \( I^\nu_k \) and it is assumed this does not happen. Then for any \( p > k \), \( Z^\nu_p (\omega) \) cannot be in any set of \( \mathcal{G}_p \) which intersects \( H_\delta \) either. If it did, you would need to have \( \omega \notin \bigcup I^\nu_p \) but all of these intervals are inside the intervals \( I^\nu_k \).) Passing to the limit as \( p \to \infty \), it follows \( X^\nu (\omega) \notin \partial A_{i_1, \ldots, i_m} \). Therefore,

\[
(X^\nu)^{-1} (\partial A_{i_1, \ldots, i_m}) \subseteq \bigcup I^\nu_k
\]

Recall that \( A_{i_1, \ldots, i_k} \subseteq B (a_{i_k}, r_k) \) and the \( r_k \to 0 \). Therefore, if \( k \) is large enough,

\[
\nu (\cup \mathcal{B}_k) < \varepsilon
\]

because \( \cup \mathcal{B}_k \) approximates \( H_\delta \) closely (In fact, \( \cap_{k=1}^\infty (\cup \mathcal{B}_k) = H_\delta \)). Therefore,

\[
m \left( (X^\nu)^{-1} (\partial A_{i_1, \ldots, i_m}) \right) \leq m (\cup I^\nu_k) = \sum_{I^\nu_{i_1, \ldots, i_k} \in \mathcal{I}^\nu_k} m (I^\nu_{i_1, \ldots, i_k}) = \sum_{A_{i_1, \ldots, i_k} \in \mathcal{B}_k} \nu (A_{i_1, \ldots, i_k}) = \nu (\cup \mathcal{B}_k) < \varepsilon.
\]
Since $\varepsilon > 0$ is arbitrary, this shows $m \left( (X^\nu)^{-1} (\partial A_{i_1, \ldots, i_m}) \right) = 0$.

If $\omega \in I_{i_1, \ldots, i_m}$, then from the construction, $Z^\nu_p(\omega) \in \text{int} (A_{i_1, \ldots, i_m})$ for all $p \geq k$. Therefore, taking a limit, as $p \to \infty$,

$$X^\nu(\omega) \in \text{int} (A_{i_1, \ldots, i_m}) \cup \partial A_{i_1, \ldots, i_m}$$

and so

$$I_{i_1, \ldots, i_m} \subseteq (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m}) \cup \partial A_{i_1, \ldots, i_m})$$

but also, if $X^\nu(\omega) \in \text{int} (A_{i_1, \ldots, i_m})$, then $Z^\nu_p(\omega) \in \text{int} (A_{i_1, \ldots, i_m})$ for all $p$ large enough and so

$$(X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \subseteq I_{i_1, \ldots, i_m} \subseteq (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m}) \cup \partial A_{i_1, \ldots, i_m})$$

Therefore,

$$m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right) \leq m \left( I_{i_1, \ldots, i_m} \right) \leq m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right) + m \left( (X^\nu)^{-1} (\partial A_{i_1, \ldots, i_m}) \right) = m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right)$$

which shows

$$m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right) = m \left( I_{i_1, \ldots, i_m} \right) = \nu (A_{i_1, \ldots, i_m}) \quad (59.4.11)$$

Also

$$m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right) \leq m \left( (X^\nu)^{-1} (A_{i_1, \ldots, i_m}) \right) \leq m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m}) \cup \partial A_{i_1, \ldots, i_m}) \right) = m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right)$$

Hence from (59.4.11)

$$\nu (A_{i_1, \ldots, i_m}) = m \left( (X^\nu)^{-1} (\text{int} (A_{i_1, \ldots, i_m})) \right) = m \left( (X^\nu)^{-1} (A_{i_1, \ldots, i_m}) \right) \quad (59.4.12)$$

Now let $U$ be an open set in $E$. Then letting

$$H_k = \{ x \in U : \text{dist} \left( x, U^C \right) \geq r_k \}$$
it follows
\[ \bigcup_k H_k = U. \]
Next consider the sets of \( G_k \) which have nonempty intersection with \( H_k, H_k \). Then \( H_k \) is covered by \( H_k \) and every set of \( H_k \) is contained in \( U \), the sets of \( H_k \) also being disjoint. Then from 59.4.12,
\[
m\left( (X')^{-1}(\bigcup H_k) \right) = \sum_{A \in H_k} m\left( (X')^{-1}(A) \right) = \sum_{A \in H_k} \nu(A) = \nu(\bigcup H_k).
\]
Therefore, letting \( k \to \infty \) and passing to the limit in the above,
\[
m\left( (X')^{-1}(U) \right) = \nu(U).
\]
Since this holds for every open set, it is routine to verify using regularity that it holds for every Borel set and so \( \mathcal{L}(X') = \nu \) as claimed.

59.5 Bochner’s Theorem In Infinite Dimensions

Let \( X \) be a real vector space and let \( X^* \) denote the space of real valued linear mappings defined on \( X \). Then you can consider each \( x \in X \) as a linear transformation defined on \( X^* \) by the convention \( x^* \mapsto x^*(x) \). Now let \( \Lambda \) be a Hamel basis. For a description of what one of these is, see Page 2787. It is just the usual notion of a basis. Thus every vector of \( X \) is a finite linear combination of vectors of \( \Lambda \) in a unique way.

Now consider \( \mathbb{R}^\Lambda \) the space of all mappings from \( \Lambda \) to \( \mathbb{R} \). In different notation, this is of the form
\[
\mathbb{R}^\Lambda = \prod_{y \in \Lambda} \mathbb{R}
\]
Since \( \Lambda \) is a Hamel basis, there exists a one to one and onto mapping, \( \theta : X^* \to \mathbb{R}^\Lambda \) defined as
\[
\theta(x^*) = \prod_{y \in \Lambda} x^*(y).
\]
Now denote by \( \sigma(X) \) the smallest \( \sigma \) algebra of sets of \( X^* \) such that each \( x \) is measurable with respect to this \( \sigma \) algebra. Thus
\[
\{ x^* : x^*(x) \in B \} \in \sigma(X)
\]
whenever \( B \) is a Borel set in \( \mathbb{R} \).

Let \( \mathcal{E} \) denote the algebra of disjoint unions of sets of \( \mathbb{R}^\Lambda \) of the form
\[
\prod_{y \in \Lambda} A_y
\]
where \( A_y = \mathbb{R} \) except for finitely many \( y \).
Lemma 59.5.1 Let $A$ denote sets of the form
\[
\{ x^* : \theta (x^*) \in U \}
\]
where $U \in \mathcal{E}$. Then $A$ is an algebra and $\sigma (A) = \sigma (X)$. Also
\[
\{ \theta^{-1}(U) : U \in \sigma (\mathcal{E}) \} = \sigma (X)
\]

Proof: Since $\mathcal{E}$ is an algebra it is clear $A$ is also an algebra. Also, $A \subseteq \sigma (X)$ because you could let $U$ have only one $A_y$ not equal to $\mathbb{R}$ and all the others equal to $\mathbb{R}$ and then
\[
\{ x^* : \theta (x^*) \in U \} = \{ x^* : y(x^*) \equiv x^*(y) \in A_y \} \in \sigma (X).
\]
Therefore, $\sigma (A) \subseteq \sigma (X)$. I need to verify that for an arbitrary $x$, it is measurable with respect to $\sigma (A)$. However, this is true because if $x$ is arbitrary, it is a linear combination of $\{ y_1, \cdots, y_n \}$, some finite set of functions in $\Lambda$ and so, $x$ being a linear combination of measurable functions implies it is itself measurable.

By definition, $\theta^{-1}(U)$ is in $A$ whenever $U \in \mathcal{E}$. Now let $G$ denote those sets, $U$ in $\sigma (\mathcal{E})$ such that $\theta^{-1}(U) \in \sigma (A)$. Then $G$ is a $\sigma$ algebra which contains $\mathcal{E}$ and so $G \supseteq \sigma (\mathcal{E}) \supseteq G$. This proves the last claim. This proves the lemma.

Definition 59.5.2 Let $\psi : X \to \mathbb{C}$. Then $\psi$ is said to be pseudo continuous if whenever $\{ x_1, \cdots, x_n \}$ is a finite subset of $X$ and $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$,
\[
a \to \psi \left( \sum_{k=1}^{n} a_k x_k \right)
\]
is continuous. $\psi$ is said to be positive definite if
\[
\sum_{j,k} \psi (x_k - x_j) \alpha_k \overline{\alpha_j} \geq 0
\]
$\psi$ is said to be a characteristic function if there exists a probability measure, $\mu$ defined on $\sigma (X)$ such that
\[
\psi (x) = \int_{X^*} e^{ix^*(x)} d\mu (x^*)
\]
Note that $x^* \to e^{ix^*(x)}$ is $\sigma (X)$ measurable.

Using Kolmogorov’s extension theorem on Page 1772, there exists a generalization of Bochner’s theorem found in [113]. For convenience, here is Kolmogorov’s theorem.

Theorem 59.5.3 (Kolmogorov extension theorem) For each finite set $J = (t_1, \cdots, t_n) \subseteq I$,
suppose there exists a Borel probability measure, \( \nu = \nu_{t_1 \cdots t_n} \) defined on the Borel sets of \( \prod_{t \in J} M_t \) for \( M_t = \mathbb{R}^{n_t} \) for \( n_t \) an integer, such that the following consistency condition holds. If 

\[
(t_1, \ldots, t_n) \subseteq (s_1, \ldots, s_p),
\]

then

\[
\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p} (G_{s_1} \times \cdots \times G_{s_p})
\]

(59.5.13)

where if \( s_i = t_j \), then \( G_{s_i} = F_{t_j} \) and if \( s_i \) is not equal to any of the indices, \( t_k \), then \( G_{s_i} = M_{s_i} \). Then for \( \mathcal{E} \) defined as in Definition 12.4.4, adjusted so that \( \pm \infty \) never appears as any endpoint of any interval, there exists a probability measure, \( P \) and a \( \sigma \)-algebra \( \mathcal{F} = \sigma (\mathcal{E}) \) such that

\[
\left( \prod_{t \in I} M_t, P, \mathcal{F} \right)
\]

is a probability space. Also there exist measurable functions, \( X_s : \prod_{t \in I} M_t \to M_s \) defined as

\[
X_s x \equiv x_s
\]

for each \( s \in I \) such that for each \( (t_1 \cdots t_n) \subseteq I, \)

\[
\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = P ([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}])
\]

\[
= P \left( (X_{t_1}, \ldots, X_{t_n}) \in \prod_{j=1}^{n} F_{t_j} \right) = P \left( \prod_{t \in I} F_t \right)
\]

(59.5.14)

where \( F_t = M_t \) for every \( t \not\in \{t_1 \cdots t_n\} \) and \( F_t \) is a Borel set. Also if \( f \) is a non-negative function of finitely many variables, \( x_{t_1}, \ldots, x_{t_n}, \) measurable with respect to \( B \left( \prod_{j=1}^{n} M_{t_j} \right) \), then \( f \) is also measurable with respect to \( \mathcal{F} \) and

\[
\int_{M_{t_1} \times \cdots \times M_{t_n}} f (x_{t_1}, \ldots, x_{t_n}) \, d\nu_{t_1 \cdots t_n}
\]

\[
= \int_{\prod_{t \in I} M_t} f (x_{t_1}, \ldots, x_{t_n}) \, dP
\]

(59.5.15)

**Theorem 59.5.4** Let \( X \) be a real vector space and let \( X^* \) be the space of linear functionals defined on \( X \). Also let \( \psi : X \to \mathbb{C} \). Then \( \psi \) is a characteristic function if and only if \( \psi (0) = 1 \) and \( \psi \) is pseudo continuous at 0.

**Proof:** Suppose first \( \psi \) is a characteristic function as just described. I need to show it is positive definite and pseudo continuous. It is obvious \( \psi (0) = 1 \) in this case. Also

\[
\psi \left( \sum_k a_k x_k \right) = \int_{X^*} \exp \left( i x^* \left( \sum_k a_k x_k \right) \right) \, d\mu (x^*)
\]
and this is obviously a continuous function of \( a \) by the dominated convergence theorem. It only remains to verify the function is positive definite. However,

$$
\sum_{k,j} \exp (ix^* (x_k - x_j)) \alpha_k \alpha_j = \sum_{k,j} e^{ix^* (x_k)} \alpha_k e^{ix^* (x_j)} \alpha_j \geq 0
$$

as in the earlier discussion of what it means to be positive definite given on Page 1988.

Next suppose the conditions hold. Define for \( t \in \mathbb{R}^n \) and \( \{y_1, \cdots, y_n\} \subseteq \Lambda \)

$$
\psi_{\{y_1, \cdots, y_n\}} (t) \equiv \psi \left( \sum_{j=1}^n t_j y_j \right).
$$

Then \( \psi_{\{y_1, \cdots, y_n\}} \) is continuous at 0, equals 1 there, and is positive definite. It follows from Bochner’s theorem, Theorem 57.21 on Page 1992 there exists a measure \( \mu_{\{y_1, \cdots, y_n\}} \) defined on the Borel sets of \( \mathbb{R}^n \) such that

$$
\psi \left( \sum_{j=1}^n t_j y_j \right) = \psi_{\{y_1, \cdots, y_n\}} (t) = \int_{\mathbb{R}^n} e^{it^* x} d\mu_{\{y_1, \cdots, y_n\}} (x). \quad (59.5.16)
$$

Thus if \( \{y_1, \cdots, y_n, y_{n+1}, \cdots, y_p\} \subseteq \Lambda \),

$$
\psi \left( \sum_{j=1}^n t_j y_j + \sum_{j=1}^{p-n} s_j y_{j+n} \right) = \psi_{\{y_1, \cdots, y_p\}} (t, s)
= \int_{\mathbb{R}^n} e^{is^* x} \int_{\mathbb{R}^n} e^{it^* x} d\mu_{\{y_1, \cdots, y_p\}} (x)
$$

I need to verify the measures are consistent to use Kolmogorov’s theorem. Specifically, I need to show

$$
\mu_{\{y_1, \cdots, y_p\}} (A \times \mathbb{R}^{p-n}) = \mu_{\{y_1, \cdots, y_n\}} (A).
$$

Letting

$$
\lambda (A) = \mu_{\{y_1, \cdots, y_p\}} (A \times \mathbb{R}^{p-n})
$$
it follows
\[
\int_{\mathbb{R}^n} e^{it \cdot x} d\lambda = \int_{\mathbb{R}^p-n} \int_{\mathbb{R}^n} e^{it \cdot x} d\mu_{\{y_1, \cdots, y_p\}}(x)
\]
\[
= \int_{\mathbb{R}^p-n} e^{i0 \cdot x} \int_{\mathbb{R}^n} e^{it \cdot x} d\mu_{\{y_1, \cdots, y_p\}}(x)
\]
\[
= \psi \left( \sum_{j=1}^{n} t_j y_j + \sum_{j=1}^{p-n} 0 y_{j+n} \right)
\]
\[
= \psi \left( \sum_{j=1}^{n} t_j y_j \right)
\]
\[
= \int_{\mathbb{R}^n} e^{it \cdot x} d\mu_{\{y_1, \cdots, y_n\}}(x)
\]
and so, by uniqueness of characteristic functions,
\[
\lambda = \mu_{\{y_1, \cdots, y_n\}}
\]
which verifies the necessary consistency condition for Kolmogorov’s theorem.

It follows there exists a probability measure \( \mu \) defined on \( \sigma(\mathcal{E}) \) and random variables, \( X_y \) for each \( y \in \Lambda \) such that whenever \( \{y_1, \cdots, y_p\} \subseteq \Lambda \),
\[
\mu_{\{y_1, \cdots, y_p\}}(A_{y_1} \times \cdots \times A_{y_n}) = \mu \left( \prod_{y \in \Lambda} A_y \right)
\]
where \( A_y = \mathbb{R} \) whenever \( y \notin \{y_1, \cdots, y_p\} \). This defines a measure on \( \sigma(\mathcal{E}) \) which consists of sets of \( \mathbb{R}^\Lambda \).

By Lemma 59.5.1 it follows \( \{ \theta^{-1}(U) : U \in \sigma(\mathcal{E}) \} = \sigma(\mathcal{A}) \) which equals \( \sigma(X) \). Define \( \nu \) on \( \sigma(X) \) by
\[
\nu(F) = \mu(\theta F).
\]
Thus \( \nu \) is a measure because \( \mu \) is and \( \theta \) is one to one.

I need to check whether \( \nu \) works. Let \( x = \sum_{k=1}^{m} t_k y_k \) and let a typical element of \( \mathbb{R}^\Lambda \) be denoted by \( z \). Then by Kolmogorov’s theorem above,
\[
\int_{X^n} \exp \left( ix^* \left( \sum_{k=1}^{m} t_k y_k \right) \right) d\nu = \int_{X^n} \exp \left( i \left( \sum_{k=1}^{m} t_k x^* (y_k) \right) \right) d\nu
\]
\[
= \int_{X^n} \exp \left( i \left( \sum_{k=1}^{m} t_k y_k (\theta x^*) \right) \right) d\nu = \int_{\mathbb{R}^\Lambda} \exp \left( i \left( \sum_{k=1}^{m} t_k y_k z \right) \right) d\mu
\]
\[
= \int_{\mathbb{R}^m} \exp \left( i (t \cdot x) \right) d\mu_{\{y_1, \cdots, y_m\}}(x) = \psi \left( \sum_{k=1}^{m} t_k y_k \right)
\]
where the last equality comes from \[59.5.16\]. Since \( \Lambda \) is a Hamel basis, it follows that for every \( x \in X \),

\[
\psi(x) = \int_{X^*} \exp (ix^*(x)) \, dv
\]

This proves the theorem.

### 59.6 The Multivariate Normal Distribution

Here I give a review of the main theorems and definitions about multivariate normal random variables. Recall that for a random vector (variable), \( X \) having values in \( \mathbb{R}^p \), \( \lambda_X \) is the law of \( X \) defined by

\[
P(|X \in E|) = \lambda_X(E)
\]

for all \( E \) a Borel set in \( \mathbb{R}^p \). In different notation, \( L(X) = \lambda_X \). Then the following definitions and theorems are proved and presented starting on Page 1958.

**Definition 59.6.1** A random vector, \( X \), with values in \( \mathbb{R}^p \) has a multivariate normal distribution written as \( X \sim N_p(\mu, \Sigma) \) if for all Borel \( E \subseteq \mathbb{R}^p \),

\[
\lambda_X(E) = \int_{\mathbb{R}^p} \chi_E(x) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2} (x-m)^* \Sigma^{-1} (x-m)} \, dx
\]

for \( \mu \) a given vector and \( \Sigma \) a given positive definite symmetric matrix.

**Theorem 59.6.2** For \( X \sim N_p(\mu, \Sigma) \), \( \mu = E(X) \) and

\[
\Sigma = E((X - \mu)(X - \mu)^*)
\]

**Theorem 59.6.3** Suppose \( X_1 \sim N_p(m_1, \Sigma_1) \), \( X_2 \sim N_p(m_2, \Sigma_2) \) and the two random vectors are independent. Then

\[
X_1 + X_2 \sim N_p(m_1 + m_2, \Sigma_1 + \Sigma_2)
\]

(59.6.17)

Also, if \( X \sim N_p(\mu, \Sigma) \) then \(-X \sim N_p(-\mu, \Sigma)\). Furthermore, if \( X \sim N_p(\mu, \Sigma) \) then

\[
E(e^{i t \cdot X}) = e^{i t \cdot \mu e^{-\frac{1}{2} t^* \Sigma t}}
\]

(59.6.18)

Also if \( a \) is a constant and \( X \sim N_p(\mu, \Sigma) \) then \( aX \sim N_p(a\mu, a^2 \Sigma) \).

Following [94] a random vector has a generalized normal distribution if its characteristic function is given as

\[
e^{i t \cdot \mu} e^{-\frac{1}{2} t^* \Sigma t}
\]

(59.6.19)

where \( \Sigma \) is symmetric and has nonnegative eigenvalues. For a random real valued variable, \( \mu \) is scalar and so is \( \Sigma \) so the characteristic function of such a generalized normally distributed random variable is

\[
e^{i t \mu} e^{-\frac{1}{2} t^2 \sigma^2}
\]

(59.6.20)
These generalized normal distributions do not require $\Sigma$ to be invertible, only that the eigenvalues be nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at $m$. In higher dimensions, it could be a mixture of such things with more familiar things. I will often not bother to distinguish between generalized normal and normal distributions.

Here are some other interesting results about normal distributions found in \[94\].

The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense. It is proved on Page \[1961\].

**Theorem 59.6.4** Let $X = (X_1, \cdots, X_p)$ where each $X_i$ is a real valued random variable. Then $X$ is normally distributed in the above generalized sense if and only if every linear combination, $\sum_{j=1}^{p} a_i X_i$ is normally distributed. In this case the mean of $X$ is

$$m = (E(X_1), \cdots, E(X_p))$$

and the covariance matrix for $X$ is

$$\Sigma_{jk} = E((X_j - m_j)(X_k - m_k))$$

where $m_j = E(X_j)$.

Also proved there is the interesting corollary listed next.

**Corollary 59.6.5** Let $X = (X_1, \cdots, X_p), Y = (Y_1, \cdots, Y_p)$ where each $X_i, Y_i$ is a real valued random variable. Suppose also that for every $a \in \mathbb{R}^p$, $a \cdot X$ and $a \cdot Y$ are both normally distributed with the same mean and variance. Then $X$ and $Y$ are both multivariate normal random vectors with the same mean and variance.

**Theorem 59.6.6** Suppose $X = (X_1, \cdots, X_p)$ is normally distributed with mean $m$ and covariance $\Sigma$. Then if $X_1$ is uncorrelated with any of the $X_i$, $E((X_1 - m_1)(X_j - m_j)) = 0$ for $j > 1$ then $X_1$ and $(X_2, \cdots, X_p)$ are both normally distributed and the two random vectors are independent. Here $m_j = E(X_j)$.

Next I will consider the question of existence of independent random variables having a given law.

**Lemma 59.6.7** Let $\mu$ be a probability measure on $\mathcal{B}(E)$, the Borel subsets of a separable real Banach space. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and two independent random variables, $X, Y$ mapping $\Omega$ to $E$ such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$.

**Proof:** First note that if $A, B$ are Borel sets of $E$ then $A \times B$ is a Borel set in $E \times E$ where the norm on $E \times E$ is given by

$$||(x, y)|| = \max(||x||, ||y||).$$
This can be proved by letting \( A \) be open and considering
\[ G \equiv \{ B \in \mathcal{B}(\mathcal{E}) : A \times B \in \mathcal{B}(A \times B) \}. \]
Show \( G \) is a \( \sigma \)-algebra and it contains the open sets. Therefore, this will show \( A \times B \) is in \( \mathcal{B}(A \times B) \) whenever \( A \) is open and \( B \) is Borel. Next repeat a similar argument to show that this is true whenever either set is Borel. Since \( \mathcal{E} \) is separable, it is completely separable and so is \( \mathcal{E} \times \mathcal{E} \). Thus every open set in \( \mathcal{E} \times \mathcal{E} \) is the union of balls from a countable set. However, these balls are of the form \( B_1 \times B_2 \) where \( B_i \) is a ball in \( \mathcal{E} \).

Now let
\[ K \equiv \{ A \times B : A, B \text{ are Borel} \} \]
Then \( K \subseteq \mathcal{B}(\mathcal{E} \times \mathcal{E}) \) as was just shown and also every open set from \( \mathcal{E} \times \mathcal{E} \) is in \( \sigma(K) \). It follows \( \sigma(K) \) equals the \( \sigma \)-algebra of product measurable sets, \( \mathcal{B}(\mathcal{E}) \times \mathcal{B}(\mathcal{E}) \) and you can consider the product measure, \( \mu \times \mu \). By Skorokhod’s theorem, Theorem 59.4.3, there exists \( (X,Y) \) a random variable with values in \( \mathcal{E} \times \mathcal{E} \) and a probability space, \( (\Omega, \mathcal{F}, \mathcal{P}) \) such that \( L((X,Y)) = \mu \times \mu \).

Then
\[ P(X \in A, Y \in B) = (\mu \times \mu)(A \times B) = \mu(A) \mu(B). \]
Also, \( P(X \in A) = P(X \in A, Y \in E) = \mu(A) \) and similarly, \( P(Y \in B) = \mu(B) \) showing \( L(X) = L(Y) = \mu \) and \( X, Y \) are independent.

Now here is an interesting theorem in [34].

**Theorem 59.6.8** Suppose \( \nu \) is a probability measure on the Borel sets of \( \mathbb{R} \) and suppose that \( \xi \) and \( \zeta \) are independent random variables such that \( L(\xi) = L(\zeta) = \nu \) and whenever \( \alpha^2 + \beta^2 = 1 \) it follows \( L(\alpha \xi + \beta \zeta) = \nu \). Then
\[ L(\xi) = N(0, \sigma^2) \]
for some \( \sigma \geq 0 \). Also if \( L(\xi) = L(\zeta) = N(0, \sigma^2) \) where \( \xi, \zeta \) are independent, then if \( \alpha^2 + \beta^2 = 1 \), it follows \( L(\alpha \xi + \beta \zeta) = N(0, \sigma^2) \).

**Proof:** Let \( \xi, \zeta \) be independent random variables with \( L(\xi) = L(\zeta) = \nu \) and whenever \( \alpha^2 + \beta^2 = 1 \) it follows \( L(\alpha \xi + \beta \zeta) = \nu \).

By independence of \( \xi \) and \( \zeta \),
\[ \phi(\nu)(t) \equiv \phi_{\alpha \xi + \beta \zeta}(t) \equiv E(e^{it(\alpha \xi + \beta \zeta)}) = E(E(e^{it\alpha \xi}) E(e^{it\beta \zeta})) = \phi(\alpha)(\nu) \phi(\beta)(\nu) \]
In simpler terms and suppressing the subscript,
\[ \phi(t) = \phi(\cos(\theta) t) \phi(\sin(\theta) t). \]
59.6. THE MULTIVARIATE NORMAL DISTRIBUTION

Since \( \nu \) is a probability measure, \( \phi(0) = 1 \). Also, letting \( \theta = \pi/4 \), this yields

\[
\phi(t) = \phi \left( \frac{\sqrt{2}}{2} t \right)^2
\]

(59.6.22)

and so if \( \phi \) has real values, then \( \phi(t) \geq 0 \).

Next I will show \( \phi \) is real. To do this, it follows from the definition of \( \phi_\nu \),

\[
\phi_\nu(-t) = \int_{\mathbb{R}} e^{-itx} \, d\nu = \int_{\mathbb{R}} e^{itx} \, d\nu = \overline{\phi_\nu(t)}.
\]

Then letting \( \theta = \pi \),

\[
\phi(t) = \phi(-t) \cdot \phi(0) = \phi(-t) = \overline{\phi(t)}
\]

showing \( \phi \) has real values. It is positive near 0 because \( \phi(0) = 1 \) and \( \phi \) is a continuous function of \( t \) thanks to the dominated convergence theorem. However, this and 59.6.22 implies it is positive everywhere. Here is why. If not, let \( t_m \) be the smallest positive value of \( t \) where \( \phi(t) = 0 \). Then \( t_m > 0 \) by continuity. Now from 59.6.22, an immediate contradiction results. Therefore, \( \phi(t) > 0 \) for all \( t > 0 \).

Similar reasoning yields the same conclusion for \( t < 0 \).

Next note that \( \phi(t) = \phi(-t) \) also implies \( \phi \) depends only on \( |t| \) because it takes the same value for \( t \) as for \( -t \). More simply, \( \phi \) depends only on \( t^2 \). Thus one can define a new function of the form \( \phi(t) = f(t^2) \) and 59.6.21 implies the following for \( \alpha \in [0,1] \).

\[
f(t^2) = f(\alpha^2 t^2) f\left((1 - \alpha^2) t^2\right).
\]

Taking ln of both sides, one obtains the following.

\[
\ln f(t^2) = \ln f(\alpha^2 t^2) + \ln f\left((1 - \alpha^2) t^2\right).
\]

Now let \( x, y \geq 0 \). Then choose \( t \) such that \( t^2 = x + y \). Then for some \( \alpha \in [0,1] \), \( x = \alpha^2 t^2 \) and so \( y = t^2 (1 - \alpha^2) \). Thus for \( x, y \geq 0 \),

\[
\ln f(x + y) = \ln f(x) + \ln f(y).
\]

Hence \( \ln f(x) = kx \) and so \( \ln f(t^2) = kt^2 \) and so \( \phi(t) = f(t^2) = e^{kt^2} \) for all \( t \).

The constant, \( k \) must be nonpositive because \( \phi(t) \) is bounded due to its definition. Therefore, the characteristic function of \( \nu \) is

\[
\phi_\nu(t) = e^{-\frac{1}{2} t^2 \sigma^2}
\]

for some \( \sigma \geq 0 \). That is, \( \nu \) is the law of a generalized normal random variable.

Note the other direction of the implication is obvious. If \( \xi, \zeta \sim N(0, \sigma) \) and they are independent, then if \( \alpha^2 + \beta^2 = 1 \), it follows

\[
\alpha \xi + \beta \zeta \sim N(0, \sigma^2)
\]
because
\[
E \left( e^{it(a\xi + \beta\zeta)} \right) = E \left( e^{ita\xi} \right) E \left( e^{it\beta\zeta} \right) \\
= e^{-\frac{1}{2}(at)^2\sigma^2} e^{-\frac{1}{2}(bt)^2\sigma^2} \\
= e^{-\frac{1}{2}t^2\sigma^2},
\]
the characteristic function for a random variable which is \( N(0, \sigma) \). This proves the theorem.

The next theorem is a useful gimmick for showing certain random variables are independent in the context of normal distributions.

**Theorem 59.6.9** Let \( \mathbf{X} \) and \( \mathbf{Y} \) be random vectors having values in \( \mathbb{R}^p \) and \( \mathbb{R}^q \) respectively. Suppose also that \( (\mathbf{X}, \mathbf{Y}) \) is multivariate normally distributed and
\[
E \left( (\mathbf{X} - E(\mathbf{X})) (\mathbf{Y} - E(\mathbf{Y}))^* \right) = 0.
\]
Then \( \mathbf{X} \) and \( \mathbf{Y} \) are independent random vectors.

**Proof:** Let \( \mathbf{Z} = (\mathbf{X}, \mathbf{Y}) \), \( m = p + q \). Then by hypothesis, the characteristic function of \( \mathbf{Z} \) is of the form
\[
E \left( e^{it\cdot\mathbf{Z}} \right) = e^{it\cdot m} e^{-\frac{1}{2} t^* \Sigma t}
\]
where \( m = (\mathbf{m}_\mathbf{X}, \mathbf{m}_\mathbf{Y}) = E(\mathbf{Z}) = E(\mathbf{X}, \mathbf{Y}) \) and
\[
\Sigma = \begin{pmatrix}
E \left( (\mathbf{X} - E(\mathbf{X})) (\mathbf{X} - E(\mathbf{X}))^* \right) & 0 \\
0 & E \left( (\mathbf{Y} - E(\mathbf{Y})) (\mathbf{Y} - E(\mathbf{Y}))^* \right)
\end{pmatrix} = \begin{pmatrix}
\Sigma_\mathbf{X} & 0 \\
0 & \Sigma_\mathbf{Y}
\end{pmatrix}.
\]
Therefore, letting \( t = (u, v) \) where \( u \in \mathbb{R}^p \) and \( v \in \mathbb{R}^q \)
\[
E \left( e^{it\cdot\mathbf{Z}} \right) = E \left( e^{i(u, v) \cdot (\mathbf{X}, \mathbf{Y})} \right) = E \left( e^{iu\mathbf{X} + iv\mathbf{Y}} \right) \\
= e^{iu \mathbf{m}_\mathbf{X}} e^{-\frac{1}{2} u^* \Sigma_\mathbf{X} u} e^{iv \mathbf{m}_\mathbf{Y}} e^{-\frac{1}{2} v^* \Sigma_\mathbf{Y} v} \\
= E \left( e^{iu\mathbf{X}} \right) E \left( e^{iv\mathbf{Y}} \right).
\]
(59.6.23)

Where the last equality needs to be justified. When this is done it will follow from Proposition 57.11.1 on Page 1938 which is proved on Page 1938 that \( \mathbf{X} \) and \( \mathbf{Y} \) are independent. Thus all that remains is to verify
\[
E \left( e^{iu\mathbf{X}} \right) = e^{iu \mathbf{m}_\mathbf{X}} e^{-\frac{1}{2} u^* \Sigma_\mathbf{X} u}, \quad E \left( e^{iv\mathbf{Y}} \right) = e^{iv \mathbf{m}_\mathbf{Y}} e^{-\frac{1}{2} v^* \Sigma_\mathbf{Y} v}.
\]
However, this follows from 59.6.23. To get the first formula, let \( v = 0 \). To get the second, let \( u = 0 \). This proves the Theorem.

Note that to verify the conclusion of this theorem, it suffices to show
\[
E \left( X_i - E(X_i) (Y_j - E(Y_j)) \right) = 0.
\]
59.7 Gaussian Measures

59.7.1 Definitions And Basic Properties

First suppose $X$ is a random vector having values in $\mathbb{R}^n$ and its distribution function is $N(\mu, \Sigma)$ where $\mu$ is the mean and $\Sigma$ is the covariance. Then the characteristic function of $X$ or equivalently, the characteristic function of its distribution is

$$e^{it \cdot \mu} e^{-\frac{1}{2}t^* \Sigma t}$$

What is the distribution of $a \cdot X$ where $a \in \mathbb{R}^n$? In other words, if you take a linear functional and do it to $X$ to get a scalar valued random variable, what is the distribution of this scalar valued random variable? Let $Y = a \cdot X$. Then

$$E(e^{itY}) = E(e^{itaX})$$

which from the above formula is

$$e^{ina \cdot \mu} e^{-\frac{1}{2}a^* \Sigma a t^2}$$

which is the characteristic function of a random variable whose distribution is $N(a \cdot \mu, a^* \Sigma a)$. In other words, it is normally distributed having mean equal to $a \cdot \mu$ and variance equal to $a^* \Sigma a$. Obviously such a concept generalizes to a Banach space in place of $\mathbb{R}^n$ and this motivates the following definition.

**Definition 59.7.1** Let $E$ be a real separable Banach space. A probability measure, $\mu$ defined on $B(E)$ is called a Gaussian measure if for every $h \in E'$, the law of $h$ considered as a random variable defined on the probability space, $(E, B(E), \mu)$ is normal. That is, for $A \subseteq \mathbb{R}$ a Borel set,

$$\lambda_h(A) \equiv \mu(h^{-1}(A))$$

is given by

$$\int_A \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x-m)^2} dx$$

for some $\sigma$ and $m$. A Gaussian measure is called symmetric if $m$ is always equal to 0.

There is another definition of symmetric. First here are a few simple conventions. For $f \in E'$, $x \rightarrow f(x)$ is normally distributed. In particular,

$$\int_E |f(x)| d\mu < \infty$$

and so it makes sense to define

$$m_{\mu}(f) \equiv \int_E f(x) d\mu.$$
Thus $m_\mu (f)$ is the mean of the random variable $x \to f(x)$. It is obvious that $f \to m_\mu (f)$ is linear. Also define the variance $\sigma^2 (f)$ by

$$\sigma^2 (f) \equiv \int_E (f(x) - m_\mu (f))^2 d\mu$$

This is finite because $x \to f(x)$ is normally distributed. The following lemma gives such an equivalent condition for $\mu$ to be symmetric.

**Lemma 59.7.2** Let $\mu$ be a Gaussian measure defined on $B(E)$. Then $\mu (F) = \mu (-F)$ for all $F$ Borel if and only if $m_\mu (f) = 0$ for all $f \in E'$. Such a Gaussian measure is called symmetric.

**Proof:** Suppose first $m_\mu (f) = 0$ for all $f \in E'$. Let

$$G \equiv f_1^{-1} (F_1) \cap f_2^{-1} (F_2) \cap \cdots \cap f_m^{-1} (F_m)$$

where $F_i$ is a Borel set of $\mathbb{R}$ and each $f_i \in E'$. Since every linear combination of the $f_i$ is in $E'$, every such linear combination is normally distributed and so $f \equiv (f_1, \ldots, f_m)$ is multivariate normal. That is, $\lambda_f$ the distribution measure, is multivariate normal. Since each $m_\mu (f) = 0$, it follows

$$\mu (G) = \lambda_f \left( \prod_{i=1}^m F_i \right) = \lambda_f \left( \prod_{i=1}^m -F_i \right) = \mu (-G) \quad (59.7.24)$$

By Lemma 19.1.4 on Page 629 there exists a countable subset, $D \equiv \{ f_k \}_{k=1}^\infty$ of the closed unit ball such that for every $x \in E$,

$$||x|| = \sup_{f \in D} |f(x)|.$$ 

Therefore, letting $D(a, r)$ denote the closed ball centered at $a$ having radius $r$, it follows

$$D(a, r) = \cap_{k=1}^\infty f_k^{-1} (D(f_k(a), r))$$

Let

$$D_n(a, r) = \cap_{k=1}^n f_k^{-1} (D(f_k(a), r))$$

Then by 59.7.4

$$\mu (D_n(a, r)) = \mu (-D_n(a, r))$$

and letting $n \to \infty$, it follows

$$\mu (D(a, r)) = \mu (-D(a, r))$$

Therefore the same is true with $D(a, r)$ replaced with an open ball. Now consider

$$D(a, r_1) \cap D(b, r_2) = \cap_{k=1}^\infty f_k^{-1} (D(f_k(a), r_1)) \cap \cap_{k=1}^\infty f_k^{-1} (D(f_k(b), r_2))$$
The intersection of these two closed balls is the intersection of sets of the form
\[
\bigcap_{k=1}^{n} f_k^{-1} (D (f_k (a), r_1)) \cap \bigcap_{k=1}^{n} f_k^{-1} (D (f_k (b), r_2))
\]
to which \(59.7.24\) applies. Therefore, by continuing this way it follows that if \(G\) is any finite intersection of closed balls,
\[
\mu (G) = \mu (-G).
\]
Let \(K\) denote the set of finite intersections of closed balls, a \(\pi\) system. Thus for \(G \in K\) the above holds. Now let
\[
G \equiv \{ F \in \sigma (K) : \mu (F) = \mu (-F) \}
\]
Thus \(G\) contains \(K\) and it is clearly closed with respect to complements and countable disjoint unions. By the \(\pi\) system lemma, \(G \supseteq \sigma (K)\) but \(\sigma (K)\) clearly contains the open sets since every open ball is the countable union of closed disks and every open set is the countable union of open balls. Therefore, \(\mu (G) = \mu (-G)\) for all Borel \(G\).

Conversely suppose \(\mu (G) = \mu (-G)\) for all \(G\) Borel. If for some \(f \in E', m_{\mu} (f) \neq 0\), then
\[
\mu \left( f^{-1} (0, \infty) \right) = \lambda_f (0, \infty) \neq \lambda_f (-\infty, 0) = \mu \left( f^{-1} (-\infty, 0) \right) = \mu \left( f^{-1} (0, \infty) \right)
\]
a contradiction. This proves the lemma.

**Lemma 59.7.3** Let \(\mu = \mathcal{L} (X)\) where \(X\) is a random variable defined on a probability space, \((\Omega, F, P)\) which has values in \(E\), a Banach space. Suppose also that for all \(\phi \in E', \phi \circ X\) is normally distributed. Then \(\mu\) is a Gaussian measure. Conversely, suppose \(\mu\) is a Gaussian measure on \(\mathcal{B} (E)\) and \(X\) is a random variable having values in \(E\) such that \(\mathcal{L} (X) = \mu\). Then for every \(h \in E', h \circ X\) is normally distributed.

**Proof:** First suppose \(\mu\) is a Gaussian measure and \(X\) is a random variable such that \(\mathcal{L} (X) = \mu\). Then if \(F\) is a Borel set in \(\mathbb{R}\), and \(h \in E'\)
\[
P \left( (h \circ X)^{-1} (F) \right) = P \left( X^{-1} (h^{-1} (F)) \right)
\]
\[
= \mu (h^{-1} (F))
\]
\[
= \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} \, dx
\]
for some \(m\) and \(\sigma^2\) showing that \(h \circ X\) is normally distributed.

Next suppose \(h \circ X\) is normally distributed whenever \(h \in E'\) and \(\mathcal{L} (X) = \mu\). Then letting \(F\) be a Borel set in \(\mathbb{R}\), I need to verify
\[
\mu (h^{-1} (F)) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} \, dx.
\]
However, this is easy because
\[
\mu (h^{-1} (F)) = P \left( X^{-1} (h^{-1} (F)) \right) = P \left( (h \circ X)^{-1} (F) \right)
\]
which is given to equal
\[
\frac{1}{\sqrt{2\pi \sigma}} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx
\]
for some \( m \) and \( \sigma^2 \). This proves the lemma.

Here is another important observation. Suppose \( X \) is as just described, a random variable having values in \( E \) such that \( \mathcal{L} (X) = \mu \) and suppose \( h_1, \ldots, h_n \) are each in \( E' \).

Then for scalars, \( t_1, \ldots, t_n \),
\[
t_1 h_1 \circ X + \cdots + t_n h_n \circ X
= (t_1 h_1 + \cdots + t_n h_n) \circ X
\]
and this last is assumed to be normally distributed because \((t_1 h_1 + \cdots + t_n h_n) \in E'\).

Therefore, by Theorem 59.6.4, \((h_1 \circ X, \ldots, h_n \circ X)\)
is distributed as a multivariate normal.

Obviously there exist examples of Gaussian measures defined on \( E \), a Banach space. Here is why. Let \( \xi \) be a random variable defined on a probability space, \((\Omega, \mathcal{F}, P)\) which is normally distributed with mean 0 and variance \( \sigma^2 \). Then let \( X (\omega) \equiv \xi (\omega) e \) where \( e \in E \).

Then let \( \mu \equiv \mathcal{L} (X) \). For \( A \) a Borel set of \( \mathbb{R} \) and \( h \in E' \),
\[
\mu \left( [h \circ (x) \in A] \right) = P \left( [X (\omega) \in [x : h \circ (x) \in A]] \right)
= P \left( [h \circ X \in A] \right) = P \left( [\xi (\omega) h (e) \in A] \right)
= \frac{1}{|h (e)|} e^{-\frac{1}{2}\frac{|h (e)|^2}{\sigma^2} x^2} dx
\]
because \( h (e) \xi \) is a random variable which has variance \( |h (e)|^2 \sigma^2 \) and mean 0. Thus \( \mu \) is indeed a Gaussian measure. Similarly, one can consider finite sums of the form
\[
\sum_{i=1}^{n} \xi_i (\omega) e_i
\]
where the \( \xi_i \) are independent normal random variables having mean 0 for convenience. However, this is a rather trivial case.

59.7.2 Fernique’s Theorem

The following is an interesting lemma.
Lemma 59.7.4 Suppose $\mu$ is a symmetric Gaussian measure on the real separable Banach space, $E$. Then there exists a probability space, $(\Omega, \mathcal{F}, P)$ and independent random variables, $X$ and $Y$ mapping $\Omega$ to $E$ such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. Also, the two random variables,

$$\frac{1}{\sqrt{2}} (X - Y), \frac{1}{\sqrt{2}} (X + Y)$$

are independent and

$$\mathcal{L} \left( \frac{1}{\sqrt{2}} (X - Y) \right) = \mathcal{L} \left( \frac{1}{\sqrt{2}} (X + Y) \right) = \mu.$$

**Proof:** Letting $X' \equiv \frac{1}{\sqrt{2}} (X + Y)$ and $Y' \equiv \frac{1}{\sqrt{2}} (X - Y)$, it follows from Theorem 57.13.2 on Page 1945 that $X'$ and $Y'$ are independent if whenever $h_1, \cdots, h_m \in E'$ and $g_1, \cdots, g_k \in E'$, the two random vectors,

$$\left( h_1 \circ X', \cdots, h_m \circ X' \right) \quad \text{and} \quad \left( g_1 \circ Y', \cdots, g_k \circ Y' \right)$$

are independent. Now consider linear combinations

$$\sum_{j=1}^{m} t_j h_j \circ X' + \sum_{i=1}^{k} s_i g_i \circ Y'.$$

This equals

$$\frac{1}{\sqrt{2}} \sum_{j=1}^{m} t_j h_j (X) + \frac{1}{\sqrt{2}} \sum_{j=1}^{m} t_j h_j (Y) + \frac{1}{\sqrt{2}} \sum_{i=1}^{k} s_i g_i (X) - \frac{1}{\sqrt{2}} \sum_{i=1}^{k} s_i g_i (Y)$$

$$= \frac{1}{\sqrt{2}} \left( \sum_{j=1}^{m} t_j h_j + \sum_{i=1}^{k} s_i g_i \right) (X) + \frac{1}{\sqrt{2}} \left( \sum_{j=1}^{m} t_j h_j - \sum_{i=1}^{k} s_i g_i \right) (Y)$$

and this is the sum of two independent normally distributed random variables so it is also normally distributed. Therefore, by Theorem 57.13.2

$$\left( h_1 \circ X', \cdots, h_m \circ X', g_1 \circ Y', \cdots, g_k \circ Y' \right)$$

is a random variable with multivariate normal distribution and by Theorem 57.13.5 the two random vectors

$$\left( h_1 \circ X', \cdots, h_m \circ X' \right) \quad \text{and} \quad \left( g_1 \circ Y', \cdots, g_k \circ Y' \right)$$
are independent if

\[ E ((h_i \circ X')(g_j \circ Y')) = 0 \]

for all \( i, j \). This is what I will show next.

\[
E ((h_i \circ X')(g_j \circ Y')) \\
= \frac{1}{4} E ((h_i (X) + h_i (Y)) (g_j (X) - g_j (Y))) \\
= \frac{1}{4} E (h_i (X) g_j (X)) - \frac{1}{4} E (h_i (X) g_j (Y)) \\
+ \frac{1}{4} E (h_i (Y) g_j (X)) - \frac{1}{4} E (h_i (Y) g_j (Y)) \\
(59.7.25)
\]

Now from the above observation after the definition of Gaussian measure
\[ h_i (X) g_j (X) \]
and \[ h_i (Y) g_j (Y) \] are both in \( L^1 \) because each term in each product is normally distributed. Therefore, by Lemma 57.15.2,

\[
E (h_i (X) g_j (X)) = \int \Omega h_i (Y) g_j (Y) dP \\
= \int_E h_i (y) g_j (y) d\mu \\
= \int \Omega h_i (X) g_j (X) dP \\
= E (h_i (Y) g_j (Y))
\]

and so (59.7.25) reduces to

\[
\frac{1}{4} (E (h_i (Y) g_j (X) - h_i (X) g_j (Y))) = 0
\]

because \( h_i (X) \) and \( g_j (Y) \) are independent due to the assumption that \( X \) and \( Y \) are independent. Thus

\[
E (h_i (X) g_j (Y)) = E (h_i (X)) E (g_j (Y)) = 0
\]

due to the assumption that \( \mu \) is symmetric which implies the mean of these random variables equals 0. The other term works out similarly. This has proved the independence of the random variables, \( X' \) and \( Y' \).

Next consider the claim they have the same law and it equals \( \mu \). To do this, I will use Theorem 57.12.9 on Page 1944. Thus I need to show

\[
E \left( e^{ih(X')} \right) = E \left( e^{ih(Y')} \right) = E \left( e^{ih(X)} \right) \\
(59.7.26)
\]

for all \( h \in E' \). Pick such an \( h \). Then \( h \circ X \) is normally distributed and has mean 0. Therefore, for some \( \sigma \),

\[
E \left( e^{ih \circ X} \right) = e^{-\frac{1}{2}t^2\sigma^2}.
\]
Now since $X$ and $Y$ are independent,

\[ E \left( e^{itho X'} \right) = E \left( e^{ith \left( \frac{1}{\sqrt{2}} \right) (X+Y)} \right) = E \left( e^{ith \left( \frac{1}{\sqrt{2}} \right) X} \right) E \left( e^{ith \left( \frac{1}{\sqrt{2}} \right) Y} \right) \]

the product of two characteristic functions of two random variables, $\frac{1}{\sqrt{2}} X$ and $\frac{1}{\sqrt{2}} Y$.

The variance of these two random variables which are normally distributed with zero mean is $\frac{1}{2} \sigma^2$ and so

\[ E \left( e^{itho X'} \right) = e^{-\frac{1}{2} \left( \frac{1}{2} \sigma^2 \right)} e^{-\frac{1}{2} \left( \frac{1}{2} \sigma^2 \right)} = e^{-\frac{1}{2} \sigma^2} = E \left( e^{itho X} \right). \]

Similar reasoning shows $E \left( e^{itho Y'} \right) = E \left( e^{itho Y} \right) = E \left( e^{itho X} \right)$. Letting $t = 1$, this yields 59.7.26. This proves the lemma.

With this preparation, here is an incredible theorem due to Fernique.

**Theorem 59.7.5** Let $\mu$ be a symmetric Gaussian measure on $B(E)$ where $E$ is a real separable Banach space. Then for $\lambda$ sufficiently small and positive,

\[ \int_E e^{\lambda ||x||^2} \, d\mu < \infty. \]

More specifically, if $\lambda$ and $r$ are chosen such that

\[ \ln \left( \frac{\mu \left( \{x : ||x|| > r\} \right)}{\mu \left( B(0,r) \right)} \right) + 25 \lambda r^2 < -1, \]

then

\[ \int_E e^{\lambda ||x||^2} \, d\mu \leq \exp \left( \lambda r^2 \right) + \frac{e^2}{e^2 - 1}. \]

**Proof:** Let $X, Y$ be independent random variables having values in $E$ such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. Then by Lemma 59.7.24

\[ \frac{1}{\sqrt{2}} (X - Y), \frac{1}{\sqrt{2}} (X + Y) \]

are also independent and have the same law. Now let $0 \leq s \leq t$ and use independence of the above random variables along with the fact they have the same law as $X$ and $Y$ to obtain

\[ P \left( ||X|| \leq s, ||Y|| > t \right) = P \left( ||X|| \leq s \right) P \left( ||Y|| > t \right) \]
\[ \begin{align*}
\text{CHAPTER 59. PROBABILITY IN INFINITE DIMENSIONS} \\
\text{=} & \ P \left( \left\| \frac{1}{\sqrt{2}} (X - Y) \right\| \leq s \right) \ P \left( \left\| \frac{1}{\sqrt{2}} (X + Y) \right\| > t \right) \\
\text{=} & \ P \left( \left\| \frac{1}{\sqrt{2}} (X - Y) \right\| \leq s, \ \left\| \frac{1}{\sqrt{2}} (X + Y) \right\| > t \right) \\
\text{\leq} & \ P \left( \frac{1}{\sqrt{2}} \left\| X \right\| - \left\| Y \right\| \leq s, \ \frac{1}{\sqrt{2}} \left( \left\| X \right\| + \left\| Y \right\| \right) > t \right). \\
\text{Now consider the following picture in which the region, R represents the points, } (\left\| X \right\|, \left\| Y \right\|) \text{ such that} \\
\frac{1}{\sqrt{2}} \left\| X \right\| - \left\| Y \right\| \leq s \text{ and } \frac{1}{\sqrt{2}} \left( \left\| X \right\| + \left\| Y \right\| \right) > t. \\
\text{Therefore, continuing with the chain of inequalities above,} \\
P \left( \left\| X \right\| \leq s \right) P \left( \left\| Y \right\| > t \right) \\
\text{\leq} \ P \left( \left\| X \right\| > \frac{t - s}{\sqrt{2}}, \left\| Y \right\| > \frac{t - s}{\sqrt{2}} \right) \\
\text{=} \ P \left( \left\| X \right\| > \frac{t - s}{\sqrt{2}} \right)^2. \\
\text{Since } X, Y \text{ have the same law, this can be written as} \\
P \left( \left\| X \right\| > t \right) \leq \frac{P \left( \left\| X \right\| > \frac{t - s}{\sqrt{2}} \right)^2}{P \left( \left\| X \right\| \leq s \right)}. \\
\text{Now define a sequence as follows. } t_0 \equiv r > 0 \text{ and } t_{n+1} \equiv r + \sqrt{2} t_n. \text{ Also, in the above inequality, let } s \equiv r \text{ and then it follows} \\
P \left( \left\| X \right\| > t_{n+1} \right) \leq \frac{P \left( \left\| X \right\| > \frac{t_{n+1} - r}{\sqrt{2}} \right)^2}{P \left( \left\| X \right\| \leq r \right)} \\
= \frac{P \left( \left\| X \right\| > t_n \right)^2}{P \left( \left\| X \right\| \leq r \right)}. \\
\end{align*} \]
Let 
\[ \alpha_n (r) \equiv \frac{P(||X|| > t_n)}{P(||X|| \leq r)}. \]

Then it follows 
\[ \alpha_{n+1} (r) \leq \alpha_n (r)^2, \]

Consequently, \( \alpha_n (r) \leq \alpha_0 (r)^2n \) and also 
\[ P(||X|| > t_n) = \alpha_n (r) P(||X|| \leq r) \]
\[ \leq P(||X|| \leq r) \alpha_0 (r)^2n \]
\[ = P(||X|| \leq r) e^{\ln(\alpha_0 (r))2n}. \] (59.7.27)

Now using the distribution function technique and letting \( \lambda > 0, \)
\[ \int_E e^{\lambda||x||^2} d\mu = \int_0^\infty \mu \left( e^{\lambda||x||^2} > t \right) dt \]
\[ = 1 + \int_1^\infty \mu \left( e^{\lambda||x||^2} > t \right) dt \]
\[ = 1 + \int_1^\infty P \left( e^{\lambda||X||^2} > t \right) dt. \] (59.7.28)

From 59.7.28,
\[ P \left( \left[ e^{\lambda||X||^2} \right] \leq P \left( ||X|| \leq r \right) e^{\ln(\alpha_0 (r))2n}. \]

Now split the above improper integral into intervals, \( \left( e^{\lambda||X||^2} \right) \) for \( n = 0, 1, \cdots \) and note that \( P \left( e^{\lambda||X||^2} > t \right) \) is decreasing in \( t. \) Then from 59.7.28,
\[ \int_E e^{\lambda||x||^2} d\mu \leq e^{\lambda r^2} + \sum_{n=0}^\infty \int_{e^{\lambda t_n^2}}^{e^{\lambda t_{n+1}^2}} P \left( e^{\lambda||X||^2} > t \right) dt \]
\[ \leq e^{\lambda r^2} + \sum_{n=0}^\infty P \left( ||X|| \leq r \right) e^{\ln(\alpha_0 (r))2n} \exp \left( \lambda t_{n+1}^2 \right) \]
\[ \leq \exp \left( \lambda r^2 \right) + \sum_{n=0}^\infty e^{\ln(\alpha_0 (r))2n} \exp \left( \lambda t_{n+1}^2 \right). \]

It remains to estimate \( t_{n+1}. \) From the description of the \( t_n, \)
\[ t_n = \left( \sum_{k=0}^n \left( \sqrt{2} \right)^k \right) r = r \left( \frac{\sqrt{2}^{n+1} - 1}{\sqrt{2} - 1} \right) \leq \frac{\sqrt{2} - 1}{\sqrt{2} - 1} r \left( \sqrt{2} \right)^n \]
and so
\[ t_{n+1} \leq 5r \left( \sqrt{2} \right)^n \]

Therefore,
\[
\int_E e^{\lambda ||x||^2} d\mu \leq \exp (\lambda r^2) + \sum_{n=0}^{\infty} e^{4\ln(\alpha_0(r))2^n + \lambda 25r^2 2^n}.
\]

Now first pick \( r \) large enough that \( \ln (\alpha_0 (r)) < -2 \) and then let \( \lambda \) be small enough that \( 25\lambda r^2 < 1 \) or some such scheme and you obtain \( \ln (\alpha_0 (r)) + \lambda 25r^2 < -1 \). Then for this choice of \( r \) and \( \lambda \), or for any other choice which makes \( \ln (\alpha_0 (r)) + \lambda 25r^2 < -1 \),
\[
\int_E e^{\lambda ||x||^2} d\mu \leq \exp (\lambda r^2) + \sum_{n=0}^{\infty} e^{2n} \leq \exp (\lambda r^2) + \frac{e^2}{e^2 - 1}.
\]

This proves the theorem.

59.8 Gaussian Measures For A Separable Hilbert Space

First recall the Kolmogorov extension theorem, Theorem 57.2.3 on Page 1906 which is stated here for convenience. In this theorem, \( I \) is an ordered index set, possibly infinite, even uncountable.

**Theorem 59.8.1** (Kolmogorov extension theorem) For each finite set
\[ J = (t_1, \ldots, t_n) \subseteq I, \]

suppose there exists a Borel probability measure, \( \nu_J = \nu_{t_1, \ldots, t_n} \) defined on the Borel sets of \( \prod_{i \in J} M_i \) where \( M_i = \mathbb{R}^{n_i} \) such that if
\[
(t_1, \ldots, t_n) \subseteq (s_1, \ldots, s_p),
\]

then
\[
\nu_{t_1, \ldots, t_n} (F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1, \ldots, s_p} (G_{s_1} \times \cdots \times G_{s_p}) \quad (59.8.29)
\]

where if \( s_i = t_j \), then \( G_{s_i} = F_{t_j} \) and if \( s_i \) is not equal to any of the indices, \( t_k \), then \( G_{s_i} = M_{s_i} \). Then there exists a probability space, \((\Omega, P, F)\) and measurable functions, \( X_t : \Omega \to M_t \) for each \( t \in I \) such that for each \((t_1 \cdots t_n) \subseteq I\),
\[
\nu_{t_1, \ldots, t_n} (F_{t_1} \times \cdots \times F_{t_n}) = P \left( [X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}] \right). \quad (59.8.30)
\]
Lemma 59.8.2 There exists a sequence, \( \{\xi_k\}_{k=1}^{\infty} \) of random variables such that

\[ L(\xi_k) = N(0,1) \]

and \( \{\xi_k\}_{k=1}^{\infty} \) is independent.

Proof: Let \( i_1 < i_2 \cdots < i_n \) be positive integers and define

\[
\mu_{i_1\cdots i_n}(F_1 \times \cdots \times F_n) \equiv \frac{1}{(\sqrt{2\pi})^n} \int_{F_1 \times \cdots \times F_n} e^{-|x|^2/2} dx.
\]

Then for the index set equal to \( \mathbb{N} \) the measures satisfy the necessary consistency condition for the Kolmogorov theorem above. Therefore, there exists a probability space, \((\Omega, P, F)\) and measurable functions, \(\xi_k : \Omega \to \mathbb{R}\) such that

\[
P(\{\xi_{i_1} \in F_{i_1} \cap \xi_{i_2} \in F_{i_2} \cdots \cap \xi_{i_n} \in F_{i_n}\}) = \mu_{i_1\cdots i_n}(F_1 \times \cdots \times F_n)
\]

which shows the random variables are independent as well as normal with mean 0 and variance 1. This proves the Lemma.

A random variable \( X \) defined on a probability space \((\Omega, F, P)\) is called Gaussian if

\[
P(\{X \in A\}) = \frac{1}{\sqrt{2\pi\sigma(v)^2}} \int_A e^{-\frac{1}{2\sigma(v)^2}(x-m(v))^2} dx
\]

for all \( A \) a Borel set in \( \mathbb{R} \). Therefore, for the probability space \((X, B(X), \mu)\) it is natural to say \( \mu \) is a Gaussian measure if every \( x^* \) in the dual space \( X' \) is a Gaussian random variable. That is, normally distributed.

Definition 59.8.3 Let \( \mu \) be a measure defined on \( B(X) \), the Borel sets of \( X \), a separable Banach space. It is called a Gaussian measure if each of the functions in the dual space \( X' \) is normally distributed. As a special case, when \( X = U \) a separable real Hilberts space, \( \mu \) is called a Gaussian measure if for each \( v \in U \), the function \( u \to (u,v)_U \) is normally distributed. That is, denoting this random variable as \( v' \), it follows for \( A \) a Borel set in \( \mathbb{R} \)

\[
\lambda_{v'}(A) \equiv \mu(\{u : v'(u) \in A\}) = \frac{1}{\sqrt{2\pi\sigma(v)^2}} \int_A e^{-\frac{1}{2\sigma(v)^2}(x-m(v))^2} dx
\]

in case \( \sigma(v) > 0 \). In case \( \sigma(v) = 0 \)

\[
\lambda_{v'} \equiv \delta_{m(v)}
\]

In other words, the random variables \( v' \) for \( v \in U \) are all normally distributed on the probability space \((U, B(U), \mu)\).
Also recall the definition of the characteristic function of a measure.

**Definition 59.8.4** The Borel sets in a topological space $X$ will be denoted by $\mathcal{B}(X)$. For a Borel probability measure $\mu$ defined on $\mathcal{B}(U)$ for $U$ a real separable Hilbert space, define its characteristic function as follows.

$$\phi_\mu(u) \equiv \hat{\mu}(u) \equiv \int_U e^{i(u,v)} d\mu(v) \quad (59.8.31)$$

More generally, if $\mu$ is a probability measure defined on $\mathcal{B}(X)$ where $X$ is a separable Banach space, then the characteristic function is defined as

$$\phi_\mu(x^*) \equiv \hat{\mu}(x^*) \equiv \int_U e^{ix^*(x)} d\mu(x)$$

One can tell whether $\mu$ is a Gaussian measure by looking at its characteristic function. In fact you can show the following theorem. One part of this theorem is that if $\mu$ is Gaussian, then $m$ and $\sigma^2$ have a certain form.

**Theorem 59.8.5** A measure $\mu$ on $\mathcal{B}(U)$ is Gaussian if and only if there exists $m \in U$ and $Q \in \mathcal{L}(U)$ such that $Q$ is nonnegative symmetric with finite trace, $\sum_k (Qe_k, e_k) < \infty$ for a complete orthonormal basis for $U$, and

$$\phi_\mu(u) = \hat{\mu}(u) = e^{it(m,u) - \frac{1}{2}(Qu,u)} \quad (59.8.32)$$

In this case $\mu$ is called $N(m, Q)$ where $m$ is the mean and $Q$ is called the covariance. The measure $\mu$ is uniquely determined by $m$ and $Q$. Also for all $h, g \in U$

$$\int (x, h)_U d\mu(x) = (m, h)_U$$

$$\int ((x, h) - (m, h))((x, g) - (m, g)) d\mu(x) = (Qh, g)$$

$$\int \|x - m\|^2_U d\mu(x) = \text{trace}(Q). \quad (59.8.35)$$

**Proof:** First of all suppose $59.8.32$ holds. Why is $\mu$ Gaussian? Consider the random variable $u'$ defined by $u'(v) \equiv (v, u)$. Why is $\lambda_u'$ a Gaussian measure on $\mathbb{R}$? By the definition in $59.8.31$.

$$\int_U e^{itu'(v)} d\mu(v) = \int_U e^{it(v,u)} d\mu(v) = \int_{\mathbb{R}} e^{ix} d\lambda_{uu'}(x)$$

$$= \int_U e^{i(v,uu')} d\mu(v) = e^{it(m,u) - \frac{1}{2}t^2(Qu,u)}$$
and this is the characteristic equation for a random variable having mean \((m, u)\) and variance \((Qu, u)\). In case \((Qu, u) = 0\), you get \(e^{it(m,u)}\) which is the characteristic function for a random variable having distribution \(\delta_{(m,u)}\). Thus if \((Qu, u) \neq 0\) holds, then \(u'\) is normally distributed as desired. Thus \(\mu\) is Gaussian by definition.

The next task is to suppose \(\mu\) is Gaussian and show the existence of \(m, Q\) which have the desired properties. This involves the following lemma.

**Lemma 59.8.6** Let \(U\) be a real separable Hilbert space and let \(\mu\) be a probability measure defined on \(\mathcal{B}(U)\). Suppose for some positive integer, \(k\)
\[
\int_U |(x, z)|^k \, d\mu(x) < \infty
\]
for all \(z \in U\). Then the transformation,
\[
(h_1, \cdots , h_k) \mapsto \int_U (h_1, x) \cdots (h_k, x) \, d\mu(x)
\]
(59.8.36)
is a continuous \(k\)-linear form.

**Proof:** I need to show that for each \(h \in U^k\), the integral in (59.8.36) exists. From this it is obvious it is \(k\)-linear, meaning linear in each argument. Then it is shown it is continuous.

First note
\[
|(h_1, x) \cdots (h_k, x)| \leq |(h_1, x)|^k + \cdots + |(h_k, x)|^k
\]
This follows from observing that one of \(|(h_j, x)|\) is largest. Then the left side is smaller than \(|(h_j, x)|^k\). Therefore, the above inequality is valid. This inequality shows the integral in (59.8.36) makes sense.

I need to establish an estimate of the form
\[
\int_U |(x, h)|^k \, d\mu(x) < C < \infty
\]
for every \(h \in U\) such that \(\|h\|\) is small enough.

Let
\[
U_n = \left\{ z \in U : \int_U |(x, z)|^k \, d\mu(x) \leq n \right\}
\]
Then by assumption \(U = \cup_{n=1}^\infty U_n\) and it is also clear from Fatou’s lemma that each \(U_n\) is closed. Therefore, by the Bair category theorem, at least one of these \(U_{n_0}\) contains an open ball, \(B(z_0, r)\). Then letting \(|y| < r\),
\[
\int_U |(x, z_0 + y)|^k \, d\mu(x), \int_U |(x, z_0)|^k \, d\mu(x) \leq n_0,
\]
and so for such \(y\),
\[
\int_U |(x, y)|^k \, d\mu = \int_U |(x, z_0 + y) - (x, z_0)|^k \, d\mu
\leq \int_U 2^k |(x, z_0 + y)|^k + 2^k |(x, z_0)|^k \, d\mu(x)
\leq 2^k (n_0 + n_0) = 2^{k+1} n_0.
\]
It follows that for arbitrary nonzero \( y \in U \)

\[
\int_U \left| \left( x, \frac{r/2}{||y||} y \right) \right|^k d\mu \leq 2^{k+1} n_0
\]

and so

\[
\int_U |(x, y)|^k d\mu \leq (2^{k+2}/r) n_0 ||y||^k \equiv C ||y||^k.
\]

Thus by Holder’s inequality,

\[
\int_U |(h_1, x) \cdot \cdots \cdot (h_k, x)| d\mu(x) \leq \prod_{j=1}^k \left( \int_U |(h_j, x)|^k d\mu(x) \right)^{1/k}
\]

\[
\leq C \prod_{j=1}^k ||h_j||
\]

This proves the lemma.

Now continue with the proof of the theorem. I need to identify \( m \) and \( Q \). It is assumed \( \mu \) is Gaussian. Recall this means \( h' \) is normally distributed for each \( h \in U \).

Then using

\[
|x| \leq |x - m(h)| + |m(h)|
\]

\[
\int_U |(x, h)|_U^1 d\mu(x) = \int_R |x| d\lambda_{h'}(x)
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2(h)}} \int_R |x| e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx
\]

\[
\leq \frac{1}{\sqrt{2\pi\sigma^2(h)}} \int_R |x - m(h)| e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx
\]

Then using the Cauchy Schwarz inequality, with respect to the probability measure

\[
\frac{1}{\sqrt{2\pi\sigma^2(h)}} e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx,
\]

\[
\leq \frac{1}{\sqrt{2\pi\sigma^2(h)}} \left( \int_R |x - m(h)|^2 e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx \right)^{1/2} + |m(h)| < \infty
\]

Thus by Lemma 59.8.6

\[
h \to \int_U (x, h) d\mu(x)
\]

is a continuous linear transformation and so by the Riesz representation theorem, there exists a unique \( m \in U \) such that

\[
(h, m)_U = \int_U (h, x) d\mu(x)
\]
Also the above says \((h, m)\) is the mean of the random variable \(x \rightarrow (x, h)\) so in the above,

\[
m(h) = (h, m)_U.
\]

Next it is necessary to find \(Q\). To do this let \(Q\) be given by \(59.8.34\). Thus

\[
(Qh, g) = \int_U ((x, h) - (m, h)) ((x, g) - (m, g)) d\mu(x)
\]

\[
= \int_U (x - m, h) (x - m, g) d\mu(x)
\]

It is clear \(Q\) is linear and the above is a bilinear form (The integral makes sense because of the assumption that \(h', g'\) are normally distributed.) but is it continuous? Does \((Qh, h) = \sigma^2(h)\)?

First, the above equals

\[
\int_U (x, h) (x - m, g) d\mu(x) - \int_U (m, h) (x - m, g) d\mu(x)
\]

\[
= \int_U (x, h) (x - m, g) d\mu(x)
\]

(59.8.37)

because from the first part,

\[
\int_U (x - m, g) d\mu(x) = \int_U (x, g) d\mu(x) - (m, g)_U = 0.
\]

Now by the first part, the term in \(59.8.37\) is

\[
\int_U (x, h) (x, g) d\mu(x) - (m, g) \int_U (x, h) d\mu(x)
\]

\[
= \int_U (x, h) (x, g) d\mu(x) - (m, g) (m, h) .
\]

Thus

\[
|(Qh, g)| \leq \int_U |(x, h) (x, g)| d\mu(x) + ||m||^2 ||h|| ||g||
\]

and since the random variables \(h'\) and \(g'\) given by \(x \rightarrow (x, h)\) and \(x \rightarrow (x, g)\) respectively are given to be normally distributed with variance \(\sigma^2(h)\) and \(\sigma^2(g)\) respectively, the above integral is finite. Also for all \(h\),

\[
\int_U |(x, h)|^2 d\mu(x) < \infty
\]

because the random variable \(h'\) is given to be normally distributed. Therefore from Lemma \(59.8.4\), there exists some constant \(C\) such that

\[
|(Qh, g)| \leq C ||h|| ||g||
\]

which shows \(Q\) is continuous as desired.
Why is $\sigma^2(h) = (Qh, h)$? This follows because from the above
\[(Qh, h) = \int_U (h, x - m)^2 d\mu(x) = \int_U ((x, h) - (h, m))^2 d\mu(x) = \int_\mathbb{R} (t - (h, m))^2 d\lambda_{h'}(t) = \frac{1}{\sqrt{2\pi \sigma^2(h)}} \int_\mathbb{R} (t - (h, m))^2 e^{-\frac{1}{2\sigma^2(h)}(t - (h, m))^2} dt = \sigma^2(h)
from a standard result for the normal distribution function which follows from an easy change of variables argument.

Why must $Q$ have finite trace? For $h \in U$, it follows from the above that $h'$ is normally distributed with mean $(h, m)$ and variance $(Qh, h)$. Therefore, the characteristic function of $h'$ is known. In fact
\[\int_U e^{it(x, h)} d\mu(x) = e^{it(h, m)} e^{-\frac{1}{2}t^2(Qh, h)}
Thus also
\[\int_U e^{it(x-m, h)} d\mu(x) = e^{-\frac{1}{2}t^2(Qh, h)}
and letting $t = 1$ this yields
\[\int_U e^{i(x-m, h)} d\mu(x) = e^{-\frac{1}{2}(Qh, h)}
From this it follows
\[\int_U \left(1 - e^{i(x-m, h)}\right) d\mu(x) = 1 - e^{-\frac{1}{2}(Qh, h)}
and since the right side is real, this implies
\[\int_U \left(1 - \cos(x - m, h)\right) d\mu(x) = 1 - e^{-\frac{1}{2}(Qh, h)}
Thus
\[1 - e^{-\frac{1}{2}(Qh, h)} \leq \int_{||(x-m)|| \leq c} \left(1 - \cos(x - m, h)\right) d\mu(x) + 2\int_{||(x-m)|| > c} d\mu(x)
Now it is routine to show
\[1 - \cos t \leq \frac{1}{2} t^2
and so
\[1 - e^{-\frac{1}{2}(Qh, h)} \leq \frac{1}{2} \int_{||x-m|| \leq c} ||(x-m, h)||^2 d\mu(x) + 2\mu(||x-m|| > c)\]
Pick \( c \) large enough that the last term is smaller than \( 1/8 \). This can be done because the sets decrease to \( \emptyset \) as \( c \to \infty \) and \( \mu \) is given to be a finite measure. Then with this choice of \( c \),

\[
\frac{7}{8} - \frac{1}{2} \int_{||x-m|| \leq c} |(x-m,h)|^2 \, d\mu(x) \leq e^{-\frac{1}{2}(Qh,h)}
\] (59.8.38)

For each \( h \) the integral in the above is finite. In fact

\[
\int_{||x-m|| \leq c} |(x-m,h)|^2 \, d\mu(x) \leq e^2 ||h||^2
\]

Let

\[
(Q_c h, h_1) \equiv \int_{||x-m|| \leq c} (x-m,h)(x-m,h_1) \, d\mu(x)
\]

and let \( A \) denote those \( h \in U \) such that

\[
(Q_c h, h) < 1.
\]

Then from (59.8.38) it follows that for \( h \in A \),

\[
\frac{3}{8} = \frac{7}{8} - \frac{1}{2} - (Q_c h, h) \leq e^{-\frac{1}{2}(Qh,h)}
\]

Therefore, for such \( h \),

\[
\frac{8}{3} \geq e^{\frac{1}{2}(Qh,h)} \geq 1 + \frac{1}{2} (Qh, h)
\]

and so for \( h \in A \),

\[
(Qh, h) \leq \left( \frac{8}{3} - 1 \right) \frac{1}{2} = \frac{10}{3}
\]

Now let \( h \) be arbitrary. Then for each \( \varepsilon > 0 \)

\[
\frac{h}{\varepsilon + \sqrt{(Q_c h, h)}} \in A
\]

and so

\[
\left( Q \left( \frac{h}{\varepsilon + \sqrt{(Q_c h, h)}} \right), \frac{h}{\varepsilon + \sqrt{(Q_c h, h)}} \right) \leq \frac{10}{3}
\]

which implies

\[
(Qh, h) \leq \frac{10}{3} \left( \varepsilon + \sqrt{(Q_c h, h)} \right)^2
\]

Since \( \varepsilon \) is arbitrary,

\[
(Qh, h) \leq \frac{10}{3} (Q_c h, h). \quad (59.8.39)
\]

However, \( Q_c \) has finite trace. To see this, let \( \{ e_k \} \) be an orthonormal basis in \( U \). Then

\[
\sum_k (Q_c e_k, e_k) = \sum_k \int_{||x-m|| \leq c} |(x-m, e_k)|^2 \, d\mu(x)
\]
= \int_{||x-m|| \leq c} \sum_k |(x-m,e_k)|^2 \, d\mu(x) = \int_{||x-m|| \leq c} ||x-m||^2 \, d\mu(x) \leq c^2

It follows from 59.8.39 that \( Q \) must also have finite trace.

That \( \mu \) is uniquely determined by \( m \) and \( Q \) follows from Theorem 57.12.9. This proves the theorem.

Suppose you have a given \( Q \) having finite trace and \( m \in U \). Does there exist a Gaussian measure on \( B(U) \) having these as the covariance and mean respectively?

**Proposition 59.8.7** Let \( U \) be a real separable Hilbert space and let \( m \in U \) and \( Q \) be a positive, symmetric operator defined on \( U \) which has finite trace. Then there exists a Gaussian measure with mean \( m \) and covariance \( Q \).

**Proof:** By Lemma 59.8.2 which comes from Kolmogorov’s extension theorem, there exists a probability space \((\Omega, F, P)\) and a sequence \( \{\xi_i\} \) of independent random variables which are normally distributed with mean 0 and variance 1. Then let

\[ X(\omega) \equiv m + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j(\omega) e_j \]

where the \( \{e_j\} \) are the eigenvectors of \( Q \) such that \( Qe_j = \lambda_j e_j \). The series in the above converges in \( L^2(\Omega; U) \) because

\[ \left\| \sum_{j=m}^{n} \sqrt{\lambda_j} \xi_j(\omega) e_j \right\|_{L^2(\Omega; U)}^2 = \int_{\Omega} \sum_{j=m}^{n} \lambda_j \xi_j^2(\omega) \, dP = \sum_{j=m}^{n} \lambda_j \]

and so the partial sums form a Cauchy sequence in \( L^2(\Omega; U) \).

Now if \( h \in U \), I need to show that \( \omega \rightarrow (X(\omega), h) \) is normally distributed. From this it will follow that \( L(X) \) is Gaussian. A subsequence

\[ \left\{ m + \sum_{j=1}^{n_k} \sqrt{\lambda_j} \xi_j(\omega) e_j \right\} \equiv \{S_{n_k}(\omega)\} \]

of the above sequence converges pointwise a.e. to \( X \).

\[ E\left(\exp(it(X,h))\right) = \lim_{k \to \infty} E\left(\exp(it(S_{n_k},h))\right) \]

\[ = \exp(\langle m, h \rangle) \lim_{k \to \infty} E \left( \exp \left( it \sum_{j=1}^{n_k} \sqrt{\lambda_j} \xi_j(\omega)(e_j, h) \right) \right) \]

Since the \( \xi_j \) are independent,

\[ = \exp(\langle m, h \rangle) \lim_{k \to \infty} \prod_{j=1}^{n_k} E \left( \exp \left( it \sqrt{\lambda_j} (e_j, h) \right) \xi_j \right) \]
59.8. GAUSSIAN MEASURES FOR A SEPARABLE HILBERT SPACE

\[ = \exp (it (m, h)) \lim_{k \to \infty} \prod_{j=1}^{n_k} e^{-\frac{1}{2} t^2 \lambda_j (e_j, h)^2} \]

\[ = \exp (it (m, h)) \lim_{k \to \infty} \exp \left( -\frac{1}{2} t^2 \sum_{j=1}^{n_k} \lambda_j (e_j, h)^2 \right). \tag{59.8.40} \]

Now

\[ (Qh, h) = \left( Q \sum_{k=1}^{\infty} (e_k, h) e_k, \sum_{j=1}^{\infty} (e_j, h) e_j \right) \]

\[ = \left( \sum_{k=1}^{\infty} (e_k, h) \lambda_k e_k, \sum_{j=1}^{\infty} (e_j, h) e_j \right) \]

\[ = \sum_{j=1}^{\infty} \lambda_j (e_j, h)^2 \]

and so, passing to the limit in \textbf{59.8.40} yields

\[ \exp (it (m, h)) \exp \left( -\frac{1}{2} t^2 (Qh, h) \right) \tag{59.8.41} \]

which implies that \( \omega \to (X (\omega), h) \) is normally distributed with mean \( (m, h) \) and variance \( (Qh, h) \).

Now let \( \mu = \mathcal{L} (X) \). That is, for all \( B \in \mathcal{B} (U) \),

\[ \mu (B) \equiv P ([X \in B]) \]

In particular, \( B \) could be the cylindrical set

\[ B \equiv [x : (x, h) \in A] \]

for \( A \) a Borel set in \( \mathbb{R} \). Then by definition, if \( h \in U \), and \( A \) is a Borel set in \( \mathbb{R} \),

\[ \mu (B) = \mu ([x : (x, h) \in A]) \equiv P ([\omega : (X (\omega), h) \in A]) \]

\[ = \int_A \frac{1}{\sqrt{2\pi (Qh, h)}} e^{-\frac{(x - (m, h))^2}{2 (Qh, h)}} \, dt \]

and so \( x \to (x, h) \) is normally distributed. Therefore by definition, \( \mu \) is a Gaussian measure.

Letting \( t = 1 \) in \textbf{59.8.41} it follows

\[ \int_U e^{i (x, h)} d\mu (x) = \int_\Omega e^{i (X (\omega), h)} dP = \exp (i (m, h)) \exp \left( -\frac{1}{2} (Qh, h) \right) \]

which is the characteristic function of a Gaussian measure on \( U \) having covariance \( Q \) and mean \( m \). This proves the proposition.
59.9 Abstract Wiener Spaces

This material follows [20], [13] and [15]. More can be found on this subject in these references. Here $H$ will be a separable real Hilbert space.

**Definition 59.9.1** Cylinder sets in $H$ are of the form

$$\{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\}$$

where $F \in \mathcal{B}(\mathbb{R}^n)$, the Borel sets of $\mathbb{R}^n$ and $\{e_k\}$ are given vectors in $H$. Denote this collection of cylinder sets as $C$.

**Proposition 59.9.2** The cylinder sets form an algebra of sets.

**Proof:** First note the complement of a cylinder set is a cylinder set.

$$\{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\}^C = \{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F^C\}.$$ 

Now consider the intersection of two of these cylinder sets. Let the cylinder sets be

$$\{x \in H : ((x, e_1), \cdots, (x, e_n)) \in E\}, \quad \{x \in H : ((x, f_1), \cdots, (x, f_m)) \in F\}$$

The first of these equals

$$\{x \in H : ((x, e_1), \cdots, (x, e_n), (x, f_1), \cdots, (x, f_m)) \in E \times \mathbb{R}^m\}$$

and the second equals

$$\{x \in H : ((x, e_1), \cdots, (x, e_n), (x, f_1), \cdots, (x, f_m)) \in \mathbb{R}^n \times F\}$$

Therefore, their intersection equals

$$\{x \in H : ((x, e_1), \cdots, (x, e_n), (x, f_1), \cdots, (x, f_m)) \in E \times \mathbb{R}^m \cap \mathbb{R}^n \times F\},$$

a cylinder set.

Now it is clear the whole of $H$ and $\emptyset$ are cylinder sets given by

$$\{x \in H : (e, x) \in \mathbb{R}\}, \quad \{x \in H : (e, x) \in \emptyset\}$$

respectively and so if $C_1, C_2$ are two cylinder sets,

$$C_1 \setminus C_2 \equiv C_1 \cap C_2^C,$$

which was just shown to be a cylinder set. Hence

$$C_1 \cup C_2 = (C_1^C \cap C_2^C)^C,$$
It is good to have a more geometrical description of cylinder sets. Letting $A$ be a cylinder set as just described, let $P$ denote the orthogonal projection onto span $(e_1, \cdots, e_n)$. Also let $\alpha : PH \to \mathbb{R}^n$ be given by

$$\alpha(x) \equiv ((x, e_1), \cdots, (x, e_n)).$$

This is continuous but might not be one to one if the $e_i$ are not a basis for example. Then consider $\alpha^{-1}(F)$, those $x \in PH$ such that

$$((x, e_1), \cdots, (x, e_n)) \in F.$$

For any $x \in H$,

$$((I - P)x, e_k) = 0$$

for each $k$ and so

$$((x, e_1), \cdots, (x, e_n)) = ((Px, e_1), \cdots, (Px, e_n)) \in F$$

Thus $Px \in \alpha^{-1}(F)$, which is a Borel set of $PH$ and

$$x = Px + (I - P)x$$

so the cylinder set is contained in

$$\alpha^{-1}(F) + (PH)^\perp$$

which is of the form

$$(\text{Borel set of } PH) + (PH)^\perp$$

On the other hand, consider a set of the form

$$G + (PH)^\perp$$

where $G$ is a Borel set in $PH$. There is a basis for $PH$ consisting of a subset of $\{e_1, \cdots, e_n\}$. For simplicity, suppose it is $\{e_1, \cdots, e_k\}$. Then let $\alpha_1 : PH \to \mathbb{R}^k$ be given by

$$\alpha_1(x) \equiv ((x, e_1), \cdots, (x, e_k))$$

Thus $\alpha$ is a homeomorphism of $PH$ and $\mathbb{R}^k$ so $\alpha_1(G)$ is a Borel set of $\mathbb{R}^k$. Now

$$\alpha^{-1}(\alpha_1(G) \times \mathbb{R}^{n-k}) = G$$

and $\alpha_1(G) \times \mathbb{R}^{n-k}$ is a Borel set of $\mathbb{R}^n$. This has proved the following important Proposition illustrated by the following picture.

\[ B + M_\perp \]
CHAPTER 59. PROBABILITY IN INFINITE DIMENSIONS

Proposition 59.9.3 The cylinder sets are sets of the form

\[ B + M^\perp \]

where \( M \) is a finite dimensional subspace and \( B \) is a Borel subset of \( M \). Furthermore, the collection of cylinder sets is an algebra.

Lemma 59.9.4 \( \sigma(C) \), the smallest \( \sigma \) algebra containing \( C \), contains the Borel sets of \( H, B(H) \).

Proof: It follows from the definition of these cylinder sets that if \( f_i(x) \equiv (x, e_i) \), so that \( f_i \in H' \), then with respect to \( \sigma(C) \), each \( f_i \) is measurable. It follows that every linear combination of the \( f_i \) is also measurable with respect to \( \sigma(C) \). However, this set of linear combinations is dense in \( H' \) and so the conclusion of the lemma follows from Lemma 57.4.2 on Page 1915. This proves the lemma.

Also note that the mapping

\[ x \rightarrow ((x, e_1), \cdots, (x, e_n)) \]

is a \( \sigma(C) \) measurable map. Restricting it to span \( (e_1, \cdots, e_n) \), it is Borel measurable. Next is a definition of a Gaussian measure defined on \( C \). While this is what it is called, it is a fake measure in general because it cannot be extended to a countably additive measure on \( \sigma(C) \). This will be shown below.

Definition 59.9.5 Let \( Q \in L(H, H) \) be self adjoint and satisfy

\[ (Qx, x) > 0 \]

for all \( x \in H, x \neq 0 \). Define \( \nu \) on the cylinder sets, \( C \) by the following rule. For \( \{e_k\}_{k=1}^n \) an orthonormal set in \( H \),

\[
\nu \left( \{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\} \right) = \frac{1}{(2\pi)^{n/2} (\det (\theta^*Q\theta))^{1/2}} \int_F e^{-\frac{1}{2}t^*\theta^*Q^{-1}\theta t} \, dt.
\]

where here

\[ \theta t \equiv \sum_{i=1}^n t_i e_i. \]

Note that the cylinder set is of the form

\[ \theta F + \text{span} (e_1, \cdots, e_n)^\perp. \]

Thus if \( B + M^\perp \) is a typical cylinder set, choose an orthonormal basis for \( M, \{e_k\}_{k=1}^n \) and do the above definition with \( F = \theta^{-1}B \).
To see this last claim which is like what was done earlier, let
\((x, e_1), \ldots, (x, e_n)) \in F\).

Then \(\theta ((x, e_1), \ldots, (x, e_n)) = \sum_i (x, e_i) e_i = Px\) and so
\[x = x - Px + Px = x - Px + \theta ((x, e_1), \ldots, (x, e_n)) \in \theta F + \text{span} (e_1, \ldots, e_n)\]

Thus
\[\{x \in H : ((x, e_1), \ldots, (x, e_n)) \in F\} \subseteq \theta F + \text{span} (e_1, \ldots, e_n)\]

To see the other inclusion, if \(t \in F\) and \(y \in \text{span} (e_1, \ldots, e_n)\), then if \(x = \theta t\), it follows
\[t_i = (x, e_i)\]
and so \(((x, e_1), \ldots, (x, e_n)) \in F\). But \((y, e_k) = 0\) for all \(k\) and so \(x + y\) is in
\[\{x \in H : ((x, e_1), \ldots, (x, e_n)) \in F\}\]

Lemma 59.9.6 The above definition is well defined.

Proof: Let \(\{f_k\}\) be another orthonormal set such that for \(F, G\) Borel sets in \(\mathbb{R}^n\),
\[A = \{x \in H : ((x, e_1), \ldots, (x, e_n)) \in F\} = \{x \in H : ((x, f_1), \ldots, (x, f_n)) \in G\}\]

I need to verify \(\nu(A)\) is the same using either \(\{f_k\}\) or \(\{e_k\}\). Let \(a \in G\). Then
\[x = \sum_{i=1}^n a_i f_i \in A\]
because \((x, f_k) = a_k\). Therefore, for this \(x\) it is also true that \(((x, e_1), \ldots, (x, e_n)) \in F\).

In other words for \(a \in G\),
\[\left(\sum_{i=1}^n (e_1, f_i) a_i, \ldots, \sum_{i=1}^n (e_n, f_i) a_i\right) \in F\]

Let \(L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\) be defined by
\[La = \sum_i L_{ij} a_i, \quad L_{ij} = (e_j, f_i)\]

Since the \(\{e_j\}\) and \(\{f_k\}\) are orthonormal, this mapping is unitary. Also this has shown that
\[LG \subseteq F\]

Similarly
\[L^* F \subseteq G\]
where $L^*$ has the $ij$ entry $L^*_{ij} = (f_i, e_j)$ as above and $L^*$ is the inverse of $L$ because $L$ is unitary. Thus $F = L(L^*(F)) \subseteq L(G) \subseteq F$

showing that $LG = F$ and $L^*F = G$.

Now let $\theta_e\cdot t = \sum t_i e_i$ with $\theta_f$ defined similarly. Then the definition of $\nu(A)$ corresponding to $\{e_i\}$ is

$$\nu(A) \equiv \frac{1}{(2\pi)^{n/2} (\det(\theta^*_e Q \theta_e))^{1/2}} \int_F e^{-\frac{1}{2} t^* \theta^*_e Q^{-1} \theta_e t} dt$$

Now change the variables letting $t = Ls$ where $s \in G$.

From the definition,

$$\theta_e Ls = \sum_j \sum_i (e_j, f_i) s_i e_j = \sum_j \left( e_j, \sum_i f_i s_i \right) e_j = \sum_j (e_j, \theta_f s) e_j$$

and so $Ls = ((e_1, \theta_f s), \ldots, (e_n, \theta_f s)) = \theta^*_e \theta_f s$

where from the definition,

$$(\theta^*_e \theta_f s, t) = \sum_i t_i \left( \sum_j s_j e_j, e_i \right)$$

$$= \sum_i t_i s_i = (s, t)$$

and so $\theta^*_e \theta_f$ is the identity on $\mathbb{R}^n$ and similar reasoning yields $\theta_e \theta^*_f$ is the identity on $\theta_e(\mathbb{R}^n)$. Then using the change of variables formula and the fact $|\det(L)| = 1$,

$$\frac{1}{(2\pi)^{n/2} (\det(\theta^*_e Q \theta_e))^{1/2}} \int_F e^{-\frac{1}{2} t^* \theta^*_e Q^{-1} \theta_e t} dt$$

$$= \frac{1}{(2\pi)^{n/2} (\det(\theta^*_e Q \theta_e))^{1/2}} \int_F e^{-\frac{1}{2} s^* L^* \theta^*_e Q^{-1} \theta_e Ls} ds$$

$$= \frac{1}{(2\pi)^{n/2} (\det(\theta^*_f Q \theta_f))^{1/2}} \int_G e^{-\frac{1}{2} s^* \theta^*_e \theta_e \theta^*_f Q^{-1} \theta_e \theta^*_f s} ds$$

$$= \frac{1}{(2\pi)^{n/2} (\det(\theta^*_f Q \theta_f))^{1/2}} \int_G e^{-\frac{1}{2} s^* \theta^*_f Q^{-1} \theta_f s} ds$$

where part of the justification is as follows.

$\det(\theta^*_f Q \theta_f) = \det(\theta^*_e \theta_e \theta^*_f Q \theta_e \theta^*_f)$
\[ = \det (\theta^* f \theta e) \det (\theta^* Q \theta e) \det (\theta^* \theta f) \]
\[ = \det (\theta^* Q \theta e) \]

because
\[ \det (\theta^* f \theta e) \det (\theta^* \theta f) = \det (\theta^* f \theta e \theta^* \theta f) = \det (\theta^* f) = 1. \]

This proves the lemma.

It would be natural to try to extend \( \nu \) to the \( \sigma \) algebra determined by \( C \) and obtain a measure defined on this \( \sigma \) algebra. However, this is always impossible in the case where \( Q = I \).

**Proposition 59.9.7** For \( Q = I \), \( \nu \) cannot be extended to a measure defined on \( \sigma (C) \) whenever \( H \) is infinite dimensional.

**Proof:** Let \( \{ e_n \} \) be a complete orthonormal set of vectors in \( H \). Then first note that \( H \) is a cylinder set.
\[
H = \{ x \in H : (x, e_1) \in \mathbb{R} \}
\]

and so
\[
\nu (H) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} dt = 1.
\]

However, \( H \) is also equal to the countable union of the sets,
\[
A_n \equiv \{ x \in H : ((x, e_1)_H, \cdots, (x, e_{a_n})_H) \in B(0,n) \}
\]
where \( a_n \to \infty \).
\[
\nu (A_n) = \left( \frac{1}{\sqrt{2\pi}} \right)^{a_n} \int_{B(0,n)} e^{-\frac{1}{2}|t|^2} dt
\leq \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} \cdots \int_{-n}^{n} e^{-|t|^2/2} dt_1 \cdots dt_{a_n}
\leq \left( \frac{\int_{-n}^{n} e^{-x^2/2} dx}{\sqrt{2\pi}} \right)^{a_n}
\]

Now pick \( a_n \) so large that the above is smaller than \( 1/2^{n+1} \). This can be done because for no matter what choice of \( n \),
\[
\int_{-n}^{n} e^{-x^2/2} dx < 1.
\]

Then
\[
\sum_{n=1}^{\infty} \nu (A_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.
\]

This proves the proposition and shows something else must be done to get a countably additive measure from \( \nu \).

However, let \( \mu (C) \equiv \nu_M (C) \) where \( C \) is a cylinder set of the form \( C = B + M^\bot \) for \( M \) a finite dimensional subspace.
Proposition 59.9.8  \( \mu \) is finitely additive on \( C \) the algebra of cylinder sets.

Proof: Let

\[
A \equiv \{ x \in H : ((x, e_1), \cdots, (x, e_n)) \in E \}, \quad B \equiv \{ x \in H : ((x, f_1), \cdots, (x, f_m)) \in F \}
\]

be two disjoint cylinder sets. Then writing them differently as was done earlier they are

\[
\{ x \in H : ((x, e_1), \cdots, (x, e_n), (x, f_1), \cdots, (x, f_m)) \in E \times \mathbb{R}^m \}
\]

and

\[
\{ x \in H : ((x, e_1), \cdots, (x, e_n), (x, f_1), \cdots, (x, f_m)) \in \mathbb{R}^n \times F \}
\]

respectively. Hence the two sets \( E \times \mathbb{R}^m, \mathbb{R}^n \times F \) must be disjoint. Then the definition yields \( \mu (A \cup B) = \mu (A) + \mu (B) \). This proves the proposition.

Definition 59.9.9 Let \( H \) be a separable Hilbert space and let ||·|| be a norm defined on \( H \) which has the following property. Whenever \( \{ e_n \} \) is an orthonormal sequence of vectors in \( H \) and \( F(\{ e_n \}) \) consists of the set of all orthogonal projections onto the span of finitely many of the \( e_k \) the following condition holds. For every \( \varepsilon > 0 \) there exists \( P_{\varepsilon} \in F(\{ e_n \}) \) such that if \( P \in F(\{ e_n \}) \) and \( PP_{\varepsilon} = 0 \), then

\[
\nu(\{ x \in H : \|Px\| > \varepsilon \}) < \varepsilon.
\]

Then ||·|| is called Gross measurable.

The following lemma is a fundamental result about Gross measurable norms. It is about the continuity of ||·||. It is obvious that with respect to the topology determined by ||·|| that \( x \rightarrow ||x|| \) is continuous. However, it would be interesting if this were the case with respect to the topology determined by the norm on \( H, |·| \), This lemma shows this is the case and so the funny condition above implies \( x \rightarrow ||x|| \) is a continuous, hence Borel measurable function.

Lemma 59.9.10 Let ||·|| be Gross measurable. Then there exists \( c > 0 \) such that

\[
||x|| \leq c \ |x|
\]

for all \( x \in H \). Furthermore, the above definition is well defined.

Proof: First it is important to consider the question whether the above definition is well defined. To do this note that on \( PH \), the two norms are equivalent because \( PH \) is a finite dimensional space. Let \( G = \{ y \in PH : \|y\| > \varepsilon \} \) so \( G \) is an open set in \( PH \). Then

\[
\{ x \in H : ||Px|| > \varepsilon \}
\]

equals

\[
\{ x \in H : Px \in G \}
which equals a set of the form
\[ \{ x \in H : ((x, e_1)_H, \ldots, (x, e_m)_H) \in G' \} \]
for \( G' \) an open set in \( \mathbb{R}^m \) and so everything makes sense in the above definition.

Now it is necessary to verify \( ||\cdot|| \leq c \cdot |\cdot| \). If it is not so, there exists \( e_1 \) such that
\[ ||e_1|| \geq 1, \quad |e_1| = 1. \]

Suppose \( \{ e_k \}_{k=1}^n \) have been chosen such that each is a unit vector in \( H \) and \( ||e_k|| \geq k \).

Then considering span \( (e_1, \ldots, e_n)^\perp \) if for every \( x \in \text{span} (e_1, \ldots, e_n)^\perp \), \( ||x|| \leq c |x| \),
then if \( z \in H \) is arbitrary, \( z = x + y \) where \( y \in \text{span} (e_1, \ldots, e_n) \) and so since the two
norms are equivalent on a finite dimensional subspace, there exists \( c' \) corresponding
\[ \|z\|^2 \leq (\|x\|^2 + \|y\|^2) \leq 2 (\|x\|^2 + \|y\|^2) \]
and the lemma is proved. Therefore it can be assumed, there exists
\[ e_{n+1} \in \text{span} (e_1, \ldots, e_n)^\perp \]
such that \( |e_{n+1}| = 1 \) and \( ||e_{n+1}|| \geq n + 1. \)

This constructs an orthonormal set of vectors, \( \{ e_k \} \). Letting \( 0 < \varepsilon < \frac{1}{2} \), it follows since \( ||\cdot|| \) is measurable, there exists \( P_\varepsilon \in \mathcal{F} (\{ e_n \}) \) such that if \( PP_\varepsilon = 0 \)
where \( P \in \mathcal{F} (\{ e_n \}) \), then
\[ \nu (\{ x \in H : ||Px|| > \varepsilon \}) < \varepsilon. \]

Say \( P_n \) is the projection onto the span of finitely many of the \( e_k \), the last one being \( e_N \). Then for \( n > N \) and \( P_n \) the projection onto \( e_n \), it follows \( P_\varepsilon P_n = 0 \) and from the definition of \( \nu \),
\[ \varepsilon > \nu (\{ x \in H : ||P_n x|| > \varepsilon \}) \]
\[ = \nu (\{ x \in H : (x, e_n) \|e_{n+1}\| > \varepsilon \}) \]
\[ = \nu (\{ x \in H : (x, e_n) > \varepsilon / \|e_{n+1}\| \}) \]
\[ \geq \nu (\{ x \in H : (x, e_n) > \varepsilon / (n + 1) \}) \]
\[ > \frac{1}{\sqrt{2\pi}} \int_{\varepsilon/(n+1)}^{\infty} e^{-x^2/2} dx \]
which yields a contradiction for all \( n \) large enough. This proves the lemma.

What are examples of Gross measurable norms defined on a separable Hilbert
space, \( H \)? The following lemma gives an important example.
Lemma 59.9.11 Let $H$ be a separable Hilbert space and let $A \in \mathcal{L}_2(H,H)$, a Hilbert Schmidt operator. Thus $A$ is a continuous linear operator with the property that for any orthonormal set, $\{e_k\}$,

$$
\sum_{k=1}^{\infty} |Ae_k|^2 < \infty.
$$

Then define $||\cdot||$ by

$$
||x|| \equiv |Ax|_H.
$$

Then if $||\cdot||$ is a norm, it is measurable\(^1\).

Proof: Let $\{e_k\}$ be an orthonormal sequence. Let $P_n$ denote the orthogonal projection onto span $(e_1, \cdots, e_n)$. Let $\varepsilon > 0$ be given. Since $A$ is a Hilbert Schmidt operator, there exists $N$ such that

$$
\sum_{k=N}^{\infty} |Ae_k|^2 < \alpha
$$

where $\alpha$ is chosen very small. In fact, $\alpha$ is chosen such that $\alpha < \varepsilon^2/r^2$ where $r$ is sufficiently large that

$$
\frac{2}{\sqrt{2\pi}} \int_r^{\infty} e^{-t^2/2} dt < \varepsilon.
$$

Let $P$ denote an orthogonal projection in $\mathcal{F}(\{e_k\})$ such that $PP_N = 0$. Thus $P$ is the projection on to span $(e_{i_1}, \cdots, e_{i_m})$ where each $i_k > N$. Then

$$
\nu \left( \{ x \in H : ||Px|| > \varepsilon \} \right) = \nu \left( \{ x \in H : |APx| > \varepsilon \} \right)
$$

Now $Px = \sum_{j=1}^{m} (x, e_{i_j}) e_{i_j}$ and the above reduces to

$$
\nu \left( \left\{ x \in H : \left| \sum_{j=1}^{m} (x, e_{i_j}) Ae_{i_j} \right| > \varepsilon \right\} \right) \leq
$$

\(^1\)If it is only a seminorm, it satisfies the same conditions.
\[ \nu \left( \left\{ x \in H : \left( \sum_{j=1}^{m} |(x,e_{i_j})|^2 \right)^{1/2} \left( \sum_{j=1}^{m} |Ae_{i_j}|^2 \right)^{1/2} > \varepsilon \right\} \right) \]
\[ \leq \nu \left( \left\{ x \in H : \left( \sum_{j=1}^{m} |(x,e_{i_j})|^2 \right)^{1/2} > \frac{\varepsilon}{\alpha^{1/2}} \right\} \right) \]
\[ = \nu \left( \left\{ x \in H : \left( \sum_{j=1}^{m} |(x,e_{i_j})|^2 \right)^{1/2} > \frac{\varepsilon}{\alpha^{1/2}} \right\} \right) \]
\[ = \nu \left( \left\{ x \in H : ((x,e_{i_1}), \ldots, (x,e_{i_m})) \in B \left( 0, \frac{\varepsilon}{\alpha^{1/2}} \right)^{C} \right\} \right) \]
\[ \leq \nu \left( \left\{ x \in H : \text{max} \left\{ \| (x,e_{i_j}) \| \right\} > \frac{\varepsilon}{\sqrt{\alpha m^{1/2}}} \right\} \right) \]

This is no larger than
\[
\frac{1}{\sqrt{2\pi}} \int_{|t_1| > \frac{\varepsilon}{\sqrt{\alpha m^{1/2}}}} \cdots \int_{|t_m| > \frac{\varepsilon}{\sqrt{\alpha m^{1/2}}}} e^{-|t|^2/2} dt_1 \cdots dt_m
\]
\[ = \left( \frac{2 \int_{\varepsilon/\sqrt{\alpha^{1/2}}} e^{-t^2/2} dt}{\sqrt{2\pi}} \right)^m \]

which by Jensen’s inequality is no larger than
\[
\frac{2 \int_{\varepsilon/\sqrt{\alpha^{1/2}}} e^{-t^2/2} dt}{\sqrt{2\pi}} \leq \frac{2 \int_{\varepsilon/\sqrt{\alpha^{1/2}}} e^{-t^2/2} dt}{\sqrt{2\pi}} < \varepsilon
\]

By \[59.9.42\] This proves the lemma.

**Definition 59.9.12** A triple, \((i,H,B)\) is called an abstract Wiener space if \(B\) is a separable Banach space and \(H\) is a separable Hilbert space such that \(H\) is dense and continuously embedded in \(B\) and the norm \(\| \cdot \|\) on \(B\) is Gross measurable.

Next consider a weaker norm for \(H\) which comes from the inner product
\[ (x,y)_{E} \equiv \sum_{k=1}^{\infty} \frac{1}{k^2} (x,e_{k})_{H} (y,e_{k})_{H} \cdot \]
Then let $E$ be the completion of $H$ with respect to this new norm. Thus $\{ke_k\}$ is a complete orthonormal basis for $E$. This follows from the density of $H$ in $E$ along with the obvious observation that in the above inner product, $\{ke_k\}$ is an orthonormal set of vectors.

**Lemma 59.9.13** There exists a countably additive Gaussian measure, $\lambda$ defined on $\mathcal{B}(E)$. This measure is the law of the random variable, 

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega)e_k,$$

where $\{\xi_k\}$ denotes a sequence of independent normally distributed random variables having mean 0 and variance 1, the series converging pointwise a.e. in $E$. Also 

$$k^2 \langle X(\omega), e_k \rangle_E = \xi_k(\omega) \text{ a.e.}$$

**Proof:** Observe that $\sum_{k=1}^{\infty} \frac{1}{k^2} (ke_k) \otimes (ke_k)$ is a nuclear operator on the Hilbert space, $E$. Letting $\{\xi_k\}$ be a sequence of independent random variables each normally distributed with mean 0 and variance 1, that

$$X(\omega) \equiv \sum_{k=1}^{\infty} \frac{1}{k}\xi_k(\omega)e_k = \sum_{k=1}^{\infty} \xi_k(\omega)e_k \quad (59.9.43)$$

is a random variable with values in $E$ and $\mathcal{L}(X)$ is a Gaussian measure on $\mathcal{B}(E)$, the series converging pointwise a.e. in $E$. Let $\lambda$ be the name of this Gaussian measure and denote the probability space on which the $\xi_k$ are defined as $(\Omega, \mathcal{F}, P)$. Thus for $F \in \mathcal{B}(E)$,

$$\lambda(F) \equiv P(\{\omega \in \Omega : X(\omega) \in F\})$$

Finally, denoting by $X_N$, the partial sum, 

$$X_N(\omega) \equiv \sum_{k=1}^{N} \xi_k(\omega)e_k,$$

the definition of $\langle \cdot, \cdot \rangle_E$ on $H$ and a simple computation yields

$$\xi_k(\omega) = \lim_{N \to \infty} k^2 \langle X_N(\omega), e_k \rangle_E = k^2 \langle X(\omega), e_k \rangle_E. \quad (59.9.44)$$

One can pass to the limit because $X_N(\omega)$ converges to $X(\omega)$ in $E$. This proves the lemma.

**Theorem 59.9.14** Let $(i, H, B)$ be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of $B$. 

---

(59.9.43)
59.9. ABSTRACT WIENER SPACES

Proof: Let $E$ be defined above as the completion of $H$ with respect to that weaker norm. Then from Lemma 59.9.13 and $X(\omega)$ given above in 59.9.43, 

$$k^2 (X(\omega), e_k)_E = \xi_k(\omega) \text{ a.e. } \omega.$$

Let $\{e_n\}$ be a complete orthonormal set for $H$. There exists an increasing sequence of projections, $\{Q_n\} \subseteq \mathcal{F}(\{e_n\})$ such that $Q_n x \rightarrow x$ in $H$ for each $x \in H$. Say $Q_n$ is the orthogonal projection onto span $(e_1, \cdots, e_{p_n})$. Then since $||\cdot||$ is measurable, these can be chosen such that if $Q$ is the orthogonal projection onto span $(e_1, \cdots, e_k)$ for some $k > p_n$ then 

$$\nu \left( \{x : ||Qx - Q_n x|| > 2^{-n}\} \right) < 2^{-n}.$$

In particular, 

$$\nu \left( \{x : ||Q_n x - Q_m x|| > 2^{-m}\} \right) < 2^{-m}$$

whenever $n \geq m$.

I would like to consider the infinite series, 

$$S(\omega) \equiv \sum_{k=1}^{\infty} k^2 (X(\omega), e_k)_E e_k \in B.$$ 

converging in $B$ but of course this might make no sense because the series might not converge. It was shown above that the series converges in $E$ but it has not been shown to converge in $B$.

Suppose the series did converge a.e. Then let $f \in B'$ and consider the random variable $f \circ S$ which maps $\Omega$ to $\mathbb{R}$. I would like to verify this is normally distributed. First note that the following finite sum is weakly measurable and separably valued so it is strongly measurable with values in $B$.

$$S_{p_n}(\omega) \equiv \sum_{k=1}^{p_n} k^2 (X(\omega), e_k)_E e_k,$$

Since $f \in B'$ which is a subset of $H'$ due to the assumption that $H$ is dense in $B$, there exists a unique $v \in H$ such that $f(x) = (x, v)$ for all $x \in H$. Then from the above sum, 

$$f(S_{p_n}(\omega)) = (S_{p_n}(\omega), v) = \sum_{k=1}^{p_n} k^2 (X(\omega), e_k)_E (e_k, v)$$

which by Lemma 59.9.13 equals 

$$\sum_{k=1}^{p_n} (e_k, v)_H \xi_k(\omega)$$

a finite linear combination of the independent $N(0, 1)$ random variables, $\xi_k(\omega)$. Then it follows 

$$\omega \rightarrow f(S_{p_n}(\omega))$$
is also normally distributed and has mean 0 and variance equal to
\[ \sum_{k=1}^{p_n} (e_k, v)_H^2. \]

Then it seems reasonable to suppose
\[
E(e^{itf \circ S}) = \lim_{n \to \infty} E(e^{itf \circ S_{p_n}}) = \lim_{n \to \infty} e^{-t^2 \sum_{k=1}^{p_n} (e_k, v)_H^2} = e^{-t^2 \sum_{k=1}^{\infty} (e_k, v)_H^2} = e^{-t^2 \|v\|_H^2}
\]

the characteristic function of a random variable which is \( N \left( 0, \|v\|_H^2 \right) \). Thus at least formally, this would imply for all \( f \in B' \), \( f \circ S \) is normally distributed and so if \( \mu = \mathcal{L}(S) \), then by Lemma 59.7.3 it follows \( \mu \) is a Gaussian measure.

What is missing to make the above a proof? First of all, there is the issue of the sum. Next there is the problem of passing to the limit in the little argument above in which the characteristic function is used.

First consider the sum. Note that
\[
\begin{align*}
P \left( \{ \omega \in \Omega : \|S_n(\omega) - S_{p_m}(\omega)\| > 2^{-m} \} \right) \\
= P \left( \left\{ \omega \in \Omega : \left\| \sum_{k=p_m+1}^{n} k^2 (X(\omega), e_k) e_k \right\| > 2^{-m} \right\} \right) \\
= P \left( \left\{ \omega \in \Omega : \left\| \sum_{k=p_m+1}^{n} \xi_k(\omega) e_k \right\| > 2^{-m} \right\} \right)
\end{align*}
\]

Let \( Q \) be the orthogonal projection onto \( \text{span} \{e_1, \cdots, e_n\} \). Define
\[ F = \{ x \in (Q - Q_m) H : \|x\| > 2^{-m} \} \]

Then continuing the chain of equalities ending with \( \nu \),
\[
\begin{align*}
P \left( \left\{ \omega \in \Omega : \left\| \sum_{k=p_m+1}^{n} \xi_k(\omega) e_k \right\| < 2^{-m} \right\} \right) \\
= P \left( \left\{ \omega \in \Omega : \left( \xi_n(\omega), \cdots, \xi_{p_m+1}(\omega) \right) \in F' \right\} \right) \\
= \nu \left( \{ x \in H : (x, e_n)_H, \cdots, (x, e_{p_m+1})_H \} \in F' \} \right) \\
= \nu \left( \{ x \in H : (Q(x) - Q_m(x)) \in F \} \right) \\
= \nu \left( \{ x \in H : \|Q(x) - Q_m(x)\| > 2^{-m} \} \right) < 2^{-m}.
\end{align*}
\]
This has shown that
\[
P \left( \{ \omega \in \Omega : \|S_n (\omega) - S_{p_m} (\omega)\| > 2^{-m} \} \right) < 2^{-m}
\] (59.9.47)
for all \(n > p_m\). In particular, the above is true if \(n = p_n\) for \(n > m\).

If \(\{S_{p_n} (\omega)\}\) fails to converge, then \(\omega\) must be contained in the set,
\[
A \equiv \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{ \omega \in \Omega : \|S_{p_n} (\omega) - S_{p_m} (\omega)\| > 2^{-m} \}
\] (59.9.48)
because if \(\omega\) is in the complement of this set,
\[
\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{ \omega \in \Omega : \|S_{p_n} (\omega) - S_{p_m} (\omega)\| \leq 2^{-m} \},
\]
it follows \(\{S_{p_n} (\omega)\}_{n=1}^{\infty}\) is a Cauchy sequence and so it must converge. However, the set in (59.9.48) is a set of measure 0 because of (59.9.47) and the observation that for all \(m\),
\[
P(A) \leq \sum_{n=m}^{\infty} P \left( \{ \omega \in \Omega : \|S_{p_n} (\omega) - S_{p_m} (\omega)\| > 2^{-m} \} \right)
\]
\[
\leq \sum_{n=m}^{\infty} \frac{1}{2^m}
\]
Thus the subsequence \(\{S_{p_n}\}\) of the sequence of partial sums of the above series does converge pointwise in \(B\) and so the dominated convergence theorem also verifies that the computations involving the characteristic function in (59.9.45) are correct.

The random variable obtained as the limit of the partial sums, \(\{S_{p_n} (\omega)\}\) described above is strongly measurable because each \(S_{p_n} (\omega)\) is strongly measurable due to each of these being weakly measurable and separably valued. Thus the measure given as the law of \(S\) defined as
\[
S (\omega) \equiv \lim_{n \to \infty} S_{p_n} (\omega)
\]
is defined on the Borel sets of \(B\). This proves the theorem.

Also, there is an important observation from the proof which I will state as the following corollary.

**Corollary 59.9.15** Let \((i, H, B)\) be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of \(B\). This Gaussian measure equals \(L (S)\) where \(S (\omega)\) is the a.e. limit of a subsequence of the sequence of partial sums,
\[
S_{p_n} (\omega) \equiv \sum_{k=1}^{p_n} \xi_k (\omega) e_k
\]
for \(\{\xi_k\}\) a sequence of independent random variables which are normal with mean 0 and variance 1 which are defined on a probability space, \((\Omega, \mathcal{F}, P)\). Furthermore, for any \(k > p_n\),
\[
P \left( \{ \omega \in \Omega : \|S_k (\omega) - S_{p_n} (\omega)\| > 2^{-n} \} \right) < 2^{-n}.
\]
59.10 White Noise

In an abstract Wiener space as discussed above there is a Gaussian measure, $\mu$ defined on the Borel sets of $B$. This measure is the law of a random variable having values in $B$ which is the limit of a subsequence of a sequence of partial sums. I will show here that the sequence of partial sums also converges pointwise a.e.

Now with this preparation, here is the theorem about white noise.

**Theorem 59.10.1** Let $(i, H, B)$ be an abstract Wiener space and $\{e_k\}$ is a complete orthonormal sequence in $H$. Then there exists a Gaussian measure on the Borel sets of $B$. This Gaussian measure equals $L(S)$ where $S(\omega)$ is the a.e. limit of the sequence of partial sums,

$$S_n(\omega) \equiv \sum_{k=1}^{n} \xi_k(\omega) e_k$$

for $\{\xi_k\}$ a sequence of independent random variables which are normal with mean 0 and variance 1, defined on a probability space, $(\Omega, \mathcal{F}, P)$

**Proof:** By Corollary 59.9.15 there is a subsequence, $\{S_{p_n}\}$ of these partial sums which converge pointwise a.e. to $S(\omega)$. However, this corollary also states that

$$P\left(\{\omega \in \Omega : ||S_k(\omega) - S_{p_n}(\omega)|| > 2^{-n}\}\right) < 2^{-n}$$

whenever $k > p_n$ and so by Lemma 59.15.6 the original sequence of partial sums also converges uniformly of a set of measure zero. The reason this lemma applies is that $\xi_k(\omega) e_k$ has symmetric distribution. This proves the theorem.

59.11 Existence Of Abstract Wiener Spaces

It turns out that if $E$ is a separable Banach space, then it is the top third of an abstract Wiener space. This is what will be shown in this section. Therefore, it follows from the above that there exists a Gaussian measure on $E$ which is the law of an a.e. convergent series as discussed above. First recall Lemma 15.4.2 on Page 443.

**Lemma 59.11.1** Let $E$ be a separable Banach space. Then there exists an increasing sequence of subspaces, $\{F_n\}$ such that $\dim(F_{n+1}) - \dim(F_n) \leq 1$ for all $n$ if the dimension of $E$ is infinite. Also $\bigcup_{n=1}^{\infty} F_n$ is dense in $E$.

**Lemma 59.11.2** Let $E$ be a separable Banach space. Then there exists a sequence $\{e_n\}$ of points of $E$ such that whenever $|\beta| \leq 1$ for $\beta \in \mathbb{F}^n$,

$$\sum_{k=1}^{n} \beta_k e_k \in B(0,1)$$

the unit ball in $E$. 

59.11. EXISTENCE OF ABSTRACT WIENER SPACES

Proof: By Lemma 59.11.1, let \( \{z_1, \ldots, z_n\} \) be a basis for \( F_n \) where \( \bigcup_{n=1}^{\infty} F_n \) is dense in \( E \). Then let \( \alpha_1 \) be such that \( e_1 \equiv \alpha_1 z_1 \in B(0,1) \). Thus \( \beta_1 e_1 \in B(0,1) \) whenever \( |\beta_1| \leq 1 \). Suppose \( \alpha_i \) has been chosen for \( i = 1, 2, \ldots, n \) such that for all \( \in D_n \equiv \{ \alpha \in \mathbb{F}^n : |\alpha| \leq 1 \} \), it follows

\[
\sum_{k=1}^{n} \beta_k \alpha_k z_k \in B(0,1).
\]

Then

\[
C_n \equiv \left\{ \sum_{k=1}^{n} \beta_k \alpha_k z_k : \beta \in D_n \right\}
\]

is a compact subset of \( B(0,1) \) and so it is at a positive distance from the complement of \( B(0,1) \), \( \delta \). Now let \( 0 < \alpha_{n+1} < \delta / \|z_{n+1}\| \). Then for \( \beta \in D_{n+1} \),

\[
\sum_{k=1}^{n} \beta_k \alpha_k z_k \in C_n
\]

and so

\[
\left\| \sum_{k=1}^{n+1} \beta_k \alpha_k z_k - \sum_{k=1}^{n} \beta_k \alpha_k z_k \right\| = \| \beta_{n+1} \alpha_{n+1} z_{n+1} \| < \| \alpha_{n+1} z_{n+1} \| < \delta
\]

which shows

\[
\sum_{k=1}^{n+1} \beta_k \alpha_k z_k \in B(0,1).
\]

This proves the lemma. Let \( e_k \equiv \alpha_k z_k \).

Now the main result is the following. It says that any separable Banach space is the upper third of some abstract Wiener space.

**Theorem 59.11.3** Let \( E \) be a real separable Banach space with norm \( \|\cdot\| \). Then there exists a separable Hilbert space, \( H \) such that \( H \) is dense in \( E \) and the inclusion map is continuous. Furthermore, if \( \nu \) is the Gaussian measure defined earlier on the cylinder sets of \( H, \|\cdot\| \) is Gross measurable.

Proof: Let \( \{e_k\} \) be the points of \( E \) described in Lemma 59.11.2. Then let \( H_0 \) denote the subspace of all finite linear combinations of the \( \{e_k\} \). It follows \( H_0 \) is dense in \( E \). Next decree that \( \{e_k\} \) is an orthonormal basis for \( H_0 \). Thus for

\[
\sum_{k=1}^{n} c_k e_k, \sum_{j=1}^{n} d_j e_j \in H_0,
\]

\[
\left( \sum_{k=1}^{n} c_k e_k, \sum_{j=1}^{n} d_j e_j \right)_{H_0} \equiv \sum_{k=1}^{n} c_k d_k
\]
this being well defined because the \{e_k\} are independent. Let the norm on \(H_0\) be denoted by \(|·|_{H_0}\). Let \(H_1\) be the completion of \(H_0\) with respect to this norm.

I want to show that \(|·|_{H_0}\) is stronger than \(|·|_{H_1}\). Suppose then that

\[
\sum_{k=1}^{n} \beta_k e_k \leq 1.
\]

It follows then from the definition of \(|·|_{H_0}\) that

\[
\sum_{k=1}^{n} \beta_k^2 e_k^2 = \sum_{k=1}^{n} \beta_k^2 \leq 1
\]

and so from the construction of the \(e_k\), it follows that

\[
\left| \sum_{k=1}^{n} \beta_k e_k \right| < 1
\]

Stated more simply, this has just shown that if \(h \in H_0\) then since \(|h|_{H_0} \leq 1\), it follows that

\[
||h||_{H_1} < |h|_{H_0}.
\]

It follows that the completion of \(H_0\) must lie in \(E\) because this shows that every Cauchy sequence in \(H_0\) is a Cauchy sequence in \(E\). Thus \(H_1\) embeds continuously into \(E\) and is dense in \(E\). Denote its norm by \(|·|_{H_1}\).

Now consider the nuclear operator,

\[
A = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k
\]

where each \(\lambda_k > 0\) and \(\sum_k \lambda_k < \infty\). This operator is clearly one to one. Also it is clear the operator is Hilbert Schmidt because \(\sum_k \lambda_k^2 < \infty\). Let

\[
H \equiv AH_1.
\]

and for \(x \in H\), define

\[
|x|_H \equiv |A^{-1} x|_{H_1}.
\]

Since each \(e_k\) is in \(H\) it follows that \(H\) is dense in \(E\). Note also that \(H \subseteq H_1\) because \(A\) maps \(H_1\) to \(H_1\).

\[
Ax = \sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k
\]
and the series converges in $H_1$ because
\[
\sum_{k=1}^{\infty} \lambda_k |(x, e_k)| \leq \left( \sum_{k=1}^{\infty} \lambda_k^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} |(x, e_k)|^2 \right)^{1/2} < \infty.
\]

Also $H$ is a Hilbert space with inner product given by
\[
(x, y)_H \equiv (A^{-1}x, A^{-1}y)_{H_1}.
\]

$H$ is complete because if $\{x_n\}$ is a Cauchy sequence in $H$, this is the same as $\{A^{-1}x_n\}$ being a Cauchy sequence in $H_1$ which implies $A^{-1}x_n \to y$ for some $y \in H_1$. Then it follows $x_n = A(A^{-1}x_n) \to Ay$ in $H$.

For $x \in H \subseteq H_1$,
\[
||x|| \leq ||x||_{H_1} = |A| ||A^{-1}x||_{H_1} \leq ||A|| |A^{-1}x|_{H_1} = ||A|| ||x||_H
\]
and so the embedding of $H$ into $E$ is continuous. Why is $||\cdot||$ a measurable norm on $H$? Note first that for $x \in H \subseteq H_1$, \[
|Ax|_H = |A| ||A^{-1}Ax||_{H_1} = |x|_{H_1} \geq ||x||_E.
\]
(59.11.49)

Therefore, if it can be shown $A$ is a Hilbert Schmidt operator on $H$, the desired measurability will follow from Lemma 59.9.11 on Page 2086.

Claim: $A$ is a Hilbert Schmidt operator on $H$.

Proof of the claim: From the definition of the inner product in $H$, it follows an orthonormal basis for $H$ is $\{\lambda_k e_k\}$. This is because
\[
(\lambda_k e_k, \lambda_j e_j)_H \equiv (\lambda_k A^{-1} e_k, \lambda_j A^{-1} e_j)_{H_1} = (e_k, e_j)_{H_1} = \delta_{jk}.
\]

To show that $A$ is Hilbert Schmidt, it suffices to show that
\[
\sum_k |A(\lambda_k e_k)|^2 < \infty
\]
because this is the definition of an operator being Hilbert Schmidt. However, the above equals
\[
\sum_k |A^{-1}A(\lambda_k e_k)|^2_{H_1} = \sum_k \lambda_k^2 < \infty.
\]

This proves the claim.

Now consider $\|\cdot\|_E$. By Lemma 59.9.4, it follows the norm $||\cdot||_E' \equiv |Ax|_H$ is Gross measurable on $H$. Therefore, $||\cdot||_E$ is also Gross measurable because it is smaller. This proves the theorem.

Using Theorem 59.11.2 and Theorem 59.11.3 this proves most of the following important corollary.
Corollary 59.11.4 Let $E$ be any real separable Banach space. Then there exists a sequence, $\{e_k\} \subseteq E$ such that for any $\{\xi_k\}$ a sequence of independent random variables such that $\mathcal{L}(\xi_k) = N(0, 1)$, it follows

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega) e_k$$

converges a.e. and its law is a Gaussian measure defined on $\mathcal{B}(E)$. Furthermore, $\|e_k\|_E \leq \lambda_k$ where $\sum_k \lambda_k < \infty$.

Proof: From the proof of Theorem 59.10.1, a basis for $H$ is $\{\lambda_k e_k\}$. Therefore, by Theorem 59.10.1, if $\{\xi_k\}$ is a sequence of independent $N(0, 1)$ random variables, then $\sum_{k=1}^{\infty} \xi_k(\omega) \lambda_k e_k$ converges a.e. to a random variable whose law is Gaussian. Also from the proof of Theorem 59.10.1, each $e_k$ in that proof has the property that $\|e_k\| \leq 1$ because if $\|e_k\| > 1$, then you could consider $\beta \equiv (0, 0, \cdots, 1)$ and from the construction of the $e_k$, you would need $1e_k \in B(0, 1)$ which is a contradiction. Thus $\|\lambda_k e_k\| \leq \lambda_k$ and changing the notation, replacing $\lambda_k e_k$ with $e_k$, this proves the corollary.
Chapter 60

Stochastic Processes

60.1 Fundamental Definitions And Properties

Here $E$ will be a separable Banach space and $B(E)$ will be the Borel sets of $E$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $I$ will be an interval of $\mathbb{R}$. A set of $E$ valued random variables, one for each $t \in I$, $\{X(t) : t \in I\}$ is called a stochastic process. Thus for each $t$, $X(t)$ is a measurable function of $\omega \in \Omega$. Set $X(t, \omega) \equiv X(t)(\omega)$. Functions $t \to X(t, \omega)$ are called trajectories. Thus there is a trajectory for each $\omega \in \Omega$. A stochastic process, $Y$ is called a version or a modification of a stochastic process, $X$ if for all $t \in I$,

$$X(t, \omega) = Y(t, \omega) \text{ a.e. } \omega$$

There are several descriptions of stochastic processes.

1. $X$ is measurable if $X(\cdot, \cdot) : I \times \Omega \to E$ is $B(I) \times \mathcal{F}$ measurable. Note that a stochastic process, $X$ is not necessarily measurable.

2. $X$ is stochastically continuous at $t_0 \in I$ means: for all $\varepsilon > 0$ and $\delta > 0$ there exists $\rho > 0$ such that

$$P(||X(t) - X(t_0)|| \geq \varepsilon) \leq \delta \text{ whenever } |t - t_0| < \rho, t \in I.$$ 

Note the above condition says that for each $\varepsilon > 0$,

$$\lim_{t \to t_0} P(||X(t) - X(t_0)|| \geq \varepsilon) = 0.$$ 

3. $X$ is stochastically continuous if it is stochastically continuous at every $t \in I$.

4. $X$ is stochastically uniformly continuous if for every $\varepsilon, \delta > 0$ there exists $\rho > 0$ such that whenever $s, t \in I$ with $|s - t| < \rho$, it follows

$$P(||X(t) - X(s)|| \geq \varepsilon) \leq \delta.$$ 

2097
5. \(X\) is mean square continuous at \(t_0 \in I\) if
   \[
   \lim_{t \to t_0} E \left( \| X(t) - X(t_0) \|^2 \right) \equiv \lim_{t \to t_0} \int_{\Omega} \| X(t)(\omega) - X(t_0)(\omega) \|^2 dP = 0.
   \]

6. \(X\) is mean square continuous in \(I\) if it is mean square continuous at every point of \(I\).

7. \(X\) is continuous with probability 1 or continuous if \(t \to X(t,\omega)\) is continuous for all \(\omega\) outside some set of measure 0.

8. \(X\) is H\"older continuous if \(t \to X(t,\omega)\) is H\"older continuous for a.e. \(\omega\).

**Lemma 60.1.1** A stochastically continuous process on \([a,b] \equiv I\) is uniformly stochastically continuous on \([a,b] \equiv I\).

**Proof:** If this is not so, there exists \(\varepsilon, \delta > 0\) and points of \(I, s_n, t_n\) such that even though
   \[
   |t_n - s_n| < \frac{1}{n},
   \]
   
   \[
   P \left( \| X(s_n) - X(t_n) \| \geq \varepsilon \right) > \delta. \quad (60.1.1)
   \]

Taking a subsequence, still denoted by \(s_n\) and \(t_n\) there exists \(t \in I\) such that the above hold and
   \[
   \lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = t.
   \]

Then
   \[
   P \left( \| X(s_n) - X(t_n) \| \geq \varepsilon \right) 
   \leq P \left( \| X(s_n) - X(t) \| \geq \varepsilon/2 \right) + P \left( \| X(t) - X(t_n) \| \geq \varepsilon/2 \right).
   \]

But the sum of the last two terms converges to 0 as \(n \to \infty\) by stochastic continuity of \(X\) at \(t\), violating 60.1.1 for all \(n\) large enough. This proves the lemma.

For a stochastically continuous process defined on a closed and bounded interval, there always exists a measurable version. This is significant because then you can do things with product measure and iterated integrals.

**Proposition 60.1.2** Let \(X\) be a stochastically continuous process defined on a closed interval, \(I \equiv [a,b]\). Then there exists a measurable version of \(X\).

**Proof:** By Lemma 60.1.1 \(X\) is uniformly stochastically continuous and so there exists a sequence of positive numbers, \(\{\rho_n\}\) such that if \(|s - t| < \rho_n\), then
   \[
   P \left( \| X(t) - X(s) \| \leq \frac{1}{2n} \right) \leq \frac{1}{2^n}. \quad (60.1.2)
   \]
60.1. FUNDAMENTAL DEFINITIONS AND PROPERTIES

Then let \( \{t^n_0, t^n_1, \cdots, t^n_m \} \) be a partition of \([a, b]\) in which \( |t^n_i - t^n_{i-1}| < \rho_n \). Now define \( X_n \) as follows:

\[
X_n (t) \equiv \sum_{i=1}^{m_n} X (t_{i-1}^n) X_{[t_{i-1}^n, t^n_i]} (t) \\
X_n (b) \equiv X (b).
\]

Then \( X_n \) is obviously \( B(I) \times F \) measurable because it is the sum of functions which are. Consider the set, \( A \) on which \( \{X_n (t, \omega)\} \) is a Cauchy sequence. This set is of the form

\[
A = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{p,q \geq m} \left[ ||X_p - X_q|| < \frac{1}{n} \right]
\]

and so it is a \( B(I) \times F \) measurable set. Now define

\[
Y (t, \omega) \equiv \begin{cases} 
\lim_{n \to \infty} X_n (t, \omega) & \text{if } (t, \omega) \in A \\
0 & \text{if } (t, \omega) \notin A
\end{cases}
\]

I claim \( Y (t, \omega) = X (t, \omega) \) for a.e. \( \omega \). To see this, consider \( \Box \). From the construction of \( X_n \), it follows that for each \( t \),

\[
P \left( \left. ||X_n (t) - X (t)|| \geq \frac{1}{2^n} \right\} \right) \leq \frac{1}{2^n}
\]

Also, for a fixed \( t \), if \( X_n (t, \omega) \) fails to converge to \( X (t, \omega) \), then \( \omega \) must be in infinitely many of the sets,

\[
B_n = \left[ ||X_n (t) - X (t)|| \geq \frac{1}{2^n} \right]
\]

which is a set of measure zero by the Borel Cantelli lemma. Recall why this is so.

\[
P \left( \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n \right) \leq \sum_{n=k}^{\infty} P (B_n) < \frac{1}{2^n}
\]

Therefore, for each \( t, (t, \omega) \in A \) for a.e. \( \omega \). Hence \( X (t) = Y (t) \) a.e. and so \( Y \) is a measurable version of \( X \).

**Lemma 60.1.3** Let \( D \) be a dense subset of an interval, \( I = [0, T] \) and suppose \( X : D \to E \) satisfies

\[
||X (d) - X (d')|| \leq C |d - d'|^\gamma
\]

for all \( d', d \in D \). Then \( X \) extends uniquely to a continuous \( Y \) defined on \([0, T]\) such that

\[
||Y (t) - Y (t')|| \leq C |t - t'|^\gamma.
\]
Proof: Let \( t \in I \) and let \( d_k \to t \) where \( d_k \in D \). Then \( \{X(d_k)\} \) is a Cauchy sequence because \( ||X(d_k) - X(d_m)|| \leq C|d_k - d_m|^{\gamma} \). Therefore, \( X(d_k) \) converges. The thing it converges to will be called \( Y(t) \). Note this is well defined, giving \( X(t) \) if \( t \in D \). Also, if \( d_k \to t \) and \( d'_k \to t \), then \( ||X(d_k) - X(d'_k)|| \leq C|d_k - d'_k|^{\gamma} \) and so \( X(d_k) \) and \( X(d'_k) \) converge to the same thing. Therefore, it makes sense to define \( Y(t) \equiv \lim_{d_k \to t} X(d) \). It only remains to verify the estimate. But letting \( |d - t| \) and \( |d' - t'| \) be small enough,

\[
||Y(t) - Y(t')|| = ||X(d) - X(d')|| + \varepsilon \\
\leq C|d' - d| + \varepsilon \leq C|t - t'| + 2\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, this proves the existence part of the lemma. Uniqueness follows from observing that \( Y(t) \) must equal \( \lim_{d \to t} X(d) \). This proves the lemma.

### 60.2 Kolmogorov Čentsov Continuity Theorem

**Lemma 60.2.1** Let \( r^m_j \) denote \( j \left( \frac{T}{2^m} \right) \) where \( j \in \{0, 1, \ldots, 2^m\} \). Also let \( D_m = \{r^m_j\}_{j=1}^{2^m} \) and \( D = \bigcup_{m=1}^{\infty} D_m \). Suppose \( X(t) \) satisfies

\[
||X(r^m_{j+1}) - X(r^m_j)|| \leq 2^{-\gamma k}
\]

(60.2.3) for all \( k \geq M \). Then if \( d, d' \in D_m \) for \( m > n \geq M \) such that \( |d - d'| \leq T2^{-n} \), then

\[
||X(d') - X(d)|| \leq 2 \sum_{j=n+1}^{m} 2^{-\gamma j}.
\]

Also, there exists a constant \( C \) depending on \( M \) such that for all \( d, d' \in D \),

\[
||X(d) - X(d')|| \leq C|d - d'|^{\gamma}.
\]

**Proof:** Suppose \( d' < d \). Suppose first \( m = n + 1 \). Then \( d = (k + 1)T2^{-(n+1)} \) and \( d' = kT2^{-(n+1)} \). Then from (60.2.3)

\[
||X(d') - X(d)|| \leq 2^{-\gamma(n+1)} \leq 2 \sum_{j=n+1}^{n+1} 2^{-\gamma j}.
\]

Suppose the claim is true for some \( m > n \) and let \( d, d' \in D_{m+1} \) with \( |d - d'| < T2^{-n} \). If there is no point of \( D_m \) between these, then \( d', d \) are adjacent points either in \( D_m \) or in \( D_{m+1} \). Consequently,

\[
||X(d') - X(d)|| \leq 2^{-\gamma m} < 2 \sum_{j=n+1}^{m+1} 2^{-\gamma j}.
\]

Assume therefore, there exist points of \( D_m \) between \( d' \) and \( d \). Let \( d' \leq d'_1 \leq d_1 \leq d \) where \( d_1, d'_1 \) are in \( D_m \) and \( d'_1 \) is the smallest element of \( D_m \) which is at least
as large as \( d' \) and \( d_1 \) is the largest element of \( D_m \) which is no larger than \( d \). Then \(|d' - d_1'| \leq T2^{-(m+1)} \) and \(|d_1 - d| \leq T2^{-(m+1)} \) while all of these points are in \( D_{m+1} \) which contains \( D_m \). Therefore, from (60.2.3) and induction,

\[
||X(d') - X(d)|| \\
\leq ||X(d') - X(d_1)|| + ||X(d_1) - X(d)|| \\
\leq 2 \times 2^{-\gamma(m+1)} + \sum_{j=n+1}^{m} 2^{-\gamma j} = 2 \sum_{j=n+1}^{m+1} 2^{-\gamma j} \\
\leq 2 \left( \frac{2^{-\gamma(n+1)}}{1 - 2^{-\gamma}} \right) \left( T2^{-(n+1)} \right)^\gamma \quad (60.2.4)
\]

It follows the above holds for any \( d, d' \in D \) such that \(|d - d'| \leq T2^{-n} \) because they are both in some \( D_m \) for \( m > n \).

Consider the last claim. Let \( d, d' \in D \), \(|d - d'| \leq T2^{-M} \). Then \( d, d' \) are both in some \( D_m \) for \( m > M \). The number \(|d - d'| \) satisfies

\[
T2^{-(n+1)} < |d - d'| \leq T2^{-n}
\]

for large enough \( n \geq M \). Just pick the first \( n \) such that \( T2^{-(n+1)} < |d - d'| \). Then from (60.2.3),

\[
||X(d') - X(d)|| \leq \left( \frac{2T^{-\gamma}}{1 - 2^{-\gamma}} \right) \left( T2^{-(n+1)} \right)^\gamma \\
\leq \left( \frac{2T^{-\gamma}}{1 - 2^{-\gamma}} \right) (|d - d'|)^\gamma
\]

Now \([0, T]\) is covered by \( 2^M \) intervals of length \( T2^{-M} \) and so for any pair \( d, d' \in D \),

\[
||X(d) - X(d')|| \leq C |d - d'|^\gamma
\]

where \( C \) is a suitable constant depending on \( 2^M \).

For \( \gamma \leq 1 \), you can show, using convexity arguments, that it suffices to have

\[
C = \left( \frac{2T^{-\gamma}}{1 - 2^{-\gamma}} \right)^{1/\gamma} (2^M)^{1-\gamma}.
\]

Of course the case where \( \gamma > 1 \) is not interesting because it would result in \( X \) being a constant.

The following is the amazing Kolmogorov Čentsov continuity theorem [41].

**Theorem 60.2.2** Suppose \( X \) is a stochastic process on \([0, T]\). Suppose also that there exists a constant, \( C \) and positive numbers, \( \alpha, \beta \) such that

\[
E(||X(t) - X(s)||^\alpha) \leq C |t - s|^{1+\beta} \quad (60.2.5)
\]

Then there exists a stochastic process \( Y \) such that for a.e. \( \omega, t \rightarrow Y(t)(\omega) \) is Hölder continuous with exponent \( \gamma < \frac{\beta}{\alpha} \) and for each \( t \), \( P(||Y(t) - Y(t)|| > 0) = 0 \). (\( Y \) is a version of \( X \).)
Proof: Let \( r^m_j \) denote \( j \left( \frac{T}{2^m} \right) \) where \( j \in \{0, 1, \ldots, 2^m \} \). Also let \( D_m = \{ r^m_j \}_{j=1}^{2^m} \) and \( D = \cup_{m=1}^{\infty} D_m \). Consider the set,

\[
\{ ||X(t) - X(s)|| > \delta \}
\]

By \( 60.2.5 \),

\[
P \left( \{ ||X(t) - X(s)|| > \delta \} \right) \delta^\alpha \leq \int_{\{ ||X(t) - X(s)|| > \delta \}} ||X(t) - X(s)||^\alpha \, dP \leq C |t-s|^{1+\beta}.
\] (60.2.6)

Letting \( t = r^k_{j+1}, s = r^k_j \) and \( \delta = 2^{-\gamma k} \) where \( \gamma = \left( 0, \frac{\beta}{\alpha} \right) \), this yields

\[
P \left( \{ ||X(r^k_{j+1}) - X(r^k_j)|| > 2^{-\gamma k} \} \right) \leq C 2^{\alpha \gamma k} (T 2^{-k})^{1+\beta} = CT^{1+\beta} 2^k (\alpha \gamma - (1+\beta))
\]

There are \( 2^k \) of these differences and so letting

\[
N_k = \cup_{j=1}^{2^k} \{ ||X(r^k_{j+1}) - X(r^k_j)|| > 2^{-\gamma k} \}
\]

it follows

\[
P (N_k) \leq C 2^{\alpha \gamma k} (T 2^{-k})^{1+\beta} 2^k = C 2^{k(\alpha \gamma - \beta)} T^{1+\beta}.
\]

Since \( \gamma < \beta / \alpha \),

\[
\sum_{k=1}^{\infty} P (N_k) \leq CT^{1+\beta} \sum_{k=1}^{\infty} 2^k (\alpha \gamma - \beta) < \infty
\]

and so by the Borel Cantelli lemma, Lemma \( 57.1.2 \), there exists a set of measure zero \( N \), such that if \( \omega \notin N \), then \( \omega \) is in only finitely many \( N_k \). In other words, for \( \omega \notin N \), there exists \( M(\omega) \) such that if \( k \geq M(\omega) \), then for each \( j \),

\[
||X(r^k_{j+1}) - X(r^k_j)|| \leq 2^{-\gamma k}.
\] (60.2.7)

It follows from Lemma \( 60.2.3 \) that \( t \to X(t)(\omega) \) is Holder continuous on \( D \) with Holder exponent \( \gamma \). Note the constant is a measurable function of \( \omega \), depending on how many measurable \( N_k \) which contain \( \omega \).

By Lemma \( 60.2.3 \), one can define \( Y(t)(\omega) \) to be the unique function which extends \( d \to X(d)(\omega) \) off \( D \) for \( \omega \notin N \) and let \( Y(t)(\omega) = 0 \) if \( \omega \in N \). Thus by Lemma \( 60.2.3 \), \( t \to Y(t)(\omega) \) is Holder continuous. Also, \( \omega \to Y(t)(\omega) \) is measurable because it is the pointwise limit of measurable functions

\[
Y(t)(\omega) = \lim_{d \to t} X(d)(\omega) X_{NC}(\omega).
\] (60.2.8)
It remains to verify the claim that \( Y(t)(\omega) = X(t)(\omega) \) a.e.

\[
\mathcal{K}_{\|Y(t)-X(t)\|>\varepsilon\cap N_c}(\omega) \leq \liminf_{d \to t} \mathcal{K}_{\|X(d)-X(t)\|>\varepsilon\cap N_c}(\omega)
\]

because if \( \omega \in N \) both sides are 0 and if \( \omega \in N_c \) then the above limit in [10.3.8] holds and so if \( \|Y(t)(\omega) - X(t)(\omega)\| > \varepsilon \), the same is true of \( \|X(d)(\omega) - X(t)(\omega)\| \) whenever \( d \) is close enough to \( t \) and so by Fatou’s lemma,

\[
P\left(\|Y(t) - X(t)\| > \varepsilon\right) = \int \mathcal{K}_{\|Y(t)-X(t)\|>\varepsilon\cap N_c}(\omega) \ dP
\]

\[
\leq \liminf_{d \to t} \int \mathcal{K}_{\|X(d)-X(t)\|>\varepsilon}(\omega) \ dP
\]

\[
\leq \liminf_{d \to t} \int \mathcal{K}_{\|X(d)-X(t)\|>\varepsilon}(\omega) \ dP
\]

\[
\leq \liminf_{d \to t} \int \mathcal{K}_{\|X(d)-X(t)\|>\varepsilon}(\omega) \ dP
\]

\[
\leq \liminf_{d \to t} \frac{C}{\varepsilon^\alpha} |d-t|^{1+\beta} = 0.
\]

Therefore,

\[
P\left(\|Y(t) - X(t)\| > 0\right)
\]

\[
= P\left(\bigcup_{k=1}^\infty \left\{\|Y(t) - X(t)\| > \frac{1}{k}\right\}\right)
\]

\[
\leq \sum_{k=1}^\infty P\left(\left\{\|Y(t) - X(t)\| > \frac{1}{k}\right\}\right) = 0. \quad \blacksquare
\]

A few observations are interesting. In the proof, the following inequality was obtained,

\[
\|X(d')(\omega) - X(d)(\omega)\| \leq \frac{2}{T^\gamma(1-2^-\gamma)} \left(T2^{-(\alpha+1)}\right)^\gamma
\]

\[
\leq \frac{2}{T^\gamma(1-2^-\gamma)} (|d-d'|)^\gamma
\]

which was so for any \( d',d \in D \) with \( |d'-d| < T2^{-(M(\omega)+1)} \). Thus the Holder continuous version of \( X \) will satisfy

\[
\|Y(t)(\omega) - Y(s)(\omega)\| \leq \frac{2}{T^\gamma(1-2^-\gamma)} (|t-s|)^\gamma
\]

provided \( |t-s| < T2^{-(M(\omega)+1)} \). Does this translate into an inequality of the form

\[
\|Y(t)(\omega) - Y(s)(\omega)\| \leq \frac{2}{T^\gamma(1-2^-\gamma)} (|t-s|)^\gamma
\]
for any pair of points $t, s \in [0, T]$? It seems it does not for any $\gamma < 1$ although it does yield

$$||Y(t)(\omega) - Y(s)(\omega)|| \leq C(|t-s|)^\gamma$$

where $C$ depends on the number of intervals having length less than $T2^{-(M(\omega)+1)}$ which it takes to cover $[0, T]$. First note that if $\gamma > 1$, the Holder continuity will imply $t \to Y(t)(\omega)$ is a constant. Therefore, the only case of interest is $\gamma < 1$.

Let $s, t$ be any pair of points and let $s = x_0 < \cdots < x_n = t$ where $|x_i - x_{i-1}| < T2^{-(M(\omega)+1)}$. Then

$$||Y(t)(\omega) - Y(s)(\omega)|| \leq \sum_{i=1}^{n} ||Y(x_i)(\omega) - Y(x_{i-1})(\omega)|| \leq \frac{2}{T^\gamma (1 - 2 - \gamma)} \sum_{i=1}^{n} (|x_i - x_{i-1}|) ^ \gamma \tag{60.2.9}$$

How does this compare to

$$\left( \sum_{i=1}^{n} |x_i - x_{i-1}| \right) ^ \gamma = |t-s|^\gamma?$$

This last expression is smaller than the right side of (60.2.9) for any $\gamma < 1$. Thus for $\gamma < 1$, the constant in the conclusion of the theorem depends on both $T$ and $\omega \notin N$.

In the case where $\alpha \geq 1$, here is another proof of this theorem. It is based on the one in the book by Stroock [109].

**Theorem 60.2.3** Suppose $X$ is a stochastic process on $[0, T]$ having values in the Banach space $E$. Suppose also that there exists a constant, $C$ and positive numbers $\alpha, \beta, \alpha \geq 1$, such that

$$E(||X(t) - X(s)||^\alpha) \leq C |t-s|^{1+\beta} \tag{60.2.10}$$

Then there exists a stochastic process $Y$ such that for a.e. $\omega, t \to Y(t)(\omega)$ is Hölder continuous with exponent $\gamma < \frac{\alpha}{\beta}$ and for each $t$, $P(||X(t) - Y(t)|| > 0) = 0$. (Y is a version of X.) Also

$$E\left( \sup_{0 \leq s < t \leq T} \frac{||Y(t) - Y(s)||}{(t-s)^\gamma} \right) \leq C$$

where $C$ depends on $\alpha, \beta, T, \gamma$.

**Proof:** The proof considers piecewise linear approximations of $X$ which are automatically continuous. These are shown to converge to $Y$ in $L^\alpha (\Omega; C([0, T], E))$ so it follows that $Y$ must be continuous for a.e. $\omega$. Finally, it is shown that $Y$ is a version of $X$ and is Hölder continuous. In the proof, I will use $C$ to denote a constant which depends on the quantities $\gamma, \alpha, \beta, T$. Let $\{t^n_k\}_{k=0}^{2^n}$ be a uniform partition of the interval $[0, T]$ so that $t^n_{k+1} - t^n_k = T2^{-n}$. Now let

$$M_n = \max_{k \leq 2^n} ||X(t^n_k) - X(t^n_{k-1})||$$
Then it follows that
\[ M_n^\alpha \leq \sum_{k=1}^{2^n} \| X(t^n_k) - X(t^n_{k-1}) \|^\alpha \]
and so
\[ E(M_n^\alpha) \leq \sum_{k=1}^{2^n} C(T2^{-n})^{1+\beta} = C2^n2^{-n(1+\beta)} = C2^{-n\beta} \quad (60.2.11) \]

Next denote by \( X_n \) the piecewise linear function which results from the values of \( X \) at the points \( t^n_k \). Consider the following picture which illustrates a part of the graphs of \( X_n \) and \( X_{n+1} \).

Then
\[ \max_{t \in [0,T]} \| X_{n+1}(t) - X_n(t) \| \leq \max_{1 \leq k \leq 2^{n+1}} \left\| X(t^{n+1}_{2k-1}) - \frac{X(t^n_k) + X(t^n_{k-1})}{2} \right\| \]
\[ \leq \max_{k \leq 2^{n+1}} \left( \frac{1}{2} \| X(t^{n+1}_{2k-1}) - X(t^{n+1}_{2k}) \| + \frac{1}{2} \| X(t^{n+1}_{2k-1}) - X(t^{n+1}_{2k-2}) \| \right) \leq M_{n+1} \]

Denote by \( \| \cdot \|_\infty \) the usual norm in \( C([0,T], E) \),
\[ \max_{t \in [0,T]} \| Z(t) \| \equiv \| Z \|_\infty , \]

Then from what was just established,
\[ E(\|X_{n+1} - X_n\|_\infty^\alpha) = \int_\Omega \|X_{n+1} - X_n\|_\infty^\alpha dP \leq E(M_{n+1}^\alpha) = C2^{-n\beta} \]
which shows that
\[ \|X_{n+1} - X_n\|_{L^\alpha(\Omega; C([0,T], E))} = \left( \int_\Omega \|X_{n+1} - X_n\|_\infty^\alpha dP \right)^{1/\alpha} \leq C\left(2^{(\beta/\alpha)}\right)^{-n} \]

Also, for \( m > n \), it follows from the assumption that \( \alpha \geq 1 \),
\[ \|X_m - X_n\|_{L^\alpha(\Omega; C([0,T], E))} \leq \]
\[ \sum_{k=n}^{\infty} C \left( 2^{(\beta/\alpha)} \right)^{-k} \leq C \left( \frac{2^{(\beta/\alpha)}}{1 - 2^{(-\beta/\alpha)}} \right)^{-n} = C \left( 2^{(\beta/\alpha)} \right)^{-n} \]  

(60.2.12)

Thus \{X_n\} is a Cauchy sequence in \( L^\alpha(\Omega; C([0,T], E)) \) and so it converges to some \( Y \) in this space, a subsequence converging pointwise. Then from Fatou’s lemma,

\[ \|Y - X_n\|_{L^\alpha(\Omega; C([0,T], E))} \leq C \left( 2^{(\beta/\alpha)} \right)^{-n} \]  

(60.2.13)

Also, for a.e. \( \omega, t \to Y(t) \) is in \( C([0,T], E) \). It remains to verify that \( Y(t) = X(t) \) a.e.

From the construction, it follows that for any \( n \) and \( m \geq n \)

\[ Y(t^n_k) = X_m(t^n_k) = X(t^n_k) \]

Thus

\[ \|Y(t) - X(t)\| \leq \|Y(t) - Y(t^n_k)\| + \|Y(t^n_k) - X(t)\| = \|Y(t) - Y(t^n_k)\| + \|X(t^n_k) - X(t)\| \]

Now from the hypotheses of the theorem,

\[ P \left( \|X(t^n_k) - X(t)\|^\alpha > \varepsilon \right) \leq \frac{1}{\varepsilon} E \left( \|X(t^n_k) - X(t)\|^\alpha \right) \leq \frac{C}{\varepsilon} |t^n_k - t|^{1+\beta} \]

Thus, there exists a sequence of mesh points \{s_n\} converging to \( t \) such that

\[ P \left( \|X(s_n) - X(t)\|^\alpha > 2^{-n} \right) \leq 2^{-n} \]

Then by the Borel Cantelli lemma, there is a set of measure zero \( N \) such that for \( \omega \notin N \),

\[ \|X(s_n) - X(t)\|^\alpha \leq 2^{-n} \]

for all \( n \) large enough. Then

\[ \|Y(t) - X(t)\| \leq \|Y(t) - Y(s_n)\| + \|X(s_n) - X(t)\| \]

which shows that, by continuity of \( Y \), for \( \omega \) not in an exceptional set of measure zero, \( \|Y(t) - X(t)\| = 0 \).

It remains to verify the assertion about Holder continuity of \( Y \). Let \( 0 \leq s < t \leq T \). Then for some \( n \),

\[ 2^{-(n+1)}T \leq t - s \leq 2^{-n}T \]  

(60.2.14)

Thus

\[ \|Y(t) - Y(s)\| \leq \|Y(t) - X_n(t)\| + \|X_n(t) - X_n(s)\| + \|X_n(s) - Y(s)\| \leq 2 \sup_{\tau \in [0,T]} \|Y(\tau) - X_n(\tau)\| + \|X_n(t) - X_n(s)\| \]  

(60.2.15)
Now
\[ \frac{\|X_n(t) - X_n(s)\|}{t - s} \leq \frac{\|X_n(t) - X_n(s)\|}{2^{-(n+1)}T} \]

From \[60.2.14\] a picture like the following must hold.

\[ t_{k+1}^{n+1} \quad s \quad t_k^{n+1} \quad t \quad t_{k+1}^{n+1} \]

Therefore, from the above,
\[ \frac{\|X_n(t) - X_n(s)\|}{t - s} \leq \frac{\|X(t_{k+1}^{n+1}) - X(t_k^{n+1})\|}{2^{-(n+1)}T} \leq C2^n M_{n+1} \]

It follows from \[60.2.15\],
\[ \|Y(t) - Y(s)\| \leq 2\|Y - X_n\|_\infty + C2^n M_{n+1} (t - s) \]

Next, letting \( \gamma < \beta/\alpha \), and using \[60.2.14\],
\[ \frac{\|Y(t) - Y(s)\|}{(t - s)^\gamma} \leq 2 (2^{-1}2^{n+1})^\gamma \|Y - X_n\|_\infty + C2^n \left(2^{-n}\right)^{1-\gamma} M_{n+1} = C2^{n\gamma} (\|Y - X_n\|_\infty + M_{n+1}) \]

The above holds for any \( s, t \) satisfying \[60.2.14\]. Then
\[ \sup_{0 \leq s < t \leq T, |t - s| \in \left[2^{-(n+1)}T, 2^{-n}T\right]} \frac{\|Y(t) - Y(s)\|}{(t - s)^\gamma} \leq C2^{n\gamma} (\|Y - X_n\|_\infty + M_{n+1}) \]

Denote by \( P_n \) the ordered pairs \((s, t)\) satisfying the above condition that
\[ 0 \leq s < t \leq T, |t - s| \in \left[2^{-(n+1)}T, 2^{-n}T\right], \]
\[ \sup_{(s, t) \in P_n} \frac{\|Y(t) - Y(s)\|}{(t - s)^\gamma} \leq C2^{n\gamma} (\|Y - X_n\|_\infty + M_{n+1}) \]

Thus for a.e. \( \omega \), and for all \( n \),
\[ \left( \sup_{(s, t) \in P_n} \frac{\|Y(t) - Y(s)\|}{(t - s)^\gamma} \right)^\alpha \leq C \sum_{k=0}^{\infty} 2^{k\alpha\gamma} (\|Y - X_k\|_\infty^\alpha + M_{k+1}^\alpha) \]

Note that \( n \) is arbitrary. Hence
\[ \sup_{0 \leq s < t \leq T} \left( \frac{\|Y(t) - Y(s)\|}{(t - s)^\gamma} \right)^\alpha \leq \]
\[
\sup_n \sup_{(s,t) \in P_n} \left( \sup_{(s,t) \in P_n} \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha \leq \sup_n \left( \sup_{(s,t) \in P_n} \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha \leq \sum_{k=0}^{\infty} C2^{k\alpha\gamma} \left( \|Y - X_k\|^\alpha + M_k \right)
\]

By continuity of \(Y\), the result on the left is unchanged if the ordered pairs are restricted to lie in \(Q \cap [0,T] \times Q \cap [0,T]\), a countable set. Thus the left side is measurable. It follows from \(60.2.11\) and \(60.2.13\) which say

\[
\|Y - X_k\|_{L^\alpha(\Omega; C([0,T],E))} \leq C \left(2^{(\beta/\alpha)}\right)^{-k}, \quad E(M_k^\alpha) \leq C2^{-k\beta}
\]

that

\[
E \left( \sup_{0 \leq s < t \leq T} \left( \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha \right) \leq \sum_{k=0}^{\infty} C2^{k\alpha\gamma}2^{-\beta k} = C < \infty
\]

because \(\alpha\gamma - \beta < 0\). By continuity of \(Y\), there are no measurability concerns in taking the above expectation. Note that this implies, since \(\alpha \geq 1\),

\[
E \left( \sup_{0 \leq s < t \leq T} \left( \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha \right) \leq \left( E \left( \sup_{0 \leq s < t \leq T} \left( \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha \right) \right)^{1/\alpha} \leq C^{1/\alpha} = C
\]

Now

\[
P \left( \sup_{0 \leq s < t \leq T} \left( \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha > 2^\alpha k \right) \leq \frac{1}{2^\alpha k} C
\]

and so there exists a set of measure zero \(N\) such that for \(\omega \notin N\),

\[
\sup_{0 \leq s < t \leq T} \left( \frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \right)^\alpha \leq 2^\alpha k
\]

for all \(k\) large enough. Pick such a \(k\), depending on \(\omega \notin N\). Then for any \(s, t\),

\[
\frac{\|Y(t) - Y(s)\|}{(t-s)^\gamma} \leq 2^k
\]

and so, this has shown that for \(\omega \notin N\),

\[
\|Y(t) - Y(s)\| \leq C(\omega)(t-s)^\gamma \quad \blacksquare
\]

Note that if \(X(t)\) is known to be continuous off a set of measure zero, then the piecewise linear approximations converge to \(X(t)\) in \(C([0,T],E)\) off this set of measure zero. Therefore, it must be that off a set of measure zero, \(Y(t) = X(t)\) and so in fact \(X(t)\) is Holder continuous off a set of measure zero and the condition on expectation also must hold, that is

\[
E \left( \sup_{0 \leq s < t \leq T} \left( \frac{|X(t) - X(s)|}{(t-s)^\gamma} \right)^\alpha \right) \leq C.
\]
60.3 Filtrations

Instead of having a sequence of $\sigma$-algebras, one can consider an increasing collection of $\sigma$-algebras indexed by $t \in \mathbb{R}$. This is called a filtration.

**Definition 60.3.1** Let $X$ be a stochastic process defined on an interval, $I = [0, T]$ or $[0, \infty)$. Suppose the probability space, $(\Omega, \mathcal{F}, P)$ has an increasing family of $\sigma$-algebras, $\{\mathcal{F}_t\}$. This is called a filtration. If for arbitrary $t \in I$ the random variable $X(t)$ is $\mathcal{F}_t$ measurable, then $X$ is said to be adapted to the filtration $\{\mathcal{F}_t\}$. Denote by $\mathcal{F}_{t+}$ the intersection of all $\mathcal{F}_s$ for $s > t$. The filtration is normal if

1. $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $P(A) = 0$
2. $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \in I$

$X$ is called progressively measurable if for every $t \in I$, the mapping

$$(s, \omega) \in [0, t] \times \Omega, \quad (s, \omega) \mapsto X(s, \omega)$$

is $B([0, t]) \times \mathcal{F}_t$ measurable.

Thus $X$ is progressively measurable means

$$(s, \omega) \mapsto X_{[0, t]}(s) X(s, \omega)$$

is $B([0, t]) \times \mathcal{F}_t$ measurable. As an example of a normal filtration, here is an example.

**Example 60.3.2** For example, you could have a stochastic process, $X(t)$ and you could define

$$\mathcal{G}_t \equiv \sigma(X(s) : s \leq t),$$

the completion of the smallest $\sigma$ algebra such that each $X(s)$ is measurable for all $s \leq t$. This gives an example of a filtration to which $X(t)$ is adapted which satisfies $\{\mathcal{G}_t\}$. More generally, suppose $X(t)$ is adapted to a filtration, $\mathcal{G}_t$. Define

$$\mathcal{F}_t \equiv \cap_{s>\mathbb{R}} \mathcal{G}_s$$

Then

$$\mathcal{F}_{t+} \equiv \cap_{s>t} \mathcal{F}_s = \cap_{s>t} \cap_{r>s} \mathcal{G}_r = \cap_{s>t} \mathcal{G}_s \equiv \mathcal{F}_t.$$

and each $X(t)$ is measurable with respect to $\mathcal{F}_t$. Thus there is no harm in assuming a stochastic process adapted to a filtration can be modified so the filtration is normal. Also note that $\mathcal{F}_t$ defined this way will be complete so if $A \in \mathcal{F}_t$ has $P(A) = 0$ and if $B \subseteq A$, then $B \in \mathcal{F}_t$ also. This is because this relation between the sets and the probability of $A$ being zero, holds for this pair of sets when considered as elements of each $\mathcal{G}_s$ for $s > t$. Hence $B \in \mathcal{G}_s$ for each $s > t$ and is therefore one of the sets in $\mathcal{F}_t$. 
What is the description of a progressively measurable set?

![Diagram](Q ∩ [0, t] × Ω) ⊆ Q

It means that for Q progressively measurable, \( Q ∩ [0, t] × Ω \) as shown in the above picture is \( B ([0, t]) × F_t \) measurable. It is like saying a little more descriptively that the function is progressively product measurable.

I shall generally assume the filtration is normal.

**Observation 60.3.3** If \( X \) is progressively measurable, then it is adapted. Furthermore the progressively measurable sets, those \( E ∩ [0, T] × Ω \) for which \( X_E \) is progressively measurable form a σ algebra.

To see why this is, consider \( X \) progressively measurable and fix \( t \). Then \( (s, ω) → X (s, ω) \) for \( (s, ω) ∈ [0, t] × Ω \) is given to be \( B ([0, t]) × F_t \) measurable, the ordinary product measure and so fixing any \( s ∈ [0, t] \), it follows the resulting function of \( ω \) is \( F_t \) measurable. In particular, this is true upon fixing \( s = t \). Thus \( ω → X (t, ω) \) is \( F_t \) measurable and so \( X (t) \) is adapted.

A set \( E ⊆ [0, T] × Ω \) is progressively measurable means that \( X_E \) is progressively measurable. That is \( X_E \) restricted to \( [0, t] × Ω \) is \( B ([0, t]) × F_t \) measurable. In other words, \( E \) is progressively measurable if

\[
E ∩ ([0, t] × Ω) \in B ([0, t]) × F_t.
\]

If \( E_i \) is progressively measurable, does it follow that \( E \equiv ∪_{i=1}^{∞} E_i \) is also progressively measurable? Yes.

\[
E ∩ ([0, t] × Ω) = ∪_{i=1}^{∞} E_i ∩ ([0, t] × Ω) \in B ([0, t]) × F_t
\]

because each set in the union is in \( B ([0, t]) × F_t \). If \( E \) is progressively measurable, is \( E^C \)?

\[
E^C ∩ ([0, t] × Ω) = ∪_{i=1}^{∞} E_i^C ∩ ([0, t] × Ω) \in B ([0, t]) × F_t
\]

and so \( E^C ∩ ([0, t] × Ω) \in B ([0, t]) × F_t \). Thus the progressively measurable sets are a σ algebra.

Another observation of interest is in the following lemma.

**Lemma 60.3.4** Suppose \( Q \) is in \( B ([0, a]) × F_r \). Then if \( b ≥ a \) and \( t ≥ r \), then \( Q \) is also in \( B ([0, b]) × F_t \).
60.3. FILTRATIONS

Proof: Consider a measurable rectangle \( A \times B \) where \( A \in B([0,a]) \) and \( B \in \mathcal{F}_r \). Is it true that \( A \times B \in B([0,b]) \times \mathcal{F}_t \)? This reduces to the question whether \( A \in B([0,b]) \). If \( A \) is an interval, it is clear that \( A \in B([0,b]) \). Consider the \( \pi \) system of intervals and let \( G \) denote those Borel sets \( A \in B([0,a]) \) such that \( A \in B([0,b]) \). If \( A \in G \), then \([0,b] \setminus A \in B([0,a]) \) by assumption (the difference of Borel sets is surely Borel). However, this set equals

\[
([0,a] \setminus A) \cup (a,b)
\]

and so

\[
[0,b] = ([0,a] \setminus A) \cup (a,b] \cup A
\]

The set on the left is in \( B([0,b]) \) and the sets on the right are disjoint and two of them are also in \( B([0,b]) \). Therefore, the third, \( ([0,a] \setminus A) \) is in \( B([0,b]) \). It is obvious that \( G \) is closed with respect to countable disjoint unions. Therefore, by Lemma 10.12.3, Dynkin’s lemma, \( G \supseteq \sigma(\text{Intervals}) = B([0,a]) \).

Therefore, such a measurable rectangle \( A \times B \) where \( A \in B([0,a]) \) and \( B \in \mathcal{F}_r \) is in \( B([0,b]) \times \mathcal{F}_t \). Now let \( K \) denote all these measurable rectangles \( A \times B \) where \( A \in B([0,a]) \) and \( B \in \mathcal{F}_r \). Let \( G \) (new \( G \)) denote those sets \( Q \) of \( B([0,a]) \times \mathcal{F}_r \) which are in \( B([0,b]) \times \mathcal{F}_t \). Then if \( Q \in G \),

\[
Q \cup ([0,a] \times \Omega \setminus Q) \cup (a,b] \times \Omega = [a,b] \times \Omega
\]

Then the sets are disjoint and all but \([0,a] \times \Omega \setminus Q\) are in \( B([0,b]) \times \mathcal{F}_t \). Therefore, this one is also in \( B([0,b]) \times \mathcal{F}_t \). If \( Q_i \in G \) and the \( Q_i \) are disjoint, then \( \cup_i Q_i \) is also in \( B([0,b]) \times \mathcal{F}_t \) and so \( G \) is closed with respect to countable disjoint unions and complements. Hence \( G \supseteq \sigma(K) = B([0,a]) \times \mathcal{F}_r \) which shows

\[
B([0,a]) \times \mathcal{F}_r \subseteq B([0,b]) \times \mathcal{F}_t \quad \blacksquare
\]

A significant observation is the following which states that the integral of a progressively measurable function is progressively measurable.

Proposition 60.3.5 Suppose \( X : [0,T] \times \Omega \to E \) where \( E \) is a separable Banach space. Also suppose that \( X(\cdot,\omega) \in L^1([0,T],E) \) for each \( \omega \). Here \( \mathcal{F}_t \) is a filtration and with respect to this filtration, \( X \) is progressively measurable. Then

\[
(t,\omega) \to \int_0^t X(s,\omega) \, ds
\]

is also progressively measurable.

Proof: Suppose \( Q \in [0,T] \times \Omega \) is progressively measurable. This means for each \( t \),

\[
Q \cap [0,t] \times \Omega \in B([0,t]) \times \mathcal{F}_t
\]

What about

\[
(s,\omega) \in [0,t] \times \Omega, \ (s,\omega) \to \int_0^s X_Q \, dr?
\]
Is that function on the right $B([0,t]) \times F_t$ measurable? We know that $Q \cap [0,s] \times \Omega$ is $B([0,s]) \times F_s$ measurable and hence $B([0,t]) \times F_t$ measurable. When you integrate a product measurable function, you do get one which is product measurable. Therefore, this function must be $B([0,t]) \times F_t$ measurable. This shows that $(t,\omega) \rightarrow \int_0^t X_Q(s,\omega) \, ds$ is progressively measurable. Here is a claim which was just used.

Claim: If $Q$ is $B([0,t]) \times F_t$ measurable, then $(s,\omega) \rightarrow \int_s^0 X_Q(r,\omega) \, dr$ is also $B([0,t]) \times F_t$ measurable.

Proof of claim: First consider $A \times B$ where $A \in B([0,t])$ and $B \in F_t$. Then

$$\int_0^s X_{A\times B} \, dr = \int_0^s X_A X_B \, dr = X_B(\omega) \int_0^s X_A(s) \, dr$$

This is clearly $B([0,t]) \times F_t$ measurable. It is the product of a continuous function of $s$ with the indicator function of a set in $F_t$. Now let

$$G = \left\{ Q \in B([0,t]) \times F_t : (s,\omega) \rightarrow \int_0^s X_Q(r,\omega) \, dr \text{ is } B([0,t]) \times F_t \text{ measurable} \right\}$$

Then it was just shown that $G$ contains the measurable rectangles. It is also clear that $G$ is closed with respect to countable disjoint unions and complements. Therefore, $G \supseteq \sigma(K_t) = B([0,t]) \times F_t$ where $K_t$ denotes the measurable rectangles $A \times B$ where $B \in F_t$ and $A \in B([0,T]) = B([0,T]) \cap [0,t]$. This proves the claim.

Thus if $Q$ is progressively measurable, it follows that $(s,\omega) \rightarrow \int_s^0 X_Q(r,\omega) \, dr \equiv f(s,\omega)$ is progressively measurable because for $(s,\omega) \in [0,t] \times \Omega$, $(s,\omega) \rightarrow f(s,\omega)$ is $B([0,t]) \times F_t$ measurable. This is what was to be proved in this special case.

Now consider the conclusion of the proposition. By considering the positive and negative parts of $\phi(X)$ for $\phi \in E'$ and using Pettis theorem, it suffices to consider the case where $X \geq 0$. Then there exists an increasing sequence of progressively measurable simple functions $\{X_n\}$ converging pointwise to $X$. From what was just shown,

$$(t,\omega) \rightarrow \int_0^t X_n ds$$

is progressively measurable. Hence, by the monotone convergence theorem, $(t,\omega) \rightarrow \int_0^t X ds$ is also progressively measurable. ■

What else can you do to something which is progressively measurable and obtain something which is progressively measurable? It turns out that shifts in time can preserve progressive measurability. Let $F_t$ be a filtration on $[0,T]$ and extend the filtration to be equal to $F_0$ and $F_T$ for $t < 0$ and $t > T$, respectively. Recall the following definition of progressively measurable sets.

**Definition 60.3.6** Denote by $\mathcal{P}$ those sets $Q$ in $F_T \times B([0,T])$ such that for $t \in [-\infty,T]$

$$\Omega \times (-\infty,t] \cap Q \in F_t \times B((-\infty,t]).$$
Lemma 60.3.7 Define $Q + h$ as

$$Q + h \equiv \{(t + h, \omega) : (t, \omega) \in Q\}.$$ 

Then if $Q \in \mathcal{P}$, it follows that $Q + h \in \mathcal{P}$.

**Proof:** This is most easily seen through the use of the following diagram. In this diagram, $Q$ is in $\mathcal{P}$ so it is progressively measurable.

By definition, $S$ in the picture is $\mathcal{B}((\infty, t-h]) \times \mathcal{F}_{t-h}$ measurable. Hence $S + h \equiv Q + h \cup \Omega \times (\infty, t]$ is $\mathcal{B}((\infty, t]) \times \mathcal{F}_t$ measurable. To see this, note that if $B \times A \in \mathcal{B}((\infty, t-h]) \times \mathcal{F}_{t-h}$, then translating it by $h$ gives a set in $\mathcal{B}((\infty, t]) \times \mathcal{F}_t$. Then if $G$ consists of sets $S$ in $\mathcal{B}((\infty, t-h]) \times \mathcal{F}_{t-h}$ for which $S + h$ is in $\mathcal{B}((\infty, t]) \times \mathcal{F}_t$, $G$ is closed with respect to countable disjoint unions and complements. Thus, $G$ equals $\mathcal{B}((\infty, t-h]) \times \mathcal{F}_{t-h}$. In particular, it contains the set $S$ just described.

Now for $h > 0$,

$$\tau_h f (t) \equiv \begin{cases} f(t-h) & \text{if } t \geq h, \\ 0 & \text{if } t < h. \end{cases}$$

Lemma 60.3.8 Let $Q \in \mathcal{P}$. Then $\tau_h \mathcal{X}_Q$ is $\mathcal{P}$ measurable.

**Proof:** If $\tau_h \mathcal{X}_Q (t, \omega) = 1$, then you need to have $(t - h, \omega) \in Q$ and so $(t, \omega) \in Q + h$. Thus

$$\tau_h \mathcal{X}_Q = \mathcal{X}_{Q+h},$$

which is $\mathcal{P}$ measurable since $Q \in \mathcal{P}$. In general,

$$\tau_h \mathcal{X}_Q = \mathcal{X}_{[h,T] \times \Omega} \mathcal{X}_{Q+h},$$

which is $\mathcal{P}$ measurable. □

This lemma implies the following.

Lemma 60.3.9 Let $f(t, \omega)$ have values in a separable Banach space and suppose $f$ is $\mathcal{P}$ measurable. Then $\tau_h f$ is $\mathcal{P}$ measurable.

**Proof:** Taking values in a separable Banach space and being $\mathcal{P}$ measurable, $f$ is the pointwise limit of $\mathcal{P}$ measurable simple functions. If $s_n$ is one of these, then from the above lemmas, $\tau_h s_n$ is $\mathcal{P}$ measurable. Then, letting $n \to \infty$, it follows that $\tau_h f$ is $\mathcal{P}$ measurable. □

The following is similar to Proposition [60.1.2]. It shows that under pretty weak conditions, an adapted process has a progressively measurable adapted version.
Proposition 60.3.10 Let $X$ be a stochastically continuous adapted process for a normal filtration defined on a closed interval, $I \equiv [0, T]$. Then $X$ has a progressively measurable adapted version.

Proof: By Lemma 60.1.1 $X$ is uniformly stochastically continuous and so there exists a sequence of positive numbers, $\{\rho_n\}$ such that if $|s - t| < \rho_n$, then

$$P \left( ||X(t) - X(s)|| \geq \frac{1}{2^n} \right) \leq \frac{1}{2^n}. \quad (60.3.16)$$

Then let $\{t^n_0, t^n_1, \cdots, t^n_{m_n}\}$ be a partition of $[0, T]$ in which $t^n_i - t^n_{i-1} < \rho_n$. Now define $X_n$ as follows:

$$X_n(t)(\omega) = \sum_{i=1}^{m_n} X(t^n_{i-1}) (\omega) X_{[t^n_{i-1}, t^n_i]}(t)$$

$$X_n(T) = X(T).$$

Then $(s, \omega) \to X_n(s, \omega)$ for $(s, \omega) \in [0, t] \times \Omega$ is obviously $B([0, t]) \times \mathcal{F}_t$ measurable. Consider the set, $A$ on which $\{X_n(t, \omega)\}$ is a Cauchy sequence. This set is of the form

$$A = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{p,q \geq m} \left[ ||X_p - X_q|| < \frac{1}{n} \right]$$

and so it is a $B(I) \times \mathcal{F}$ measurable set and $A \cap [0, t] \times \Omega$ is $B([0, t]) \times \mathcal{F}_t$ measurable for each $t \leq T$ because each $X_q$ in the above has the property that its restriction to $[0, t] \times \Omega$ is $B([0, t]) \times \mathcal{F}_t$ measurable. Now define

$$Y(t, \omega) = \begin{cases} \lim_{n \to \infty} X_n(t, \omega) & \text{if } (t, \omega) \in A \\ 0 & \text{if } (t, \omega) \notin A \end{cases}$$

I claim that for each $t$, $Y(t, \omega) = X(t, \omega)$ for a.e. $\omega$. To see this, consider 60.3.10. From the construction of $X_n$, it follows that for each $t$,

$$P \left( ||X_n(t) - X(t)|| \geq \frac{1}{2^n} \right) \leq \frac{1}{2^n}.$$

Also, for a fixed $t$, if $X_n(t, \omega)$ fails to converge to $X(t, \omega)$, then $\omega$ must be in infinitely many of the sets,

$$B_n = \left[ ||X_n(t) - X(t)|| \geq \frac{1}{2^n} \right]$$

which is a set of measure zero by the Borel Cantelli lemma. Recall why this is so.

$$P \left( \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n \right) \leq \sum_{n=k}^{\infty} P(B_n) < \frac{1}{2^{k-1}}$$

Therefore, for each $t, (t, \omega) \in A$ for a.e. $\omega$. Hence $X(t) = Y(t)$ a.e. and so $Y$ is a measurable version of $X$. $Y$ is adapted because the filtration is normal and hence
\( F_t \) contains all sets of measure zero. Therefore, \( Y(t) \) differs from \( X(t) \) on a set which is \( F_t \) measurable. ■

There is a more specialized situation in which the measurability of a stochastic process automatically implies it is adapted. Furthermore, this can be defined easily in terms of a \( \pi \) system of sets.

**Definition 60.3.11** Let \( F_t \) be a filtration on \((\Omega, F, P)\) and denote by \( P_\infty \) the smallest \( \sigma \)-algebra of sets of \([0, \infty) \times \Omega \) containing the sets

\[
(s, t] \times F, F \in F_s, \quad \{0\} \times F, \quad F \in F_0.
\]

This is called the predictable \( \sigma \)-algebra, and the sets in this \( \sigma \)-algebra are called the predictable sets. Denote by \( P_T \) the intersection of \( P_\infty \) to \([0, T] \times \Omega \). A stochastic process \( X \) which maps either \([0, T] \times \Omega \) or \([0, \infty) \times \Omega \) to \( E \), a separable real Banach space is called predictable if for every Borel set \( A \in \mathcal{B}(E) \), it follows \( X^{-1}(A) \in P_T \) or \( P_\infty \).

This is a lot like product measure except one of the \( \sigma \) algebras is changing.

**Proposition 60.3.12** Let \( F_t \) be a filtration as above and let \( X \) be a predictable stochastic process. Then \( X \) is \( F_t \) adapted.

**Proof:** Let \( s_0 > 0 \) and define

\[
G_{s_0} \equiv \{ S \in P_\infty : S_{s_0} \in F_{s_0} \}
\]

where

\[
S_{s_0} \equiv \{ \omega \in \Omega : (s_0, \omega) \in S \}.
\]

It is clear \( G_{s_0} \) is a \( \sigma \) algebra. The next step is to show \( G_{s_0} \) contains the sets

\[
(s, t] \times F, \quad F \in F_s \tag{60.3.17}
\]

and

\[
\{0\} \times F, \quad F \in F_0. \tag{60.3.18}
\]

It is clear \( \{0\} \times F \) is contained in \( G_{s_0} \) because \((\{0\} \times F)_{s_0} = 0 \in F_{s_0} \). Similarly, if \( s \geq s_0 \) or if \( s, t < s_0 \) then \((s, t] \times F)_{s_0} = 0 \in F_{s_0} \). The only case left is for \( s < s_0 \) and \( t \geq s_0 \). In this case, letting \( A_s \in F_s \), \((s, t] \times A_s)_{s_0} = A_s \in F_s \subseteq F_{s_0} \). Therefore, \( G_{s_0} \) contains all the sets of the form given in (60.3.17) and (60.3.18) and so
CHAPTER 60. STOCHASTIC PROCESSES

since $\mathcal{P}_\infty$ is the smallest $\sigma$ algebra containing these sets, it follows $\mathcal{P}_\infty = \mathcal{G}_{s_0}$. The case where $s_0 = 0$ is entirely similar but shorter.

Therefore, if $X$ is predictable, letting $A \in \mathcal{B}(E)$, $X^{-1}(A) \in \mathcal{P}_\infty$ or $\mathcal{P}_T$ and so

$$(X^{-1} (A))_s = \{ \omega \in \Omega : X(s, \omega) \in A \} = X(s)^{-1} (A) \in \mathcal{F}_s$$

showing $X(t)$ is $\mathcal{F}_t$ adapted. This proves the proposition.

Another way to see this is to recall the progressively measurable functions are adapted. Then show the predictable sets are progressively measurable.

**Proposition 60.3.13** Let $\mathcal{P}$ denote the predictable $\sigma$ algebra and let $\mathcal{R}$ denote the progressively measurable $\sigma$ algebra. Then $\mathcal{P} \subseteq \mathcal{R}$.

**Proof:** Let $\mathcal{G}$ denote those sets of $\mathcal{P}$ such that they are also in $\mathcal{R}$. Then $\mathcal{G}$ clearly contains the $\pi$ system of sets $\{0\} \times A, A \in \mathcal{F}_0$, and $(s,t] \times A, A \in \mathcal{F}_s$. Furthermore, $\mathcal{G}$ is closed with respect to countable disjoint unions and complements. It follows $\mathcal{G}$ contains the $\sigma$ algebra generated by this $\pi$ systems which is $\mathcal{P}$. This proves the proposition.

**Proposition 60.3.14** Let $X(t)$ be a stochastic process having values in $E$ a complete metric space and let it be $\mathcal{F}_t$ adapted and left continuous. Then it is predictable. Also, if $X(t)$ is stochastically continuous and adapted on $[0,T]$, then it has a predictable version.

**Proof:** Define $I_{m,k} \equiv ((k-1)2^{-m}T, k2^{-m}T]$ if $k \geq 1$ and $I_{m,0} = \{0\}$ if $k = 1$. Then define

$$X_m(t) = \sum_{k=1}^{2^m} X(T(k-1)2^{-m}) \mathcal{X}_{((k-1)2^{-m}T, k2^{-m}T]}(t)$$

$$+ X(0) \mathcal{X}_{[0,0]}(t)$$

Here the sum means that $X_m(t)$ has value $X(T(k-1)2^{-m})$ on the interval $((k-1)2^{-m}T, k2^{-m}T]$.

Thus $X_m$ is predictable because each term in the sum is. Thus

$$X_m^{-1}(U) = \bigcup_{k=1}^{2^m} (X(T(k-1)2^{-m}) \mathcal{X}_{((k-1)2^{-m}T, k2^{-m}T]}^{-1}(U))$$

$$= \bigcup_{k=1}^{2^m} ((k-1)2^{-m}T, k2^{-m}T] \times (X(T(k-1)2^{-m}))^{-1}(U),$$

a finite union of predictable sets. Since $X$ is left continuous,

$$X(t, \omega) = \lim_{m \to \infty} X_m(t, \omega)$$

and so $X$ is predictable.
Next consider the other claim. Since $X$ is stochastically continuous on $[0, T]$, it is uniformly stochastically continuous on this interval by Lemma 60.1. Therefore, there exists a sequence of partitions of $[0, T]$, the $m$th being

$$ 0 = t_{m,0} < t_{m,1} < \cdots < t_{m,n(m)} = T $$

such that for $X_m$ defined as above, then for each $t$

$$ P \left( \left[ d(X_m(t), X(t)) \geq 2^{-m} \right] \right) \leq 2^{-m} \quad (60.3.19) $$

Then as above, $X_m$ is predictable. Let $A$ denote those points of $\mathcal{P}_T$ at which $X_m(t, \omega)$ converges. Thus $A$ is a predictable set because it is just the set where $X_m(t, \omega)$ is a Cauchy sequence. Now define the predictable function $Y$

$$ Y(t, \omega) \equiv \begin{cases} \lim_{m \to \infty} X_m(t, \omega) & \text{if } (t, \omega) \in A \\ 0 & \text{if } (t, \omega) \notin A \end{cases} $$

From (60.3.19) it follows from the Borel Cantelli lemma that for fixed $t$, the set of $\omega$ which are in infinitely many of the sets,

$$ \left[ d(X_m(t), X(t)) \geq 2^{-m} \right] $$

has measure zero. Therefore, for each $t$, there exists a set of measure zero, $N(t)$ such that for $\omega \notin N(t)$ and all $m$ large enough

$$ d(X_m(t, \omega), X(t, \omega)) < 2^{-m} $$

Hence for $\omega \notin N(t)$, $(t, \omega) \in A$ and so $X_m(t, \omega) \to Y(t, \omega)$ which shows

$$ d(Y(t, \omega), X(t, \omega)) = 0 \text{ if } \omega \notin N(t). $$

The predictable version of $X(t)$ is $Y(t)$. $\blacksquare$

Here is a summary of what has been shown above.

\begin{align*}
\text{adapted and left continuous} & \quad \Downarrow \\
\text{predictable} & \quad \Downarrow \\
\text{progressively measurable} & \quad \Downarrow \\
\text{adapted} & \\
\end{align*}

Also

stochastically continuous and adapted $\implies$ progressively measurable version
60.4 Martingales

Definition 60.4.1 Let $X$ be a stochastic process defined on an interval, $I$ with values in a separable Banach space, $E$. It is called integrable if $E(||X(t)||) < \infty$ for each $t \in I$. Also let $\mathcal{F}_t$ be a filtration. An integrable and adapted stochastic process $X$ is called a martingale if for $s \leq t$

$$E(X(t) | \mathcal{F}_s) = X(s) \text{ P a.e. } \omega.$$ 

Recalling the definition of conditional expectation this says that for $F \in \mathcal{F}_s$

$$\int_F X(t) \, dP = \int_F E(X(t) | \mathcal{F}_s) \, dP = \int_F X(s) \, dP$$

for all $F \in \mathcal{F}_s$. A real valued stochastic process is called a submartingale if whenever $s \leq t$,

$$E(X(t) | \mathcal{F}_s) \geq X(s) \text{ a.e.}$$

and a supermartingale if

$$E(X(t) | \mathcal{F}_s) \leq X(s) \text{ a.e.}$$

Example 60.4.2 Let $\mathcal{F}_t$ be a filtration and let $Z$ be in $L^1(\Omega, \mathcal{F}_T, P)$. Then let $X(t) = E(Z|\mathcal{F}_t)$.

This works because for $s < t$, $E(X(t) | \mathcal{F}_s) = E(E(Z|\mathcal{F}_t) | \mathcal{F}_s) = E(Z|\mathcal{F}_s) = X(s)$.

Proposition 60.4.3 The following statements hold for a stochastic process defined on $[0,T] \times \Omega$ having values in a real separable Banach space, $E$.

1. If $X(t)$ is a martingale then $||X(t)||, t \in [0,T]$ is a submartingale.

2. If $g$ is an increasing convex function from $[0, \infty)$ to $[0, \infty)$ and $E(g(||X(t)||)) < \infty$ for all $t \in [0,T]$ then then $g(||X(t)||), t \in [0,T]$ is a submartingale.

Proof: Let $s \leq t$

$$||X(s)|| = ||E(X(s) - X(t) | \mathcal{F}_s) + E(X(t) | \mathcal{F}_s)||$$

$$= 0 \text{ a.e.}$$

$$\leq \sqrt{||E(X(s) - X(t) | \mathcal{F}_s)||^2 + ||E(X(t) | \mathcal{F}_s)||^2}$$

$$\leq ||E(X(t) | \mathcal{F}_s)||.$$

Now by Theorem 60.1.1 on Page 2051

$$||E(X(t) | \mathcal{F}_s)|| \leq E(||X(t)|| | \mathcal{F}_s).$$

Thus $||X(s)|| \leq E(||X(t)|| | \mathcal{F}_s)$ which shows $||X||$ is a submartingale as claimed.
Consider the second claim. Recall Jensen’s inequality for submartingales, Theorem 60.1.6 on Page 1996. From the first part
\[ ||X(t)|| \leq E(||X(t)|| |\mathcal{F}_s) \text{ a.e.} \]
and so from Jensen’s inequality,
\[ g(||X(s)||) \leq g(E(||X(t)|| |\mathcal{F}_s)) \leq E(g(||X(t)||) |\mathcal{F}_s) \text{ a.e.,} \]
showing that \( g(||X(t)||) \) is also a submartingale. This proves the proposition.

### 60.5 Some Maximal Estimates

Martingales and submartingales have some very interesting maximal estimates. I will present some of these here. The proofs are fairly general and do not require the filtration to be normal.

**Lemma 60.5.1** Let \( \{\mathcal{F}_t\} \) be a filtration and let \( \{X(t)\} \) be a nonnegative valued submartingale for \( t \in [S,T] \). Then for \( \lambda > 0 \) and any \( p \geq 1 \), if \( A_t \) is a \( \mathcal{F}_t \) measurable subset of \( \{X(t) \geq \lambda\} \), then
\[ P(A_t) \leq \frac{1}{\lambda^p} \int_{A_t} X(T)^p \, dP. \]

**Proof:** From Jensen’s inequality,
\[ \lambda^p P(A_t) \leq \int_{A_t} X(t)^p \, dP \leq \int_{A_t} E(X(T) |\mathcal{F}_t)^p \, dP \leq \int_{A_t} E(X(T)^p |\mathcal{F}_t) \, dP = \int_{A_t} X(T)^p \, dP \]
and this proves the lemma.

The following theorem is the main result.

**Theorem 60.5.2** Let \( \{\mathcal{F}_t\} \) be a filtration and let \( \{X(t)\} \) be a nonnegative valued right continuous submartingale for \( t \in [S,T] \). Then for all \( \lambda > 0 \) and \( p \geq 1 \), for \( X^* \equiv \sup_{t \in [S,T]} X(t) \),
\[ P(\{X^* \geq \lambda\}) \leq \frac{1}{\lambda^p} \int_{\Omega} X^*|\{X^* \geq \lambda\}|X(T)^p \, dP \]
In the case that \( p > 1 \), it is also true that
\[ E((X^*)^p) \leq \left( \frac{p}{p-1} \right) E(X(T)^p)^{1/p} E((X^*)^p)^{1/p'} \]
Also there are no measurability issues related to the above \( \sup_{t \in [S,T]} X(t) \equiv X^* \)
\[ ^{1}t \mapsto M(t)(\omega) \text{ is continuous from the right for a.e. } \omega. \]
Proof: Let \( S \leq t_0^m < t_1^m < \cdots < t_{m+1}^m = T \) where \( t_{j+1}^m - t_j^m = (T - S) 2^{-m} \).

First consider \( m = 1 \).

\[
A_{t_0} \equiv \{ \omega \in \Omega : X (t_0^1) (\omega) \geq \lambda \}, \quad A_{t_1} \equiv \{ \omega \in \Omega : X (t_1^1) (\omega) \geq \lambda \} \backslash A_{t_0} \quad A_{t_2} \equiv \{ \omega \in \Omega : X (t_2^1) (\omega) \geq \lambda \} \backslash (A_{t_1} \cup A_{t_0}) .
\]

Do this type of construction for \( m = 2, 3, 4, \ldots \) yielding disjoint sets, \( \{ A_{t_j^m} \}_{j=0}^{2^m} \) whose union equals
\[
\bigcup_{t \in D_m} [X (t) \geq \lambda]
\]
where \( D_m = \{ t_j^m \}_{j=0}^{2^m} \). Thus \( D_m \subseteq D_{m+1} \). Then also, \( D \equiv \bigcup_{m=1}^{\infty} D_m \) is dense and countable. From Lemma \[60.5.1\]
\[
P (\bigcup_{t \in D_m} [X (t) \geq \lambda]) = P (\sup_{t \in D_m} X (t) \geq \lambda) = \sum_{j=0}^{2^m} P (A_{t_j^m}) 
\leq \frac{1}{\lambda^p} \sum_{j=0}^{2^m} \int_{A_{t_j^m}} X (\sup_{t \in D_m} X(t) \geq \lambda) X (T)^p \ dP
\leq \frac{1}{\lambda^p} \int_{\Omega} X (\sup_{t \in D} X(t) \geq \lambda) X (T)^p \ dP.
\]

Let \( m \to \infty \) in the above to obtain
\[
P (\bigcup_{t \in D} [X (t) \geq \lambda]) = P \left( \left[ \sup_{t \in D} X (t) \geq \lambda \right] \right) \leq \frac{1}{\lambda^p} \int_{\Omega} X (\sup_{t \in D} X(t) \geq \lambda) X (T)^p \ dP.
\]

(60.5.20)

From now on, assume that for a.e. \( \omega \in \Omega \), \( t \to X (t) (\omega) \) is right continuous. Then with this assumption, the following claim holds.

\[
\sup_{t \in [S, T]} X (t) \equiv X^* = \sup_{t \in D} X (t)
\]

which verifies that \( X^* \) is measurable. Then from \[60.5.2\],
\[
P (|X^*| \geq \lambda) = P \left( \left[ \sup_{t \in D} X (t) |X^*| \geq \lambda \right] \right)
\leq \frac{1}{\lambda^p} \int_{\Omega} X (\sup_{t \in D} X(t) |X^*| \geq \lambda) X (T)^p \ dP
\leq \frac{1}{\lambda^p} \int_{\Omega} X (|X^*| \geq \lambda) X (T)^p \ dP.
\]

Now consider the other inequality. Using the distribution function technique and the above estimate obtained in the first part,
\[
E (|X^*|^p) = \int_0^\infty \alpha^{p-1} P (|X^*| > \alpha) \ d\alpha
\leq \int_0^\infty \alpha^{p-1} P (|X^*| \geq \alpha) \ d\alpha.
\]
60.5. SOME MAXIMAL ESTIMATES

\[ \int_0^\infty p^\alpha p^{-1} \frac{1}{\alpha} \int_\Omega X_{[X^* \geq \alpha]} X(T) dP d\alpha \]

\[ = p \int_\Omega \int_0^{X^*} \alpha^{p-2} d\alpha X(T) dP \]

\[ = \frac{p}{p-1} \int_\Omega (X^*)^{p-1} X(T) dP \]

\[ \leq \frac{p}{p-1} \left( \int_\Omega (X^*)^p \right)^{1/p'} \left( \int_\Omega X(T)^p \right)^{1/p} \]

\[ = \frac{p}{p-1} E(X(T)^p)^{1/p} E((X^*)^{p/p'}). \blacksquare \]

Of course it would be nice to divide both sides by \( E((X^*)^{p/p'}) \) but we don’t know that this is finite. One can use a stopped submartingale which will have \( X(t) \) bounded, divide, and then let the stopping time increase to \( \infty \). However, this is discussed later.

With Theorem 60.5.2, here is an important maximal estimate for martingales having values in \( E \), a real separable Banach space.

**Theorem 60.5.3** Let \( X(t) \) for \( t \in I = [0,T] \) be an \( E \) valued right continuous martingale with respect to a filtration, \( \mathcal{F}_t \). Then for \( p \geq 1 \),

\[ P \left( \sup_{t \in I} ||X(t)|| \geq \lambda \right) \leq \frac{1}{\lambda^p} E(||X(T)||^p). \] (60.5.21)

If \( p > 1 \),

\[ E \left( \left( \sup_{t \in [S,T]} ||X(t)|| \right)^p \right) \leq \frac{p}{p-1} E(||X(T)||^{p/p}) \left( \sup_{t \in [S,T]} ||X(t)|| \right)^{p/p}. \] (60.5.22)

**Proof:** By Proposition 60.4.3, \( ||X(t)||, t \in I \) is a submartingale and so from Theorem 60.5.2, it follows (60.5.21) and (60.5.22) hold. \( \blacksquare \)

**Definition 60.5.4** Let \( K \) be a set of functions of \( L^1(\Omega, \mathcal{F}, P) \). Then \( K \) is called **equi integrable** if

\[ \lim_{\lambda \to \infty} \sup_{f \in K} \int_{||f|| \geq \lambda} |f| dP = 0. \]

Recall that from Corollary 18.9.6 on Page 623 such an equi integrable set of functions is weakly sequentially precompact in \( L^1(\Omega, \mathcal{F}, P) \) in the sense that if \( \{f_n\} \subseteq K \), there exists a subsequence, \( \{f_{n_k}\} \) and a function, \( f \in L^1(\Omega, \mathcal{F}, P) \) such that for all \( g \in L^1(\Omega, \mathcal{F}, P)^{p'} \),

\[ g(f_{n_k}) \to g(f). \]
60.6 Optional Sampling Theorems

60.6.1 Stopping Times And Their Properties

The optional sampling theorem is very useful in the study of martingales and submartingales as will be shown.

First it is necessary to define the notion of a stopping time.

**Definition 60.6.1** Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(\{\mathcal{F}_n\}_{n=1}^{\infty}\) be an increasing sequence of \(\sigma\)-algebras each contained in \(\mathcal{F}\), called a discrete filtration. A stopping time is a measurable function, \(\tau\) which maps \(\Omega\) to \(\mathbb{N}\), such that for all \(n \in \mathbb{N}\),
\[
[\tau \leq n] \in \mathcal{F}_n,
\]
and for all \(n \in \mathbb{N}\),
\[
[\tau = n] \in \mathcal{F}_n.
\]
Note this is equivalent to saying
\[
[\tau = n] \in \mathcal{F}_n
\]
because
\[
[\tau = n] = [\tau \leq n] \setminus [\tau \leq n-1].
\]
For \(\tau\) a stopping time define \(\mathcal{F}_\tau\) as follows.
\[
\mathcal{F}_\tau = \{ A \in \mathcal{F} : A \cap [\tau \leq n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N} \}
\]
These sets in \(\mathcal{F}_\tau\) are referred to as “prior” to \(\tau\).

The most important example of a stopping time is the first hitting time.

**Example 60.6.2** The first hitting time of an adapted process \(X(n)\) of a Borel set \(G\) is a stopping time. This is defined as
\[
\tau = \min \{ k : X(k) \in G \}
\]
To see this, note that
\[
[\tau = n] = \cap_{k<n} [X(k) \in G^c] \cap [X(n) \in G] \in \mathcal{F}_n.
\]

**Proposition 60.6.3** For \(\tau\) a stopping time, \(\mathcal{F}_\tau\) is a \(\sigma\)-algebra and if \(Y(k)\) is \(\mathcal{F}_k\) measurable for all \(k\), \(Y(k)\) having values in a separable Banach space \(E\), then
\[
\omega \to Y(\tau(\omega))
\]
is \(\mathcal{F}_\tau\) measurable.
Proof: Let $A_n \in \mathcal{F}_\tau$. I need to show $\bigcup_n A_n \in \mathcal{F}_\tau$. In other words, I need to show that

$$\bigcup_n A_n \cap [\tau \leq k] \in \mathcal{F}_k$$

The left side equals

$$\bigcup_n (A_n \cap [\tau \leq k])$$

which is a countable union of sets of $\mathcal{F}_k$ and so $\mathcal{F}_\tau$ is closed with respect to countable unions. Next suppose $A \in \mathcal{F}_\tau$.

$$(A^c \cap [\tau \leq k]) \cup (A \cap [\tau \leq k]) = \Omega \cap [\tau \leq k]$$

and $\Omega \cap [\tau \leq k] \in \mathcal{F}_k$. Therefore, so is $A^c \cap [\tau \leq k]$.

It remains to verify the last claim. Let $B$ be an open set in $E$. Is

$$[Y(\tau) \in B] \in \mathcal{F}_\tau?$$

Is

$$[Y(\tau) \in B] \cap [\tau \leq k] \in \mathcal{F}_k \text{ for all } k?$$

This equals

$$\bigcup_{i=1}^k [Y(\tau) \in B] \cap [\tau = i] = \bigcup_{i=1}^k [Y(i) \in B] \cap [\tau = i] \in \mathcal{F}_k$$

Therefore, $Y(\tau)$ must be $\mathcal{F}_\tau$ measurable. $\blacksquare$

The following lemma contains the fundamental properties of stopping times for discrete filtrations.

Lemma 60.6.4 In the situation of Definition 60.6.1, let $\sigma, \tau$ be two stopping times. Then

1. $\tau$ is $\mathcal{F}_\tau$ measurable
2. $\mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$
3. $\mathcal{F}_\tau = \mathcal{F}_k$ on $[\tau = k]$ for all $k$. That is if $A \in \mathcal{F}_k$, then $A \cap [\tau = k] \in \mathcal{F}_\tau$ and if $A \in \mathcal{F}_\tau$, then $A \cap [\tau = k] \in \mathcal{F}_k$. In other words, the two $\sigma$ algebras

$$[\tau = k] \cap \mathcal{F}_\tau, [\tau = k] \cap \mathcal{F}_k$$

are equal. Letting $\mathcal{G}$ denote this $\sigma$ algebra, if $g$ is either $\mathcal{F}_\tau$ or $\mathcal{F}_k$ measurable then its restriction to $[\tau = k]$ is $\mathcal{G}$ measurable. Also if $A \in \mathcal{F}_\tau$, and $Y \in L^1(\Omega; E)$,

$$\int_{A \cap [\tau = k]} E(Y|\mathcal{F}_\tau) \, dP = \int_{A \cap [\tau = k]} E(Y|\mathcal{F}_k) \, dP$$

and

$$E(Y|\mathcal{F}_\tau) = E(Y|\mathcal{F}_k) \text{ a.e.}$$

on $[\tau = k]$. 


Proof: Consider the first claim. \( [\tau \leq l] \cap [\tau \leq m] = [\tau \leq \lfloor l \rfloor] \cap [\tau \leq m] \in \mathcal{F}_{\lfloor l \rfloor \wedge m} \subseteq \mathcal{F}_m \) and so \( \tau \) is \( \mathcal{F}_\tau \) measurable. Here \( \lfloor l \rfloor \) is the greatest integer less than or equal to \( l \). Next note that \( \sigma \wedge \tau \) is a stopping time because
\[
[\sigma \wedge \tau \leq k] = [\sigma \leq k] \cup [\tau \leq k] \in \mathcal{F}_k
\]

Next consider the second claim. Let \( A \in \mathcal{F}_\sigma \). I want to show
\[
A \cap [\sigma \leq \tau] \in \mathcal{F}_{\tau \wedge \sigma}
\]
(60.6.23)
In other words, I want to show
\[
A \cap [\sigma \leq \tau] \cap [\tau \wedge \sigma \leq k] \in \mathcal{F}_k
\]
(60.6.24)
for all \( k \). However, the set on the left equals
\[
A \cap [\sigma \leq \tau] \cap [\sigma \leq k] = \bigcup_{j=1}^{\lfloor k \rfloor} A \cap [\sigma = j] \cap [\tau \geq j] \cap [\sigma \leq k] = \bigcup_{j=1}^{\lfloor k \rfloor} A \cap [\sigma = j] \cap [\tau \leq \sigma - 1] \cap [\sigma \leq k] \in \mathcal{F}_k
\]

Now let \( A \in \mathcal{F}_{\sigma \wedge \tau} \). I want to show it is in both \( \mathcal{F}_\tau \) and \( \mathcal{F}_\sigma \). To show it is in \( \mathcal{F}_\tau \) I need to show that
\[
A \cap [\tau \leq k] \in \mathcal{F}_k
\]
for all \( k \). However,
\[
A \cap [\tau = k] = \bigcup_{i=1}^{\infty} A \cap [\sigma = i] \cap [\tau = k]
\]
\[
= \bigcup_{i=1}^{\infty} A \cap [\sigma \wedge \tau = i] \cap [\tau = k] \cup \bigcup_{i=1}^{\infty} A \cap [\sigma = i] \cap [\tau = k]
\]
\[
= \bigcup_{i=1}^{\infty} A \cap [\sigma \wedge \tau = i] \cap [\tau = k] \cup A \cap [\sigma \wedge \tau = k] \cap [\tau = k]
\]
and so this is in \( \mathcal{F}_k \). Thus \( A \cap [\tau \leq k] \in \mathcal{F}_k \) being the finite union of sets which are. Similarly \( A \cap [\sigma \leq k] \in \mathcal{F}_k \) for all \( k \) and so \( A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma \).

Next let \( A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma \). Then is it in \( \mathcal{F}_{\sigma \wedge \tau} \)? Is \( A \cap [\sigma \wedge \tau \leq k] \in \mathcal{F}_k \)? Of course this is so because
\[
A \cap [\sigma \wedge \tau \leq k] = A \cap ([\sigma \leq k] \cup [\tau \leq k])
\]
\[
= (A \cap [\sigma \leq k]) \cup (A \cap [\tau \leq k]) \in \mathcal{F}_k
\]
since both \( \sigma, \tau \) are stopping times. This proves part 2.).

Now consider part 3.). Note that \( [\tau = k] \) is in both \( \mathcal{F}_k \) and \( \mathcal{F}_\tau \). First consider the claim it is in \( \mathcal{F}_\tau \).
\[
[\tau = k] \cap [\tau \leq l] = \emptyset \text{ if } l < k
\]
which is in \( F \). If \( l \geq k \), it reduces to \([\tau = k] \in F_k \subseteq F \) so it is in \( F_\tau \). \([\tau = k] \) is obviously in \( F_k \).

I need to show

\[
F_\tau \cap [\tau = k] = F_k \cap [\tau = k]
\]

where \( H \cap [\tau = k] \) means all sets of the form \( A \cap [\tau = k] \) where \( A \in H \). Let \( A \in F_\tau \). Then

\[
A \cap [\tau = k] = (A \cap [\tau \leq k]) \setminus (A \cap [\tau \leq k - 1]) \in F_k
\]

Therefore, there exists \( B \in F_k \) such that \( B = A \cap [\tau = k] \) and so

\[
B \cap [\tau = k] = A \cap [\tau = k]
\]

which shows \( F_\tau \cap [\tau = k] \subseteq F_k \cap [\tau = k] \). Now let \( A \in F_k \) so that

\[
A \cap [\tau = k] \in F_k \cap [\tau = k]
\]

Then

\[
A \cap [\tau = k] \cap [\tau \leq j] \in F_j
\]

because in case \( j < k \), the set on the left is \( \emptyset \) and if \( j \geq k \) it reduces to \( A \cap [\tau = k] \) and both \( A \) and \([\tau = k]\) are in \( F_k \subseteq F_j \). Thus \( A \cap [\tau = k] = B \in F_\tau \) and so

\[
A \cap [\tau = k] = B \cap [\tau = k] \in F_\tau \cap [\tau = k].
\]

Therefore, the two \( \sigma \) algebras of subsets of \([\tau = k]\),

\[
F_\tau \cap [\tau = k], F_k \cap [\tau = k]
\]

are equal. Thus for \( A \) in either \( F_\tau \) or \( F_k \), \( A \cap [\tau = k] \) is a set of both \( F_\tau \) and \( F_k \) because if \( A \in F_k \), then from the above, there exists \( B \in F_\tau \) such that

\[
e_{F_\tau \cap F_k} A \cap [\tau = k] = B \cap [\tau = k] \in F_\tau
\]

with similar reasoning holding if \( A \in F_\tau \). In other words, if \( g \) is \( F_\tau \) or \( F_k \) measurable, then the restriction of \( g \) to \([\tau = k]\) is measurable with respect to \( F_\tau \cap [\tau = k] \) and \( F_k \cap [\tau = k] \). Let \( Y \) be an arbitrary random variable in \( L^1 (\Omega, \mathcal{F}) \). It follows, since \( A \cap [\tau = k] \) is in both \( F_\tau \) and \( F_k \),

\[
\int_{A \cap [\tau = k]} E(Y|F_\tau) \, dP = \int_{A \cap [\tau = k]} Y \, dP = \int_{A \cap [\tau = k]} E(Y|F_k) \, dP
\]

Since this holds for an arbitrary set in \( F_\tau \cap [\tau = k] = F_k \cap [\tau = k] \), it follows

\[
E(Y|F_\tau) = E(Y|F_k) \text{ a.e. on } [\tau = k] \]

The assertion that 
\[ E (Y | \mathcal{F}_\tau) = E (Y | \mathcal{F}_k) \] a.e. 
onumber on \([\tau = k]\) and that a function \(g\) which is \(\mathcal{F}_\tau\) or \(\mathcal{F}_k\) measurable when restricted to \([\tau = k]\) is \(\mathcal{G}\) measurable for 
\[ \mathcal{G} = [\tau = k] \cap \mathcal{F}_\tau = [\tau = k] \cap \mathcal{F}_k \] 

is the main result in the above lemma and this fact leads to the amazing Doob optional sampling theorem below. Also note that if \(Y (k)\) is any process defined on the positive integers \(k\), then by definition, \(Y (k) (\omega) = Y (\tau (\omega)) (\omega)\) on the set \([\tau = k]\) because \(\tau\) is constant on this set.

### 60.6.2 Doob Optional Sampling Theorem

With this lemma, here is a major theorem, the optional sampling theorem of Doob. This one is for martingales having values in a Banach space. To begin with, consider the case of a martingale defined on a countable set.

**Theorem 60.6.5** Let \(\{ M (k) \}\) be a martingale having values in \(E\) a separable real Banach space with respect to the increasing sequence of \(\sigma\) algebras, \(\{ \mathcal{F}_k \}\) and let \(\sigma, \tau\) be two stopping times such that \(\tau\) is bounded. Then \(M (\tau)\) defined as 
\[ \omega \rightarrow M (\tau (\omega)) \] is integrable and 
\[ M (\sigma \wedge \tau) = E (M (\tau) | \mathcal{F}_\sigma) . \]

**Proof:** By Proposition 60.6.3 \(M (\tau)\) is \(\mathcal{F}_\tau\) measurable. Next note that since \(\tau\) is bounded by some \(l\),
\[
\int_\Omega \| M (\tau (\omega)) \| \, dP \leq \sum_{i=1}^l \int_{[\tau = i]} \| M (i) \| \, dP < \infty.
\]
This proves the first assertion and makes possible the consideration of conditional expectation.

Let \(l \geq \tau\) as described above. Then for \(k \leq l\), by Lemma 60.6.3, 
\[ \mathcal{F}_k \cap [\tau = k] = \mathcal{F}_\tau \cap [\tau = k] \equiv \mathcal{G} \]
implying that if \(g\) is either \(\mathcal{F}_k\) measurable or \(\mathcal{F}_\tau\) measurable, then its restriction to \([\tau = k]\) is \(\mathcal{G}\) measurable and so if \(A \in \mathcal{F}_k \cap [\tau = k] = \mathcal{F}_\tau \cap [\tau = k]\), 
\[
\int_A E (M (l) | \mathcal{F}_\tau) \, dP \equiv \int_A M (l) \, dP \\
= \int_A E (M (l) | \mathcal{F}_k) \, dP \\
= \int_A M (k) \, dP \\
= \int_A M (\tau) \, dP \text{ (on } A, \tau = k) \]
Therefore, since $A$ was arbitrary,

$$E(M(l) | \mathcal{F}_\tau) = M(\tau) \text{ a.e.}$$

on $[\tau = k]$ for every $k \leq l$. It follows

$$E(M(l) | \mathcal{F}_\tau) = M(\tau) \text{ a.e.}$$

since it is true on each $[\tau = k]$ for all $k \leq l$.

Now consider $E(M(\tau) | \mathcal{F}_\sigma)$ on the set $[\sigma = i] \cap [\tau = j]$. By Lemma 60.6.4, on this set,

$$E(M(\tau) | \mathcal{F}_\sigma) = E(M(\tau) | \mathcal{F}_i) = E(E(M(l) | \mathcal{F}_\tau) | \mathcal{F}_i) = E(E(M(l) | \mathcal{F}_j) | \mathcal{F}_i)$$

If $j \leq i$, this reduces to

$$E(M(l) | \mathcal{F}_j) = M(j) = M(\sigma \wedge \tau)$$

If $j > i$, this reduces to

$$E(M(l) | \mathcal{F}_i) = M(i) = M(\sigma \wedge \tau)$$

and since this exhausts all possibilities for values of $\sigma$ and $\tau$, it follows

$$E(M(\tau) | \mathcal{F}_\sigma) = M(\sigma \wedge \tau) \text{ a.e.} \quad \blacksquare$$

You can also give a version of the above to submartingales. This requires the following very interesting decomposition of a submartingale into the sum of an increasing stochastic process and a martingale.

**Theorem 60.6.6** Let $\{X_n\}$ be a submartingale. Then there exists a unique stochastic process, $\{A_n\}$ and martingale, $\{M_n\}$ such that

1. $A_n(\omega) \leq A_{n+1}(\omega), A_1(\omega) = 0,$
2. $A_n$ is $\mathcal{F}_{n-1}$ adapted for all $n \geq 1$ where $\mathcal{F}_0 \equiv \mathcal{F}_1$.

and also $X_n = M_n + A_n$.

**Proof:** Let $A_1 \equiv 0$ and define

$$A_n \equiv \sum_{k=2}^{n} E(X_k - X_{k-1} | \mathcal{F}_{k-1}).$$

It follows $A_n$ is $\mathcal{F}_{n-1}$ measurable. Since $\{X_k\}$ is a submartingale, $A_n$ is increasing because

$$A_{n+1} - A_n = E(X_{n+1} - X_n | \mathcal{F}_n) \geq 0 \quad (60.6.25)$$
CHAPTER 60. STOCHASTIC PROCESSES

It is a submartingale because

\[
E(A_n|\mathcal{F}_{n-1}) = E\left(\sum_{k=2}^{n} E(X_k - X_{k-1}|\mathcal{F}_{k-1})|\mathcal{F}_{n-1}\right)
\]

\[
= \sum_{k=2}^{n} E(X_k - X_{k-1}|\mathcal{F}_{k-1}) \equiv A_n \geq A_{n-1}
\]

Now let \( M_n \) be defined by

\[
X_n = M_n + A_n.
\]

Then from Theorem 60.6.25,

\[
E(M_{n+1}|\mathcal{F}_n) = E(X_{n+1}|\mathcal{F}_n) - E(A_{n+1}|\mathcal{F}_n)
\]

\[
= E(X_{n+1}|\mathcal{F}_n) - E(A_{n+1} - A_n|\mathcal{F}_n) - A_n
\]

\[
= E(X_{n+1}|\mathcal{F}_n) - E(E(X_{n+1} - X_n|\mathcal{F}_n)|\mathcal{F}_n) - A_n
\]

\[
= E(X_n|\mathcal{F}_n) - A_n
\]

\[
= X_n - A_n \equiv M_n
\]

This proves the existence part.

It remains to verify uniqueness. Suppose then that

\[
X_n = M_n + A_n = M'_n + A'_n
\]

where \( \{A_n\} \) and \( \{A'_n\} \) both satisfy the conditions of the theorem and \( \{M_n\} \) and \( \{M'_n\} \) are both martingales. Then

\[
M_n - M'_n = A'_n - A_n
\]

and so, since \( A'_n - A_n \) is \( \mathcal{F}_{n-1} \) measurable and \( \{M_n - M'_n\} \) is a martingale,

\[
M_{n-1} - M'_{n-1} = E(M_n - M'_n|\mathcal{F}_{n-1})
\]

\[
= E(A'_n - A_n|\mathcal{F}_{n-1})
\]

\[
= A'_n - A_n = M_n - M'_n.
\]

Continuing this way shows \( M_n - M'_n \) is a constant. However, since \( A'_1 - A_1 = 0 = M_1 - M'_1 \), it follows \( M_n = M'_n \) and this proves uniqueness.

Now here is a version of the optional sampling theorem for submartingales.

Theorem 60.6.7 Let \( \{X(k)\} \) be a real valued submartingale with respect to the increasing sequence of \( \sigma \) algebras, \( \{\mathcal{F}_k\} \) and let \( \sigma \leq \tau \) be two stopping times such that \( \tau \) is bounded. Then \( M(\tau) \) defined as

\[
\omega \rightarrow X(\tau(\omega))
\]


60.7. DOOB OPTIONAL SAMPLING CONTINUOUS CASE

is integrable and

\[ X(\sigma) \leq E(X(\tau)|F_\sigma). \]

Without assuming \( \sigma \leq \tau \), one can write

\[ X(\sigma \wedge \tau) \leq E(X(\tau)|F_\sigma). \]

**Proof:** That \( \omega \rightarrow X(\tau(\omega)) \) is integrable follows right away as in the optional sampling theorem for martingales. You just consider the finitely many values of \( \tau \).

Use Theorem 60.6.6 above to write

\[ X(n) = M(n) + A(n) \]

where \( M \) is a martingale and \( A \) is increasing with \( A(n) \) being \( F_{n-1} \) measurable and \( A(0) = 0 \) as discussed in Theorem 60.6.6. Then

\[ E(X(\tau)|F_\tau) = E(M(\tau) + A(\tau)|F_\sigma) \]

Now since \( A \) is increasing, you can use the optional sampling theorem for martingales, Theorem 60.6.6 to conclude that, since \( F_{\sigma \wedge \tau} \subseteq F_\sigma \) and \( A(\sigma \wedge \tau) \) is \( F_{\sigma \wedge \tau} \) measurable,

\[ E(M(\tau) + A(\sigma \wedge \tau)|F_\sigma) = E(M(\tau)|F_\sigma) + A(\sigma \wedge \tau) \]

In summary, the main results for stopping times for discrete filtrations are the following definitions and theorems.

\[ [\tau \leq m] \in F_m \]

\( A \in F_\tau \) means \( A \cap [\tau \leq m] \in F_m \) for any \( m \)

\( X \) adapted implies \( X(\tau) \) is \( F_\tau \) measurable

\[ F_{\sigma \wedge \tau} = F_\sigma \cap F_\tau \]

\[ [\tau = k] \cap F_k = [\tau = k] \cap F_\tau \]

This last theorem implies the following amazing result. From these fundamental properties, we obtain the optional sampling theorem for martingales and submartingales.

\[ E(Y|F_\tau) = E(Y|F_k) \text{ a.e. on } [\tau = k] \]

60.7 Doob Optional Sampling Continuous Case

60.7.1 Stopping Times

With continuous processes, the discrete filtration is replaced by a normal filtration. Also we tend to feature right continuous or continuous processes. As in the case of discrete martingales, there is something called a stopping time.
**Definition 60.7.1** Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \(\mathcal{F}_t\) be a filtration. A measurable function, \(\tau: \Omega \to [0, \infty]\) is called a stopping time if

\[ [\tau \leq t] \in \mathcal{F}_t \]

for all \(t \geq 0\). Associated with a stopping time is the \(\sigma\)-algebra, \(\mathcal{F}_\tau\) defined by

\[ \mathcal{F}_\tau \equiv \{ A \in \mathcal{F} : A \cap [\tau \leq t] \in \mathcal{F}_t \ \text{for all} \ t \geq 0 \} . \]

These sets are also called those “prior” to \(\tau\).

Note that \(\mathcal{F}_\tau\) is obviously closed with respect to countable unions. If \(A \in \mathcal{F}_\tau\), then

\[ A^C \cap [\tau \leq t] = [\tau \leq t] \setminus (A \cap [\tau \leq t]) \in \mathcal{F}_t \]

Thus \(\mathcal{F}_\tau\) is a \(\sigma\)-algebra.

**Proposition 60.7.2** Let \(B\) be an open subset of topological space \(E\) and let \(X(t)\) be a right continuous \(\mathcal{F}_t\) adapted stochastic process such that \(\mathcal{F}_t\) is normal. Then define

\[ \tau(\omega) \equiv \inf \{ t > 0 : X(t)(\omega) \in B \} . \]

This is called the first hitting time. Then \(\tau\) is a stopping time. If \(X(t)\) is continuous and adapted to \(\mathcal{F}_t\), a normal filtration, then if \(H\) is a nonempty closed set such that \(H = \bigcap_{n=1}^{\infty} B_n\) for \(B_n\) open, \(B_n \supseteq B_{n+1}\),

\[ \tau(\omega) \equiv \inf \{ t > 0 : X(t)(\omega) \in H \} \]

is also a stopping time.

**Proof:** Consider the first claim. \(\omega \in [\tau = a]\) implies that for each \(n \in \mathbb{N}\), there exists \(t \in [a, a + \frac{1}{n}]\) such that \(X(t) \in B\). Also for \(t < a\), you would need \(X(t) \notin B\). By right continuity, this is the same as saying that \(X(d) \notin B\) for all rational \(d < a\). (If \(t < a\), you could let \(d_n \downarrow t\) where \(X(d_n) \in B^C\), a closed set. Then it follows that \(X(t)\) is also in the closed set \(B^C\).) Thus, aside from a set of measure zero, for each \(m \in \mathbb{N}\),

\[ [\tau = a] = \left( \bigcap_{n=m}^{\infty} \bigcup_{t \in [a, a + \frac{1}{n}]} [X(t) \in B] \right) \cap \left( \bigcap_{t \in [0, a]} [X(t) \in B^C] \right) \]

Since \(X(t)\) is right continuous, this is the same as

\[ \left( \bigcap_{n=m}^{\infty} \bigcup_{d \in \mathbb{Q} \cap [a, a + \frac{1}{n}]} [X(d) \in B] \right) \cap \left( \bigcap_{d \in \mathbb{Q} \cap [0, a]} [X(d) \in B^C] \right) \in \mathcal{F}_{a+\frac{1}{n}} \]

Thus, since the filtration is normal,

\[ [\tau = a] \in \cap_{n=1}^{\infty} \mathcal{F}_{a+\frac{1}{n}} = \mathcal{F}_{a+} = \mathcal{F}_a \]
Now what of \([\tau < a]\)? This is equivalent to saying that \(X(t) \in B\) for some \(t < a\). Since \(X\) is right continuous, this is the same as saying that \(X(t) \in B\) for some \(t \in \mathbb{Q}, t < a\). Thus
\[
[\tau < a] = \cup_{d \in \mathbb{Q}, d < a} [X(d) \in B] \in \mathcal{F}_a
\]
It follows that \([\tau \leq a] = [\tau < a] \cup [\tau = a] \in \mathcal{F}_a\).

Now consider the claim involving the additional assumption that \(X(t)\) is continuous and it is desired to hit a closed set \(H = \cap_{n=1}^{\infty} B_n\) where \(B_n\) is open, \(B_n \supseteq B_{n+1}\). (Note that if the topological space is a metric space, this is always possible so this is not a big restriction.) Then let \(\tau_n\) be the first hitting time of \(B_n\) by \(X(t)\). Then it can be shown that
\[
[\tau \leq a] = \cap_n [\tau_n \leq a] \in \mathcal{F}_a
\]
To show this, first note that \(\omega \in [\tau \leq a]\) if and only if there exists \(t \leq a\) such that \(X(t)(\omega) \in H\). This follows from continuity and the fact that \(H\) is closed. Thus \(\tau_n \leq a\) for all \(n\) because for some \(t \leq a\), \(X(t) \in H \subseteq B_n\) for all \(n\). Next suppose \(\omega \in [\tau_n \leq a]\) for all \(n\). Then for \(\delta_n \downarrow 0\), there exists \(t_n \in [0, a + \delta_n]\) such that \(X(t_n)(\omega) \in B_n\). It follows there is a subsequence, still denoted by \(t_n\), such that \(t_n \to t \in [0, a]\). By continuity of \(X\), it must be the case that \(X(t)(\omega) \in H\) and so \(\omega \in [\tau \leq a]\). This shows the above formula. Now by the first part, each \([\tau_n \leq a] \in \mathcal{F}_a\) and so \([\tau \leq a] \in \mathcal{F}_a\) also. ■

Another useful result for real valued stochastic process is the following.

**Proposition 60.7.3** Let \(X(t)\) be a real valued stochastic process which is \(\mathcal{F}_t\) adapted for a normal filtration \(\mathcal{F}_t\), with the property that off a set of measure zero in \(\Omega\), \(t \to X(t)\) is lower semicontinuous. Then
\[
\tau \equiv \inf \{t : X(t) > a\}
\]
is a stopping time.

**Proof:** As above
\[
[\tau = a] = \left(\cap_{n=m}^{\infty} \cup_{t \in [a, a+\frac{1}{n}]} [X(t) > a]\right) \cap \left(\cap_{t \in [0, a]} [X(t) \leq a]\right)
\]
Now
\[
\cap_{t \in [0, a]} [X(t) \leq a] \subseteq \cap_{t \in [0, a], t \in \mathbb{Q}} [X(t) \leq a]
\]
If \(\omega\) is in the right side, then for arbitrary \(t < a\), let \(t_n \downarrow t\) where \(t_n \in \mathbb{Q}\) and \(t < a\). Then \(X(t) \leq \liminf_{n \to \infty} X(t_n) < a\) and so \(\omega\) is in the left side also. Thus
\[
\cap_{t \in [0, a]} [X(t) \leq a] = \cap_{t \in [0, a], t \in \mathbb{Q}} [X(t) \leq a]
\]
\[
\cup_{t \in [a, a+\frac{1}{n}]} [X(t) > a] \supseteq \cup_{t \in [a, a+\frac{1}{n}], t \in \mathbb{Q}} [X(t) > a]
\]
If \(\omega\) is in the left side, then for some \(t\) in the given interval, \(X(t) > a\). If for all \(s \in [a, a+\frac{1}{n}] \cap \mathbb{Q}\) you have \(X(s) \leq a\), then you could take \(s_n \to t\) where
\(X(s_n) \leq a\) and conclude that \(X(t) \leq a\) also by lower semicontinuity. Thus there is some rational \(s\) where \(X(s) > a\) and so the two sides are equal. Hence,

\[
\left\{ \tau = a \right\} = \left( \cap_{n=m}^{\infty} \cup_{t \in [a, a + \frac{1}{n}], t \in \mathbb{Q}} [X(t) > a] \right) \cap \left( \cap_{t \in [0, a], t \in \mathbb{Q}} [X(t) \leq a] \right)
\]

The first set on the right is in \(F_{a + (1/m)}\) and so is the next set on the right. Hence \(\left\{ \tau = a \right\} \in \cap_{m} F_{a + (1/m)} = F_a\). What of \(\left\{ \tau < a \right\}\)? This equals \(\cup_{t \in [0, a]} [X(t) > a] = \cup_{t \in [0, a] \cap \mathbb{Q}} [X(t) > a] \in F_a\), the equality following from lower semicontinuity. Thus \(\left\{ \tau \leq a \right\} = \left\{ \tau = a \right\} \cup \left\{ \tau < a \right\} \in F_a\).

Thus there do exist stopping times, the first hitting time above being an example. When dealing with continuous stopping times on a normal filtration, one uses the following discrete stopping times

\[
\tau_n = \sum_{k=1}^{\infty} \Delta_{[\tau \in (t^n_k, t^n_{k+1})]} t^n_{k+1}
\]

where here \(|t^n_k - t^n_{k+1}| = r_n\) for all \(k\) where \(r_n \to 0\). Then here is an important lemma.

**Lemma 60.7.4** \(\tau_n\) is a stopping time \((\left\{ \tau_n \leq t \right\} \in F_t\). Also \(F_\tau \subseteq F_{\tau_n}\) and for each \(\omega, \tau_n(\omega) \downarrow \tau(\omega)\).

**Proof:** Say \(t \in (t^n_{k-1}, t^n_k]\). Then \(\left\{ \tau_n \leq t \right\} = \left\{ \tau \leq t^n_{k-1} \right\}\) if \(t < t^n_k\) and it equals \(\left\{ \tau \leq t^n_k \right\}\) if \(t = t^n_k\). Either way \(\left\{ \tau_n \leq t \right\} \in F_t\) so it is a stopping time. Also from the definition, it follows that \(\tau_n \geq \tau\) and \(|\tau_n(\omega) - \tau(\omega)| \leq r_n\) which is given to converge to 0. Now suppose \(A \in F_\tau\) and say \(t \in (t^n_{k-1}, t^n_k]\) as above. Then

\[
A \cap \left\{ \tau_n \leq t \right\} = A \cap \left\{ \tau \leq t^n_{k-1} \right\} \in F_{t^n_{k-1}} \subseteq F_t\] if \(t < t^n_k\)

and

\[
A \cap \left\{ \tau_n \leq t \right\} = A \cap \left\{ \tau \leq t^n_k \right\} \in F_{t^n_k} = F_t\] if \(t = t^n_k\)

Thus \(F_\tau \subseteq F_{\tau_n}\) as claimed.

Next is the claim that if \(X(t)\) is adapted to \(F_t\), then \(X(\tau)\) is adapted to \(F_\tau\) just like the discrete case.

**Proposition 60.7.5** Let \((\Omega, F, P)\) be a probability space and let \(\sigma \leq \tau\) be two stopping times with respect to a filtration, \(F_t\). Then \(F_\sigma \subseteq F_\tau\). If \(X(t)\) is a right continuous stochastic process adapted to a normal filtration \(F_t\) and \(\tau\) is a stopping time, \(\omega \rightarrow X(\tau(\omega))\) is \(F_\tau\) measurable.

**Proof:** Let \(A \in F_\sigma\). Then \(A \cap [\sigma \leq t] \in F_t\) for all \(t \geq 0\). Since \(\sigma \leq \tau\),

\[
A \cap [\tau \leq t] = \left( \bigcap_{s \in F_t} [\sigma \leq s] \right) \cap [\tau \leq t] \in F_t
\]

Thus \(A \in F_\tau\) and so \(F_\sigma \subseteq F_\tau\).
Consider the following approximation of $\tau$ in which $t^n_k = k2^{-n}$.

$$\tau_n = \sum_{k=1}^{\infty} X[\tau \in (t^n_k, t^n_{k+1}]) t^n_{k+1}$$

Thus $\tau_n \downarrow \tau$. Consider for $U$ an open set, $X(\tau_n)^{-1}(U) \cap [\tau_n < t]$. Say $t \in (t^n_k, t^n_{k+1}]$. Then from the above definition of $\tau_n$,

$$[\tau_n < t] = [\tau \leq t^n_k] \in \mathcal{F}_{t^n_k} \subseteq \mathcal{F}_t$$

It follows that

$$X(\tau_n)^{-1}(U) \cap [\tau_n < t] = \bigcup_{j=1}^{k} X(t^n_j)^{-1}(U) \cap [\tau_n = t^n_j]$$

and so this set is in $\mathcal{F}_{t^n_k} \subseteq \mathcal{F}_t$. The reason $[\tau_n = t^n_j] \in \mathcal{F}_{t^n_j}$ is that it equals $[\tau \in (t^n_{j-1}, t^n_j)] \in \mathcal{F}_{t^n_j}$ by assumption that $\tau$ is a stopping time.

By right continuity of $X$, it follows that

$$X(\tau)^{-1}(U) \cap [\tau < t] = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} X(\tau_n)^{-1}(U) \cap [\tau_n < t] \in \mathcal{F}_t$$

It follows that for every $m$,

$$X(\tau)^{-1}(U) \cap [\tau \leq t] = \bigcap_{n=m}^{\infty} X(\tau)^{-1}(U) \cap \left[ \tau < t + \frac{1}{n} \right] \in \mathcal{F}_{t+\frac{1}{n}}$$

Since the filtration is normal, it follows that

$$X(\tau)^{-1}(U) \cap [\tau \leq t] \in \mathcal{F}_{t+} = \mathcal{F}_t.$$

Now consider an increasing family of stopping times, $\tau(t)$ ($\omega \to \tau(t)(\omega)$). It turns out this is a submartingale.

**Example 60.7.6** Let $\{\tau(t)\}$ be an increasing family of stopping times. Then $\tau(t)$ is adapted to the $\sigma$-algebras $\mathcal{F}_{\tau(t)}$ and $\{\tau(t)\}$ is a submartingale adapted to these $\sigma$-algebras.

First I need to show that a stopping time, $\tau$ is $\mathcal{F}_r$ measurable. Consider $[\tau \leq s]$. Is this in $\mathcal{F}_r$? Is $[\tau \leq s] \cap [\tau \leq r] \in \mathcal{F}_r$ for each $r$? This is obviously so if $s \leq r$ because the intersection reduces to $[\tau \leq s] \in \mathcal{F}_s \subseteq \mathcal{F}_r$. On the other hand, if $s > r$ then the intersection reduces to $[\tau \leq r] \in \mathcal{F}_r$ and so it is clear that $\tau$ is $\mathcal{F}_r$ measurable. It remains to verify it is a submartingale.

Let $s < t$ and let $A \in \mathcal{F}_{\tau(s)}$

$$\int_A E(\tau(t) | \mathcal{F}_{\tau(s)}) dP \equiv \int_A \tau(t) dP \geq \int_A \tau(s) dP$$

and this shows $E(\tau(t) | \mathcal{F}_{\tau(s)}) \geq \tau(s)$.

Now here is an important example. First note that for $\tau$ a stopping time, so is $t \lor \tau$. Here is why.

$$[t \lor \tau \leq s] = [t \leq s] \cap [\tau \leq s] \in \mathcal{F}_s.$$
Example 60.7.7 Let \( \tau \) be a stopping time and let \( X \) be continuous and adapted to the filtration \( \mathcal{F}_t \). Then for \( a > 0 \), define \( \sigma \) as

\[
\sigma(\omega) \equiv \inf \{ t > \tau(\omega) : ||X(t)(\omega) - X(\tau(\omega))|| = a \}
\]

Then \( \sigma \) is also a stopping time.

To see this is so, let

\[
Y(t)(\omega) = ||X(t \vee \tau)(\omega) - X(\tau(\omega))||
\]

Then \( Y(t) \) is \( \mathcal{F}_{t \vee \tau} \) measurable. It is desired to show that \( Y(t) \) is \( \mathcal{F}_t \) adapted. Hence if \( U \) is open in \( \mathbb{R} \), then

\[
Y(t)^{-1}(U) = \left( Y(t)^{-1}(U) \cap [\tau \leq t] \right) \cup \left( Y(t)^{-1}(U) \cap [\tau > t] \right)
\]

The second set in the above union on the right equals either \( \emptyset \) or \( [\tau > t] \) depending on whether \( 0 \in U \). If \( \tau > t \), then \( Y(t) = 0 \) and so the second set equals \( [\tau > t] \) if \( 0 \in U \). If \( 0 \notin U \), then the second set equals \( \emptyset \). Thus the second set above is in \( \mathcal{F}_t \). It is necessary to show the first set is also in \( \mathcal{F}_t \). The first set equals

\[
Y(t)^{-1}(U) \cap [\tau \leq t] = Y(t)^{-1}(U) \cap [\tau \vee t \leq t]
\]

because \( [\tau \vee t \leq t] = [\tau \leq t] \). However, \( Y(t)^{-1}(U) \in \mathcal{F}_{t \vee \tau} \) and so the set on the right in the above is in \( \mathcal{F}_t \). Therefore, \( Y(t) \) is adapted. Then \( \sigma \) is just the first hitting time for \( Y(t) \) to equal the closed set \( a \). Therefore, \( \sigma \) is a stopping time by Proposition 60.7.2.

60.7.2 The Optional Sampling Theorem Continuous Case

Next I want a version of the Doob optional sampling theorem which applies to martingales defined on \([0, L], L \leq \infty\). First recall Theorem 60.1.2 part of which is stated as the following lemma.

Lemma 60.7.8 Let \( f \in L^1(\Omega; E, \mathcal{F}) \) where \( E \) is a separable Banach space. Then if \( \mathcal{G} \) is a \( \sigma \) algebra \( \mathcal{G} \subseteq \mathcal{F} \),

\[
||E(f|\mathcal{G})|| \leq E(||f|| |\mathcal{G}||).
\]

Here is a lemma which is the main idea for the proofs of the optional sampling theorem for the continuous case.

Lemma 60.7.9 Let \( X(t) \) be a right continuous nonnegative submartingale such that the filtration \( \{\mathcal{F}_t\} \) is normal. Recall this includes

\[
\mathcal{F}_t = \cap_{s > t} \mathcal{F}_s.
\]
Also let \( \tau \) be a stopping time with values in \([0, T]\). Let \( \mathcal{P}_n = \{t^n_k\}_{k=1}^{m_n+1} \) be a sequence of partitions of \([0, T]\) which have the property that
\[
\mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \quad \lim_{n \to \infty} ||\mathcal{P}_n|| = 0,
\]
where
\[
||\mathcal{P}_n|| \equiv \sup \left\{ |t^n_k - t^n_{k+1}| : k = 1, 2, \ldots, m_n \right\}
\]
Then let \( \tau_n(\omega) \equiv \sum_{k=0}^{m_n} t^n_{k+1} \mathcal{X}_{\tau-1}\left(\left(\omega\right)\right) \)
It follows that \( \tau_n \) is a stopping time and also the functions \( |X(\tau_n)| \) are uniformly integrable. Furthermore, \( |X(\tau)| \) is integrable.

**Proof:** First of all, say \( t \in (t^n_k, t^n_{k+1}] \). If \( t < t^n_{k+1} \), then
\[
[\tau_n \leq t] = [\tau \leq t^n_k] \in \mathcal{F}_t \subseteq \mathcal{F}_t
\]
and if \( t = t^n_{k+1} \), then
\[
[\tau_n \leq t^n_{k+1}] = [\tau \leq t^n_k] \in \mathcal{F}_{t^n_{k+1}} = \mathcal{F}_t
\]
and so \( \tau_n \) is a stopping time. It follows from Proposition \[10.7.2\] that \( X(\tau_n) \) is \( \mathcal{F}_{\tau_n} \) measurable.

Now from Lemma \[8.3.4\] or Theorem \[10.6.7\], \( X(0), X(\tau_n), X(T) \) is a submartingale. Then
\[
\int_{\{X(\tau_n) \geq \lambda\}} X(\tau_n) \, dP \leq \int_{\{X(\tau_n) \geq \lambda\}} E(X(T) | \mathcal{F}_{\tau_n}) \, dP
\]
\[
= \int_{\Omega} E(X(\tau_n) \geq \lambda | X(T)) \, X(T) \, dP
\]
\[
= \int_{\{X(\tau_n) \geq \lambda\}} X(T) \, dP
\]
From maximal estimates, for example Theorem \[8.2.8\],
\[
P(\{X(\tau_n) \geq \lambda\}) \leq \frac{1}{\lambda} \int_{\Omega} X(T) \, dP = \frac{1}{\lambda} \int_{\Omega} X(T) \, dP
\]
and now it follows from the above that the random variables \( X(\tau_n) \) are equiintegrable. Recall this means that
\[
\lim_{\lambda \to \infty} \sup_n \int_{\{X(\tau_n) \geq \lambda\}} X(\tau_n) \, dP = 0
\]
Hence they are uniformly integrable.
To verify that \(|X(\tau)|\) is integrable, note that by right continuity, \(X(\tau_n) \to X(\tau)\) pointwise. Apply the Vitali convergence theorem to obtain

\[
\int_\Omega |X(\tau)| \, dP = \lim_{n \to \infty} \int_\Omega |X(\tau_n)| \, dP \leq \int_\Omega X(T) \, dP < \infty. \square
\]

In fact, you do not need to assume \(X\) is nonnegative.

**Lemma 60.7.10** Let \(X(t)\) be a right continuous submartingale such that the filtration \(\{\mathcal{F}_t\}\) is normal. Recall this includes \(\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s\).

Also let \(\tau\) be a stopping time with values in \([0,T]\). Let \(P_n = \{t^n_k\}_{k=1}^{m_n+1}\) be a sequence of partitions of \([0,T]\) which have the property that

\[
P_n \subseteq P_{n+1}, \quad \lim_{n \to \infty} ||P_n|| = 0,
\]

where

\[
||P_n|| \equiv \sup \{ |t^n_k - t^n_{k+1}| : k = 1, 2, \ldots, m_n \}
\]

Then let

\[
\tau_n(\omega) \equiv \sum_{k=0}^{m_n} t^n_{k+1} X_{\tau_n^{-1}((t^n_k, t^n_{k+1}])}(\omega)
\]

It follows that \(\tau_n\) is a stopping time and also the functions \(|X(\tau_n)|\) are uniformly integrable. Furthermore, \(|X(\tau)|\) is integrable.

**Proof:** It was shown above that \(\tau_n\) is a stopping time. Also, \(t^n_k \to X(t^n_k)\) is a discrete submartingale. Then by Theorem 60.6.6 there is a martingale \(t^n_k \to M(t^n_k)\) and an increasing submartingale \(t^n_k \to A(t^n_k)\) such that \(A \geq 0\) and is increasing

\[
X(t^n_k) = M(t^n_k) + A(t^n_k)
\]

You define \(A(t^n_0) \equiv 0\) and for \(n \geq 1,\)

\[
A(t^n_m) \equiv \sum_{k=1}^{m} E \left( X(t^n_k) - X(t^n_{k-1}) | \mathcal{F}_{t^n_{k-1}} \right)
\]

and repeat the arguments in that theorem. You know that \(A(0), A(\tau_n), A(T)\) is a submartingale by the optional sampling theorem given earlier, Theorem 60.6.7, and so

\[
P(A(\tau_n) > \lambda) \leq \frac{1}{\lambda} \int_{[A(\tau_n) > \lambda]} A(\tau_n) \, dP \leq \frac{1}{\lambda} \int_{[A(\tau_n) > \lambda]} A(T) \, dP \leq \frac{\|A(T)\|_{L^1}}{\lambda}
\]

It also follows from the definition of \(A\) that

\[
\|A(T)\|_{L^1} = \int_\Omega |X(T)| - X(0) \, dP < \infty
\]
Hence
\[ \lim_{\lambda \to \infty} \int_{[A(\tau_n) > \lambda]} A(\tau_n) \, dP \leq \lim_{\lambda \to \infty} \int_{[A(\tau_n) > \lambda]} A(T) \, dP = 0 \]

Because \( P(A(\tau_n) > \lambda) \to 0 \) and a single function in \( L^1 \) is uniformly integrable. Thus these functions \( A(\tau_n) \) are equi-integrable. Hence they are uniformly integrable. Now \( t_k^n \to |M(t_k^n)| \) is also a nonnegative submartingale. Thus
\[ |M(0)|, |M(\tau_n)|, |M(T)| \]
is a submartingale by the optional sampling theorem for discrete submartingales given earlier. Therefore,
\[ P(\{|M(\tau_n)| > \lambda\}) \leq \frac{1}{\lambda} \int_{|[M(\tau_n)| > \lambda]} |M(\tau_n)| \, dP \leq \frac{1}{\lambda} \int_{|[M(\tau_n)| > \lambda]} |M(T)| \, dP \leq \frac{\|M(T)\|_{L^1}}{\lambda} \]

Of course \( \|M(T)\|_{L^1} \) is finite because it is dominated by
\[ \int_{\Omega} A(T) + |X(T)| \, dP < \infty \]

Hence
\[ \lim_{\lambda \to \infty} \sup_n \int_{|[M(\tau_n)| > \lambda]} |M(\tau_n)| \, dP \leq \lim_{\lambda \to \infty} \sup_n \int_{|[M(\tau_n)| > \lambda]} |M(T)| \, dP = 0 \]
because a single function in \( L^1 \) is uniformly integrable and the above estimate shows that \( P(\{|M(\tau_n)| > \lambda\}) \to 0 \) uniformly in \( n \). Thus, in fact \( X(\tau_n) \) must be uniformly integrable since it is the sum of two which are. 

**Theorem 60.7.11** Let \( \{M(t)\} \) be a right continuous martingale having values in \( E \) a separable real Banach space with respect to the increasing sequence of \( \sigma \) algebras, \( \{\mathcal{F}_t\} \) which is assumed to be a normal filtration satisfying,
\[ \mathcal{F}_t = \cap_{s > t} \mathcal{F}_s, \]
for \( t \in [0, L], L \leq \infty \) and let \( \sigma, \tau \) be two stopping times with \( \tau \) bounded. Then \( M(\tau) \) defined as
\[ \omega \to M(\tau(\omega)) \]
is integrable and
\[ M(\sigma \land \tau) = E(M(\tau)|\mathcal{F}_\sigma). \]

**Proof:** Since \( M(t) \) is a martingale, \( \|M(t)\| \) is a submartingale. Let
\[ \tau_n(\omega) = \sum_{k=0}^\infty 2^{-n}(k+1)T X_{\tau_n^{-1}(k2^{-n}T,(k+1)2^{-n}T)}(\omega). \]
By Lemma 60.7.10, \( \tau_n \) is a stopping time and the functions \( ||M(\tau_n)|| \) are uniformly integrable. Also \( ||M(\tau)|| \) is integrable. Similarly \( ||M(\tau_n \wedge \sigma_n)|| \) are uniformly integrable where \( \sigma_n \) is defined similarly to \( \tau_n \).

Consider the main claim now. Letting \( \sigma, \tau \) be stopping times with \( \tau \) bounded, it follows that for \( \sigma_n \) and \( \tau_n \) as above, it follows from Theorem 60.6.5

\[
M(\sigma_n \wedge \tau_n) = E(M(\tau_n) | F_{\sigma_n})
\]

Thus, taking \( A \in F_{\sigma} \) and recalling \( \sigma \leq \sigma_n \) so that by Proposition 60.7.5, \( F_{\sigma} \subseteq F_{\sigma_n} \),

\[
\int_A M(\sigma_n \wedge \tau_n) \, dP = \int_A E(M(\tau_n) | F_{\sigma_n}) \, dP = \int_A M(\tau_n) \, dP.
\]

Now passing to a limit as \( n \to \infty \), the Vitali convergence theorem, Theorem 6.5.3 on Page 244 and the right continuity of \( M \) implies one can pass to the limit in the above and conclude

\[
\int_A M(\sigma \wedge \tau) \, dP = \int_A M(\tau) \, dP.
\]

By Proposition 60.7.8, \( M(\sigma \wedge \tau) \) is \( F_{\sigma \wedge \tau} \subseteq F_{\sigma} \) measurable showing

\[
E(M(\tau) | F_{\sigma}) = M(\sigma \wedge \tau).
\]

A similar theorem is available for submartingales defined on \([0,L], L \leq \infty\).

**Theorem 60.7.12** Let \( \{X(t)\} \) be a right continuous submartingale with respect to the increasing sequence of \( \sigma \) algebras, \( \{\mathcal{F}_t\} \) which is assumed to be a normal filtration,

\[
\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s,
\]

for \( t \in [0,L], L \leq \infty \) and let \( \sigma, \tau \) be two stopping times with \( \tau \) bounded. Then \( X(\tau) \) defined as

\[
\omega \to X(\tau(\omega))
\]

is integrable and

\[
X(\sigma \wedge \tau) \leq E(X(\tau) | F_{\sigma}).
\]

**Proof:** Let

\[
\tau_n(\omega) \equiv \sum_{k \geq 0} 2^{-n} (k + 1) \mathbb{I}_{(X_{\tau_n^{-1}(\{\omega \mid T_{2^{-n}}(\omega) \mid T_{2^{-n}}\}))}(\omega).
\]

Then by Lemma 60.7.11 \( \tau_n \) is a stopping time, the functions \( |X(\tau_n)| \) are uniformly integrable, and \( |X(\tau)| \) is also integrable. For \( \sigma_n \) defined similarly to \( \tau_n \), it also follows \( |X(\tau_n \wedge \sigma_n)| \) are uniformly integrable.

Let \( A \in \mathcal{F}_\sigma \). Since \( \sigma \leq \sigma_n \), it follows that \( \mathcal{F}_\sigma \subseteq \mathcal{F}_{\sigma_n} \). By the discrete optional sampling theorem for submartingales, Theorem 60.6.3,

\[
X(\sigma_n \wedge \tau_n) \leq E(X(\tau_n) | F_{\sigma_n})
\]
and so
\[ \int_A X (\sigma_n \wedge \tau_n) \, dP \leq \int_A E (X (\tau_n) \mid \mathcal{F}_{\sigma_n}) \, dP = \int_A X (\tau_n) \, dP \]
and now taking \( \lim_{n \to \infty} \) of both sides and using the Vitali convergence theorem along with the right continuity of \( X \), it follows
\[ \int_A X (\sigma \wedge \tau) \, dP \leq \int_A X (\tau) \, dP \equiv \int_A E (X (\tau) \mid \mathcal{F}_\sigma) \, dP \]
By Proposition 60.7.5, \( \mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma \), and so since \( A \in \mathcal{F}_\sigma \) was arbitrary,
\[ E (X (\tau) \mid \mathcal{F}_\sigma) \geq X (\sigma \wedge \tau) \text{ a.e.} \]
Note that a function defined on a countable ordered set such as the integers or equally spaced points is right continuous.

Here is an interesting lemma.

**Lemma 60.7.13** Suppose \( E (|X_n|) < \infty \) for all \( n \), \( X_n \) is \( \mathcal{F}_n \) measurable, \( \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \) for all \( n \in \mathbb{N} \), and there exist \( X_\infty \mathcal{F}_\infty \) measurable such that \( \mathcal{F}_\infty \subseteq \mathcal{F}_n \) for all \( n \) and \( X_0 \mathcal{F}_0 \) measurable such that \( \mathcal{F}_0 \supseteq \mathcal{F}_n \) for all \( n \) such that for all \( n \in \{0,1,\cdots\} \),
\[ E (X_n \mid \mathcal{F}_{n+1}) \geq X_{n+1}, \quad E (X_n \mid \mathcal{F}_\infty) \geq X_\infty. \]
Then \( \{X_n : n \in \mathbb{N}\} \) is uniformly integrable.

**Proof:**
\[ E (X_{n+1}) \leq E (E (X_n \mid \mathcal{F}_{n+1})) = E (X_n) \]
Therefore, the sequence \( E (X_n) \) is a decreasing sequence bounded below by \( E (X_\infty) \) so it has a limit. Let \( k \) be large enough that
\[ \left| E (X_k) - \lim_{m \to \infty} E (X_m) \right| < \varepsilon \quad (60.7.26) \]
and suppose \( n > k \). Then if \( \lambda > 0 \),
\[ \int_{|X_n| \geq \lambda} |X_n| \, dP = \int_{|X_n| \geq \lambda} X_n \, dP + \int_{|X_n| \leq -\lambda} (-X_n) \, dP \]
\[ = \int_{|X_n| \geq \lambda} X_n \, dP + \int_\Omega (-X_n) \, dP - \int_{|X_n| < \lambda} (-X_n) \, dP \]
\[ = \int_{|X_n| \geq \lambda} X_n \, dP - \int_\Omega X_n \, dP + \int_{|X_n| < \lambda} X_n \, dP \]
From \( (60.7.26) \)
\[ \leq \int_{|X_n| \geq \lambda} X_n \, dP - \int_\Omega X_k \, dP + \varepsilon + \int_{|X_n| < \lambda} X_n \, dP \]
By assumption,  
\[ E(X_k | \mathcal{F}_n) \geq X_n \]
and so
\[
\int_{[X_n \geq \lambda]} E(X_k | \mathcal{F}_n) \, dP - \int \Omega X_k dP + \varepsilon + \int_{[-X_n < \lambda]} E(X_k | \mathcal{F}_n) \, dP \\
= \int_{[X_n \geq \lambda]} X_k dP - \int \Omega X_k dP + \varepsilon + \int_{[-X_n < \lambda]} X_k dP \\
= \int_{[X_n \geq \lambda]} X_k dP - \int \Omega X_k dP + \varepsilon + \int_{[X_n > -\lambda]} X_k dP \\
= \int_{[X_n \geq \lambda]} X_k dP + \left( \int \Omega (-X_k) dP - \int_{[X_n > -\lambda]} (-X_k) dP \right) + \varepsilon \\
= \int_{[X_n \geq \lambda]} X_k dP + \int_{[X_n \leq -\lambda]} (-X_k) dP + \varepsilon = \int_{|X_n| \geq \lambda} |X_k| dP + \varepsilon
\]
Applying the maximal inequality for submartingales, Theorem 58.5.4.  
\[ P(\max \{|X_j| : j = n, \cdots, 1\} \geq \lambda) \leq \frac{1}{\lambda} (E(|X_0|) + E(|X_\infty|)) \leq \frac{C}{\lambda} \]
and taking sup for all \( n \),  
\[ P(\sup \{|X_j|\} \geq \lambda) \leq \frac{C}{\lambda} \]
It follows that for all \( \lambda \) large enough,  
\[ \int_{|X_n| \geq \lambda} |X_n| dP \leq 2\varepsilon \]
and since \( \varepsilon \) is arbitrary, this shows \( \{X_n\} \) for \( n > k \) is equiintegrable. Since there are only finitely many \( X_j \) for \( j \leq k \), this shows \( \{X_n\} \) is equiintegrable. Hence \( \{X_n\} \) is uniformly integrable.

### 60.8 Right Continuity Of Submartingales

The following theorem is an attempt to consider the question of right continuity. It turns out that you can always assume right continuity of a submartingale by going to a suitable version and this theorem is a first step in this direction.

**Theorem 60.8.1** Let \( \{X(t)\} \) be a real valued submartingale adapted to the filtration \( \mathcal{F}_t \). Then there exists a set of measure zero \( N, P(N) = 0 \), such that if \( \omega \notin N \), then,
\[
\lim_{r \to t^+, r \in \mathbb{Q}} X(r, \omega), \quad \lim_{r \to t^-, r \in \mathbb{Q}} X(r, \omega)
\]
both exist. There also exists a set of measure zero $N$ such that for $\mathbb{Q}^+$ the nonnegative rationals and $\omega \notin N$,

$$\sup_{t \in \mathbb{Q}^+ \cap [0, M]} |X(t, \omega)| < \infty$$

is bounded for each $M \in \mathbb{N}$. $\mathbb{Q}$ can be replaced with any countable dense subset of $\mathbb{R}$.

**Proof:** Let $\{r_k\}_{k=1}^\infty$ be an enumeration of the nonnegative rationals. Let $t > 0$ be given. Then let $\{s_1, s_2, \cdots, s_n\}$ be such that these are in order and $\{s_2, \cdots, s_{n-1}\}$ are the first $n - 2$ rationals less than $t$ listed in order and $s_1 = 0$ while $s_n = t$. Then let $Y_k \equiv X(s_k)$. It follows $\{Y_k\}$ is a submartingale and so from the maximal inequality in Theorem 60.8.4,

$$P\left(\max_{1 \leq k \leq n} |Y_k| \geq 2^m\right) \leq \frac{1}{2^m} (2E(|Y_n| + |Y_1|)) = 2^{-m} (2E(|X(t)| + |X(0)|))$$

Then letting $n \to \infty$, it follows upon letting $C_t = 2E(|X(t)| + |X(0)|)$,

$$P\left(\sup_{r \in \mathbb{Q}^+ \cap [0, t]} |X(r)| \geq 2^m\right) \leq 2^{-m} C_t.$$

By the Borel Cantelli lemma, there exists a set of measure 0, $N_t$ such that for $\omega \notin N_t$, $\omega$ is contained in only finitely many of the sets

$$\left[\sup_{r \in \mathbb{Q}^+ \cap [0, t]} |X(r)| \geq 2^m\right]$$

which shows that for $\omega \notin N_t$, $\sup_{r \in \mathbb{Q}^+ \cap [0, t]} |X(r)|$ is bounded. Now let $N = \cup_{j=1}^\infty N_j$. This proves the second claim.

Next consider the first claim. By the upcrossing estimate, Theorem 60.8.5 or Lemma 60.8.4, and letting $a < b$ and $U_{[a, b]}^{\infty}[c, d]$ the upcrossings of $Y_k$ from $a$ to $b$ on $[c, d]$ for $d \leq t$ and $c \geq 0$,

$$E\left(U_{[a, b]}^{\infty}[0, t]\right) \leq \frac{1}{b-a} E\left((Y_n-a)^+\right) = \frac{1}{b-a} E\left((X(t)-a)^+\right).$$

Hence

$$P\left(U_{[a, b]}^{\infty}[0, t] \geq M\right) \leq \frac{1}{M} \left(\frac{1}{b-a} E\left((X(t)-a)^+\right)\right). \quad (60.8.27)$$

Suppose for some $s < t$,

$$\lim_{r \to s+} \sup_{r \in \mathbb{Q}} X(r, \omega) > b > a > \lim_{r \to s+} \inf_{r \in \mathbb{Q}} X(r, \omega). \quad (60.8.28)$$
If this is so, then in \((s, t) \cap \mathbb{Q}\) there must be infinitely many values of \(r \in \mathbb{Q}\) such that \(X(r, \omega) \geq b\) as well as infinitely many values of \(r \in \mathbb{Q}\) such that \(X(r, \omega) \leq a\). Note this involves the consideration of a limit from one side. Thus, since it is a limit from one side only, there are an arbitrarily large number of upcrossings between \(s\) and \(t\). Therefore, letting \(M\) be a large positive number, it follows that for all \(n\) sufficiently large,

\[
U^a_{[a, b]}[0, t] (\omega) \geq M
\]

which implies

\[
\omega \in \left[ U^a_{[a, b]}[0, t] \geq M \right]
\]

which from (60.8.28) is a set of measure no more than

\[
\frac{1}{M} \left( \frac{1}{b-a} E \left( (X(t) - a)^+ \right) \right).
\]

This has shown that the set of \(\omega\) such that for some \(s \in [0, t)\) (60.8.28) holds is contained in the set

\[
N_{[a, b]} \equiv \cap_{n=1}^{\infty} \cup_{n=1}^{\infty} \left[ U^n_{[a, b]}[0, t] \geq M \right]
\]

Now the sets,

\[
\left[ U^n_{[a, b]}[0, t] \geq M \right]
\]

are increasing in \(n\) and each has measure less than

\[
\frac{1}{M} \left( \frac{1}{b-a} E \left( (X(t) - a)^+ \right) \right)
\]

and so

\[
P \left( \cup_{n=1}^{\infty} \left[ U^n_{[a, b]}[0, t] \geq M \right] \right) \leq \frac{1}{M} \left( \frac{1}{b-a} E \left( (X(t) - a)^+ \right) \right).
\]

which shows that

\[
P \left( N_{[a, b]} \right) \leq \frac{1}{M} \left( \frac{1}{b-a} E \left( (X(t) - a)^+ \right) \right)
\]

for every \(M\) and therefore, \(P \left( N_{[a, b]} \right) = 0\).

Therefore, corresponding to \(a < b\), there exists a set of measure 0, \(N_{[a, b]}\) such that for \(\omega \notin N_{[a, b]}\) (60.8.28) is not true for any \(s \in [0, t)\). Let \(N \equiv \cup_{a, b \in \mathbb{Q}} N_{[a, b]}\), a set of measure 0 with the property that if \(\omega \notin N\), then (60.8.28) fails to hold for any pair of rational numbers, \(a < b\) for any \(s \in [0, t)\). Thus for \(\omega \notin N\),

\[
\lim_{r \to s^+, r \in \mathbb{Q}} X(r, \omega)
\]

exists for all \(s \in [0, t)\). Similar reasoning applies to show the existence of the limit

\[
\lim_{r \to s^-, r \in \mathbb{Q}} X(r, \omega).
\]
60.8. RIGHT CONTINUITY OF SUBMARTINGALES

for all \( s \in (0, t] \) whenever \( \omega \) is outside of a set of measure zero. Of course, this exceptional set depends on \( t \). However, if this exceptional set is denoted as \( N_t \), one could consider \( N \equiv \bigcup_{n=1}^{\infty} N_n \). It is obvious there is no change if \( \mathbb{Q} \) is replaced with any countable dense subset. This proves the theorem.

Of course the above theorem does not say the left and right limits are equal, just that they exist in some way for \( \omega \) not in some set of measure zero. Also it has not been shown that \( \lim_{r \to s^+, r \in \mathbb{Q}} X(r, \omega) = X(r, \omega) \) for a.e. \( \omega \).

**Corollary 60.8.2** In the situation of Theorem 60.8.1, let \( s > 0 \) and let \( D_1 \) and \( D_2 \) be two countable dense subsets of \( \mathbb{R} \). Then

\[
\lim_{r \to s^-, r \in D_1} X(r, \omega) = \lim_{n \to \infty} X(r_n, \omega) = \lim_{r \to s^-, r \in D_2} X(r, \omega) \quad \text{a.e. } \omega
\]

\[
\lim_{r \to s^+, r \in D_1} X(r, \omega) = \lim_{n \to \infty} X(r_n, \omega) = \lim_{r \to s^+, r \in D_2} X(r, \omega) \quad \text{a.e. } \omega
\]

**Proof:** Let \( \{r^i_n\} \) be an increasing sequence from \( D_i \) converging to \( s \) and let \( N \) be the exceptional set corresponding to the countable dense set \( D_1 \cup D_2 \). Then for \( \omega \notin N \), and \( i = 1, 2, \)

\[
\lim_{r \to s^-, r \in D_1} X(r, \omega) = \lim_{n \to \infty} X(r^i_n, \omega) = \lim_{r \to s^-, r \in D_i} X(r, \omega)
\]

The other claim is similar. This proves the corollary.

Now here is an impressive lemma about submartingales and uniform integrability.

**Lemma 60.8.3** Let \( X(t) \) be a submartingale adapted to a filtration \( \mathcal{F}_t \). Let \( \{r_k\} \subseteq [s, t) \) be a decreasing sequence converging to \( s \). Then \( \{X(r_j)\}_{j=1}^{\infty} \) is uniformly integrable.

**Proof:** First I will show the sequence is equiintegrable. I need to show that for all \( \varepsilon > 0 \) there exists \( \lambda \) large enough that for all \( n \)

\[
\int_{|X(r_n)| \geq \lambda} |X(r_n)| \, dP < \varepsilon.
\]

Let \( \varepsilon > 0 \) be given. Since \( \{X(r)\}_{r \geq 0} \) is a submartingale, \( E(X(r_n)) \) is a decreasing sequence bounded below by \( E(X(s)) \). This is because for \( r_n < r_k \),

\[
E(X(r_n)) \leq E(E(X(r_k) | \mathcal{F}_n)) = E(X(r_k))
\]

Pick \( k \) such that

\[
E(X(r_k)) - \lim_{n \to \infty} E(X(r_n)) = \left| E(X(r_k)) - \lim_{n \to \infty} E(X(r_n)) \right| < \varepsilon/2.
\]
Then for \( n > k \),
\[
\int_{|X(r_n)| \geq \lambda} |X(r_n)| \, dP = \int_{|X(r_n) - \lambda| \geq \lambda} X(r_n) \, dP + \int_{|X(r_n) - \lambda| < \lambda} -X(r_n) \, dP
\]
\[
= \int_{|X(r_n)| \geq \lambda} X(r_n) \, dP + \int_{|X(r_n)| > \lambda} -X(r_n) \, dP - \int_{\Omega} X(r_n) \, dP
\]
\[
\leq \int_{|X(r_n)| \geq \lambda} X(r_n) \, dP + \int_{|X(r_n)| < \lambda} E(X(r_k) | F_n) \, dP - \int_{\Omega} X(r_n) \, dP
\]
\[
\leq \int_{|X(r_n)| \geq \lambda} X(r_n) \, dP + \int_{|X(r_n)| < \lambda} X(r_k) \, dP - \int_{\Omega} X(r_k) \, dP + \varepsilon/2
\]
\[
= \int_{|X(r_n)| \geq \lambda} X(r_n) \, dP - \int_{|X(r_n)| < \lambda} X(r_k) \, dP + \varepsilon/2
\]
\[
= \int_{|X(r_n)| \geq \lambda} X(r_k) \, dP + \varepsilon/2
\]
\[
\leq \int_{\sup \{|X(r)| \geq \lambda : r \in \{r_j\}_{j=1}^\infty \}} |X(r_k)| \, dP + \varepsilon/2
\]
(60.8.29)

From maximal inequalities of Theorem 58.5.4
\[
P \left( \sup_{r \in \{r_n, r_{n-1}, \ldots, r_1\}} |X(r)| \geq \lambda \right) \leq \frac{2E(|X(t)| + |X(0)|)}{\lambda} = \frac{C}{\lambda}
\]
and so, letting \( n \to \infty \),
\[
P \left( \sup_{r \in \{r_n\}_{n=1}^\infty} |X(r)| \geq \lambda \right) \leq \frac{C}{\lambda}
\]

It follows that for \( \lambda \) sufficiently large the first term in (60.8.29) is smaller than \( \varepsilon/2 \) because \( k \) is fixed. Now this shows there is a choice of \( \lambda \) such that for all \( n > k \),
\[
\int_{|X(r_n)| \geq \lambda} |X(r_n)| \, dP < \varepsilon
\]

There are only finitely many \( r_n \) for \( n \leq k \) and by choosing \( \lambda \) sufficiently large the above formula can be made to hold for these also, thus showing \( \{X(r_n)\} \) is equi integrable.

Now this implies the sequence of random variables is uniformly integrable as well. Let \( \varepsilon > 0 \) be given and choose \( \lambda \) large enough that for all \( n \),
\[
\int_{|X(r_n)| \geq \lambda} |X(r_n)| \, dP < \varepsilon/2
\]
Then let $A$ be a measurable set.

$$
\int_A |X(r_n)| \, dP = \int_{A \cap \{|X(r_n)| \geq \lambda\}} |X(r_n)| \, dP + \int_{A \cap \{|X(r_n)| < \lambda\}} |X(r_n)| \, dP \\
< \varepsilon/2 + \int_{A \cap \{|X(r_n)| < \lambda\}} |X(r_n)| \, dP \leq \frac{\varepsilon}{2} + \lambda P(A)
$$

and now you see that if $P(A)$ is sufficiently small then for all $n$,

$$
\int_A |X(r_n)| \, dP < \varepsilon
$$

which shows the set of functions is uniformly integrable as claimed. This proves the lemma.

You can often consider a submartingale to be right continuous. This is the importance of the following theorem.

**Theorem 60.8.4** Let $\{X(t)\}$ be a submartingale adapted to a normal filtration $F_t$. There exists a right continuous submartingale having left limits, $\{Y(t)\}$ such that $Y(t) = X(t)$ a.e. for every $t \in \mathbb{Q}^+$. Furthermore $\{X(t)\}$ has a right continuous left limits version if and only if

$$
t \rightarrow E(X(t))
$$

is right continuous. More generally, $Y(t) = X(t)$ a.e. at every point where the above function is right continuous.

**Proof:** From Theorem 60.8.1, there exists a set of measure zero, $N$ such that for $\omega \notin N$, left and right limits of the following form exist.

$$
\lim_{r \to t^+, r \in \mathbb{Q}} X(r, \omega), \lim_{r \to t^-, r \in \mathbb{Q}} X(r, \omega).
$$

Then define for each $t$ and $\omega \notin N$,

$$
Y(t, \omega) \equiv \lim_{r \to t^+, r \in \mathbb{Q}} X(r, \omega).
$$

and for $\omega \in N$,

$$
Y(t, \omega) \equiv 0
$$

Thus $Y(t)(\omega) = X(t)(\omega)$ a.e. for $t \in \mathbb{Q}^+$. For each $\omega \notin N$, there exists $\delta > 0$ such that if $r \in \mathbb{Q}$, $t < r < t + 2\delta$, then

$$
|Y(t, \omega) - X(r, \omega)| < \varepsilon/2.
$$

Now suppose $s \in (t, t + \delta)$. Then there exists $\delta_1 < \delta$ such that if $s < r < s + \delta_1$ then

$$
|Y(s, \omega) - X(r, \omega)| < \varepsilon/2.
$$
pick \( r \in \mathbb{Q} \cap (s, t + \delta) \). Then both of the above two inequalities hold and so it follows

\[
|Y (t, \omega) - Y (s, \omega)| < |Y (t, \omega) - X (r, \omega)| + |X (r, \omega) - Y (s, \omega)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, \( t \to Y (t, \omega) \) is right continuous as claimed.

From the definition of \( Y (t, \omega) \), it follows \( \omega \to Y (t, \omega) \) is measurable in \( \mathcal{F}_{t+} \) because it is the limit of a sequence, \( X (r_n, \omega) X_{N_C} \) where \( r_n \to t + \). Now each \( X (r_n, \cdot) \) is \( \mathcal{F}_{r_n} \) measurable and so \( Y (t, \cdot) \) is \( \mathcal{F}_{r_n} \) measurable also for each \( r_n \). Thus \( Y (t, \cdot) \) is \( \mathcal{F}_{t+} \) measurable. Since the filtration is normal, \( \mathcal{F}_t = \mathcal{F}_{t+} \) and it follows \( Y (t, \cdot) \) is \( \mathcal{F}_t \) measurable. Why is \( Y (t, \cdot) \in L^1 (\Omega) \)?

From Lemma 60.8.3, the collection \( \{ X (r_n) \} \) is uniformly integrable. Therefore, from the Vitali convergence theorem, Theorem 9.5.3 on Page 244,

\[
\lim_{n \to \infty} \int_{\Omega} |Y (s) - X (r_n)| \, dP = 0 \quad (60.8.30)
\]

and \( Y (s) \in L^1 (\Omega) \).

It remains to verify \( \{ Y (s) \} \) is a submartingale. For \( s < t \), is it true that

\[
E (Y (t) | \mathcal{F}_s) \geq Y (s)
\]

Fix \( A \in \mathcal{F}_s \). From the above construction, there exists \( w \in \mathbb{Q} \) and \( w \geq t \) such that

\[
\int_A Y (t) \, dP \geq \int_A X (w) \, dP - \varepsilon.
\]

Then also, there exists \( r \in \mathbb{Q} \cap (s, t) \) such that

\[
\int_A X (r) \, dP \geq \int_A Y (s) \, dP - \varepsilon.
\]

Now

\[
\int_A E (Y (t) | \mathcal{F}_s) \, dP = \int_A Y (t) \, dP \geq \int_A X (w) \, dP - \varepsilon
\]

\[
= \int_A E (X (w) | \mathcal{F}_r) \, dP - \varepsilon
\]

\[
\geq \int_A X (r) \, dP - \varepsilon \geq \int_A Y (s) \, dP - 2\varepsilon.
\]

Since \( \varepsilon \) was arbitrary, this shows

\[
\int_A E (Y (t) | \mathcal{F}_s) \, dP \geq \int_A Y (s) \, dP
\]

for any \( A \in \mathcal{F}_s \) and so this verifies since \( A \) is arbitrary that

\[
E (Y (t) | \mathcal{F}_s) \geq Y (s)
\]
60.9. SOME MAXIMAL INEQUALITIES

so $Y$ is a submartingale.

By Theorem 60.8.1 there exists a set of measure 0, $N$ such that the left limits of $Y(r, \omega)$ exist through rational numbers if $\omega \notin N$.

Is $\{Y(t)\}$ a version of $\{X(t)\}$? This is where the assumption $t \to E(X(t))$ is continuous is used. We know $E(X(r_n) | F_s) \geq X(s)$ and also for $A \in F_s$

$$\int_A X(r_n) \, dP = \int_A E(X(r_n) | F_s) \, dP \geq \int_A X(s) \, dP.$$ 

Hence taking a limit yields

$$\int_A Y(s) \, dP \geq \int_A X(s) \, dP$$

and since $A$ is arbitrary, $Y(s) \geq X(s)$. Now since $t \to E(X(t))$ is continuous,

$$\int \Omega |Y(s) - X(s)| \, dP = E(Y(s)) - E(X(s)) = \lim_{n \to \infty} (E(X(r_n)) - E(X(r_n))) = 0.$$

It only remains to verify the only way $X(t)$ has a right continuous version is for $t \to E(X(t))$ to be continuous. Suppose then that $\{X(t)\}$ has a right continuous version, $\{Y(t)\}$. Letting $r_n \downarrow s$, Lemma 60.8.3 implies $\{Y(r_n)\}$ is uniformly integrable. Also $Y(s)(\omega) = \lim_{n \to \infty} Y(r_n)(\omega)$ a.e. and so by the Vitali convergence theorem,

$$\lim_{n \to \infty} \int \Omega |X(r_n) - X(s)| \, dP = \lim_{n \to \infty} \int \Omega |Y(r_n) - Y(s)| \, dP = 0.$$

This proves the theorem.

Note that the condition $t \to E(X(t))$ being continuous holds for any martingale. Therefore, every martingale has a right continuous version. The condition that $t \to E(X(t))$ is right continuous is not a very stringent assumption. For $\{X(t)\}$ a submartingale, this is an increasing function. Therefore, the only points where the condition might not hold comprise a countable set.

60.9 Some Maximal Inequalities

As in the case of discrete martingales and submartingales, there are maximal inequalities available.

**Lemma 60.9.1** Let $X$ be right continuous and adapted such that the given filtration is complete in the sense that $F_0$ contains all sets $A$ of $F$ such that $P(A) = 0$. Then there exists a set of measure zero $N$ and a $F \times B(\mathbb{R})$ measurable function $Y$ such that if $\omega \notin N$, then $Y(t)(\omega) = X(t)(\omega)$. Also, if $f$ is $F$ measurable and nonnegative then $(\lambda, \omega) \to X_{[f > \lambda]}$ is $F \times B(\mathbb{R})$ measurable.
**Proof:** Let \( \{ t^n_0, t^n_1, \ldots, t^n_m \} \) be a partition of \([0, T]\) in which \( |t^n_i - t^n_{i-1}| < \rho_n \) where \( \rho_n \to 0 \). Now define \( X_n \) as follows:

\[
X_n(t)(\omega) = \sum_{i=1}^{m_n} X(t^n_i)(\omega) \mathcal{X}_{[t^n_{i-1}, t^n_i)}(\omega) \\
X_n(0) = X(0).
\]

then each \( X_n \) is obviously product measurable because it is the sum of functions which are. By right continuity, \( X_n \) converges pointwise to \( X \) for \( \omega \notin N \) where \( N \) is a set of measure zero and so if \( Y(t)(\omega) \equiv X(t)(\omega) \) for all \( \omega \notin N \) and \( Y(t)(\omega) = 0 \) for all \( \omega \in N \), this is the desired product measurable function.

To see the last claim, let \( s \) be a nonnegative simple function, \( s(\omega) = \sum_{k=1}^{n_n} c_k \mathcal{X}_{E_k}(\omega) \) where the \( c_k \) are strictly increasing in \( k \). Also let \( F_k = \bigcup_{i=k}^{n_k} E_i \). Then

\[
\mathcal{X}_{[s > \lambda]} = \sum_{k=1}^{n_n} \mathcal{X}_{[c_k-1, c_k)}(\lambda) \mathcal{X}_{F_k}(\omega)
\]

which is clearly product measurable. For arbitrary \( f \geq 0 \) and measurable, there is an increasing sequence of simple functions \( s_n \) converging pointwise to \( f \). Therefore,

\[
\lim_{n \to \infty} \mathcal{X}_{[s_n > \lambda]} = \mathcal{X}_{[f > \lambda]}
\]

and so \( \mathcal{X}_{[f > \lambda]} \) is product measurable. 

**Definition 60.9.2** Let \( X(t) \) be a right continuous submartingale for \( t \in I \) and let \( \{ \tau_n \} \) be a sequence of stopping times such that \( \lim_{n \to \infty} \tau_n = \infty \). Then \( X^{\tau_n} \) is called the stopped submartingale and it is defined by

\[
X^{\tau_n}(t) \equiv X(t \wedge \tau_n).
\]

**Proposition 60.9.3** The stopped submartingale just defined is a submartingale.

**Proof:** By the optional sampling theorem for submartingales, Theorem 60.7.12, it follows that for \( s < t \),

\[
E(X^{\tau_n}(t) | F_s) \equiv E(X(t \wedge \tau_n) | F_s) \geq X(t \wedge \tau_n \wedge s) = X(\tau_n \wedge s) \equiv X^{\tau_n}(s).
\]

**Theorem 60.9.4** Let \( \{ X(t) \} \) be a right continuous nonnegative submartingale adapted to the normal filtration \( F_t \) for \( t \in [0, T] \). Let \( p \geq 1 \). Define

\[
X^*(t) \equiv \sup \{ X(s) : 0 < s < t \}, \quad X^*(0) \equiv 0.
\]

Then for \( \lambda > 0 \), if \( X(t)^p \) is in \( L^1(\Omega) \) for each \( t \),

\[
P(\{ X^*(T) > \lambda \} \leq \frac{1}{\lambda^p} \int_{\mathcal{X}[X^*(T) > \lambda]} X(T)^p dP
\]

(60.9.31)
If \( X (t) \) is continuous, the above inequality holds without this assumption. In case \( p > 1 \), and \( X (t) \) continuous, then for each \( t \leq T \),

\[
\left( \int_{\Omega} |X^* (t)|^p \, dP \right)^{1/p} \leq \frac{p}{p - 1} \left( \int_{\Omega} X^p (T) \, dP \right)^{1/p} \tag{60.9.32}
\]

**Proof:** The first inequality follows from Theorem 60.5.2. However, it can also be obtained a different way using stopping times.

Define the stopping time

\[
\tau \equiv \inf \{ t > 0 : X^* (t) > \lambda \} \wedge T.
\]

(The infimum over an empty set will equal \( \infty \).) This is a stopping time by 60.7.2 because it is just a continuous function of the first hitting time of an open set. Also from the definition of \( X^* \) in which the supremum is taken over an open interval,

\[
[\tau < t] = [X^* (t) > \lambda]
\]

Note this also shows \( X^* (t) \) is \( F_t \)-measurable. Then it follows that \( X^p (t) \) is also a submartingale since \( r^p \) is increasing and convex. By the optional sampling theorem, \( X^p (t) \), \( X^p (\tau) \), \( X^p (T) \) is a submartingale. Also \( [\tau < T] \in \mathcal{F}_\tau \) and so

\[
\int_{[\tau < T]} X (\tau)^p \, dP \leq \int_{[\tau < T]} E (X (T)^p | \mathcal{F}_\tau) \, dP = \int_{[\tau < T]} X (T)^p \, dP
\]

By right continuity, on \( [\tau < T], X (\tau) \geq \lambda \). Therefore,

\[
\lambda^p P ([X^* (T) > \lambda]) = \lambda^p P ([\tau < T]) \leq \int_{[\tau < T]} X (\tau)^p \, dP \leq \int_{[X^* (T) > \lambda]} X (T)^p \, dP
\]

Next suppose \( X (t) \) is continuous and let \( \{ \tau_n \} \) be a localizing sequence,

\[
\tau_n \equiv \inf \{ t : X^* (t) > n \}.
\]

Then by continuity, \( X^{\tau_n} \) is bounded because \( X (\tau_n \wedge t) \leq n \), and so from what was just shown,

\[
\lambda^p P \left( \left( (X^{\tau_n})^* (T) > \lambda \right) \right) \leq \int_{[(X^{\tau_n})^* (T) > \lambda]} (X^{\tau_n})^p (T) \, dP
\]

Then \( (X^{\tau_n}) (T) \) is increasing as \( \tau_n \to \infty \) so the result follows from the monotone convergence theorem. This proves the first part.

Let \( X^{\tau_n} \) be as just defined. Thus it is a bounded submartingale. To save on notation, the \( X \) in the following argument is really \( X^{\tau_n} \). This is done so that all the integrals are finite. If \( p > 1 \), then from the first part using the case of \( p = 1 \),

\[
\int_{\Omega} |X^* (t)|^p \, dP \leq \int_{\Omega} |X^* (T)|^p \, dP = \int_{0}^{\infty} p \lambda^{p-1} \tilde{P} ([X^* (T) > \lambda]) \, d\lambda
\]
CHAPTER 60. STOCHASTIC PROCESSES

\[ \begin{align*}
\leq & \ p \int_0^\infty \lambda^{p-1} \frac{1}{\lambda} \int X_{[X^*(T) > \lambda]} X(T) \, dP \, d\lambda \\
= & \ p \int_\Omega X(T) \int_0^{X^*(T)} \lambda^{p-2} \, d\lambda \, dP \\
= & \ p \int_\Omega X(T) \frac{X^*(T)^{p-1}}{p-1} \, dP \\
\leq & \ \frac{p}{p-1} \left( \int_\Omega X^*(T)^p \, dP \right)^{1/p} \left( \int_\Omega X(T)^p \, dP \right)^{1/p}
\end{align*} \]

Now divide both sides by \( \left( \int_\Omega X^*(T)^p \, dP \right)^{1/p} \). Substituting \( X^\tau_n \) for \( X \)

\[ \left( \int_\Omega |X^\tau_n*(t)|^p \, dP \right)^{1/p} \leq \left( \int_\Omega X^\tau_n*(T)^p \, dP \right)^{1/p} \leq \frac{p}{p-1} \left( \int_\Omega X^\tau_n(T)^p \, dP \right)^{1/p} \]

Now let \( n \to \infty \) and use the monotone convergence theorem to obtain the inequality of the theorem. This establishes \( 60.9.32 \). The use of Fubini's theorem follows from Lemma \( 60.9.1 \).

Here is another sort of maximal inequality in which \( X(t) \) is not assumed non-negative.

**Theorem 60.9.5** Let \( \{ X(t) \} \) be a right continuous submartingale adapted to the normal filtration \( F_t \) for \( t \in [0,T] \) and \( X^*(t) \) defined as in Theorem \( 60.9.4 \)

\[ X^*(t) \equiv \sup \{ X(s) : 0 < s < t \}, \quad X^*(0) \equiv 0, \]

For \( t > 0 \), let

\[ X_*(t) = \inf \{ X(s) : s < t \}. \]

Then

\[ P([X_*(T) < -\lambda]) \leq \frac{1}{\lambda} E(|X(T)|) \]

(60.9.33)

Also

\[ P(\sup \{|X(s)| : s < T\} > \lambda) \leq \frac{2}{\lambda} E(|X(T)| + |X(0)|) \]

(60.9.34)

**Proof:** The function \( f(r) = r^+ \equiv \frac{1}{2}(|r| + r) \) is convex and increasing. Therefore, \( X^+(t) \) is also a submartingale but this one is nonnegative. Also

\[ [X^*(T) > \lambda] = \left( [X^+(T)^* > \lambda] \right) \]

and so from Theorem \( 60.9.3. \)

\[ P([X^*(T) > \lambda]) = P\left( [\left( X^+(T)^* > \lambda \right)] \right) \leq \frac{1}{\lambda} E\left( X^+(T) \right) \leq \frac{1}{\lambda} E\left( |X(T)| \right). \]
Next let
\[ \tau = \min(\inf \{ t : X(t) < -\lambda \}, T) \]
then as before, \( X(0), X(\tau), X(T) \) is a submartingale and so
\[
\int_{[\tau<T]} X(\tau) \, dP + \int_{[\tau=T]} X(\tau) \, dP = \int_{\Omega} X(\tau) \, dP \geq \int_{\Omega} X(0) \, dP
\]
Now for \( \omega \in [\tau<T] \), \( X(t)(\omega) < -\lambda \) for some \( t < T \) and so by right continuity, \( X(\tau)(\omega) \leq -\lambda \). therefore,
\[
-\lambda \int_{[\tau<T]} dP \geq -\int_{[\tau=T]} X(T) \, dP + \int_{\Omega} X(0) \, dP
\]
If \( X_*(T) < -\lambda \), then from the definition given above, there exists \( t < T \) such that \( X(t) < -\lambda \) and so \( \tau < T \). If \( \tau < T \), then by definition, there exists \( t < T \) such that \( X(t) < -\lambda \) and so \( X_*(T) < -\lambda \). Hence \( [\tau < T] = [X_*(T) < -\lambda] \). It follows that
\[
P([X_*(T) < -\lambda]) = P([\tau < T]) \leq \frac{1}{\lambda} \int_{[\tau=T]} X(T) \, dP - \frac{1}{\lambda} \int_{\Omega} X(0) \, dP
\]
and this proves [60.9.34].

Finally, combining the above two inequalities,
\[
P(\sup \{|X(s)| : s < T \} > \lambda)
\]
\[
= P([X_*(T) < -\lambda]) + P([X^*(T) > \lambda]) \leq \frac{2}{\lambda} E(|X(T)| + |X(0)|)
\]

60.10 Continuous Submartingale Convergence Theorem

In this section, \( \{Y(t)\} \) will be a continuous submartingale and \( a < b \). Let \( X(t) \equiv (Y(t) - a)_+ + a \) so \( X(0) \geq a \). Then \( X \) is also a submartingale. It is an increasing convex function of one. If \( Y(t) \) has an upcrossing of \( [a,b] \), then \( X(t) \) starts off at \( a \) and ends up at least as large as \( b \). If \( X(t) \) has an upcrossing of \( [a,b] \) then it must start off at \( a \) since it cannot be smaller and it ends up at least as large as \( b \). Thus we can count the upcrossings of \( Y(t) \) by considering the upcrossings of \( X(t) \).
The next task is to consider an upcrossing estimate as was done before for discrete submartingales.

$$
\begin{align*}
\tau_0 & = \min (\inf \{ t > 0 : X(t) = a \}, M), \\
\tau_1 & = \min (\inf \{ t > 0 : (X(t \lor \tau_0) - X(\tau_0))_+ = b - a \}, M), \\
\tau_2 & = \min (\inf \{ t > 0 : (X(\tau_1) - X(t \lor \tau_1))_+ = b - a \}, M), \\
\tau_3 & = \min (\inf \{ t > 0 : (X(t \lor \tau_2) - X(\tau_2))_+ = b - a \}, M), \\
\tau_4 & = \min (\inf \{ t > 0 : (X(\tau_3) - X(t \lor \tau_3))_+ = b - a \}, M), \\
& \vdots
\end{align*}
$$

If $X(t)$ is never $a$, then $\tau_0 = M$ and there are no upcrossings. It is obvious $\tau_1 \geq \tau_0$ since otherwise, the inequality could not hold. Thus the evens have $X(\tau_{2k}) = a$ and $X(\tau_{2k+1}) = b$.

Lemma 60.10.1 The above $\tau_i$ are stopping times for $t \in [0, M]$.

**Proof:** It is obvious that $\tau_0$ is a stopping time because it is the minimum of $M$ and the first hitting time of a closed set by a continuous adapted process. Consider a stopping time $\eta \leq M$ and let

$$\sigma = \inf \{ t > 0 : (X(t \lor \eta) - X(\eta))_+ = b - a \}$$

I claim that $t \to X(t \lor \eta) - X(\eta)$ is adapted to $\mathcal{F}_t$. Suppose $\alpha \geq 0$ and consider

$$[(X(t \lor \eta) - X(\eta))_+ > \alpha]$$

The above set equals

$$\left( [(X(t \lor \eta) - X(\eta))_+ > \alpha] \cap [\eta \leq t] \right) \cap \left( [(X(t \lor \eta) - X(\eta))_+ > \alpha] \cap [\eta > t] \right)$$

Consider the second of the above two sets. Since $\alpha \geq 0$, this set is $\emptyset$. This is because for $\eta > t$, $X(t \lor \eta) - X(\eta) = 0$. Now consider the first. It equals

$$[(X(t \lor \eta) - X(\eta))_+ > \alpha] \cap [\eta \lor t \leq t],$$

a set of $\mathcal{F}_{t \lor \eta}$ intersected with $[\eta \lor t \leq t]$ and so it is in $\mathcal{F}_t$ from properties of stopping times.

If $\alpha < 0$, then reduces to $\Omega$, also in $\mathcal{F}_t$. Therefore, by Proposition 60.7.2, $\sigma$ is a stopping time because it is the first hitting time of a closed set of a continuous adapted process. It follows that $\sigma \land M$ is also a stopping time. Similarly $t \to X(\eta) - X(t \lor \eta)$ is adapted and

$$\sigma = \inf \{ t > 0 : (X(\eta) - X(t \lor \eta))_+ = b - a \}$$

is also a stopping time from the same reasoning. It follows that the $\tau_i$ defined above are all stopping times.
Note that in the above, if \( \eta = M \), then \( \sigma = M \) also. Thus in the definition of the \( \tau_i \), if any \( \tau_i = M \), it follows that also \( \tau_{i+1} = M \) and so there is no change in the stopping times. Also note that these stopping times \( \tau_i \) are increasing as \( i \) increases.

Let

\[
U^n_{[a,b]} = \lim_{\varepsilon \to 0} \sum_{k=0}^{n} \frac{X(\tau_{2k+1}) - X(\tau_{2k})}{X(\tau_{2k+1}) - X(\tau_{2k})}
\]

Note that if an upcrossing occurs after \( \tau_{2k} \) on \([0,M]\), then \( \tau_{2k+1} > \tau_{2k} \) because there exists \( t \) such that

\[
(X(t \lor \tau_{2k}) - X(\tau_{2k}))_+ = b - a
\]

However, you could have \( \tau_{2k+1} > \tau_{2k} \) without an upcrossing occurring. This happens when \( \tau_{2k} < M \) and \( \tau_{2k+1} = M \) which may mean that \( X(t) \) never again climbs to \( b \). You break the sum into those terms where \( X(\tau_{2k+1}) - X(\tau_{2k}) = b - a \) and those where this is less than \( b - a \). Suppose for a fixed \( \omega \), the terms where the difference is \( b - a \) are for \( k \leq m \). Then there might be a last term for which \( X(\tau_{2k+1}) - X(\tau_{2k}) < b - a \) because it fails to complete the up crossing. There is only one of these at \( k = m + 1 \). Then the above sum is

\[
\leq \frac{1}{b - a} \sum_{k=0}^{m} X(\tau_{2k+1}) - X(\tau_{2k}) + \frac{X(M) - a}{\varepsilon + X(M) - a}
\]

\[
\leq \frac{1}{b - a} \sum_{k=0}^{n} X(\tau_{2k+1}) - X(\tau_{2k}) + \frac{X(M) - a}{\varepsilon + X(M) - a}
\]

\[
\leq \frac{1}{b - a} \sum_{k=0}^{n} X(\tau_{2k+1}) - X(\tau_{2k}) + 1
\]

Then \( U^n_{[a,b]} \) is clearly a random variable which is at least as large as the number of upcrossings occurring for \( t \leq M \) using only \( 2n + 1 \) of the stopping times. From the optional sampling theorem,

\[
E(X(\tau_{2k})) - E(X(\tau_{2k-1})) = \int_{\Omega} X(\tau_{2k}) - X(\tau_{2k-1}) \, dP
\]

\[
= \int_{\Omega} E(X(\tau_{2k})|\mathcal{F}_{\tau_{2k-1}}) - X(\tau_{2k-1}) \, dP
\]

\[
\geq \int_{\Omega} X(\tau_{2k-1}) - X(\tau_{2k-1}) \, dP = 0
\]

Note that, \( X(\tau_{2k}) = a \) while \( X(\tau_{2k-1}) = b \) so the above may seem surprising. However, the two stopping times can both equal \( M \) so this is actually possible. For example, it could happen that \( X(t) = a \) for all \( t \in [0,M] \).

Next, take the expectation of both sides,

\[
E\left(U^n_{[a,b]}\right) \leq \frac{1}{b - a} \sum_{k=0}^{n} E(X(\tau_{2k+1})) - E(X(\tau_{2k})) + 1
\]
≤ \frac{1}{b-a} \sum_{k=0}^{n} E(X(\tau_{2k+1})) - E(X(\tau_{2k})) + \frac{1}{b-a} \sum_{k=1}^{n} E(X(\tau_{2k})) - E(X(\tau_{2k-1})) + 1
\]

= \frac{1}{b-a} \left( E(X(\tau_1)) - E(X(\tau_0)) \right) + \frac{1}{b-a} \sum_{k=1}^{n} E(X(\tau_{2k+1})) - E(X(\tau_{2k-1})) + 1
\]

≤ \frac{1}{b-a} \left( E(X(\tau_{2n+1})) - E(X(\tau_0)) \right) + 1
\]

≤ \frac{1}{b-a} \left( E(X(M)) - a \right) + 1
\]

which does not depend on \(n\). The last inequality follows because \(0 \leq \tau_{2n+1} \leq M\) and \(X(t)\) is a submartingale. Let \(n \to \infty\) to obtain

\[ E\left( U_{[a,b]}^M \right) \leq \frac{1}{b-a} \left( E(X(M)) - a \right) + 1 \]

where \(U_{[a,b]}^M\) is an upper bound to the number of upcrossings of \(\{X(t)\}\) on \([0,M]\).

This proves the following interesting upcrossing estimate.

**Lemma 60.10.2** Let \(\{Y(t)\}\) be a continuous submartingale adapted to a normal filtration \(\mathcal{F}_t\) for \(t \in [0,M]\). Then if \(U_{[a,b]}^M\) is defined as the above upper bound to the number of upcrossings of \(\{Y(t)\}\) for \(t \in [0,M]\), then this is a random variable and since \(a \leq X(0)\),

\[ E\left( U_{[a,b]}^M \right) \leq \frac{1}{b-a} \left( (Y(M) - a)_+ + a - a \right) + 1 \]

\[ = \frac{1}{b-a} (Y(M) - a)_+ + 1 \]

With this it is easy to prove a continuous submartingale convergence theorem.

**Theorem 60.10.3** Let \(\{X(t)\}\) be a continuous submartingale adapted to a normal filtration such that

\[ \sup_t \{ E(|X(t)|) \} = C < \infty. \]

Then there exists \(X_\infty \in L^1(\Omega)\) such that

\[ \lim_{t \to \infty} X(t)(\omega) = X_\infty(\omega) \text{ a.e. } \omega. \]

**Proof:** Let \(U_{[a,b]}\) be defined by

\[ U_{[a,b]} = \lim_{M \to \infty} U_{[a,b]}^M. \]

Thus the random variable \(U_{[a,b]}\) is an upper bound for the number of upcrossings. From Lemma 60.10.2 and the assumption of this theorem, there exists a constant \(C\) independent of \(M\) such that

\[ E\left( U_{[a,b]}^M \right) \leq \frac{C}{b-a} + 1. \]
Letting $M \to \infty$, it follows from monotone convergence theorem that
\[ E \left( U_{[a,b]} \right) \leq \frac{C}{b-a} + 1 \]
also. Therefore, there exists a set of measure 0 $N_{ab}$ such that if $\omega \notin N_{ab}$, then $U_{[a,b]}(\omega) < \infty$. That is, there are only finitely many upcrossings. Now let
\[ N = \cup \{ N_{ab} : a, b \in \mathbb{Q} \}. \]

It follows that for $\omega \notin N$, it cannot happen that
\[ \limsup_{t \to \infty} X(t)(\omega) - \liminf_{t \to \infty} X(t)(\omega) > 0 \]
because if this expression is positive, there would be arbitrarily large values of $t$ where $X(t)(\omega) > b$ and arbitrarily large values of $t$ where $X(t)(\omega) < a$ where $a, b$ are rational numbers chosen such that
\[ \limsup_{t \to \infty} X(t)(\omega) > b > a > \liminf_{t \to \infty} X(t)(\omega) \]
Thus there would be infinitely many upcrossings which is not allowed for $\omega \notin N$. Therefore, the limit $\lim_{t \to \infty} X(t)(\omega)$ exists for a.e. $\omega$. Let $X(\infty)(\omega)$ equal this limit for $\omega \notin N$ and let $X(\infty)(\omega) = 0$ for $\omega \in N$. Then $X(\infty)$ is measurable and by Fatou’s lemma,
\[ \int_{\Omega} |X(\infty)(\omega)| \, dP \leq \liminf_{n \to \infty} \int_{\Omega} |X(n)(\omega)| \, dP < C. \]

Now here is an interesting result due to Doob.

**Theorem 60.10.4** Let $\{ M(t) \}$ be a continuous real martingale adapted to the normal filtration $\mathcal{F}_t$. Then the following are equivalent.

1. The random variables $M(t)$ are equiintegrable.
2. There exists $M(\infty) \in L^1(\Omega)$ such that $\lim_{t \to \infty} \| M(\infty) - M(t) \|_{L^1(\Omega)} = 0$.

In this case, $M(t) = E(M(\infty) | \mathcal{F}_t)$ and convergence also takes place pointwise.

**Proof:** Suppose the equiintegrable condition. Then there exists $\lambda$ large enough that for all $t$,
\[ \int_{\| M(t) \| \geq \lambda} |M(t)| \, dt < 1. \]
It follows that for all $t$,
\[ \int_{\Omega} |M(t)| \, dP = \int_{\| M(t) \| \geq \lambda} |M(t)| \, dP + \int_{\| M(t) \| < \lambda} |M(t)| \, dP \leq 1 + \lambda. \]
Since the martingale is bounded in \( L^1 \), by Theorem 60.10.3 there exists \( M(\infty) \in L^1(\Omega) \) such that \( \lim_{t \to \infty} M(t)(\omega) = M(\infty)(\omega) \) pointwise a.e. By the assumption \( \{M(t)\} \) are equiintegrable, it follows these functions are uniformly integrable. Letting \( \delta > 0 \) be such that if \( P(E) < \delta \), then
\[
\int_E |M(t)| \, dP < \frac{\varepsilon}{5},
\]
and \( t_n \to \infty \), Egoroff’s theorem implies that there exists a set \( E \) of measure less than \( \delta \) such that on \( E^C \), the convergence of the \( M(t_n) \) is uniform. Thus
\[
\int_\Omega |M(t_m) - M(t_n)| \, dP = \int_E |M(t_m) - M(t_n)| \, dP + \int_{E^C} |M(t_m) - M(t_n)| \, dP \\
\leq \frac{2\varepsilon}{5} + \int_{E^C} |M(t_m) - M(t_n)| \, dP < \varepsilon
\]
whenever \( m,n \) are large enough. Therefore, the sequence \( \{M(t_n)\} \) is Cauchy in \( L^1(\Omega) \) which implies it converges to something in \( L^1(\Omega) \) which must equal \( M(\infty) \) a.e.

Next suppose there is a function \( M(\infty) \) to which \( M(t) \) converges in \( L^1(\Omega) \). Then for \( t \) fixed and \( A \in \mathcal{F}_t \), then as \( s \to \infty, s > t \)
\[
\int_A M(t) \, dP = \int_A E(M(s) \, |\mathcal{F}_t) \, dP = \int_A M(s) \, dP \\
\to \int_A M(\infty) \, dP = \int_A E(M(\infty) \, |\mathcal{F}_t)
\]
which shows \( E(M(\infty) \, |\mathcal{F}_t) = M(t) \) a.e. since \( A \in \mathcal{F}_t \) is arbitrary. By Lemma 60.7.8
\[
\int_{|M(t)| \geq \lambda} |M(t)| \, dP = \int_{|M(t)| \geq \lambda} |E(M(\infty) \, |\mathcal{F}_t)| \, dP \\
\leq \int_{|M(t)| \geq \lambda} E(|M(\infty)| \, |\mathcal{F}_t) \, dP \\
= \int_{|M(t)| \geq \lambda} |M(\infty)| \, dP
\]
Now from this,
\[
\lambda P(|M(t)| \geq \lambda) \leq \int_{|M(t)| \geq \lambda} |M(t)| \, dP \leq \int_\Omega |E(M(\infty) \, |\mathcal{F}_t)| \, dP \\
\leq \int_\Omega E(|M(\infty)| \, |\mathcal{F}_t) \, dP = \int_\Omega |M(\infty)| \, dP
\]
and so
\[
P(|M(t)| \geq \lambda) \leq \frac{C}{\lambda}
\]
From 60.10.37, this shows \( \{M(t)\} \) is uniformly integrable because this is true of the single function \( |M(\infty)| \). By the submartingale convergence theorem, the convergence to \( M(\infty) \) also takes place pointwise. ■
60.11 Hitting This Before That

Let \( \{M(t)\} \) be a real valued martingale for \( t \in [0, T] \) where \( T \leq \infty \) and \( M(0) = 0 \).
In case \( T = \infty \), assume the conditions of Theorem 60.10.4 are satisfied. Thus there exists \( M(\infty) \) and the \( M(t) \) are equiintegrable. With the Doob optional sampling theorem it is possible to estimate the probability that \( M(t) \) hits \( b \) before it hits \( a \) where \( a < 0 < b \). There is no loss of generality in assuming \( T = \infty \) since if it is less than \( \infty \), you could just let \( M(t) \equiv M(T) \) for all \( t > T \). In this case, the equiintegrability of the \( M(t) \) follows because for \( t < T \),

\[
\int_{[|M(t)| > \lambda]} |M(t)| \, dP = \int_{[|M(t)| > \lambda]} |E(M(T) | \mathcal{F}_t)| \, dP \\
\leq \int_{[|M(t)| > \lambda]} |M(T)| \, dP
\]

and from Theorem 60.9.5,

\[
P(|M(t)| > \lambda) \leq P([M^* (t) > \lambda]) \leq \frac{1}{\lambda} \int_{\Omega} |M(T)| \, dP.
\]

**Definition 60.11.1** Let \( M \) be a process adapted to the filtration \( \mathcal{F}_t \) and let \( \tau \) be a stopping time. Then \( M^\tau \), called the stopped process is defined by

\[
M^\tau (t) \equiv M(\tau \wedge t).
\]

With this definition, here is a simple lemma.

**Lemma 60.11.2** Let \( M \) be a right continuous martingale adapted to the normal filtration \( \mathcal{F}_t \) and let \( \tau \) be a stopping time. Then \( M^\tau \) is also a martingale adapted to the filtration \( \mathcal{F}_t \).

**Proof:** Let \( s < t \). By the Doob optional sampling theorem,

\[
E(M^\tau (t) | \mathcal{F}_s) = E(M(\tau \wedge t) | \mathcal{F}_s) = M(\tau \wedge t \wedge s) = M^\tau (s).
\]

**Theorem 60.11.3** Let \( \{M(t)\} \) be a continuous real valued martingale adapted to the normal filtration \( \mathcal{F}_t \) and let

\[
M^* \equiv \sup \{|M(t)|: t \geq 0\}
\]

and \( M(0) = 0 \). Letting

\[
\tau_x \equiv \inf \{t > 0: M(t) = x\}
\]

Then if \( a < 0 < b \) the following inequalities hold.

\[
(b - a) \, P([\tau_a \leq \tau_b]) \geq -a \, P([M^* > 0]) \geq (b - a) \, P([\tau_b < \tau_a])
\]

and

\[
(b - a) \, P([\tau_a < \tau_b]) \leq b \, P([M^* > 0]) \leq (b - a) \, P([\tau_a \leq \tau_b]).
\]

In words, \( P([\tau_b \leq \tau_a]) \) is the probability that \( M(t) \) hits \( b \) no later than when it hits \( a \). (Note that if \( \tau_a = \infty = \tau_b \) then you would have \( [\tau_a = \tau_b] \).


Proof: For $x \in \mathbb{R}$, define
\[ \tau_x \equiv \inf \{ t \in \mathbb{R} \text{ such that } M(t) = x \} \]
with the usual convention that $\inf(\emptyset) = \infty$. Let $a < 0 < b$ and let
\[ \tau = \tau_a \wedge \tau_b \]
Then the following claim will be important.

Claim: $E(M(\tau)) = 0$.

Proof of the claim: Let $t > 0$. Then by the Doob optional sampling theorem,
\[
E(M(\tau \wedge t)) = E(E(M(t)|\mathcal{F}_\tau)) = E(M(t))
\]
(60.11.38)
\[
= E(E(M(t)|\mathcal{F}_0)) = E(M(0)) = 0.
\]
(60.11.39)

Observe the martingale $M^\tau$ must be bounded because it is stopped when $M(t)$ equals either $a$ or $b$. There are two cases according to whether $\tau = \infty$. If $\tau = \infty$, then $M(t)$ never hits $a$ or $b$ so $M(t)$ has values between $a$ and $b$. In this case $M^\tau(t) = M(t) \in [a,b]$. On the other hand, you could have $\tau < \infty$. Then in this case $M^\tau(t)$ is eventually equal to either $a$ or $b$ depending on which it hits first.

In either case, the martingale $M^\tau$ is bounded and by the martingale convergence theorem, Theorem 60.10.3, there exists $M^\tau(\infty)$ such that
\[
\lim_{t \to \infty} M^\tau(t)(\omega) = M^\tau(\infty)(\omega) = M(\tau)(\omega)
\]
and since the $M^\tau(t)$ are bounded, the dominated convergence theorem implies
\[
E(M(\tau)) = \lim_{t \to \infty} E(M(\tau \wedge t)) = 0.
\]
This proves the claim.

Recall
\[
M^*(\omega) \equiv \sup \{|M(t)(\omega)| : t \in [0,\infty]|.
\]
Also note that $[\tau_a = \tau_b] = [\tau = \infty]$. Now from the claim,
\[
0 = E(M(\tau)) = \int_{[\tau_a < \tau_b]} M(\tau) \, dP
\]
\[
+ \int_{[\tau_b < \tau_a]} M(\tau) \, dP + \int_{[\tau_a = \tau_b]} M(\infty) \, dP
\]
\[
+ \int_{[\tau_a = \tau_b] \cap [M^* > 0]} M(\infty) \, dP
\]
(60.11.40)

The last term equals 0. By continuity, $M(\tau)$ is either equal to $a$ or $b$ depending on whether $\tau_a < \tau_b$ or $\tau_b < \tau_a$. Thus
\[
0 = E(M(\tau)) = aP([\tau_a < \tau_b])
\]
Consider this last term. By the definition, \([\tau_a = \tau_b]\) corresponds to \(M(t)\) never hitting either \(a\) or \(b\). Since \(M(0) = 0\), this can only happen if \(M(t)\) has values in \([a, b]\). Therefore, this last term satisfies
\[
aP([\tau_a = \tau_b] \cap [M^* > 0])
\]
\[
\leq \int_{[\tau_a = \tau_b] \cap [M^* > 0]} M(\infty) dP
\]
(60.11.42)

It follows
\[
aP([\tau_a = \tau_b] \cap [M^* > 0]) + aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a]) \leq 0 \leq bP([\tau_a = \tau_b] \cap [M^* > 0]) + aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a])
\]
(60.11.43)

Note that \([\tau_b < \tau_a], [\tau_a < \tau_b] \subseteq [M^* > 0]\) and so
\[
[\tau_b < \tau_a] \cup [\tau_a < \tau_b] \cup ([\tau_a = \tau_b] \cap [M^* > 0]) = [M^* > 0]
\]
(60.11.44)

The following diagram may help in keeping track of the various substitutions.

\[
\begin{array}{ccc}
[\tau_a < \tau_b] & [\tau_b < \tau_a] & [\tau_b = \tau_a] \cap [M^* > 0] \\
\end{array}
\]

Left side of \(0\):

From \(0\), this yields on substituting for \(P([\tau_a < \tau_b])\)
\[
0 \geq aP([\tau_a = \tau_b] \cap [M^* > 0]) + aP([M^* > 0]) - P([\tau_a \geq \tau_b] \cap [M^* > 0]) + bP([\tau_b < \tau_a])
\]
and so since \([\tau_a \neq \tau_b] \subseteq [M^* > 0]\),
\[
0 \geq aP([M^* > 0]) - P([\tau_a > \tau_b]) + bP([\tau_b < \tau_a])
\]
(60.11.45)

Next use \(0\) to substitute for \(P([\tau_b < \tau_a])\)
\[
0 \geq aP([\tau_a = \tau_b] \cap [M^* > 0]) + aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a])
\]
\[
= aP([\tau_a = \tau_b] \cap [M^* > 0]) + aP([\tau_a < \tau_b]) + bP([M^* > 0]) - P([\tau_a \geq \tau_b] \cap [M^* > 0])
\]
\[
= aP([\tau_a \leq \tau_b] \cap [M^* > 0]) + bP([M^* > 0]) - P([\tau_a \leq \tau_b] \cap [M^* > 0])
\]
and so
\[
(b - a) P([\tau_a \leq \tau_b]) \geq bP([M^* > 0])
\]
(60.11.46)
Right side of

From (60.11.43), used to substitute for \( P(\tau_a < \tau_b) \) this yields

\[
0 \leq bP(\tau_a = \tau_b \cap [M^* > 0]) + aP(\tau_a < \tau_b) + bP(\tau_b < \tau_a)
\]

\[
= bP(\tau_a = \tau_b \cap [M^* > 0]) + a(P([M^* > 0]) - P(\tau_a \geq \tau_b \cap [M^* > 0]))
+ bP(\tau_b < \tau_a)
\]

\[
= bP(\tau_a \geq \tau_b \cap [M^* > 0]) + aP([M^* > 0]) - P(\tau_a \geq \tau_b \cap [M^* > 0])
\]

and so

\[
\frac{(b - a)}{P(\tau_a \geq \tau_b)} \geq -aP([M^* > 0]) \tag{60.11.47}
\]

Next use (60.11.44) to substitute for the term \( P(\tau_b < \tau_a) \) and write

\[
0 \leq bP(\tau_a = \tau_b \cap [M^* > 0]) + aP(\tau_a < \tau_b) + bP(\tau_b < \tau_a)
\]

\[
= bP(\tau_a = \tau_b \cap [M^* > 0]) + aP(\tau_a < \tau_b)
+ b[P([M^* > 0]) - P(\tau_a \leq \tau_b \cap [M^* > 0])]
\]

\[
= aP(\tau_a < \tau_b) + bP([M^* > 0]) - bP(\tau_a < \tau_b \cap [M^* > 0])
\]

\[
= aP(\tau_a < \tau_b) + bP([M^* > 0]) - bP(\tau_a < \tau_b)
\]

and so

\[
\frac{(b - a)}{P(\tau_a < \tau_b)} \leq bP([M^* > 0]) \tag{60.11.48}
\]

Now the boxed in formulas in (60.11.47 - 60.11.48) yield the conclusion of the theorem. This proves the theorem.

Note \( P(\tau_a < \tau_b) \) means \( M(t) \) hits \( a \) before it hits \( b \) with other occurrences of similar expressions being defined similarly.

### 60.12 The Space \( \mathcal{M}_T^p(E) \)

Here \( p \geq 1 \).

**Definition 60.12.1** Let \( M \) be an \( E \) valued martingale. Then \( M \in \mathcal{M}_T^p(E) \) if \( t \to M(t)(\omega) \) is continuous for a.e. \( \omega \) and

\[
E\left( \sup_{t \in [0,T]} ||M(t)||^p \right) < \infty
\]

Here \( E \) is a separable Banach space.
Proposition 60.12.2 Define a norm on $\mathcal{M}^p_T(E)$ by

$$||M||_{\mathcal{M}^p_T(E)} \equiv E \left( \sup_{t \in [0,T]} ||M(t)||^p \right)^{1/p}.$$ 

Then with this norm, $\mathcal{M}^p_T(E)$ is a Banach space.

Proof: First it is good to observe that $\sup_{t \in [0,T]} ||M(t)||^p$ is measurable. This follows because of the continuity of $t \to M(t)$. Let $D$ be a dense countable set in $[0,T]$. Then by continuity,

$$\sup_{t \in [0,T]} ||M(t)||^p = \sup_{t \in D} ||M(t)||^p$$

and the expression on the right is measurable because $D$ is countable.

Next it is necessary to show this is a norm. It is clear that

$$||M||_{\mathcal{M}^p_T(E)} \geq 0$$

and equals 0 only if

$$0 = E \left( \sup_{t \in [0,T]} ||M(t)||^p \right)$$

which requires $M(t) = 0$ for all $t$ for $\omega$ off a set of measure zero so that $M = 0$. It is also clear that

$$||\alpha M||_{\mathcal{M}^p_T(E)} = |\alpha||M||_{\mathcal{M}^p_T(E)}.$$ 

It remains to check the triangle inequality. Let $M, N \in \mathcal{M}^p_T(E)$.

$$||M + N||_{\mathcal{M}^p_T(E)} \equiv E \left( \sup_{t \in [0,T]} ||M(t) + N(t)||^p \right)^{1/p}$$

$$\leq E \left( \sup_{t \in [0,T]} (||M(t)|| + ||N(t)||)^p \right)^{1/p}$$

$$\leq E \left( \left( \sup_{t \in [0,T]} ||M(t)|| + \sup_{t \in [0,T]} ||N(t)|| \right)^p \right)^{1/p}$$

$$= \left( \int_\Omega \left( \sup_{t \in [0,T]} ||M(t)|| + \sup_{t \in [0,T]} ||N(t)|| \right)^p dP \right)^{1/p}$$

$$\leq \left( \int_\Omega \left( \sup_{t \in [0,T]} ||M(t)|| \right)^p dP \right)^{1/p} + \left( \int_\Omega \left( \sup_{t \in [0,T]} ||N(t)|| \right)^p dP \right)^{1/p}$$

$$= ||M||_{\mathcal{M}^p_T(E)} + ||N||_{\mathcal{M}^p_T(E)}$$
Next consider the claim that $\mathcal{M}^p_T (E)$ is a Banach space. Let $\{M_n\}$ be a Cauchy sequence. Then

$$E \left( \sup_{t \in [0, T]} ||M_n(t) - M_m(t)||^p \right) \to 0 \quad (60.12.49)$$

as $m, n \to \infty$. From continuity,

$$\sup_{t \in [0, T]} ||M_n(t) - M_m(t)|| = \sup_{t \in (0, T)} ||M_n(t) - M_m(t)||$$

Then from theorem 60.5.3 or 60.9.4,

$$P \left( \sup_{t \in [0, T]} ||M_n(t) - M_m(t)|| > \lambda \right) \leq \frac{1}{\lambda^p} E \left( ||M_n(T) - M_m(T)||^p \right)$$

Therefore, one can extract a subsequence $\{M_{n_k}\}$ such that

$$P \left( \sup_{t \in [0, T]} ||M_{n_k}(t) - M_{n_{k+1}}(t)|| > 2^{-k} \right) \leq 2^{-k}.$$ 

By the Borel Cantelli lemma, it follows $\{M_{n_k}(t)(\omega)\}$ converges uniformly on $[0, T]$ for a.e. $\omega$. Denote by $M(t)(\omega)$ the thing to which it converges, a continuous process because of the uniform convergence. Also, because it is the pointwise limit off a set of measure zero, $\omega \to M(t)(\omega)$ is $\mathcal{F}_t$ measurable. Also, from 60.12.49 and Fatou’s lemma

$$\int \sup_{t \in [0, T]} ||M_n(t) - M(t)||^p dP \leq \liminf_{k \to \infty} \int \sup_{t \in [0, T]} ||M_n(t) - M_{n_k}(t)||^p dP \leq \varepsilon$$

whenever $n$ is large enough, this from the assumption that $\{M_n\}$ is Cauchy. Thus

$$\lim_{n \to \infty} E \left( \sup_{t \in [0, T]} ||M_n(t) - M(t)||^p \right) = 0$$

and so for each $t, M_n(t) \to M(t)$ in $L^p(\Omega)$. This also shows that for large, $n$

$$E \left( \sup_{t \in [0, T]} ||M(t)||^p \right) \leq E \left( \sup_{t \in [0, T]} \left( ||M(t) - M_n(t)|| + ||M_n(t)|| \right)^p \right)$$

$$\leq 2^{p-1} E \left( \sup_{t \in [0, T]} ||M(t) - M_n(t)||^p + \sup_{t \in [0, T]} \left( ||M_n(t)|| \right)^p \right) < \infty$$
It only remains to verify $M$ is a martingale. Let $s \leq t$ and let $B \in \mathcal{F}_s$. For each $s$, $M_n(s) \to M(s)$ in $L^p(\Omega)$. Then from the above, $\omega \to M(s)(\omega)$ is $\mathcal{F}_s$ measurable. Then it follows that

$$\int_B M(s) \, dP = \lim_{n \to \infty} \int_B M_n(s) \, dP = \lim_{n \to \infty} \int_B E(M_n(t) | \mathcal{F}_s) \, dP$$

$$= \lim_{n \to \infty} \int_B M_n(t) \, dP = \int_B M(t) \, dP$$

and so by definition, $E(M(t) | \mathcal{F}_s) = M(s)$ which shows $M$ is a martingale. 

**Proposition 60.12.3** The functions $M(t)$ for each $M \in \mathcal{M}_T^p(E)$ are equi integrable.

**Proof:** This follows because

$$\int_{\|M(t)\| \geq \lambda} \|M(t)\|^p \, dP \leq \int_{\left[\sup_{t \in [0,T]} \|M(t)\| \geq \lambda\right]} \left(\sup_{t \in [0,T]} \|M(t)\|^p\right) \, dP \quad (60.12.50)$$

which converges to 0 due to the definition of $\mathcal{M}_T^p(E)$ which requires that

$$\sup_{t \in [0,T]} \|M(t)\|^p \in L^1(\Omega, \mathcal{F}, P).$$

Since the sets $\left[\sup_{t \in [0,T]} \|M(t)\| \geq \lambda\right]$ decrease to $\emptyset$ as $\lambda \to \infty$, the dominated convergence theorem implies the integral on the right in (60.12.50) converges to 0.
Chapter 61

The Quadratic Variation Of A Martingale

61.1 How To Recognize A Martingale

The main ideas are most easily understood in the special case where it is assumed the martingale is bounded. Then one can extend to more general situations using a localizing sequence of stopping times.

Let \( \{M(t)\} \) be a continuous martingale having values in a separable Hilbert space. The idea is to consider the submartingale, \( \{||M(t)||^2\} \) and write it as the sum of a martingale and a submartingale. An important part of the argument is the following lemma which gives a checkable criterion for a stochastic process to be a martingale.

**Lemma 61.1.1** Let \( \{X(t)\} \) be a stochastic process adapted to the filtration \( \{\mathcal{F}_t\} \) for \( t \geq 0 \). Then it is a martingale for the given filtration if for every stopping time \( \sigma \) it follows

\[
E(X(t)) = E(X(\sigma)).
\]

In fact, it suffices to check this on stopping times which have two values.

**Proof:** Let \( s < t \) and \( A \in \mathcal{F}_s \). Define a stopping time

\[
\sigma(\omega) \equiv sX_A(\omega) + tX_AC(\omega)
\]

This is a stopping time because \([\sigma \leq l] = \Omega \) if \( l \geq t \). Also \([\sigma \leq l] = A \in \mathcal{F}_s \) if \( l \in [s,t) \) and \([\sigma \leq l] = \emptyset \) if \( l < s \). Then by assumption,

\[
\int_A X(t) dP + \int_{AC} X(t) dP =
\]

by assumption

\[
\int X(t) dP = \int X(\sigma) dP = \int_A X(s) dP + \int_{AC} X(t) dP
\]
Therefore,
\[ \int_A X(t) \, dP = \int_A X(s) \, dP \]
and since \( X(s) \) is \( \mathcal{F}_s \) measurable, it follows \( E(X(t) \mid \mathcal{F}_s) = X(s) \) a.e. and this shows \( \{X(t)\} \) is a martingale. ■

Note that if \( t \in [0, T] \), it suffices to check the expectation condition for stopping times which have two values no larger than \( T \).

The following lemma will be useful.

**Lemma 61.1.2** Suppose \( X_n \to X \) in \( L^1(\Omega, \mathcal{F}, P; E) \) where \( E \) is a separable Banach space. Then letting \( G \) be a \( \sigma \) algebra contained in \( \mathcal{F} \),

\[ E(X_n \mid G) \to E(X \mid G) \]

in \( L^1(\Omega) \).

**Proof:** This follows from the definitions and Theorem 59.1.1 on Page 2031.

\[
\int_\Omega \|E(X \mid G) - E(X_n \mid G)\| \, dP = \int_\Omega \|E(X_n - X \mid G)\| \, dP \\
\leq \int_\Omega E(\|X_n - X\| \mid G) \, dP \\
= \int_\Omega \|X_n - X\| \, dP
\]

**Corollary 61.1.3** Let \( X, Y \) be in \( L^2(\Omega, \mathcal{F}, P; H) \) where \( H \) is a separable Hilbert space and let \( X \) be \( G \) measurable where \( G \subseteq \mathcal{F} \). Then

\[ E((X, Y) \mid G) = (X, E(Y \mid G)) \ a.e. \]

**Proof:** First let \( X = a X_B \) where \( B \in G \). Then for \( A \in G \),

\[
\int_A E((aX_B, Y) \mid G) \, dP = \int_A X_B E((a, Y) \mid G) \, dP = \int_A X_B (a, Y) \, dP \\
= \int_{A \cap B} (a, Y) \, dP = \left( a, \int_{A \cap B} Y \, dP \right)
\]

\[
\int_A (a X_B, E(Y \mid G)) \, dP = \int_A X_B (a, E(Y \mid G)) \, dP \\
= \left( a, \int_A X_B E(Y \mid G) \, dP \right) = \left( a, \int_{A \cap B} Y \, dP \right)
\]

It follows that the formula holds for \( X \) simple.

Therefore, letting \( X_n \) be a sequence of \( G \) measurable simple functions converging pointwise to \( X \) and also in \( L^2(\Omega) \),

\[ E((X_n, Y) \mid G) = (X_n, E(Y \mid G)) \]
61.1. HOW TO RECOGNIZE A MARTINGALE

Now the desired formula holds from Lemma 61.1.2. The following is related to something called a martingale transform. It is a lot like what will happen later with the Ito integral.

**Proposition 61.1.4** Let \( \{\tau_k\} \) be an increasing sequence of stopping times for the normal filtration \( \{F_t\} \) such that

\[
\lim_{k \to \infty} \tau_k = \infty, \quad \tau_0 = 0.
\]

Also let \( \xi_k \) be \( F_{\tau_k} \) measurable with values in \( H \), a separable Hilbert space and let \( M(t) \) be a right continuous martingale adapted to the normal filtration \( F_t \) which has the property that \( M(t) \in L^2(\Omega; H) \) for all \( t, M(0) = 0 \). Then if \( |\xi_k| \leq C \),

\[
E \left( \left( \sum_{k \geq 0} (\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t)))^2 \right)^2 \right) \leq C^2 E \left( |M(t)|^2 \right)
\]

(61.1.1)

**Proof:** First of all, the sum converges because eventually \( \tau_k \wedge t = t \). Therefore, for large enough \( k \),

\[
M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t) \equiv \Delta M_k = 0.
\]

Consider first the finite sum, \( k \leq q \).

\[
E \left( \left( \sum_{k=0}^q (\xi_k, \Delta M_k) \right)^2 \right)
\]

(61.1.2)

When the sum is multiplied out, you get mixed terms. Consider one of these mixed terms, \( j < k \)

\[
E \left( (\xi_k, \Delta M_k) (\xi_j, \Delta M_j) \right)
\]

Using Corollary 61.1.3 and Doob’s optional sampling theorem, Theorem 58.4.4, this equals

\[
E \left( E \left( (\xi_k, \Delta M_k) (\xi_j, \Delta M_j) | F_{\tau_k} \right) \right) = E \left( (\xi_j, \Delta M_j) E \left( (\xi_k, \Delta M_k) | F_{\tau_k} \right) \right)
\]

\[
= E \left( (\xi_j, \Delta M_j) (\xi_k, E (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t) | F_{\tau_k}) \right) = E \left( (\xi_j, \Delta M_j) (\xi_k, 0) \right) = 0
\]

Note that in using the optional sampling theorem, the stopping time \( \tau_{k+1} \wedge t \) is bounded.

Therefore, the only terms which survive in (61.1.2) are the non mixed terms and so this expression reduces to

\[
\sum_{k=0}^q E (\xi_k, \Delta M_k)^2 \leq C^2 \sum_{k=0}^q E \left( ||\Delta M_k||^2 \right)
\]

\[
= C^2 \sum_{k=0}^q E \left( ||M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t)||^2 \right)
\]
Consider the term $E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t)))$. By Doob’s optional sampling theorem for martingales and Corollary 61.1.3 again, this equals

$$E(E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t)) | F_{\tau_k})) = E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t \wedge \tau_k)))$$

$$= E\left(||M(\tau_k \wedge t)||^2\right)$$

It follows equals

$$C^2 \sum_{k=0}^{q} E\left(||M(\tau_{k+1} \wedge t)||^2\right) - E\left(||M(\tau_k \wedge t)||^2\right) \leq C^2 E\left(||M(t)||^2\right).$$

Then from Fatou’s lemma,

$$E\left(\left(\sum_{k \geq 0} (\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t)))\right)^2\right) \leq$$

$$\lim_{q \to \infty} \inf E\left(\left(\sum_{k=0}^{q} (\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t)))\right)^2\right) \leq C^2 E\left(||M(t)||^2\right)$$

Now here is an interesting lemma which will be used to prove uniqueness in the main result.

**Lemma 61.1.5** Let $\mathcal{F}_t$ be a normal filtration and let $A(t), B(t)$ be adapted to $\mathcal{F}_t$, continuous, and increasing with $A(0) = B(0) = 0$ and suppose $A(t) - B(t)$ is a martingale. Then $A(t) - B(t) = 0$ for all $t$.

**Proof:** I shall show $A(l) = B(l)$ where $l$ is arbitrary. Let $M(t)$ be the name of the martingale. Define a stopping time

$$\tau \equiv \inf \{t > 0 : |M(t)| > C\} \wedge l \wedge \inf \{t > 0 : A(t) > C\}$$

$$\wedge \inf \{t > 0 : B(t) > C\}$$

where $\inf(\emptyset) \equiv \infty$ and denote the stopped martingale

$$M^\tau(t) \equiv M(t \wedge \tau).$$
Then I claim this is also a martingale with respect to the filtration $\mathcal{F}_t$ because by Doob’s optional sampling theorem for martingales, if $s < t$, 

$$E(M^\tau(t) | \mathcal{F}_s) \equiv E(M(\tau \land t) | \mathcal{F}_s) = M(\tau \land t \land s) = M(\tau \land s) = M^\tau(s)$$

Note the bounded stopping time is $\tau \land t$ and the other one is $\sigma = s$ in this theorem. Then $M^\tau$ is a continuous martingale which is also uniformly bounded. It equals $A^\tau - B^\tau$. The stopping time ensures $A^\tau$ and $B^\tau$ are uniformly bounded by $C$. Thus all of $|M^\tau(t)|, B^\tau(t), A^\tau(t)$ are bounded by $C$ on $[0, \ell]$. Now let $\mathcal{P}_n \equiv \{t_k\}_{k=1}^n$ be a uniform partition of $[0, \ell]$ and let $M^\tau(\mathcal{P}_n)$ denote

$$M^\tau(\mathcal{P}_n) = \max \{|M^\tau(t_{i+1}) - M^\tau(t_i)|\}_{i=1}^n.$$

Then

$$E\left(\left(M^\tau(l)\right)^2\right) = E\left(\sum_{k=0}^{n-1} (M^\tau(t_{k+1}) - M^\tau(t_k))^2\right)$$

Now consider a mixed term in the sum where $j < k$.

$$E\left((M^\tau(t_{k+1}) - M^\tau(t_k)) (M^\tau(t_{j+1}) - M^\tau(t_j))\right)$$

$$= E\left(E\left((M^\tau(t_{k+1}) - M^\tau(t_k))(M^\tau(t_{j+1}) - M^\tau(t_j)) | \mathcal{F}_{t_k}\right)\right)$$

$$= E\left((M^\tau(t_{j+1}) - M^\tau(t_j)) E\left((M^\tau(t_{k+1}) - M^\tau(t_k)) | \mathcal{F}_{t_k}\right)\right)$$

$$= E\left((M^\tau(t_{j+1}) - M^\tau(t_j))(M^\tau(t_k) - M^\tau(t_k))\right) = 0$$

It follows

$$E\left(M^\tau(l)^2\right) = E\left(\sum_{k=0}^{n-1} (M^\tau(t_{k+1}) - M^\tau(t_k))^2\right)$$

$$\leq E\left(\sum_{k=0}^{n-1} M^\tau(\mathcal{P}_n) |M^\tau(t_{k+1}) - M^\tau(t_k)|\right)$$

$$\leq E\left(\sum_{k=0}^{n-1} M^\tau(\mathcal{P}_n) (|A^\tau(t_{k+1}) - A^\tau(t_k)| + |B^\tau(t_{k+1}) - B^\tau(t_k)|)\right)$$

$$\leq E\left(M^\tau(\mathcal{P}_n) \sum_{k=0}^{n-1} (|A^\tau(t_{k+1}) - A^\tau(t_k)| + |B^\tau(t_{k+1}) - B^\tau(t_k)|)\right)$$

$$\leq E\left(M^\tau(\mathcal{P}_n) 2C\right)$$

the last step holding because $A$ and $B$ are increasing. Now letting $n \to \infty$, the right side converges to 0 by the dominated convergence theorem and the observation that for a.e. $\omega$,

$$\lim_{n \to \infty} M^\tau(\mathcal{P}_n)(\omega) = 0$$
because of continuity of $M$. Thus for $\tau = \tau_C$ given above,

$$M (l \wedge \tau_C) = 0 \text{ a.e.}$$

Now let $C \in \mathbb{N}$ and let $N_C$ be the exceptional set off which $M (l \wedge \tau_C) = 0$. Then letting $N_1$ denote the union of all these exceptional sets for $C \in \mathbb{N}$, it is also a set of measure zero and for $\omega$ not in this set, $M (l \wedge \tau_C) = 0$ for all $C$. Since the martingale is continuous, it follows for each such $\omega$, eventually $\tau_C > l$ and so $M (l) = 0$. Thus for $\omega \notin N_1$,

$$M (l) (\omega) = 0$$

Now let $N = \cup_{l \in \mathbb{Q} \cap [0, \infty)} N_l$. Then for $\omega \notin N$, $M (l) (\omega) = 0$ for all $l \in \mathbb{Q} \cap [0, \infty)$ and so by continuity, this is true for all positive $l$. □

Note this shows a continuous martingale is not of bounded variation unless it is a constant.

### 61.2 The Quadratic Variation

This section is on the quadratic variation of a martingale. Actually, you can also consider the quadratic variation of a local martingale which is more general. Therefore, this concept is defined first. We will generally assume $M (0) = 0$ since there is no real loss of generality in doing so. One can simply subtract $M (0)$ otherwise.

**Definition 61.2.1** Let $\{M (t)\}$ be adapted to the normal filtration $\mathcal{F}_t$ for $t > 0$. Then $\{M (t)\}$ is a local martingale (submartingale) if there exist stopping times $\tau_n$ increasing to infinity such that for each $n$, the process $M^{\tau_n} (t) \equiv M (t \wedge \tau_n)$ is a martingale (submartingale) with respect to the given filtration. The sequence of stopping times is called a localizing sequence. The martingale $M^{\tau_n}$ is called the stopped martingale. Exactly the same convention applies to a localized submartingale.

**Proposition 61.2.2** If $M (t)$ is a continuous local martingale (submartingale) for a normal filtration as above, $M (0) = 0$, then there exists a localizing sequence $\tau_n$ such that for each $n$ the stopped martingale(submartingale) $M^{\tau_n}$ is uniformly bounded. Also if $M$ is a martingale, then $M^\tau$ is also a martingale (submartingale). If $\tau_n$ is an increasing sequence of stopping times such that $\lim_{n \to \infty} \tau_n = \infty$, and for each $\tau_n$ and real valued stopping time $\delta$, there exists a function $X$ of $\tau_n \wedge \delta$ such that $X (\tau_n \wedge \delta)$ is $\mathcal{F}_{\tau_n \wedge \delta}$ measurable, then $\lim_{n \to \infty} X (\tau_n \wedge \delta) \equiv X (\delta)$ exists for each $\omega$ and $X (\delta)$ is $\mathcal{F}_\delta$ measurable.

**Proof:** First consider the claim about $M^\tau$ being a martingale (submartingale) when $M$ is. By optional sampling theorem,

$$E (M^\tau (t) | \mathcal{F}_s) = E (M (\tau \wedge t) | \mathcal{F}_s) = M (\tau \wedge t \wedge s) = M^\tau (s).$$

The case where $M$ is a submartingale is similar.

Next suppose $\sigma_n$ is a localizing sequence for the local martingale(submartingale) $M$. Then define

$$\eta_n \equiv \inf \{t > 0 : ||M (t)|| > n\}.$$
61.2. THE QUADRATIC VARIATION

Therefore, by continuity of $M$, $||M (η_n)|| \leq n$. Now consider $τ_n \equiv η_n \wedge σ_n$. This is an increasing sequence of stopping times. By continuity of $M$, it must be the case that $η_n \to \infty$. Hence $σ_n \wedge η_n \to \infty$.

Finally, consider the last claim. Pick $ω$. Then $X (τ_n (ω) \wedge δ (ω)) (ω)$ is eventually constant as $n \to \infty$ because for all $n$ large enough, $τ_n (ω) > δ (ω)$ and so this sequence of functions converges pointwise. That which it converges to, denoted by $X (δ)$, is $F_δ$ measurable because each function $ω \to X (τ_n (ω) \wedge δ (ω)) (ω)$ is $F_δ \wedge τ_n \subseteq F_δ$ measurable. ■

One can also give a generalization of Lemma 61.2.4 to conclude a local martingale must be constant or else they must fail to be of bounded variation.

**Corollary 61.2.3** Let $F_t$ be a normal filtration and let $A (t), B (t)$ be adapted to $F_t$, continuous, and increasing with $A (0) = B (0) = 0$ and suppose $A (t) - B (t) \equiv M (t)$ is a local martingale. Then $M (t) = A (t) - B (t) = 0$ a.e. for all $t$.

**Proof:** Let $\{τ_n\}$ be a localizing sequence for $M$. For given $n$, consider the martingale,

$$M^τ_n (t) = A^τ_n (t) - B^τ_n (t)$$

Then from Lemma 61.2.4, it follows $M^τ_n (t) = 0$ for all $t$ for all $ω \notin N_n$, a set of measure 0. Let $N = \bigcup_n N_n$. Then for $ω \notin N$, $M (τ_n (ω) \wedge t) (ω) = 0$. Let $n \to \infty$ to conclude that $M (t) (ω) = 0$. Therefore, $M (t) (ω) = 0$ for all $t$. ■

Recall Example 61.2.7 on Page 2133. For convenience, here is a version of what it says.

**Lemma 61.2.4** Let $X (t)$ be continuous and adapted to a normal filtration $F_t$ and let $η$ be a stopping time. Then if $K$ is a closed set with $0 \notin K$,

$$τ \equiv \inf \{t > η : X (t) \in K\}$$

is also a stopping time.

**Proof:** First consider $Y (t) = X (t \lor η) - X (η)$. I claim that $Y (t)$ is adapted to $F_t$. Consider $U$ and open set and $[Y (t) \in U]$. Is it in $F_t$? We know it is in $F_{t \lor η}$. It equals

$$([Y (t) \in U] \cap [η \leq t]) \cup ([Y (t) \in U] \cap [η > t])$$

Consider the second of these sets. It equals

$$([X (η) - X (η) \in U] \cap [η > t])$$

If $0 \in U$, then it reduces to $[η > t] \in F_t$. If $0 \notin U$, then it reduces to $0$ still in $F_t$.

Next consider the first set. It equals

$$[X (t \lor η) - X (η) \in U] \cap [η \leq t] = [X (t \lor η) - X (η) \in U] \cap [t \lor η \leq t] \in F_t$$

from the definition of $F_{t \lor η}$. (You know that $[X (t \lor η) - X (η) \in U] \in F_{t \lor η}$ and so when this is intersected with $[t \lor η \leq t]$ one obtains a set in $F_t$. This is what it means to be in $F_{t \lor η}$.) Now $τ$ is just the first hitting time of $Y (t)$ of the closed set. ■
Proposition 61.2.5 Let $M(t)$ be a continuous local martingale for $t \in [0,T]$ having values in $H$ a separable Hilbert space adapted to the normal filtration $\{F_t\}$ such that $M(0) = 0$. Then there exists a unique continuous, increasing, nonnegative, local submartingale $[M](t)$ called the quadratic variation such that

$$||M(t)||^2 - [M](t)$$

is a real local martingale and $[M](0) = 0$. Here $t \in [0,T]$. If $\delta$ is any stopping time $[M^\delta] = [M]^{\delta}$

Proof: First it is necessary to define some stopping times. Define stopping times $\tau_n^0 \equiv \eta^0\equiv 0$. 

$$\eta_{k+1}^n \equiv \inf \{ s > \eta_k^n : ||M(s) - M(\eta_k^n)|| = 2^{-n} \}, \quad \tau_k^n \equiv \eta_k^n \land T$$

where $\inf \emptyset \equiv \infty$. These are stopping times by Example 60.7.4 on Page 2134. See also Lemma 61.2.4. Then for $t > 0$ and $\delta$ any stopping time, and fixed $\omega$, for some $k$, 

$$t \land \delta \in I_k(\omega), \quad I_0(\omega) \equiv [\tau_0^n(\omega), \tau_1^n(\omega)], \quad I_k(\omega) \equiv (\tau_k^n(\omega), \tau_{k+1}^n(\omega))$$

some $k$

Here is why. The sequence $\{\tau_k^n(\omega)\}_{k=1}^\infty$ eventually equals $T$ for all $n$ sufficiently large. This is because if it did not, it would converge, being bounded above by $T$ and then by continuity of $M$, $\{M(\tau_k^n(\omega))\}_{k=1}^\infty$ would be a Cauchy sequence contrary to the requirement that 

$$||M(\tau_{k+1}^n(\omega)) - M(\tau_k^n(\omega))||$$

$$= ||M(\eta_{k+1}^n(\omega)) - M(\eta_k^n(\omega))|| = 2^{-n}.$$ 

Note that if $\delta$ is any stopping time, then 

$$||M(t \land \delta \land \tau_{k+1}^n) - M(t \land \delta \land \tau_k^n)||$$

$$= ||M^{\delta}(t \land \tau_{k+1}^n) - M^{\delta}(t \land \tau_k^n)|| \leq 2^{-n}.$$ 

You can see this is the case by considering the cases, $t \land \delta \geq \tau_{k+1}^n, t \land \delta \in [\tau_k^n, \tau_{k+1}^n)$, and $t \land \delta < \tau_k^n$. It is only this approximation property and the fact that the $\tau_k^n$ partition $[0,T]$ which is important in the following argument.

Now let $\alpha_n$ be a localizing sequence such that $M^{\alpha_n}$ is bounded as in Proposition 61.2.5. Thus $M^{\alpha_n}(t) \in L^2(\Omega)$ and this is all that is needed. In what follows, let $\delta$ be a stopping time and denote $M^{\alpha_n \land \delta}$ by $M$ to save notation. Thus $M$ will be uniformly bounded and from the definition of the stopping times $\tau_k^n$, for $t \in [0,T]$, 

$$M(t) = \sum_{k \geq 0} M(t \land \tau_{k+1}^n) - M(t \land \tau_k^n), \quad (61.2.4)$$
61.2. THE QUADRATIC VARIATION

and the terms of the series are eventually 0, as soon as \( \eta^n_k = \infty \).

Therefore,

\[
\| M(t) \|^2 = \left\| \sum_{k \geq 0} M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k) \right\|^2
\]

Then this equals

\[
\sum_{k \geq 0} \left\| M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k) \right\|^2
\]

\[
+ \sum_{j \neq k} ((M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k)), (M(t \wedge \tau^n_{j+1}) - M(t \wedge \tau^n_j))) \tag{61.2.5}
\]

Consider the second sum. It equals

\[
2 \sum_{k \geq 0} \sum_{j=0}^{k-1} ((M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k)), (M(t \wedge \tau^n_{j+1}) - M(t \wedge \tau^n_j)))
\]

\[
= 2 \sum_{k \geq 0} \left( (M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k)), \sum_{j=0}^{k-1} (M(t \wedge \tau^n_{j+1}) - M(t \wedge \tau^n_j)) \right)
\]

\[
= 2 \sum_{k \geq 0} ((M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k)), M(t \wedge \tau^n_k))
\]

This last sum equals \( P_n(t) \) defined as

\[
2 \sum_{k \geq 0} (M(\tau^n_k), (M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k))) \equiv P_n(t) \tag{61.2.6}
\]

This is because in the \( k^{th} \) term, if \( t \geq \tau^n_k \), then it reduces to

\[
(M(\tau^n_k), (M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k)))
\]

while if \( t < \tau^n_k \), then the term reduces to 0 which is also the same as

\[
(M(\tau^n_k), (M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k))).
\]

This is a finite sum because eventually, for large enough \( k \), \( \tau^n_k = T \). However the number of nonzero terms depends on \( \omega \). This is not a good thing. However, a little more can be said. In fact the sum also converges in \( L^2(\Omega) \). Say \( ||M(t,\omega)|| \leq C \).

\[
E \left( \left( \sum_{k \geq p} (M(\tau^n_k), (M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k))) \right)^2 \right)
\]
= \sum_{k \geq p}^q E \left( \left( M (\tau_k^n), (M (t \wedge \tau_{k+1}^n) - M (t \wedge \tau_k^n))^2 \right) \right) + \text{mixed terms} \quad \text{(61.2.7)}

Consider one of these mixed terms for \( j < k \).

\[
E \left( \left( M (\tau_j^n), \left( M (t \wedge \tau_{j+1}^n) - M (t \wedge \tau_j^n) \right) \right) \cdot \left( M (\tau_k^n), \left( M (t \wedge \tau_{k+1}^n) - M (t \wedge \tau_k^n) \right) \right) \right)
\]

Then it equals

\[
E \left( E \left( \left( M (\tau_j^n), \Delta_j \right) \left( M (\tau_k^n), \Delta_k \right) \mid \mathcal{F}_{\tau_k} \right) \right)
= E \left( \left( M (\tau_j^n), \Delta_j \right) E \left( \left( M (\tau_k^n), \Delta_k \right) \mid \mathcal{F}_{\tau_k} \right) \right)
= E \left( \left( E \left( \left( M (\tau_j^n), \Delta_j \right) \mid \mathcal{F}_{\tau_k} \right) \right) \left( E \left( \left( M (\tau_k^n), E (\Delta_k \mid \mathcal{F}_{\tau_k}) \right) \right) \right) = 0
\]

Now since the mixed terms equal 0, it follows from (61.2.7), that expression is dominated by

\[
C^2 \sum_{k \geq p}^q E \left( \left| \left| M (t \wedge \tau_{k+1}^n) - M (t \wedge \tau_k^n) \right| \right|^2 \right)
\]

Using a similar manipulation to what was just done to show the mixed terms equal 0, this equals

\[
C^2 \sum_{k \geq p}^q E \left( \left| \left| M (t \wedge \tau_{k+1}^n) \right| \right|^2 \right) - E \left( \left| \left| M (t \wedge \tau_k^n) \right| \right|^2 \right)
\leq C^2 E \left( \left| \left| M (t \wedge \tau_{q+1}^n) \right| \right|^2 - \left| \left| M (t \wedge \tau_p^n) \right| \right|^2 \right)
\]

The integrand converges to 0 as \( p, q \to \infty \) and the uniform bound on \( M \) allows a use of the dominated convergence theorem. Thus the partial sums of the series of

\[
\sum_{k \geq p}^q E \left( \left| \left| M (t \wedge \tau_{k+1}^n) \right| \right|^2 \right)
\]

converge in \( L^2 (\Omega) \) as claimed.

By adding in the values of \( \{ \tau_{k+1}^n \} \) \( P_n (t) \) can be written in the form

\[
2 \sum_{k \geq 0} \left( M (\tau_{k+1}^{n+1}), (M (t \wedge \tau_{k+1}^{n+1}) - M (t \wedge \tau_{k+1}^{n+1})) \right)
\]

where \( \tau_{k+1}^{n+1} \) has some repeats. From the construction,

\[
\left| \left| M (\tau_{k+1}^{n+1}) - M (\tau_{k+1}^{n+1}) \right| \right| \leq 2^{-(n+1)}
\]

Thus

\[
P_n (t) - P_{n+1} (t) = 2 \sum_{k \geq 0} \left( M (\tau_{k+1}^{n+1}) - M (\tau_{k+1}^{n+1}), (M (t \wedge \tau_{k+1}^{n+1}) - M (t \wedge \tau_{k+1}^{n+1})) \right)
\]
and so from Proposition \(61.2.8\) applied to \(\xi_k \equiv M(\tau_k^{n+1}) - M(\tau_k^n)\),

\[
E\left(\|P_n(t) - P_{n+1}(t)\|^2\right) \leq 2^{-2n} E\left(\|M(t)\|^2\right), \tag{61.2.8}
\]

Now \(t \to P_n(t)\) is continuous because it is a finite sum of continuous functions. It is also the case that \(\{P_n(t)\}\) is a martingale. To see this use Lemma \(61.2.9\). Let \(\sigma\) be a stopping time having two values. Then using Corollary \(61.2.10\) and the Doob optional sampling theorem, Theorem \(61.2.11\)

\[
E \left( \sum_{k=0}^{q} (M(\tau_k^n), (M(\sigma \land \tau_{k+1}^n) - M(\sigma \land \tau_k^n))) \right)
= \sum_{k=0}^{q} E \left( (E (M(\tau_k^n), (M(\sigma \land \tau_{k+1}^n) - M(\sigma \land \tau_k^n))) | \mathcal{F}_{\tau_k^n}) \right)
= \sum_{k=0}^{q} E \left( (M(\tau_k^n), E (M(\sigma \land \tau_{k+1}^n) - M(\sigma \land \tau_k^n)) | \mathcal{F}_{\tau_k^n}) \right)
= \sum_{k=0}^{q} E \left( (M(\tau_k^n), E (M(\sigma \land \tau_{k+1}^n \land \tau_k^n) - M(\sigma \land \tau_k^n))) \right) = 0
\]

Note the Doob theorem applies because \(\sigma \land \tau_{k+1}^n\) is a bounded stopping time due to the fact \(\sigma\) has only two values. Similarly

\[
E \left( \sum_{k=0}^{q} (M(\tau_k^n), (M(t \land \tau_{k+1}^n) - M(t \land \tau_k^n))) \right)
= \sum_{k=0}^{q} E \left( (E (M(\tau_k^n), (M(t \land \tau_{k+1}^n) - M(t \land \tau_k^n))) | \mathcal{F}_{\tau_k^n}) \right)
= \sum_{k=0}^{q} E \left( (M(\tau_k^n), E (M(t \land \tau_{k+1}^n) - M(t \land \tau_k^n)) | \mathcal{F}_{\tau_k^n}) \right)
= \sum_{k=0}^{q} E \left( (M(\tau_k^n), E (M(t \land \tau_{k+1}^n \land \tau_k^n) - M(t \land \tau_k^n))) \right) = 0
\]

It follows each partial sum for \(P_n(t)\) is a martingale. As shown above, these partial sums converge in \(L^2(\Omega)\) and so it follows that \(P_n(t)\) is also a martingale. Note the Doob theorem applies because \(t \land \tau_{k+1}^n\) is a bounded stopping time.
I want to argue that $P_n$ is a Cauchy sequence in $\mathcal{M}^2_T(\mathbb{R})$. By Theorem 61.2.3 and continuity of $P_n$
\[ E \left( \sup_{t \leq T} |P_n(t) - P_{n+1}(t)| \right)^2 \leq 2E \left( |P_n(T) - P_{n+1}(T)|^2 \right)^{1/2} \]
By 61.2.3
\[ \leq 2^{-n} E \left( |M(T)|^2 \right)^{1/2} \]
which shows $\{P_n\}$ is indeed a Cauchy sequence in $\mathcal{M}^2_T(\mathbb{R})$.

Therefore, by Proposition 61.2.4, there exists $\{N(t)\} \in \mathcal{M}^2_T(\mathbb{R})$ such that $P_n \to N$ in $\mathcal{M}^2_T(H)$. That is
\[ \lim_{n \to \infty} E \left( \sup_{t \in [0,T]} |P_n(t) - N(t)|^2 \right)^{1/2} = 0. \]
Since $\{N(t)\} \in \mathcal{M}^2_T(\mathbb{R})$, it is a continuous martingale and $N(t) \in L^2(\Omega)$, and $N(0) = 0$ because this is true of each $P_n(0)$. From the above
\[ ||M(t)||^2 = Q_n(t) + P_n(t) \]
where
\[ Q_n(t) = \sum_{k \geq 0} ||M(t \wedge \tau^n_{k+1}) - M(t \wedge \tau^n_k)||^2 \]
and $P_n(t)$ is a martingale. Then from 61.2.4, $Q_n(t)$ is a submartingale and converges for each $t$ to something, denoted as $[M](t)$ in $L^1(\Omega)$ uniformly in $t \in [0,T]$. This is because $P_n(t)$ converges uniformly on $[0,T]$ to $N(t)$ in $L^2(\Omega)$ and $||M(t)||^2$ does not depend on $n$. Then also $[M]$ is a submartingale which equals 0 at 0 because this is true of $Q_n$ and because if $A \in \mathcal{F}_s$ where $s < t$,
\[ \int_A E([M](t) | \mathcal{F}_s) dP = \lim_{n \to \infty} \int_A [M](t) dP = \lim_{n \to \infty} \int_A \left( ||M(t)||^2 - P_n(t) \right) dP \]
\[ = \lim_{n \to \infty} \int_A E\left( ||M(t)||^2 - P_n(t) | \mathcal{F}_s \right) dP \geq \lim_{n \to \infty} \int_A ||M(s)||^2 - P_n(s) dP \]
\[ = \lim_{n \to \infty} \int_A Q_n(s) dP = \int_A [M](s) dP. \]
Note that $Q_n(t)$ is increasing because as $t$ increases, the definition allows for the possibility of more nonzero terms in the sum. Therefore, $[M](t)$ is also increasing in $t$. The function $t \to [M](t)$ is continuous because $||M(t)||^2 = [M](t) + N(t)$ and $t \to N(t)$ is continuous as is $t \to ||M(t)||^2$. That is, off a set of measure zero, these are both continuous functions of $t$ and so the same is true of $[M]$.

Now put back in $M^{\alpha_p \wedge \delta}$ in place of $M$. From the above, this has shown
\[ ||M^{\alpha_p \wedge \delta}(t)||^2 = [M^{\alpha_p \wedge \delta}](t) + N_p(t) \]
where \( N_p \) is a martingale and

\[
[M^{\alpha_p \wedge \delta}] (t) = \lim_{n \to \infty} \sum_{k \geq 0} \left| |M^{\alpha_p \wedge \delta} (t \wedge \tau^p_{k+1}) - M^{\alpha_p \wedge \delta} (t \wedge \tau^p_k)|^2 \right|
\]

\[
= \lim_{n \to \infty} \sum_{k \geq 0} \left| |M (t \wedge \tau^p_{k+1} \wedge \alpha_p \wedge \delta) - M (t \wedge \tau^p_k \wedge \alpha_p \wedge \delta)|^2 \right| \quad \text{in } L^1(\Omega), \quad (61.2.10)
\]

the convergence being uniform on \([0, T]\). The above formula shows that \([M^{\alpha_p \wedge \delta}] (t)\) is a \( \mathcal{F}_{t \wedge \delta \wedge \alpha_p} \) measurable random variable which depends on \( t \wedge \delta \wedge \alpha_p \). (Note that \( t \wedge \delta \) is a real valued stopping time even if \( \delta = \infty \).) Therefore, by Proposition 61.2.9, there exists a random variable, denoted as \([M^\delta] (t)\) which is the pointwise limit as \( p \to \infty \) of these random variables which is \( \mathcal{F}_{t \wedge \delta} \) measurable because, for a given \( \omega \), when \( \alpha_p \) becomes larger than \( t \), the sum in (61.2.10) loses its dependence on \( p \). Thus from pointwise convergence in (61.2.10)

\[
[M^\delta] (t) \equiv \lim_{n \to \infty} \sum_{k \geq 0} \left| |M (t \wedge \delta \wedge \tau^p_{k+1}) - M (t \wedge \delta \wedge \tau^p_k)|^2 \right|
\]

In case \( \delta = \infty \), the above gives an \( \mathcal{F}_t \) measurable random variable denoted by \([M] (t)\) such that

\[
[M] (t) \equiv \lim_{n \to \infty} \sum_{k \geq 0} \left| |M (t \wedge \tau^p_{k+1}) - M (t \wedge \tau^p_k)|^2 \right|
\]

Now stopping with the stopping time \( \delta \), this shows that

\[
[M^\delta] (t) \equiv \lim_{n \to \infty} \sum_{k \geq 0} \left| |M (t \wedge \delta \wedge \tau^p_{k+1}) - M (t \wedge \delta \wedge \tau^p_k)|^2 \right| = [M]^\delta (t)
\]

That is, the quadratic variation of the stopped local martingale makes sense a.e. and equals the stopped quadratic variation of the local martingale.

This has now shown that

\[
||M^{\alpha_n} (t)||^2 - [M]^{\alpha_n} (t) = ||M^{\alpha_n} (t)||^2 - [M^\alpha_n] (t) = N_n(t) , \quad N_n(t) \text{ a martingale}
\]

and both of the random variables on the left converge pointwise as \( n \to \infty \) to a function which is \( \mathcal{F}_t \) measurable. Hence so does \( N_n(t) \). Of course \( N_n(t) \) is likewise a function of \( \alpha_n \wedge t \) and so by Proposition 61.2.8 again, it converges pointwise to a \( \mathcal{F}_t \) measurable function called \( N(t) \) and \( N(t) \) is a continuous local martingale.

It remains to consider the claim about the uniqueness. Suppose then there are two which work, \([M] \), \([M]_1 \). Then \([M] - [M]_1 \) equals a local martingale \( G \) which is 0 when \( t = 0 \). Thus the uniqueness assertion follows from Corollary 61.2.8.

Here is a corollary which tells how to manipulate stopping times. It is contained in the above proposition, but it is worth emphasizing it from a different point of view.
Corollary 61.2.6 In the situation of Proposition 61.2.5 let \( \tau \) be a stopping time. Then

\[
[M^\tau] = [M]^\tau.
\]

Proof: 

\[
[M^\tau](t) + N_1(t) = \left( ||M||^2 \right)^\tau(t) = ||M^\tau||^2(t) = [M^\tau](t) + N_2(t)
\]

where \( N_i \) is a local martingale. Therefore,

\[
[M^\tau](t) - [M^\tau](t) = N_2(t) - N_1(t),
\]

a local martingale. Therefore, by Corollary 61.2.3 this shows \([M^\tau](t) - [M^\tau](t) = 0\).

\[\blacksquare\]

61.3 The Covariation

Definition 61.3.1 The covariation of two continuous \( H \) valued local martingales for \( H \) a separable Hilbert space \( M, N, M(0) = 0 = N(0) \), is defined as follows.

\[
[M, N] \equiv \frac{1}{4}([M + N] - [M - N])
\]

Lemma 61.3.2 The following hold for the covariation.

\[
[M] = [M, M]
\]

\[
[M, N] = \text{local martingale} + \frac{1}{4} \left( ||M + N||^2 - ||M - N||^2 \right)
\]

\[
= (M, N) + \text{local martingale}.
\]

Proof: From the definition of covariation,

\[
[M] = ||M||^2 - N_1
\]

\[
[M, M] = \frac{1}{4}([M + M] - [M - M]) = \frac{1}{4} \left( ||M + M||^2 - N_2 \right)
\]

\[
= ||M||^2 - \frac{1}{4} N_2
\]

where \( N_i \) is a local martingale. Thus \([M] - [M, M] \) is equal to the difference of two increasing continuous adapted processes and it also equals a local martingale. By Corollary 61.2.3, this process must equal 0. Now consider the second claim.

\[
[M, N] = \frac{1}{4}([M + N] - [M - N]) = \frac{1}{4} \left( ||M + N||^2 - ||M - N||^2 + N \right)
\]

\[
= (M, N) + \frac{1}{4} N
\]

where \( N \) is a local martingale. \[\blacksquare\]
Corollary 61.3.3 Let $M, N$ be two continuous local martingales, $M(0) = N(0) = 0$, as in Proposition 61.2.5. Then $[M, N]$ is of bounded variation and

$$(M, N)_H - [M, N]$$

is a local martingale. Also for $\tau$ a stopping time,

$$[M, N]^\tau = [M^\tau, N^\tau] = [M^\tau, N] = [M, N^\tau].$$

In addition to this,

$$[M - M^\tau] = [M] - [M^\tau] \leq [M]$$

and also

$$(M, N) \rightarrow [M, N]$$

is bilinear and symmetric.

Proof: Since $[M, N]$ is the difference of increasing functions, it is of bounded variation.

$$(M, N)_H - [M, N] = \frac{1}{4} \left( \frac{(M, N)_H}{[M, N]} - \frac{1}{4} (|M + N| - |M - N|) \right)$$

which equals a local martingale from the definition of $[M + N]$ and $[M - N]$. It remains to verify the claim about the stopping time. Using Corollary

$$[M, N]^\tau = \frac{1}{4} (|M + N| - [M - N])$$

$$= \frac{1}{4} ([M + N]^\tau - [M - N]^\tau)$$

$$= \frac{1}{4} ([M^\tau + N^\tau] - [M^\tau - N^\tau]) = [M^\tau, N^\tau].$$

The really interesting part is the next equality. This will involve Corollary

$$[M, N]^\tau - [M^\tau, N] = [M^\tau, N^\tau] - [M^\tau, N]$$

$$\equiv \frac{1}{4} ([M^\tau + N^\tau] - [M^\tau - N^\tau]) - \frac{1}{4} ([M^\tau + N] - [M^\tau - N])$$

$$= \frac{1}{4} ([M^\tau + N^\tau] + [M^\tau - N]) - \frac{1}{4} ([M^\tau + N] + [M^\tau - N^\tau]), (61.3.11)$$

the difference of two increasing adapted processes. Also, this equals

local martingale $- (M^\tau, N) + (M^\tau, N^\tau)$
Claim: \((M^\sigma, N) - (M^\sigma, N^\tau) = (M^\sigma, N - N^\tau)\) is a local martingale. Let \(\sigma_n\) be a localizing sequence for both \(M\) and \(M\). Such a localizing sequence is of the form \(\tau^M_n \wedge \tau^N_n\) where these are localizing sequences for the indicated local submartingale. Then obviously,

\[-(M^\sigma, N) + (M^\sigma, N^\tau) = -(M^\sigma_{\tau^M_n}, N^\sigma_{\tau^N_n}) + (M^\sigma_{\tau^M_n} \wedge \tau^N_n, N^\sigma_{\tau^N_n})\]

where \(N^\sigma_{\tau^N_n}\) and \(M^\sigma_{\tau^M_n}\) are martingales. To save notation, denote these by \(M\) and \(N\) respectively. Now use Lemma 61.1.1. Let \(\sigma\) be a stopping time with two values. 

\[E((M^\sigma (\sigma), N (\sigma) - N^\tau (\sigma))) = E(E((M^\tau (\sigma), N (\sigma) - N^\tau (\sigma))|\mathcal{F}_\tau))\]

Now \(M^\tau (\sigma)\) is \(M (\sigma \wedge \tau)\) which is \(\mathcal{F}_\tau\) measurable and so by the Doob optional sampling theorem,

\[= E(M^\tau (\sigma), E(N (\sigma) - N^\tau (\sigma)|\mathcal{F}_\tau))\]

while

\[E((M^\tau (t), N (t) - N^\tau (t))) = E(E((M^\tau (t), N (t) - N^\tau (t))|\mathcal{F}_\tau))\]

Since \(M^\tau (t)\) is \(\mathcal{F}_\tau\) measurable,

\[= E((M^\tau (t), E(N (t) - N^\tau (t)|\mathcal{F}_\tau)))\]

\[= E((M^\tau (t), E(N (t \wedge \tau) - N (t \wedge \tau))) = 0\]

This shows the claim is true.

Now from \(61.3.11\) and Corollary \(61.3.3\),

\([M, N]^\tau - [M^\tau, N] = 0.\)

Similarly

\([M, N]^\tau - [M, N^\tau] = 0\)

Now consider the next claim that \([M - M^\tau] = [M] - [M^\tau]\). From the definition, it follows

\([M - M^\tau] - ([M] + [M^\tau] - 2[M, M^\tau])\]

\[= ||M - M^\tau||^2 - \left(||M||^2 + ||M^\tau||^2 - 2(M, M^\tau)\right) + \text{local martingale}\]

By the first part of the corollary which ensures \([M, M^\tau]\) is of bounded variation, the left side is the difference of two increasing adapted processes and so by Corollary \(61.2.3\) again, the left side equals 0. Thus from the above,

\([M - M^\tau] = [M] + [M^\tau] - 2[M, M^\tau]\]
\[= [M] + [M^\tau] - 2[M^\tau, M^\tau]\]
\[= [M] + [M^\tau] - 2[M]\]
\[= [M] - [M^\tau] \leq [M]\]
Finally consider the claim that \([M, N]\) is bilinear. From the definition, letting \(M_1, M_2, N\) be \(H\) valued local martingales,

\[
(aM_1 + bM_2, N)_H = [aM_1 + bM_2, N] + \text{local martingale}
\]

\[
a(M_1, N) + b(M_2, N)_H = a[M_1, N] + b[M_2, N] + \text{local martingale}
\]

Hence

\[
[aM_1 + bM_2, N] - (a[M_1, N] + b[M_2, N]) = \text{local martingale}.
\]

The left side can be written as the difference of two increasing functions thanks to \([M, N]\) of bounded variation and so by Lemma 61.1.5 it equals 0. \([M, N]\) is obviously symmetric from the definition.

### 61.4 The Burkholder Davis Gundy Inequality

Define

\[
M^* (\omega) \equiv \sup \{||M (t) (\omega)|| : t \in [0, T]\}.
\]

The Burkholder Davis Gundy inequality is an amazing inequality which involves \(M^*\) and \(|M| (T)\).

Before presenting this, here is the good lambda inequality, Theorem 10.7.1 on Page 281 listed here for convenience.

**Theorem 61.4.1** Let \((\Omega, F, \mu)\) be a finite measure space and let \(F\) be a continuous increasing function defined on \([0, \infty)\) such that \(F(0) = 0\). Suppose also that for all \(\alpha > 1\), there exists a constant \(C_\alpha\) such that for all \(x \in [0, \infty)\),

\[
F(\alpha x) \leq C_\alpha F(x).
\]

Also suppose \(f, g\) are nonnegative measurable functions and there exists \(\beta > 1, 0 < r \leq 1, \text{ such that for all } \lambda > 0\) and \(1 > \delta > 0\),

\[
\mu ([f > \beta \lambda] \cap [g \leq r \delta \lambda]) \leq \phi (\delta) \mu ([f > \lambda]) \quad (61.4.12)
\]

where \(\lim_{\delta \to 0^+} \phi (\delta) = 0\) and \(\phi\) is increasing. Under these conditions, there exists a constant \(C\) depending only on \(\beta, \phi, r\) such that

\[
\int_{\Omega} F(f(\omega)) d\mu(\omega) \leq C \int_{\Omega} F(g(\omega)) d\mu(\omega).
\]

The proof of this important inequality also will depend on the hitting this before that theorem which is listed next for convenience.

**Theorem 61.4.2** Let \(\{M(t)\}\) be a continuous real valued martingale adapted to the normal filtration \(\mathcal{F}_t\) and let

\[
M^* \equiv \sup \{|M(t) : t \geq 0\}
\]
and $M(0) = 0$. Letting 
\[ \tau_x \equiv \inf \{ t > 0 : M(t) = x \} \]
Then if $a < 0 < b$ the following inequalities hold.
\[
(b - a) P(\lfloor \tau_b \leq \tau_a \rfloor) \geq -a P(\lfloor M^* > 0 \rfloor) \geq (b - a) P(\lfloor \tau_b < \tau_a \rfloor)
\]
and
\[
(b - a) P(\lfloor \tau_a < \tau_b \rfloor) \leq b P(\lfloor M^* > 0 \rfloor) \leq (b - a) P(\lfloor \tau_a \leq \tau_b \rfloor).
\]
In words, $P(\lfloor \tau_b \leq \tau_a \rfloor)$ is the probability that $M(t)$ hits $b$ no later than when it hits $a$. (Note that if $\tau_a = \infty = \tau_b$ then you would have $\lfloor \tau_a = \tau_b \rfloor$.

Then the Burkholder Davis Gundy inequality is as follows. Generalizations will be presented later.

**Theorem 61.4.3** Let $\{M(t)\}$ be a continuous $H$ valued martingale which is uniformly bounded, $M(0) = 0$, where $H$ is a separable Hilbert space and $t \in [0, T]$. Then if $F$ is a function of the sort described in the good lambda inequality above, there are constants, $C$ and $c$ independent of such martingales $M$ such that
\[
c \int \Omega F(\lfloor [M(T)]^{1/2} \rfloor) dP \leq \int \Omega F(M^*) dP \leq C \int \Omega F(\lfloor [M(T)]^{1/2} \rfloor) dP
\]
where
\[ M^*(\omega) \equiv \sup \{ ||M(t)(\omega)|| : t \in [0, T] \} \]

**Proof:** Using Corollary let
\[ N(t) \equiv ||M(t) - M^*(t)||^2 - [M - M^*](t) \]
\[ = ||M(t) - M^*(t)||^2 - [M](t) + [M^*](t) \]
where
\[ \tau \equiv \inf \{ t \in [0, T] : ||M(t)|| > \lambda \} \]
Thus $N$ is a martingale and $N(0) = 0$. In fact $N(t) = 0$ as long as $t \leq \tau$. As usual $\inf(\emptyset) \equiv \infty$. Note
\[ [\tau < \infty] = [M^* > \lambda] \supseteq [N^* > 0]. \]
This is because to say $\tau < \infty$ is to say there exists $t < T$ such that $||M(t)|| > \lambda$ which is the same as saying $M^* > \lambda$. Thus the first two sets are equal. If $\tau = \infty$, then from the formula for $N(t)$ above, $N(t) = 0$ for all $t \in [0, T]$ and so it can’t happen that $N^* > 0$. Thus the third set is contained in $[\tau < \infty]$ as claimed.

Let $\beta > 2$ and let $\delta \in (0, 1)$. Then
\[ \beta - 1 > 1 > \delta > 0 \]
Consider the following which is set up to use the good lambda inequality.
\[ S_r \equiv [M^* > \beta\lambda] \cap \lfloor ([M(T)]^{1/2} \leq r\delta \lambda \rfloor] \]
where $0 < r < 1$. It is shown that $S_r$ corresponds to hitting “this before that” and there is an estimate for this which involves $P(\{N^* > 0\})$ which is bounded above by $P(\{M^* > \lambda\})$ as discussed above. This will satisfy the hypotheses of the good lambda inequality.

Claim: For $\omega \in S_r$, $N(t)$ hits $\lambda^2 \left(1 - \delta^2\right)$.

Proof of claim: For $\omega \in S_r$, there exists a $t < T$ such that $\|M(t)\| > \beta \lambda$ and so using Corollary 61.3.3,

\[
N(t) \geq \|M(t)\| - \|M^*(t)\|^2 - [M - M^*](t) \geq |\beta \lambda - \lambda|^2 - \|M(t)\|
\]

which shows that $N(t)$ hits $(\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2$ for $\omega \in S_r$. By the intermediate value theorem, it also hits $\lambda^2 (1 - \delta^2)$. This proves the claim.

Claim: $N(t)(\omega)$ never hits $-\delta^2 \lambda^2$ for $\omega \in S_r$.

Proof of claim: Suppose $t$ is the first time $N(t)$ reaches $-\delta^2 \lambda^2$. Then $t > \tau$ and so

\[
N(t) = -\delta^2 \lambda^2 \geq \|M(t)\| - \lambda^2 - [M(t) + M^*(t)]\]

\[
\geq -r^2 \lambda^2 \delta^2,
\]

a contradiction since $r < 1$. This proves the claim.

Therefore, for all $\omega \in S_r$, $N(t)(\omega)$ reaches $\lambda^2 \left(1 - \delta^2\right)$ before it reaches $-\delta^2 \lambda^2$. It follows

\[
P(S_r) \leq P(N(t) \text{ reaches } \lambda^2 \left(1 - \delta^2\right) \text{ before } -\delta^2 \lambda^2)
\]

and because of Theorem 61.3.3 this is no larger than

\[
P(\{N^* > 0\}) \frac{\delta^2 \lambda^2}{\lambda^2 (1 - \delta^2) - (-\delta^2 \lambda^2)} = P(\{N^* > 0\}) \delta^2 \leq \delta^2 P(\{M^* > \lambda\})
\]

Thus

\[
P(\{M^* > \beta \lambda \cap \left(\left([M(T)]^{1/2} \leq r \delta \lambda\right)\right) \leq P(\{M^* > \lambda\}) \delta^2
\]

By the good lambda inequality,

\[
\int_\Omega F(M^*) dP \leq C \int_\Omega F\left(\left([M(T)]^{1/2}\right)\right) dP
\]

which is one half the inequality.

Now consider the other half. This time define the stopping time $\tau$ by

\[
\tau \equiv \inf \left\{ t \in [0, T] : ([M(t)]^{1/2} > \lambda) \right\}
\]

and let

\[
S_r \equiv \left(\left([M(T)]^{1/2} > \beta \lambda\right) \cap \left(2M^* \leq r \delta \lambda\right)\right).
\]

Then there exists $t < T$ such that $[M(t)]^{1/2} > \beta^2 \lambda^2$. This time, let

\[
N(t) \equiv [M(t)] - [M^*(t)] - [M(t) - M^*(t)]^{1/2}
\]
CHAPTER 61. THE QUADRATIC VARIATION OF A MARTINGALE

This is still a martingale since by Corollary 61.3.3

\[ [M](t) - [M^\tau](t) = [M - M^\tau](t) \]

Claim: \( N(t)(\omega) \) hits \( \lambda^2 \left(1 - \delta^2\right) \) for some \( t < T \) for \( \omega \in S_r \).

Proof of claim: Fix such a \( \omega \in S_r \). Let \( t < T \) be such that \( [M](t) > \beta^2 \lambda^2 \).

Then \( t > \tau \) and so for that \( \omega \),

\[ N(t) > \beta^2 \lambda^2 - \lambda^2 - ||M(t) - M(\tau)||^2 \]
\[ \geq (\beta - 1)^2 \lambda^2 - (||M(t)|| + ||M(\tau)||)^2 \]
\[ \geq (\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2 \geq \lambda^2 - \delta^2 \lambda^2 \]

By the intermediate value theorem, it hits \( \lambda^2 \left(1 - \delta^2\right) \). This proves the claim.

Claim: \( N(t)(\omega) \) never hits \( -\delta^2 \lambda^2 \) for \( \omega \in S_r \).

Proof of claim: By Corollary 61.3.3, if it did at \( t \), then \( t > \tau \) because \( N(t) = 0 \) for \( t \leq \tau \), and so

\[ 0 \leq [M](t) - [M^\tau](t) = ||M(t) - M(\tau)||^2 - \delta^2 \lambda^2 \]
\[ \leq (||M(t)|| + ||M(\tau)||)^2 - \delta^2 \lambda^2 \leq r^2 \delta^2 \lambda^2 - \delta^2 \lambda^2 < 0, \]

a contradiction. This proves the claim.

It follows that for each \( r \in (0, 1) \),

\[ P(S_r) \leq P( N(t) \text{ hits } \lambda^2 \left(1 - \delta^2\right) \text{ before } -\delta^2 \lambda^2) \]

By Theorem 60.11.3 this is no larger than

\[ P\left(\left\{ N^* > 0 \right\}\right) \frac{\delta^2 \lambda^2}{\lambda^2 \left(1 - \delta^2\right) + \delta^2 \lambda^2} = P\left(\left\{ N^* > 0 \right\}\right) \delta^2 \]
\[ \leq P\left(\left\{ \tau < \infty \right\}\right) \delta^2 = P\left(\left(\left\{ [M](T)\right\}^{1/2} > \lambda \right)\right) \delta^2 \]

Now by the good lambda inequality, there is a constant \( k \) independent of \( M \) such that

\[ \int_{\Omega} F\left(\left\{ [M](T)\right\}^{1/2}\right) dP \leq k \int_{\Omega} F\left(2M^*\right) dP \leq kC_2 \int_{\Omega} F\left(M^*\right) dP \]

by the assumptions about \( F \). Therefore, combining this result with the first part,

\[ (kC_2)^{-1} \int_{\Omega} F\left(\left\{ [M](T)\right\}^{1/2}\right) dP \leq \int_{\Omega} F\left(M^*\right) dP \leq C \int_{\Omega} F\left(\left\{ [M](T)\right\}^{1/2}\right) dP \]

Of course, everything holds for local martingales in place of martingales.
Theorem 61.4.4 Let \( \{M(t)\} \) be a continuous \( H \) valued local martingale, \( M(0) = 0 \), where \( H \) is a separable Hilbert space and \( t \in [0, T] \). Then if \( F \) is a function of the sort described in the good lambda inequality, that is,

\[
F(0) = 0, \ F \text{ continuous, } F \text{ increasing,}
\]

there are constants, \( C \) and \( c \) independent of such local martingales \( M \) such that

\[
c \int_{\Omega} F \left( \left\lfloor M \right\rfloor (T)^{1/2} \right) dP \leq \int_{\Omega} F (M^*) dP \leq C \int_{\Omega} F \left( \left\lfloor M \right\rfloor (T)^{1/2} \right) dP
\]

where

\[
M^* (\omega) \equiv \sup \{||M(t)(\omega)|| : t \in [0, T]\}.
\]

**Proof:** Let \( \{\tau_n\} \) be an increasing localizing sequence for \( M \) such that \( M^{\tau_n} \) is uniformly bounded. Such a localizing sequence exists from Proposition 61.2.2. Then from Theorem 61.4.3 there exist constants \( c, C \) independent of \( \tau_n \) such that

\[
c \int_{\Omega} F \left( \left\lfloor M^{\tau_n} \right\rfloor (T)^{1/2} \right) dP \leq \int_{\Omega} F (M^{\tau_n})^* dP \leq C \int_{\Omega} F \left( \left\lfloor M^{\tau_n} \right\rfloor (T)^{1/2} \right) dP
\]

By Corollary 61.3.3, this implies

\[
c \int_{\Omega} F \left( \left\lfloor M^{\tau_n} \right\rfloor (T)^{1/2} \right) dP \leq \int_{\Omega} F (M^{\tau_n})^* dP \leq C \int_{\Omega} F \left( \left\lfloor M^{\tau_n} \right\rfloor (T)^{1/2} \right) dP
\]

and now note that \( \left\lfloor M^{\tau_n} \right\rfloor (T)^{1/2} \) and \( (M^{\tau_n})^* \) increase in \( n \) to \( \lfloor M \rfloor (T)^{1/2} \) and \( M^* \) respectively. Then the result follows from the monotone convergence theorem. 

Here is a corollary.

**Corollary 61.4.5** Let \( \{M(t)\} \) be a continuous \( H \) valued local martingale and let \( \varepsilon, \delta \in (0, \infty) \). Then there is a constant \( C \), independent of \( \varepsilon, \delta \) such that

\[
P \left( \sup_{t \in [0, T]} ||M(t)|| \geq \varepsilon \right) \leq \frac{C}{\varepsilon} E \left( \lfloor M \rfloor^{1/2} (T) \wedge \delta \right) + P \left( \lfloor M \rfloor^{1/2} (T) > \delta \right)
\]

**Proof:** Let the stopping time \( \tau \) be defined by

\[
\tau \equiv \inf \{t > 0 : \lfloor M \rfloor^{1/2} (t) > \delta\}
\]
Then

\[ P ([M^* \geq \varepsilon]) = P ([M^* \geq \varepsilon] \cap [\tau = \infty]) + P ([M^* \geq \varepsilon] \cap [\tau < \infty]) \]

On the set where \([\tau = \infty]\), \(M^\tau = M\) and so \(P ([M^* \geq \varepsilon]) \leq \frac{1}{\varepsilon} \int_\Omega (M^*)^* dP + P ([M^* \geq \varepsilon] \cap [M^{1/2}(T) > \delta])\)

By Theorem 61.4.4 and Corollary 61.3.3,

\[ \leq C \varepsilon \int_\Omega [M^T]^{1/2}(T) dP + P ([M^* \geq \varepsilon] \cap [M^{1/2}(T) > \delta]) \]

\[ = C \varepsilon \int_\Omega [M^T]^{1/2}(T) dP + P ([M^* \geq \varepsilon] \cap [M^{1/2}(T) > \delta]) \]

\[ \leq C \varepsilon \int_\Omega [M^{1/2}(T) \wedge \delta] dP + P ([M^* \geq \varepsilon] \cap [M^{1/2}(T) > \delta]) \]

\[ \leq C \varepsilon \int_\Omega [M^{1/2}(T) \wedge \delta] dP + P ([M^{1/2}(T) > \delta]) \]

The Burkholder Davis Gundy inequality along with the properties of the covariation implies the following amazing proposition.

**Proposition 61.4.6** The space \(M^2_T (H)\) is a Hilbert space. Here \(H\) is a separable Hilbert space.

**Proof:** We already know from Proposition 61.12.2 that this space is a Banach space. It is only necessary to exhibit an equivalent norm which makes it a Hilbert space. However, you can let \(F (\lambda) = \lambda^2\) in the Burkholder Davis Gundy theorem and obtain for \(M \in M^2_T (H)\), the two norms

\[ \left( \int_\Omega [M](T) dP \right)^{1/2} = \left( \int_\Omega [M, M](T) dP \right)^{1/2} \]

and

\[ \left( \int_\Omega (M^*)^2 dP \right)^{1/2} \]

are equivalent. The first comes from an inner product since from Corollary 61.4.3, \([, \cdot, \cdot]\) is bilinear and symmetric and nonnegative. If \([M, M](T) = [M](T) = 0\) in \(L^1 (\Omega)\), then from the Burkholder Davis Gundy inequality, \(M^* = 0\) in \(L^2 (\Omega)\) and so \(M = 0\). Hence

\[ \int_\Omega [M, N](T) dP \]

is an inner product which yields the equivalent norm. □
Example 61.4.7 An example of a real martingale is the Wiener process, \( W(t) \). It has the property that whenever \( t_1 < t_2 < \cdots < t_n \), the increments \( \{ W(t_i) - W(t_{i-1}) \} \) are independent and whenever \( s < t \), \( W(t) - W(s) \) is normally distributed with mean 0 and variance \( t - s \). For the Wiener process, we let

\[
F_t \equiv \cap_{u > t} \sigma \{ W(s) - W(r) : r < s \leq u \}
\]

and it is with respect to this normal filtration that \( W \) is a continuous martingale.

What is the quadratic variation of such a process?

The quadratic variation of the Wiener process is just \( t \). This is because if \( A \in F_s, s < t \),

\[
E \left( \mathcal{X}_A \left( |W(t)|^2 - t \right) \right) =
E \left( \mathcal{X}_A \left( |W(t) - W(s)|^2 + |W(s)|^2 + 2 \langle W(s), W(t) - W(s) \rangle - (t - s + s) \right) \right)
\]

Now

\[
E \left( \mathcal{X}_A \left( 2 \langle W(s), W(t) - W(s) \rangle \right) \right) = P(A) E(2W(s)) E(W(t) - W(s)) = 0
\]

by the independence of the increments. Thus the above reduces to

\[
E \left( \mathcal{X}_A \left( |W(t) - W(s)|^2 + |W(s)|^2 - (t - s + s) \right) \right)
\]

\[
= E \left( \mathcal{X}_A \left( |W(t) - W(s)|^2 - (t - s) \right) \right) + E \left( \mathcal{X}_A \left( |W(s)|^2 - s \right) \right)
\]

\[
= P(A) E \left( |W(t) - W(s)|^2 - (t - s) \right) + E \left( \mathcal{X}_A \left( |W(s)|^2 - s \right) \right)
\]

\[
= E \left( \mathcal{X}_A \left( |W(s)|^2 - s \right) \right)
\]

and so \( E \left( |W(t)|^2 - t \right| F_s \right) = |W(s)|^2 - s \) showing that \( t \rightarrow |W(t)|^2 - t \) is a martingale. Hence, by uniqueness, \( [W](t) = t \).

61.5 The Quadratic Variation And Stochastic Integration

Let \( F_t \) be a normal filtration and let \( \{M(t)\} \) be a continuous local martingale adapted to \( F_t \) having values in \( U \) a separable real Hilbert space.

Definition 61.5.1 Let \( F_t \) be a normal filtration and let

\[
f(t) = \sum_{k=0}^{n-1} f_k \mathcal{X}_{(t_k, t_{k+1}]} (t)
\]
where \( \{ t_k \}_{k=0}^n \) is a partition of \([0,T]\) and each \( f_k \) is \( \mathcal{F}_{t_k} \) measurable, \( f_k M^* \in L^2(\Omega) \).

\[
M^*(\omega) = \sup_{t \in [0,T]} ||M(t)(\omega)||
\]

Such a function is called an elementary function. Also let \( \{ M(t) \} \) be a local martingale adapted to \( \mathcal{F}_t \) which has values in a separable real Hilbert space \( U \) such that \( M(0) = 0 \). For such an elementary real valued function define

\[
\int_0^t f dM = \sum_{k=0}^{n-1} f_k (M(t \wedge t_{k+1}) - M(t \wedge t_k)).
\]

Then with this definition, here is a wonderful lemma.

**Lemma 61.5.2** For \( f \) an elementary function as above, \( \left\{ \int_0^t f dM \right\} \) is a continuous local martingale and

\[
E \left( \left\| \int_0^t f dM \right\|^2_U \right) = \int_\Omega \int_0^t f(s)^2 d[M](s) dP. \tag{61.5.13}
\]

If \( N \) is another continuous local martingale adapted to \( \mathcal{F}_t \) and both \( f, g \) are elementary functions such that for each \( k \),

\[
f_k M^*, g_k N^* \in L^2(\Omega),
\]

then

\[
E \left( \left( \int_0^t f dM, \int_0^t g dN \right)_U \right) = \int_\Omega \int_0^t f g [M,N] dP. \tag{61.5.14}
\]

and both sides make sense.

**Proof:** Let \( \{ \tau_l \} \) be a localizing sequence for \( M \) such that \( M^{\tau_l} \) is a bounded martingale. Then from the definition, for each \( \omega \)

\[
\int_0^t f dM = \lim_{l \to \infty} \int_0^t f dM^{\tau_l} = \lim_{l \to \infty} \left( \int_0^t f dM \right)^{\tau_l}
\]

and it is clear that \( \left\{ \int_0^t f dM^{\tau_l} \right\} \) is a martingale because it is just the sum of some martingales. Thus \( \{ \tau_l \} \) is a localizing sequence for \( \int_0^t f dM \). It is also clear \( \int_0^t f dM \) is continuous because it is a finite sum of continuous random variables.

Next consider the formula which is really a version of the Ito isometry. There is no loss of generality in assuming the mesh points are the same for the two elementary functions because if not, one can simply add in points to make this happen. It suffices to consider \( \int_0^t f dM^{\tau_l} \) because the other formula is a special case. To begin with, let \( \{ \tau_l \} \) be a localizing sequence which makes both \( M^{\tau_l} \) and \( N^{\tau_l} \) into bounded martingales. Consider the stopped process.

\[
E \left( \left( \int_0^t f dM^{\tau_l}, \int_0^t g dN^{\tau_l} \right)_U \right)
\]
where

\[ \sum_{k=0}^{n-1} f_k (M^{\tau_i} (t \land t_{k+1}) - M^{\tau_i} (t \land t_k)) \]

\[ + \sum_{k=0}^{n-1} g_k (N^{\tau_i} (t \land t_{k+1}) - N^{\tau_i} (t \land t_k)) \]

To save on notation, write \( M^{\tau_i} (t \land t_{k+1}) - M^{\tau_i} (t \land t_k) \equiv \Delta M_k (t) \), similar for \( \Delta N_k \). Thus

\[ \Delta M_k = M^{\tau_i \land t_{k+1}} - M^{\tau_i \land t_k}, \]

similar for \( \Delta N_k \). Then the above equals

\[ E \left( \sum_{k=0}^{n-1} \left( f_k \Delta M_k, \sum_{k=0}^{n-1} g_k \Delta N_k \right) \right) = E \left( \sum_{k,j} f_k g_j (\Delta M_k, \Delta N_j) \right) \]

Now consider one of the mixed terms with \( j < k \).

\[ E ((f_k \Delta M_k, g_j \Delta N_j)) = E (E ((f_k \Delta M_k, g_j \Delta N_j) | \mathcal{F}_{t_k})) \]

\[ = E (g_j \Delta N_j, f_k E (\Delta M_k | \mathcal{F}_{t_k})) = 0 \]

since \( E (\Delta M_k | \mathcal{F}_{t_k}) = E ((M^{\tau_i} (t \land t_{k+1}) - M^{\tau_i} (t \land t_k)) | \mathcal{F}_{t_k}) = 0 \) by the Doob optional sampling theorem. Thus

\[ E \left( \left( \int_0^t f dM^{\tau_i}, \int_0^t g dN^{\tau_i} \right) \right) = \]

\[ = \sum_{k=0}^{n-1} E (f_k g_k (\Delta M_k, \Delta N_k)) = \sum_{k=0}^{n-1} E (f_k g_k ([\Delta M_k + \Delta N_k] + \Delta N_k) \]

(61.5.15)

where \( N_k \) is a martingale such that \( N_k (t) = 0 \) for all \( t \leq t_k \). This is because the martingale \( (N^{\tau_i})^{t_{k+1}} - (N^{\tau_i})^{t_k} = \Delta N_k \) equals 0 for such \( t \); and so \( E (N_k (t)) = 0 \). Thus \( f_k g_k N_k \) is a martingale which equals zero when \( t = 0 \). Therefore, its expectation also equals 0. Consequently the above reduces to

\[ = \sum_{k=0}^{n-1} E (f_k g_k [\Delta M_k, \Delta N_k]) \]

At this point, recall the definition of the covariation. The above equals

\[ \frac{1}{4} \sum_{k=0}^{n-1} E (f_k g_k ([\Delta M_k + \Delta N_k] - [\Delta M_k - \Delta N_k])) \]

Rewriting this yields

\[ = \frac{1}{4} \sum_{k=0}^{n-1} E \left( f_k g_k \left( \left( M^{\tau_i})^{t_{k+1}} + (N^{\tau_i})^{t_{k+1}} - \left( (M^{\tau_i})^{t_k} + (N^{\tau_i})^{t_k} \right) \right) \right. \]

\[ \left. - \left( (M^{\tau_i})^{t_{k+1}} - (N^{\tau_i})^{t_{k+1}} - \left( (M^{\tau_i})^{t_k} - (N^{\tau_i})^{t_k} \right) \right) \right) \]
To save on notation, denote

\[(M^{\tau_l})^{t_k+1} + (N^{\tau_l})^{t_k+1} - (M^{\tau_l})^{t_k} - (N^{\tau_l})^{t_k}\]  
\[(M^{\tau_l})^{t_k+1} - (N^{\tau_l})^{t_k+1} - (M^{\tau_l})^{t_k} + (N^{\tau_l})^{t_k}\]  
\[= \Delta_k (M^{\tau_l} + N^{\tau_l})\]  
\[= \Delta_k (M^{\tau_l} - N^{\tau_l})\]

Thus the above equals

\[\frac{1}{4} \sum_{k=0}^{n-1} E (f_k g_k ([\Delta_k (M^{\tau_l} + N^{\tau_l})] - [\Delta_k (M^{\tau_l} - N^{\tau_l})]))\]

Now from Corollary 61.3.3,

\[= \frac{1}{4} \sum_{k=0}^{n-1} E (f_k g_k ([M + N]^{\tau_l} - [M - N]^{\tau_l}))\]

Letting \(l \to \infty\), this reduces to

\[= \frac{1}{4} \sum_{k=0}^{n-1} E (f_k g_k ([M + N] - [M - N]))\]

\[= \frac{1}{4} \left( \int_0^t \int_0^t f g (d [M + N] - d [M - N]) \right)\]

\[= \int_0^t \int_0^t f g d[M,N]\]

Now consider the left side of 61.5.16.

\[E \left( \left( \int_0^t f dM^{\tau_l}, \int_0^t g dN^{\tau_l} \right)_{U} \right)\]

\[= \int_{\Omega} \sum_{k,j} f_k g_j ((M^{\tau_l} (t \wedge t_k+1) - M^{\tau_l} (t \wedge t_k)), (N^{\tau_l} (t \wedge t_{j+1}) - N^{\tau_l} (t \wedge t_j))) dP\]

Then for each \(\omega\), the integrand converges as \(l \to \infty\) to

\[\sum_{k,j} f_k g_j ((M (t \wedge t_k+1) - M (t \wedge t_k)), (N (t \wedge t_{j+1}) - N (t \wedge t_j)))\]

But also you can do a sloppy estimate which will allow the use of the dominated convergence theorem.

\[\left\| \sum_{k,j} f_k g_j (M^{\tau_l} (t \wedge t_k+1) - M^{\tau_l} (t \wedge t_k)), (N^{\tau_l} (t \wedge t_{j+1}) - N^{\tau_l} (t \wedge t_j)) \right\|\]
by assumption. Thus the left side of \(61.5.16\) converges as \(l \to \infty\) to
\[
\int_\Omega \sum_{k,j} f_k g_j \left( (M(t \wedge t_{k+1}) - M(t \wedge t_k)) , (N(t \wedge t_{j+1}) - N(t \wedge t_j)) \right) dP
\]
\[
= \int_\Omega \left( \int_0^t f dM, \int_0^t g dN \right) U \ dP \quad \Box
\]

Note for each \(\omega\), the inside integral in \(61.5.13\) is just a Stieltjes integral taken with respect to the increasing integrating function \([M]\).

Of course, with this estimate it is obvious how to extend the integral to a larger class of functions.

**Definition 61.5.3** Let \(\nu(\omega)\) denote the Radon measure representing the functional
\[
\Lambda(\omega)(g) = \int_0^T g \ [M](t)(\omega)
\]
\((t \to [M](t)(\omega)\) is a continuous increasing function and \(\nu(\omega)\) is the measure representing the Stieltjes integral, one for each \(\omega\).) Then let \(G_M\) denote functions \(f(s,\omega)\) which are the limit of such elementary functions in the space \(L^2(\Omega; L^2([0,T], \nu(\omega)))\), the norm of such functions being
\[
||f||_G^2 = \int_\Omega \int_0^T f(s)^2 \ [M](s) \ dP
\]
For \(f \in G\) just defined,
\[
\int_0^t f dM \equiv \lim_{n \to \infty} \int_0^t f_n dM
\]
where \(\{f_n\}\) is a sequence of elementary functions converging to \(f\) in
\[
L^2(\Omega; L^2([0,T], \nu(\omega))).
\]

Now here is an interesting lemma.

**Lemma 61.5.4** Let \(M, N\) be continuous local martingales, \(M(0) = N(0) = 0\) having values in a separable Hilbert space, \(U\). Then
\[
[M + N]^{1/2} \leq \left( |M|^{1/2} + [N]^{1/2} \right)
\]  
(61.5.17)
\[
|M + N| \leq 2 (|M| + [N])
\]  
(61.5.18)
Also, letting \(\nu_{M+N}\) denote the measure obtained from the increasing function \([M+N]\) and \(\nu_N, \nu_M\) defined similarly,
\[
\nu_{M+N} \leq 2 (\nu_M + \nu_N)
\]  
(61.5.19)
on all Borel sets.
**Proof:** Since \((M, N) \rightarrow [M, N]\) is bilinear and satisfies

\[
[M, N] = [N, M]
\]
\[
aM + bM_1, N = a[M, N] + b[M_1, N]
\]
\[
[M, M] \geq 0
\]

which follows from Corollary 61.3.3, the usual Cauchy Schwarz inequality holds and so

\[
|[M, N]| \leq [M]^{1/2} [N]^{1/2}
\]

Thus

\[
[M + N] = [M + N, M + N] = [M, M] + [N, N] + 2 [M, N]
\]
\[
\leq [M] + [N] + 2 [M]^{1/2} [N]^{1/2} = \left([M]^{1/2} + [N]^{1/2}\right)^2
\]

This proves 61.5.17. Now square both sides. Then the right side is no larger than

\[
2 ([M] + [N])
\]

and this shows 61.5.18.

Now consider the claim about the measures. It was just shown that

\[
[(M + N) - (M + N)^*] \leq 2 ([M - M^*] + [N - N^*])
\]

and from Corollary 61.3.3 this implies that for \(t > s\)

\[
[M + N] (t) - [M + N] (s \wedge t)
\]
\[
= [M + N] (t) - [M + N]^* (t)
\]
\[
= [M + N - (M^* + N^*]) (t)
\]
\[
= [M - M^* + (N - N^*]) (t)
\]
\[
\leq 2 [M - M^*] (t) + 2 [N - N^*] (t)
\]
\[
\leq 2 ([M] (t) - [M] (s)) + 2 ([N] (t) - [N] (s))
\]

Thus

\[
\nu_{M+N} ([s, t]) \leq 2 (\nu_M ([s, t]) + \nu_N ([s, t]))
\]

By regularity of the measures, this continues to hold with any Borel set \(F\) in place of \([s, t]\). 

**Theorem 61.5.5** The integral is well defined and has a continuous version which is a local martingale. Furthermore it satisfies the Itô isometry,

\[
E \left( \left\| \int_0^t f dM \right\|_V^2 \right) = \int_0^t \int f \, d[M] (s) \, dP
\]
Let the norm on $\mathcal{G}_N \cap \mathcal{G}_M$ be the maximum of the norms on $\mathcal{G}_N$ and $\mathcal{G}_M$ and denote by $\mathcal{E}_N$ and $\mathcal{E}_M$ the elementary functions corresponding to the martingales $N$ and $M$ respectively. Define $\mathcal{G}_{NM}$ as the closure in $\mathcal{G}_N \cap \mathcal{G}_M$ of $\mathcal{E}_N \cap \mathcal{E}_M$. Then for $f, g \in \mathcal{G}_{NM}$,

$$E \left( \left( \int_0^t f \, dM, \int_0^t g \, dN \right) \right) = \int_\Omega \int_0^t f g \, d[M,N] \quad (61.5.20)$$

**Proof:** It is clear the definition is well defined because if $\{f_n\}$ and $\{g_n\}$ are two sequences of elementary functions converging to $f$ in $L^2(\Omega; L^2([0,T], \nu(\cdot)))$ and if $\int_0^t f \, dM$ is the integral which comes from $\{g_n\}$,

$$\int_\Omega \left\| \int_0^t f \, dM - \int_0^t g \, dM \right\|^2 \, dP$$

$$= \lim_{n \to \infty} \int_\Omega \left\| \int_0^t g_n \, dM - \int_0^t f_n \, dM \right\|^2 \, dP$$

$$\leq \lim_{n \to \infty} \int_\Omega \int_0^T \left\| g_n - f_n \right\|^2 \, d\nu \, dP = 0.$$

Consider the claim the integral has a continuous version. Recall Theorem 61.5.6, part of which is listed here for convenience.

**Theorem 61.5.6** Let $\{X(t)\}$ be a right continuous nonnegative submartingale adapted to the normal filtration $\mathcal{F}_t$ for $t \in [0,T]$. Let $p \geq 1$. Define

$$X^*(t) \equiv \sup\{X(s) : 0 \leq s \leq t\}, \quad X^*(0) \equiv 0.$$

Then for $\lambda > 0$

$$P \left( [X^*(T) > \lambda] \right) \leq \frac{1}{\lambda^p} \int_\Omega X(T)^p \, dP \quad (61.5.21)$$

Let $\{f_n\}$ be a sequence of elementary functions converging to $f$ in

$$L^2(\Omega; L^2([0,T], \nu(\cdot))).$$

Then letting

$$X_{n,m}^\tau(t) = \left\| \int_0^t (f_n - f_m) \, dM^\tau \right\|_U,$$

$$X_{n,m}(t) = \left\| \int_0^t (f_n - f_m) \, dM \right\|_U$$

$$= \left\| \int_0^t f_n dM - \int_0^t f_m dM \right\|_U$$

It follows $X_{n,m}^\tau$ is a continuous nonnegative submartingale and from Theorem 61.5.6 just listed,

$$P \left( [X_{n,m}^\tau(T) > \lambda] \right) \leq \frac{1}{\lambda^p} \int_\Omega X_{n,m}(T)^p \, dP$$
CHAPTER 61. THE QUADRATIC VARIATION OF A MARTINGALE

\[
\leq \frac{1}{\lambda^2} \int_0^T \int_0^T |f_n - f_m|^2 \, d[M^r] \, dP \\
\leq \frac{1}{\lambda^2} \int_0^T |f_n - f_m|^2 \, d[M] \, dP
\]

Letting \( l \to \infty \),

\[
P \left( \left[ X^*_{n,m} (T) > \lambda \right] \right) \leq \frac{1}{\lambda^2} \int_0^T |f_n - f_m|^2 \, d[M] \, dP
\]

Therefore, there exists a subsequence, still denoted by \( \{f_n\} \) such that

\[
P \left( \left[ X^*_{n,n+1} (T) > 2^{-n} \right] \right) < 2^{-n}
\]

Then by the Borel Cantelli lemma, the \( \omega \) in infinitely many of the sets

\[
\left[ X^*_{n,n+1} (T) > 2^{-n} \right]
\]

has measure 0. Denoting this exceptional set as \( N \), it follows that for \( \omega \notin N \), there exists \( n(\omega) \) such that for \( n > n(\omega) \),

\[
\sup_{t \in [0,T]} \left\| \int_0^t f_n dM - \int_0^t f_{n+1} dM \right\| \leq 2^{-n}
\]

and this implies uniform convergence of \( \left\{ \int_0^t f_n dM \right\} \). Letting

\[
G(t) = \lim_{n \to \infty} \int_0^t f_n dM,
\]

for \( \omega \notin N \) and \( G(t) = 0 \) for \( \omega \in N \), it follows that for each \( t \), the continuous adapted process \( G(t) \) equals \( \int_0^t f dM \) a.e. Thus \( \left\{ \int_0^t f dM \right\} \) has a continuous version.

It suffices to verify \ref{61.5.20}. Let \( \{f_n\} \) and \( \{g_n\} \) be sequences of elementary functions converging to \( f \) and \( g \) in \( \mathcal{G}_M \cap \mathcal{G}_N \). By Lemma \ref{61.5.2},

\[
E \left( \left( \int_0^t f_n dM, \int_0^t g_n dN \right) \right) = \int_0^t f_n g_n d[M,N]
\]

Then by the Holder inequality and the above definition,

\[
\lim_{n \to \infty} E \left( \left( \int_0^t f_n dM, \int_0^t g_n dN \right) \right) = E \left( \left( \int_0^t f dM, \int_0^t g dN \right) \right)
\]

Consider the right side which equals

\[
\frac{1}{4} \int_\Omega \int_0^T f_n g_n d[M + N] \, dP - \frac{1}{4} \int_\Omega \int_0^T f_n g_n d[M - N] \, dP
\]
Now from Lemma 61.5.4,

\[
\left| \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M + N] dP - \int_{\Omega} \int_{0}^{t} f g d[M + N] dP \right|
\]

\[
= \left| \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d\nu_{M+N} dP - \int_{\Omega} \int_{0}^{t} f g d\nu_{M+N} dP \right|
\]

\[
\leq 2 \left( \int_{\Omega} \int_{0}^{t} |f_{n} g_{n} - f g| d\nu_{M} dP + \int_{\Omega} \int_{0}^{t} |f_{n} g_{n} - f g| d\nu_{N} dP \right)
\]

and by the choice of the \( f_{n} \) and \( g_{n} \), these both converge to 0. Similar considerations apply to

\[
\left| \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M - N] dP - \int_{\Omega} \int_{0}^{t} f g d[M - N] dP \right|
\]

and show

\[
\lim_{n \to \infty} \int_{\Omega} \int_{0}^{t} f_{n} g_{n} d[M, N] = \int_{\Omega} \int_{0}^{t} f g d[M, N] \quad \blacksquare
\]

### 61.6 Another Limit For Quadratic Variation

The problem to consider first is to define an integral

\[
\int_{0}^{t} f dM
\]

where \( f \) has values in \( H' \) and \( M \) is a continuous martingale having values in \( H \). For the sake of simplicity assume \( M(0) = 0 \). The process of definition is the same as before. First consider an elementary function

\[
f(t) \equiv \sum_{k=0}^{m-1} f_{k} \chi_{[t_{k}, t_{k+1}]}(t)
\]

where \( f_{k} \) is measurable into \( H' \) with respect to \( F_{t_{k}} \). Then define

\[
\int_{0}^{t} f dM \equiv \sum_{k=0}^{m-1} f_{k} (M(t \wedge t_{k+1}) - M(t \wedge t_{k})) \in \mathbb{R}
\]

**Lemma 61.6.1** The \( k^{th} \) term in the above sum is a martingale and the integral is also a martingale.

**Proof:** Let \( \sigma \) be a stopping time with two values. Then

\[
E \left( f_{k} (M(\sigma \wedge t_{k+1}) - M(\sigma \wedge t_{k})) \right)
\]

\[
= E \left( E \left( f_{k} (M(\sigma \wedge t_{k+1}) - M(\sigma \wedge t_{k})) \mid F_{t_{k}} \right) \right)
\]

\[
= E \left( f_{k} E \left( (M(\sigma \wedge t_{k+1}) - M(\sigma \wedge t_{k})) \mid F_{t_{k}} \right) \right) = 0
\]
and it works the same with $\sigma$ replaced with $t$. Hence by the lemma about recognizing martingales, Lemma 61.1.1, each term is a martingale and so it follows that the integral $\int_0^t f dM$ is also a martingale.

Note also that, since $M$ is continuous, this is a continuous martingale.

As before, it is important to estimate this.

$$E \left( \left| \int_0^t f dM \right|^2 \right) \leq \ ?$$

Consider a mixed term. For $j < k$, it follows from measurability considerations that

$$E((f_k (M(t \wedge t_{k+1}) - M(t \wedge t_k))) (f_j (M(t \wedge t_{j+1}) - M(t \wedge t_j))))$$

$$= E(E((f_k (M(t \wedge t_{k+1}) - M(t \wedge t_k))) (f_j (M(t \wedge t_{j+1}) - M(t \wedge t_j)))) | F_{t_k}))$$

$$= E((f_j (M(t \wedge t_{j+1}) - M(t \wedge t_j))) f_k E((M(t \wedge t_{k+1}) - M(t \wedge t_k)) | F_{t_k})) = 0$$

Therefore,

$$E \left( \left| \int_0^t f dM \right|^2 \right) = E \left( \sum_{k=0}^{m-1} |f_k (M(t \wedge t_{k+1}) - M(t \wedge t_k))|^2 \right)$$

$$\leq E \left( \sum_{k=0}^{m-1} \|f_k\|^2 |M(t \wedge t_{k+1}) - M(t \wedge t_k)|^2 \right)$$

$$= E \left( \sum_{k=0}^{m-1} \|f_k\|^2 \left( [M^{t_{k+1}} - M^{t_k}] (t) + N_k (t) \right) \right)$$

$$= E \left( \sum_{k=0}^{m-1} \|f_k\|^2 \left( [M^{t_{k+1}}] (t) - [M^{t_k}] (t) + N_k (t) \right) \right)$$

$$= E \left( \sum_{k=0}^{m-1} \|f_k\|^2 \left( [M] (t \wedge t_{k+1}) - [M] (t \wedge t_k) + N_k (t) \right) \right)$$

where $N_k$ is a martingale which equals 0 for $t \leq t_k$. The above equals

$$E \left( \int_0^t \|f\|^2 d[M] \right) = E \left( \int_0^t \|f\|^2 d\nu \right)$$

the integral inside being the ordinary Lebesgue Stieltjes integral for the step function where $\nu$ is the measure determined by the positive linear functional

$$Ag = \int_0^T g d[M]$$
where the integral on the right is the ordinary Stieltjes integral. Thus, the following
inequality is obtained.

\[ E \left( \left| \int_0^t f \, dM \right|^2 \right) \leq E \left( \int_0^t \|f\|^2 \, d[M] \right), \tag{61.6.24} \]

Now what would it take for

\[ E \left( \left| \int_0^t f \, dM \right|^2 \right) \tag{61.6.25} \]

to be well defined? A convenient condition would be to insist that each \( \|f_k\| M^* \)
is in \( L^2(\Omega) \) where

\[ M^*(\omega) \equiv \sup_{t \in [0,T]} |M(t)(\omega)|_H \]

Is this condition also sufficient for the above integral \( 61.6.25 \) to be finite? From
the above, that integral equals

\[
E \left( \sum_{k=0}^{m-1} \|f_k\|^2 |M(t \wedge t_{k+1}) - M(t \wedge t_k)|^2 \right) \\
\leq E \left( 4 \sum_{k=0}^{m-1} \|f_k\|^2 (M^*)^2 \right)
\]

Thus the condition that for each \( k, \|f_k\| M^* \in L^2(\Omega) \) is sufficient for all of the above
to consist of real numbers and be well defined.

**Definition 61.6.2** A function \( f \) is called an elementary function if it is a step
function of the form given in \( 61.6.22 \) where each \( f_k \) is \( F_{t_k} \) measurable and for each
\( k, \|f_k\| M^* \in L^2(\Omega) \). Define \( G_M \) to be the collection of functions \( f \) having values
in \( H' \) which have the property that there exists a sequence of elementary functions
\( \{f_n\} \) with \( f_n \to f \) in the space

\[ L^2(\Omega; L^2([0,T], \nu)) \]

Then picking such an approximating sequence,

\[ \int_0^t f \, dM \equiv \lim_{n \to \infty} \int_0^t f_n \, dM \]

the convergence happening in \( L^2(\Omega) \).

The inequality \( 61.6.24 \) shows that this definition is well defined. So what are the
properties of the integral just defined? Each \( \int_0^t f_n \, dM \) is a continuous martingale
because it is the sum of continuous martingales. Since convergence happens in
It follows that \( \int_0^t f \, dM \) is also a martingale. Is it continuous? By the maximal inequality Theorem 60.9.4, it follows that
\[
P\left( \sup_{t \in [0,T]} \left| \int_0^t f_n \, dM - \int_0^t f_m \, dM \right| > \lambda \right) \leq \frac{1}{\lambda^2} E \left( \int_0^T (f_m - f_n) \, dM \right)^2
\]
and it follows that there exists a subsequence, still called \( n \) such that for all \( p \) positive,
\[
P\left( \sup_{t \in [0,T]} \left| \int_0^t f_n + \delta \, dM - \int_0^t f_n \, dM \right| > \frac{1}{n} \right) < 2^{-n}
\]
By the Borel Cantelli lemma, there exists a set of measure zero \( N \) such that for \( \omega \notin N \), \( \left\{ \int_0^t f \, dM \right\} \) is a Cauchy sequence. Thus, what it converges to is continuous in \( t \) for each \( \omega \notin N \) and for each \( t \), it equals \( \int_0^t f \, dM \) a.e. Hence we can regard \( \int_0^t f \, dM \) as this continuous version.

**What is an example of such a function in \( G_M \)?**

**Lemma 61.6.3** Let \( R : H \to H' \) be the Riesz map.
\[
\langle Rf, g \rangle = \langle f, g \rangle_H.
\]
Also suppose \( M \) is a uniformly bounded continuous martingale with values in \( H \). Then \( RM \in \mathcal{G}_M \).

**Proof:** I need to exhibit an approximating sequence of elementary functions as described above. Consider
\[
M_n(t) = \sum_{i=0}^{m_n-1} M(t_i) \mathcal{X}_{(t_i, t_{i+1})}(t)
\]
Then clearly \( RM_n(t_i) M^* \in L^\infty(\Omega) \) and so in particular it is in \( L^2(\Omega) \). Here
\[
\lim_{n \to \infty} \max_{i=0} \| t_i^n - t_{i+1}^n \|_{i=0, \ldots, m_n} = 0.
\]
Say \( M^*(\omega) \leq C \). Furthermore, I claim that
\[
\lim_{n \to \infty} E \left( \int_0^T \| RM_n - RM \|^2 \, d[M] \right) = 0. \tag{61.6.26}
\]
This requires a little proof. Recall the description of \( [M](t) \). It was as follows. You considered
\[
P_n(t) = 2 \sum_{k \geq 0} \left( (M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n), M(t \wedge \tau_k^n) \right)
\]
where the stopping times were defined such that $\tau_{k+1}^n$ is the first time $t > \tau_k^n$ such that $|M(t) - M(\tau_k^n)|^2 = 2^{-n}$ and $\tau_0^n = 0$. Recall that $\lim_{k \to \infty} \tau_k^n = \infty$ or $T$ in the way it was formulated earlier. Then it was shown that $P_n(t)$ converged to a martingale $P(t)$ in $L^1(\Omega)$. Then by the usual procedure using the Borel Cantelli lemma, a subsequence converges to $P(t)$ uniformly off a set of measure zero. It is easy to estimate $P_n(t)$:

$$|P_n(t)| \leq \sum_{k \geq 0} |M(t \wedge \tau_{k+1}^n)|^2 - |M(t \wedge \tau_k^n)|^2 = |M(t)|^2 \leq M^*$$

This follows from the observation that $(M(t \wedge \tau_{k+1}^n), M(t \wedge \tau_k^n)) \leq \frac{1}{2} \left( |M(t \wedge \tau_{k+1}^n)|^2 + |M(t \wedge \tau_k^n)|^2 \right)$

Then it follows that $\sup_{t \in [0,T]} |P(t)(\omega)| \leq M^*(\omega) \leq C$ for a.e. $\omega$. The quadratic variation $[M]$ was defined as

$$[M](t)^2 = P(t) + [M](t)$$

Thus $[M](t) \leq 2(M^*)^2$. Now consider the above limit in 61.6.26. From the assumption that $M$ is uniformly bounded,

$$\int_0^T \|RM_n - RM\|^2 \, d[M] \leq \int_0^T 4C^2 \, d[M] = 4C^2 \, [M](T) \leq 4C^2 \, (2C^2) < \infty$$

Also, by the continuity of the martingale, for each $\omega$,

$$\lim_{n \to \infty} \|RM_n - RM\|^2 = 0$$

By the dominated convergence theorem, and the fact that the integrand is bounded,

$$\lim_{n \to \infty} \int_0^T \|RM_n - RM\|^2 \, d[M] = 0.$$

Then from the above estimate and the dominated convergence theorem again, 61.6.26 follows. Thus $RM \in \mathcal{G}_M$. ■

From the above lemma, it makes sense to speak of

$$\int_0^t (RM) \, dM$$

and this is a continuous martingale having values in $\mathbb{R}$. Also from the above argument, if $\{t^n_k\}_{k=0}^{m_n}$ is a sequence of partitions such that

$$\lim_{n \to \infty} \max \left\{ |t^n_i - t^n_{i+1}| : i = 0, \ldots, m_n \right\} = 0,$$

then it follows that

$$\sum_{i=0}^{m_n-1} RM(t_i)(M(t \wedge t_{k+1}) - M(t \wedge t_k)) \to \int_0^t (RM) \, dM$$

in $L^2(\Omega)$, this for each $t \in [0,T]$. Now here is the main result.
CHAPTER 61. THE QUADRATIC VARIATION OF A MARTINGALE

Theorem 61.6.4 Let $H$ be a Hilbert space and suppose $(M, F_t), t \in [0, T]$ is a uniformly bounded continuous martingale with values in $H$. Also let $\{t^n_k\}_{k=1}^{m_n}$ be a sequence of partitions satisfying

$$\lim_{n \to \infty} \max \{|t^n_i - t^n_{i+1}|, i = 0, \ldots, m_n\} = 0, \quad \{t^n_k\}_{k=1}^{m_n} \subseteq \{t^{n+1}_k\}_{k=1}^{m_n+1}.$$  

Then

$$[M](t) = \lim_{n \to \infty} m_n^{-1} \sum_{k=0}^{m_n-1} |M(t \wedge t^n_{k+1}) - M(t \wedge t^n_k)|^2$$

the limit taking place in $L^2(\Omega)$. In case $M$ is just a continuous local martingale, the above limit happens in probability.

Proof: First suppose $M$ is uniformly bounded.

$$= \sum_{k=0}^{m_n-1} |M(t \wedge t^n_{k+1})|^2 - |M(t \wedge t^n_k)|^2 - 2 \sum_{k=0}^{m_n-1} (M(t \wedge t^n_k), M(t \wedge t^{n+1}_k) - M(t \wedge t^n_k))$$

$$= |M(t)|^2_H - 2 \sum_{k=0}^{m_n-1} (M(t \wedge t^n_k), M(t \wedge t^{n+1}_k) - M(t \wedge t^n_k))$$

$$= |M(t)|^2_H - 2 \sum_{k=0}^{m_n-1} RM(t^n_k) (M(t \wedge t^{n+1}_k) - M(t \wedge t^n_k))$$

Then by Lemma 61.6.3, the right side converges to

$$|M(t)|^2_H - 2 \int_0^t (RM) dM$$

Therefore, in $L^2(\Omega)$,

$$\lim_{n \to \infty} \sum_{k=0}^{m_n-1} |M(t \wedge t^n_{k+1}) - M(t \wedge t^n_k)|^2 + 2 \int_0^t (RM) dM = |M(t)|^2_H$$

That term on the left involving the limit is increasing and equal to 0 when $t = 0$. Therefore, it must equal $[M](t)$.

Next suppose $M$ is only a continuous local martingale. By Proposition 61.2.2 there exists an increasing localizing sequence $\{\tau_k\}$ such that $M^{\tau_k}$ is a uniformly bounded martingale. Then

$$P(\cup_{k=1}^\infty [\tau_k = \infty]) = 1$$
61.7. DOOB MEYER DECOMPOSITION

To save notation, let

\[ Q_n(t) \equiv \sum_{k=0}^{m_n-1} \left| M(t \wedge t^n_k) - M(t \wedge t^n_{k+1}) \right|^2_H \]

Let \( \eta, \varepsilon > 0 \) be given. Then there exists \( k \) large enough that \( P(\tau_k = \infty) > 1 - \eta/2 \). This is because the sets \( \tau_k = \infty \) increase to \( \Omega \) other than a set of measure zero.

Then,

\[ \left| Q_n - [M]^{\tau_k}(t) \right| > \varepsilon \cap \tau_k = \infty = \left| Q_n - [M](t) \right| > \varepsilon \cap \tau_k = \infty \]

Thus

\[ P\left(\left| Q_n - [M](t) \right| > \varepsilon\right) \leq P\left(\left| Q_n - [M](t) \right| > \varepsilon \cap \tau_k = \infty\right) + P(\tau_k < \infty) \]

\[ \leq P\left(\left| Q_n^{\tau_k} - [M]^{\tau_k}(t) \right| > \varepsilon\right) + \eta/2 \]

From the first part, the convergence in probability of \( Q_n^{\tau_k} (t) \) to \([M]^{\tau_k}(t)\) follows from the convergence in \( L^2(\Omega) \) and so if \( n \) is large enough, the right side of the above inequality is less than \( \eta/2 + \eta/2 = \eta \). Since \( \eta \) was arbitrary, this proves convergence in probability.

61.7 Doob Meyer Decomposition

This section is on the Doob Meyer decomposition which is a way of starting with a submartingale and writing it as the sum of a martingale and an increasing adapted stochastic process of a certain form. This is more general than what was done above with the submartingales \( ||M(t)||^2 \) for \( M(t) \in M^2_x(H) \) where \( M \) is a continuous martingale. There are two forms for this theorem, one for discrete martingales and one for martingales defined on an interval of the real line which is much harder. According to [67], this material is found in [71] however, I am following [67] for the continuous version of this theorem.

Theorem 61.7.1 Let \( \{X_n\} \) be a submartingale. Then there exists a unique stochastic process, \( \{A_n\} \) and martingale, \( \{M_n\} \) such that

1. \( A_n(\omega) \leq A_{n+1}(\omega), \ A_1(\omega) = 0, \)
2. \( A_n \) is \( F_{n-1} \) adapted for all \( n \geq 1 \) where \( F_0 \equiv F_1 \).

and also \( X_n = M_n + A_n \).

Proof: Let \( A_1 \equiv 0 \) and define

\[ A_n \equiv \sum_{k=2}^{n} E(X_k - X_{k-1}|F_{k-1}) \].
It follows $A_n$ is $\mathcal{F}_{n-1}$ measurable. Since $\{X_k\}$ is a submartingale, $A_n$ is increasing because
\[
A_{n+1} - A_n = E(X_{n+1} - X_n | \mathcal{F}_n) \geq 0 \tag{61.7.27}
\]
It is a submartingale because
\[
E(A_n | \mathcal{F}_{n-1}) = A_n \geq A_{n-1}.
\]
Now let $M_n$ be defined by
\[
X_n = M_n + A_n.
\]
Then from (61.7.27),
\[
E(M_{n+1} | \mathcal{F}_n) = E(X_{n+1} | \mathcal{F}_n) - E(A_{n+1} | \mathcal{F}_n)
= E(X_{n+1} | \mathcal{F}_n) - E(A_{n+1} - A_n | \mathcal{F}_n) - A_n
= E(X_{n+1} | \mathcal{F}_n) - E(E(X_{n+1} - X_n | \mathcal{F}_n) | \mathcal{F}_n) - A_n
= E(X_{n+1} | \mathcal{F}_n) - E(X_{n+1} - X_n | \mathcal{F}_n) - A_n
= E(X_n | \mathcal{F}_n) - A_n
= X_n - A_n \equiv M_n
\]
This proves the existence part.
It remains to verify uniqueness. Suppose then that
\[
X_n = M_n + A_n = M'_n + A'_n
\]
where $\{A_n\}$ and $\{A'_n\}$ both satisfy the conditions of the theorem and $\{M_n\}$ and $\{M'_n\}$ are both martingales. Then
\[
M_n - M'_n = A'_n - A_n
\]
and so, since $A'_n - A_n$ is $\mathcal{F}_{n-1}$ measurable and $\{M_n - M'_n\}$ is a martingale,
\[
M_{n-1} - M'_{n-1} = E(M_n - M'_n | \mathcal{F}_{n-1})
= E(A'_n - A_n | \mathcal{F}_{n-1})
= A'_n - A_n = M_n - M'_n.
\]
Continuing this way shows $M_n - M'_n$ is a constant. However, since $A'_1 - A_1 = 0 = M_1 - M'_1$, it follows $M_n = M'_n$ and this proves uniqueness. This proves the theorem.

**Definition 61.7.2** A stochastic process, $\{A_n\}$ which satisfies the conditions of Theorem 61.7.1,
\[
A_n (\omega) \leq A_{n+1} (\omega)
\]
and
A_n is \( \mathcal{F}_{n-1} \) adapted for all \( n \geq 1 \)

where \( \mathcal{F}_0 = \mathcal{F}_1 \) is said to be natural.

The Doob Meyer theorem needs to be extended to continuous submartingales and this will require another description of what it means for a stochastic process to be natural. To get an idea of what this condition should be, here is a lemma.

**Lemma 61.7.3** Let a stochastic process, \( \{A_n\} \), be natural. Then for every martingale, \( \{M_n\} \),

\[
E(M_n A_n) = E \left( \sum_{j=1}^{n-1} M_j (A_{j+1} - A_j) \right)
\]

**Proof:** Start with the right side.

\[
E \left( \sum_{j=1}^{n-1} M_j (A_{j+1} - A_j) \right) = E \left( \sum_{j=2}^{n} M_{j-1} A_j - \sum_{j=1}^{n-1} M_j A_j \right)
\]

\[
= E \left( \sum_{j=2}^{n-1} A_j (M_{j-1} - M_j) \right) + E(M_{n-1} A_n)
\]

Then the first term equals zero because since \( A_j \) is \( \mathcal{F}_{j-1} \) measurable,

\[
\int_{\Omega} A_j M_{j-1} dP - \int_{\Omega} A_j M_j dP = \int_{\Omega} A_j E(M_j | \mathcal{F}_{j-1}) dP - \int_{\Omega} A_j M_j dP
\]

\[
= \int_{\Omega} E(A_j M_j | \mathcal{F}_{j-1}) dP - \int_{\Omega} A_j M_j dP
\]

\[
= \int_{\Omega} A_j M_j dP - \int_{\Omega} A_j M_j dP = 0.
\]

The last term equals

\[
\int_{\Omega} M_{n-1} A_n dP = \int_{\Omega} E(M_n | \mathcal{F}_{n-1}) A_n dP
\]

\[
= \int_{\Omega} E(M_n A_n | \mathcal{F}_{n-1}) dP = E(M_n A_n).
\]

This proves the lemma.

**Definition 61.7.4** Let \( A \) be an increasing function defined on \( \mathbb{R} \). By Theorem 3.3.4 on Page 48 there exists a positive linear functional, \( L \) defined on \( C_c(\mathbb{R}) \) given by

\[
Lf = \int_{a}^{b} f dA \text{ where } \text{spt}(f) \subseteq [a, b]
\]
where the integral is just the Riemann Stieltjes integral. Then by the Riesz representation theorem, Theorem 10.3.2 on Page 269, there exists a unique Radon measure, \( \mu \) which extends this functional, as described in the Riesz representation theorem. Then for \( B \) a measurable set, I will write either

\[
\int_B f d\mu \text{ or } \int_B f dA
\]

to denote the Lebesgue integral,

\[
\int_X B f d\mu.
\]

Lemma 61.7.5 Let \( f \) be right continuous. Then \( f \) is Borel measurable. Also, if the limit from the left exists, then \( f^- (x) \equiv f (x)_- \equiv \lim_{y \to x^-} f (y) \) is also Borel measurable. If \( A \) is an increasing right continuous function and \( f \) is right continuous and \( f^- \), the left limit function exists, then if \( f \) is bounded, on \([a,b]\), and if

\[
\{x_0^p, \ldots, x_{n_p}^p\}_{p=1}^\infty
\]

is a sequence of partitions of \([a,b]\) such that

\[
\lim_{p \to \infty} \max \{ |x_k^p - x_{k-1}^p| : k = 1, 2, \ldots, n_p \} = 0 \quad (61.7.28)
\]

then

\[
\int_{(a,b]} f^- dA = \lim_{p \to \infty} \sum_{k=1}^{n_p} f (x_{k-1}^p) (A (x_k^p) - A (x_{k-1}^p)) \quad (61.7.29)
\]

More generally, let

\[
D \equiv \bigcup_{p=1}^\infty \{x_0^p, \ldots, x_{n_p}^p\}_{p=1}^\infty
\]

and

\[
f^- (t) = \lim_{s \to t-} s \in D f (s).
\]

Then (61.7.29) holds.

**Proof:** For \( x \in f^- ((a, \infty)) \), denote by \( I_x \) the union of all intervals containing \( x \) such that \( f (y) \) is larger than \( a \) for all \( y \) in the interval. Since \( f \) is right continuous, each \( I_x \) has positive length. Now if \( I_x \) and \( I_y \) are two of these intervals, then either they must have empty intersection or they are the same interval. Thus \( f^- ((a, \infty)) \) is of the form \( \bigcup_{x \in f^- ((a, \infty))} I_x \) and there can only be countably many distinct intervals because each has positive length and \( \mathbb{R} \) is separable. Hence \( f^- ((a, \infty)) \) equals the countable union of intervals and is therefore, Borel measurable. Now

\[
f^- (x) = \lim_{n \to \infty} f (x - r_n) \equiv \lim_{n \to \infty} f_{r_n} (x)
\]

where \( r_n \) is a decreasing sequence converging to 0. Now each \( f_{r_n} \) is Borel measurable by the first part of the proof because it is right continuous and so it follows the same is true of \( f^- \).
Finally consider the claim about the integral. Since \( A \) is right continuous, a simple argument involving the dominated convergence theorem and approximating \((c, d]\) with a piecewise linear continuous function nonzero only on \((c, d + h)\) which approximates \( \mathcal{X}_{(c, d]} \) will show that for \( \mu \) the measure of Definition 61.7.4
\[
\mu ((c, d]) = A (d) - A (c).
\]
Therefore, the sum in 61.7.29 is of the form
\[
\sum_{k=1}^{n_p} f \left( x_{k-1}^p \right) \mu ((x_{k-1}, x_k]) = \int_{(a, b]} \sum_{k=1}^{n_p} f \left( x_{k-1}^p \right) \mathcal{X}_{(x_{k-1}, x_k]} \text{d}\mu
\]
and by 61.7.28
\[
\lim_{p \to \infty} \sum_{k=1}^{n_p} f \left( x_{k-1}^p \right) \mathcal{X}_{(x_{k-1}, x_k]} (x) = f_-(x)
\]
for each \( x \in (a, b] \). Therefore, since \( f \) is bounded, 61.7.29 follows from the dominated convergence theorem. The last claim follows the same way. This proves the lemma.

**Definition 61.7.6** An increasing stochastic process, \( \{ A(t) \} \) which is right continuous is said to be natural if \( A(0) = 0 \) and whenever \( \{ \xi(t) \} \) is a bounded right continuous martingale,
\[
E \left( A(t) \xi(t) \right) = E \left( \int_{(0, t]} \xi_- (s) \text{d}A (s) \right),
\] (61.7.30)
Here
\[
\xi_- (s, \omega) \equiv \lim_{r \to s^- \in D} \xi (r, \omega)
\]
a.e. where \( D \) is a countable dense subset of \([0, t] \). By Corollary 60.8.2 the right side of 61.7.30 is not dependent on the choice of \( D \) since if \( \xi_- \) is computed using two different dense subsets, the two random variables are equal a.e.

Some discussion is in order for this definition. Pick \( \omega \in \Omega \). Then since \( A \) is right continuous, the function \( t \to A(t, \omega) \) is increasing and right continuous. Therefore, one can do the Lebesgue Stieltjes integral defined in Definition 61.7.4 for each \( \omega \) whenever \( f \) is Borel measurable and bounded. Now it is assumed \( \{ \xi(t) \} \) is bounded and right continuous. By Lemma 61.7.5 \( \xi_- (t) \equiv \lim_{r \to t^- \in D} \xi (r) \) is measurable and by this lemma,
\[
\int_{(0, t]} \xi_- (s) \text{d}A (s) = \lim_{p \to \infty} \sum_{k=1}^{n_p} \xi \left( t_{k-1}^p \right) \left( A \left( t_k^p \right) - A \left( t_{k-1}^p \right) \right)
\]
where \( \{ t_k^p \}_{k=1}^{n_p} \) is a sequence of partitions of \([0, t] \) such that
\[
\lim_{p \to \infty} \max \left\{ |t_k^p - t_{k-1}^p| : k = 1, 2, \ldots, n_p \right\} = 0.
\] (61.7.31)
and \( D \equiv \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{n_p} \{ t^p_k \} \).

Also, if \( t \to A(t, \omega) \) is right continuous, hence Borel measurable, then for \( \xi(t) \) the above bounded right continuous martingale, it follows it makes sense to write

\[
\int_{(0,t]} \xi(s) \, dA(s).
\]

Consider the right sum,

\[
\sum_{k=1}^{n_p} \xi(t^p_k) \left( A(t^p_k) - A(t^p_{k-1}) \right)
\]

This equals

\[
\int_{(0,t]} \sum_{k=1}^{n_p} \xi(t^p_k) \lambda([t^p_{k-1}, t^p_k]) \, dA(s)
\]

and by right continuity, it follows

\[
\lim_{p \to \infty} \sum_{k=1}^{n_p} \xi(t^p_k) \lambda([t^p_{k-1}, t^p_k]) (s) = \xi(s)
\]

and so the dominated convergence theorem applies and it follows

\[
\lim_{p \to \infty} \sum_{k=1}^{n_p} \xi(t^p_k) \left( A(t^p_k) - A(t^p_{k-1}) \right) = \int_{(0,t]} \xi(s) \, dA(s)
\]

where this is a random variable. Thus

\[
E \left( \int_{(0,t]} \xi(s) \, dA(s) \right) = \int_{\Omega} \left( \lim_{p \to \infty} \int_{(0,t]} \sum_{k=1}^{n_p} \xi(t^p_k) \lambda([t^p_{k-1}, t^p_k]) \, dA(s) \right) \, dP
\]

(61.7.32)

Now as mentioned above,

\[
\int_{(0,t]} \sum_{k=1}^{n_p} \xi(t^p_k) \lambda([t^p_{k-1}, t^p_k]) \, dA(s) = \sum_{k=1}^{n_p} \xi(t^p_k) \left( A(t^p_k) - A(t^p_{k-1}) \right)
\]

and since \( A \) is increasing, this is bounded above by an expression of the form \( CA(t) \), a function in \( L^1 \). Therefore, by the dominated convergence theorem, (61.7.32) reduces
to

\[
\lim_{p \to \infty} \int \int_{(0,t]} \sum_{k=1}^{n_p} \xi(t_k^p) \chi_{(t_{k-1}^p, t_k^p]}(s) dA(s) dP
\]

\[
= \lim_{p \to \infty} \int_{\Omega} \sum_{k=1}^{n_p} \xi(t_k^p) (A(t_k^p) - A(t_{k-1}^p)) dP
\]

\[
= \lim_{p \to \infty} \int_{\Omega} \left( \sum_{k=1}^{n_p} \xi(t_k^p) A(t_k^p) - \sum_{k=0}^{n_p-1} \xi(t_{k+1}^p) A(t_k^p) \right) dP
\]

\[
= \lim_{p \to \infty} \sum_{k=1}^{n_p-1} \int_{\Omega} (\xi(t_k^p) - \xi(t_{k+1}^p)) A(t_k^p) dP + \int_{\Omega} \xi(t) A(t) dP \tag{61.7.33}
\]

Since \(\xi\) is a martingale,

\[
\int_{\Omega} \xi(t_{k+1}^p) A(t_k^p) dP = \int_{\Omega} E(\xi(t_{k+1}^p) A(t_k^p) \mid F_{t_k^p}) dP
\]

\[
= \int_{\Omega} E(\xi(t_{k+1}^p) \mid F_{t_k^p}) A(t_k^p) dP
\]

\[
= \int_{\Omega} A(t_k^p) \xi(t_k^p) dP
\]

and so in \(61.7.33\) the term with the sum equals 0 and it reduces to

\[
E(\xi(t) A(t)).
\]

This is sufficiently interesting to state as a lemma.

**Lemma 61.7.7** Let \(A\) be an increasing adapted stochastic process which is right continuous. Also let \(\xi(t)\) be a bounded right continuous martingale. Then

\[
E(\xi(t) A(t)) = E \left( \int_{[0,t]} \xi(s) dA(s) \right)
\]

and \(A\) is natural, if and only if for all such bounded right continuous martingales,

\[
E(\xi(t) A(t)) = E \left( \int_{[0,t]} \xi(s) dA(s) \right) = E \left( \int_{[0,t]} \xi_-(s) dA(s) \right)
\]

**Lemma 61.7.8** Let \((\Omega, F, P)\) be a probability space and let \(G\) be a \(\sigma\) algebra contained in \(F\). Suppose also that \(\{f_n\}\) is a sequence in \(L^1(\Omega)\) which converges weakly to \(f\) in \(L^1(\Omega)\). That is, for every \(h \in L^\infty(\Omega)\),

\[
\int_{\Omega} f_n h dP \to \int_{\Omega} f h dP.
\]

Then \(E(f_n|G)\) converges weakly in \(L^1(\Omega)\) to \(E(f|G)\).
CHAPTER 61. THE QUADRATIC VARIATION OF A MARTINGALE

**Proof:** First note that if \( h \in L^\infty(\Omega, \mathcal{F}) \), then \( E(h|\mathcal{G}) \in L^\infty(\Omega, \mathcal{G}) \) because if \( A \in \mathcal{G} \),

\[
\int_A |E(h|\mathcal{G})| \, dP \leq \int_A |E|h| \, dP = \int_A |h| \, dP
\]

and so if \( A = [||E(h|\mathcal{G})|| > ||h||_\infty] \), then if \( P(A) > 0 \),

\[
||h||_\infty P(A) < \int_A |E(h|\mathcal{G})| \, dP \leq \int_A |h| \, dP \leq ||h||_\infty P(A),
\]

a contradiction. Hence \( P(A) = 0 \) and so \( E(h|\mathcal{G}) \in L^\infty(\Omega, \mathcal{G}) \) as claimed. Let \( h \in L^\infty(\Omega, \mathcal{G}) \).

\[
\int_\Omega E(f_n|\mathcal{G}) \, h dP = \int_\Omega E(E(f_n|\mathcal{G}) \, h|\mathcal{G}) \, dP
\]

\[
= \int_\Omega E(f_n|\mathcal{G}) \, E(h|\mathcal{G}) \, dP
\]

\[
= \int_\Omega E(f_n \, E(h|\mathcal{G})|\mathcal{G}) \, dP
\]

\[
= \int_\Omega f_n \, E(h|\mathcal{G}) \, dP
\]

and so

\[
\lim_{n \to \infty} \int_\Omega E(f_n|\mathcal{G}) \, h dP = \lim_{n \to \infty} \int_\Omega f_n \, E(h|\mathcal{G}) \, dP
\]

\[
= \int_\Omega f \, E(h|\mathcal{G}) \, dP
\]

\[
= \int_\Omega E(f \, E(h|\mathcal{G})|\mathcal{G}) \, dP
\]

\[
= \int_\Omega E(h|\mathcal{G}) \, E(f|\mathcal{G}) \, dP
\]

\[
= \int_\Omega E(E(f|\mathcal{G}) \, h|\mathcal{G}) \, dP
\]

\[
= \int_\Omega E(f|\mathcal{G}) \, h dP
\]

and this proves the lemma.

Next suppose \( \{X(t)\} \) is a real submartingale and suppose \( X(t) = M(t) + A(t) \) where \( A(t) \) is an increasing stochastic process adapted to \( \mathcal{F}_t \) such that \( A(0) = 0 \) and \( \{M(t)\} \) is a martingale adapted to \( \mathcal{F}_t \). Also let \( T \) be a stopping time bounded above by \( a \). Then by the optional sampling theorem, and the observation that \( \{|M(t)|\} \)
is a submartingale
\[
\int_{|X(T)| \geq \lambda} |X(T)| \, dP \\
\leq \int_{|X(T)| \geq \lambda} |M(T)| \, dP + \int_{|X(T)| \geq \lambda} A(T) \, dP \\
\leq \int_{|X(T)| \geq \lambda} E(|M(a)| |\mathcal{F}_T) \, dP + \int_{|X(T)| \geq \lambda} E(A(a) |\mathcal{F}_T) \, dP \\
\leq \int_{|X(T)| \geq \lambda} |M(a)| \, dP + \int_{|X(T)| \geq \lambda} A(a) \, dP
\]

Now by Theorem 58.5.4,
\[
P(|X_T| \geq \lambda) \leq \frac{2}{\lambda} E(|X(0)| + |X(a)|)
\]
and so \( P(|X_T| \geq \lambda) \to 0 \) uniformly for \( T \) a stopping time bounded by \( a \) as \( \lambda \to \infty \) and so this shows equi integrability of \( \{X(T)\} \) because \( A(t, \omega) \geq 0 \).

This motivates the following definition.

**Definition 61.7.9** A stochastic process, \( \{X(t)\} \) is called DL if for all \( a > 0 \), the set of random variables, \( \{X(T)\} \) for \( T \) a stopping time bounded by \( a \) is equi integrable.

**Example 61.7.10** Let \( \{M(t)\} \) be a continuous martingale. Then \( \{M(t)\} \) is of class DL.

To show this, let \( a > 0 \) be given and let \( T \) be a stopping time bounded by \( a \). Then by the optional sampling theorem, \( M(0), M(T), M(a) \) is a martingale and so
\[
E(M(a) |\mathcal{F}_T) = M(T)
\]
and so by Jensen’s inequality, \( |M(T)| \leq E(|M(a)| |\mathcal{F}_T) \). Therefore,
\[
\int_{|M(T)| \geq \lambda} |M(T)| \, dP \leq \int_{|M(T)| \geq \lambda} E(|M(a)| |\mathcal{F}_T) \, dP = \int_{|M(T)| \geq \lambda} |M(a)| \, dP.
\]

Now by Theorem 60.5.3,
\[
P(|M(T)| \geq \lambda) \leq \frac{1}{\lambda} E(|M(a)|)
\]
and so since a given \( L^1 \) function is uniformly integrable, there exists \( \delta \) such that if \( P(A) < \delta \) then
\[
\int_A |M(a)| \, dP < \epsilon.
\]
Now choose \( \lambda \) large enough that
\[
\frac{1}{\lambda} E (|M (a)|) < \delta.
\]
Then for such \( \lambda \), it follows from 61.7.34 that for any stopping time bounded by \( a \),
\[
\int_{|M(T)|\geq \lambda} |M(T)| dP < \varepsilon.
\]
This shows \( M \) is DL.

**Example 61.7.11** Let \( \{X(t)\} \) be a nonnegative submartingale with \( t \to E (X(t)) \) right continuous so \( \{X(t)\} \) can be considered right continuous. Then \( \{X(t)\} \) is DL.

To show this, let \( T \) be a stopping time bounded by \( a > 0 \). Then by the optional sampling theorem,
\[
\int_{X(T)\geq \lambda} X(T) dP \leq \int_{X(T)\geq \lambda} X(a) dP
\]
and now by Theorem 58.5.4 on Page 2016
\[
P (|X(T) \geq \lambda|) \leq \frac{1}{\lambda} E (X^+_a).
\]
Thus if \( \varepsilon > 0 \) is given, there exists \( \lambda \) large enough that for any stopping time, \( T \leq a \),
\[
\int_{X(T)\geq \lambda} X(T) dP \leq \varepsilon
\]
Thus the submartingale is DL.

Now with this preparation, here is the Doob Meyer decomposition.

**Theorem 61.7.12** Let \( \{X(t)\} \) be a submartingale of class DL. Then there exists a martingale, \( \{M(t)\} \) and an increasing submartingale, \( \{A(t)\} \) such that for each \( t \),
\[
X(t) = M(t) + A(t).
\]
If \( \{A(t)\} \) is chosen to be natural and \( A(0) = 0 \), then with this condition, \( \{M(t)\} \) and \( \{A(t)\} \) are unique.

**Proof:** First I will show uniqueness. Suppose then that
\[
X(t) = M(t) + A(t) = M'(t) + A'(t)
\]
where \( M, M' \) and \( A, A' \) satisfy the given conditions. Let \( t > 0 \) and consider \( s \in [0,t] \). Then
\[
A(s) - A'(s) = M'(s) - M(s)
\]
Since $A, A'$ are natural, it follows that for $\xi(t)$ a right continuous bounded martingale,

$$E (\xi(t)(A(t) - A'(t))) = E \left( \int_{[0,t]} \xi_- (s) dA(s) \right) - E \left( \int_{[0,t]} \xi_- (s) dA'(s) \right)$$

$$= E \left( \lim_{n \to \infty} \sum_{k=1}^{m_n} \xi\left(t_{k-1}^n\right) \left(A(t_k^n) - A(t_{k-1}^n)\right) - \sum_{k=1}^{m_n} \xi\left(t_{k-1}^n\right) \left(A'(t_k^n) - A'(t_{k-1}^n)\right) \right)$$

where $\{t_k\}_{k=0}^{m_n}$ is a sequence of partitions of $[0,t]$ such that these are equally spaced points, $\lim_{n \to \infty} t_k^n - t_k = 0$, and $\{t_k\}_{k=0}^{m_n} \leq \{t_k^{n+1}\}_{k=0}^{m_n+1}$. Then since $A(t)$ and $A'(t)$ are increasing, the absolute value of each sum is bounded above by an expression of the form

$$CA(t) \text{ or } CA'(t)$$

and so the dominated convergence theorem can be applied to get the above expression to equal

$$\lim_{n \to \infty} E \left( \sum_{k=1}^{m_n} \xi\left(t_{k-1}^n\right) \left(M(t_k^n) - M(t_{k-1}^n)\right) - \sum_{k=1}^{m_n} \xi\left(t_{k-1}^n\right) \left(M'(t_k^n) - M'(t_{k-1}^n)\right) \right).$$

Now using $X = A + M$ and $X = A' + M'$

$$= \lim_{n \to \infty} E \left( \sum_{k=1}^{m_n} \xi\left(t_{k-1}^n\right) \left(M(t_k^n) - M(t_{k-1}^n)\right) - \sum_{k=1}^{m_n} \xi\left(t_{k-1}^n\right) \left(M'(t_k^n) - M'(t_{k-1}^n)\right) \right).$$

Both terms in the above equal 0. Here is why.

$$E (\xi(t_{k-1}^n) M(t_k^n)) = E \left( E (\xi(t_{k-1}^n) M(t_k^n) | F_{t_{k-1}^n}) \right)$$

$$= E \left( \xi(t_{k-1}^n) E (M(t_k^n) | F_{t_{k-1}^n}) \right)$$

$$= E (\xi(t_{k-1}^n) M(t_k^n)) \cdot$$

Thus the expected value of the first sum equals 0. Similarly, the expected value of the second sum equals 0. Hence this has shown that for any bounded right continuous martingale, $\{\xi(t)\}$ and $t > 0$,

$$E (\xi(t)(A(t) - A'(t))) = 0.$$

Now let $\xi$ be a bounded random variable and let $\xi(t)$ be a right continuous version of the martingale $E (\xi|F_t)$. Then

$$0 = E (E (\xi|F_t) (A(t) - A'(t))) = E (E (\xi(A(t) - A'(t)) | F_t))$$

$$= E (\xi(A(t) - A'(t)))$$

and since $\xi$ is arbitrary, it follows that $A(t) = A'(t)$ a.e. which proves uniqueness.
Because of the uniqueness assertion, it suffices to prove the theorem on an arbitrary interval, \([0, a]\).

Without loss of generality, it can be assumed \(X(0) = 0\) since otherwise, you could simply consider \(X(t) - X(0)\) in its place and then at the end, add \(X(0)\) to \(M(t)\). Let \(\{t^n_k\}_{k=0}^{m_n}\) be a sequence of partitions of \([0, a]\) such that these are equally spaced points, \(\lim_{n \to \infty} t^n_{k+1} - t^n_k = 0\), and \(\{t^n_k\}_{k=0}^{m_n} \subseteq \{t^{n+1}_k\}_{k=0}^{m_{n+1}}\). Then consider the submartingale, \(\{X(t^n_k)\}_{k=0}^{m_n}\). Theorem 61.7.1 implies there exists a unique martingale, and increasing submartingale, \(\{M_t\}_{t=0}^{m_n}\) and \(\{A_t\}_{t=0}^{m_n}\) respectively such that \(M(0) = 0 = A(0)\),

\[
X(t^n_k) = M^n(t^n_k) + A^n(t^n_k).
\]

and \(A^n(t^n_k)\) is \(\mathcal{F}_{t^n_{k-1}}\) measurable. Recall how these were defined.

\[
A^n(t^n_k) = \sum_{j=1}^{k} E\left( X(t^n_j) - X(t^n_{j-1}) \mid \mathcal{F}_{t^n_{j-1}} \right), \quad A^n(0) = 0
\]

\[
M^n(t^n_k) = X(t^n_k) - A^n(t^n_k).
\]

I want to show that \(\{A^n(a)\}\) is equi integrable. From this there will be a weakly convergent subsequence and nice things will happen. Define \(T^n(\omega)\) to equal \(t^n_{j-1}\) where \(t^n_j\) is the first time where \(A^n(t^n_j, \omega) \geq \lambda\) or \(T^n(\omega) = a\) if this never happens. I want to say that \(T^n\) is a stopping time and so I need to verify that \(T^n \leq t^n_j\) \(\in \mathcal{F}_{t^n_j}\) for each \(j\). If \(\omega \in [T^n \leq t^n_j]\), then this means the first time, \(t^n_k\), where \(A^n(t^n_k, \omega) \geq \lambda\) is such that \(t^n_k \leq t^n_{j+1}\). Since \(A^n_k\) is increasing in \(k\),

\[
[T^n \leq t^n_j] = \cup_{k=0}^{j+1} [A^n(t^n_k) \geq \lambda] = [A^n(t^n_{j+1}) \geq \lambda] \in \mathcal{F}_{t^n_j}.
\]

Note \(T^n\) only has the values \(t^n_k\). Thus for \(t \in [t^n_{j-1}, t^n_j]\),

\[
[T^n \leq t] = [T^n \leq t^n_{j-1}] \in \mathcal{F}_{t^n_{j-1}} \subseteq \mathcal{F}_t.
\]

Thus \(T^n\) is one of those stopping times bounded by \(a\). Since \(\{X(t)\}\) is DL, this shows \(\{X(T^n)\}\) is equi integrable. Now from the definition of \(T^n\), it follows

\[
A^n(T^n) \leq \lambda.
\]

Recall \(T^n(\omega) = t^n_{j-1}\) where \(t^n_j\) is the first time where \(A^n(t^n_j, \omega) \geq \lambda\) or \(T^n(\omega) = a\) if this never happens. Thus \(T^n\) is such that it is before \(A^n\) gets larger than \(\lambda\). Thus,

\[
\int_{\{A^n(a) \geq 2\lambda\}} \frac{1}{2} A^n(a) dP \leq \int_{\{A^n(a) \geq 2\lambda\}} (A^n(a) - \lambda) dP \leq \int_{\{A^n(a) \geq 2\lambda\}} (A^n(a) - A^n(T^n)) dP.
\]
18.9.6

there exists a subsequence

It also follows that

Thus it is obvious from properties of conditional expectation that

The second of the integrals on the right is such that for \( \omega \) in this set, \( T^n(\omega) = a \) and so the second integral equals 0. Hence from the above,

\[
\int \Omega (M^n(a) - M^n(T^n)) dP = 0.
\]

Remember \( \{M^n(t^n_k)\}_{k=0}^{m_n} \) was a martingale.

\[
\int \Omega (X(a) - X(T^n)) dP = \int_{[A^n(a) \geq \lambda]} (X(a) - X(T^n)) dP
\]

\[
+ \int_{[A^n(a) < \lambda]} (X(a) - X(T^n)) dP.
\]

The second of the integrals on the right is such that for \( \omega \) in this set, \( T^n(\omega) = a \) and so the second integral equals 0. Hence from the above,

\[
\int_{[A^n(a) \geq 2\lambda]} \frac{1}{2} A^n(a) dP \leq \int_{[A^n(a) \geq \lambda]} (X(a) - X(T^n)) dP
\]

and since \( \{X(t)\} \) is DL, this shows \( \{A^n(a)\}_{n=1}^{\infty} \) is equi integrable.

By Corollary 18.5.1 on Page 2213 there exists a subsequence \( \{A^{n_k}(a)\}_{k=1}^{\infty} \) which converges weakly in \( L^1(\Omega) \) to \( A(a) \). By Lemma 18.5.2 it also follows that \( E(A^{n_k}(a) | F_t) \) converges weakly to \( E(A(a) | F_t) \) in \( L^1(\Omega) \). Now define

\[
M(t) = E(X(a) - A(a) | F_t).
\]

Thus it is obvious from properties of conditional expectation that \( \{M(t)\} \) is a martingale adapted to \( F_t \) and without loss of generality, it is a right continuous version. Let

\[
A(t) = X(t) - M(t).
\]

Then since \( \{X(t)\} \) is a submartingale, it follows \( \{A(t)\} \) is also a submartingale.

It remains to show several things. First, it is necessary to show \( A(t) \) is increasing in \( t \) and \( A(0) = 0 \). To see this, let \( s < t, s,t \in \cup_{n=1}^{\infty} \cup_{k=0}^{m_n} t^n_k \). Then letting \( n \) large enough both \( s, t \) are in \( \cup_{k=0}^{m_n} t^n_k \). Only consider such \( n \). Let \( t = t^n_{k(t)}, s = t^n_{k(s)} \) and let \( h \in L^\infty(\Omega), h \geq 0 \). Then

\[
\int \Omega (A(t) - A(s)) hdP = \int \Omega (X(t) - M(t) - (X(s) - M(s))) hdP
\]

\[
\int \Omega (X(t) - E(X(a) - A(a) | F_t) - (X(s) - E(X(a) - A(a) | F_s))) hdP.
\]

(61.7.35)
Now by Lemma 61.7.8, the following weak limit holds.

\[ E(X(a) - A(a) \mid F_t) = \lim_{k \to \infty} E\left(\frac{M^{n_k}(a)}{X(a) - A^{n_k}(a)} \mid F_t\right) = \lim_{k \to \infty} M^{n_k}(t) \]

A similar formula holds for \( s \) in place of \( t \). Then the expression in 61.7.35 equals

\[ \lim_{k \to \infty} \int_{(0,t]} (X(t) - M^{n_k}(t) - (X(s) - M^{n_k}(s))) \, h \, dP = \lim_{k \to \infty} \int_{(0,t]} (A^{n_k}(t) - A^{n_k}(s)) \, h \, dP \geq 0 \]

Since \( h \geq 0 \) is arbitrary, this shows \( A(t) - A(s) \geq 0 \) a.e. Not requiring \( h \geq 0 \), the above argument also shows that for \( s, t \in \cup_{n=1}^{\infty} \cup_{k=0}^{m_n} t^n_k \),

\[ A(t) - A(s) = \text{weak lim}_{p \to \infty} A^{n_p}(t) - A^{n_p}(s). \quad (61.7.36) \]

Now consider the claim that \( A(0) = 0 \). Recall

\[ A(0) \equiv X(0) - E(X(a) - A(a) \mid F_0) = -E(X(a) - A(a) \mid F_0) \]

and so

\[ A(0) = \lim_{k \to \infty} -E(X(a) - A^{n_k}(a) \mid F_0) = \lim_{k \to \infty} -E(M^{n_k}(a) \mid F_0) = \lim_{k \to \infty} -M^{n_k}(0) = 0. \]

This proves the theorem except for the claim that \( A(t) \) is natural. Let \( \xi(t) \) be a bounded right continuous martingale. I need to consider

\[ E\left(\int_{(0,t]} \xi_-(s) \, dA(s)\right) \]

and show it equals \( \xi(t) A(t) \). First consider the case \( t = a \). By Lemma 61.7.8

\[ E\left(\int_{(0,a]} \xi_-(s) \, dA(s)\right) = E\left(\lim_{k \to \infty} \sum_{j=1}^{m_{n_k}} \xi(t_{j-1}^{n_k}) (A(t_j^{n_k}) - A(t_{j-1}^{n_k}))\right) \quad (61.7.37) \]

Since \( \xi \) is bounded, you can take the limit outside. This follows from the dominated convergence theorem and the fact, shown above that \( A \) is increasing and nonnegative. Here is why.

\[ 0 \leq |\xi(t_{j-1}^{n_k})| A(t_j^{n_k}) \leq A(a) C \]
61.7. DOOB MEYER DECOMPOSITION

where $C$ is a constant larger than the values of $\xi$. Thus the above equals

$$\lim_{k \to \infty} E \left( \sum_{j=1}^{m_n} \xi (t_{j-1}^{n_k}) \left( A (t_j^{n_k}) - A (t_{j-1}^{n_k}) \right) \right)$$

$$= \lim_{k \to \infty} E \left( \sum_{j=1}^{m_n} \xi (t_{j-1}^{n_k}) \left( X (t_j^{n_k}) - M (t_j^{n_k}) - (X (t_{j-1}^{n_k}) - M (t_{j-1}^{n_k})) \right) \right)$$

$$= \lim_{k \to \infty} E \left( \sum_{j=1}^{m_n} \xi (t_{j-1}^{n_k}) \left( X (t_j^{n_k}) - X (t_{j-1}^{n_k}) \right) \right)$$

(61.7.38)

because

$$E \left( \xi (t_{j-1}^{n_k}) M (t_j^{n_k}) \right) = E \left( E \left( \xi (t_{j-1}^{n_k}) M (t_j^{n_k}) \mid \mathcal{F}_{t_{j-1}^{n_k}} \right) \right)$$

$$= E \left( E (t_{j-1}^{n_k}) E \left( M (t_j^{n_k}) \mid \mathcal{F}_{t_{j-1}^{n_k}} \right) \right)$$

$$= E \left( \xi (t_{j-1}^{n_k}) M (t_j^{n_k}) \right)$$

since $M$ is a martingale. Now by a similar trick, this time using that $\{M^{n_k} (t_j^{n_k})\}_{j=0}^{m_n}$ is a martingale, it equals

$$\lim_{k \to \infty} E \left( \sum_{j=1}^{m_n} \xi (t_{j-1}^{n_k}) \left( A^{n_k} (t_j^{n_k}) - A^{n_k} (t_{j-1}^{n_k}) \right) \right)$$

(61.7.39)

and now recall that $A^{n_k} (t_j^{n_k})$ is $\mathcal{F}_{t_{j-1}^{n_k}}$ measurable. This will now be used to change the subscript of $t_{j-1}^{n_k}$ in $\xi (t_{j-1}^{n_k})$ to a $j$. It equals

$$= \lim_{k \to \infty} \sum_{j=1}^{m_n} E \left( E \left( \xi (t_j^{n_k}) \mid \mathcal{F}_{t_{j-1}^{n_k}} \right) \left( A^{n_k} (t_j^{n_k}) - A^{n_k} (t_{j-1}^{n_k}) \right) \right)$$

$$= \lim_{k \to \infty} \sum_{j=1}^{m_n} E \left( E \left( \xi (t_j^{n_k}) \left( A^{n_k} (t_j^{n_k}) - A^{n_k} (t_{j-1}^{n_k}) \right) \mid \mathcal{F}_{t_{j-1}^{n_k}} \right) \right)$$

$$= \lim_{k \to \infty} \sum_{j=1}^{m_n} E \left( \xi (t_j^{n_k}) \left( A^{n_k} (t_j^{n_k}) - A^{n_k} (t_{j-1}^{n_k}) \right) \right)$$

$$= \lim_{k \to \infty} E \left( \sum_{j=1}^{m_n} \xi (t_j^{n_k}) \left( A^{n_k} (t_j^{n_k}) - A^{n_k} (t_{j-1}^{n_k}) \right) \right)$$
From this all that remains is to write the above as

\[
\lim_{k \to \infty} E \left( \sum_{j=1}^{m_{n_k}} \xi (t_{j+1}^{n_k}) A^{n_k} (t_j^{n_k}) - \sum_{j=0}^{m_{n_k}-1} \xi (t_{j+1}^{n_k}) A^{n_k} (t_j^{n_k}) \right)
\]

\[
= \lim_{k \to \infty} \left( E (\xi (a) A^{n_k} (a)) + E \left( \sum_{j=1}^{m_{n_k}-1} (\xi (t_j^{n_k}) - \xi (t_{j+1}^{n_k})) A^{n_k} (t_j^{n_k}) \right) \right)
\]

Now using the fact \( \xi \) is a martingale, this last term equals 0. Here is why.

\[
E (\xi (t_{j+1}^{n_k}) A^{n_k} (t_j^{n_k})) = E \left( E \left( \xi (t_{j+1}^{n_k}) A^{n_k} (t_j^{n_k}) \mid \mathcal{F}_{t_j^{n_k}} \right) \right)
\]

\[
= E \left( A^{n_k} (t_j^{n_k}) \xi (t_{j+1}^{n_k}) \mid \mathcal{F}_{t_j^{n_k}} \right)
\]

The first term converges to \( E (\xi (a) A (a)) \) because this was how \( A (a) \) was obtained, as a weak limit in \( L^1 (\Omega) \) of \( A^{n_k} (a) \). Also by Lemma 61.7.7,

\[
E (\xi (a)) = E \left( \int_{[0, a]} \xi (s) dA (s) \right)
\]

From 61.7.37 this has now shown that

\[
E (\xi (a) A (a)) = E \left( \int_{[0, a]} \xi (s) dA (s) \right)
\]

To get the desired result on \((0, t]\), apply what was just shown to a “stopped martingale”,

\[
\xi^t (s) \equiv \begin{cases} 
\xi (s) & \text{if } s \leq t \\
\xi (t) & \text{if } s > t
\end{cases}
\]

\[
E \left( \int_{[0, t]} \xi (s) dA (s) \right) + (A (a) - A (t)) E (\xi (t))
\]

\[
= E \left( \int_{[0, a]} \xi^t (s) dA (s) \right)
\]

From what was shown above,

\[
= E \left( \int_{[0, a]} \xi^t_-(s) dA (s) \right)
\]

\[
= E \left( \int_{[0, t]} \xi^- (s) dA (s) + \int_{[t, a]} \xi (t) sA (s) \right)
\]

\[
= E \left( \int_{[0, t]} \xi^- (s) dA (s) \right) + (A (a) - A (t)) E (\xi (t))
\]
and so

\[ E \left( \int_{[0,t]} \xi(s) dA(s) \right) = E \left( \int_{[0,t]} \xi^-(s) dA(s) \right) \]

which shows \( A \) is natural by Lemma 61.7.7. This proves the theorem.

There is another interesting variation of the above theorem. It involves the following definition.

**Definition 61.7.13** A submartingale, \( \{X(t)\} \) is said to be \( D \) if

\[ \{X_T : T < \infty \text{ is a stopping time}\} \]

is equi integrable.

In this case, you can consider partitions of the entire positive real line and the martingales, \( \{M(t^n_k)\} \) and \( \{A(t^n_k)\} \) as before. This time you don’t stop at \( m_n \). By the submartingale convergence theorem, you can argue there exists \( A^n = \lim_{k \to \infty} A(t^n_k) \). Then repeat the above argument using \( A^n \) in place of \( A^n(a) \). This time you get \( \{A(t)\} \) equi integrable. Thus the following corollary is obtained.

**Corollary 61.7.14** Let \( \{X(t)\} \) be a right continuous submartingale of class \( D \). Then there exists a right continuous martingale, \( \{M(t)\} \) and a right continuous increasing submartingale, \( \{A(t)\} \) such that for each \( t \),

\[ X(t) = M(t) + A(t) \]

If \( \{A(t)\} \) is chosen to be natural and \( A(0) = 0 \), then with this condition, \( \{M(t)\} \) and \( \{A(t)\} \) are unique. Furthermore \( \{M(t)\} \) and \( \{A(t)\} \) are equi integrable on \([0, \infty)\).

In the above theorem, \( \{X(t)\} \) was a submartingale and so it has a right continuous version. What if \( \{X(t)\} \) is actually continuous? Can one conclude that \( A(t) \) and \( M(t) \) are also continuous? The answer is yes.

**Theorem 61.7.15** Let \( \{X(t)\} \) be a right continuous submartingale of class \( DL \). Then there exists a right continuous martingale, \( \{M(t)\} \) and a right continuous increasing submartingale, \( \{A(t)\} \) such that for each \( t \),

\[ X(t) = M(t) + A(t) \]

If \( \{A(t)\} \) is chosen to be natural and \( A(0) = 0 \), then with this condition, \( \{M(t)\} \) and \( \{A(t)\} \) are unique. Also, if \( \{X(t)\} \) is continuous, \( t \to X(t, \omega) \) is continuous for a.e. \( \omega \) then the same is true of \( \{A(t)\} \) and \( \{M(t)\} \).

**Proof:** The first part is done above. Let \( \{X(t)\} \) be continuous. As before, let \( \{t^n_k\} \) be a sequence of partitions of \([0, a]\) such that these are equally spaced points, \( \lim_{n \to \infty} t^n_{k+1} - t^n_k = 0 \), and \( \{t^n_k\}_{k=0}^{m_n} \subseteq \{t^{n+1}_k\}_{k=0}^{m_{n+1}} \) where here \( a > 0 \) is an arbitrary positive number and let \( \lambda > 0 \) be an arbitrary positive number. Define

\[ \xi^n(t) = E(\min(\lambda, A(t^n_j)) | \mathcal{F}_s) \] for \( t^j_{j-1} < t \leq t^j_n \)
Thus on \((t_{j-1}^n, t_j^n]\) \(\xi^n_t\) is a bounded martingale. Assuming we are dealing with a right continuous version of this martingale so there are no measurability questions, it follows since \(A\) is natural,

\[
E \left( \int_{(t_{j-1}^n, t_j^n]} \xi^n_t \, dA(s) \right) = E \left( \int_{(t_{j-1}^n, t_j^n]} \xi^n_s \, dA(s) \right)
\]

where here

\[
\xi^n_{s, \omega} = \lim_{r \to s^-} \xi^n_{r, \omega} \quad \text{a.e.}
\]

for \(D = \cup_{n=1}^{\infty} \cup_{k=1}^{m_n} \{ t_k^n \}_{k=0}^{m_n} \). Thus, adding these up for all the intervals, \((t_{j-1}^n, t_j^n]\) yields

\[
E \left( \int_{(0,a]} \xi^n_t \, dA(s) \right) = E \left( \int_{(0,a]} \xi^n_s \, dA(s) \right)
\]

I want to show that for a.e. \(\omega, \xi^{n_k}(t, \omega)\) converges uniformly to

\[
\min(\lambda, A(t, \omega)) = \lambda \wedge A(t, \omega)
\]

on \((0,a]\). From this it will follow

\[
E \left( \int_{(0,a]} \lambda \wedge A(s, \omega) \, dA(s) \right) = E \left( \int_{(0,a]} \lambda \wedge A_- (s, \omega) \, dA(s) \right)
\]

Now since \(s \to A(s, \omega)\) is increasing, there is no problem in writing \(A_- (s, \omega)\) and the above equation will suffice to show with simple considerations that for a.e. \(\omega, s \to A(s, \omega)\) is left continuous. Since \(\{A(s)\}\) is a submartingale already, it has a right continuous version which we are using in the above. Thus for a.e. \(\omega\) it must be the case that \(s \to A(s, \omega)\) is continuous. Let \(t \in (t_{j-1}^n, t_j^n]\). Then since \(\lambda \wedge A(t)\) is \(\mathcal{F}_t\) measurable,

\[
\xi^n_t - \lambda \wedge A(t) = E \left( \lambda \wedge A(t_{j}^n) - \lambda \wedge A(t) | \mathcal{F}_t \right) \geq 0
\]

because \(A(t)\) is increasing.

Now define a stopping time, \(T^n(\varepsilon)\) for \(\varepsilon > 0\) by letting \(T^n(\varepsilon)\) be the infimum of all \(t \in [0,a]\) with the property that

\[
\xi^n_t - \lambda \wedge A(t) > \varepsilon
\]

or if this does not happen, then \(T^n(\varepsilon) = a\). Thus

\[
T^n(\varepsilon) = a \wedge \inf \{ t \in [0,a] : \xi^n_t - \lambda \wedge A(t, \omega) > \varepsilon \}
\]

I need to verify \(T^n(\varepsilon)\) really is a stopping time. Letting \(s < a\), it follows that if \(\omega \in [T^n(\varepsilon) \leq s]\), then for each \(N\), there exists \(t \in [s, s + \frac{1}{N}]\) such that \(\xi^n(t, \omega) - \lambda \wedge A(t, \omega) > \varepsilon\). Then by right continuity it follows there exists \(r \in D \cap [s, s + \frac{1}{N}]\) such that

\[
\xi^n(r, \omega) - \lambda \wedge A(r, \omega) > \varepsilon
\]
61.7. DOOB MEYER DECOMPOSITION

Thus

\[ [T^n (\varepsilon) \leq s] = \bigcap_{N=1}^{\infty} \bigcup_{r \in D \cap [s, s+\frac{1}{N}]} [\xi^n (r, \omega) - \lambda \wedge A (r, \omega) > \varepsilon] \]

and each \( \bigcup_{r \in D \cap [s, s+\frac{1}{N}]} [\xi^n (r, \omega) - \lambda \wedge A (r, \omega) > \varepsilon] \in \mathcal{F}_{s+1/N} \) and so \([T^n (\varepsilon) \leq s] \in \bigcap_{r \in D, r \geq s} \mathcal{F}_r = \mathcal{F}_s^+ = \mathcal{F}_s \) due to the assumption that the filtration is normal. What if \( s \geq \tilde{a} \)? Then from the definition, \([T^n (\varepsilon) \leq a] = \Omega \in \mathcal{F}_a \). Thus this really is a stopping time.

Now let \( B_j \equiv [t^n_{j-1} < T^n (\varepsilon) \leq t^n_j] \). Note that \([T^n (\varepsilon) \wedge t^n_j] \) is also a stopping time.

\[
\int_{\Omega} \xi^n_{T^n (\varepsilon)} d\mathbb{P} = \sum_{j=1}^{m_n} \int_{B_j} \xi^n_{T^n (\varepsilon)} d\mathbb{P} = \sum_{j=1}^{m_n} \int_{B_j} \xi^n_{T^n (\varepsilon) \wedge t^n_j} d\mathbb{P}
\]

This is because \( B_j \in \mathcal{F}_{T^n (\varepsilon) \wedge t^n_j} \). Thus from the definition, the above equals

\[
= \sum_{j=1}^{m_n} \int_{B_j} E \left( E \left( \lambda \wedge A (t^n_j) \mid \mathcal{F}_{T^n (\varepsilon) \wedge t^n_j} \right) \right) d\mathbb{P}
\]

\[
= \sum_{j=1}^{m_n} \int_{B_j} \lambda \wedge A (t^n_j) d\mathbb{P} = \int_{\Omega} \lambda \wedge A (\lfloor T^n (\varepsilon) \rfloor) d\mathbb{P} \quad (61.7.40)
\]

where on \((t^n_{j-1}, t^n_j), [T^n (\varepsilon)] = t^n_j\). Now \([T^n (\varepsilon)] \) is also a bounded stopping time. Here is why. Suppose \( s \in (t^n_{j-1}, t^n_j) \). Then

\[
\lfloor T^n (\varepsilon) \rfloor \leq s = \lfloor T^n (\varepsilon) \leq t^n_{j-1} \rfloor \in \mathcal{F}_{t^n_{j-1}} \subseteq \mathcal{F}_s.
\]

Now let

\[
Q_n = \sup_{t \in [0, a]} |\xi^n (t) - \lambda \wedge A (t)|.
\]

Then first note that

\[
[Q_n > \varepsilon] = \left[ \sup_{t \in [0, a]} |\xi^n (t) - \lambda \wedge A (t)| > \varepsilon \right]
\]

because \( Q_n (a) = 0 \) follows from the definition of \( \xi^n (t) \) as

\[
E (\lambda \wedge A (t^n_j) \mid \mathcal{F}_t) \text{ for } t^n_{j-1} < t \leq t^n_j
\]

and so

\[
\xi^n (a) = E (\lambda \wedge A (a) \mid \mathcal{F}_a) = \lambda \wedge A (a).
\]
Thus it suffices to take the supremum over the half open interval, \([0, a)\). It follows
\[
[Q_n > \varepsilon] = [T^n(\varepsilon) < a]
\]
By right continuity,
\[
\xi^n(T^n(\varepsilon)) - \lambda \land A(T^n(\varepsilon)) \geq \varepsilon
\]
on \([Q_n > \varepsilon]\).

\[
\varepsilon P([Q_n > \varepsilon]) = \varepsilon P([T^n(\varepsilon) < a])
\]
\[
\leq \int_{[Q_n > \varepsilon]} (\xi^n(T^n(\varepsilon)) - \lambda \land A(T^n(\varepsilon))) dP
\]
\[
\leq \int \Omega (\xi^n(T^n(\varepsilon)) - \lambda \land A(T^n(\varepsilon))) dP
\]

Therefore, from (61.7.40),
\[
P([Q_n > \varepsilon]) \leq \frac{1}{\varepsilon} \int \Omega (\lambda \land A([T^n(\varepsilon)]) - \lambda \land A(T^n(\varepsilon))) dP
\]
\[
\leq \frac{1}{\varepsilon} \int \Omega (A([T^n(\varepsilon)]) - A(T^n(\varepsilon))) dP
\]  

(61.7.41)

By optional sampling theorem,
\[
E(M(T^n(\varepsilon))) = E(M(0)) = 0
\]
and also
\[
E(M([T^n(\varepsilon)])) = E(M(0)) = 0.
\]
Therefore, (61.7.41) reduces to
\[
P([Q_n > \varepsilon]) \leq \frac{1}{\varepsilon} \int \Omega (X([T^n(\varepsilon)]) - X(T^n(\varepsilon))) dP
\]

By the assumption that \(\{X(t)\}\) is DL, it follows the functions in the above integrand are equi integrable and so since \(\lim_{n \to \infty} X([T^n(\varepsilon)]) - X(T^n(\varepsilon)) = 0\), the above integral converges to 0 as \(n \to \infty\) by Vitali’s convergence theorem, Theorem 9.5.3 on Page 244. It follows that there is a subsequence, \(n_k\) such that
\[
P([Q_{n_k} > 2^{-k}]) \leq 2^{-k}
\]
and so from the definition of \(Q_n\),
\[
\lim_{k \to \infty} \sup_{t \in [0,a]} |\xi^{n_k}(t) - \lambda \land A(t)|
\]
giving uniform convergence. Now recall that
\[
E \left( \int_{[0,a]} \xi^{n_k}(s) dA(s) \right) = E \left( \int_{[0,a]} \xi^{n_k}(s) dA(s) \right)
\]
and so passing to the limit as \( k \to \infty \) with the uniform convergence yields

\[
E \left( \int_{(0,a]} \lambda \wedge A(s) \, dA(s) \right) = E \left( \int_{(0,a]} \lambda \wedge A_-(s) \, dA(s) \right)
\]

Now let \( \lambda \to \infty \). Then from the monotone convergence theorem,

\[
E \left( \int_{(0,a]} A(s) \, dA(s) \right) = E \left( \int_{(0,a]} A_-(s) \, dA(s) \right)
\]

and so for a.e. \( \omega \),

\[
\int_{(0,a]} (A(s) - A_-(s)) \, dA(s) = 0.
\]

Thus letting the measure associated with this Lebesgue integral be denoted by \( \mu \),

\[
A(s) - A_-(s) = 0 \text{ a.e.}
\]

Suppose then that \( A(s) - A_-(s) > 0 \). Then \( \mu (\{s\}) = 0 = A(s) - A(s-) \), a contradiction. Hence \( A(s) - A_-(s) = 0 \) for all \( s \). It is already the case that \( s \to A(s) \) is right continuous. Therefore, this proves the theorem.

**Example 61.7.16** Suppose \( \{M(t)\} \) is a continuous martingale. Assume

\[
\sup_{t \in [0,a]} ||M(t)||_{L^2(\Omega)} < \infty
\]

Then \( \{||M(t)||\} \) is a submartingale and so is \( \{||M(t)||^2\} \). By Example 61.7.14, this is DL. Then there exists a unique Doob Meyer decomposition,

\[
||M(t)||^2 = Y(t) + \langle ||M(t)|| \rangle
\]

where \( Y(t) \) is a martingale and \( \{||M(t)||\} \) is a submartingale which is continuous, natural, increasing and equal to 0 when \( t = 0 \). This submartingale is called the quadratic variation.

### 61.8 Levy’s Theorem

This remarkable theorem has to do with when a martingale is a Wiener process. The proof I am giving here follows [11].

**Definition 61.8.1** Let \( W(t) \) be a stochastic process which has the properties that whenever \( t_1 < t_2 < \cdots < t_m \), the increments \( \{W(t_i) - W(t_{i-1})\} \) are independent and whenever \( s < t \), it follows \( W(t) - W(s) \) is normally distributed with variance \( t - s \) and mean 0. Also \( t \to W(t) \) is Holder continuous with every exponent \( \gamma < 1/2 \) and \( W(0) = 0 \). This is called a Wiener process.
First here is a lemma.

**Lemma 61.8.2** Let \( \{X(t)\} \) be a real martingale adapted to the filtration \( \mathcal{F}_t \) for \( t \in [a, b] \) some interval such that for all \( t \in [a, b] \), \( E(X(t)^2) < \infty \). Then \( \{X(t)^2 - t\} \) is also a martingale if and only if whenever \( s < t \),

\[
E \left( (X(t) - X(s))^2 \middle| \mathcal{F}_s \right) = t - s.
\]

**Proof:** Suppose first \( \{X(t)^2 - t\} \) is a real martingale. Then since \( \{X(t)\} \) is a martingale,

\[
E \left( (X(t) - X(s))^2 \middle| \mathcal{F}_s \right) = E \left( X(t)^2 - 2X(t)X(s) + X(s)^2 \middle| \mathcal{F}_s \right)
\]

\[
= E \left( X(t)^2 \middle| \mathcal{F}_s \right) - 2E(X(t)X(s) \middle| \mathcal{F}_s) + X(s)^2
\]

\[
= E \left( X(t)^2 \middle| \mathcal{F}_s \right) - 2X(s)E(X(t) \middle| \mathcal{F}_s) + X(s)^2
\]

\[
= E \left( X(t)^2 - t \middle| \mathcal{F}_s \right) - 2X(s)^2 + X(s)^2
\]

\[
= E \left( X(t)^2 - t \middle| \mathcal{F}_s \right) + t - X(s)^2
\]

\[
= X(s)^2 - s + t - X(s)^2 = t - s
\]

Next suppose \( E \left( (X(t) - X(s))^2 \middle| \mathcal{F}_s \right) = t - s \). Then since \( \{X(t)\} \) is a martingale,

\[
t - s = E \left( X(t)^2 - X(s)^2 \middle| \mathcal{F}_s \right)
\]

\[
= E \left( X(t)^2 - t \middle| \mathcal{F}_s \right) + t - X(s)^2
\]

and so

\[
0 = E \left( X(t)^2 - t \middle| \mathcal{F}_s \right) - \left( X(s)^2 - s \right)
\]

which proves the converse.

**Theorem 61.8.3** Suppose \( \{X(t)\} \) is a real stochastic process which satisfies all the conditions of a real Wiener process except the requirement that it be continuous. Then both \( \{X(t)\} \) and \( \{X(t)^2 - t\} \) are martingales.

**Proof:** First define the filtration to be

\[
\mathcal{F}_t \equiv \sigma \left( X(s) - X(r) : r \leq s \leq t \right).
\]

**Claim:** If \( A \in \mathcal{F}_s \), then

\[
\int \Omega X_A (X(t) - X(s)) \, dP = P(A) \int \Omega (X(t) - X(s)) \, dP.
\]
**Proof of claim:** Let $\mathcal{G}$ denote those sets of $\mathcal{F}_s$ for which the above formula holds. Then it is clear that $\mathcal{G}$ is closed with respect to countable unions of disjoint sets and complements. Let $\mathcal{K}$ denote those sets which are finite intersections of sets of the form $(X(u) - X(r))^{-1}(B)$ where $B$ is a Borel set and $r \leq u \leq s$. Say a set, $A$ of $\mathcal{K}$ is of the form

$$\bigcap_{i=1}^{m} (X(u_i) - X(r_i))^{-1}(B_i)$$

Then since disjoint increments are independent, linear combinations of the random variables, $X(u_i) - X(r_i)$ are normally distributed. Consequently, $(X(u_1) - X(r_1), \ldots, X(u_m) - X(r_m), X(t) - X(s))$ is multivariate normal. The covariance matrix is of the form

$$\begin{pmatrix} A & 0 \\ 0 & t-s \end{pmatrix}$$

and so the random vector, $(X(u_1) - X(r_1), \ldots, X(u_m) - X(r_m))$ and the random variable $X(t) - X(s)$ are independent. Consequently, $X_A$ is independent of $X(t) - X(s)$ for any $A \in \mathcal{K}$. Then by the lemma on $\pi$ systems, Lemma 10.12.3 on Page 61.8, $\mathcal{F}_s \supseteq \mathcal{G} \supseteq \sigma(\mathcal{K}) = \mathcal{F}_s$. This proves the claim.

Thus

$$\int_A (X(t) - X(s)) dP = \int_\Omega (X(t) - X(s)) X_A dP$$

$$= P(A) \int_\Omega (X(t) - X(s)) dP = 0$$

which shows that since $A \in \mathcal{F}_s$ was arbitrary,

$$E (X(t) | \mathcal{F}_s) = X(s)$$

and $\{X(t)\}$ is a martingale.

Now consider whether $\{X(t)^2 - t\}$ is a martingale. By assumption,

$$\mathcal{L}(X(t) - X(s)) = \mathcal{L}(X(t-s)) = N(0, t-s).$$

Then for $A \in \mathcal{F}_s$, the independence of $X_A$ and $X(t) - X(s)$ shows

$$\int_A E \left( (X(t) - X(s))^2 | \mathcal{F}_s \right) dP = \int_A (X(t) - X(s))^2 dP$$

$$= P(A) (t-s) = \int_A (t-s) dP$$

and since $A \in \mathcal{F}_s$ is arbitrary,

$$E \left( (X(t) - X(s))^2 | \mathcal{F}_s \right) = t-s$$

and so the result follows from Lemma 61.8.2. This proves the theorem.

The next lemma is the main result from which Levy’s theorem will be established.
Lemma 61.8.4 Let \( \{X(t)\} \) be a real continuous martingale adapted to the filtration \( \mathcal{F}_t \) for \( t \in [a, b] \) some interval such that for all \( t \in [a, b] \), \( E\left(X(t)^2\right) < \infty \). Suppose also that \( \{X(t)^2 - t\} \) is a martingale. Then for \( \lambda \) real, 
\[
E(e^{i\lambda X(b)}) = E(e^{i\lambda X(a)}) e^{-(b-a)\lambda^2/2}
\]

Proof: Let \( \lambda \in [-p, p] \) where for most of the proof, \( p \) is fixed but arbitrary. Let \( \{t^n_k\}_{k=0}^{2^n} \) be uniform partitions such that \( t^n_k - t^n_{k-1} = \delta_n \equiv (b-a)/2^n \). Now for \( \varepsilon > 0 \) define a stopping time \( \tau_{\varepsilon, n} \) to be the first time, \( t \) such that there exist \( s_1, s_2 \in [a, t] \) with \( |s_1 - s_2| < \delta_n \) but 
\[
|X(s_1) - X(s_2)| = \varepsilon.
\]

If no such time exists, then \( \tau_{\varepsilon, n} \equiv b \).

Then \( \tau_{\varepsilon, n} \) really is a stopping time because from continuity of \( X(t) \) and denoting by \( r, r_1 \) elements of \( \mathcal{Q} \), then 
\[
[\tau_{\varepsilon, n} > t] = \bigcup_{m=1}^{\infty} \bigcap_{0 \leq r_1, r_2 \leq t, |r_1 - r_2| \leq \delta_n} \left[ |X(r_1) - X(r_2)| \leq \varepsilon - \frac{1}{m} \right] \in \mathcal{F}_t
\]
because to be in \( [\tau_{\varepsilon, n} > t] \) it means that by \( t \) the absolute value of the differences must always be less than \( \varepsilon \). Hence \( [\tau_{\varepsilon, n} \leq t] = \Omega \setminus [\tau_{\varepsilon, n} > t] \in \mathcal{F}_t \).

Now consider \( [\tau_{\varepsilon, n} = b] \) for various \( n \). By continuity, it follows that for each \( \omega \in \Omega \),
\[
\tau_{\varepsilon, n}(\omega) = b
\]
for all \( n \) large enough. Thus 
\[
\emptyset = \cap_{n=1}^{\infty} [\tau_{\varepsilon, n} < b],
\]
the sets in the intersection decreasing. Thus there exists \( n(\varepsilon) \) such that 
\[
P\left([\tau_{\varepsilon, n(\varepsilon)} < b]\right) < \varepsilon. \tag{61.8.42}
\]

Denote \( \tau_{\varepsilon, n(\varepsilon)} \) as \( \tau_\varepsilon \) for short and it will always be assumed that \( n(\varepsilon) \) is at least this large and that \( \lim_{\varepsilon \to 0^+} n(\varepsilon) = \infty \). In addition to this, \( n(\varepsilon) \) will also be large enough that 
\[
1 - \frac{\lambda^2}{2} \delta_{n(\varepsilon)} > 0
\]
for all \( \lambda \in [-p, p] \). To save on notation, \( t_j \) will take the place of \( t^n_j \). Then consider the stopping times \( \tau_\varepsilon \land t_j \) for \( j = 0, 1, \ldots, 2^{n(\varepsilon)} \).

Let \( y_j = X(\tau_\varepsilon \land t_j) - X((\tau_\varepsilon \land t_j) - 1) \), it follows from the definition of the stopping time that 
\[
|y_j| \leq \varepsilon \tag{61.8.43}
\]
because both $\tau_\varepsilon \land t_j$ and $\tau_\varepsilon \land t_{j-1}$ are less than $\tau_\varepsilon$ and closer together than $\delta_n(\varepsilon)$ and so if $|y_j| > \varepsilon$, then $\tau_\varepsilon \leq t_j, t_{j-1}$ and so $y_j$ would need to equal 0.

By the optional stopping theorem, $\{X (\tau_\varepsilon \land t_j)\}_j$ is a martingale as is also

$$\{X (\tau_\varepsilon \land t_j) − \tau_\varepsilon \land t_j\}_j.$$

Thus for $A \in \mathcal{F}_{\tau_\varepsilon \land t_{j-1}}$,

$$\int_A E (y_j^2 | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}}) \, dP = \int_A E \left( \left( X (\tau_\varepsilon \land t_j) − X (\tau_\varepsilon \land t_{j-1}) \right)^2 | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \, dP$$

$$= \int_A E \left( \left( X (\tau_\varepsilon \land t_j) \right)^2 | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) + X (\tau_\varepsilon \land t_{j-1})^2$$

$$− 2X (\tau_\varepsilon \land t_{j-1}) E \left( X (\tau_\varepsilon \land t_j) | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \, dP$$

$$= \int_A E \left( \left( X (\tau_\varepsilon \land t_j) \right)^2 − \tau_\varepsilon \land t_j | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \, dP + \int_A E \left( \tau_\varepsilon \land t_j | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \, dP$$

$$+ \int_A (\tau_\varepsilon \land t_{j-1})^2 \, dP − 2 \int_A (\tau_\varepsilon \land t_{j-1})^2 \, dP$$

$$= \int_A X (\tau_\varepsilon \land t_{j-1})^2 \, dP − \int_A \tau_\varepsilon \land t_{j-1} \, dP + \int_A E \left( \tau_\varepsilon \land t_j | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \, dP$$

$$+ \int_A X (\tau_\varepsilon \land t_{j-1})^2 \, dP − 2 \int_A (\tau_\varepsilon \land t_{j-1})^2 \, dP$$

$$= \int_A E \left( \tau_\varepsilon \land t_j | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \, dP − \int_A \tau_\varepsilon \land t_{j-1} \, dP$$

$$= \int_A (\tau_\varepsilon \land t_j − \tau_\varepsilon \land t_{j-1}) \, dP \leq \int_A t_j − t_{j-1} \, dP.$$

Thus, since $A$ is arbitrary,

$$\sigma_j^2 \equiv \int_A E (y_j^2 | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}}) \, dP =$$

$$E \left( \left( X (\tau_\varepsilon \land t_j) − X (\tau_\varepsilon \land t_{j-1}) \right)^2 | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) \leq t_j − t_{j-1} = \delta_n(\varepsilon) \quad (61.8.44)$$

Also,

$$E (y_j | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}}) = E \left( X (\tau_\varepsilon \land t_j) − X (\tau_\varepsilon \land t_{j-1}) | \mathcal{F}_{\tau_\varepsilon \land t_{j-1}} \right) = 0. \quad (61.8.45)$$

Now it is time to find $E \left( e^{i\lambda X(\tau_\varepsilon \land t_j)} \right)$,

$$E \left( e^{i\lambda X(\tau_\varepsilon \land t_j)} \right) = E \left( e^{i\lambda (X(\tau_\varepsilon \land t_{j-1}) + y_j)} \right)$$
\[ E \left( e^{i\lambda X(\tau \wedge t_{j-1})} E \left( e^{i\lambda y_j | \mathcal{F}_{\tau \wedge t_{j-1}}} \right) \right). \]  

(61.8.46)

Now let \( o(1) \) denote any quantity which converges to 0 as \( \varepsilon \to 0 \) for all \( \lambda \in [-p, p] \) and \( O(1) \) is a quantity which is bounded as \( \varepsilon \to 0 \). Then from 61.8.45 and 61.8.46 you can consider the power series for \( e^{i\lambda y_j} \) which converges uniformly due to 61.8.43 and write 61.8.46 as

\[ E \left( e^{i\lambda X(\tau \wedge t_{j-1})} \left( 1 - \frac{\lambda^2}{2} \sigma_j^2 (1 + o(1)) \right) \right). \]

then noting that from 61.8.44 which shows \( \sigma_j^2 \) is \( o(1) \), it is routine to verify

\[ 1 - \frac{\lambda^2}{2} \sigma_j^2 (1 + o(1)) = e^{-\frac{\lambda^2}{2} \sigma_j^2(1+o(1))}. \]

Now this shows

\[ E \left( e^{i\lambda X(\tau \wedge t_{j})} \right) = E \left( e^{i\lambda X(\tau \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \sigma_j^2(1+o(1))} \right) \]

Recall that \( \sigma_j^2 \leq \delta_n = t_j - t_{j-1} \). Consider

\[
\left| E \left( e^{i\lambda X(\tau \wedge t_{j})} \right) - E \left( e^{i\lambda X(\tau \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \sigma_j^2(1+o(1))} \right) \right|
\]

\[
= \left| E \left( e^{i\lambda X(\tau \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \sigma_j^2(1+o(1))} \right) - E \left( e^{i\lambda X(\tau \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \right) \right|
\]

\[
= \left| E \left( e^{i\lambda X(\tau \wedge t_{j-1})} \left( e^{-\frac{\lambda^2}{2} \sigma_j^2(1+o(1))} - e^{-\frac{\lambda^2}{2} \delta_n} \right) \right) \right|
\]

\[
= \left| E \left( e^{i\lambda X(\tau \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \left( e^{-\frac{\lambda^2}{2} \sigma_j^2(1+o(1))} - e^{-\frac{\lambda^2}{2} \delta_n} \right) \right) \right|
\]

\[
\leq \left| \left( e^{-\frac{\lambda^2}{2} \delta_n} \right) e^{-\frac{\lambda^2}{2} \delta_n} \right| - 1 \right|
\]

Everything in the exponent is \( o(1) \) and so the above expression is bounded by

\[ O(1) \left( \left| \frac{\lambda^2}{2} \right| \delta_n - \sigma_j^2 + \sigma_j^2 o(1) \right) \]

\[
\leq O(1) \left( \left| \delta_n - \sigma_j^2 \right| + \delta_n \right) \]

\[ = O(1) \left[ \delta_n - E \left( y_j^2 \right) + \delta_n o(1) \right]. \]  

(61.8.47)

Therefore,

\[
\left| E \left( e^{i\lambda X(\tau \wedge t_{j})} \right) - E \left( e^{i\lambda X(\tau \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \right) \right| \]

\[ \leq O(1) \left[ \delta_n - E \left( y_j^2 \right) + \delta_n o(1) \right] \]
LEVY’S THEOREM

and so it also follows
\[ \left| E \left( e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) e^{\frac{\lambda^2}{2} t_j} - E \left( e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{\frac{\lambda^2}{2} t_{j-1}} \right) \right| \]
\[ \leq O(1) \left[ \delta_n - E (y_j^2) + \delta_n o(1) \right] \]

Now also remember
\[ y_j = X(\tau_\varepsilon \wedge t_j) - X(\tau_\varepsilon \wedge t_{j-1}) \]
and that \( \{ X(\tau_\varepsilon \wedge t_j) \}_j \) is a martingale. Therefore it is routine to show,
\[ E (y_j^2) = E \left( X(\tau_\varepsilon \wedge t_j)^2 \right) - E \left( X(\tau_\varepsilon \wedge t_{j-1})^2 \right) \]
and so
\[ \left| E \left( e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) e^{\frac{\lambda^2}{2} t_j} - E \left( e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{\frac{\lambda^2}{2} t_{j-1}} \right) \right| \]
\[ \leq O(1) \left[ \delta_n - \left( E \left( X(\tau_\varepsilon \wedge t_j)^2 \right) - E \left( X(\tau_\varepsilon \wedge t_{j-1})^2 \right) \right) + \delta_n o(1) \right] \]
and so, summing over all \( j = 1, \cdots, 2^n(\varepsilon) \),
\[ \left| E \left( e^{i\lambda X(\tau_\varepsilon \wedge b)} \right) e^{\frac{\lambda^2}{2} b} - E \left( e^{i\lambda X(\tau_\varepsilon \wedge a)} e^{\frac{\lambda^2}{2} a} \right) \right| \]
\[ \leq O(1) \left( (1 + o(1)) (b - a) - \left( E \left( X(\tau_\varepsilon \wedge b)^2 \right) - E \left( X(a)^2 \right) \right) \right) \]

(61.8.48)

Now recall \( \ref{58.5.4} \) which said
\[ P \left( [\tau_\varepsilon < b] \right) < \varepsilon. \]

Let \( \varepsilon_k \equiv 2^{-k} \) and then by the Borel Cantelli lemma,
\[ X(\tau_\varepsilon \wedge b) \to X(b) \]
a.e. since if \( \omega \) is such that convergence does not take place, \( \omega \) must be in infinitely many of the sets, \( [\tau_{\varepsilon_k} < b] \), a set of measure 0. Also since \( \{ X(\tau_\varepsilon \wedge t_j) \}_j \) is a martingale, it follows from optional sampling theorem that \( \{ X(a)^2, X(\tau_\varepsilon \wedge b)^2, X(b)^2 \} \) is a submartingale and so
\[ \int_{[X(\tau_\varepsilon \wedge b)^2 \geq \alpha]} X(\tau_\varepsilon \wedge b)^2 dP \leq \int_{[X(\tau_\varepsilon \wedge b)^2 \geq \alpha]} X(b)^2 dP \]
and also from the maximal inequalities, Theorem \( \ref{58.5.4} \) on Page \( \ref{2016} \) it follows
\[ P \left( [X(\tau_\varepsilon \wedge b)^2 \geq \alpha] \right) \leq \frac{1}{\alpha} E \left( X(b)^2 \right) \]
2228  CHAPTER 61. THE QUADRATIC VARIATION OF A MARTINGALE

and so the functions, \( \left\{ X(\tau_n \wedge b) \right\}_n \), are uniformly integrable which implies by the Vitali convergence theorem, Theorem 61.8.4 on Page 244, that you can pass to the limit as \( \varepsilon_k \to 0 \) in the inequality, 61.8.4 and conclude

\[
E \left( e^{i\lambda X(b)} \right) e^{\frac{2}{\lambda}(b-a)} - E \left( e^{i\lambda X(a)} e^{\frac{2}{\lambda}(a)} \right) \leq O(1) \left\{ (b-a) - \left( E \left( X(b)^2 \right) - E \left( X(a)^2 \right) \right) \right\} = 0.
\]

Therefore,

\[
E \left( e^{i\lambda X(b)} \right) = E \left( e^{i\lambda X(a)} \right) e^{-\frac{2}{\lambda}(b-a)}
\]

This proves the lemma because \( p \) was arbitrary.

Now from this lemma, it is not hard to establish Levy’s theorem.

**Theorem 61.8.5** Let \( \{X(t)\} \) be a real continuous martingale adapted to the filtration \( \mathcal{F}_t \) for \( t \in [0, a] \) some interval such that for all \( t \in [0, a] \), \( E \left( X(t)^2 \right) < \infty \).

Suppose also that \( \{X(t)^2 - t\} \) is a martingale. Then for \( s < t \), \( X(t) - X(s) \) is normally distributed with mean 0 and variance \( t-s \). Also if \( 0 \leq t_0 < t_1 < \cdots < t_m \leq b \), then the increments \( \{X(t_j) - X(t_{j-1})\} \) are independent.

**Proof:** Let the \( t_j \) be as described above and consider the interval \([t_{m-1}, t_m]\) in place of \([a, b]\) in Lemma 61.8.4. Also let \( \lambda_k \) for \( k = 1, 2, \cdots, m \) be given. For \( t \in [t_{m-1}, t_m] \), and \( \lambda_m \neq 0 \),

\[
Z_{\lambda_m}(t) = \frac{1}{\lambda_m} \sum_{j=1}^{m-1} \lambda_j \left( X(t_j) - X(t_{j-1}) \right) + \left( X(t) - X(t_{m-1}) \right)
\]

Then it is clear that \( \{Z_{\lambda_m}(t)\} \) is a martingale on \([t_{m-1}, t_m]\). What is possibly less clear is that \( \{Z_{\lambda_m}(t)^2 - t\} \) is also a martingale. Note that \( Z_{\lambda_m}(t) = X(t) + Y \) where \( Y \) is measurable in \( \mathcal{F}_{t_{m-1}} \). Therefore, for \( s < t, s \in [t_{m-1}, t_m] \),

\[
E \left( Z_{\lambda_m}(t)^2 - t | \mathcal{F}_s \right) = E \left( X(t)^2 + 2X(t)Y + Y^2 - t | \mathcal{F}_s \right)
\]

\[
= X(s)^2 - s + 2E(\left(X(t)Y\right) | \mathcal{F}_s) + Y^2
\]

\[
= X(s)^2 - s + 2YX(s) + Y^2 = Z_{\lambda_m}(s)^2 - s
\]

and so Lemma 61.8.4 can be applied to conclude

\[
E \left( e^{i\lambda Z_{\lambda_m}(t_m)} \right) = E \left( e^{i\lambda Z_{\lambda_m}(t_{m-1})} \right) e^{-\frac{\lambda^2}{2}(t_m-t_{m-1})}.
\]

Now letting \( \lambda = \lambda_m \),

\[
E \left( e^{i \sum_{j=1}^{m} \lambda_j (X(t_j) - X(t_{j-1}))} \right) = E \left( e^{i \sum_{j=1}^{m-1} \lambda_j (X(t_j) - X(t_{j-1}))} \right) e^{-\frac{\lambda^2}{2}(t_m-t_{m-1})}.
\]
By continuity, this equation continues to hold for $\lambda_m = 0$. Then iterate this, using a similar argument on the first factor of the right side to eventually obtain

$$E \left( e^{i \sum_{j=1}^{m} \lambda_j (X(t_j) - X(t_{j-1}))} \right) = \prod_{j=1}^{m} e^{- \lambda_j^2 / 2 (t_j - t_{j-1})}.$$ 

Then letting all but one $\lambda_j$ equal zero, this shows the increment, $X(t_j) - X(t_{j-1})$ is a random variable which is normally distributed having variance $t_j - t_{j-1}$ and mean 0. The above formula also shows from Proposition 57.11.1 on Page [insert page number] that the increments are independent. This proves the theorem.
A real valued random variable $X$ is normally distributed with mean 0 and variance $\sigma^2$ if

$$P(X \in A) = \frac{1}{\sqrt{2\pi\sigma}} \int_A e^{-\frac{1}{2} \frac{x^2}{\sigma^2}} dx$$

Consider the characteristic function. By definition it is

$$\phi_X(\lambda) = \int e^{i\lambda x} d\lambda X(x)$$

where $\lambda X$ is the distribution measure for this random variable. Thus the characteristic function of this random variable is

$$\frac{1}{\sqrt{2\pi\sigma}} \int e^{i\lambda x} e^{-\frac{1}{2} \frac{x^2}{\sigma^2}} dx$$

One can then show through routine arguments that this equals

$$\exp\left(-\frac{1}{2} \sigma^2 \lambda^2\right)$$

### 62.1 Real Wiener Processes

Here is the definition of a Wiener process.

**Definition 62.1.1** Let $W(t)$ be a stochastic process which has the properties that whenever $t_1 < t_2 < \cdots < t_m$, the increments $\{W(t_i) - W(t_{i-1})\}$ are independent and whenever $s < t$, it follows $W(t) - W(s)$ is normally distributed with variance $t - s$ and mean 0. Also $t \to W(t)$ is Holder continuous with every exponent $\gamma < 1/2$ and $W(0) = 0$. This is called a Wiener process.

Do Wiener processes exist? Yes, they do. First here is a simple lemma which has really been done before. It depends on the Kolmogorov extension theorem, Theorem 57.2.3 on Page 1906.
Lemma 62.1.2 There exists a sequence, \( \{\xi_k\}_{k=1}^\infty \) of random variables such that
\[
\mathcal{L}(\xi_k) = N(0, 1)
\]
and \( \{\xi_k\}_{k=1}^\infty \) is independent.

Proof: Let \( i_1 < i_2 \cdots < i_n \) be positive integers and define
\[
\mu_{i_1 \cdots i_n}(F_1 \times \cdots \times F_n) \equiv \frac{1}{\sqrt{2\pi}} \int_{F_1 \times \cdots \times F_n} e^{-|x|^2/2} dx.
\]
Then for the index set equal to \( \mathbb{N} \) the measures satisfy the necessary consistency condition for the Kolmogorov theorem. Therefore, there exists a probability space, \( (\Omega, P, \mathcal{F}) \) and measurable functions, \( \xi_k : \Omega \to \mathbb{R} \) such that
\[
P\left( [\xi_{i_1} \in F_{i_1}] \cap [\xi_{i_2} \in F_{i_2}] \cdots \cap [\xi_{i_n} \in F_{i_n}] \right)
= \mu_{i_1 \cdots i_n}(F_1 \times \cdots \times F_n)
= P\left( [\xi_{i_1} \in F_{i_1}] \right) \cdots P\left( [\xi_{i_n} \in F_{i_n}] \right)
\]
which shows the random variables are independent as well as normal with mean 0 and variance 1.

Recall that the sum of independent normal random variables is normal. The Wiener process is just an infinite weighted sum of the above independent normal random variables, the weights depending on \( t \). Therefore, if the sum converges, it is not too surprising that the result will be normally distributed and the variance will depend on \( t \). This is the idea behind the following theorem.

Theorem 62.1.3 There exists a real Wiener process as defined in Definition 62.1.1. Furthermore, the distribution of \( W(t) - W(s) \) is the same as the distribution of \( W(t-s) \) and \( W \) is Holder continuous with exponent \( \gamma \) for any \( \gamma < 1/2 \). Also for each \( \alpha > 1 \),
\[
E(|W(t) - W(s)|^\alpha) \leq C_\alpha |t-s|^{\alpha/2} E(|W(1)|^\alpha)
\]

Proof: Let \( \{g_m\}_{m=1}^\infty \) be a complete orthonormal set in \( L^2(0, \infty) \). Thus, if \( f \in L^2(0, \infty) \),
\[
f = \sum_{i=1}^\infty \langle f, g_i \rangle_{L^2} g_i.
\]
The Wiener process is defined as
\[
W(t, \omega) \equiv \sum_{i=1}^\infty \langle X_{(0,i)}(t), g_i \rangle_{L^2} \xi_i(\omega)
\]
where the random variables, \( \{\xi_i\} \) are as described in Lemma 62.1.2. The series converges in \( L^2(\Omega) \) where \( (\Omega, \mathcal{F}, P) \) is the probability space on which the random
variables, ξ_i are defined. This will first be shown. Note first that from the indepen-
dence of the ξ_i,
\[ \int_{\Omega} \xi_i \xi_j dP = 0 \]
Therefore,
\[ \int_{\Omega} \left| \sum_{i=m}^{n} (\mathcal{X}_{(0,t)}, g_i)_{L^2} \xi_i (\omega) \right|^2 dP = \sum_{i=m}^{n} (\mathcal{X}_{(0,t)}, g_i)_{L^2}^2 \int_{\Omega} |\xi_i|^2 dP \]
\[ = \sum_{i=m}^{n} (\mathcal{X}_{(0,t)}, g_i)_{L^2}^2 \]
which converges to 0 as m, n → ∞. Thus the partial sums are a Cauchy sequence in \( L^2 (\Omega, P) \).

It just remains to verify this definition satisfies the desired conditions. First I
will show that \( \omega \to W(t, \omega) \) is normally distributed with mean 0 and variance t.
That it should be normally distributed is not surprising since it is just a sum of
independent random variables which are this way. Selecting a suitable subsequence,
\( \{ n_k \} \) it can be assumed
\[ W(t, \omega) = \lim_{k \to \infty} \sum_{i=1}^{n_k} (\mathcal{X}_{(0,t)}, g_i)_{L^2} \xi_i (\omega) \text{ a.e.} \]
and so from the dominated convergence theorem and the independence of the ξ_i,
\[ E \left( \exp (i \lambda W(t)) \right) = \lim_{k \to \infty} E \left( \exp \left( i \lambda \sum_{j=1}^{n_k} (\mathcal{X}_{(0,t)}, g_j)_{L^2} \xi_j (\omega) \right) \right) \]
\[ = \lim_{k \to \infty} E \left( \prod_{j=1}^{n_k} \exp \left( i \lambda (\mathcal{X}_{(0,t)}, g_j)_{L^2} \xi_j (\omega) \right) \right) \]
\[ = \lim_{k \to \infty} \prod_{j=1}^{n_k} E \left( \exp \left( i \lambda (\mathcal{X}_{(0,t)}, g_j)_{L^2} \xi_j (\omega) \right) \right) \]
\[ = \lim_{k \to \infty} \prod_{j=1}^{n_k} e^{-\frac{1}{2} \lambda^2 (\mathcal{X}_{(0,t)}, g_j)_{L^2}^2} \]
\[ = \lim_{k \to \infty} \exp \left( \sum_{j=1}^{n_k} -\frac{1}{2} \lambda^2 (\mathcal{X}_{(0,t)}, g_j)_{L^2}^2 \right) \]
\[ = \exp \left( -\frac{1}{2} \lambda^2 ||\mathcal{X}_{(0,t)}||_{L^2}^2 \right) = \exp \left( -\frac{1}{2} \lambda^2 t \right) , \]
the characteristic function of a normally distributed random variable having vari-
ance t and mean 0.
It is clear $W(0) = 0$. It remains to verify the increments are independent. To do this, consider

$$E \left( \exp \left( i [\lambda (W(t) - W(s)) + \mu (W(s) - W(r))] \right) \right)$$  \hspace{1cm} (62.1.1)

Is this equal to

$$E \left( \exp \left( i [\lambda (W(t) - W(s))] \right) \right) E \left( \exp \left( i [\mu (W(s) - W(r))] \right) \right)?$$  \hspace{1cm} (62.1.2)

Letting $n_k \to \infty$ such that convergence happens pointwise for each function of interest, and using the independence of the $\xi_i$, and the dominated convergence theorem as needed,

$$E \left( \exp \left( i \sum_{i=1}^{\infty} \lambda (X(s,t), g_i)_{L^2} \xi_i + \sum_{i=1}^{\infty} \mu (X(r,s), g_i)_{L^2} \xi_i \right) \right) = \lim_{k \to \infty} E \left( \exp \left( i \sum_{j=1}^{n_k} (\lambda (X(s,t), g_j)_{L^2} + \mu (X(r,s), g_j)_{L^2}) \xi_j \right) \right)$$

$$= \lim_{k \to \infty} E \left( \prod_{j=1}^{n_k} \exp \left( i (\lambda (X(s,t), g_j)_{L^2} + \mu (X(r,s), g_j)_{L^2}) \xi_j \right) \right)$$

$$= \lim_{k \to \infty} \prod_{j=1}^{n_k} E \left( \exp \left( i (\lambda (X(s,t), g_j)_{L^2} + \mu (X(r,s), g_j)_{L^2}) \xi_j \right) \right)$$

$$= \lim_{k \to \infty} \prod_{j=1}^{n_k} \exp \left( -\frac{1}{2} (\lambda X(s,t) + \mu X(r,s), g_j)_{L^2}^2 \right)$$

$$= \lim_{k \to \infty} \exp \left( -\frac{1}{2} \sum_{j=1}^{n_k} (\lambda X(s,t) + \mu X(r,s), g_j)_{L^2}^2 \right)$$

$$= \exp \left( -\frac{1}{2} \sum_{j=1}^{\infty} (\lambda X(s,t) + \mu X(r,s), g_j)_{L^2}^2 \right) = \exp \left( -\frac{1}{2} \|\lambda X(s,t) + \mu X(r,s)\|^2_{L^2} \right)$$

$$= \exp \left( -\frac{1}{2} \left[ \lambda^2 \|X(s,t)\|^2_{L^2} + \mu^2 \|X(r,s)\|^2_{L^2} \right] \right)$$

because the functions $\lambda X(s,t), \mu X(r,s)$ are orthogonal. Then this equals

$$= \exp \left( -\frac{1}{2} \left[ \lambda^2 (t-s) + \mu^2 (s-r) \right] \right)$$

$$= \exp \left( -\frac{1}{2} (t-s) \lambda^2 \right) \exp \left( -\frac{1}{2} (s-r) \mu^2 \right)$$
which equals and this shows the increments are independent. Obviously, this same argument shows this holds for any finite set of disjoint increments.

From the definition, if \( t > s \)

\[
W(t - s) = \sum_{k=1}^{\infty} (X_{(0, t-s)}, g_k)_{L^2} \xi_k
\]

while

\[
W(t) - W(s) = \sum_{k=1}^{\infty} (X_{(s, t)}, g_k)_{L^2} \xi_k.
\]

Then the same argument given above involving the characteristic function to show \( W(t) \) is normally distributed shows both of these random variables are normally distributed with mean 0 and variance \( t - s \) because they have the same characteristic function.

For example, ignoring the limit questions and proceeding formally,

\[
E \left( \exp \left( i\lambda (W(t) - W(s)) \right) \right) = E \left( \exp \left( i\lambda \left( \sum_{k=1}^{\infty} (X_{(s, t)}, g_k)_{L^2} \xi_k \right) \right) \right)
\]

\[
= E \left( \prod_{k=1}^{\infty} \exp \left( i\lambda (X_{(s, t)}, g_k)_{L^2} \xi_k \right) \right)
\]

\[
= \prod_{k=1}^{\infty} E \left( \exp \left( i\lambda (X_{(s, t)}, g_k)_{L^2} \xi_k \right) \right)
\]

\[
= \prod_{k=1}^{\infty} e^{-\frac{1}{2}\lambda^2 (X_{(s, t)}, g_k)_{L^2}^2}
\]

\[
= \exp \left( -\frac{1}{2} \lambda^2 \sum_{k=1}^{\infty} (X_{(s, t)}, g_k)_{L^2}^2 \right)
\]

\[
= \exp \left( -\frac{1}{2} \lambda^2 (t - s) \right)
\]

which is the characteristic function of a random variable having mean 0 and variance \( t - s \).

Finally note the distribution of \( W(t - s) \) is the same as the distribution of

\[
W(1)(t - s)^{1/2} = \sum_{k=1}^{\infty} (X_{(0, 1)}, g_k)_{L^2} \xi_k (t - s)^{1/2}
\]

because the characteristic function of this last random variable is the same as the characteristic function of \( W(t - s) \) which is \( e^{-\frac{1}{2}\lambda^2(t-s)} \) which follows from a simple computation. Since \( W(1) \) is a normally distributed random variable with mean 0 and variance 1,

\[
E \left( \exp \left( i\lambda W(1)(t - s)^{1/2} \right) \right) = e^{-\frac{1}{2}\lambda^2(t-s)}
\]
which is the same as the characteristic function of \( W(t-s) \).

Hence for any positive \( \alpha \),

\[
E(|W(t) - W(s)|^\alpha) = E(|W(t-s)|^\alpha) = E\left(|(t-s)^{1/2} W(1)|^\alpha\right) = |t-s|^{\alpha/2} E(|W(1)|^\alpha)
\]

(62.1.3)

It follows from Theorem 62.2.2 that \( W(t) \) is Holder continuous with exponent \( \gamma \) where \( \gamma \) is any positive number less than \( \beta/\alpha \) where \( \alpha/2 = 1 + \beta \). Thus \( \gamma \) is any constant less than \( \alpha^2 - 1/\alpha \).

The proof of the theorem, which only depended on \( \{\xi_i\}_{i=1}^\infty \) being independent random variables each normal with mean 0 and variance 1, implies the following corollary.

**Corollary 62.1.4** Let \( \{\xi_i\}_{i=1}^\infty \) be independent random variables each normal with mean 0 and variance 1. Then

\[
W(t, \omega) \equiv \sum_{i=1}^\infty (\mathcal{X}[0,t], g_i)_{L^2} \xi_i(\omega)
\]

is a real Wiener process. Furthermore, the distribution of \( W(t) - W(s) \) is the same as the distribution of \( W(t-s) \) and \( W \) is Holder continuous with exponent \( \gamma \) for any \( \gamma < 1/2 \). Also for each \( \alpha > 1 \),

\[
E(|W(t) - W(s)|^\alpha) \leq C_\alpha |t-s|^{\alpha/2} E(|W(1)|^\alpha)
\]

### 62.2 Nowhere Differentiability Of Wiener Processes

If \( W(t) \) is a Wiener process, it turns out that \( t \to W(t, \omega) \) is nowhere differentiable for a.e. \( \omega \). This fact is based on the independence of the increments and the fact that these increments are normally distributed.

First note that \( W(t) - W(s) \) has the same distribution as \( (t-s)^{1/2} W(1) \). This is because they have the same characteristic function. Next it follows that because of the independence of the increments and what was just noted that,

\[
P \left( \bigcap_{r=1}^5 \left| W(t + r\delta) - W(t + (r-1)\delta) \right| \leq K\delta \right) = \prod_{r=1}^5 P \left( \left| W(t + r\delta) - W(t + (r-1)\delta) \right| \leq K\delta \right) = \prod_{r=1}^5 \left( \frac{1}{\sqrt{2\pi}} \int_{-K\sqrt{\delta}}^{K\sqrt{\delta}} e^{-\frac{1}{2}t^2} dt \right)^5 \leq C\delta^{5/2}. \quad (62.2.4)
\]
With this observation, here is the proof which follows \cite{[108]} and according to this reference is due to Payley, Wiener and Zygmund and the proof is like one given by Dvoretsky, Erdős and Kakutani.

**Theorem 62.2.1** Let $W(t)$ be a Wiener process. Then there exists a set of measure 0, $N$ such that for all $\omega \notin N$,

$$t \rightarrow W(t, \omega)$$

is nowhere differentiable.

**Proof:** Let $[0, a]$ be an interval. If for some $\omega, t \rightarrow W(t, \omega)$ is differentiable at some $s$, then for some $n, p > 0$,

$$\left| \frac{W(t, \omega) - W(s, \omega)}{t - s} \right| \leq p$$

whenever $|t - s| < 5a2^{-n} \equiv 5\delta_n$. Define $C_{np}$ by

$$\{ \omega : \text{for some } s \in [0, a], \left| \frac{W(t, \omega) - W(s, \omega)}{t - s} \right| \leq p \text{ if } |t - s| \leq 5\delta_n \}.$$  \hfill (62.2.5)

Thus $\bigcup_{n,p \in \mathbb{N}} C_{np}$ contains the set of $\omega$ such that $t \rightarrow W(t, \omega)$ is differentiable for some $s \in [0, a)$.

Now define uniform partitions of $[0, a)$, \(\{t^n_k\}_{k=0}^{2^n}\) such that

$$|t^n_k - t^n_{k-1}| = a2^{-n} \equiv \delta_n$$

Let

$$D_{np} = \bigcup_{n=0}^{2^n-1} \left( \bigcap_{r=1}^{5} \left[ |W(t^n_k + r\delta_n, \omega) - W(t^n_k + (r - 1)\delta_n, \omega)| \leq 10p\delta_n \right] \right)$$

If $\omega \in C_{np}$, then for some $s \in [0, a)$, the condition of \cite{[108]} holds. Suppose $k$ is the number such that $s \in [t^n_{k-1}, t^n_k)$. Then for $r \in \{1, 2, 3, 4, 5\}$,

$$|W(t^n_k + r\delta_n, \omega) - W(t^n_k + (r - 1)\delta_n, \omega)| \leq |W(t^n_k + r\delta_n, \omega) - W(s, \omega)| + |W(s, \omega) - W(t^n_k + (r - 1)\delta_n, \omega)|$$

$$\leq 5p\delta_n + 5p\delta_n = 10p\delta_n$$

Thus $C_{np} \subseteq D_{np}$. Now from \cite{[108]}

$$P(D_{np}) \leq 2^nC_{np}^{\sqrt{2}/2} = Ca^{\sqrt{2}/2}2^n (2^{-n})^{\sqrt{2}/2} = C \left( \sqrt{a} \right)^{\sqrt{2}} 2^{-\sqrt{2} n}$$  \hfill (62.2.6)

Let

$$C_p = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} C_{kp} \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} D_{kp}.$$  

It was just shown in \cite{[108]} that $P(\bigcap_{k=n}^{\infty} D_{kp}) = 0$ and so $C_p$ has measure 0. Thus $\bigcup_{p=1}^{\infty} C_p$, the set of points, $\omega$ where $t \rightarrow W(t, \omega)$ could have a derivative has
measure 0. Taking the union of the exceptional sets corresponding to intervals $[0, n)$ for $n \in \mathbb{N}$, this proves the theorem.

This theorem on nowhere differentiability is very important because it shows it is doubtful one can define an integral $\int f(s) \, dW(s)$ by simply fixing $\omega$ and then doing some sort of Stieltjes integral in time. The reason for this is that the nowhere differentiability of $W$ implies it is also not of bounded variation on any interval since if it were, it would equal the difference of two increasing functions and would therefore have a derivative at a.e. point.

I have presented the theorem on nowhere differentiability for one dimensional Wiener processes but the same proof holds with minor modifications if you have defined the Wiener process in $\mathbb{R}^n$ or you could simply consider the components and apply the above result.

62.3 Wiener Processes In Separable Banach Space

Here is an important lemma on which the existence of Wiener processes will be based.

Lemma 62.3.1 There exists a sequence of real Wiener processes, $\{\psi_k(t)\}_{k=1}^{\infty}$ which have the following properties. Let $t_0 < t_1 < \cdots < t_n$ be an arbitrary sequence. Then the random variables

$$\{\psi_k(t_q) - \psi_k(t_{q-1}) : (q,k) \in (1, 2, \cdots, n) \times (k_1, \cdots, k_m)\} \quad (62.3.7)$$

are independent. Also each $\psi_k$ is Holder continuous with exponent $\gamma$ for any $\gamma < 1/2$ and for each $m \in \mathbb{N}$ there exists a constant $C_m$ independent of $k$ such that

$$\int_{\Omega} |\psi_k(t) - \psi_k(s)|^2 dP \leq C_m |t - s|^m \quad (62.3.8)$$

Proof: First, there exists a sequence $\{\xi_{ij}\}_{i,j}^{\infty}$ such that the $\{\xi_{ij}\}$ are independent and each normally distributed with mean 0 and variance 1. This follows from Lemma 62.1.2. Let $\{\xi_{i,j}\}_{i=1}^{\infty}$ be independent and normally distributed with mean 0 and variance 1. (Let $\theta$ be a one to one and onto map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Then define $\xi_{ij} \equiv \xi_{\theta^{-1}(i,j)}$.)

Let

$$\psi_k(t) = \sum_{j=1}^{\infty} \langle X_{[0,1], g_j}, L^2 \xi_{kj} \rangle \quad (62.3.9)$$

where $\{g_j\}$ is a orthonormal basis for $L^2(0, \infty)$. By Corollary 62.1.3, this defines a real Wiener process satisfying 62.3.8. It remains to show that the random variables

$$\psi_k(t_q) - \psi_k(t_{q-1}) \quad (62.3.10)$$

are independent.
Let
\[ P = \sum_{q=1}^{n} \sum_{r=1}^{m} s_{qr} (\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1})) \]
and consider \( E(e^{iP}) \). I want to use Proposition 57.11.1 on Page 1938. To do this I need to show \( E(e^{iP}) \) equals
\[ \prod_{q=1}^{n} \prod_{r=1}^{m} E\left( e^{is_{qr} (\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1}))} \right). \]

Using 62.3.9, \( E(e^{iP}) \) equals
\[ E\left( \exp\left( \sum_{q=1}^{n} \sum_{r=1}^{m} s_{qr} \sum_{j=1}^{\infty} (X_{[t_{q-1}, t_q]}(\xi_{k_r}), g_j)_{L^2} \xi_{k_r,j} \right) \right) \]
\[ = \lim_{N \to \infty} E\left( \exp\left( \sum_{q=1}^{n} \sum_{r=1}^{m} s_{qr} \sum_{j=1}^{N} (X_{[t_{q-1}, t_q]}(\xi_{k_r}), g_j)_{L^2} \xi_{k_r,j} \right) \right) \]

Now the \( \xi_{k_r,j} \) are independent by construction. Therefore, the above equals
\[ = \lim_{N \to \infty} \prod_{q=1}^{n} \prod_{r=1}^{m} \prod_{j=1}^{N} E\left( e^{is_{qr} (X_{[t_{q-1}, t_q]}(\xi_{k_r}), g_j)_{L^2} \xi_{k_r,j}} \right) \]
\[ = \lim_{N \to \infty} \prod_{q=1}^{n} \prod_{r=1}^{m} \exp\left( \frac{-1}{2} s_{qr}^2 (X_{[t_{q-1}, t_q]}(\xi_{k_r}), g_j)_{L^2}^2 \right) \]
\[ = \prod_{q=1}^{n} \prod_{r=1}^{m} \lim_{N \to \infty} \exp\left( \frac{-1}{2} s_{qr}^2 \sum_{j=1}^{N} (X_{[t_{q-1}, t_q]}(\xi_{k_r}), g_j)_{L^2}^2 \right) \]
\[ = \prod_{q=1}^{n} \prod_{r=1}^{m} \exp\left( \frac{-1}{2} s_{qr}^2 (t_q - t_{q-1}) \right) \]
\[ = \prod_{q=1}^{n} \prod_{r=1}^{m} E\left( e^{is_{qr} (\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1}))} \right) \]

because \( \psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1}) \) is normally distributed with variance \( t_q - t_{q-1} \) and mean 0. By Proposition 57.11.1 on Page 1938, it follows the random variables of 62.3.10 are independent. Note that as a special case, this also shows the random variables, \( \{\psi_k(t)\}_{k=1}^{\infty} \) are independent due to the fact \( \psi_k(0) = 0 \).

Recall Corollary 59.11.4 which is stated here for convenience.
Corollary 62.3.2 Let $E$ be any real separable Banach space. Then there exists a sequence, \( \{e_k\} \subseteq E \) such that for any \( \{\xi_k\} \) a sequence of independent random variables such that \( L(\xi_k) = N(0,1) \), it follows

\[
X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega)e_k
\]

converges a.e. and its law is a Gaussian measure defined on \( \mathcal{B}(E) \). Furthermore, \( \|e_k\|_E \leq \lambda_k \) where \( \sum \lambda_k < \infty \).

Now let \( \{\psi_k(t)\} \) be the sequence of Wiener processes described in Lemma 62.3.1. Then define a process with values in \( E \) by

\[
W(t) \equiv \sum_{k=1}^{\infty} \psi_k(t)e_k \tag{62.3.11}
\]

Then \( \psi_k(t)/\sqrt{t} \) is \( N(0,1) \) and so by Corollary 59.11.4 the law of

\[
W(t)/\sqrt{t} = \sum_{k=1}^{\infty} \left( \psi_k(t)/\sqrt{t} \right) e_k
\]

is a Gaussian measure. Therefore, the same is true of \( W(t) \). Similar reasoning applies to the increments, \( W(t) - W(s) \) to conclude the law of each of these is Gaussian. Consider the question whether the increments are independent. Let \( 0 \leq t_0 < t_1 < \cdots < t_m \) and let \( \phi_j \in E' \). Then by the dominated convergence theorem and the properties of the \( \{\psi_k\} \),

\[
E \left( \exp \left( i \sum_{j=1}^{m} \phi_j (W(t_j) - W(t_{j-1})) \right) \right)
\]
### 62.3. WIENER PROCESSES IN SEPARABLE BANACH SPACE

\[
\begin{align*}
&= \lim_{n \to \infty} E \left( \prod_{j=1}^{m} \exp \left( i \sum_{k=1}^{n} (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
&= \lim_{n \to \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E \left( \exp \left( i (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
&= \lim_{n \to \infty} \prod_{j=1}^{m} E \left( \exp \left( i \sum_{k=1}^{n} (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
&= \prod_{j=1}^{m} E \left( \exp \left( i \sum_{k=1}^{\infty} (\psi_k(t_j) - \psi_k(t_{j-1})) e_k \right) \right) \\
&= \prod_{j=1}^{m} E \left( \exp \left( i \phi_j \left( \sum_{k=1}^{\infty} (\psi_k(t_j) - \psi_k(t_{j-1})) e_k \right) \right) \right) \\
&= \prod_{j=1}^{m} E \left( \exp \left( i \phi_j \left( W(t_j) - W(t_{j-1}) \right) \right) \right)
\end{align*}
\]

which shows by Theorem 57.13.3 on Page 1945 that the random vectors,

\[
\{W(t_j) - W(t_{j-1})\}_{j=1}^{m}
\]

are independent.

It is also routine to verify using properties of the \(\psi_k\) and characteristic functions that \(L(W(t) - W(s)) = L(W(t - s))\). To see this, let \(\phi \in E\)

\[
E \left( \exp \left( i \phi \left( W(t) - W(s) \right) \right) \right)
\]

\[
= E \left( \exp \left( i \phi \sum_{k=1}^{\infty} (\psi_k(t) - \psi_k(s)) e_k \right) \right)
\]

\[
= \lim_{n \to \infty} E \left( \exp \left( i \phi \sum_{k=1}^{n} (\psi_k(t) - \psi_k(s)) e_k \right) \right)
\]

\[
= \lim_{n \to \infty} \prod_{k=1}^{n} E \left( \exp \left( i \phi (e_k) (\psi_k(t) - \psi_k(s)) \right) \right)
\]

\[
= \lim_{n \to \infty} \prod_{k=1}^{n} E \left( \exp \left( -\frac{1}{2} \phi(e_k)^2 (t - s) \right) \right)
\]

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( -\frac{1}{2} \phi(e_k)^2 (t - s) \right)
\]
which is the same as the result for

\[ E(\exp(i\phi(W(t-s)))) \]

and

\[ E(\exp(i\phi(\sqrt{t-s}W(1)))) \]

This has proved the following lemma.

**Lemma 62.3.3** Let \( E \) be a real separable Banach space. Then there exists an \( E \) valued stochastic process, \( W(t) \) such that \( \mathcal{L}(W(t)) \) and \( \mathcal{L}(W(t) - W(s)) \) are Gaussian measures and the increments, \( \{W(t) - W(s)\} \) are independent. Furthermore, the increment \( W(t) - W(s) \) has the same distribution as \( W(t-s) \) and \( W(t) \) has the same distribution as \( \sqrt{t}W(1) \).

Now I want to consider the question of Holder continuity of the functions, \( t \to W(t,\omega) \).

\[
\int_{\Omega} ||W(t) - W(s)||^\alpha dP = \int_{E} ||x||^\alpha d\mu_{W(t)-W(s)} = \int_{E} ||x||^\alpha d\mu_{\sqrt{t-s}W(1)} = \int_{\Omega} ||\sqrt{t-s}W(1)||^\alpha dP = |t-s|^\alpha/2 \int_{\Omega} ||W(1)||^\alpha dP = C_\alpha |t-s|^\alpha/2
\]

by Fernique’s theorem, Theorem 59.7.5. From the Kolmogorov Čentsov theorem, Theorem 60.2.2, it follows \( \{W(t)\} \) is Holder continuous with exponent \( \gamma < (\frac{\alpha}{2} - 1)/\alpha \).

This completes the proof of the following theorem.

**Theorem 62.3.4** Let \( E \) be a separable real Banach space. Then there exists a stochastic process, \( \{W(t)\} \) such that the distribution of \( W(t) \) and every increment, \( W(t) - W(s) \) is Gaussian. Furthermore, the increments corresponding to disjoint intervals are independent, \( \mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t-s)) = \mathcal{L}(\sqrt{t-s}W(1)) \).

Also for a.e. \( \omega, t \to W(t,\omega) \) is Holder continuous with exponent \( \gamma < 1/2 \).

### 62.4 An Example Of Martingales, Independent Increments

Here is an interesting lemma.

**Lemma 62.4.1** Let \( (W(t), \mathcal{F}_t) \) be a stochastic process which has independent increments having values in \( E \) a real separable Banach space. Let

\[ A \in \mathcal{F}_s = \sigma(W(u) - W(r) : 0 \leq r < u \leq s) \]
Suppose \( g(W(t) - W(s)) \in L^1(\Omega; E) \). Then the following formula holds.

\[
\int_\Omega XAg(W(t) - W(s)) \, dP = P(A) \int_\Omega g(W(t) - W(s)) \, dP \tag{62.4.12}
\]

**Proof:** Let \( G \) denote the set of all \( A \in \mathcal{F}_s \) such that \( 62.4.12 \) holds. Then it is obvious \( G \) is closed with respect to complements and countable disjoint unions. Let \( \mathcal{K} \) denote those sets which are finite intersections of the form

\[
A = \cap_{i=1}^m A_i
\]

where each \( A_i \) is in a set of \( \sigma(W(u_i) - W(r_i)) \) for some \( 0 \leq r_i < u_i \leq s \). For such \( A \), it follows

\[
A \in \sigma(W(u_i) - W(r_i), i = 1, \cdots, m).
\]

Now consider the random vector having values in \( E^{m+1} \),

\[
(W(u_1) - W(r_1), \cdots, W(u_m) - W(r_m), g(W(t) - W(s)))
\]

Let \( t^* \in (E')^m \) and \( s^* \in E' \).

\[
t^* \cdot (W(u_1) - W(r_1), \cdots, W(u_m) - W(r_m))
\]

can be written in the form \( g^* \cdot (W(\tau_1) - W(\eta_1), \cdots, W(\tau_l) - W(\eta_l)) \) where the intervals, \( (\eta_j, \tau_j) \) are disjoint and each \( \tau_j \leq s \). For example, suppose you have

\[
a(W(2) - W(1)) + b(W(2) - W(0)) + c(W(3) - W(1)),
\]

where obviously the increments are not disjoint. Then you would write the above expression as

\[
a(W(2) - W(1)) + b(W(2) - W(1)) + b(W(1) - W(0))
\]

\[
+ c(W(3) - W(2)) + c(W(2) - W(1))
\]

and then you would collect the terms to obtain

\[
b(W(1) - W(0)) + (a + b + c)(W(2) - W(1)) + c(W(3) - W(2))
\]

and now these increments are disjoint.

Therefore, by independence of the increments,

\[
E(\exp i(t^* \cdot (W(u_1) - W(r_1), \cdots, W(u_m) - W(r_m)) + s^* (g(W(t) - W(s)))))
\]

\[
= E(\exp i(g^* \cdot (W(\tau_1) - W(\eta_1), \cdots, W(\tau_l) - W(\eta_l)) + s^* (g(W(t) - W(s)))))
\]

\[
= \prod_{j=1}^l E(\exp (ig_j (W(\tau_j) - W(\eta_j)))) E(\exp (is^* (g(W(t) - W(s)))))
\]

\[
= E(\exp (i(t^* \cdot (W(u_1) - W(r_1), \cdots, W(u_m) - W(r_m))))) \cdot E(\exp (is^* (g(W(t) - W(s)))))
\]
Let
\[ t > s \]
the normal filtration defined by
\[ \mathcal{X}_t \]
Banach space which has the property that if
\[ t > s \]
are independent and integrable and
\[ E \left( W(t) - W(s) \right) = 0 \]. Suppose also that
\[ W(t) \]
is right continuous, meaning that for \( \omega \) off a set of measure zero, \( t \rightarrow W(t)(\omega) \) is right continuous. Also suppose that for some \( q > 1 \)
\[ \lVert W(t) - W(s) \rVert_{L^q(\Omega)} \]
is bounded independent of \( s \leq t \). Then \( \{ W(t) \} \) is also a martingale with respect to the normal filtration defined by
\[ \mathcal{F}_s \equiv \cap_{t > s} \sigma(W(u) - W(r) : 0 \leq r < u \leq t) \]
where this denotes the intersection of the completions of the \( \sigma \) algebras
\[ \sigma(W(u) - W(r) : 0 \leq r < u \leq t) \]
Also, in the same situation but without the assumption that \( E(W(t) - W(s)) = 0 \), if \( t > s \) and \( A \in \mathcal{F}_s \) it follows that if \( g \) is a continuous function such that
\[ \lVert g(W(t) - W(s)) \rVert_{L^q(\Omega)} \] (62.4.13)
is bounded independent of \( s \leq t \) for some \( q > 1 \) then for \( t > s \),
\[ \int_{\Omega} \mathcal{X}_t g(W(t) - W(s)) dP = P(A) \int_{\Omega} g(W(t) - W(s)) dP. \] (62.4.14)

\textbf{Proof:} Consider first the claim, \[ \mathcal{F}_s \] To begin with I show that if \( A \in \mathcal{F}_s \) then for all \( \varepsilon > 0 \),
\[ \int_{\Omega} \mathcal{X}_t g(W(t) - W(s + \varepsilon)) dP = P(A) \int_{\Omega} g(W(t) - W(s + \varepsilon)) dP \] (62.4.15)
\footnote{Note how the \( \sigma \) algebra \( \mathcal{F}_s \) are defined, as the intersection of completions of \( \sigma \) algebras corresponding to \( t \) strictly larger than \( s \).}
This will happen if $X_A$ and $g(W(t) - W(s + \varepsilon))$ are independent. First note that from the definition

$$A \in \sigma(W(u) - W(r) : 0 \leq r < u \leq s + \varepsilon)$$

and so from the process of completion of a measure space, there exists

$$B \in \sigma(W(u) - W(r) : 0 \leq r < u \leq s + \varepsilon)$$

such that $B \supseteq A$ and $P(B \setminus A) = 0$. Therefore, letting $\phi \in E'$,

$$E(\exp(itX_A + i\phi(g(W(t) - W(s + \varepsilon)))) = E(\exp(itX_B + i\phi(g(W(t) - W(s + \varepsilon))))
= E(\exp(itX_B))E(\exp(i\phi(g(W(t) - W(s + \varepsilon))))$$

because $X_B$ is independent of $g(W(t) - W(s + \varepsilon))$ by Lemma 62.4.1 above. Then the above equals

$$= E(\exp(itX_A))E(\exp(i\phi(g(W(t) - W(s + \varepsilon))))$$

Now by Theorem 57.13.3 follows. Next pass to the limit in both sides of 62.4.15 as $\varepsilon \to 0$. One can do this because of 62.4.13 which implies the functions in the integrands are uniformly integrable and Vitali’s convergence theorem, Theorem 19.5.7. This yields 62.4.14.

Now consider the part about the stochastic process being a martingale. Let $g$ be the identity map. If $A \in F_s$, the above implies

$$\int_A E(W(t)|F_s)\,dP = \int_A W(t)\,dP = \int_A (W(t) - W(s))\,dP + \int_A W(s)\,dP$$

and so since $A$ is arbitrary, $E(W(t)|F_s) = W(s)$. ■

Note this implies immediately from Lemma 61.1.5 that Wiener process is not of bounded variation on any interval. This is because this lemma implies if it were of bounded variation, then it would be constant which is not the case due to

$$\mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t - s)) = \mathcal{L}(\sqrt{t-s}W(1)).$$

Here is an interesting theorem about approximation.

**Theorem 62.4.3** Let $\{W(t)\}$ be a Wiener process having values in a separable Banach space as described in Theorem 62.3.4. There exists a set of measure 0, $N$ such that for $\omega \notin N$, the sum in 62.3.11 converges uniformly to $W(t, \omega)$ on any interval, $[0, T]$. That is, for each $\omega$ not in a set of measure zero, the partial sums of the sum in that formula converge uniformly to $t \to W(t, \omega)$ on $[0, T]$. 

Proof: By Lemma 62.4.2, the independence of the increments imply
\[ \sum_{k=m}^{n} \psi_k(t) e_k \]
is a martingale and so by Theorem 60.5.3,
\[ P \left( \sup_{t \in [0,T]} \left| \sum_{k=m}^{n} \psi_k(t) e_k \right| \geq \alpha \right) \leq \frac{1}{\alpha} \int_{\Omega} \left| \sum_{k=m}^{n} \psi_k(T) e_k \right| dP \]
From Corollary 62.3.2,
\[ \int_{\Omega} \left| \sum_{k=m}^{n} \psi_k(T) e_k \right| dP \leq \sum_{k=m}^{n} \int_{\Omega} |\psi_k(T)| dP \lambda_k \leq \sum_{k=m}^{n} \lambda_k \]
which shows that there exists a subsequence, \( m_l \) such that whenever \( n > m_l \),
\[ P \left( \sup_{t \in [0,T]} \left| \sum_{k=m_l}^{n} \psi_k(t) e_k \right| \geq 2^{-k} \right) \leq 2^{-k}. \]
Recall Lemma 57.15.4 stated below for convenience.

Lemma 62.4.4 Let \( \{ \zeta_k \} \) be a sequence of random variables having values in a separable real Banach space, \( E \) whose distributions are symmetric. Letting \( S_k \equiv \sum_{i=1}^{k} \zeta_i \), suppose \( \{ S_{n_k} \} \) converges a.e. Also suppose that for every \( m > n_k \),
\[ P \left( \left| \| S_m - S_{n_k} \|_E > 2^{-k} \right| \right) < 2^{-k}. \] (62.4.16)
Then in fact,
\[ S_k(\omega) \to S(\omega) \] a.e.\( \omega \) (62.4.17)
Apply this lemma to the situation in which the Banach space, \( E \) is \( C([0,T];E) \) and \( \zeta_k = \psi_k e_k \). Then you can conclude uniform convergence of the partial sums,
\[ \sum_{k=1}^{m} \psi_k(t) e_k. \]
This proves the theorem.
Why is \( C([0,T];E) \) separable? You can assume without loss of generality that the interval is \( [0,1] \) and consider the Bernstein polynomials
\[ p_n(t) = \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n} \right) t^k (1-t)^{n-k} \]
These converge uniformly to $f$. Now look at all polynomials of the form

$$\sum_{k=0}^{n} a_k t^k (1 - t^k)$$

where the $a_k$ is one of the countable dense set and $n \in \mathbb{N}$. Each Bernstein polynomial uniformly close to one of these and also uniformly close to $f$. Hence polynomials of this sort are countable and dense in $C([0,T];E)$.

### 62.5 Hilbert Space Valued Wiener Processes

Next I will consider the case of Hilbert space valued Wiener processes. This will include the case of $\mathbb{R}^n$ valued Wiener processes. I will present this material independent of the more general case of $E$ valued Wiener processes.

**Definition 62.5.1** Let $W(t)$ be a stochastic process with values in $H$, a real separable Hilbert space which has the properties that $t \rightarrow W(t, \omega)$ is continuous, whenever $t_1 < t_2 < \cdots < t_m$, the increments $\{W(t_i) - W(t_{i-1})\}$ are independent, $W(0) = 0$, and whenever $s < t$,

$$\mathcal{L}(W(t) - W(s)) = N(0, (t-s)Q)$$

which means that whenever $h \in H$,

$$\mathcal{L}((h, W(t) - W(s))) = N(0, (t-s)(Qh, h))$$

Also

$$E((h_1, W(t) - W(s))(h_2, W(t) - W(s))) = (Qh_1, h_2) (t-s).$$

Here $Q$ is a nonnegative trace class operator. Recall this means

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$$

where $\{e_i\}$ is a complete orthonormal basis, $\lambda_i \geq 0$, and

$$\sum_{i=1}^{\infty} \lambda_i < \infty$$

Such a stochastic process is called a $Q$ Wiener process. In the case where these have values in $\mathbb{R}^n$, $tQ$ ends up being the covariance matrix of $W(t)$.

Note the characteristic function of a $Q$ Wiener process is

$$E\left(e^{ih(W(t))}\right) = e^{-\frac{1}{2}t^2(Qh,h)}$$  \hspace{1cm} (62.5.18)
Note that by Theorem 59.8.5 if you simply say that the distribution measure of \( W(t) \) is Gaussian, then it follows there exists a trace class operator \( Q_t \) and \( m_t \in H \) such that this measure is \( N(m_t, Q_t) \). Thus for \( W(t) \) a Wiener process, \( Q_t = tQ \) and \( m_t = 0 \). In addition, the increments are independent so this is much more specific than the earlier definition of a Gaussian measure.

What is a \( Q \) Wiener process if the Hilbert space is \( \mathbb{R}^n \)? In particular, what is \( Q \)? It is given that \( \mathcal{L}((h, W(t) - W(s))) = N(0, (t - s)(Qh, h)) \)

In this case everything is a vector in \( \mathbb{R}^n \) and so for \( h \in \mathbb{R}^n \),

\[
E(e^{i \lambda (h, W(t) - W(s))}) = e^{-\frac{1}{2} \lambda^2 (t - s)(Qh, h)}
\]

In particular, letting \( \lambda = 1 \) this shows \( W(t) - W(s) \) is normally distributed with covariance \( (t - s)Q \) because its characteristic function is \( e^{-\frac{1}{2} h^*(t - s)Qh} \).

With this and definition, one can describe Hilbert space valued Wiener processes in a fairly general setting.

**Theorem 62.5.2** Let \( U \) be a real separable Hilbert space and let \( J : U_0 \to U \) be a Hilbert Schmidt operator where \( U_0 \) is a real separable Hilbert space. Then let \( \{g_k\} \) be a complete orthonormal basis for \( U_0 \) and define for \( t \in [0, T] \)

\[
W(t) = \sum_{k=1}^{\infty} \psi_k(t) Jg_k
\]

Then \( W(t) \) is a \( Q \) Wiener process for \( Q = JJ^* \) as in Definition 62.5.1. Furthermore, the distribution of \( W(t) - W(s) \) is the same as the distribution of \( W(t - s) \), and \( W \) is Holder continuous with exponent \( \gamma \) for any \( \gamma < 1/2 \). There also is a subsequence denoted by \( N \) such that the convergence of the series

\[
\sum_{k=1}^{N} \psi_k(t) Jg_k
\]

is uniform for all \( \omega \) not in some set of measure zero.

**Proof:** First it is necessary to show the series converges in \( L^2(\Omega; U) \) for each \( t \). For convenience I will consider the series for \( W(t) - W(s) \). (Always, it is assumed \( t > s \).) Then since \( \psi_k(t) - \psi_k(s) \) is normal with mean 0 and variance \( (t - s) \) and \( \psi_k(t) - \psi_k(s) \) and \( \psi_l(t) - \psi_l(s) \) are independent,

\[
\int_{\Omega} \left| \sum_{k=m}^{n} (\psi_k(t) - \psi_k(s)) Jg_k \right|^2 dP
\]

\[
= \int_{\Omega} \sum_{k,l=m}^{n} ((\psi_k(t) - \psi_k(s)) Jg_k, (\psi_l(t) - \psi_l(s)) Jg_l)
\]
which converges to 0 as \( m, n \to \infty \) thanks to the assumption that \( J \) is Hilbert Schmidt. It follows the above sum converges in \( L^2(\Omega; \mathcal{U}) \). Now letting \( m < n \), it follows by the maximal estimate, Theorem 60.5.3, and the above

\[
P \left( \left| \sup_{t \in [0,T]} \left( \sum_{k=1}^{m} \psi_k(t) Jg_k - \sum_{k=1}^{n} \psi_k(t) Jg_k \right) \right|_{\mathcal{U}} \geq \lambda \right) \leq \frac{1}{\lambda^2} E \left( \sum_{k=m+1}^{n} \psi_k(T) Jg_k \right)^2 \leq \frac{1}{\lambda^2 T} \sum_{k=m}^{n} \|Jg_k\|_{\mathcal{U}}^2
\]

and so there exists a subsequence \( n_l \) such that for all \( p \geq 0 \),

\[
P \left( \left| \sup_{t \in [0,T]} \left( \sum_{k=1}^{n_l} \psi_k(t) Jg_k - \sum_{k=1}^{n_l+p} \psi_k(t) Jg_k \right) \right|_{\mathcal{U}} \geq 2^{-l} \right) < 2^{-l}
\]

Therefore, by Borel Cantelli lemma, there is a set of measure zero such that for \( \omega \) not in this set,

\[
\lim_{l \to \infty} \sum_{k=1}^{n_l} \psi_k(t) Jg_k = \sum_{k=1}^{\infty} \psi_k(t) Jg_k
\]

is uniform on \([0,T]\). From now on denote this subsequence by \( N \) to save on notation.

I need to consider the characteristic function of \( (h, W(t) - W(s))_{\mathcal{U}} \) for \( h \in \mathcal{U} \). Then

\[
E (\exp(\imath r h, (W(t) - W(s))_{\mathcal{U}})) = \lim_{N \to \infty} E \left( \exp \left( \imath r \sum_{j=1}^{N} (\psi_j(t) - \psi_j(s))(h, Jg_j) \right) \right) = \lim_{N \to \infty} E \left( \prod_{j=1}^{N} e^{\imath r(\psi_j(t) - \psi_j(s))(h, Jg_j)} \right)
\]

Since the random variables \( \psi_j(t) - \psi_j(s) \) are independent,

\[
= \lim_{N \to \infty} \prod_{j=1}^{N} E \left( e^{\imath r(h, Jg_j)(\psi_j(t) - \psi_j(s))} \right)
\]

Since \( \psi_j(t) - \psi_j(s) \) is a Gaussian random variable having mean 0 and variance \( (t-s) \), the above equals

\[
= \lim_{N \to \infty} \prod_{j=1}^{N} e^{-\frac{1}{2} r^2(h, Jg_j)^2(t-s)}
\]
\[ \lim_{N \to \infty} \exp \left( \sum_{j=1}^{N} -\frac{1}{2} r^2 (h, Jg_j)^2 (t - s) \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \sum_{j=1}^{\infty} (h, Jg_j)^2_U \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \sum_{j=1}^{\infty} (J^* h, g_j)^2_{U_0} \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \| J^* h \|_{U_0}^2 \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \langle Q h, h \rangle_U \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \| J^* h \|_{U_0}^2 \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \langle Q h, h \rangle_U \right) \]
\[ = \exp \left( -\frac{1}{2} r^2 (t - s) \langle J^* h, h \rangle_U \right) \]

which shows \( (h, W(t) - W(s))_U \) is normally distributed with mean 0 and variance \( (t - s) \langle Q h, h \rangle_U \) where \( Q = JJ^* \). It is obvious from the definition that \( W(0) = 0 \).

Note that \( Q \) is of trace class because if \( \{e_k\} \) is an orthonormal basis for \( U \),
\[ \sum_k \langle Q e_k, e_k \rangle_U = \sum_k \| J^* e_k \|_{U_0}^2 = \sum_k \sum_l \langle J^* e_k, g_l \rangle_{U_0}^2 \]
\[ = \sum_k \sum_l \langle e_k, J^* g_l \rangle_U^2 = \sum_l \| J^* g_l \|_{U_0}^2 < \infty \]

To find the covariance, consider
\[ E \left( (h_1, W(t) - W(s)) (h_2, W(t) - W(s)) \right), \]

This equals
\[ E \left( \sum_{k=1}^{\infty} (\psi_k(t) - \psi_k(s)) (h_1, Jg_k) \sum_{j=1}^{\infty} (\psi_j(t) - \psi_j(s)) (h_2, Jg_j) \right). \]

Since the series converge in \( L^2(\Omega; U) \), the independence of the \( \psi_k(t) - \psi_k(s) \) implies the above equals
\[ = \lim_{n \to \infty} E \left( \sum_{k=1}^{n} (\psi_k(t) - \psi_k(s)) (h_1, Jg_k) \sum_{j=1}^{n} (\psi_j(t) - \psi_j(s)) (h_2, Jg_j) \right). \]
Next consider the claim that the increments are independent. Let $W_N(t)$ be given by the appropriate partial sum and let \( \{h_j\}_{j=1}^m \) be a finite list of vectors of $U$. Then from the independence properties of $\psi_j$ explained above,

\[
E \left( \exp \sum_{j=1}^m i \left( h_j, W_N(t_j) - W_N(t_{j-1}) \right) \right)
\]

\[
E \left( \exp \sum_{j=1}^m \sum_{k=1}^N i (h_j, Jg_k(U_0(\psi_k(t_j) - \psi_k(t_{j-1})))) \right)
\]

\[
= E \left( \prod_{j,k} \exp \left( i (h_j, Jg_k(U_0(\psi_k(t_j) - \psi_k(t_{j-1})))) \right) \right)
\]

This can be done because of the independence of the random variables $\{\psi_k(t_j) - \psi_k(t_{j-1})\}_{j,k}$.

Thus the above equals

\[
\prod_{j,k} \exp \left( -\frac{1}{2} (h_j, Jg_k)^2 U_0 (t_j - t_{j-1}) \right)
\]

\[
= \prod_{j=1}^m \exp \left( -\frac{1}{2} \sum_{k=1}^N (h_j, Jg_k)^2 U_0 (t_j - t_{j-1}) \right)
\]
because \( \psi_k(t_j) - \psi_k(t_{j-1}) \) is normally distributed having variance \( t_j - t_{j-1} \). Now letting \( N \to \infty \), this implies

\[
E \left( \exp \left( \sum_{j=1}^{m} i (h_j, W(t_j) - W(t_{j-1})) \right) \right)
\]

\[
= \prod_{j=1}^{m} \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} (h_j, J g_k)_{U}^2 (t_j - t_{j-1}) \right)
\]

\[
= \prod_{j=1}^{m} \exp \left( -\frac{1}{2} (t_j - t_{j-1}) \sum_{k=1}^{\infty} (J^* h_j, g_k)_{U}^2 \right)
\]

\[
= \prod_{j=1}^{m} \exp \left( -\frac{1}{2} (t_j - t_{j-1}) \| J^* h_j \|_{U_0}^2 \right)
\]

\[
= \prod_{j=1}^{m} \exp \left( i (h_j, W(t_j) - W(t_{j-1})) \right)
\]

(62.5.20)

from 62.5.19, letting \( r = 1 \). By Theorem 57.13.3 on Page 1933, this shows the increments are independent.

It remains to verify the Holder continuity. Recall

\[
W(t) = \sum_{k=1}^{\infty} J g_k \psi_k(t)
\]

where \( \psi_k \) is a real Wiener process.

Next consider the claim about Holder continuity. It was shown above that

\[
E \left( \exp \left( ir \left( h, (W(t) - W(s))_{U} \right) \right) \right) = \exp \left( -\frac{1}{2} r^2 (t - s) (Q h, h)_{U} \right)
\]

Therefore, taking a derivative with respect to \( r \) two times yields

\[
E \left( \left( - (h, (W(t) - W(s))_{U}^2 \right) \exp \left( ir \left( h, (W(t) - W(s))_{U} \right) \right) \right)
\]

\[
= - (t - s) (Q h, h) \exp \left( -\frac{1}{2} r^2 (t - s) (Q h, h)_{U} \right) +
\]

\[
r^2 (t - s)^2 (Q h, h)_{U}^2 \exp \left( -\frac{1}{2} r^2 (t - s) (Q h, h)_{U} \right)
\]

Now plug in \( r = 0 \) to obtain

\[
E \left( (h, (W(t) - W(s))_{U}^2 \right) = (t - s) (Q h, h).
\]
Similarly, taking 4 derivatives, it follows that an expression of the following form holds.

\[ E \left( (h, (W(t) - W(s)))^4 \right) = C_2 (Qh, h)^2 (t - s)^2, \]

and in general,

\[ E \left( (h, (W(t) - W(s)))^{2m} \right) = C_m (Qh, h)^m (t - s)^m. \]

Now it follows from Minkowsky’s inequality applied to the two integrals \( \sum_{i=1}^{\infty} \) and \( \int_{\Omega} \) that

\[
\left[ E \left( |W(t) - W(s)|^{2m} \right) \right]^{1/m} = \left[ E \left( \left( \sum_{k=1}^{\infty} (e_k, W(t) - W(s))^2 \right)^m \right) \right]^{1/m} \\
\leq \sum_{k=1}^{\infty} \left[ E \left( (e_k, W(t) - W(s))^{2m} \right) \right]^{1/m} \\
= \sum_{k=1}^{\infty} [C_m (Qe_k, e_k)^m (t - s)^m]^{1/m} \\
= C_m^1 |t - s| \left( \sum_{k=1}^{\infty} (Qe_k, e_k) \right) \equiv C_m' |t - s|.
\]

Hence there exists a constant \( C_m \) such that

\[ E \left( |W(t) - W(s)|^{2m} \right) \leq C_m |t - s|^m \]

By the Kolmogorov Čentsov Theorem, Theorem 60.2.2, it follows that off a set of measure 0, \( t \to W(t, \omega) \) is Holder continuous with exponent \( \gamma \) such that

\[ \gamma < \frac{m - 1}{2m}, \quad m > 2. \]

Finally, from 60.5.14 with \( r = 1, \)

\[ E \left( \exp i (h, W(t) - W(s))_{\mathcal{U}} \right) = \exp \left( -\frac{1}{2} (t - s) (Qh, h) \right) \]

which is the same as \( E \left( \exp i (h, W(t - s))_{\mathcal{U}} \right) \) due to the fact \( W(0) = 0. \]

The above has shown that \( W(t) \) satisfies the conditions of Lemma 60.4.3 and so it is a martingale with respect to the filtration given there. What is its quadratic variation?

\[ E \left( ||W(t)||^2 \right) = \sum_{k=1}^{\infty} E \left( (W(t), e_k)(W(t), e_k) \right) = \sum_{k=1}^{\infty} (Qe_k, e_k) t = \text{trace}(Q) t \]
Is it the case that $[W] (t) = \text{trace} (Q) t$? Let the filtration be as in Lemma 62.4.2 and let $A \in \mathcal{F}_s$. Then using the result of that lemma,

$$
\int_A \left( ||W (t)||^2 - t \text{trace} (Q) |\mathcal{F}_s| \right) dP
$$

$$
= \int_A \left( ||W (t) - W (s)||^2 + 2 (W (t), W (s)) - ||W (s)||^2 - (t-s) \text{trace} Q - \text{trace} Q s |\mathcal{F}_s| \right) dP
$$

$$
= P (A) \int_\Omega ||W (t) - W (s)||^2 - (t-s) \text{trace} Q dP
$$

$$
+ \int_A \left(2 (W (t), W (s)) - ||W (s)||^2 - \text{trace} (Q) s |\mathcal{F}_s| \right) dP
$$

$$
= \int_A 2 (W (s), E (W (t)|\mathcal{F}_s)) dP - \int_A ||W (s)||^2 dP - \int_A s \text{trace} Q dP
$$

$$
= \int_A \left( ||W (s)||^2 - s \text{trace} Q \right) dP
$$

and this shows that the quadratic variation $[W] (t) = t \text{trace} (Q)$ by uniqueness of the quadratic variation.

Now suppose you start with a nonnegative trace class operator $Q$. Then in this case also one can define a $Q$ Wiener process. It is possible to get this theorem from Theorem 62.5.2 but this will not be done here.

**Theorem 62.5.3** Let $U$ be a real separable Hilbert space and let $Q$ be a nonnegative trace class operator defined on $U$. Then there exists a $Q$ Wiener process as defined in Definition 62.5.1. Furthermore, the distribution of $W (t) - W (s)$ is the same as the distribution of $W (t - s)$ and $W$ is Holder continuous with exponent $\gamma$ for any $\gamma < 1/2$.

**Proof:** One can obtain this theorem as a corollary of Theorem 62.5.2 but this will not be done here.

Let

$$
Q = \sum_{i=1}^\infty \lambda_i e_i \otimes e_i
$$

where $\{e_i\}$ is a complete orthonormal set and $\lambda_i \geq 0$ and $\sum \lambda_i < \infty$. Now the definition of the $Q$ Wiener process is

$$
W (t) \equiv \sum_{k=1}^\infty \sqrt{\lambda_k} e_k \psi_k (t) \quad (62.5.21)
$$

where $\{\psi_k (t)\}$ are the real Wiener processes defined in Lemma 62.3.1.
Now consider. From this formula, if \( s < t \)

\[
W(t) - W(s) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k (\psi_k(t) - \psi_k(s)) \quad (62.5.22)
\]

First it is necessary to show this sum converges. Since \( \psi_j(t) \) is a Wiener process,

\[
\int_{\Omega} \left| \sum_{j=m}^{n} \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s)) e_j \right|^2 dP
= \int_{\Omega} \sum_{j=m}^{n} \lambda_j (\psi_j(t) - \psi_j(s))^2 dP
= (t-s) \sum_{j=m}^{n} \lambda_j
\]

and this converges to 0 as \( m, n \to \infty \) because it was given that

\[
\sum_{j=1}^{\infty} \lambda_j < \infty
\]

so the series in (62.5.22) converges in \( L^2(\Omega; U) \).

Therefore, there exists a subsequence

\[
\left\{ \sum_{k=1}^{N} \sqrt{\lambda_k} e_k (\psi_k(t) - \psi_k(s)) \right\}
\]

which converges pointwise a.e. to \( W(t) - W(s) \) as well as in \( L^2(\Omega; U) \) as \( N \to \infty \). Then letting \( h \in U \),

\[
(h, W(t) - W(s))_U = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (\psi_k(t) - \psi_k(s)) (h, e_k) \quad (62.5.23)
\]

Then by the dominated convergence theorem,

\[
E(\exp(i r (h, (W(t) - W(s)))_U))
= \lim_{N \to \infty} E\left( \exp \left( i r \left( \sum_{j=1}^{N} \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s)) (h, e_j) \right) \right) \right)
= \lim_{N \to \infty} E\left( \prod_{j=1}^{N} e^{ir \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s)) (h, e_j)} \right)
\]
Since the random variables \( \psi_j(t) - \psi_j(s) \) are independent,
\[
= \lim_{N \to \infty} \prod_{j=1}^{N} E \left( e^{ir\sqrt{\lambda_j} (\psi_j(t) - \psi_j(s))(h,e_j)} \right)
\]

Since \( \psi_j(t) \) is a real Wiener process,
\[
= \lim_{N \to \infty} \prod_{j=1}^{N} e^{-\frac{1}{2} r^2 \lambda_j (t-s)(h,e_j)^2} \]
\[
= \exp \left( -\frac{1}{2} r^2 (t-s) \sum_{j=1}^{\infty} \lambda_j (h,e_j)^2 \right) \quad (62.5.24)
\]

Thus \((h, W(t) - W(s))\) is normally distributed with mean 0 and variance \((t-s)(Qh, h)\).

It is obvious from the definition that \(W(0) = 0\).

Also to find the covariance, consider
\[
E \left( (h_1, W(t) - W(s))(h_2, W(t) - W(s)) \right),
\]
and use \(62.5.23\) to obtain this is equal to
\[
E \left( \sum_{k=1}^{\infty} \sqrt{\lambda_k} (\psi_k(t) - \psi_k(s))(h_1, e_k) \sum_{j=1}^{\infty} \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s))(h_2, e_j) \right)
\]
\[
= \lim_{n \to \infty} E \left( \sum_{k=1}^{n} \sqrt{\lambda_k} (\psi_k(t) - \psi_k(s))(h_1, e_k) \sum_{j=1}^{n} \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s))(h_2, e_j) \right)
\]
\[
= \lim_{n \to \infty} (t-s) \sum_{k=1}^{n} \lambda_k (h_1, e_k)(h_2, e_j) = (t-s)(Qh_1, h_2)
\]
(Recall \(Q \equiv \sum_k \lambda_k e_k \otimes e_k\).)

Next I show the increments are independent. Let \(N\) be the subsequence defined above and let \(W^N(t)\) be given by the appropriate partial sum and let \(\{h_j\}_{j=1}^{m}\) be a finite list of vectors of \(U\). Then from the independence properties of \(\psi_j\) explained above,
\[
E \left( \exp \sum_{j=1}^{m} i(h_j, W^N(t_j) - W^N(t_{j-1}))_{U} \right)
\]
62.5. HILBERT SPACE VALUED WIENER PROCESSES

\[ E \left( \exp \left( \sum_{j=1}^{m} \left( \sum_{k=1}^{N} \sqrt{\lambda_k} e_k (\psi_k (t_j) - \psi_k (t_{j-1})) \right) \right) \right) \]

\[ = E \left( \exp \left( \sum_{j=1}^{m} \sum_{k=1}^{N} i \sqrt{\lambda_k} (h_j, e_k) \right) \left( \psi_k (t_j) - \psi_k (t_{j-1}) \right) \right) \]

\[ = \prod_{j,k} E \left( \exp \left( i \sqrt{\lambda_k} (h_j, e_k) \right) \left( \psi_k (t_j) - \psi_k (t_{j-1}) \right) \right) \]

This can be done because of the independence of the random variables \( \{\psi_k (t_j) - \psi_k (t_{j-1})\}_{j,k} \).

Thus the above equals

\[ = \prod_{j=1}^{m} \exp \left( -\frac{1}{2} \sum_{k=1}^{N} \lambda_k (h_j, e_k)^2 (t_j - t_{j-1}) \right) \]

because \( \psi_k (t_j) - \psi_k (t_{j-1}) \) is normally distributed having variance \( t_j - t_{j-1} \) and mean 0. Now letting \( N \to \infty \), this implies

\[ E \left( \exp \left( \sum_{j=1}^{m} i (h_j, W (t_j) - W (t_{j-1})) \right) \right) \]

\[ = \prod_{j=1}^{m} \exp \left( -\frac{1}{2} (t_j - t_{j-1}) \sum_{k=1}^{\infty} \lambda_k (h_j, e_k)^2 \right) \]

\[ = \prod_{j=1}^{m} \exp \left( -\frac{1}{2} (t_j - t_{j-1}) (Q h, h) \right) \]

\[ = \prod_{j=1}^{m} \exp \left( i (h_j, W (t_j) - W (t_{j-1})) \right) \]

(62.5.25)

because of the fact shown above that \( (h, W (t) - W (s)) \) is normally distributed with mean 0 and variance \( (t - s) (Q h, h) \). By Theorem 57.13.3 on Page 1945, this shows the increments are independent.

Next consider the continuity assertion. Recall

\[ W (t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \psi_k (t) \]
where \( \psi_k \) is a real Wiener process. Therefore, letting \( 2m > 2, m \in \mathbb{N} \) and using (62.5.24) for \( \psi_k \) and Jensen’s inequality along with Lemma 62.3.1,

\[
E \left( \left| W(t) - W(s) \right|^{2m} \right) = E \left( \left( \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k (\psi_k(t) - \psi_k(s)) \right)^{2m} \right)
\]
\[
= E \left( \left( \sum_{k=1}^{\infty} \lambda_k |\psi_k(t) - \psi_k(s)|^2 \right)^m \right)
\]
\[
\leq E \left( \left( \sum_{k=1}^{\infty} \lambda_k \right)^{m-1} \sum_{k=1}^{\infty} \lambda_k |\psi_k(t) - \psi_k(s)|^{2m} \right)
\]
\[
\leq C_m \left( \sum_{k=1}^{\infty} \lambda_k \right) E \left( |\psi_k(t) - \psi_k(s)|^{2m} \right) \quad (62.5.26)
\]
\[
\leq C_m |t-s|^m \quad (62.5.27)
\]

By the Kolmogorov Čentsov Theorem, Theorem 60.2.2, it follows that off a set of measure 0, \( t \to W(t, \omega) \) is Holder continuous with exponent \( \gamma \) such that

\[
\gamma < \frac{m - 1}{2m}.
\]

Finally, from (62.5.24) taking \( r = 1 \),

\[
E \left( \exp i (h, W(t) - W(s))_{U} \right) = \exp \left( -\frac{1}{2} (t-s) (Qh, h) \right)
\]

which is the same as \( E \left( \exp i (h, W(t - s))_{U} \right) \) due to the fact \( W(0) = 0 \). This proves the theorem.

The above shows there exists \( Q \) Wiener processes in any separable Hilbert space. Next I will show the way described above is the only way it can happen.

**Theorem 62.5.4** Suppose \( \{W(t)\} \) is a \( Q \) Wiener process in \( U \), a real separable Hilbert space. Then letting

\[
Q = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k
\]

where the \( \{e_k\} \) are orthonormal, \( \lambda_k \geq 0 \), and \( \sum_{k=1}^{\infty} \lambda_k < \infty \), it follows

\[
W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \psi_k(t) e_k \quad (62.5.28)
\]

where

\[
\psi_k(t) = \begin{cases} \frac{1}{\sqrt{\lambda_k}} (W(t), e_k)_{U} & \text{if } \lambda_k \neq 0 \\ 0 & \text{if } \lambda_k = 0 \end{cases}
\]
then \( \{\psi_k(t)\} \) is a Wiener process and for \( t_0 < t_1 < \cdots < t_n \) the random variables
\[
\{\psi_k(t_q) - \psi_k(t_{q-1}) : (q, k) \in (1, 2, \cdots, n) \times (k_1, \cdots, k_m)\}
\]
are independent. Furthermore, the sum in (62.5.28) converges uniformly for a.e. \( \omega \) on any closed interval, \([0, T]\).

**Proof:** First of all, the fact that \( W(t) \) has values in \( U \) and that \( \{e_k\} \) is an orthonormal basis implies the sum in (62.5.28) converges for each \( \omega \). Consider
\[
E(\exp(\imath \psi_k(t))) = E\left(\exp\left(\frac{\imath}{\sqrt{\lambda_k}} (W(t), e_k)_U\right)\right)
\]
Since \( W(t) \) is given to be a \( Q \) Wiener process, \((W(t), e_k)_U\) is normally distributed with mean 0 and variance \( t(Qh, h) \). Therefore, the above equals
\[
e^{-\frac{1}{2} t^2 \frac{1}{\lambda_k}((Qe_k, e_k) = e^{-\frac{1}{2} q^2 \lambda_k} = e^{-\frac{1}{2} t^2},
\]
the characteristic function for a random variable which is \( N(0, t) \). The independence of the increments for a given \( \psi_k(t) \) follows right away from the independence of the increments of \( W(t) \) and the distribution of the increments being \( N(0, (t-s)) \) follows similarly to the above.

For \( t_1 < t_2 < \cdots < t_n \), why are the random variables,
\[
\{(W(t_q), e_k)_U - (W(t_{q-1}), e_k)_U : (q, k) \in (1, 2, \cdots, n) \times (k_1, \cdots, k_m)\}
\]
(62.5.29)
independent? Let
\[
P = \sum_{q=1}^{n} \sum_{j=1}^{m} s_{qj} \left( (W(t_q), e_k)_U - (W(t_{q-1}), e_k)_U \right)
\]
and consider \( E(e^{iP}) \). This equals
\[
e^{iP} = E\left(\exp\left(i \sum_{q=1}^{n} \sum_{j=1}^{m} s_{qj} \left( (W(t_q), e_k)_U - (W(t_{q-1}), e_k)_U \right)\right)\right) \quad (62.5.30)
\]
\[
e^{iP} = E\left(\exp\left(i \sum_{q=1}^{n} \left( (W(t_q), \sum_{j=1}^{m} s_{qj} e_k)_U - (W(t_{q-1}), \sum_{j=1}^{m} s_{qj} e_k)_U \right)\right)\right)
\]
\[
e^{iP} = E\left(\prod_{q=1}^{n} \exp\left(i \left( W(t_q) - W(t_{q-1}) \right), \sum_{j=1}^{m} s_{qj} e_k \right)_U\right)\right)
\]
Now recall that by assumption the increments \( W(t) - W(s) \) are independent. Therefore, the above equals
\[
\prod_{q=1}^{n} E\left(\exp\left(i \left( W(t_q) - W(t_{q-1}) \right), \sum_{j=1}^{m} s_{qj} e_k \right)_U\right)\right)
\]
Recall that by assumption \((W(t) - W(s), h)_U\) is normally distributed with variance \((t-s)(Qh, h)\) and mean 0. Therefore, the above equals

\[
\prod_{q=1}^n \exp \left( -\frac{1}{2} (t_q - t_{q-1}) \left( Q \sum_{j=1}^m s_{qj} e_{kj}, \sum_{j=1}^m s_{qj} e_{kj} \right) \right)
\]

\[
= \prod_{q=1}^n \exp \left( -\frac{1}{2} (t_q - t_{q-1}) \sum_{j=1}^m s_{qj}^2 \lambda_{kj} \right)
\]

\[
= \exp \left( -\frac{1}{2} \sum_{q=1}^n \sum_{j=1}^m (t_q - t_{q-1}) s_{qj}^2 \lambda_{kj} \right)
\]

(62.5.31)

Also

\[
\prod_{q=1}^n \prod_{j=1}^n E \left( \exp \left( is_{qj} \left( (W(t_q), e_{kj})_U - (W(t_{q-1}), e_{kj})_U \right) \right) \right)
\]

\[
= \prod_{q=1}^n \prod_{j=1}^n E \left( \exp \left( is_{qj} \left( (W(t_q) - W(t_{q-1}), e_{kj})_U \right) \right) \right)
\]

\[
= \prod_{q=1}^n \prod_{j=1}^n \exp \left( -\frac{1}{2} (t_q - t_{q-1}) s_{qj}^2 (Qe_{kj}, e_{kj}) \right)
\]

\[
= \prod_{q=1}^n \prod_{j=1}^n \exp \left( -\frac{1}{2} (t_q - t_{q-1}) s_{qj}^2 \lambda_{kj} \right)
\]

\[
= \exp \left( -\frac{1}{2} \sum_{q=1}^n \sum_{j=1}^m (t_q - t_{q-1}) s_{qj}^2 \lambda_{kj} \right)
\]

(62.5.32)

Therefore, \(e^{IP}\) equals the expression in (62.5.31) because both equal the expression in (62.5.32) and it follows from Proposition 57.11.1 on Page 1938 that the random variables of (62.5.29) are independent.

What about the claim of uniform convergence? By the independence of the increments, it follows from Lemma 54.4.2 that \(\{W(t)\}\) is a martingale and each real valued function, \((W(t), e_k)_U\) is also a martingale. Therefore, Theorem 60.5.3 can be applied to conclude

\[
P \left( \sup_{t \in [0,T]} \left| \sum_{k=m}^n (W(t), e_k)_U e_k \right| \geq \alpha \right) \leq \frac{1}{\alpha} \int_{\Omega} \left| \sum_{k=m}^n (W(T), e_k)_U e_k \right| dP
\]
\[ \alpha \sum_{k=m}^{n} (W(T), e_k) U e_k \leq \int_{\Omega} \left( \sum_{k=m}^{n} (W(T), e_k) U e_k \right)^2 dP = \int_{\Omega} (W(T), e_k)^2 U dP \]
\[ = \frac{1}{\alpha} \sum_{k=m}^{n} (Qe_k, e_k) T = \frac{T}{\alpha} \sum_{k=m}^{n} \lambda_k \leq \frac{T}{\alpha} \sum_{k=m}^{\infty} \lambda_k \]

Since \( \sum_{k=1}^{\infty} \lambda_k < \infty \), there exists a sequence, \( \{m_l\} \) such that if \( n > m_l \)
\[ P \left( \left\{ \sup_{t \in [0, T]} \left| \sum_{k=m_l}^{n} (W(t), e_k) U e_k \right| > 2^{-k} \right\} \right) < 2^{-k} \]

and so by the Borel Cantelli lemma, off a set of measure 0 the partial sums
\[ \left\{ \sum_{k=1}^{m_l} (W(t), e_k) U e_k \right\} \]
converge uniformly on \([0, T]\). This is very interesting but more can be said. In fact the original partial sums converge.

Recall Lemma 62.5.5 stated below for convenience.

**Lemma 62.5.5** Let \( \{\zeta_k\} \) be a sequence of random variables having values in a separable real Banach space, \( E \) whose distributions are symmetric. Letting \( S_k \equiv \sum_{i=1}^{k} \zeta_i \), suppose \( \{S_n\} \) converges a.e. Also suppose that for every \( m > n \),
\[ P \left( \|S_m - S_n\|_E > 2^{-k} \right) < 2^{-k}. \]

Then in fact,
\[ S_k(\omega) \rightarrow S(\omega) \ a.e. \omega \]

Apply this lemma to the situation in which the Banach space, \( E \) is \( C([0, T]; U) \).
Then you can conclude uniform convergence of the partial sums,
\[ \sum_{k=1}^{m} (W(t), e_k) U e_k. \]

This proves the theorem.

### 62.6 Wiener Processes, Another Approach

#### 62.6.1 Lots Of Independent Normally Distributed Random Variables

You can use the Kolmogorov extension theorem to prove the following corollary. It is Corollary 57.20.3 on Page 1984.
Corollary 62.6.1 Let $H$ be a real Hilbert space. Then there exist real valued random variables $W(h)$ for $h \in H$ such that each is normally distributed with mean 0 and for every $h, g, (W(f), W(g))$ is normally distributed and

$$E(W(h)W(g)) = (h, g)_H$$

Furthermore, if $\{e_i\}$ is an orthogonal set of vectors of $H$, then $\{W(e_i)\}$ are independent random variables. Also for any finite set $\{f_1, f_2, \cdots, f_n\}$, 

$$(W(f_1), W(f_2), \cdots, W(f_n))$$

is normally distributed.

Corollary 62.6.2 The map $h \to W(h)$ is linear. Also, $\{W(h) : h \in H\}$ is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$ where $\mathcal{F} = \sigma(W(h) : h \in H)$.

Proof: This follows from the above description.

$$E\left(|W(g + h) - (W(g) + W(h))|^2\right) = E\left(W(g + h)^2\right)$$

$$+ E\left((W(g) + W(h))^2\right) - 2E(W(g + h)(W(g) + W(h)))$$

$$= |g + h|^2 + |g|^2 + |h|^2 + 2(g, h) - 2(g + h, g) - 2(g + h, h)$$

$$= |g|^2 + |h|^2 + 2(g, h) + 2(g, h) + |g|^2$$

$$+ |h|^2 - 2|g|^2 - 2(g, h) - 2(g, h) - 2|h|^2 = 0$$

Hence $W(h + g) = W(g) + W(h)$.

$$E\left((W(\alpha f) - \alpha W(f))^2\right) = E\left(W(\alpha f)^2\right) + E\left(\alpha^2 W(f)^2\right) - 2E(W(\alpha f) \alpha W(f))$$

$$= \alpha^2 |f|^2 + \alpha^2 |f|^2 - 2\alpha(\alpha, f) = 0.$$  

Why is $\{W(h) : h \in H\}$ a subspace? This is obvious because $W$ is linear. Why is it closed? Say $W(h_n) \to f \in L^2(\Omega)$. This requires that $\{h_n\}$ is a Cauchy sequence. Thus $h_n \to h$ and so

$$E\left(|f - W(h)|^2\right) \leq 2 \lim_{n \to \infty} E\left(|f - W(h_n)|^2\right) + E\left(|W(h_n) - W(h)|^2\right)$$

$$= 2 \lim_{n \to \infty} E\left(|W(h_n) - W(h)|^2\right) = 2 \lim_{n \to \infty} |h_n - h|^2 = 0$$

and so $f = W(h)$ showing that this is indeed a closed subspace. ■

Next is a technical lemma which will be of considerable use.
**Lemma 62.6.3** Let $X \geq 0$ and measurable. Also define a finite measure on $\mathcal{B}(\mathbb{R}^p)$

$$\nu (B) \equiv \int_\Omega X \chi_B (Y) dP$$

Then let $f : \mathbb{R}^p \to [0, \infty)$ be Borel measurable. Then

$$\int_\Omega f (Y) X dP = \int_{\mathbb{R}^p} f (y) \, d\nu (y)$$

where here $Y$ is a given measurable function with values in $\mathbb{R}^p$. Formally, $X dP = d\nu$.

Note that $Y$ is given and $X$ is just some random variable which here has non-negative values. Of course similar things will work without this stipulation.

**Proof:** First say $X = X_D$ and replace $f (Y)$ with $X Y - 1(B)$. Then

$$\int_\Omega X_D X Y^{-1}(B) dP = P (D \cap Y^{-1}(B))$$

$$\int_{\mathbb{R}^p} X_B (y) \, d\nu (y) \equiv \nu (B) \equiv \int_\Omega X_D X_B (Y) dP$$

$$= \int_\Omega X_D X Y^{-1}(B) dP = P (D \cap Y^{-1}(B))$$

Thus

$$\int_\Omega X_D X Y^{-1}(B) dP = \int_\Omega X_D X_B (Y) dP \int_{\mathbb{R}^p} X_B (y) \, d\nu (y)$$

Now let $s_n (y) \uparrow f (y)$, and let $s_n (y) = \sum_{k=1}^m c_k X_{B_k} (y)$ where $B_k$ is a Borel set. Then

$$\int_{\mathbb{R}^p} s_n (y) \, d\nu (y) = \int_{\mathbb{R}^p} \sum_{k=1}^m c_k X_{B_k} (y) \, d\nu (y) = \sum_{k=1}^m c_k \int_{\mathbb{R}^p} X_{B_k} (y) \, d\nu (y)$$

$$= \sum_{k=1}^m c_k P (D \cap Y^{-1}(B_k))$$

$$\int_\Omega s_n (Y) X_D dP = \sum_{k=1}^m c_k \int_\Omega X_D X_{B_k} (Y) dP = \sum_{k=1}^m c_k P (D \cap Y^{-1}(B_k))$$

which is the same thing. Therefore,

$$\int_\Omega s_n (Y) X_D dP = \int_{\mathbb{R}^p} s_n (y) \, d\nu (y)$$

Now pass to a limit using the monotone convergence theorem to obtain

$$\int_\Omega f (Y) X_D dP = \int_{\mathbb{R}^p} f (y) \, d\nu (y)$$
Next replace $\mathcal{X}_D$ with $\sum_{k=1}^m d_k \mathcal{X}_{D_k} \equiv s_n(\omega)$, a simple function.

$$
\int_{\Omega} f(Y) \sum_{k=1}^m d_k \mathcal{X}_{D_k} dP = \sum_{k=1}^m d_k \int_{\Omega} f(Y) \mathcal{X}_{D_k} dP
$$

$$
= \sum_{k=1}^m d_k \int_{\mathbb{R}^p} f(Y) d\nu_k
$$

where $\nu_k(B) = \int_{\Omega} \mathcal{X}_{D_k} \mathcal{X}_B(Y) dP$. Now let

$$
\nu_n(B) = \int_{\Omega} \sum_{k=1}^m d_k \mathcal{X}_{D_k} \mathcal{X}_B(Y) = \int_{\Omega} s_n \mathcal{X}_B(Y) dP
$$

It is indexed with $n$ thanks to $s_n$. Then

$$
\nu_n(B) = \sum_{k=1}^m d_k \int_{\Omega} \mathcal{X}_{D_k} \mathcal{X}_B(Y) dP = \sum_{k=1}^m d_k \nu_k(B)
$$

Hence

$$
\int_{\Omega} f(Y) s_n dP = \int_{\Omega} f(Y) \sum_{k=1}^m d_k \mathcal{X}_{D_k} dP = \sum_{k=1}^m d_k \int_{\mathbb{R}^p} f(Y) d\nu_k
$$

$$
= \int_{\mathbb{R}^p} f(Y) \sum_{k=1}^m d_k d\nu_k = \int_{\mathbb{R}^p} f(Y) d\nu_n
$$

($s_n dP = d\nu_n$ so to speak.) Then let $s_n(\omega) \uparrow X(\omega)$. Clearly $\nu_n \ll \nu$ and so by the Radon Nikodym theorem $d\nu_n = h_n d\nu$ where $h_n \uparrow 1$. It follows from the monotone convergence theorem that one can pass to a limit in the above and obtain

$$
\int_{\Omega} f(Y) X dP = \int_{\mathbb{R}^p} f(Y) d\nu
$$

The interest here is to let $f(Y) \equiv e^{\lambda Y}$ so $f(y) = e^{\lambda y}$. To remember this, $XdP = d\nu$ in a sort of sloppy way then the above formula holds.

**Lemma 62.6.4** Each $e^{W(h)}$ is in $L^p(\Omega)$ for every $h \in H$ and for every $p \geq 1$. In fact,

$$
\int_{\Omega} \left( e^{W(h)} \right)^p dP = \int_{\Omega} e^{W(p h)} dP = e^{\frac{1}{2} |ph|^2 h}.
$$

In addition to this,

$$
\sum_{k=0}^n \frac{W(h)^k}{k!} \to e^{W(h)} \text{ in } L^p(\Omega, F, P), \ p > 1
$$
Proof: It suffices to verify this for all positive integers $p$. Let $p$ be such an integer. Note that from the linearity of $W$, $(e^{W(h)})^p = e^{pW(h)} = e^{W(ph)}$ and so it suffices to verify that for each $h \in H$, $e^{W(h)}$ is in $L^1(\Omega)$. From Lemma 64.1,2.2.

\[ \int_{\Omega} e^{W(h)}dP = \int_{\mathbb{R}} e^y d\nu(y) \]

where $\nu (B) \equiv \int_{\Omega} \mathcal{X}_B (W (h))dP = \int_{\mathbb{R}} \mathcal{X}_B (y) d\nu (y)$. In using this lemma, $Y = W (h), X = 1$. Thus

\[ \int_{\Omega} e^{W(h)}dP = \int_0^{\infty} \nu (e^y > \lambda) d\lambda = \int_0^{\infty} \frac{1}{\sqrt{2\pi |h|}} \int_{[y>\ln (\lambda)]} e^{-\frac{1}{2} \frac{y^2}{\lambda}} dyd\lambda, u = \ln (\lambda), \]

\[ = \frac{1}{\sqrt{2\pi |h|}} \int_{-\infty}^{\infty} e^u \int_u^{\infty} e^{-\frac{1}{2} \frac{v^2}{\lambda}} dvdu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_u^{\lambda} e^{\frac{1}{2} \frac{v^2}{\lambda}} dvdu \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{|h|^2}{2u}} e^{\frac{1}{2} \frac{v^2}{u}} dvdu < \infty \]

If $h = 0$, $W (h)$ would be 0 because by the construction, $E (W (0)^2) = (0, 0)_H = 0$. Then

\[ \int_{\Omega} e^{W(h)}dP = \int_{\Omega} e^0 dP = 1 \]

Consider the last claim. It is enough to assume $p$ is an integer.

\[ \left| \sum_{k=0}^{n} \frac{W(h)^k}{k!} - e^{W(h)} \right| = \left| \sum_{k=n+1}^{\infty} \frac{W(h)^k}{k!} \right| = \left| W(h)^{n+1} \right| \left| \sum_{k=0}^{\infty} \frac{W(h)^k}{(n+1+k)!} \right| \]

\[ = \left| W(h)^{n+1} \right| \left| \sum_{k=0}^{\infty} \frac{W(h)^k}{k!} \frac{k!}{(n+1+k)!} \right| \]

\[ \leq \left| W(h)^{n+1} \right| \frac{1}{(n+1)!} \left| \sum_{k=0}^{\infty} \frac{W(h)^k}{k!} \right| = \left| W(h)^{n+1} \right| \frac{1}{(n+1)!} e^{W(h)} \]

This converges to 0 for each $\omega$ because it says nothing more than that the $n^{th}$ term of a convergent sequence converges to 0.

\[ \int_{\Omega} \left( \left| \frac{W(h)^{n+1}}{(n+1)!} \right| e^{W(h)} \right)^{2p} dP = \int_{\Omega} \left( \left| \frac{W(h)^{n+1}}{(n+1)!} \right| e^{W(h)} \right)^{2p} dP \]

\[ = \left( \frac{1}{(n+1)!} \right)^{2p} \frac{1}{\sqrt{2\pi |h|}} \int_{\mathbb{R}} e^{-\frac{1}{2} \frac{x^2}{\lambda}} e^{2px} d2p(n+1)dx \]
the ratio test, \( \sum \) argument are as follows. The ratio, after simplifying is

\[
\frac{1}{(n+1)!} \left( \frac{2^p}{\sqrt{2\pi|h|}} \right)^{2p} \frac{1}{e^{2p|h|^2}} x^{2p(n+1)} dx \leq \frac{1}{(n+1)!} \left( \frac{2^p 2^{2p(n+1)}}{\sqrt{2\pi|h|}} \right)^{2p} \frac{1}{e^{2p|h|^2}} x^{2p(n+1)} dx
\]

\[
+ \left( \frac{1}{(n+1)!} \right)^{2p} \frac{2^p 2^{2p(n+1)}}{\sqrt{2\pi|h|}} e^{2p|h|^2} \int_{\mathbb{R}} e^{-\frac{1}{2|h|^2} (x-2p|h|^2)^2} dx
\]

This becomes

\[
2 \left( \frac{1}{(n+1)!} \right)^{2p} \frac{2^p 2^{2p(n+1)}}{\sqrt{2\pi}} \frac{e^{2p|h|^2}}{e^{2p|h|^2}} \int_{0}^{\infty} e^{-t^2} t^{2p(n+1)} dt
\]

\[
= C(h) \left( \frac{2|\mathcal{h}|^{2p(n+1)}}{(n+1)!} \right)^{2p} \frac{1}{(n+1)!} \frac{1}{((n+1)!)^{2p-1}} \Gamma \left( \frac{p(n+1)}{2} \right)
\]

\[
geq C(h) \left( \frac{2|\mathcal{h}|^{2p(n+1)}}{(n+1)!} \right)^{2p} \frac{1}{(n+1)!} \frac{1}{((n+1)!)^{2p-1}} \Gamma \left( \frac{p(n+1)}{2} \right)
\]

\[
= C(h) \left( \frac{2|\mathcal{h}|^{2p(n+1)}}{(n+1)!} \right)^{2p} \frac{1}{(n+1)!} \frac{1}{((n+1)!)^{2p-1}} \Gamma \left( \frac{p(n+1)}{2} \right)
\]

\[
= C(h) \left( \frac{2|\mathcal{h}|^{2p(n+1)}}{(n+1)!} \right)^{2p} \frac{1}{(n+1)!} \frac{1}{((n+1)!)^{2p-1}} \Gamma \left( \frac{p(n+1)}{2} \right)
\]

\[
\sum_{n} \frac{(p(n+1))!}{((n+1)!)^{2p-1}} < \infty
\]

so this also converges to 0. The details of this ratio test argument are as follows. The ratio, after simplifying is

\[
\frac{p}{(p+2p)(p+2p-1)\cdots(p+n+1)} \leq \frac{p^p(n+p)^p}{(n+2)^{2p-1}}
\]
which clearly converges to 0 since $2p - 1 > p$ since $p$ is an integer larger than 1.

Therefore, $\left\{ \left| \frac{W(h)^{n+1}}{(n+1)!} e^{W(h)} \right| \right\}_{n=1}^{\infty}$ is bounded in $L^2(\Omega)$. Then

$$\int_\Omega \left| \sum_{k=0}^{n} \frac{W(h)^k}{k!} - e^{W(h)} \right|^p dP \to 0$$

because the integrand is bounded by $\left( \left| \frac{W(h)^{n+1}}{(n+1)!} e^{W(h)} \right| \right)^p$ and it was just shown that these functions are bounded in $L^2(\Omega)$. Therefore, the claimed convergence follows from the Vitali convergence theorem. \[\blacksquare\]

The following lemma shows that the functions $e^{W(h)}$ are dense in $L^p(\Omega)$ for every $p > 1$.

**Lemma 62.6.5** Let $\mathcal{F}$ be the $\sigma$ algebra determined by the random variables $W(h)$. If $X \in L^p(\Omega, \mathcal{F}, P)$, $p > 1$ and $\int_\Omega X e^{W(h)} dP = 0$ for every $h \in H$, then $X = 0$.

**Proof:** Let $h_1, \cdots, h_p$ be given. Then for $t_i \in \mathbb{R}$,

$$\sum_{i} t_i h_i \in H$$

and so since $W$ is linear,

$$\int_\Omega X e^{t_i W(h)} dP = 0, \quad W(h) \equiv (W(h_1), \cdots, W(h_p))$$

Now by Lemma 62.4.3,

$$\int_\Omega X^+ e^{t_i (W(h_1), \cdots, W(h_p))} dP = \int_{\mathbb{R}^p} e^{t_i y} d\nu_+(y)$$

where $\nu_+(B) = E\left( X^+ \mathcal{X}_B(W(h)) \right)$. From Lemma 62.4.3, this function of $t$ is finite for all $t \in \mathbb{R}^p$. Similarly,

$$\int_\Omega X^- e^{t_i (W(h_1), \cdots, W(h_p))} dP = \int_{\mathbb{R}^p} e^{t_i y} d\nu_-(y)$$

where $\nu_-(B) = E\left( X^- \mathcal{X}_B(W(h)) \right)$. Thus for $\nu$ equal to the signed measure $\nu \equiv \nu_+ - \nu_-$,

$$f(t) \equiv \int_{\mathbb{R}^p} e^{t_i y} d\nu(y) = 0$$

for $t \in \mathbb{R}^p$. Also

$$\int_\Omega X^+ e^{it (W(h))} dP = \int_{\mathbb{R}^p} e^{it_1 y} d\nu_+(y)$$

with a similar formula holding for $X^-$. Thus

$$f(t) \equiv \int_{\mathbb{R}^p} e^{t_i y} d\nu(y) \in \mathbb{C}$$
is well defined for all $t \in \mathbb{C}$. Consider

$$\int_{\mathbb{R}^p} e^{ty} d\nu(y)$$

Is this function analytic in each $t_k$? Take a difference quotient. It equals for $h \in \mathbb{C}$,

$$\int_{\Omega} X^+ \left( e^{(t+he_k)\cdot(W(h))} - e^{t\cdot(W(h))} \right) dP = \int_{\Omega} X^+ e^{t\cdot(W(h))} \frac{(e^{he_k\cdot(W(h))} - 1)}{h} dP$$

In case $e_k \cdot W(h) = 0$ there is nothing to show. Assume then that this is not 0. Then this equals

$$\int_{\Omega} X^+ e_k \cdot (W(h)) e^{t\cdot(W(h))} e^{(he_k\cdot(W(h)) - 1)} dP$$

Now

$$\frac{e^z - 1}{z} = \left| \frac{1}{z} \sum_{k=1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=0}^{\infty} \left| e^z \right|$$

and so the integrand is dominated by

$$\left| X^+ e_k \cdot (W(h)) e^{t\cdot(W(h))} e^{(he_k\cdot(W(h)) - 1)} \right| \leq X^+ \left| e_k \cdot (W(h)) e^{(t+he_k\cdot(W(h)) - 1)} \right|$$

From Lemma D.4.3, which says that $e^{W(h)}$ is in $L^q(\Omega)$ for each $q > 1$, this is in particular true for $q = mp$ where $m$ is an arbitrary positive integer satisfying

$$p > \frac{m+1}{m}$$

Then the integrand is of the form $fg_h$ where $f \in L^p$ and $g_h$ is bounded in $L^{mp}$. Therefore,

$$\alpha \equiv (pm) / (m+1) > 1$$

and

$$\int_{\Omega} |f| g_h^\alpha \, dP = \int_{\Omega} |f| g_h^\alpha \, dP \leq \left( \int_{\Omega} |f|^p \, dP \right)^{m/(m+1)} \left( \int_{\Omega} |g_h|^{pm} \, dP \right)^{1/(m+1)}$$

which is bounded. By the Vitali convergence theorem,

$$\lim_{h \to 0} \int_{\Omega} X^+ \left( e^{(t+he_k\cdot(W(h)) - e^{t\cdot(W(h))}} \right) dP = \int_{\Omega} X^+ e_k \cdot (W(h)) e^{t\cdot(W(h))} dP$$

and so this function of $t_k$ is analytic. Similarly one can do the same thing for the integral involving $X^-$. Thus

$$0 = \int_{\mathbb{R}^p} e^{ty} d\nu(y)$$
whenever \( t_j \in \mathbb{R} \) for all \( j \) and \( t_1 \to \int_{\mathbb{R}^p} e^{it_1y_1 + t_2y_2 + \cdots + t_py_p} d\nu(y) \) is analytic on \( \mathbb{C} \). Thus this analytic function of \( t_1 \) is zero for all \( t_1 \in \mathbb{C} \) since it is zero on a set which has a limit point, and in particular

\[
\int_{\mathbb{R}^p} e^{it_1y_1 + t_2y_2 + \cdots + t_py_p} d\nu(y) = 0
\]

where each \( t_j \) is real. Now repeat the argument with respect to \( t_2 \) and conclude that

\[
\int_{\mathbb{R}^p} e^{it_1y_1 + it_2y_2 + \cdots + t_py_p} d\nu(y) = 0,
\]

and continue this way to conclude that

\[
0 = \int_{\mathbb{R}^p} e^{it\cdot y} d\nu(y)
\]

which shows that the inverse Fourier transform of \( \nu \) is 0. Thus \( \nu = 0 \). To see this, let \( \psi \in \mathcal{S} \), the Schwartz class. Then neglecting troublesome constants in the Fourier transform,

\[
0 = \int_{\mathbb{R}^p} \psi(t) \int_{\mathbb{R}^p} e^{it\cdot y} d\nu(y) dt = \int_{\mathbb{R}^p} \psi(t) e^{it\cdot y} dt d\nu(y) = \nu(F^{-1}\psi)
\]

Now \( F^{-1} \) maps \( \mathcal{S} \) onto \( \mathcal{S} \) and so this reduces to

\[
\int_{\mathbb{R}^p} \psi dt d\nu = 0
\]

for all \( \psi \in \mathcal{S} \). By density of \( \mathcal{S} \) in \( C_0(\mathbb{R}^p) \), it follows that the above holds for all \( \psi \in C_0(\mathbb{R}^p) \) and so \( \nu = 0 \).

It follows that for every \( B \) Borel and for every such description of \( \mathbf{W}(h) \),

\[
0 = \int_{\Omega} XX_B(\mathbf{W}(h)) dP = \int_{\Omega} XX_{\mathbf{W}(h)^{-1}(B)} dP
\]

Let \( \mathcal{K} \) be sets of the form \( \mathbf{W}(h)^{-1}(B) \) where \( B \) is of the form \( B_1 \times \cdots \times B_p, B_i \) open, this for some \( p \). Then this is clearly a \( \pi \) system because the intersection of any two of them is another one and

\[
0, \Omega = \mathbf{W}(h)^{-1}(\mathbb{R}^p)
\]

are both in \( \mathcal{K} \). Also \( \sigma(\mathcal{K}) = \mathcal{F} \). Let \( \mathcal{G} \) be those sets \( F \) of \( \mathcal{F} \) such that

\[
0 = \int_{\Omega} XX_F dP \tag{62.6.35}
\]

This is true for \( F \in \mathcal{K} \). Now it is clear that \( \mathcal{G} \) is closed with respect to complements and countable disjoint unions. It is closed with respect to complements because

\[
\int_{\Omega} XX_{F \cap} dP = \int_{\Omega} X (1 - X_F) dP = \int_{\Omega} X dP - \int_{\Omega} XX_F dP = 0
\]

By Dynkin’s lemma, \( \mathcal{G} = \mathcal{F} \) and so \( (62.6.35) \) holds for all \( F \in \mathcal{F} \) which requires \( X = 0 \). 

\[\blacksquare\]
62.6.2 The Wiener Processes

Recall the definition of the Wiener process.

**Definition 62.6.6** Let \( W(t) \) be a stochastic process which has the properties that whenever \( t_1 < t_2 < \cdots < t_m \), the increments \( \{W(t_i) - W(t_{i-1})\} \) are independent and whenever \( s < t \), it follows \( W(t) - W(s) \) is normally distributed with variance \( t-s \) and mean 0. Also \( t \to W(t) \) is Holder continuous with every exponent \( \gamma < 1/2 \), \( W(0) = 0 \). This is called a Wiener process.

Now in the definition of \( W \) above, you begin with a Hilbert space \( H \). There exists a probability space \((\Omega, \mathcal{F}, P)\) and a linear mapping \( W \) such that

\[
E(W(f) W(g)) = (f, g)
\]

and \( (W(f_1), W(f_2), \ldots, W(f_n)) \) is normally distributed with mean 0. Next define \( \mathcal{F} = \sigma(W(h) : h \in H) \).

Consider the special example where \( H = L^2(0, \infty; \mathbb{R}) \), real valued functions which are square integrable with respect to Lebesgue measure. Note that for each \( t \in [0, \infty) \), \( \mathcal{X}_{[0,t]} \in H \). Let

\[
W(t) = W(\mathcal{X}_{[0,t]})
\]

Then from definition, if \( t_1 < t_2 < \cdots < t_m \), the increments \( \{W(t_i) - W(t_{i-1})\} \) are independent. This is because, due to the linearity of \( W \), each of these equals \( W(\mathcal{X}_{[0,t_i]} - \mathcal{X}_{[0,t_{i-1}]} = W(\mathcal{X}_{[t_{i-1},t_i]}) \) and from Corollary \[24.5.1\], the random vector \( (W(\mathcal{X}_{[t_1,t_2]}), \ldots, W(\mathcal{X}_{[t_{m-1},t_m]})) \) is normally distributed with covariance equal to a diagonal matrix. Also \( E(W(t)^2) = E(W(\mathcal{X}_{[0,t]}))^2 = \int_0^\infty \mathcal{X}_{[0,t]}^2 ds = t \). More generally,

\[
W(t) - W(s) = W(\mathcal{X}_{[0,t]}) - W(\mathcal{X}_{[0,s]}) = W(\mathcal{X}_{[s,t]})
\]

\[
W(t - s) = W(\mathcal{X}_{[0,t-s]})
\]

so both \( W(t) - W(s) \) and \( W(t - s) \) are normally distributed with mean 0 and variance \( t-s \). What about the Holder continuity? The characteristic function of \( W(t) - W(s) \) is

\[
E(e^{i\lambda(W(t-s))}) = e^{\frac{1}{2}\lambda^2 |t-s|}
\]

Consider a few derivatives of the right side with respect to \( \lambda \) and then let \( \lambda = 0 \). This will yield \( E((W(t) - W(s))^n) \) for \( n = 1, 2, 3, 4 \).

\[
0, |s-t|, 0, 3|s-t|^2
\]

You see the pattern. By induction, you can show that \( E((W(t) - W(s))^{2m}) = C_m |t-s|^{m} \). By the Kolmogorov Centsov theorem, Theorem \[10.3.5\].

\[
E\left( \sup_{0 \leq s < t \leq T} \frac{\|W(t) - W(s)\|}{(t-s)^2} \right) \leq C_m
\]
62.6. WIENER PROCESSES, ANOTHER APPROACH

whenever \( \gamma < \beta/\alpha = \frac{m-1}{2m} \). Thus the above is true whenever \( \gamma < 1/2 \). It follows that there exists a set of measure zero off which \( t \to W(t) \) is Holder continuous with exponent \( \gamma < 1/2 \).

Thus this gives a construction of the real Wiener process. Now consider the normal filtration

\[
\mathcal{F}_x = \cap_{t>s} \sigma (W(u) - W(r) : 0 \leq r < u \leq t)
\]

By Lemma \[62.4.2\], \( \{W(t)\} \) is a martingale with respect to this filtration, because of the independence of the increments.

Of course you could also take an arbitrary \( f \in L^2(0, \infty) \) and consider \( W(t) \equiv W(X(0,t)f) \). You could consider this as an integral and write it in the notation

\[
W(t) \equiv \int_0^t f dW \equiv W(fX(0,t))
\]

Then from the construction,

\[
E \left( \left( \int_0^t f dW \right)^2 \right) = E \left( W(fX(0,t))^2 \right) = \int_0^t f^2 X(0,t) \, ds = \int_0^t |f|^2 \, ds = E \left( \int_0^t |f|^2 \, ds \right)
\]

because \( f \) does not depend on \( \omega \). This of course is formally the Ito isometry.

62.6.3 Q Wiener Processes In Hilbert Space

Now let \( U \) be a real separable Hilbert space. Let an orthonormal basis for \( U \) be \( \{g_i\} \). Now let \( L^2(0, \infty, U) \) be \( H \) in the above construction. For \( h, g \in L^2(0, \infty, U) \).

\[
E(W(h)W(g)) = (h, g)_{L^2(0, \infty, U)} = (h, g)_H
\]

Here each \( W(g) \) will be a real valued normal random variable, the variance of \( W(g) \) is \( |g|^2_{L^2(0, \infty, U)} \) and its mean is 0, every vector \( (W(h_1), \cdots, W(h_n)) \) being generalized multivariate normal. Let

\[
\psi_k(t) = W(X(0,t)g_k).
\]

Then this is a real valued random variable. Disjoint increments are obviously independent in the same way as before. Also

\[
E(\psi_k(t)\psi_j(s)) = E(W(X(0,t)g_k)W(X(0,s)g_j)) = \int_0^\infty X(0,t\wedge s)(g_k, g_j)_U \, dt = 0
\]

if \( j \neq k \). Thus the random variables \( \psi_k(t) \) and \( \psi_j(s) \) are independent. This is because, from the construction, \( (\psi_k(t), \psi_j(s)) \) is normally distributed and the covariance is a diagonal matrix. Also

\[
\psi_k(t) - \psi_k(s) = W(X(0,t)Jg_k) - W(X(0,s)Jg_k) = W(X(s,t)Jg_k)
\]
\[
\psi_k(t-s) \equiv W(\mathcal{A}_{(0,t-s)}Jg_k)
\]
so \(\psi_k(t-s)\) has the same mean, 0 and variance, \(|t-s|\), as \(\psi_k(t) - \psi_k(s)\). Thus these have the same distribution because both are normally distributed.

Now let \(J\) be a Hilbert Schmidt map from \(U\) to \(H\). Then consider
\[
W(t) = \sum_k \psi_k(t) Jg_k. \tag{62.6.37}
\]
This has values in \(H\). It is shown below that the series converges in \(L^2(\Omega; H)\).

Definition 62.6.7 Let \(W(t)\) be a stochastic process with values in \(H\), a real separable Hilbert space which has the properties that \(t \to W(t,\omega)\) is continuous, whenever \(t_1 < t_2 < \cdots < t_m\), the increments \(\{W(t_i) - W(t_{i-1})\}\) are independent, \(W(0) = 0\), and whenever \(s < t\),
\[
\mathcal{L}(W(t) - W(s)) = N(0, (t-s)Q)
\]
which means that whenever \(h \in H\),
\[
\mathcal{L}((h, W(t) - W(s))) = N(0, (t-s)(Qh,h))
\]
Also
\[
E((h_1, W(t) - W(s))(h_2, W(t) - W(s))) = (Qh_1, h_2) (t-s).
\]
Here \(Q\) is a nonnegative trace class operator. Recall this means
\[
Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i
\]
where \(\{e_i\}\) is a complete orthonormal basis, \(\lambda_i \geq 0\), and
\[
\sum_{i=1}^{\infty} \lambda_i < \infty
\]
Such a stochastic process is called a \(Q\) Wiener process. In the case where these have values in \(\mathbb{R}^n\), \(tQ\) ends up being the covariance matrix of \(W(t)\).

Proposition 62.6.8 The process defined in 62.6.37 is a \(Q\) Wiener process in \(H\) where \(Q = JJ^*\).

Proof: First, why does the sum converge? Consider the sum for an increment in time. Let \(t_{i-1} = 0\) to obtain the convergence of the sum for a given \(t\). Consider
the difference of two partial sums.
\[
E \left( \sum_{k,l=m}^{n} (\psi_k(t_i) - \psi_k(t_{i-1})) Jg_k, (\psi_l(t_i) - \psi_l(t_{i-1})) Jg_l \right)
\]
\[
= E \left( \sum_{k,l=m}^{n} (J^* Jg_k, g_l) (\psi_k(t_i) - \psi_k(t_{i-1})) (\psi_l(t_i) - \psi_l(t_{i-1})) \right)
\]
\[
= \sum_{k,l=m}^{n} (J^* Jg_k, g_l) E \left( (\psi_k(t_i) - \psi_k(t_{i-1})) (\psi_l(t_i) - \psi_l(t_{i-1})) \right)
\]
\[
= \sum_{k=m}^{n} (J^* Jg_k, g_k) E \left( \psi_k(t_i) - \psi_k(t_{i-1}) \right)^2 = \sum_{k=m}^{n} (J^* Jg_k,g_k) (t_i - t_{i-1})
\]
\[
= \sum_{k=m}^{n} |Jg_k|^2_H (t_i - t_{i-1})
\]
and this converges to 0 as \( m, n \rightarrow \infty \) since \( J \) is Hilbert Schmidt. Thus the sum converges in \( L^2(\Omega, H) \). Why are the disjoint increments independent?

Let \( \lambda_k \in H \). Consider \( t_0 < t_1 < \cdots < t_n \).
\[
E \left( \exp i \sum_{k=1}^{n} (\lambda_k, W(t_k) - W(t_{k-1})) \right) = \prod_{k=1}^{n} E \left( \exp (i (\lambda_k, W(t_k) - W(t_{k-1}))) \right)
\]
(62.6.38)

Start with the left. There are finitely many increments concerned and so it can be assumed that for each \( k \) one can have \( m \rightarrow \infty \) such that the partial sums up to \( m \) in the definition of \( W(t_k) - W(t_{k-1}) \) converge pointwise a.e. Thus
\[
E \left( \exp i \sum_{k=1}^{n} (\lambda_k, W(t_k) - W(t_{k-1})) \right)
\]
\[
= \lim_{m \rightarrow \infty} E \left( \exp i \sum_{k=1}^{m} \left( \lambda_k, \sum_{j=1}^{m} (\psi_j(t_k) - \psi_j(t_{k-1})) Jg_j \right) \right)
\]
\[
= \lim_{m \rightarrow \infty} E \left( \exp \sum_{k=1}^{m} \sum_{j=1}^{m} i (\lambda_k, (\psi_j(t_k) - \psi_j(t_{k-1})) Jg_j) \right)
\]
\[
= \lim_{m \rightarrow \infty} E \left( \prod_{j=1}^{m} \exp \left( \sum_{k=1}^{m} i (\lambda_k, (\psi_j(t_k) - \psi_j(t_{k-1})) Jg_j) \right) \right)
\]
Now from 62.4.39 \( \{ \sum_{k=1}^{m} i (\lambda_k, (\psi_j(t_k) - \psi_j(t_{k-1})) Jg_j) \}_{j=1}^{m} \) are independent. Hence the above equals
\[
= \lim_{m \rightarrow \infty} \prod_{j=1}^{m} E \left( \exp \left( \sum_{k=1}^{m} i (\lambda_k, (\psi_j(t_k) - \psi_j(t_{k-1})) Jg_j) \right) \right)
\]
CHAPTER 62. WIENER PROCESSES

\[
\lim_{m \to \infty} \prod_{j=1}^{m} E \left( \prod_{k=1}^{n} \exp \left( i \left( \lambda_k, (\psi_j (t_k) - \psi_j (t_{k-1})) Jg_j \right) \right) \right)
\]

Now from independence of the increments for the \(\psi_j\), this equals

\[
\lim_{m \to \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E \left( \exp \left( i \left( \lambda_k, (\psi_j (t_k) - \psi_j (t_{k-1})) Jg_j \right) \right) \right)
\]

\[
= \lim_{m \to \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} E \left( \exp \left( i \left( \lambda_k, Jg_j \right) (\psi_j (t_k) - \psi_j (t_{k-1})) \right) \right)
\]

\[
= \lim_{m \to \infty} \prod_{j=1}^{m} \prod_{k=1}^{n} e^{-\frac{1}{2} (\lambda_k, Jg_j)^2 (t_k - t_{k-1})} = \lim_{m \to \infty} \prod_{j=1}^{m} e^{-\frac{1}{2} \sum_{k=1}^{n} (\lambda_k, Jg_j)^2 (t_k - t_{k-1})}
\]

\[
= \lim_{m \to \infty} \exp \left( -\frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{\infty} (J^* \lambda_k, g_j)^2 (t_k - t_{k-1}) \right)
\]

\[
= \exp \left( -\frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{\infty} (J^* \lambda_k, g_j)^2 (t_k - t_{k-1}) \right)
\]

(62.6.39)

What is the right side of (62.6.39)

\[
\prod_{k=1}^{n} E \left( \exp \left( i \left( \lambda_k, W (t_k) - W (t_{k-1}) \right) \right) \right) = \prod_{k=1}^{n} E \left[ \exp \left( i \left( \lambda_k, \sum_{j=1}^{\infty} (\psi_j (t_k) - \psi_j (t_{k-1})) Jg_j \right) \right) \right]
\]

\[
= \lim_{m \to \infty} \prod_{k=1}^{m} E \left[ \exp \left( i \left( \lambda_k, \sum_{j=1}^{m} (\psi_j (t_k) - \psi_j (t_{k-1})) Jg_j \right) \right) \right]
\]

\[
= \lim_{m \to \infty} \prod_{k=1}^{m} E \left[ \exp \left( i \sum_{j=1}^{m} (Jg_j, \lambda_k) (\psi_j (t_k) - \psi_j (t_{k-1})) \right) \right]
\]

\[
= \lim_{m \to \infty} \prod_{k=1}^{m} \prod_{j=1}^{m} \left( \exp \left( i (J^* \lambda_k, g_j) (\psi_j (t_k) - \psi_j (t_{k-1})) \right) \right)
\]
and by independence,
\[
\lim_{m \to \infty} \prod_{k=1}^{n} \prod_{j=1}^{m} E \left[ i (J^* \lambda_k, g_j) \left( \psi_j (t_k) - \psi_j (t_{k-1}) \right) \right] = \lim_{m \to \infty} \prod_{k=1}^{n} \exp \left( -\frac{1}{2} \sum_{j=1}^{m} (J^* \lambda_k, g_j)^2 (t_k - t_{k-1}) \right)
\]

which is exactly the same thing as 62.6.39. Thus the disjoint increments are independent.

You could also do something like the following. Let \( W_m (t) \) denote the partial sum for \( W (t) \) and since there are only finitely many increments, we can assume the partial sums converge a.e. Then we need to consider the random variables
\[
\{ (W_m (t_k) - W_m (t_{k-1})) \}_{k=1}^{m} = \left\{ \left( \sum_{i=1}^{m} (\psi_i (t_k) - \psi_i (t_{k-1})) (Jg_i) \right) \right\}_{k=1}^{m}
\]
Then for any \( h \in H \), you could consider
\[
\left\{ \left( \sum_{i=1}^{m} (\psi_i (t_k) - \psi_i (t_{k-1})) (Jg_i, h)_H \right) \right\}_{k=1}^{m}
\]
and the vector whose \( k^{th} \) component is \( \sum_{i=1}^{m} (\psi_i (t_k) - \psi_i (t_{k-1})) (Jg_i, h)_H \) for \( k = 1, 2, \ldots, n \) is normally distributed and the covariance is a diagonal matrix. Hence these are independent random variables as hoped. Now you can pass to a limit as \( m \to \infty \). Since this is true for any \( h \in H \) that the random variables \( (W (t_k) - W (t_{k-1}), h)_H \) are independent, it follows that the random variables \( W (t_k) - W (t_{k-1}) \) are also.

What of the Holder continuity? In the above computation for independence, as a special case, for \( \lambda \in H \),
\[
E \left( \exp i \left( \lambda, W (t) - W (s) \right) \right) = \exp \left( -\frac{1}{2} |J^* \lambda|_{U}^2 (t-s) \right) \tag{62.6.40}
\]
In particular, replacing \( \lambda \) with \( r \lambda \) for \( r \) real,
\[
E \left( \exp ir \left( \lambda, W (t) - W (s) \right) \right) = \exp \left( -\frac{1}{2} r^2 |J^* \lambda|_{U}^2 (t-s) \right)
\]
Now we differentiate with respect to \( r \) and then take \( r = 0 \) as before to obtain finally that
\[
E \left( (\lambda, W (t) - W (s))^{2m} \right) \leq C_m |J^* \lambda|^{2m} |t-s|^m = C_m (Q \lambda)^m |t-s|^m
\]
Then letting \( \{ h_k \} \) be an orthonormal basis for \( H \), and using the above inequality with Minkowski’s inequality,

\[
\left( E \left( |W(t) - W(s)|^{2m} \right)^{1/m} \right)^{1/m} \leq \sum_{k=1}^{\infty} \left( E \left( |W(t) - W(s), h_k|^2 \right)^{m} \right)^{1/m}
\]

\[
\leq \sum_{k=1}^{\infty} \left( E \left( |W(t) - W(s), h_k|^2 \right) \right)^{1/m} \leq \sum_{k=1}^{\infty} \left( C_m (t - s)^m |J^* h_k|_{U}^{2m} \right)^{1/m}
\]

\[
= C_m^{1/m} |t - s| \sum_{k=1}^{\infty} |J^* h_k|^2 = C_m^{1/m} |t - s| \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (J^* h_k, g_j)^2
\]

\[
= C_m^{1/m} |t - s| \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (h_k, Jg_j)^2 = |t - s| C_m^{1/m} \sum_{j=1}^{\infty} |Jg_j|^2_H
\]

and since \( J \) is Hilbert Schmidt, modifying the constant yields

\[
E \left( |W(t) - W(s)|^{2m} \right) \leq C_m |t - s|^m
\]

By the Kolmogorov Centsov theorem, Theorem 60.2.3, whenever \( \gamma < \beta / \alpha = \frac{m-1}{2m} \). Thus the above is true whenever \( \gamma < 1/2 \). Hence off a set of measure zero, \( t \to W(t) \) is Holder continuous.

What of the covariance condition? From letting \( f, g \) be two elements of \( H \),

\[
E \left( \exp i (\alpha f + \beta g, W(t) - W(s)) \right) = \exp \left( -\frac{1}{2} (Q(\alpha f + \beta g), \alpha f + \beta g) (t - s) \right)
\]

Differentiate with respect to \( \alpha \)

\[
E (i (\alpha f + \beta g, W(t) - W(s)) \exp i (\alpha f + \beta g, W(t) - W(s)))
\]

\[
= - |\alpha (Qf, f) + (Qf, \beta g)| (t - s) \exp \left( -\frac{1}{2} (Q(\alpha f + \beta g), \alpha f + \beta g) (t - s) \right)
\]

Let \( \alpha = 0. \)

\[
E (i (f, W(t) - W(s)) \exp i (\beta g, W(t) - W(s)))
\]

\[
= - [(Qf, \beta g)] (t - s) \exp \left( -\frac{1}{2} (Q(\beta g), \beta g) (t - s) \right)
\]

Now differentiate with respect to \( \beta \)

\[
E (- (f, W(t) - W(s)) (g, W(t) - W(s)) \exp i (\beta g, W(t) - W(s)))
\]
= −[(Qf, g)](t − s) \exp\left(-\frac{1}{2} (Q(\beta g), \beta g)(t − s)\right) + \{(Qf, \beta g)\}(t − s) \text{(something)}

Now let \( \beta = 0 \).

\[
E \left( (f, W(t)) - W(s) \right) = (Qf, g)(t − s)
\]

Finally, \( Q = JJ^* \). It is self-adjoint and nonnegative and so there is a complete orthonormal basis \( \{e_i\} \) such that \( Qe_i = \lambda_i e_i \). Then \( \lambda_i = (Qe_i, e_i)_H \) and so

\[
\sum_i \lambda_i = \sum_i (Qe_i, e_i) = \sum_i |J^*e_i|_U^2 < \infty
\]

because \( J \) and hence \( J^* \) are both Hilbert Schmidt operators. \( \square \)

Recall the notion of the Hilbert space \( LU \) in Definition 17.2.1.

What if you have a given \( Q \in \mathcal{L}(H,H) \) which is trace class, \( Q = Q^* \), and nonnegative. Does there exist a \( Q \) Wiener process of the sort just described? It appears this amounts to obtaining a Hilbert Schmidt map \( J \) from some Hilbert space \( U \) to \( H \) such that \( Q = JJ^* \).

Since \( Q \) is trace class and is self-adjoint, it follows that there is an orthonormal basis \( \{e_i\} \), \( Qe_i = \lambda_i e_i \), where \( \lambda_i \) is positive for \( i \leq L \) or positive for all \( i \). Then

\[
Q^{1/2} = \sum_{i=1}^L \sqrt{\lambda_i} e_i \otimes e_i
\]

and

\[
Q^{1/2}e_i = \sqrt{\lambda_i} e_i.
\]

Then also on \( Q^{1/2}H \),

\[
\left(Q^{1/2}e_i, Q^{1/2}e_j\right)_{Q^{1/2}H} = (e_i, e_j)_H
\]

and so an orthonormal basis in \( Q^{1/2}H \) is \( \{\sqrt{\lambda_i}e_i\}_{i=1}^L \). Then define \( J : Q^{1/2}H \to H \)

\[
Jx = \sum_{k=1}^L x, \sqrt{\lambda_k} e_k \bigg|_{Q^{1/2}H} \sqrt{\lambda_k} e_k
\]

It follows from the above that

\[
J e_j = \sum_{k=1}^L \frac{1}{\sqrt{\lambda_j}} \sqrt{\lambda_k} e_j \bigg|_{Q^{1/2}H} \sqrt{\lambda_k} e_k = e_j
\]

Then

\[
\sum_{i=1}^L \left|J \sqrt{\lambda_i} e_i\right|_H^2 = \sum_{i=1}^L \sum_{k=1}^L \left(\sqrt{\lambda_i} e_i, \sqrt{\lambda_k} e_k\right)_{Q^{1/2}H} \sqrt{\lambda_k} e_k \bigg|_H^2 = \sum_{i=1}^L \lambda_i < \infty
\]
Thus it is clear that $J$ is Hilbert Schmidt. Is $JJ^* = Q$? For $y \in Q^{1/2} H, x \in H$,

$$ (J^* x, y)_{Q^{1/2} H} \equiv (x, J(y))_H = \left( x, \sum_{k=1}^L \left( y, \sqrt{\lambda_k} e_k \right)_{Q^{1/2} H} \sqrt{\lambda_k} e_k \right)_H $$

$$ = \sum_{k=1}^L (x, \sqrt{\lambda_k} e_k)_H \left( y, \sqrt{\lambda_k} e_k \right)_{Q^{1/2} H} $$

Thus for $y \in H, x \in H$,

$$ (J^* x, J^* y)_{Q^{1/2} H} = \sum_{k=1}^L (x, \sqrt{\lambda_k} e_k)_H \left( J^* y, \sqrt{\lambda_k} e_k \right)_{Q^{1/2} H} $$

$$ = \sum_{k=1}^L \lambda_k (x, e_k)_H (y, e_k)_H = (Qx, y) $$

and so $(JJ^* x, y) = (Qx, y)$ showing that $JJ^* = Q$. This shows the following.

**Proposition 62.6.9** Let $Q \in \mathcal{L}(H, H)$ where $H$ is a real separable Hilbert space and $(Qx, x) \geq 0$ and is trace class. Then there exists a one to one Hilbert Schmidt map $J : Q^{1/2} H \rightarrow H$ such that $JJ^* = Q$. Then the $Q$ Wiener process is $W(t) = \sum_{k=1}^\infty \psi_k(t) J g_k$ where $\{g_k\}$ is a complete orthonormal basis for the Hilbert space $Q^{1/2} H$.

Note that in case $H$ is $\mathbb{R}^p$ and $Q$ is any symmetric $p \times p$ matrix, having non-negative eigenvalues, this is automatically trace class and so the above conclusion holds. In particular, the covariance condition says in this case that

$$ E\left( ((e_i, W(t) - W(s))) (e_j, W(t) - W(s)) \right) = E\left( (W_i(t) - W_i(s)) (W_j(t) - W_j(s)) \right) = (Qe_i, e_j) = Q_{ij} $$

This is a $p$ dimensional Wiener process.

### 62.6.4 Levy’s Theorem In Hilbert Space

Recall the concept of quadratic variation. Let $W(t)$ be a $Q$ Wiener process. Does it follow $\{W(t)\} \in \mathcal{M}^2_T (H)$? The Wiener process is continuous. Furthermore,

$$ E\left( |W(t)|^2_H \right) < \infty $$

for each $t \in [0, T]$. Since $\{W(t)\}$ is a martingale, Theorem [16.3] can be applied to conclude

$$ E\left( \left| W(t) \right|^2_H \right)^{1/2} \leq E\left( \left( \sup_{t \in [0, T]} \left| W(t) \right| \right)^2 \right)^{1/2} \leq 2 E\left( \left| W(T) \right|^2_H \right)^{1/2} $$
and so \( \{ W(t) \} \in \mathcal{M}_2^H \). Therefore, by the Doob Meyer decomposition, Theorem 61.7.15, there exists an increasing natural process, \( A(t) \) and a martingale, \( Y(t) \) such that
\[
|W(t)|_H^2 = Y(t) + A(t).
\]
What is \( A(t) \)? Consider the process
\[
|W(t)|^2
\]
From Theorem 62.5.4 this equals
\[
\sum_{k=1}^{\infty} \lambda_k \psi_k(t)^2
\]
where \( \psi_k(t) \) is a one dimensional Wiener process and
\[
Q = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k, \sum_{k=1}^{\infty} \lambda_k < \infty.
\]
By Lemma 62.4.2, \( \{ W(t) \} \) is a martingale. Therefore, for \( s < t \) and \( A \in \mathcal{F}_s \), it follows since \( X_A \) is independent of \( W(t) - W(s) \) as in the proof of Lemma 62.4.2 that the following holds.
\[
\int_A E \left( |W(t)|^2 |\mathcal{F}_s \right) - |W(s)|^2 \, dP
\]
\[
= \int_A E \left( |W(t)|^2 + |W(s)|^2 - 2W(t) \cdot W(s) |\mathcal{F}_s \right) \, dP
\]
\[
= \int_A E \left( |W(t) - W(s)|^2 \right) \, dP = \int_A |W(t) - W(s)|^2 \, dP
\]
\[
= P(A) \int_{\Omega} |W(t) - W(s)|^2 \, dP
\]
\[
= P(A) \sum_{k=1}^{\infty} \lambda_k E \left( (\psi_k(t) - \psi_k(s))^2 \right)
\]
\[
= P(A) (t - s) \sum_{k=1}^{\infty} \lambda_k = P(A) (t - s) \text{tr}(Q).
\]
Therefore,
\[
\int_A E \left( |W(t)|^2 - t \text{tr}(Q) \right) |\mathcal{F}_s \right) - \left( |W(s)|^2 - s \text{tr}(Q) \right) \, dP = 0
\]
and since \( A \in \mathcal{F}_s \) is arbitrary, this shows \( \{ |W(t)|^2 - t \text{tr}(Q) \} \) is a martingale. Hence the Doob Meyer decomposition for \( |W(t)|^2 \) is
\[
|W(t)|^2 = Y(t) + t \text{tr}(Q).
\]
where $Y(t)$ is a martingale.

There is a generalization of Levy's theorem to Hilbert space valued Wiener processes.

**Theorem 62.6.10** Let \( \{W(t)\} \in \mathcal{M}_2^2(H), E(W(t)) = 0 \), where $H$ is a real separable Hilbert space. Then for $Q$ a nonnegative symmetric trace class operator, \{W(t)\} is a $Q$ Wiener process if and only if both \{W(t)\} and \((W(t),h)^2 - t(Qh,h)\) are martingales for every $h \in H$.

**Proof:** First suppose \{W(t)\} is a $Q$ Wiener process. Then defining the filtration to be
\[
\mathcal{F}_t \equiv \sigma(W(s) - W(u) : u \leq s \leq t),
\]
it follows from Lemma 62.4.2 that \{W(t)\} is a martingale. Consider
\[
\{(W(t),h)^2 - t(Qh,h)\}.
\]
Let $A \in \mathcal{F}_s$ where $s \leq t$. Then using the fact \{W(t)\} is a martingale,
\[
\int_A E\left((W(t) - W(s),h)^2 | \mathcal{F}_s\right) dP
\]
\[
= \int_A E\left((W(t),h)^2 + (W(s),h)^2 - 2(W(t),h)(W(s),h) | \mathcal{F}_s\right) dP
\]
\[
= \int_A E\left((W(t),h)^2 | \mathcal{F}_s\right) + (W(s),h)^2 - E(2(W(t),h)(W(s),h) | \mathcal{F}_s) dP
\]
\[
= \int_A E\left((W(t),h)^2 | \mathcal{F}_s\right) dP + \int_A (W(s),h)^2 dP
\]
\[
- \int_A (W(s),h) E(2(W(t),h) | \mathcal{F}_s) dP
\]
\[
= \int_A E\left((W(t),h)^2 | \mathcal{F}_s\right) dP - \int_A (W(s),h)^2 dP.
\]
Also since $X_A$ is independent of $(W(t) - W(s),h)^2$ as in the proof of Lemma 62.4.2, and \{W(t)\} is a $Q$ Wiener process,
\[
\int_A E\left((W(t) - W(s),h)^2 | \mathcal{F}_s\right) dP
\]
\[
= \int_A (W(t) - W(s),h)^2 dP
\]
\[
= P(A) \int_\Omega (W(t) - W(s),h)^2 dP
\]
\[
= P(A)(t - s)(Qh,h).
\]
Thus, this has shown that for all $A \in \mathcal{F}_s$,
\[
\int_A E \left( (W(t), h)^2 | \mathcal{F}_s \right) dP - \int_A (W(s), h)^2 dP = P(A) (t-s) (Qh, h) = \int_A (t-s) (Qh, h) dP
\]
and since $A \in \mathcal{F}_s$ is arbitrary, this proves
\[
E \left( (W(t), h)^2 - t (Qh, h) | \mathcal{F}_s \right) = (W(s), h)^2 - s (Qh, h)
\]
This proves one half of the theorem.

Next suppose both $\{W(t)\}$ and $\{(W(t), h)^2 - t (Qh, h)\}$ are martingales for any $h \in H$. It follows that both $\{(W(t), h)\}$ and $\{(W(t), h)^2 - t (Qh, h)\}$ are martingales also. Therefore, by Levy’s theorem, Theorem 61.8.5, $\{(W(t), h)\}$ is a Wiener process with the property that its variance at $t$ equals $(Qh, h) t$ instead of $t$. Thus the time increments are normal and independent. I need to verify that $\{W(t)\}$ is a $Q$ Wiener process. One of the things which needs to be shown is that
\[
E ((W(t) - W(s), h_1) (W(t) - W(s), h_2)) = (Qh_1, h_2) (t-s) . \tag{62.6.41}
\]
I have just shown
\[
E ((W(t) - W(s), h)^2) = (t-s) (Qh, h) \tag{62.6.42}
\]
which follows from Levy’s theorem which concludes $\{(W(t), h)\}$ is a Wiener process. Therefore,
\[
E ((W(t) - W(s), h_1 + h_2) (W(t) - W(s), h_2 + h_1)) = (Q (h_1 + h_2), (h_1 + h_2)) (t-s)
\]
Now using 62.6.42, it follows from this that
\[
E ((W(t) - W(s), h_1) (W(t) - W(s), h_2)) = (Qh_1, h_2) (t-s)
\]
which shows 62.6.42. This completes the proof.
Chapter 63

Stochastic Integration

63.1 Integrals Of Elementary Processes

Stochastic integration starts with a $Q$ Wiener process having values in a separable Hilbert space $U$. Thus it satisfies the following definition.

**Definition 63.1.1** Let $W(t)$ be a stochastic process with values in $U$, a real separable Hilbert space which has the properties that $t \to W(t, \omega)$ is continuous. Whenever $t_1 < t_2 < \cdots < t_m$, the increments $\{W(t_i) - W(t_{i-1})\}$ are independent, $W(0) = 0$, and whenever $s < t$,

$$\mathcal{L}(W(t) - W(s)) = N(0, (t - s) Q)$$

which means that whenever $h \in H$,

$$\mathcal{L}((h, W(t) - W(s))) = N(0, (t - s) (Qh, h))$$

Also

$$E((h_1, W(t) - W(s)) (h_2, W(t) - W(s))) = (Qh_1, h_2) (t - s).$$

Here $Q$ is a nonnegative trace class operator. Recall this means

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$$

where $\{e_i\}$ is a complete orthonormal basis, $\lambda_i \geq 0$, and

$$\sum_{i=1}^{\infty} \lambda_i < \infty$$

Such a stochastic process is called a $Q$ Wiener process.
Recall that such Wiener processes are always of the form
\[ \sum_{k=1}^{\infty} \psi_k(t) Jg_k \]
where \( J \) is a Hilbert Schmidt operator from a suitable space \( U_0 \) to \( U \) and the \( \psi_k \) are real independent Wiener processes described earlier. This follows from Theorem 62.5.4 where you let \( U_0 \subseteq U \) be such that for \( J \) the inclusion map, \( Je_k = \sqrt{\lambda_k} e_k \) for \( Q = \sum \lambda_k e_k \otimes e_k \), the \( e_k \) an orthonormal set in \( U \). Thus
\[
(Qx, y) = \left( \sum_k \lambda_k e_k (x, e_k), y \right) = \sum_k \left( x, \sqrt{\lambda_k} e_k \right) \left( y, \sqrt{\lambda_k} e_k \right) = \sum_k (x, Je_k) (y, Je_k) = \left( J^* x, e_k \right) \left( J^* y, e_k \right) = (JJ^* x, y)
\]
so it follows that \( Q = JJ^* \). Of course in finite dimensions, there is no issue because the identity map is Hilbert Schmidt.

Recall the definition of \( L^2(U, H) \equiv \mathcal{L}_2 \) the space of Hilbert Schmidt operators. \( \Psi \in L^2(U, H) \) means \( \Psi \) has the property that for some (equivalently all) orthonormal basis of \( U \) \( \{e_k\} \), it follows
\[
\sum_{k=1}^{\infty} \|\Psi(e_k)\|^2 < \infty
\]
and the inner product for two of these, \( \Psi, \Phi \) is given by
\[
(\Psi, \Phi)_{L_2} = \sum_k (\Psi(e_k), \Phi(e_k))
\]
Then for such a Hilbert Schmidt operator, the norm in \( \mathcal{L}_2 \) is given by
\[
\left( \sum_{k=1}^{\infty} \|\Psi(e_k)\|^2 \right)^{1/2} \equiv \|\Psi\|_{\mathcal{L}_2}.
\]
Note this is the same as
\[
\left( \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\Psi(e_k), f_j)^2 \right)^{1/2}
\]
(63.1.1)
where \( \{f_j\} \) is an orthonormal basis for \( H \). This is the analog of the Frobenius norm for matrices obtained as
\[
\text{trace}(MM^*)^{1/2} = \left( \sum_{i} (MM^*)_{ii} \right)^{1/2} = \left( \sum_{i,j} M^2_{ij} \right)^{1/2}
\]
Also shows right away that if \( \Psi \in L^2(U,H) \), then
\[
\|\Psi\|_{L^2(U,H)}^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\Psi e_k, f_j)_H^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (e_k, \Psi^* f_j)_U^2 = \|\Psi^*\|_{L^2(H,U)}^2
\]
and that \( \Psi \) and \( \Psi^* \) are Hilbert Schmidt together.

The filtration will continue to be denoted by \( F_t \). It will be defined as the following normal filtration in which
\[
\sigma(W(s) - W(r) : 0 \leq r < s \leq u)
\]
is the completion of \( \sigma(W(s) - W(r) : 0 \leq r < s \leq u) \).
\[
F_t \equiv \cap_{u>t} \sigma(W(s) - W(r) : 0 \leq r < s \leq u).
\tag{63.1.2}
\]
and \( \sigma(W(s) - W(r) : 0 \leq r < s \leq u) \) denotes the \( \sigma \) algebra of all sets of the form
\[
(W(s) - W(r))^{-1} \text{(Borel)}
\]
where \( 0 \leq r < s \leq u \).

**Definition 63.1.2** Let \( \Phi(t) \in L(U,H) \) be constant on each interval, \( (t_m, t_{m+1}] \) determined by a partition of \( [a,T] \), \( 0 \leq a = t_0 < t_1 \cdots < t_n = T \). Then \( \Phi(t) \) is said to be elementary if also \( \Phi(t_m) \) is \( F_{t_m} \) measurable and \( \Phi(t_m) \) equals a sum of the form
\[
\Phi(t_m)(\omega) = \sum_{j=1}^{m} \Phi_j \chi_{A_j}
\]
where \( \Phi_j \in L(U,H) \), \( A_j \in F_{t_m} \). What does the measurability assertion mean? It means that if \( O \) is an open (Borel) set in the topological space \( L(U,H) \), \( \Phi(t_m)^{-1}(O) \in F_{t_m} \). Thus an elementary function is of the form
\[
\Phi(t) = \sum_{k=0}^{n-1} \Phi(t_k) \chi_{(t_k,t_{k+1}]}(t).
\]
Then for \( \Phi \) elementary, the stochastic integral is defined by
\[
\int_a^t \Phi(s) dW(s) = \sum_{k=0}^{n-1} \Phi(t_k) (W(t \wedge t_{k+1}) - W(t \wedge t_k)).
\]
It is also sometimes denoted by \( \Phi \cdot W(t) \).
The above definition is the same as saying that for $t \in (t_m, t_{m+1}]$,
\[
\int_a^t \Phi(s) \, dW(s) = \sum_{k=0}^{m-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) \\
+ \Phi(t_m) (W(t) - W(t_m)).
\] (63.1.3)

The following lemma will be useful.

**Lemma 63.1.3** Let $f, g \in L^2(\Omega; H)$ and suppose $g$ is $\mathcal{G}$ measurable and $f$ is $\mathcal{F}$ measurable where $\mathcal{F} \supseteq \mathcal{G}$. Then
\[
E((f, g)_H | \mathcal{G}) = (E(f | \mathcal{G}), g)_H \text{ a.e.}
\]

Similarly if $\Phi$ is $\mathcal{G}$ measurable as a map into $L(U, H)$ with
\[
\int_\Omega ||\Phi||^2 \, dP < \infty
\]
and $f$ is $\mathcal{F}$ measurable as a map into $U$ such that $f \in L^2(\Omega; H)$, then
\[
E(\Phi f | \mathcal{G}) = \Phi E(f | \mathcal{G}).
\]

**Proof:** Let $A \in \mathcal{G}$. Let $\{g_n\}$ be a sequence of simple functions, measurable with respect to $\mathcal{G}$,
\[
g_n(\omega) \equiv \sum_{k=1}^{m_n} a_{n,k} X_{E_k^n}(\omega)
\]
which converges in $L^2(\Omega; H)$ and pointwise to $g$. Then
\[
\int_A (E(f | \mathcal{G}), g)_H \, dP = \lim_{n \to \infty} \int_A (E(f | \mathcal{G}), g_n)_H \, dP
\]
\[
= \lim_{n \to \infty} \int_A \sum_{k=1}^{m_n} (E(f | \mathcal{G}), a_{n,k} X_{E_k^n})_H \, dP = \lim_{n \to \infty} \int_A \sum_{k=1}^{m_n} E((f, a_{n,k} X_{E_k^n})_H | \mathcal{G}) X_{E_k^n} \, dP
\]
\[
= \lim_{n \to \infty} \int_A E((f, g_n)_H | \mathcal{G}) \, dP = \lim_{n \to \infty} \int_A (f, g_n)_H \, dP = \int_A (f, g)_H \, dP
\]
which shows
\[
(E(f | \mathcal{G}), g)_H = E((f, g)_H | \mathcal{G})
\]
as claimed.

Consider the other claim. Let
\[
\Phi_n(\omega) = \sum_{k=1}^{m_n} \Phi_{k,n} X_{E_k^n}(\omega), \quad E_k^n \in \mathcal{G}
\]
63.1. INTEGRALS OF ELEMENTARY PROCESSES

where \( \Phi_{n}^{k} \in \mathcal{L}(U, H) \) be such that \( \Phi_{n} \) converges to \( \Phi \) pointwise in \( \mathcal{L}(U, H) \) and also

\[
\int_{\Omega} ||\Phi_{n} - \Phi||^{2} dP \to 0.
\]

Then letting \( A \in \mathcal{G} \) and using Corollary 19.2.6 as needed,

\[
\int_{A} \Phi E(f|\mathcal{G}) dP = \lim_{n \to \infty} \int_{A} \Phi_{n} E(f|\mathcal{G}) dP = \lim_{n \to \infty} \sum_{k=1}^{m} \Phi_{n}^{k} E(f|\mathcal{G}) dP
\]

Since \( A \in \mathcal{G} \) is arbitrary, this proves the lemma. ■

**Lemma 63.1.4** Let \( J : U_{0} \to U \) be a Hilbert Schmidt operator and let \( W(t) \) be the resulting Wiener process

\[
W(t) = \sum_{k=1}^{\infty} \psi_{k}(t) Jg_{k}
\]

where \( \{g_{k}\} \) is an orthonormal basis for \( U_{0} \). Let \( f \in H \). Then considering one of the terms of the integral defined above,

\[
E \left( \left( \Phi(t_{k}) (W(t \wedge t_{k+1}) - W(t \wedge t_{k})) , f \right)^{2} \right) = E \left( \left( (W(t \wedge t_{k+1}) - W(t \wedge t_{k})) , \Phi(t_{k})^{*} f \right)^{2} \right)
\]

\[
= (t \wedge t_{k+1} - t \wedge t_{k}) E \left( ||J^{*} \Phi(t_{k})^{*} f ||_{L_{0}}^{2} \right).
\]

**Proof:** For simplicity, write \( \Delta W_{k}(t) \) for \( W(t \wedge t_{k+1}) - W(t \wedge t_{k}) \) and \( \Delta_{k}(t) = (t \wedge t_{k+1}) - (t \wedge t_{k}) \). If \( \Phi(t_{k}) \) were a constant, then the result would follow right away from the fact that \( W(t) \) is a Wiener process. Therefore, suppose for disjoint \( E_{i} \),

\[
\Phi(t_{k})(\omega) = \sum_{i=1}^{m} \Phi_{i} \mathcal{X}_{E_{i}}(\omega)
\]

where \( \Phi_{i} \in \mathcal{L}(U, H) \) and \( E_{i} \in \mathcal{F}_{t_{k}} \). Then, since the \( E_{i} \) are disjoint,

\[
E \left( \left( \Phi(t_{k}) (W(t \wedge t_{k+1}) - W(t \wedge t_{k})) , f \right)^{2} \right)
\]
\[ \sum_{i=1}^{m} \mathbb{E} \left( \left( (\Delta_k W(t), \Phi_i^* f X_i) \right)^2 \right) = \sum_{i=1}^{m} \int_{\Omega} \mathbb{E} \left( \left( (\Delta_k W(t), \Phi_i^* f) \right)^2 \right) dP \]

Thus each \( E_i \) is \( F_{t_k} \)-measurable. By Lemma 62.4.2, and the properties of the Wiener process, this equals

\[ \sum_{i=1}^{m} P(\Omega) \int_{\Omega} \mathbb{E} \left( \left( (\Delta_k W(t), \Phi_i^* f) \right)^2 \right) dP = \sum_{i=1}^{m} P(\Omega) \Delta_k t \left( Q \Phi_i^* f, \Phi_i^* f \right)_U \]

where \( Q = JJ^* \). Then the above reduces to

\[ (t \wedge t_{k+1} - t \wedge t_k) R \left( \left| J \Phi(t_k) f \right|^2 \right) \]

Now here is a major result on the integral of elementary functions. The last assertion in the following proposition is called the Ito isometry.

**Proposition 63.1.5** Let \( \Phi(t) \) be an elementary process as defined in Definition 63.1.2 and let \( W(t) \) be a Wiener process.

\[ W(t) = \sum_{k=1}^{\infty} \psi_k(t) J g_k \]

where \( J : U_0 \to U \) is Hilbert Schmidt and the \( \psi_k \) are real independent Wiener processes as described above.

\[ U_0 \xrightarrow{J} U \xrightarrow{\Phi} H \]

Then \( \int_{a}^{t} \Phi(s) dW(s) \) is a continuous square integrable \( H \)-valued martingale with respect to the \( \sigma \)-algebras of Definition 63.1.3 on \([0, T]\) and

\[ E \left( \left| \int_{a}^{t} \Phi(s) dW(s) \right|^2_H \right) = \int_{a}^{t} E \left( \left| \Phi \circ J \right|^2_{L_2(U_0, H)} \right) ds \]

**Proof:** Start with the left side. Denote by \( \Delta_k W(t) \equiv W(t \wedge t_{k+1}) - W(t \wedge t_k) \).

Then

\[ E \left( \left| \int_{a}^{t} \Phi(s) dW(s) \right|^2_H \right) = E \left( \sum_{k=0}^{n-1} \Phi(t_k) \Delta_k W(t) \right)^2_H \]

Consider a mixed term for \( j < k \). Using Lemma 63.1.3 and the fact that \( W(t) \) is a martingale,

\[ E \left( (\Phi(t_k) \Delta_k W(t), \Phi(t_j) \Delta_j W(t))_H \right) \]

\[ = E \left( E \left( \left( \Phi(t_k) \Delta_k W(t), \Phi(t_j) \Delta_j W(t) \right)_H \right) \mid \mathcal{F}_{t_k} \right) \]

\[ = E \left( \left( \Phi(t_j) \Delta_j W(t), E \left( \Phi(t_k) \Delta_k W(t) \mid \mathcal{F}_{t_k} \right) \right) \right) \]

\[ = E \left( \left( \Phi(t_j) \Delta_j W(t), \Phi(t_k) E \left( \Delta_k W(t) \mid \mathcal{F}_{t_k} \right) \right) \right) \]

\[ = E \left( \left( \Phi(t_j) \Delta_j W(t), \Phi(t_k) 0 \right) \right) = 0. \]
Therefore, from Lemma 63.1.4 and letting \( \{f_j\} \) be an orthonormal basis for \( H \),
\[
E \left( \int_a^t \Phi (s) dW \right)^2_H = \sum_{k=0}^{n-1} E \left( (\Phi (t_k) \Delta_k W(t), \Phi (t_k) \Delta_k W(t)) \right)
\]
\[
= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (\Phi (t_k) \Delta_k W(t), f_j)^2 = \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} E \left( (\Phi (t_k) \Delta_k W(t), f_j)^2 \right)
\]
\[
= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (t \wedge t_{k+1} - t \wedge t_k) E \left( \|J^* \Phi (t_k)^* f_j\|_{U_0}^2 \right)
\]
\[
= \sum_{k=0}^{n-1} (t \wedge t_{k+1} - t \wedge t_k) E \left( \|J^* \Phi (t_k)^*\|^2_{L_2(U,H)} \right)
\]
\[
= \sum_{k=0}^{n-1} (t \wedge t_{k+1} - t \wedge t_k) E \left( \|\Phi (t_k) J\|^2_{L_2(U_0,H)} \right)
\]
\[
= \int_a^t E \left( \|\Phi \circ J\|^2_{L_2(U_0,H)} \right) ds
\]
It is obvious that \( \int_a^t \Phi (s) dW \) is a continuous square integrable martingale from the definition, because it is just a finite sum of such things. \( \blacksquare \)

Of course this is a version of the Ito isometry. The presence of the \( J \) is troublesome but it is hidden in the definition of \( W \) on the left side of the conclusion of the proposition. In finite dimensions one could just let \( J = I \) and this fussy detail would not be there to cause confusion. The next task is to generalize the above integral to a more general class of functions and obtain a process which is not explicitly dependent on \( J \).

### 63.2 Different Definition Of Elementary Functions

What if elementary functions had been defined in terms of \( \mathcal{X}_{[t_k,t_{k+1}]} \)? That is, what if the elementary functions had been of the form
\[
\Phi (t) = \sum_{k=0}^{n-1} \Phi (t_k) \mathcal{X}_{[t_k,t_{k+1}]} (t)
\]
Would anything change? If you go over the arguments given, it is clear that nothing would change at all. Furthermore, this elementary function equals the one described above off a finite set of mesh points so the convergence properties in \( L^2 ([0, T] \times \Omega, L_2 (Q^{1/2} U, H)) \), which will be important in what follows are exactly the same. Thus it does not matter whether we give elementary functions in this form or in the form described above. However, some arguments given later about localization depend on it being in the earlier form.
63.3 Approximating With Elementary Functions

Here is a really surprising result about approximating with step functions which is due to Doob. See [71] which is where I found this lemma. This is based on continuity of translation in the $L^p(\mathbb{R}; E)$.

**Lemma 63.3.1** Let $\Phi : [0, T] \times \Omega \to E$, be $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable and suppose

$$\Phi \in K \equiv L^p([0, T] \times \Omega; E), \quad p \geq 1$$

Then there exists a sequence of nested partitions, $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$,

$$\mathcal{P}_k \equiv \{ t^k_0, \ldots, t^k_{m_k} \}$$

such that the step functions given by

$$\Phi^r_k(t) = \sum_{j=1}^{m_k} \Phi(t^k_j) \chi_{[t^k_{j-1}, t^k_{j})}(t)$$

$$\Phi^l_k(t) = \sum_{j=1}^{m_k} \Phi(t^k_{j-1}) \chi_{[t^k_{j-1}, t^k_{j})}(t)$$

both converge to $\Phi$ in $K$ as $k \to \infty$ and

$$\lim_{k \to \infty} \max \{ |t^k_{j} - t^k_{j+1}| : j \in \{0, \ldots, m_k\} \} = 0.$$  

Also, each $\Phi(t^k_j), \Phi(t^k_{j-1})$ is in $L^p(\Omega; E)$. One can also assume that $\Phi(0) = 0$. The mesh points $\{ t^k_j \}_{j=0}^{m_k}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$|t^k_j - t^k_{j-1}| = 2^{-n_k}$$

except for the case where $j = 1$ or $j = m_{n_k}$ when this is so, you could have $|t^k_j - t^k_{j-1}| < 2^{-n_k}$.

Note that it would make no difference in terms of the conclusion of this lemma if you defined

$$\Phi^l_k(t) = \sum_{j=1}^{m_k} \Phi(t^k_{j-1}) \chi_{[t^k_{j-1}, t^k_{j})}(t)$$

because the modified function equals the one given above off a countable subset of $[0, T]$, the union of the mesh points. One could change $\Phi^l_k$ similarly with no change in the conclusion.

**Proof:** For $t \in \mathbb{R}$ let $\gamma_n(t) = k/2^n, \delta_n(t) = (k + 1)/2^n$, where $t \in (k/2^n, (k + 1)/2^n]$, and $2^{-n} < T/4$. Also suppose $\Phi$ is defined to equal 0 on $[0, T]^C \times \Omega$. There exists
a set of measure zero $N$ such that for $\omega \notin N, t \to \|\Phi(t, \omega)\|$ is in $L^p(\mathbb{R})$. Therefore by continuity of translation, as $n \to \infty$ it follows that for $\omega \notin N$, and $t \in [0, T]$,

$$
\int_{\mathbb{R}} \|\Phi(\gamma_n(t) + s) - \Phi(t + s)\|_E^p \, ds \to 0
$$

The above is dominated by

$$
\int_{\mathbb{R}} 2^{p-1} (\|\Phi(s)\|^p + \|\Phi(s)\|^p) \chi_{[-2T,2T]}(s) \, ds
$$

$$
= \int_{-2T}^{2T} 2^{p-1} (\|\Phi(s)\|^p + \|\Phi(s)\|^p) \, ds < \infty
$$

Consider

$$
\int_{\Omega} \int_{\mathbb{R}} \left( \int_{-2T}^{2T} \|\Phi(\gamma_n(t) + s) - \Phi(t + s)\|_E^p \, ds \right) \, dt \, dP
$$

By the dominated convergence theorem, this converges to 0 as $n \to \infty$. This is because the integrand with respect to $\omega$ is dominated by

$$
\int_{-2T}^{2T} \left( \int_{\mathbb{R}} 2^{p-1} (\|\Phi(s)\|^p + \|\Phi(s)\|^p) \chi_{[-2T,2T]}(s) \, ds \right) \, dt
$$

and this is in $L^1(\Omega)$ by assumption that $\Phi \in K$. Now Fubini. This yields

$$
\int_{\Omega} \int_{\mathbb{R}} \int_{-2T}^{2T} \|\Phi(\gamma_n(t) + s) - \Phi(t + s)\|_E^p \, dt \, ds \, dP
$$

Change the variables on the inside.

$$
\int_{\Omega} \int_{\mathbb{R}} \int_{-2T}^{2T} \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p \, dt \, ds \, dP
$$

Now by definition, $\Phi(t)$ vanishes if $t \notin [0, T]$, thus the above reduces to

$$
\int_{\Omega} \int_{\mathbb{R}} \int_{0}^{T} \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p \, dt \, ds \, dP
$$

$$
+ \int_{\Omega} \int_{\mathbb{R}} \int_{-2T}^{2T} \chi_{[0,T]}(s) \|\Phi(\gamma_n(t - s) + s)\|_E^p \, dt \, ds \, dP
$$

$$
= \int_{\Omega} \int_{\mathbb{R}} \int_{0}^{T} \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p \, dt \, ds \, dP
$$

$$
+ \int_{\Omega} \int_{\mathbb{R}} \int_{-2T}^{2T} \chi_{[0,T]}(s) \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p \, dt \, ds \, dP
$$
Also by definition, \( \gamma_n (t - s) + s \) is within \( 2^{-n} \) of \( t \) and so the integrand in the integral on the right equals 0 unless \( t \in [-2^{-n} - T, T + 2^{-n}] \subseteq [-2T, 2T] \). Thus the above reduces to

\[
\int_\Omega \int_\mathbb{R} \int_{-2T}^{2T} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p \, dt \, ds \, dP.
\]

Now Fubini again.

\[
\int_\mathbb{R} \int_\Omega \int_{-2T}^{2T} \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p \, dt \, ds \, dP
\]

This converges to 0 as \( n \to \infty \) as was shown above. Therefore,

\[
\int_0^T \int_\Omega \int_0^T \| \Phi (\gamma_n (t - s) + s) - \Phi (t) \|_E^p \, dt \, ds \, dP
\]

also converges to 0 as \( n \to \infty \). The only problem is that \( \gamma_n (t - s) + s \geq t - 2^{-n} \) and so \( \gamma_n (t - s) + s \) could be less than 0 for \( t \in [0, 2^{-n}] \). Since this is an interval whose measure converges to 0 it follows

\[
\int_0^T \int_\Omega \int_0^T \| \Phi (\gamma_n (t - s) + s)^+ - \Phi (t) \|_E^p \, dt \, ds \, dP
\]

converges to 0 as \( n \to \infty \). Let

\[
m_n (s) = \int_\Omega \int_0^T \| \Phi (\gamma_n (t - s) + s)^+ - \Phi (t) \|_E^p \, dt \, ds
\]

Then letting \( \mu \) denote Lebesgue measure,

\[
\mu (\{m_n (s) > \lambda \}) \leq \frac{1}{\lambda} \int_0^T m_n (s) \, ds.
\]

It follows there exists a subsequence \( n_k \) such that

\[
\mu \left( \left\{ m_{n_k} (s) > \frac{1}{k} \right\} \right) < 2^{-k}
\]

Hence by the Borel Cantelli lemma, there exists a set of measure zero \( N \) such that for \( s \notin N \),

\[
m_{n_k} (s) \leq 1/k
\]

for all \( k \) sufficiently large. Pick such an \( s \). Then consider \( t \to \Phi (\gamma_{n_k} (t - s) + s)^+ \).

For \( n_k, t \to (\gamma_{n_k} (t - s) + s)^+ \) has jumps at points of the form \( 0, s + l2^{-n_k} \) where \( l \) is an integer. Thus \( \mathcal{P}_{n_k} \) consists of points of \( [0, T] \) which are of this form and these partitions are nested. Define \( \Phi_k^l (0) = 0, \Phi_k^l (t) = \Phi (\gamma_{n_k} (t - s) + s)^+ \)\). Now suppose \( N_1 \) is a set of measure zero. Can \( s \) be chosen such that all jumps for all
partitions occur off $N_1$? Let $(a, b)$ be an interval contained in $[0, T]$. Let $S_j$ be the points of $(a, b)$ which are translations of the measure zero set $N_1$ by $t^j_i$ for some $j$. Thus $S_j$ has measure 0. Now pick $s \in (a, b) \setminus \cup_j S_j$.

It will be assumed that all these mesh points miss the set of all $t$ such that $\omega \to \Phi(t, \omega)$ is not in $L^p(\Omega; E)$. To get the other sequence of step functions, the right step functions, just use a similar argument with $\gamma_n$ in place of $\delta_n$. Just apply the argument to a subsequence of $n_k$ so that the same $s$ can hold for both. ■

The following proposition says that elementary functions can be used to approximate progressively measurable functions under certain conditions.

**Proposition 63.3.2** Let $\Phi \in L^p([0, T] \times \Omega, E)$, $p \geq 1$, be progressively measurable. Then there exists a sequence of elementary functions which converges to $\Phi$ in $L^p([0, T] \times \Omega, E)$.

These elementary functions have values in $E_0$, a dense subset of $E$. If $\varepsilon_n \to 0$, and

$$\Phi_n(t) = \sum_{k=1}^{m_n} \Psi^*_k \chi_{(t_k, t_{k+1})}(t)$$

$\Psi^*_k$ having values in $E_0$, it can be assumed that

$$\sum_{k=1}^{m_n} ||\Psi^*_k - \Phi(t_k)||_{L^p(\Omega; E)} < \varepsilon_n. \quad (63.3.4)$$

**Proof:** By Lemma 63.3.1 there exists a sequence of step functions

$$\Phi^*_k(t) = \sum_{j=1}^{m_k} \Phi(t_{j-1}^k) \chi_{(t_{j-1}^k, t_j^k)}(t)$$

which converges to $\Phi$ in $L^p([0, T] \times \Omega, E)$ where at the left endpoint $\Phi(0)$ can be modified as described above. Now each $\Phi(t_{j-1}^k)$ is in $L^p(\Omega, E)$ and is $\mathcal{F}(t_{j-1}^k)$ measurable and so it can be approximated as closely as desired in $L^p(\Omega)$ with a simple function

$$s(t_{j-1}^k) \equiv \sum_{i=1}^{m_k} c_i^j \chi_{F_i}(\omega), \quad F_i \in \mathcal{F}(t_{j-1}^k).$$

Furthermore, by density of $E_0$ in $E$, it can be assumed each $c_i^j \in E_0$ and the condition 63.3.4 holds. Replacing each $\Phi(t_{j-1}^k)$ with $s(t_{j-1}^k)$, the result is an elementary function which approximates $\Phi^*_k$. ■

Of course everything in the above holds with obvious modifications replacing $[0, T]$ with $[a, T]$ where $a < T$.

Here is another interesting proposition about the time integral being adapted.
Proposition 63.3.3 Suppose \( f \geq 0 \) is progressively measurable and \( \mathcal{F}_t \) is a filtration. Then
\[
\omega \rightarrow \int_a^t f(s, \omega) \, ds
\]
is \( \mathcal{F}_t \) adapted.

Proof: This follows right away from the fact \( f \) is \( \mathcal{B}([a, t]) \times \mathcal{F}_t \) measurable. This is just product measure and so the integral from \( a \) to \( t \) is \( \mathcal{F}_t \) measurable. See also Proposition 60.3.5.

63.4 Some Hilbert Space Theory

Recall the following definition which makes \( LU \) into a Hilbert space where \( L \in \mathcal{L}(U, H) \).

Definition 63.4.1 Let \( L \in \mathcal{L}(U, H) \), the bounded linear maps from \( U \) to \( H \) for \( U, H \) Hilbert spaces. For \( y \in L(U) \), let \( L^{-1}y \) denote the unique vector in
\[
\{ x : Lx = y \} \equiv M_{y}
\]
which is closest in \( U \) to \( 0 \).

Note this is a good definition because \( \{ x : Lx = y \} \) is closed thanks to the continuity of \( L \) and it is obviously convex. Thus Theorem 17.1.8 applies. With this definition define an inner product on \( L(U) \) as follows. For \( y, z \in L(U) \),
\[
(y, z)_{L(U)} \equiv (L^{-1}y, L^{-1}z)_{U}
\]
Thus it is obvious that \( L^{-1} : LU \rightarrow U \) is continuous. The notation is abominable because \( L^{-1}(y) \) is the normal notation for \( M_{y} \).

With this definition, here is one of the main results. It is Theorem 17.2.3 proved earlier.

Theorem 63.4.2 Let \( U, H \) be Hilbert spaces and let \( L \in \mathcal{L}(U, H) \). Then Definition 63.4.1 makes \( L(U) \) into a Hilbert space. Also \( L : U \rightarrow L(U) \) is continuous and \( L^{-1} : L(U) \rightarrow U \) is continuous. Furthermore there is a constant \( C \) independent of \( x \in U \) such that
\[
\|L\|_{\mathcal{L}(U,H)} \|Lx\|_{L(U)} \geq \|Lx\|_H \quad (63.4.5)
\]
If $U$ is separable, so is $L(U)$. Also $(L^{-1}(y), x) = 0$ for all $x \in \text{ker}(L)$, and $L^{-1} : L(U) \to U$ is linear. Also, in case that $L$ is one to one, both $L$ and $L^{-1}$ preserve norms.

Let $U$ be a separable Hilbert space and let $Q$ be a positive self adjoint operator. Then consider

$$J : Q^{1/2}U \to U_1,$$

a one to one Hilbert Schmidt operator, where $U_1$ is a separable real Hilbert space. First of all, there is the obvious question whether there are any examples.

**Lemma 63.4.3** Let $A \in \mathcal{L}(U, U)$ be a bounded linear transformation defined on $U$ a separable real Hilbert space. There exists a one to one Hilbert Schmidt operator $J : AU \to U_1$ where $U_1$ is a separable real Hilbert space. In fact you can take $U_1 = U$.

**Proof:** Let $\alpha_k > 0$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Then let $\{g_k\}_{k=1}^{L}$ be an orthonormal basis for $AU$, the inner product and norm given in Definition 63.4.1 above, and let

$$Jx \equiv \sum_{k=1}^{L} (x, g_k)_{AU} \alpha_k g_k.$$

Then it is clear that $J \in \mathcal{L}(AU, U)$. This is because,

$$||Jx||_U \leq \sum_{k=1}^{L} |(x, g_k)_{AU}| \alpha_k ||g_k||_U$$

$$\leq C \sum_{k=1}^{L} |(x, g_k)_{AU}| \alpha_k \|g_k\|_{AU}$$

$$\leq C \left( \sum_{k=1}^{L} |(x, g_k)_{AU}|^2 \right)^{1/2} \left( \sum_{k=1}^{L} \alpha_k^2 \right)^{1/2}$$

$$= C \left( \sum_{k=1}^{L} \alpha_k^2 \right)^{1/2} ||x||_{AU}$$

Also, from the definition, $Jg_j = \alpha_j g_j$. Say $g_j = Af_j$ where $f_j \in U$ and $1 = ||g_j||_{AU} = ||f_j||_U$. Since $A$ is continuous,

$$||g_j||_U = ||Af_j||_U \leq ||A|| ||f_j||_U = ||A|| ||g_j||_{AU} = ||A|| \equiv C^{1/2}$$

Thus

$$\sum_{j=1}^{L} ||Jg_j||_U^2 = \sum_{j=1}^{L} \alpha_j^2 ||g_j||_U^2 \leq C \sum_{j=1}^{L} \alpha_j^2 < \infty$$

and so $J$ is also a Hilbert Schmidt operator which maps $AU$ to $U$. It is clear that $J$ is one to one because each $\alpha_k > 0$. If $AU$ is finite dimensional, $L < \infty$ and so the above sum is finite. \qed
Definition 63.4.4  Let $U_1, U, H$ be real separable Hilbert spaces and let $Q$ be a nonnegative self adjoint operator, $Q \in \mathcal{L}(U, U)$. Let $Q^{1/2} U$ be the Hilbert space described in Definition 63.4.1. Let $J$ be a one to one Hilbert Schmidt map from $Q^{1/2} U$ to $U_1$.

$$U_1 \overset{J}{\rightarrow} Q^{1/2} U \overset{\Phi}{\rightarrow} H$$

Then denote by $L_0(U_1, H)$ the space of restrictions of elements of $L(U_1, H)$ to the Hilbert space $JQ^{1/2} U \subseteq U_1$.

Here is a diagram to keep this straight.

$$
\begin{array}{ccc}
U & \downarrow & Q^{1/2} \\
U_1 & \overset{J}{\rightarrow} & Q^{1/2} U \\
\Phi \downarrow & & \Phi \\
H & & \\
\end{array}
$$

Lemma 63.4.5  In the context of the above definition, $L(U_1, H)_0$ is dense in $L_2(JQ^{1/2} U, H)$, the Hilbert Schmidt operators from $JQ^{1/2} U$ to $H$. That is, if $f \in L_2(JQ^{1/2} U, H)$, there exists $g \in L(U_1, H)_0$, $\|g - f\|_{L_2(JQ^{1/2} U, H)} < \varepsilon$.

Proof:  The operator $JJ^* = Q_1 : U_1 \rightarrow U_1$ is self adjoint and nonnegative. It is also compact because $J$ is Hilbert Schmidt. Therefore, by Theorem 19.3.9 on Page 649,

$$Q_1 = \sum_{k=1}^{L} \lambda_k e_k \otimes e_k$$

where the $\lambda_k$ are decreasing and positive, the $\{e_k\}$ are an orthonormal basis for $U_1$, and $\lambda_L$ is the last positive $\lambda_j$. (This is a lot like the singular value matrix in linear algebra.) Thus also

$$Q_1 e_k = \lambda_k e_k$$

If the $\lambda_k$ are all positive, then $L \equiv \infty$. Then for $k \leq L$ if $L < \infty$, $k < \infty$ otherwise,

$$
\left( \frac{J^* e_k}{\sqrt{\lambda_k}}, \frac{J^* e_j}{\sqrt{\lambda_j}} \right)_{Q^{1/2}(U)} = \left( \frac{J J^* e_k}{\sqrt{\lambda_k}}, \frac{e_j}{\sqrt{\lambda_j}} \right)_{U_1} = \left( \frac{\lambda_k e_k}{\sqrt{\lambda_k}}, \frac{e_j}{\sqrt{\lambda_j}} \right)_{U_1} = \frac{\sqrt{\lambda_k}}{\sqrt{\lambda_j}} \delta_{kj} = \delta_{kj}
$$

Now in case $L < \infty$, $J(Q^{1/2}(U)) \subseteq \text{span}(e_1, \ldots, e_L)$. Here is why. First note that $Q_1$ is one to one on $\text{span}(e_1, \ldots, e_L)$ and maps this space onto itself because $Q_1$ maps $e_k$ to a nonzero multiple of $e_k$. Hence its restriction to this subspace has
an inverse which does the same. It also maps all of $U_1$ to span $(e_1, \cdots, e_L)$. This follows from the definition of $Q_1$ given in the above sum. For $x \in Q^{1/2}(U)$, $Jx \in U_1$ and so

$$J^*Jx = Q_1(Jx) \in \text{span}(e_1, \cdots, e_L)$$

Hence $Jx \in Q^{-1}_1(\text{span}(e_1, \cdots, e_L)) \in \text{span}(e_1, \cdots, e_L)$. Recall that $J$ is one to one so there is only one element of $J^{-1}x$.

Then for $x \in Q^{1/2}U$,

$$\sum_{j=1}^{L} \sqrt{\lambda_j} e_j \otimes_{Q^{1/2}U} \frac{J^*e_j}{\sqrt{\lambda_j}}(x) = \sum_{j=1}^{L} \sqrt{\lambda_j} e_j \left( \frac{J^*e_j}{\sqrt{\lambda_j}}, x \right)_{Q^{1/2}(U)}$$

$$= \sum_{j=1}^{L} \sqrt{\lambda_j} e_j \left( \frac{e_j}{\sqrt{\lambda_j}}, Jx \right)_{U_1} = \sum_{j=1}^{L} e_j(Jx)_{U_1}$$

$$= \sum_{j=1}^{\infty} e_j(Jx)_{U_1} = Jx. \quad (J \left(Q^{1/2}(U) \right) \subseteq \text{span}(e_1, \cdots, e_L) \text{ if } L < \infty$$

Thus,

$$J = \sum_{j=1}^{L} \sqrt{\lambda_j} e_j \otimes_{Q^{1/2}U} \frac{J^*e_j}{\sqrt{\lambda_j}}$$

It follows that an orthonormal basis in $JQ^{1/2}U$ is $\left\{ \frac{J^*e_j}{\sqrt{\lambda_j}} \right\}_{j=1}^{L}$. This is because an orthonormal basis for $Q^{1/2}U$ is $\left\{ \frac{e_k}{\sqrt{\lambda_k}} \right\}_{k=1}^{L}$. Since $J$ is one to one, it preserves norms between $Q^{1/2}U$ and $JQ^{1/2}U$. Let $\Phi \in L_2(JQ^{1/2}U, H)$. Then by the discussion of Hilbert Schmidt operators given earlier, in particular the demonstration that these operators are compact,

$$\Phi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \phi_{ij} f_i \otimes_{Q^{1/2}U} \frac{J^*e_j}{\sqrt{\lambda_j}}$$

where $\{f_i\}$ is an orthonormal basis for $H$. In fact, $\left\{ f_i \otimes \frac{J^*e_j}{\sqrt{\lambda_j}} \right\}_{i,j}$ is an orthonormal basis for $L_2(JQ^{1/2}U, H)$ and $\sum_i \sum_j \phi_{ij}^2 < \infty$, the $\phi_{ij}$ being the Fourier coefficients of $\Phi$. Then consider

$$\Phi_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \phi_{ij} f_i \otimes_{Q^{1/2}U} \frac{J^*e_j}{\sqrt{\lambda_j}} \quad (63.4.6)$$

Consider one of the finitely many operators in this sum. For $x \in Q^{1/2}U$, since $J$ preserves norms,

$$f_i \otimes_{Q^{1/2}U} \frac{J^*e_j}{\sqrt{\lambda_j}}(x) = f_i \left( \frac{J^*e_j}{\sqrt{\lambda_j}}, x \right)_{Q^{1/2}U} = f_i \left( J^*e_j, J^{-1}x \right)_{Q^{1/2}U}$$
= f_i \left( \frac{e_j}{\sqrt{\lambda_j}}, JJ^{-1}x \right)_{U_1} = f_i \left( \frac{e_j}{\sqrt{\lambda_j}}, x \right)_{U_1} \equiv \Lambda_{ij}(x)

Recall how, since $J$ is one to one, it preserves norms and inner products. Now $\Lambda_{ij}$ makes sense from the above formula for all $x \in U_1$ and is also a continuous linear map from $U_1$ to $H$ because

$$\left\| f_i \left( \frac{e_j}{\sqrt{\lambda_j}}, x \right)_{U_1} \right\|_H \leq \|f_i\|_H \frac{1}{\sqrt{\lambda_j}} \|x\|_{U_1}$$

Thus each term in the finite sum of $63.4.6$ is in $\mathcal{L}(U_1, H)_{0}$ and this proves the lemma.

It is interesting to note that $Q_1^{1/2} U_1 = J \left( Q_1^{1/2} (U) \right)$.

$$\sum_{j=1}^{L} \sqrt{\lambda_j} e_j \left( J^* e_j, x \right)_{Q_1^{1/2}(U)} = Jx$$

and $\left\{ J^* e_j, x \right\}_{Q_1^{1/2}(U)}$ are an orthonormal set in $Q_1^{1/2} (U)$. Therefore, the sum of the squares of $\left( J^* e_j, x \right)_{Q_1^{1/2}(U)}$ is finite. Hence you can define $y \in U_1$ by

$$y \equiv \sum_{j=1}^{L} \left( J^* e_j, x \right)_{Q_1^{1/2}(U)} e_j$$

Also

$$\sum_{i=1}^{L} \sqrt{\lambda_i} e_i \otimes e_i (y) = \sum_{i=1}^{L} \sqrt{\lambda_i} e_i \left( J^* e_i, x \right)_{Q_1^{1/2}(U)} = \sum_{i=1}^{L} e_i (e_i, Jx)_{U_1} = Jx$$

Now you can show that $Q_1^{1/2} = \sum_{i=1}^{L} \sqrt{\lambda_i} e_i \otimes e_i$. You do this by showing that it works and commutes with every operator which commutes with $Q_1$. Thus $Jx = Q_1^{1/2} y$. This shows that $J \left( Q_1^{1/2} (U) \right) \subseteq Q_1^{1/2} (U_1)$. However, you can also turn the inclusion around. Thus if you start with $y \in U_1$ and form

$$Q_1^{1/2} y = \sum_{i=1}^{L} \sqrt{\lambda_i} e_i \otimes e_i (y) = \sum_{i=1}^{L} \sqrt{\lambda_i} e_i (y, e_i),$$

then the $(y, e_i)^2_{U_1}$ has a finite sum because the $\{e_i\}$ are orthonormal. Thus you can form

$$x \equiv \sum_{i=1}^{L} (y, e_i)_{U_1} \frac{J^* e_i}{\sqrt{\lambda_i}} \in Q_1^{1/2} (U)$$
Then since the \( \left\{ \frac{J^* e_j}{\sqrt{\lambda_j}} \right\} \) are orthonormal,

\[
J(x) = \sum_{j=1}^{L} \sqrt{\lambda_j} e_j \left( \frac{J^* e_j}{\sqrt{\lambda_j}}, x \right)_{Q^{1/2}(U)} = \sum_{j=1}^{L} \sqrt{\lambda_j} e_j (y, e_j)_{U_1} = \sum_{j=1}^{L} \sqrt{\lambda_j} e_j \otimes e_j (y) = Q_1^{1/2} (y)
\]

It follows that \( Q_1^{1/2} (U_1) \subseteq J (Q^{1/2} (U)) \).

One can also show that \( W(t) \equiv \sum_{k=1}^{L} \psi_k (t) J g_k \) where the \( \psi_k (t) \) are the real Wiener processes described earlier and \( \{ g_k \} \) is an orthonormal basis for \( Q^{1/2} (U) \), is a \( Q_1 \) Wiener process. To see this, recall the above definition of a Wiener process in terms of Hilbert Schmidt operators, the convergence happening in \( U_1 \) in this case. Then by independence of the \( \psi_j \),

\[
E \left( \left( h, \sum_{k=1}^{L} \psi_k (t-s) J g_k \right) \left( l, \sum_{j=1}^{L} \psi_j (t-s) J g_j \right) \right) = E \left( \sum_{k} (h, J g_k) (l, J g_j) \psi_k (t-s) \psi_j (t-s) \right) = \sum_{k} (h, J g_k) (l, J g_k) E \left( \psi_k^2 (t-s) \right) = (t-s) \sum_{k} (h, J g_k) (l, J g_k) = (t-s) \sum_{k} (J^* h, g_k)_{Q^{1/2}(U)} (J^* l, g_k)_{Q^{1/2}(U)} = (t-s) (J^* h, J^* l)_{Q^{1/2}(U)} = (t-s) (J J^* h, l)_{U_1} \equiv (t-s) (Q_1 h, l)_{U_1}
\]

63.5 The General Integral

It is time to generalize the integral. The following diagram illustrates the ingredients of the next lemma.

\[
W(t) \in U_1 \xleftarrow{J} Q^{1/2} U \xrightarrow{\Phi} H
\]

\[
U_1 \supseteq J Q^{1/2} U \xrightarrow{\Phi_{1-1}} Q^{1/2} U \downarrow \Phi H
\]
Lemma 63.5.1 Let $\Phi \in L^2([a,T] \times \Omega; \mathcal{L}_2(Q^{1/2}U, H))$ and suppose also that $\Phi$ is progressively measurable with respect to the usual filtration associated with the Wiener process

$$W(t) = \sum_{k=1}^{L} \psi_k(t) Jg_k$$

which has values in $U_1$ for $U_1$ a separable real Hilbert space such that $J : Q^{1/2}U \to U_1$ is Hilbert Schmidt and one to one, $\{g_k\}$ an orthonormal basis in $Q^{1/2}U$. Then letting $J^{-1} : JQ^{1/2}U \to Q^{1/2}U$ be the map described in Definition 63.4.1, it follows that

$$\Phi \circ J^{-1} \in L^2([a,T] \times \Omega; \mathcal{L}_2(JQ^{1/2}U, H)).$$

Also there exists a sequence of elementary functions $\{\Phi_n\}$ having values in $L(U_1, H)$ which converges to $\Phi \circ J^{-1}$ in $L^2([a,T] \times \Omega; \mathcal{L}_2(JQ^{1/2}U, H)).$

Proof: First, why is $\Phi \circ J^{-1} \in L^2([a,T] \times \Omega; \mathcal{L}_2(JQ^{1/2}U, H))$? This follows from the observation that $A$ is Hilbert Schmidt if and only if $A^*$ is Hilbert Schmidt. In fact, the Hilbert Schmidt norms of $A$ and $A^*$ are the same. Now since $\Phi$ is Hilbert Schmidt, it follows that $\Phi^*$ is and since $J^{-1}$ is continuous, it follows $(J^{-1})^* \Phi^* = (\Phi \circ J^{-1})^*$ is Hilbert Schmidt. Also letting $\mathcal{L}_2$ be the appropriate space of Hilbert Schmidt operators,

$$\left\| (J^{-1})^* \|\Phi\|_{\mathcal{L}_2} = \left\| (J^{-1})^* \|\Phi^*\|_{\mathcal{L}_2} \right\| \geq \left\| (\Phi \circ J^{-1})^* \right\|_{\mathcal{L}_2} = \left\| \Phi \circ J^{-1} \right\|_{\mathcal{L}_2}$$

Thus $\Phi \circ J^{-1}$ has values in $\mathcal{L}_2(JQ^{1/2}U, H)$. This also shows that

$$\Phi \circ J^{-1} \in L^2([a,T] \times \Omega; \mathcal{L}_2(JQ^{1/2}U, H)).$$

Since $\Phi$ is given to be progressively measurable, so is $\Phi \circ J^{-1}$. Therefore, the existence of the desired sequence of elementary functions follows from Proposition and Lemma.

Definition 63.5.2 Let $\Phi \in L^2([a,T] \times \Omega; \mathcal{L}_2(Q^{1/2}U, H))$ and be progressively measurable where $Q$ is a self adjoint nonnegative operator defined on $U$. Let $J : Q^{1/2}U \to U_1$ be Hilbert Schmidt. Then the stochastic integral

$$\int_{a}^{t} \Phi dW$$

is defined as

$$\lim_{n \to \infty} \int_{a}^{t} \Phi_n dW$$

in $L^2(\Omega; H)$

where $W(t)$ is a Wiener process

$$\sum_{k=1}^{\infty} \psi_k(t) Jg_k, \{g_k\} \text{ orthonormal basis in } Q^{1/2}U.$$
and \( \Phi_n \) is an elementary function which has values in \( L(U_1, H) \) and converges to \( \Phi \circ J^{-1} \) in
\[
L^2 \left( [a, T] \times \Omega; \mathcal{L}_2 \left( JQ^{1/2}U, H \right) \right),
\]
such a sequence exists by Lemma 63.4.5 and Proposition 63.3.2.

\[
U_1 \supseteq JQ^{1/2}U \quad \frac{1}{1-1} \quad Q^{1/2}U
\]

\[
\Phi_n \downarrow \Phi
\]

It is necessary to show that this is well defined and does not depend on the choice of \( U_1 \) and \( J \).

**Theorem 63.5.3** The stochastic integral \( \mathbb{E} \left( \int_a T \Phi (s) dW_H \right) \) is well defined. It also is a continuous martingale and does not depend on the choice of \( J \) and \( U_1 \). Furthermore,
\[
\mathbb{E} \left( \int_a T \Phi (s) dW_H \right) = \int_a T \mathbb{E} \left( ||\Phi||^2_{L^2(JQ^{1/2}U, H)} \right) ds
\]

**Proof:** First of all, it is obvious that it is well defined in the sense that the same stochastic process is obtained from two different sequences of elementary functions. This follows from the isometry of Proposition 63.5.4 with \( U_1 \) in place of \( U \) and \( Q^{1/2}U \) in place of \( U_0 \). Thus if \( \{\Psi_n\} \) and \( \{\Phi_n\} \) are two sequences of elementary functions converging to \( \Phi \circ J^{-1} \) in \( L^2 \left( [a, T] \times \Omega; \mathcal{L}_2 \left( JQ^{1/2}U, H \right) \right) \),
\[
\mathbb{E} \left( \int_a T (\Phi_n(s) - \Psi_n(s)) dW_H \right)^2 = \int_a T \mathbb{E} \left( ||\Phi_n - \Psi_n||^2_{L^2(JQ^{1/2}U, H)} \right) ds
\]
(63.5.8)

Now for \( \Phi \in \mathcal{L}_2(U_1, H) \) and \( \{g_k\} \) an orthonormal basis for \( Q^{1/2}U \),
\[
||\Phi \circ J||^2_{L^2(JQ^{1/2}U, H)} = \sum_{k=1}^{\infty} ||\Phi (J(g_k))||^2_H = ||\Phi||^2_{L^2(JQ^{1/2}U, H)}
\]

because, by definition, \( \{Jg_k\} \) is an orthonormal basis in \( JQ^{1/2}U \). Hence (63.5.8) reduces to
\[
\int_a T \mathbb{E} \left( ||\Phi_n - \Psi_n||^2_{L^2(JQ^{1/2}U, H)} \right) ds
\]
which is given to converge to 0. This reasoning also shows that the sequence \( \left\{ \int_a T \Phi_n dW \right\} \) is indeed a Cauchy sequence in \( L^2 (\Omega, H) \).
CHAPTER 63. STOCHASTIC INTEGRATION

Why is \( \int_a^t \phi dW \) a continuous martingale? The integrals \( \int_a^t \Phi_n dW \) are martingales and so, by the maximal estimate of Theorem 60.5.3,

\[
P \left( \sup_{t \in [a,T]} \left| \int_a^t \Phi_n dW - \int_a^t \Phi_m dW \right|_H \geq \lambda \right) \leq \frac{1}{\lambda^2} \int_a^T E \left( \int_a^T (\Phi_n - \Phi_m)^2 dW \right) \]

which is given to converge to 0 as \( m, n \to \infty \). Therefore, there exists a subsequence \( \{n_k\} \) such that

\[
P \left( \sup_{t \in [a,T]} \left| \int_a^t \Phi_{n_k} dW - \int_a^t \Phi_{n_k+1} dW \right|_H \geq 2^{-k} \right) \leq 2^{-k}.
\]

Consequently, by the Borel Cantelli lemma, there is a set of measure zero \( N \) such that if \( \omega \notin N \), then the convergence of \( \int_a^t \Phi_{n_k} dW \) to \( \int_a^t \Phi dW \) is uniform on \( [a,T] \).

Hence \( t \to \int_a^t \phi dW \) is continuous as claimed.

Why is it a martingale? Let \( s < t \) and \( A \in \mathcal{F}_s \).

\[
\int_A \left( \int_a^t \phi dW \right) dP = \lim_{n \to \infty} \int_A \left( \int_a^t \Phi_n dW \right) dP = \lim_{n \to \infty} \int_A E \left( \left( \int_a^t \Phi_n dW \right) | \mathcal{F}_s \right) dP
\]

\[
= \lim_{n \to \infty} \int_A \left( \int_a^s \Phi_n dW \right) dP = \int_A \left( \int_a^s \phi dW \right) dP
\]

Hence this is a martingale as claimed.

It remains to verify that the stochastic process does not depend on \( J \) and \( U_1 \).

Let the approximating sequence of elementary functions be

\[
\Phi_n(t) = \sum_{j=0}^{m_n} f^n_j \chi_{[t^n_j, t^n_{j+1})} (t)
\]

where \( f_j^n \) is \( \mathcal{F}_{t^n_j} \) measurable and has finitely many values in \( \mathcal{L}(U_1, H)_0 \), the restrictions of things in \( \mathcal{L}(U_1, H) \) to \( JQ^{1/2} U \). These are the elementary functions which converge to \( \Phi \circ J^{-1} \).

Also let the partitions be such that

\[
\Phi_n \circ J^{-1} = \sum_{j=0}^{m_n} \Phi \left( t^n_j \right) \circ J^{-1} \chi_{[t^n_j, t^n_{j+1})}
\]

which converges to \( \Phi \circ J^{-1} \) in \( L^2 ([a,T] \times \Omega; \mathcal{L}_2 (JQ^{1/2} U, H)) \). Then by definition,

\[
\int_a^t \Phi_n dW = \sum_{j=0}^{m_n} f_j^n \left( W (t \wedge t^n_{j+1}) - W (t \wedge t^n_j) \right)
\]
where \( \{g_k\} \) is an orthonormal basis for \( Q^{1/2} U \). The infinite sum converges in \( L^2 (\Omega; U) \) and \( f_j^n \) is continuous on \( U_1 \). Therefore, \( f_j^n \) can go inside the infinite sum, and this last expression equals

\[
= \sum_{j=0}^{m_n} \sum_{k=1}^{\infty} (\psi_k (t \wedge t_{j+1}^n) - \psi_k (t \wedge t_j^n)) J g_k
\]

the infinite sum converging in \( L^2 (\Omega, H) \).

Now consider the left sum \( 63.5.10 \). Since \( \Phi (t_j^n) \in L^2 (Q^{1/2} U, H) \), it follows that the sum

\[
\sum_{k=1}^{\infty} (\psi_k (t \wedge t_{j+1}^n) - \psi_k (t \wedge t_j^n)) \Phi (t_j^n) g_k
= \sum_{k=1}^{\infty} (\psi_k (t \wedge t_{j+1}^n) - \psi_k (t \wedge t_j^n)) \Phi (t_j^n) \circ J^{-1} (J g_k)
\]

must converge in \( L^2 (\Omega, H) \). Let’s review why this is.

**Diversion** The reason the series converges goes as follows. Estimate

\[
E \left( \left| \sum_{k=p}^{q} (\psi_k (t \wedge t_{j+1}^n) - \psi_k (t \wedge t_j^n)) \Phi (t_j^n) g_k \right|^2 \right)
\]

First consider the mixed terms. Let \( \Delta \psi_k = \psi_k (t \wedge t_{j+1}^n) - \psi_k (t \wedge t_j^n) \). For \( l < k \),

\[
E (\Delta \psi_k \Phi (t_j^n) g_k, \Delta \psi_l \Phi (t_j^n) g_l)
= E (\Delta \psi_k \Delta \psi_l (\Phi (t_j^n) g_k), (\Phi (t_j^n) g_l))
\]

Now by independence, this equals

\[
E (\Delta \psi_k) E (\Delta \psi_l) E ((\Phi (t_j^n) g_k, \Phi (t_j^n) g_l)) = 0
\]

Thus you only need to consider the non mixed terms, and the thing you want to estimate is of the form

\[
\sum_{k=p}^{q} E \left( \left| (\psi_k (t \wedge t_{j+1}^n) - \psi_k (t \wedge t_j^n)) \Phi (t_j^n) g_k \right|^2 \right)
\]
CHAPTER 63. STOCHASTIC INTEGRATION

Now by independence again, this equals
\[
\sum_{k=p}^{q} E \left( (\Delta \psi_k \Phi (t^n_j) g_k, \Delta \psi_k \Phi (t^n_j) g_k) \right)
\]
\[
= \sum_{k=p}^{q} E \left( \Delta \psi_k^2 \left( \Phi (t^n_j) g_k, \Phi (t^n_j) g_k \right) \right)
\]
\[
= \sum_{k=p}^{q} E \left( \Delta \psi_k^2 \right) E \left( \Phi (t^n_j) g_k, \Phi (t^n_j) g_k \right)
\]
\[
= \left( (t \wedge t^n_{j+1}) - (t \wedge t^n_j) \right) \sum_{k=p}^{q} E \left( |\Phi (t^n_j) g_k|^2_H \right)
\]

and this sum is just a part of the convergent infinite sum for
\[
\int_{\Omega} \| \Phi (t^n_j) \|^2_{L^2(Q^{1/2} U, H)} dP < \infty
\]

Therefore, this converges to 0 as \( p, q \to \infty \) and so the sum converges in \( L^2 (\Omega, H) \) as claimed.

End of diversion

The \( J \) and the \( J^{-1} \) cancel in \( \mathbb{E} \sum \) because \( J \) is one to one. It follows that
\[
\sum_{j=0}^{m_n} \sum_{k=1}^{\infty} (\psi_k (t \wedge t^n_{j+1}) - \psi_k (t \wedge t^n_j)) \Phi (t^n_j) g_k +
\]
\[
\sum_{j=0}^{m_n} \sum_{k=1}^{\infty} (\psi_k (t \wedge t^n_{j+1}) - \psi_k (t \wedge t^n_j)) ((f^n_j - \Phi (t^n_j) \circ J^{-1}) (J g_k))
\]

The first expression does not depend on \( J \) or \( U_1 \). I need only argue that the second expression converges to 0 as \( n \to \infty \). The infinite sum converges in \( L^2 (\Omega; H) \) and also, as in the above diversion, the independence of the \( \psi_k \) implies that
\[
\mathbb{E} \left( \sum_{j=0}^{m_n} \sum_{k=1}^{\infty} (\psi_k (t \wedge t^n_{j+1}) - \psi_k (t \wedge t^n_j)) ((f^n_j - \Phi (t^n_j) \circ J^{-1}) (J g_k)) \right)^2_H
\]
\[
= \sum_{j=0}^{m_n} \sum_{k=1}^{\infty} E \left( (f^n_j - \Phi (t^n_j) \circ J^{-1}) (J g_k) \right)^2_H
\]
\[
= \sum_{j=0}^{m_n} (t \wedge t^n_{j+1} - t \wedge t^n_j) \sum_{k=1}^{\infty} E \left( (f^n_j - \Phi (t^n_j) \circ J^{-1}) (J g_k) \right)^2_H
\]
\[
= \sum_{j=0}^{m_n} (t \wedge t^n_{j+1} - t \wedge t^n_j) E \left( ||f^n_j - \Phi (t^n_j) \circ J^{-1}||^2_{L^2(Q^{1/2} U, H)} \right)
\]
\[
= \int_{a}^{t} E \left( ||\Phi_n - \Phi^n \circ J^{-1}||^2_{L^2(Q^{1/2} U, H)} \right) ds
\]
which is given to converge to 0 since both converge to $\Phi \circ J^{-1}$. Consequently, the stochastic integral defined above does not depend on $J$ or $U_1$.

It is interesting to note that in the above definition, the approximate problems do appear to depend on $J$ and $U_1$ but the limiting stochastic process does not. Since it is the case that the stochastic integral is independent of $U_1$ and $J$, it can only be dependent on $Q^{1/2}U$ and $U$, and so we refer to $W(t)$ as a cylindrical process on $U$. By Lemma 63.4.3, you can take $U_1 = U$ and so you can consider the finite sums defining the Wiener process to be in $U$ itself. From the proof of this lemma, you can even have $J$ being the identity on the span of the first $n$ vectors in the orthonormal basis for $Q^{1/2}U$. The case where $Q$ is trace class follows in the next section. In this case, $W$ is an actual $Q$ Wiener process on $U$.

The following corollary follows right away from the above theorem.

**Corollary 63.5.4** Let $\Phi, \Psi \in L^2 ([a,T] \times \Omega; L_2 (Q^{1/2}U, H))$ and suppose they are both progressively measurable. Then

$$E \left( \left( \int_a^t \Phi dW, \int_a^t \Psi dW \right)_H \right) = E \left( \int_a^t (\Phi, \Psi)_{L_2(Q^{1/2}U,H)} ds \right)$$

Also if $L$ is in $L^\infty (\Omega, L(H,H))$ and is $F_a$ measurable, then

$$L \int_a^t \Phi dW = \int_a^t L\Phi dW$$

and

$$E \left( \left( L \int_a^t \Phi dW, \int_a^t \Psi dW \right)_H \right) = E \left( \int_a^t (L\Phi, \Psi)_{L_2(Q^{1/2}U,H)} ds \right).$$

**Proof:** First note that

$$\left( \int_a^t \Phi dW, \int_a^t \Psi dW \right)_H = \frac{1}{4} \left[ \left| \int_a^t (\Phi + \Psi) dW \right|_H^2 - \left| \int_a^t (\Phi - \Psi) dW \right|_H^2 \right]$$

and so from the above theorem,

$$E \left( \left( \int_a^t \Phi dW, \int_a^t \Psi dW \right)_H \right) =$$

$$E \left( \frac{1}{4} \left[ \left| \int_a^t (\Phi + \Psi) dW \right|_H^2 - \left| \int_a^t (\Phi - \Psi) dW \right|_H^2 \right] \right)$$

$$\frac{1}{4} E \left( \int_a^t \|\Phi + \Psi\|^2_{L_2(Q^{1/2}U,H)} ds \right) + \frac{1}{4} E \left( \int_a^t \|\Phi - \Psi\|^2_{L_2(Q^{1/2}U,H)} ds \right)$$

$$= \frac{1}{4} E \left( \int_a^t 1 \left[ \|\Phi + \Psi\|^2_{L_2(Q^{1/2}U,H)} + \|\Phi - \Psi\|^2_{L_2(Q^{1/2}U,H)} \right] ds \right)$$

$$= E \left( \int_a^t (\Phi, \Psi)_{L_2(Q^{1/2}U,H)} ds \right).$$
Now consider the last claim. First suppose \( L = lX_A \) where \( A \in \mathcal{F}_a \), and \( l \in \mathcal{L}(H,H) \). Also suppose \( \Phi \) is an elementary function

\[
\Phi = \sum_{i=0}^{n} \psi_i X_{(s_i,s_{i+1})}
\]

Then

\[
L \int_a^t \Phi dW = lX_A \sum_{i=0}^{n} \psi_i (W(t \wedge s_{i+1}) - W(t \wedge s_i)) = \sum_{i=0}^{n} lX_A \psi_i (W(t \wedge s_{i+1}) - W(t \wedge s_i))
\]

Thus also holds for \( L \) a simple function which is \( \mathcal{F}_a \) measurable. For general \( L \in L^\infty(\Omega, \mathcal{L}(H,H)) \), approximating with a sequence of such simple functions \( L_n \) yields

\[
L \int_a^t \Phi dW = \lim_{n \to \infty} L_n \int_a^t \Phi dW = \lim_{n \to \infty} \int_a^t L_n \Phi dW = \int_a^t L \Phi dW
\]

because \( L_n \Phi \to L \Phi \) in \( L^2([a,T] \times \Omega; \mathcal{L}_2(Q^{1/2}U,H)) \). Now what about general \( \Phi \)? Let \( \{ \Phi_n \} \) be elementary functions converging to \( \Phi \circ J^{-1} \) in \( L^2([a,T] \times \Omega; \mathcal{L}_2(JQ^{1/2}U,H)) \).

Then by definition of the integral,

\[
L \int_a^t \Phi dW = \lim_{n \to \infty} L \int_a^t \Phi_n dW = \lim_{n \to \infty} \int_a^t L \Phi_n dW = \int_a^t L \Phi dW
\]

The remaining claim now follows from the first part of the proof.

The above has discussed the integral of \( \Phi \in L^2([a,T] \times \Omega; \mathcal{L}_2(Q^{1/2}U,H)) \). An obvious case to consider is when

\[
\Phi = \sum_{k=0}^{n-1} \Phi_k \mathcal{X}_{(t_k,t_{k+1})}(t)
\]

and \( \Phi_k \in L^2(\Omega; \mathcal{L}_2(Q^{1/2}U,H)) \) with \( \Phi_k \) measurable with respect to \( \mathcal{F}_{t_k} \). What is \( \int_a^t \Phi dW \)? First note that \( \Phi_k \circ J^{-1} \in L^2(\Omega; \mathcal{L}_2(JQ^{1/2}U,H)) \). Let \( \lim_{m \to \infty} \Phi_k^m \to \Phi_k \circ J^{-1} \) in \( L^2(\Omega; \mathcal{L}_2(JQ^{1/2}U,H)) \) where \( \Phi_k^m \) is \( \mathcal{F}_{t_k} \) measurable and is a simple function having values in \( \mathcal{L}(U_1,H) \). Thus

\[
\Phi_m = \sum_{k=0}^{n-1} \Phi_k^m \mathcal{X}_{(t_k,t_{k+1})}(t)
\]
is an elementary function and it converges to $\Phi \circ J^{-1}$ in $L^2 \left( [a, T] \times \Omega; \mathcal{L}_2 \left( JQ^{1/2}U, H \right) \right)$. It follows that

$$
\int_a^t \Phi dW \equiv \lim_{m \to \infty} \int_a^t \Phi_m dW \equiv \lim_{m \to \infty} \sum_{k=0}^{n-1} \Phi_k^m (W(t \wedge t_{k+1}) - W(t \wedge t_k))
$$

$$
= \sum_{k=0}^{n-1} \Phi_k \circ J^{-1} (W(t \wedge t_{k+1}) - W(t \wedge t_k)).
$$

Note again how it appears to depend on $J$ but really doesn’t because there is a $J$ in the definition of $W$.

### 63.6 The Case That $Q$ Is Trace Class

In this special case, you have a $Q$ Wiener process with values in $U$ and still you have

$$
\Phi \in L^2 \left( [a, T] \times \Omega; \mathcal{L}_2 \left( Q^{1/2}U, H \right) \right)
$$

with $\Phi$ progressively measurable. The difference here is that in fact, $Q$ is trace class.

$$
Q = \sum_{i=1}^L \lambda_i e_i \otimes e_i
$$

where $\lambda_i > 0$, $\sum_i \lambda_i < \infty$, and the $e_i$ form an orthonormal set of vectors. $L$ is either a positive integer or $\infty$. Then let $U_0 = Q^{1/2}U$. Then $Q^{1/2} = \sum_{i=1}^L \sqrt{\lambda_i} e_i \otimes e_i$ because this works, and the square root is unique. Hence $Q^{1/2}e_i = \sqrt{\lambda_i}e_i$ and so an orthonormal basis for $U_0 = Q^{1/2}U$ is $\{\sqrt{\lambda_i}e_i\}_{i=1}^L$. Now consider $J = \sum_{i=1}^L \sqrt{\lambda_i} (e_i \otimes \sqrt{\lambda_i}e_i)$, $J : U_0 \to U$, where the tensor product is defined in the usual way,

$$
u \otimes v (w) \equiv (u, v)_U w.
$$

Then $J^* = \sum_{i=1}^L \sqrt{\lambda_i} (\sqrt{\lambda_i}e_i \otimes e_i)$ and $JJ^* = \sum_{i=1}^L \lambda_i e_i \otimes e_i = Q$. Also, $J$ is a Hilbert Schmidt map into $U$ from $U_0$.

$$
\sum_{i=1}^L \left\| J \left( \sqrt{\lambda_i}e_i \right) \right\|_U^2 = \sum_{i=1}^L \left\| \sqrt{\lambda_i}e_i \right\|_U^2 = \sum_{i=1}^L \lambda_i < \infty
$$

and so $J$ is a Hilbert Schmidt mapping. In addition to this, from the construction, the span of $\{e_i\}_{i=1}^L$ is dense in $U_0$ and $Je_i = e_i$ because

$$
Je_k = \sum_{i=1}^L \sqrt{\lambda_i} \left( e_i \otimes \sqrt{\lambda_i}e_i \right) (e_k) = e_k \sqrt{\lambda_k} \left( e_k, \sqrt{\lambda_k}e_k \right)_{U_0} = e_k
$$
so in fact $J$ is just the injection map of $U_0$ into $U$. Hence $J^{-1}$ must also be the identity map. Now we can let $U_1 = U$ with $J$ the injection map. Thus, in this case, the elementary functions $\Phi_n$ simply converge to $\Phi$ in

$$L^2([a,T] \times \Omega; L_2(J U_0, H))$$

Note that $\|J \sqrt{\lambda_i} e_i\|_{U} = \sqrt{\lambda_i}$ whereas $\|\sqrt{\lambda_i} e_i\|_{Q^{1/2} U} = 1$, and so $J$ definitely does not preserve norms. That is, the norm in $U_0$ is not the same as the norm in $U$. Then everything else is the same. In particular

$$E\left(\left|\int_a^t \Phi dW\right|^2_H\right) = \int_a^t E\left(\|\Phi\|^2_{L_2(U_0, H)}\right) ds.$$ 

### 63.7 A Short Comment On Measurability

It will also be important to consider the composition of functions. The following is the main result. With the explanation of progressively measurable given, it says the composition of progressively measurable functions is progressively measurable.

**Proposition 63.7.1** Let $A : [a,T] \times V \times \Omega \to U$ where $V, U$ are topological spaces and suppose $A$ satisfies its restriction to $[a,t] \times V \times \Omega$ is $\mathcal{B}([a,t]) \times \mathcal{B}(V) \times \mathcal{F}_t$ measurable. This will be referred to as $A$ is progressively measurable. Then if $X : [a,T] \times \Omega \to V$ is progressively measurable, then so is the map

$$(t, \omega) \to A(t, X(t, \omega), \omega)$$

**Proof:** Consider the restriction of this map to $[a,t_0] \times \Omega$. For such $(t, \omega)$, to say

$$A(t, X(t, \omega), \omega) \in O$$

for $O$ a Borel set in $U$ is to say that

$$X(t, \omega) \in \{v : (t, v, \omega) \in A^{-1}(O), t \leq t_0\} \equiv A^{-1}(O)_{t_0}$$

Consider the set

$$\{(t, \omega) \in [a, t_0] \times \Omega : X(t, \omega) \in A^{-1}(O)_{t_0}\}$$

Is this in $\mathcal{B}([a, t_0]) \times \mathcal{F}_{t_0}$? This is what needs to be checked. Since $A$ is progressively measurable,

$$A^{-1}(O) \cap [a, t_0] \times V \times \Omega \in \mathcal{B}([a, t_0]) \times \mathcal{B}(V) \times \mathcal{F}_{t_0} \equiv \mathcal{P}_{t_0}$$

because $A^{-1}(O)$ is a progressively measurable set. So let

$$\mathcal{G} = \{S \in \mathcal{P}_{t_0} : \{(t, \omega) \in [a, t_0] \times \Omega : X(t, \omega) \in S_{t_0}\} \in \mathcal{B}([a, t_0]) \times \mathcal{F}_{t_0}\}$$
63.8. LOCALIZATION FOR ELEMENTARY FUNCTIONS

It is clear that \( \mathcal{G} \) contains the \( \pi \) system composed of sets of the form \( I \times B \times W \) where \( I \) is an interval in \([a, t_0]\), \( B \) is Borel, and \( W \in \mathcal{F}_{t_0} \). This is because for \( S \) of this form, \( S_{t\omega} = B \) or \( \emptyset \). Thus if not empty,

\[
\{(t, \omega) \in [a, t_0] \times \Omega : X(t, \omega) \in S_{t\omega}\} = X^{-1}(B) \cap [0, t_0] \times \Omega \in \mathcal{B}([a, t_0]) \times \mathcal{F}_{t_0}
\]

because \( X \) is given to be progressively measurable. Now if \( S \in \mathcal{G} \), what about \( S^C \)? You have \( (S^C)_{t\omega} = (S_{t\omega})^C \) thus

\[
\{(t, \omega) \in [a, t_0] \times \Omega : X(t, \omega) \in (S^C)_{t\omega}\} = \{(t, \omega) \in [a, t_0] \times \Omega : X(t, \omega) \in (S_{t\omega})^C\}
\]

which is the complement with respect to \([a, t_0] \times \Omega\) of a set in \( \mathcal{B}([a, t_0]) \times \mathcal{F}_{t_0} \). Therefore, \( \mathcal{G} \) is closed with respect to complements. It is clearly closed with respect to countable disjoint unions. It follows, \( \mathcal{G} = \mathcal{P}_{t_0} \). Thus

\[
\{(t, \omega) \in [a, t_0] \times \Omega : X(t, \omega) \in S_{t\omega}\} \in \mathcal{B}([a, t_0]) \times \mathcal{F}_{t_0}
\]

where \( S = A^{-1}(O) \cap [a, t_0] \times V \times \Omega \). In other words,

\[
\{(t, \omega), t \leq t_0 : A(t, X(t, \omega), \omega) \in O\} \in \mathcal{B}([0, t_0]) \times \mathcal{F}_{t_0}
\]

and so \((t, \omega) \to A(t, X(t, \omega), \omega)\) is progressively measurable. ■

63.8 Localization For Elementary Functions

It is desirable to extend everything to stochastically square integrable functions. This will involve localization using a suitable stopping time. First it is necessary to understand localization for elementary functions. As above, we are in the situation described by the following diagram.

The elementary functions \( \{\Phi_n\} \) have values in \( \mathcal{L}(U_1, H)_{t_0} \) meaning they are restrictions of functions in \( \mathcal{L}(U_1, H) \) to \( JQ^{1/2}U \) and converge to \( \Phi \circ J^{-1} \) in

\[
L^2\left([a, T] \times \Omega; \mathcal{L}_2\left(JQ^{1/2}U, H\right)\right)
\]
where \( \Phi \in L^2 \left( [a, T] \times \Omega; \mathcal{L}_2 \left( Q^{1/2} U, H \right) \right) \) is given. Let

\[
\Phi (t) = \sum_{k=0}^{n-1} \Phi (t_k) \mathcal{X}_{[t_k, t_{k+1}]} (t)
\]

be an elementary function. In particular, let \( \Phi (t_k) \) be \( \mathcal{F}_{t_k} \) measurable as a map into \( \mathcal{L} (U_1, H) \), and has finitely many values. As just mentioned, the topic of interest is the elementary functions \( \Phi_n \) in the above diagram. Thus \( \Phi \) will be one of these elementary functions.

Let \( \tau \) be a stopping time having values from the set of mesh points \( \{ t_k \} \) for the elementary function. Then from the definition of the integral for elementary functions,

\[
\int_a^{t \wedge \tau} \Phi dW = \sum_{k=0}^{n-1} \Phi (t_k) \left( W (t \wedge \tau \wedge t_{k+1}) - W (t \wedge \tau \wedge t_k) \right)
\]

If \( \omega \) is such that \( \tau (\omega) = t_j \), then to get something nonzero, you must have \( t_j > t_k \) so \( k \leq j - 1 \). Thus the above on the right reduces to

\[
\sum_{k=0}^{j-1} \Phi (t_k) \left( W (t \wedge t_{k+1}) - W (t \wedge t_k) \right)
\]

It clearly is 0 if \( j = 0 \). Define \( \sum_{k=0}^{-1} \equiv 0 \). Thus the integral equals

\[
\sum_{j=0}^{n} \mathcal{X}_{\tau = t_j} \sum_{k=0}^{j-1} \Phi (t_k) \left( W (t \wedge t_{k+1}) - W (t \wedge t_k) \right)
\]

Interchanging the order of summation, \( k \leq j - 1 \) so \( j \geq k + 1 \) and this equals

\[
\sum_{k=0}^{n-1} \sum_{j=k+1}^{n} \mathcal{X}_{\tau = t_j} \Phi (t_k) \left( W (t \wedge t_{k+1}) - W (t \wedge t_k) \right)
\]

\[
= \sum_{k=0}^{n-1} \mathcal{F}_{t_k} \text{measurable} \sum_{j=k+1}^{n} \mathcal{X}_{\tau > t_k} \Phi (t_k) \left( W (t \wedge t_{k+1}) - W (t \wedge t_k) \right)
\]

Therefore

\[
\int_a^{t \wedge \tau} \Phi dW = \int_a^{t} \sum_{k=0}^{n-1} \mathcal{X}_{\tau > t_k} \Phi (t_k) \mathcal{X}_{[t_k, t_{k+1}]} dW \quad (63.8.15)
\]
63.9. **Localization In General**

Now observe

\[
X_{[a,\tau]}(t) \Phi(t) = \sum_{k=0}^{n-1} X_{[a,\tau]}(t_k) \Phi(t_k) X(t_{k+1})
\]

\[
= \sum_{k=0}^{n-1} X_{[\tau \geq t_k]} \Phi(t_k) X(t_{k+1})
\]

\[
= \sum_{k=0}^{n-1} X_{[\tau > t_k]} \Phi(t_k) X(t_{k+1})
\]

(63.8.16)

The last step occurs because of the following reasoning. The \(k\)th term of the sum in the middle expression above equals \(\Phi(t_k)\) if and only if \(\tau \geq t\) and \(\tau \geq t_k\). If the two conditions do not hold, then the \(k\)th term equals 0. As to the third line, if \(\tau > t_k\) and \(t \in (t_k, t_{k+1})\), then \(\tau \geq t_{k+1} \geq t\) which is the same as the situation in the second line. The term equals \(\Phi(t_k)\). Note that \(X_{[\tau > t_k]}(\omega)\) is \(\mathcal{F}_{t_k}\)-measurable, because \([\tau > t_k]\) is the complement of \([\tau \leq t_k]\). Therefore, this is an elementary function. Thus, from 63.8.15 - 63.8.16, \(X_{[a,\tau]}(t) \Phi(t)\) is an elementary function and

\[
\int_a^{t \wedge \tau} \Phi dW = \int_a^t \sum_{k=0}^{n-1} X_{[\tau > t_k]} \Phi(t_k) X(t_{k+1}) dW = \int_a^t X_{[a,\tau]}(t) \Phi(t) dW
\]

From Proposition 63.1.5, if you have \(\Phi, \Psi\) two of these elementary functions

\[
E \left( \left\| \int_a^t X_{[a,\tau]}(t) \Phi(t) dW - \int_a^t X_{[a,\tau]}(t) \Psi(t) dW \right\|_H^2 \right) =
\]

\[
\int_a^t \int_\Omega \left\| (\Phi(s) - \Psi(s)) \circ J \right\|^2_{\mathcal{L}^2(\mathcal{Q}^{1/2} U, H)} dPds
\]

\[
\leq \int_a^t \int_\Omega \left\| (\Phi(s) - \Psi(s)) \circ J \right\|^2_{\mathcal{L}^2(\mathcal{Q}^{1/2} U, H)} dPds
\]

(63.8.17)

### 63.9 Localization In General

Next, what about the general case where \(\Phi \in L^2([a, T] \times \Omega; \mathcal{L}_2(\mathcal{Q}^{1/2} U, H))\) and is progressively measurable? Is it the case that for an arbitrary stopping time \(\tau\),

\[
\int_a^{t \wedge \tau} \Phi dW = \int_a^t X_{[a,\tau]} \Phi dW
\]

This is the sort of thing which would be expected for an ordinary Stieltjes integral which of course this isn’t. Let

\[L^2([a, T] \times \Omega; \mathcal{L}_2(JQ^{1/2} U, H)) = K\]
From Doob’s result Proposition 63.3 and Lemma 63.3, there exists a sequence of elementary functions \( \{\Phi_k\} \)

\[
\Phi_k(t) = \sum_{j=0}^{m_k-1} \Phi(t^k_j) X_{(t^k_j,t^k_{j+1})}(t)
\]

which converges to \( \Phi \circ J^{-1} \) in \( K \) where also the lengths of the sub intervals converge uniformly to 0 as \( k \to \infty \).

Now let \( \tau \) be an arbitrary stopping time. The partition points corresponding to \( \Phi_k \) are \( \{t^k_j\}_{j=0}^{m_k} \). Let \( \tau_k = t^k_{j+1} \) on \( \tau^{-1}(t^k_j, t^k_{j+1}) \). Then \( \tau_k \) is a stopping time because

\[
[\tau_k \leq t] \in \mathcal{F}_t
\]

Here is why. If \( t \in (t^k_j, t^k_{j+1}) \), then if \( t = t^k_{j+1} \), it would follow that \( \tau_k(\omega) \leq t \) would be the same as saying \( \omega \in [\tau \leq t^k_{j+1}] = [\tau \leq t] \in \mathcal{F}_t \). On the other hand, if \( t < t^k_{j+1} \), then \( [\tau_k \leq t] = [\tau \leq t^k_j] \subseteq \mathcal{F}_t \) because \( \tau_k \) only takes the values \( t^k_j \).

Consider \( X_{[a,\tau_k]} \Phi_k \). It is given that \( \Phi_k \to \Phi \circ J^{-1} \) in \( K \). Does it follow that \( X_{[a,\tau_k]} \Phi_k \to X_{[a,\tau]} \Phi \circ J^{-1} \) in \( K \)? Consider first the indicator function. Let \( \tau(\omega) \in (t^k_j, t^k_{j+1}) \). Fixing \( t \), if \( X_{[a,\tau_k]}(t) = 1 \), then also \( X_{[a,\tau_k]}(t) = 1 \) because \( \tau_k \geq \tau \). Therefore, in this case \( \lim_{k \to \infty} X_{[a,\tau_k]}(t) = X_{[a,\tau]}(t) \). Next suppose \( X_{[a,\tau_k]}(t) = 0 \) so that \( \tau(\omega) < t \). Since the intervals defined by the partition points have lengths which converge to 0, it follows that for all \( k \) large enough, \( \tau_k(\omega) < t \) also and so \( X_{[a,\tau_k]}(t) = 0 \). Therefore,

\[
\lim_{k \to \infty} X_{[a,\tau_k]}(\omega)(t) = X_{[a,\tau]}(\omega)(t).
\]

It follows that \( X_{[a,\tau_k]} \Phi_k \to X_{[a,\tau]} \Phi \circ J^{-1} \) in \( K \). Now from Proposition 63.3, the function \( X_{[a,\tau_k]} \Phi_k \) is progressively measurable. Therefore, the same is true of \( X_{[a,\tau]} \Phi \circ J^{-1} \).

From the proof of Theorem 63.3, the part depending on maximal estimates and the fact that \( \int_a^t X_{[a,\tau_k]} \Phi_k dW \) is a continuous martingale, there is a set of measure zero \( N \), such that off this set, a suitable subsequence satisfies

\[
\int_a^t X_{[a,\tau_k]} \Phi_k dW \to \int_a^t X_{[a,\tau]} \Phi dW
\]

uniformly on \([a, T] \). But also, since \( \Phi_k \to \Phi \circ J^{-1} \) in \( K \), a suitable subsequence satisfies,

\[
\int_a^t \Phi_k dW \to \int_a^t \Phi dW
\]

uniformly on \([a, T] \) a.e. \( \omega \). In particular, \( \int_a^{t \wedge \tau_k} \Phi_k dW \to \int_a^{t \wedge \tau} \Phi dW \). Therefore,

\[
\int_a^t X_{[a,\tau]} \Phi dW = \lim_{k \to \infty} \int_a^t X_{[a,\tau_k]} \Phi_k dW
\]

\[
= \lim_{k \to \infty} \int_a^{t \wedge \tau_k} \Phi_k dW
\]

\[
= \int_a^{t \wedge \tau} \Phi dW
\]
This has proved the following major localization lemma. This is a marvelous result. It says that the stochastic integral acts algebraically like an ordinary Stieltjes integral, one for each \( \omega \) off a set of measure zero.

**Lemma 63.9.1** Let \( \Phi \) be progressively measurable and in

\[
L^2 \left( [a, T] \times \Omega; L_2 \left( Q^{1/2} U, H \right) \right)
\]

Let \( W(t) \) be a cylindrical Wiener process as described above. Then for \( \tau \) a stopping time, \( \mathcal{X}_{[a, \tau]} \Phi \) is progressively measurable, in \( K \), and

\[
\int_a^{t \wedge \tau} \Phi \, dW = \int_a^t \mathcal{X}_{[a, \tau]} \Phi \, dW.
\]

### 63.10 The Stochastic Integral As A Local Martingale

With Lemma 63.9.1, it becomes possible to define the stochastic integral on functions which are only stochastically square integrable.

**Definition 63.10.1** \( \Phi \) is stochastically square integrable in \( L_2 \left( Q^{1/2} U, H \right) \) if \( \Phi \) is progressively measurable and

\[
P \left( \int_a^T \| \Phi(s) \|^2_{L_2(Q^{1/2}U,H)} \, ds < \infty \right) = 1
\]

Thus equivalently, there exists \( N \) such that \( P(N) = 0 \) and for \( \omega \notin N \),

\[
\int_a^T \| \Phi(s, \omega) \|^2_{L_2(Q^{1/2}U,H)} \, ds < \infty.
\]

**Lemma 63.10.2** Suppose \( \Phi \) is \( L_2 \left( Q^{1/2} U, H \right) \) progressively measurable and

\[
P \left( \int_a^T \| \Phi \|^2_{L_2(Q^{1/2}U,H)} \, ds < \infty \right) = 1.
\]

Define

\[
\tau_n(\omega) \equiv \inf \left\{ t \in [a, T] : \int_a^t \| \Phi \|^2_{L_2(Q^{1/2}U,H)} \, ds \geq n \right\},
\]

By convention, let \( \inf \emptyset = \infty \). Then \( \tau_n \) is a stopping time. Furthermore, \( \tau_n \) has the following properties.

1. \( \{ \tau_n \} \) is an increasing sequence and for \( \omega \) outside a set of measure zero \( N \), for every \( t \in [a, T] \) there exists \( n \) such that \( \tau_n(\omega) > t \). (It is a localizing sequence of stopping times.)
2. For each \( n \), \( X_{[a,\tau_n]}\Phi \) is progressively measurable and

\[
E \left( \int_a^T \|X_{[a,\tau_n]}\Phi\|^2_{L_2(Q^{1/2}U,H)} \, dt \right) < \infty
\]

**Proof:** It follows from Proposition 60.7.2 that \( \tau_n \) is a stopping time because it is the first hitting time of a closed set by an adapted continuous process.

It remains to verify the two claims. There exists a set of measure 0, \( N \) such that for \( \omega \not\in N \)

\[
\int_a^T \|\Phi\|^2_{L_2(Q^{1/2}U,H)} \, dt < \infty
\]

Therefore, for such \( \omega \), there exists \( n \) large enough that

\[
\int_a^t \|\Phi\|^2_{L_2(Q^{1/2}U,H)} \, ds < n
\]

and so \( \tau_n(\omega) \geq t \). Now consider the second claim.

\[
E \left( \int_a^T \|X_{[a,\tau_n]}\Phi\|^2_{L_2(Q^{1/2}U,H)} \, dt \right) = E \left( \int_{\tau_n(\omega)\wedge T}^{\tau_n(\omega)\wedge T} \|\Phi\|^2_{L_2(Q^{1/2}U,H)} \, dt \right) \leq E(n) = n.
\]

With this lemma, it is possible to give the following definition.

**Definition 63.10.3** Suppose \( \Phi \) is \( L_2(Q^{1/2}U,H) \) progressively measurable and

\[
P \left( \int_a^T \|\Phi\|^2_{L_2(Q^{1/2}U,H)} \, ds < \infty \right) = 1. \quad (63.10.18)
\]

More generally, suppose there exists a localizing sequence of stopping times \( \tau_n \) having the two properties of Lemma 63.10.2. Then for all \( \omega \) not in the exceptional set \( N \).

\[
\int_a^t \Phi dW \equiv \lim_{n \to \infty} \int_a^t X_{[a,\tau_n]}\Phi dW
\]

**Lemma 63.10.4** The above definition is well defined. For all \( \omega \) not in a set of measure zero,

\[
\int_a^t \Phi dW (\omega) \equiv \lim_{n \to \infty} \int_a^t X_{[a,\tau_n]}\Phi dW (\omega)
\]

the function on the right being constant for all \( n \) large enough for a given \( \omega \). The random variable \( \int_a^t \Phi dW \) is also \( \mathcal{F}_t \) adapted.
Proof: Let \( \{ \tau_n \} \) be a sequence of stopping times as described in \( \text{(1)} \) and \( \text{(2)} \) of Lemma 63.10.2. Such a sequence exists by Lemma 63.10.2. It makes sense to define the random variable

\[
\int_a^t \mathcal{X}_{[a, \tau_n]} \Phi \, dW
\]

Now what if both \( \tau_m \) and \( \tau_n \) are at least as large as \( t \) for some \( \omega \)? Do the two random variables coincide at that value of \( \omega \)? Say \( m > n \) so that \( \tau_m(\omega) \geq \tau_n(\omega) > t \). For the given \( \omega \),

\[
\int_a^t \mathcal{X}_{[a, \tau_m]} \Phi \, dW = \int_a^{t \wedge \tau_m} \mathcal{X}_{[a, \tau_m]} \Phi \, dW
\]

For the particular \( \omega \) of interest,

\[
= \int_a^{t \wedge \tau_n} \mathcal{X}_{[a, \tau_n]} \Phi \, dW
\]

and this equals

\[
= \int_a^t \mathcal{X}_{[a, \tau_n]} \mathcal{X}_{[a, \tau_m]} \Phi \, dW = \int_a^t \mathcal{X}_{[a, \tau_n]} \Phi \, dW
\]

for all \( \omega \), in particular for the given \( \omega \). Therefore, for the particular \( \omega \) of interest,

\[
\int_a^t \mathcal{X}_{[a, \tau_n]} \Phi \, dW = \int_a^t \mathcal{X}_{[a, \tau_m]} \Phi \, dW
\]

Thus the limit exists because for all \( n \) large enough, the integral is eventually constant. Then \( \int_a^t \Phi \, dW \) is \( \mathcal{F}_t \) adapted because for \( U \) an open set in \( H \),

\[
\left( \int_a^t \Phi \, dW \right)^{-1}(U) = \bigcup_{n=1}^{\infty} \left( \left( \int_a^t \mathcal{X}_{[a, \tau_n]} \Phi \, dW \right)^{-1}(U) \cap [\tau_n > t] \right) \in \mathcal{F}_t. \]
CHAPTER 63. STOCHASTIC INTEGRATION

Page 2316. Then by definition,

\[
\int_{a}^{t \wedge \sigma} \Phi dW (s) \equiv \lim_{n \to \infty} \int_{a}^{t \wedge \tau_{n} \wedge \sigma} \Phi dW (s)
\]

\[
= \lim_{n \to \infty} \int_{a}^{t \wedge \tau_{n}} \chi_{[a, \sigma]} \Phi dW (s) = \int_{a}^{t} \chi_{[a, \sigma]} \Phi dW (s)
\]

since \( t \wedge \tau_{n} = t \) for all \( n \) large enough. ■

63.11 The Quadratic Variation Of The Stochastic Integral

An important corollary of Lemma 63.9.1 concerns the quadratic variation of \( \int_{a}^{t} \Phi dW \).

It is convenient here to use the notation \( \int_{a}^{t} \Phi dW \equiv \Phi \cdot W (t) \). Recall this is a local submartingale \( [\Phi \cdot W] \) such that

\[
||\Phi \cdot W (t)||_{L_{2}}^{2} = [\Phi \cdot W] (t) + N (t)
\]

where \( N \) is a local martingale. Recall the quadratic variation is unique so that if it acts like the quadratic variation, then it is the quadratic variation. Recall also why this was so. If you have a local martingale equal to the difference of increasing adapted processes which equals 0 when \( t = 0 \), then the local martingale was equal to 0. Of course you can substitute \( a \) for 0.

**Corollary 63.11.1** Suppose \( \Phi \) is \( L_{2} (Q^{1/2}U, H) \) progressively measurable and has the localizing sequence with the two properties in Lemma 63.10.2. Then the quadratic variation, \( [\Phi \cdot W] \) is given by the formula

\[
[\Phi \cdot W] (t) = \int_{a}^{t} ||\Phi (s)||_{L_{2}(Q^{1/2}U,H)}^{2} ds
\]

**Proof:** By the above discussion, \( \int_{a}^{t} \Phi dW \) is a local martingale. Let \( \{\tau_{n}\} \) be a localizing sequence for which the stopped local martingale is a martingale and \( \Phi \chi_{[a, \tau_{n}]} \) is in \( L^{2} ([a, T] \times \Omega, L_{2} (Q^{1/2}U, H)) \). Also let \( \sigma \) be a stopping time with two values no larger than \( T \). Then from Lemma 63.10.2,

\[
E \left( \int_{a}^{\tau_{n} \wedge \sigma} \Phi dW \right)^{2}_{H} - \int_{a}^{\tau_{n} \wedge \sigma} ||\Phi (s)||_{L_{2}(Q^{1/2}U,H)}^{2} ds
\]

\[
E \left( \int_{a}^{T \wedge \tau_{n} \wedge \sigma} \Phi dW \right)^{2}_{H} - \int_{a}^{T \wedge \tau_{n} \wedge \sigma} ||\Phi (s)||_{L_{2}(Q^{1/2}U,H)}^{2} ds
\]
63.11. THE QUADRATIC VARIATION OF THE STOCHASTIC INTEGRAL

\[
E \left( \int_a^T X_{\tau_n \wedge \tau}^0 X_{\tau_n \wedge \tau}^0 \Phi \, dW \right)^2 - \int_a^T \left\| X_{\tau_n \wedge \tau}^0 X_{\tau_n \wedge \tau}^0 \Phi \right\|_{L_2}^2 \, ds
\]

\[
= E \left( \int_a^T \left\| X_{\tau_n \wedge \tau}^0 X_{\tau_n \wedge \tau}^0 \Phi \right\|_{L_2}^2 \, dt \right) - E \left( \int_a^T \left\| X_{\tau_n \wedge \tau}^0 X_{\tau_n \wedge \tau}^0 \Phi \right\|_{L_2}^2 \, ds \right) = 0
\]

thanks to the Ito isometry. There is also no change in letting \( \sigma = t \). You still get 0. It follows from Lemma 63.11.1, the lemma about recognizing a martingale when you see one, that

\[
t \to \left| \int_a^{t \wedge \tau_n} \Phi \, dW \right|_{H}^2 - \int_a^{t \wedge \tau_n} \left\| \Phi (s) \right\|_{L_2}^2 (Q^{1/2} U, H) \, ds
\]

is a martingale. Therefore,

\[
\left| \int_a^t \Phi \, dW \right|_{H}^2 - \int_a^t \left\| \Phi (s) \right\|_{L_2}^2 (Q^{1/2} U, H) \, ds
\]

is a local martingale and so, by uniqueness of the quadratic variation,

\[
[\Phi \cdot W] (t) = \int_a^t \left\| \Phi (s) \right\|_{L_2}^2 (Q^{1/2} U, H) \, ds \]

Here is an interesting little lemma which seems to be true.

**Lemma 63.11.2** Let \( \Phi, \Phi_n \) all be in \( L^2 ([0, T], L_2 (Q^{1/2} U, H)) \) off some set of measure zero. These are all progressively measurable. Thus there are all stochastically square integrable.

\[
P \left( \int_0^T \left\| \Phi \right\|^2 \, ds \right) = 1
\]

Suppose also that for each \( \omega \notin N \), the exceptional set,

\[
\int_0^T \left\| \Phi_n - \Phi \right\|_{L_2}^2 \, dt \to 0
\]

Then there exists a set of measure zero, still denoted as \( N \) and a subsequence, still denoted as \( n \) such that for each \( \omega \notin N \),

\[
\lim_{n \to \infty} \int_0^T \Phi_n \, dW = \int_0^T \Phi \, dW
\]

**Proof:** Define stopping times

\[
\tau_{np} = \inf \left\{ t \in [0, T] : \int_0^t \left\| \Phi_n \right\|^2 \, ds > p \right\}
\]
Let \( \tau_p \) be similar but defined with reference to \( \Phi \). Then by Ito isometry,

\[
E \left( \left| \int_0^T X_{[0,\tau_p]} \Phi_n dW - \int_0^T X_{[0,\tau_p]} \Phi dW \right|^2 \right) = E \left( \int_0^T \left\| X_{[0,\tau_p]} \Phi_n - X_{[0,\tau_p]} \Phi \right\|^2_{L^2} dt \right) \tag{63.11.19}
\]

The integrand in the right side is bounded by \( 2p^2 \). Also this integrand converges to 0 for each \( \omega \) as \( n \to \infty \). This is shown next.

\[
\int_0^T \left\| X_{[0,\tau_p]} \Phi_n - X_{[0,\tau_p]} \Phi \right\|^2_{L^2} dt \\
\leq 2 \int_0^T \left( \left\| X_{[0,\tau_p]} \Phi_n - X_{[0,\tau_p]} \Phi \right\|^2_{L^2} + \left\| X_{[0,\tau_p]} \Phi - X_{[0,\tau_p]} \Phi \right\|^2_{L^2} \right) dt \\
\leq 2 \int_0^T \left\| \Phi_n - \Phi \right\|^2_{L^2} dt + 2 \int_0^T \left| X_{[0,\tau_p]} (t) - X_{[0,\tau_p]} (t) \right|^2 \left\| \Phi \right\|^2_{L^2} dt
\]

The first converges to 0 by assumption. Problem is, it does not look like this second integral converges to 0. We do know that \( \int_0^t \left\| \Phi_n \right\|^2 ds \to \int_0^t \left\| \Phi \right\|^2 ds \) uniformly so \( \tau_{np} \to \tau_p \) is likely. However, this does not imply \( X_{[0,\tau_{np}]} \to X_{[0,\tau_p]} \). However, it would converge in \( L^2 (0, T) \) and so there is a subsequence such that convergence takes place a.e. \( t \). Then restricting to this subsequence, the second integral converges to 0. Actually, it may be easier than this. \( X_{[0,\tau_p]} \) has a single point of discontinuity and convergence takes place at every other point. Thus it appears that the integrand in \( 63.11.19 \) converges to 0 for each \( \omega \). Thus, by dominated convergence theorem the whole expectation converges to 0.

Now consider

\[
P \left( \left| \int_0^T X_{[0,\tau_p]} \Phi_n dW - \int_0^T X_{[0,\tau_p]} \Phi dW \right|^2 > \lambda \right) \leq E \left( \int_0^T \left| X_{[0,\tau_p]} \Phi_n dW - \int_0^T X_{[0,\tau_p]} \Phi dW \right|^2 \right) / \lambda
\]

and so, there exists a subsequence, still denoted as \( n \) such that

\[
P \left( \left| \int_0^T X_{[0,\tau_p]} \Phi_n dW - \int_0^T X_{[0,\tau_p]} \Phi dW \right|^2 > \frac{1}{n} \right) < 2^{-k}
\]

It follows that \( N \) can be enlarged so that for \( \omega \notin N_p \)

\[
\left| \int_0^T X_{[0,\tau_p]} \Phi_n dW - \int_0^T X_{[0,\tau_p]} \Phi dW \right|^2 \leq \frac{1}{n}
\]
for all \( n \) large enough. Now obtain a succession of subsequences for \( p = 1, 2, \ldots \), each a subsequence of the preceeding one such that the above convergence takes place and let \( N \) include \( \bigcup_p N_p \). Then for \( \omega \notin N \), and letting \( n \) denote the diagonal sequence, it follows that for all \( p \),

\[
\lim_{n \to \infty} \left| \int_0^T X_{[0, \tau_{np}]} \Phi_n dW - \int_0^T X_{[0, \tau_p]} \Phi dW \right| = 0
\]

For \( \omega \notin N \), there is a \( p \) such that \( \tau_p = \infty \). Then this means \( \int_0^T \| \Phi \|^2 ds < p \). It follows that the same is true for \( \Phi_n \) for all \( n \) large enough. Hence \( \tau_{np} = \infty \) also.

Thus, for large enough \( n \),

\[
\int_0^T \Phi_n dW - \int_0^T \Phi dW = \left| \int_0^T X_{[0, \tau_{np}]} \Phi_n dW - \int_0^T X_{[0, \tau_p]} \Phi dW \right|
\]

and the latter was just shown to converge to 0.

\[ \blacksquare \]

### 63.12 The Holder Continuity Of The Integral

Let \( \Phi \in L^2 \left( [0, T] \times \Omega, L_2 \left( Q^{1/2} U, H \right) \right) \). Then you can consider the stochastic integral as described above and it yields a continuous function off a set of measure zero. What if \( \Phi \in L^\infty \left( [0, T] \times \Omega, L_2 \left( Q^{1/2} U, H \right) \right) \)? Can you say more? The short answer is yes. You obtain a Holder condition in addition to continuity. This is a consequence of the Burkholder Davis Gundy inequality and Corollary 63.11.1 above.

Let \( \alpha > 2 \). Let \( \| \Phi \|_\infty \) denote the norm in \( L^\infty \left( [0, T] \times \Omega, L_2 \left( Q^{1/2} U, H \right) \right) \). By the Burkholder Davis Gundy inequality,

\[
\int_\Omega \left( \left| \int_s^t \Phi dW \right| \right) \, dP \leq 
\int_\Omega \left( \sup_{r \in [s, t]} \left| \int_s^r \Phi dW \right| \right)^\alpha \, dP \leq C \int_\Omega \left( \int_s^t \| \Phi \|^2 \, d\tau \right)^{\alpha/2} \, dP
\]

\[
\leq C \| \Phi \|_\infty^\alpha \int_\Omega \left( \int_s^t d\tau \right)^{\alpha/2} \, dP = C \| \Phi \|_\infty^\alpha |t-s|^{\alpha/2}
\]

By the Kolmogorov Čentsov theorem, Theorem 63.2.2, this shows that \( t \to \int_0^t \Phi dW \) is Holder continuous with exponent

\[
\gamma < \frac{(\alpha/2) - 1}{\alpha} = \frac{1}{2} - \frac{1}{\alpha}
\]

Since \( \alpha > 2 \) is arbitrary, this shows that for any \( \gamma < 1/2 \), the stochastic integral is Holder continuous with exponent \( \gamma \). This is exactly the same kind of continuity possessed by the Wiener process.
Theorem 63.12.1 Suppose $\Phi \in L^\infty \left([0,T] \times \Omega, \mathcal{L}_2 \left(Q^{1/2}U, H\right)\right)$ and is progressively measurable. Then if $\gamma < 1/2$, there exists a set of measure zero such that off this set,

$$t \to \int_0^t \Phi dW$$

is Hölder continuous with exponent $\gamma$.

### 63.13 Taking Out A Linear Transformation

When is

$$L \int_a^T \Phi dW = \int_a^T L\Phi dW?$$

It is assumed $L \in \mathcal{L}(H, H_1)$ where $H_1$ is another separable real Hilbert space. First of all, here is a lemma which shows $\int_a^t L\Phi dW$ at least makes sense.

**Proposition 63.13.1** Suppose $\Phi$ is $\mathcal{L}_2 \left(Q^{1/2}U, H\right)$ progressively measurable and

$$P\left(\int_a^T \|\Phi\|^2_{\mathcal{L}_2 \left(Q^{1/2}U, H\right)} ds < \infty\right) = 1.$$  

Then the same is true of $L\Phi$. Furthermore, for each $t \in [a, T]$

$$\int_a^t L\Phi dW = L \int_a^t \Phi dW$$

**Proof:** First note that if $\Phi \in \mathcal{L}_2 \left(Q^{1/2}U, H\right)$, then $L\Phi \in \mathcal{L}_2 \left(Q^{1/2}U, H_1\right)$ and that the map $\Phi \rightarrow L\Phi$ is continuous. It follows $L\Phi$ is $\mathcal{L}_2 \left(Q^{1/2}U, H_1\right)$ progressively measurable. All that remains is to check the appropriate integral.

$$\int_a^T \|L\Phi\|^2_{\mathcal{L}_2 \left(Q^{1/2}U, H_1\right)} dt \leq \int_a^T \|L\Phi\|^2_{\mathcal{L}_2 \left(Q^{1/2}U, H\right)} dt$$

and so this proves $L\Phi$ satisfies the same conditions as $\Phi$, being stochastically square integrable.

It follows one can consider

$$\int_a^T L\Phi dW.$$  

Assume to begin with that $\Phi \in L^2 \left([a, T] \times \Omega; \mathcal{L}_2 \left(Q^{1/2}U, H\right)\right)$. Next recall the situation in which the definition of the integral is considered.
Letting \( \{ \Phi_n \} \) be an approximating sequence of elementary functions satisfying
\[
E \left( \int_a^T \left\| \Phi_n - \Phi \circ J^{-1} \right\|^2_{L_2(JQ^{1/2}U,H)} \, dt \right) \to 0,
\]
it is also the case that
\[
E \left( \int_a^T \left\| L\Phi_n - L\Phi \circ J^{-1} \right\|^2_{L_2(JQ^{1/2}U_1,H_1)} \, dt \right) \to 0.
\]
By the definition of the integral, for each \( t \)
\[
\int_a^t L\Phi dW = \lim_{n \to \infty} \int_a^t L\Phi_n dW
= \lim_{n \to \infty} L \int_a^t \Phi_n dW
= L \lim_{n \to \infty} \int_a^t \Phi_n dW = L \int_a^t \Phi dW
\]
The second equality is obvious for elementary functions.
Now consider the case where \( \Phi \) is only stochastically square integrable so that all is known is that
\[
P \left( \int_a^T ||\Phi||^2_{L_2(Q^{1/2}U,H)} \, dt < \infty \right) = 1.
\]
Then define \( \tau_n \) as above
\[
\tau_n \equiv \inf \left\{ t : \int_a^t ||\Phi||^2_{L_2(Q^{1/2}U,H)} \, dt \geq n \right\}
\]
This sequence of stopping times works for \( L\Phi \) also. Recall there were two conditions the sequence of stopping times needed to satisfy. The first is obvious. Here is why the second holds.
\[
\int_a^T \left\| \mathcal{X}_{[a,\tau_n]} L\Phi \right\|^2_{L_2(Q^{1/2}U,H_1)} \, dt \leq ||L||^2 \int_a^T \left\| \mathcal{X}_{[a,\tau_n]} \Phi \right\|^2_{L_2(Q^{1/2}U,H)} \, dt
= ||L||^2 \int_a^{\tau_n} \left\| \Phi \right\|^2_{L_2(Q^{1/2}U,H)} \, dt \leq ||L||^2 n
\]
Then let \( t \) be given and pick \( n \) such that \( \tau_n (\omega) \geq t \). Then from the first part, for that \( \omega \),
\[
L \int_a^t \Phi dW = L \int_a^t \mathcal{X}_{[a,\tau_n]} \Phi dW
= \int_a^t L\mathcal{X}_{[a,\tau_n]} \Phi dW
= \int_a^t \mathcal{X}_{[a,\tau_n]} L\Phi dW = \int_a^t L\Phi dW \]
63.14 A Technical Integration By Parts Result

Let $Z \in L^2([0,T] \times \Omega, \mathcal{L}_2(Q^{1/2}U,H))$ where this has reference to the usual dia-

gram

\[ \begin{array}{c}
  U \\
  \downarrow \quad Q^{1/2} \\
  U_1 \supseteq JQ^{1/2}U \\
  \downarrow \quad \Phi \\\n  H \\
\end{array} \]

Also suppose $X \in L^2([0,T] \times \Omega, H)$, both $X$ and $Z$ being progressively measurable. Let \( \{t^n_j\}_{j=1}^{m_n} \) denote a sequence of partitions of the sort discussed earlier where

\[
X_n(t) = \sum_{j=0}^{m_n-1} X(t^n_j) X(t^n_j, t^n_{j+1})(t)
\]

converges to $X$ in $L^2([0,T] \times \Omega, H)$. Thus $X_n(t)$ is right continuous. Let

\[
\tau^n_p = \inf \{t : |X_n(t)|_H > p\}.
\]

This is the first hitting time of a right continuous adapted process so it is a stopping

time. Also there exists a set of measure zero $N$ such that for $\omega \notin N$, then given $t$,

\[
\tau^n_p \geq t
\]

if $p$ is large enough because of the assumption on $X$. Here is why. There exists a set of measure 0 $N$ such that if $\omega \notin N$, then

\[
\int_0^T |X_n(t)|^2_H dt = \sum_{j=0}^{m_n-1} |X(t^n_j)|^2_H (t^n_{j+1} - t^n_j) < \infty.
\]

It follows that there exists an upper bound, depending on $\omega$ which dominates each
of the values $|X(t^n_j)|^2_H$. Then if $p$ is larger than this upper bound, $\tau^n_p = \infty > t$.

Next consider the expression

\[
\sum_{j=0}^{m-1} \left( \int_{t^n_{j+1} \wedge t}^{t^n_j \wedge t} Z(u) \, dW, X(t^n_j) \right)_H.
\]

This expression is a function of $\omega$.

I want to write this in the form of a stochastic integral. To begin with, consider
one of the terms. For simplicity of notation, consider

\[
\left( \int_a^b Z(u) \, dW, X(u) \right)_H
\]
where $Z \in L^2([a, b] \times \Omega, \mathcal{L}_2(Q^{1/2}U, H))$ and $X(a) \in L^2(\Omega, H)$. Also assume the function of $\omega, [X(a)]_H$ is bounded. There is an Itô integral involved in the above. Let $Z_n$ be a sequence of elementary functions defined on $[a, b]$ which converges to $Z \circ J^{-1}$ in $L^2 ([a, b] \times \Omega, \mathcal{L}_2(JQ^{1/2}U, H))$. Then by the definition of the integral,

$$\left\| \int_a^t Z(u) \, dW - \int_a^t Z_n(u) \, dW \right\|_{L^2(\Omega, H)} \to 0$$

Also, by the use of a maximal inequality and the fact that the two integrals above are martingales, there is a subsequence, still called $n$ and a set of measure zero $N$ such that for $\omega \notin N$, the convergence

$$\int_a^t Z_n(u) \, dW(\omega) \to \int_a^t Z(u) \, dW(\omega)$$

is uniform for $t \in [a, b]$. Therefore, for such $\omega$,

$$\left( \int_a^t Z(u) \, dW, X(a) \right)_H = \lim_{n \to \infty} \left( \int_a^t Z_n(u) \, dW, X(a) \right)_H$$

Say $Z_n(u) = \sum_{k=0}^{m_n-1} Z^n_k \mathcal{X}_{(t^n_k, t^n_{k+1})} (u)$ where $Z^n_k$ has finitely many values in $\mathcal{L}(U_1, H)_0$, the restrictions of $\mathcal{L}(U_1, H)$ to $JQ^{1/2}U$. Then the inner product in the above formula on the right is of the form

$$\sum_{k=0}^{m_n-1} (Z^n_k W(t \wedge t^n_{k+1}) - W(t \wedge t^n_k), X(a))_H = \sum_{k=0}^{m_n-1} (\mathcal{R}((Z^n_k)^* X(a))(W(t \wedge t^n_{k+1}) - W(t \wedge t^n_k))$$

$$= \int_a^t \mathcal{R}(Z^n a) \, dW$$

where $\mathcal{R}$ is the Riesz map from $U_1$ to $U'_1$. Note that $\mathcal{R}(Z^n a)$ has values in $\mathcal{L}(U_1, \mathbb{R}) \subseteq \mathcal{L}_2(JQ^{1/2}U, \mathbb{R})$.

Now let $\{g_i\}$ be an orthonormal basis for $Q^{1/2}U$, so it follows that $\{Jg_i\}$ is an orthonormal basis for $JQ^{1/2}U$. Then

$$\sum_i \mathcal{R}\left(Z^n a - (Z \circ J^{-1})^* a \right) (Jg_i)^2$$

$$= \sum_i \left( (Z^n a - (Z \circ J^{-1})^* a, Jg_i)_{U_1} \right)^2 = \sum_i \left( (X(a), (Z_n - Z \circ J^{-1}) Jg_i)_H \right)^2$$
\[
\leq \sum_i |X(a)|_H^2 |(Z_n - Z \circ J^{-1}) Jg_1|_H^2 = |X(a)|_H^2 \|Z_n - Z \circ J^{-1}\|_{L^2(JQ^{1/2}U,H)}^2
\]

When integrated over \([a, b] \times \Omega\), it is given that this converges to 0. This has shown that
\[
\mathcal{R}(Z_n^* X(a)) \to \mathcal{R}\left((Z \circ J^{-1})^* X(a)\right)
\]
in \(L_2(JQ^{1/2}U, \mathbb{R})\). In other words
\[
\mathcal{R}(Z_n^* X(a)) \to \left(\mathcal{R}\left((Z \circ J^{-1})^* X(a)\right) \circ J\right) \circ J^{-1}
\]

It follows that
\[
\left(\int_a^t Z(u) \, dW, X(a)\right)_H = \int_a^t \mathcal{R}\left((Z \circ J^{-1})^* X(a)\right) \circ J \, dW
\]

From localization,
\[
\left(\int_{a \wedge \tau^n_p}^{b \wedge \tau^n_p} Z(u) \, dW, X(a)\right)_H = \left(\int_a^b \mathcal{X}_{[0,\tau^n_p]} Z(u) \, dW, X(a)\right)_H
\]
\[
= \int_a^b \mathcal{X}_{[0,\tau^n_p]} \mathcal{R}\left((Z \circ J^{-1})^* X(a)\right) \circ J \, dW
\]
\[
= \int_{a \wedge \tau^n_p}^{b \wedge \tau^n_p} \mathcal{R}\left((Z \circ J^{-1})^* X(a)\right) \circ J \, dW
\]

Then it follows that, using the stopping time,
\[
\sum_{j=0}^{m-1} \left(\int_{t_j \wedge \tau^n_p \wedge t}^{t_{j+1} \wedge \tau^n_p \wedge t} Z(u) \, dW, X(t^n_j)\right)_H = \sum_{j=0}^{m-1} \int_{t_j \wedge \tau^n_p \wedge t}^{t_{j+1} \wedge \tau^n_p \wedge t} \mathcal{R}\left((Z \circ J^{-1})^* X_n(t^n_j)\right) \circ J \, dW
\]
\[
= \int_0^{t \wedge \tau^n_p} \mathcal{R}\left((Z \circ J^{-1})^* (X_n^I)\right) \circ J \, dW
\]

where \(X_n^I\) is the step function
\[
X_n^I(t) = \sum_{k=0}^{m_n-1} X(t^n_k) \mathcal{X}_{[t^n_k, t^n_{k+1})}(t)
\]

By localization, this is
\[
\int_0^t \mathcal{X}_{[0,\tau^n_p]} \mathcal{R}\left((Z \circ J^{-1})^* (X_n^I)\right) \circ J \, dW
\]

If \(\omega\) is not in a suitable set of measure zero, then \(\tau^n_p(\omega) \geq t\) provided \(p\) is large enough. Thus, for such \(\omega\), if \(p\) is large enough,
\[
\sum_{j=0}^{m_n-1} \left(\int_{t_j \wedge \tau^n_p \wedge t}^{t_{j+1} \wedge \tau^n_p \wedge t} Z(u) \, dW, X(t^n_j)\right)_H = \int_0^t \mathcal{X}_{[0,\tau^n_p]} \mathcal{R}\left((Z \circ J^{-1})^* (X_n^I)\right) \circ J \, dW
\]
\[
= \int_0^t \mathcal{R}\left((Z \circ J^{-1})^* (X_n^I)\right) \circ J \, dW
\]
This shows that the expression is a local martingale. Also note that the expression on the left does not depend on $J$ or $U_1$ so the same must be true of the expression on the right although it does not look that way. This has proved the following important theorem.

**Theorem 63.14.1** Let $Z \in L^2 ([0, T] \times \Omega, \mathcal{L}_2 (Q^{1/2} U, H))$ and let $X \in L^2 ([0, T] \times \Omega, H)$, both $X, Z$ progressively measurable. Also let $\{t^n_j\}_{j=1}^{m_n}$ be a sequence of partitions of $[0, T]$ such that each $X (t^n_j)$ is in $L^2 (\Omega, H)$. Then

$$
\sum_{j=0}^{m-1} \left( \int_{t^n_j \wedge t}^{t^n_{j+1} \wedge t} Z (u) \, dW, X (t^n_j) \right)_{H} \quad (63.14.21)
$$

is a stochastic integral of the form

$$
\int_0^t R \left( (Z \circ J^{-1})^* (X^-_n) \right) \circ J dW
$$

where $\{\tau^n_p\}_{p=1}^\infty$ is a localizing sequence used to define the above integral whose integrand is only stochastically square integrable. Here $X^-_n$ is the step function defined by

$$X^-_n (t) \equiv \sum_{k=0}^{m_n-1} X (t^n_k) \mathcal{X}_{[t^n_k, t^n_{k+1})} (t)$$

In particular, $63.14.21$ is a local martingale.

Of course it would be very interesting to see what happens in the case where $X^-_n \to X$ in $L^2 ([0, T] \times \Omega, H)$. Is it the case that convergence to

$$
\int_0^t R \left( (Z \circ J^{-1})^* (X) \right) \circ J dW \quad (63.14.22)
$$

can happen in some sense? Also, does the above stochastic integral even make sense? First of all, consider the question whether it makes sense. It would be nice to define a stopping time

$$\tau_n \equiv \inf \{ t : |X (t)|_H > n \}$$

because then $\mathcal{X}_{[0, \tau_n]} R \left( (Z \circ J^{-1})^* (X) \right) \circ J$ would end up being integrable in the right way and you could define the stochastic integral provided $\tau_n > t$ whenever $n$ is large enough. However, this is problematic because $t \to X (t)$ is not known to be continuous. Therefore, some other condition must be assumed.

**Lemma 63.14.2** Suppose $t \to X (t)$ is weakly continuous into $H$ for a.e. $\omega$, and that $X$ is adapted. Then the $\tau_n$ described above is a stopping time.
CHAPTER 63. STOCHASTIC INTEGRATION

Proof: Let $B \equiv \{ x \in H : |x| > n \}$. Then the complement of $B$ is a closed convex set. It follows that $B^c$ is also weakly closed. Hence $B$ must be weakly open. Now $t \to X(t)$ is adapted as a function mapping into the topological space consisting of $H$ with the weak topology because it is in fact adapted into the strong topology. Therefore, the above $\tau_n$ is just the first hitting time of an open set by a continuous process so $\tau_n$ is a stopping time by Proposition 60.7.2. Also, by the assumption that $t \to X(t)$ is weakly continuous, it follows that $X(t)$ for $t \in [0, T]$ is weakly bounded. Hence, for each $\omega$ off a set of measure zero, $|X(t)|$ is bounded for $t \in [0, T]$. This follows from the uniform boundedness theorem. It follows that $\tau_n = \infty$ for $n$ large enough.

Hence the weak continuity of $t \to X(t)$ suffices to define the stochastic integral in 63.14.22. It remains to verify some sort of convergence in the case that

$$\lim_{k \to \infty} \mathbb{E} \left[ \max_{t \in [0,T]} |X(t) - X_k(t)|^2 \right] = 0$$

**Lemma 63.14.3** Let $X(s) - X^k(s) = \Delta_k(s)$. Here $Z \in L^2([0, T] \times \Omega, L^2(Q^{1/2}U, H))$ and let $X \in L^2([0, T] \times \Omega, H)$ with both $X$ and $Z$ progressively measurable, $t \to X(t)$ being weakly continuous into $H$,

$$\lim_{k \to \infty} \|X - X^k\|_{L^2([0, T] \times \Omega, H)} = 0$$

Then the integral

$$\int_0^t \mathcal{R} \left( (Z \circ J^{-1})^* (X) \right) \circ JdW$$

exists as a local martingale and the following limit occurs for a suitable subsequence, still called $k$.

$$\lim_{k \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \Delta_k(s) \right) \circ JdW(s) \right| \geq \varepsilon \right] = 0. \ (63.14.23)$$

That is,

$$\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* (X(s) - X^k(s)) \right) \circ JdW(s) \right|$$

converges to 0 in probability.

**Proof:** Let $k$ denote a subsequence for which $X^k$ also converges pointwise to $X$.

The existence of the integral follows from Lemma 63.14.2. From the assumption of weak continuity, $\sup_{t \in [0, T]} |X(t)| \leq C(\omega)$ for a.e. $\omega$. For the first part of the argument, assume $C$ does not depend on $\omega$ off a set of measure zero. Let

$$M(t) \equiv \int_0^t ZdW$$
Let \( \{ e_k \} \) be an orthonormal basis for \( H \) and let \( P_n \) be the orthogonal projection onto \( \text{span}(e_1, \ldots, e_n) \). For each \( e_i \)
\[
\lim_{k \to \infty} \left| \langle X(s) - X^k(s), e_i \rangle \right| = 0
\]
and so, by weak continuity,
\[
\lim_{k \to \infty} P_n (X(s) - X^k(s)) = 0 \text{ for } a.e. \omega
\]
Then
\[
\lim_{k \to \infty} \int_\Omega \int_0^T \left| P_n (X(s) - X^k(s)) \right|^2 \| Z(s) \|_{L^2}^2 ds dP = 0
\]
because you can apply the dominated convergence theorem with respect to the measure \( \| Z(s) \|_{L^2}^2 ds dP \).
Therefore,
\[
\lim_{k \to \infty} P \left( \left[ \sup_{t \in [0,T]} \left| \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* P_n \Delta_k(s) \right) \circ JdW(s) \right| \geq \varepsilon/2 \right] \right) = 0
\]
(63.14.24)
Here is why. By the Burkholder Davis Gundy theorem, Theorem 61.4.4 and Corollary 63.11.1 which describes the quadratic variation of the stochastic integral,
\[
\int_\Omega \left( \sup_{t \in [0,T]} \left| \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* P_n \Delta_k(s) \right) dW(s) \right| \right) dP
\]
\[
\leq C \int_\Omega \left( \int_0^T \left| P_n (X(s) - X^k(s)) \right|^2 \| Z(s) \|_{L^2}^2 ds \right)^{1/2} dP
\]
Consider the following two probabilities.
\[
P \left( \left[ \sup_{t \in [0,T]} \left| \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* (I - P_n) X(s) \right) \circ JdW(s) \right| \geq \varepsilon/2 \right] \right)
\]
(63.14.25)
\[
P \left( \left[ \sup_{t \in [0,T]} \left| \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* (I - P_n) X^k(s) \right) \circ JdW(s) \right| \geq \varepsilon/2 \right] \right)
\]
(63.14.26)
By Corollary 63.11.1 which depends on the Burkholder Davis Gundy inequality and Corollary 63.11.1 which describes the quadratic variation of the stochastic integral, \( \frac{C}{\varepsilon} E \left( \left( \int_0^T \| Z(s) \|^2 ds \right)^{1/2} \wedge \delta \right) \) is dominated by
\[
\frac{C}{\varepsilon} E \left( \left( \int_0^T \| Z(s) \|^2 ds \right)^{1/2} \wedge \delta \right) + P \left( \left( \int_0^T \| Z(s) \|^2 ds \right)^{1/2} > \delta \right)
\]
\[ \leq \frac{C\delta}{\varepsilon} + P \left( \left[ \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X(s) \right|^2 \, ds \right)^{1/2} > \delta \right] \right) \] (63.14.27)

Let \( \eta > 0 \) be given. Then let \( \delta \) be small enough that the first term is less than \( \eta \). Fix such a \( \delta \).

Consider the second of the above terms.

\[
P \left( \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X(s) \right|^2 \, ds \right)^{1/2} > \delta \right) \leq \frac{1}{\delta} \left( E \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X(s) \right|^2 \, ds \right) \right)^{1/2}
\]

and this converges to 0 because \( (I - P_n) X(s) \) is assumed to be bounded and converges to 0. Next consider (63.14.26). By similar reasoning, we end up with having to estimate

\[
\frac{1}{\delta} \left( E \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X^k(s) \right|^2 \, ds \right) \right)^{1/2}.
\]

But this is dominated by

\[
\frac{2}{\delta} \left( E \left( \int_0^T \|Z(s)\|^2 \left| X^k(s) - X(s) \right|^2 \, ds \right) \right)^{1/2}
\]

\[
+ \frac{2}{\delta} \left( E \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X(s) \right|^2 \, ds \right) \right)^{1/2}
\]

The first term is no larger than \( \eta \) provided \( k \) is large enough, independent of \( n \) thanks to the pointwise convergence and the assumption that \( X \) is bounded. Thus, there exists \( K \) such that if \( k > K \), then the term in (63.14.26) is dominated by

\[
2\eta + \frac{2}{\delta} \left( E \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X(s) \right|^2 \, ds \right) \right)^{1/2}
\]

It follows that for \( k > K \), the sum of (63.14.25) and (63.14.26) is dominated by

\[
3\eta + \frac{3}{\delta} \left( E \left( \int_0^T \|Z(s)\|^2 \left| (I - P_n) X(s) \right|^2 \, ds \right) \right)^{1/2}
\]

This is then no larger than \( 4\eta \) provided \( n \) is large enough. Pick such an \( n \). Then for all \( k > K \), this has shown that

\[
P \left( \sup_{t \in [0,T]} \left| \int_0^t R \left( (Z(s) \circ J^{-1})^* \Delta_k(s) \right) \circ JdW(s) \right| \geq \varepsilon \right)
\]
\[ \leq P \left( \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* P_n \Delta_k(s) \right) \circ JdW(s) \geq \varepsilon/2 \right) + P \left( \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* (I - P_n) \Delta_k(s) \right) \circ JdW(s) \geq \varepsilon/2 \right) \]

\[ \leq P \left( \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* P_n \Delta_k(s) \right) \circ JdW(s) \geq \varepsilon/2 \right) + 4\eta \]

By this whole thing is less than \( 5\eta \) provided \( k \) is large enough. This has proved that under the assumption that \( X \) is bounded uniformly off a set of measure zero,

\[ \lim_{k \to \infty} P \left( \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \Delta_k(s) \right) \circ JdW(s) \geq \varepsilon \right) = 0 \]

This is what was desired to show. It remains to remove the extra assumption that \( X \) is bounded.

Now to finish the argument, define the stopping time

\[ \tau_m = \inf \{ t > 0 : |X(t)|_H > m \} \]

As observed in Lemma, this is a valid stopping time. Also define \( \Delta_k^{\tau_m} = X^{\tau_m} - (X_k^{\tau_m})^{\tau_m} \). Using this stopping time on \( X \) and \( X_k^{\tau_m} \) does not affect the pointwise convergence to 0 as \( k \to \infty \) of \( \Delta_k^{\tau_m} \) on which the above argument depends.

Consider

\[ A_{k\varepsilon} = \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \Delta_k(s) \right) \circ JdW(s) \geq \varepsilon \]

Then

\[ P (A_{k\varepsilon} \cap \{ \tau_m = \infty \}) \leq P \left( \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \Delta_k^{\tau_m}(s) \right) \circ JdW(s) \geq \varepsilon \right) \]

which converges to 0 as \( k \to \infty \) by the first part of the argument. This is because \( |X^{\tau_m}| \) and \( |(X_k^{\tau_m})^{\tau_m}| \) are both bounded by \( m \) and the same pointwise convergence condition still holds. Now

\[ A_{k\varepsilon} = \bigcup_{m=1}^\infty A_{k\varepsilon} \cap \{ \tau_m = \infty \} \cap \{ \tau_{m-1} < \infty \} \]

Thus

\[ P (A_{k\varepsilon}) = \sum_{m=1}^\infty P (A_{k\varepsilon} \cap \{ \tau_m = \infty \} \cap \{ \tau_{m-1} < \infty \}) \]  

(63.14.28)

Also

\[ P (A_{k\varepsilon} \cap \{ \tau_m = \infty \} \cap \{ \tau_{m-1} < \infty \}) \leq P (\tau_m = \infty \cap \tau_{m-1} < \infty) \]
which is summable because these are disjoint sets. Hence one can apply the domi-
nated convergence theorem in \ref{63.14.28} and conclude

$$
\lim_{k \to \infty} P(A_{k\varepsilon}) = \sum_{m=1}^{\infty} \lim_{k \to \infty} P(A_{k\varepsilon} \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty])) = 0 \blacksquare
$$
Chapter 64

The Integral $\int_0^t (Y, dM)_H$

First the integral is defined for elementary functions.

Definition 64.0.4 Let an elementary function be one which is of the form

$$\sum_{i=0}^{m-1} Y_i \mathcal{X}_{(t_i, t_{i+1})}(t)$$

where $Y_i$ is $\mathcal{F}_{t_i}$ measurable with values in $H$ a separable real Hilbert space for $0 = t_0 < t_1 < \cdots < t_m = T$.

Definition 64.0.5 Now let $M$ be a $H$ valued continuous local martingale, $M(0) = 0$. Then for $Y$ a simple function as above,

$$\int_0^t (Y, dM) \equiv \sum_{i=0}^{m-1} (Y_i, M(t \wedge t_{i+1}) - M(t \wedge t_i))_H$$

Assumption 64.0.6 We will always assume that $d[M]$ is absolutely continuous with respect to Lebesgue measure. Thus $d[M] = kdt$ where $k \geq 0$ and is in $L^1([0, T] \times \Omega)$. This is done to avoid technical questions related to whether $t \rightarrow \int_0^t d[M]$ is continuous and also to make it easier to get examples of a certain class of functions.

This includes the usual stochastic integral $M(t) = \int_0^t \Phi dW$ where $[M](t) = \int_0^t ||\Phi||^2_{L^2} ds$ so $d[M] = ||\Phi||^2 dt$.

Next is to consider how this relates to stopping times which have values in the \{t_i\}. Let $\tau$ be a stopping time which takes the values \{t_i\}. Then

$$\int_0^{t \wedge \tau} (Y, dM) \equiv \sum_{i=0}^{m-1} (Y_i, M(t \wedge t_{i+1} \wedge \tau) - M(t \wedge t_i \wedge \tau))_H \quad (64.0.1)$$
Now consider $X_{[0,\tau]}Y$. Is it also an elementary function?

$$X_{[0,\tau]}Y = \sum_{i=0}^{m-1} X_{[0,\tau]}(t) Y_i X_{[t_i, t_{i+1}]}(t)$$

To get the $i$th term to be non zero, you must have $\tau \geq t$ and $t \in (t_i, t_{i+1}]$. Thus it must be the case that $\tau > t_i$. Also, if $\tau > t_j$ and $t \in (t_i, t_{i+1}]$, then $\tau \geq t_{i+1}$ because $\tau$ has only the values $t_i$. Hence also $\tau \geq t$. Thus the above sum reduces to

$$\sum_{i=0}^{m-1} X_{[\tau > t_i]} (\omega) Y_i X_{[t_i, t_{i+1}]}(t)$$

This shows that $X_{[0,\tau]}Y$ is of the right sort, the sum of $F_{t_i}$ measurable functions times $X_{[t_i, t_{i+1}]}(t)$. Thus from the definition of this funny integral,

$$\int_0^T (X_{[0,\tau]}Y, dM) \equiv \sum_{i=0}^{m-1} \left( \int X_{[\tau > t_i]} (\omega) Y_i X_{[t_i, t_{i+1}]}(t) M(t \wedge t_i) - M(t \wedge t_i) \right)$$

(64.0.2)

Are the right sides of (64.0.1) and (64.0.2) equal?

Begin with the right side of (64.0.1) and consider $\tau = t_j$. Then to get something nonzero in the terms of the sum in (64.0.1), you would need to have $t_j \geq t_{i+1}$. Otherwise, $t_j \leq t_i$ and the difference involving $M$ would give 0. Hence, for such $\omega$ you would need to have the sum in (64.0.1) equal to

$$\sum_{i=0}^{j-1} (Y_i, M(t \wedge t_{i+1}) - M(t \wedge t_i))_H$$

Thus this sum in (64.0.1) equals

$$\sum_{j=0}^{m} \sum_{i=0}^{j-1} (Y_i, M(t \wedge t_{i+1}) - M(t \wedge t_i))_H$$

Of course when $j = 0$ the term in the sum in (64.0.1) equals 0 so there is no harm in defining $\sum_{i=0}^{-1} = 0$. Then from the sum, you have $i \leq j - 1$ and so when you interchanges the order, you get that $\int_0^{t \wedge \tau} (Y, dM) =

$$\sum_{i=0}^{m-1} \sum_{j=i+1}^{m} (X_{[\tau > t_j]} (\omega) Y_i M(t \wedge t_{i+1}) - M(t \wedge t_i))_H$$

$$= \sum_{i=0}^{m-1} (X_{[\tau > t_i]} (\omega) Y_i M(t \wedge t_{i+1}) - M(t \wedge t_i))_H$$
Thus the right side of \( \text{64.0.1} \) equals the right side of \( \text{64.0.2} \):

\[
\int_0^t (X_{[0,\tau]} Y, dM) = \sum_{i=0}^{m-1} (X_{[\tau=t_i]} (\omega) Y_i, M(t \wedge t_{i+1}) - M(t \wedge t_i)) = \int_0^{t \wedge \tau} (Y, dM)
\]

This has proved the first part of the following lemma.

**Lemma 64.0.7** For an elementary function \( Y \), and a stopping time \( \tau \) having values in the \( \{t_i\} \), the points of discontinuity of \( Y \), it follows that \( X_{[0,\tau]} Y \) is also an elementary function and

\[
\int_0^{t \wedge \tau} (Y, dM) = \int_0^t (X_{[0,\tau]} Y, dM) = \int_0^t (Y, dM^\tau)
\]

**Proof:** Consider the second equal sign. By definition,

\[
\int_0^{t \wedge \tau} (Y, dM) = \sum_{i=0}^{m-1} (Y_i, M(t \wedge t_{i+1} \wedge \tau) - M(t \wedge t_i \wedge \tau))_H
\]

\[
= \sum_{i=0}^{m-1} (Y_i, M^\tau(t \wedge t_{i+1}) - M^\tau(t \wedge t_i))_H \equiv \int_0^t (Y, dM^\tau) \quad \blacksquare
\]

Next is another lemma about these integrals of elementary functions. First recall the following definition

\[
M^* \equiv \sup \{\|M(t)\| : t \in [0,T]\}
\]

**Lemma 64.0.8** Let \( M \) be a local martingale on \([0,T]\) where \( M(0) = 0 \) and \( M \) is continuous. Let \( 0 < r < s < T \) and consider \( (Y, (M^r_s - M^{r+\tau})^\tau(t)) \) where \( Y(M^r)^* \in L^2(\Omega) \) and \( Y \) is \( F_r \) measurable and \( \tau_p \) is a localizing sequence of stopping times for which \( M^r \) is a \( L^2 \) martingale. Then this is a martingale on \([0,T]\) which equals 0 at \( t = 0 \) and

\[
\begin{align*}
[Y, (M^r_s - M^{r+\tau})^\tau(t)] & \leq \|Y\|^2 \|M^r_s - M^{r+\tau}\|^\tau(t) \\
& = \|Y\|^2 \|M^r\|^s(t) - \|M^r\|^\tau(t) \\
& = \|Y\|^2 \|M^r\|^s(t \wedge s) - \|M^r\|^\tau(t \wedge r)
\end{align*}
\]

It follows that for \( Y \) an elementary function where each \( Y_i (M^r)^* \) is in \( L^2(\Omega) \),

\[
\int_0^t (Y, dM)
\]

is a local martingale.

**Proof:** To save notation, \( M \) is written in place of \( M^r \). It is clear that \( (Y, (M^s - M^\tau)(t)) = 0 \) if \( t \leq r \). Is it a martingale?

\[
E ((Y, (M^s - M^\tau)(t))) = E (E ((Y, (M^s - M^\tau)(t)) | F_r))
\]

\[
= E ((Y, E ((M (s \wedge t) - M (r \wedge t)) | F_r))) = 0
\]
because $M$ is a martingale. Now let $\sigma$ be a bounded stopping time with two values. Then using the optional sampling theorem where needed,

$$E \left( (Y, (M^s - M^r) (\sigma)) \right) = E \left( \left( E \left( (Y, (M^s - M^r) (\sigma)) \mid \mathcal{F}_r \right) \right) \right)$$

$$= E \left( (Y, E \left( (M (s \wedge \sigma) - M (r \wedge \sigma)) \mid \mathcal{F}_r \right)) \right)$$

$$= E \left( (Y, M (s \wedge \sigma \wedge r) - M (r \wedge \sigma)) \right)$$

$$= E \left( (Y, M (\sigma \wedge r) - M (r \wedge \sigma)) \right) = 0$$

It follows that this is indeed a martingale as claimed.

By the definition of the quadratic variation,

$$|(Y, (M^s - M^r) (t))|^2 \leq \|Y\|^2 \|\left( (M^s - M^r) (t) \right)\|^2$$

$$= \|Y\|^2 \left( ([M^s - M^r]) (t) + \|Y\|^2 \hat{N} (t) \right)$$

where $\hat{N} (t)$ is a martingale. It equals 0 if $t \leq r$. By similar reasoning to the above,

$$\|Y\|^2 \hat{N} (t)$$

is a martingale. To see this,

$$E \left( \|Y\|^2 \hat{N} (\sigma) \right) = E \left( E \left( \|Y\|^2 \hat{N} (\sigma) \mid \mathcal{F}_r \right) \right)$$

$$= E \left( \|Y\|^2 E \left( \hat{N} (\sigma) \mid \mathcal{F}_r \right) \right)$$

$$= E \left( \|Y\|^2 N (\sigma \wedge r) \right) = 0$$

One also sees that $E \left( \|Y\|^2 \hat{N} (t) \right) = 0$.

Now it follows from Corollary 61.3.3 that

$$[(M^s - M^r)] = [M^s - M^r] = [M]^s - [M]^r$$

Hence

$$[(Y, (M^s - M^r)) (t) \leq \|Y\|^2 \left( [M^s - M^r] (t) = \|Y\|^2 \left( [M]^s (t) - [M]^r (t) \right)\right)$$

as claimed.

The last claim is easy. Let $\tau_p$ be a localizing sequence for which $M^{r_p}$ is a martingale. Then

$$\int_0^{t \wedge \tau_p} (Y, dM) = \sum_{i=0}^{m-1} \left( Y_{i+1} M (t \wedge t_{i+1} \wedge \tau_p) - M (t \wedge t_i \wedge \tau_p) \right)$$

$$= \sum_{i=0}^{m-1} \left( Y_{i+1} M^{r_p} (t \wedge t_{i+1}) - M^{r_p} (t \wedge t_i) \right)$$

a finite sum of martingales. \(\blacksquare\)

Note that this is just a definition and did not use the above localization lemma. In particular, $\tau_p$ is not restricted to having only the partition points as values.
Next one needs to generalize past the elementary functions.

Continue writing $M$ in place of $M^{t^p}$ in what follows. Consider an elementary function

$$Y \equiv \sum_{k=0}^{m_n-1} Y_k \chi_{[t_k, t_{k+1}]} (t)$$

where $Y_k M^* \in L^2 (\Omega)$. Consider

$$\int_0^t (Y, dM) \equiv \sum_{k=0}^{m_n-1} (Y_k, M (t \wedge t_{k+1}) - M (t \wedge t_k)) \quad (64.0.3)$$

Then it is routine to verify that

$$E \left( \left( \sum_{k=0}^{m_n-1} (Y_k, M (t \wedge t_{k+1}) - M (t \wedge t_k)) \right)_H \right)^2 = \sum_{k=0}^{m_n-1} E \left( (Y_k, M (t \wedge t_{k+1}) - M (t \wedge t_k))_H^2 \right) \quad (64.0.4)$$

This is because the mixed terms all vanish. This follows from the following reasoning. Let $t_j < t_k$

$$E \left( (Y_k, M (t \wedge t_{k+1}) - M (t \wedge t_k))_H \right) \left( Y_j, M (t \wedge t_{j+1}) - M (t \wedge t_{j+1}) \right)_H \right)$$

$$= E \left( E \left( (Y_k, \Delta_k M (t))_H \left( Y_j, \Delta_j M (t) \right)_H \mid \mathcal{F}_{t_k} \right) \right)$$

$$= E \left( (Y_j, \Delta_j M (t))_H E \left( (Y_k, \Delta_k M (t))_H \mid \mathcal{F}_{t_k} \right) \right)$$

$$= E \left( (Y_j, \Delta_j M (t))_H \left( Y_k, E \left( (\Delta_k M (t))_H \mid \mathcal{F}_{t_k} \right) \right) \right)$$

$$= E \left( (Y_j, \Delta_j M (t))_H \left( Y_k, 0 \right)_H \right) = 0$$

Now

$$\sum_{k=0}^{m_n-1} E \left( (Y_k, M (t \wedge t_{k+1}) - M (t \wedge t_k))_H^2 \right) = \sum_{k=0}^{m_n-1} E \left( (Y_k, (M^{t_{k+1}} - M^{t_k}) (t))_H^2 \right)$$

It follows from (64.0.4)

$$E \left( \left( \sum_{k=0}^{m_n-1} (Y_k, M (t \wedge t_{k+1}) - M (t \wedge t_k))_H \right)^2 \right) = \sum_{k=0}^{m_n-1} E \left( (Y_k, (M^{t_{k+1}} - M^{t_k}) (t))_H^2 \right)$$

$$= \sum_{k=0}^{m_n-1} E \left( \left[ (Y_k, (M^{t_{k+1}} - M^{t_k}) (t)) \right] + N_k (t) \right)$$
CHAPTER 64. THE INTEGRAL $\int_0^T (Y, DM)_H$

where $N_k$ is a martingale equal to 0 for $t \leq t_k$. Then this equals

$$\sum_{k=0}^{m_n-1} E \left( \left[ (Y_k, (M_{t_{k+1}}^{t_k} - M_{t_k}^{t_k}) (t) ) \right] \right)$$

From Lemma 64.0.8

$$\leq E \left( \sum_{k=0}^{m_n-1} \|Y_k\|_H^2 \left( [M]_{t_{k+1}}^{t_k} (t) - [M]_{t_k}^{t_k} (t) \right) \right)$$

$$= E \left( \sum_{k=0}^{m_n-1} \|Y_k\|_H^2 \left( [M]_{t_{k+1}}^{t_k} (t) - [M]_{t_k}^{t_k} (t) \right) \right)$$

$$= E \left( \int_0^t \|Y\|_H^2 d [M]_{\tau_p} \right) = E \left( \int_0^t \|Y\|_H^2 d [M]_{\tau_p} \right)$$

Note that everything makes sense because it is assumed that $\|Y_k\| M^* \in L^2(\Omega)$. This proves the following lemma.

**Lemma 64.0.9** Let $\|Y (t)\| (M^{\tau_p})^* \in L^2 (\Omega)$ for each $t$, where $Y$ is an elementary function and let $\tau_p$ be a stopping time for which $M^{\tau_p}$ is a $L^2$ martingale. Then

$$E \left( \int_0^t (Y, dM^{\tau_p}) \right)^2 \leq E \left( \int_0^t \|Y\|_H^2 d [M]^{\tau_p} \right)$$

The condition that $\|Y (t)\| (M^{\tau_p})^* \in L^2 (\Omega)$ ensures that

$$E \left( (Y_k, M^{\tau_p} (t \wedge t_{k+1}) - M^{\tau_p} (t \wedge t_{k+1})))_H^2 \right)$$

always is finite.

**Definition 64.0.10** Let $G$ denote those functions $Y$ which are adapted and have the property that for each $p$,

$$\lim_{n \to \infty} E \left( \int_0^T \|Y - Y^n\|_H^2 d [M]^{\tau_p} \right) = 0$$

for some sequence $Y^n$ of elementary functions for which $\|Y^n (t)\| M^* \in L^2 (\Omega)$ for each $t$. Here $d [M]^{\tau_p}$ signifies the Lebesgue Stieltjes measure determined by the increasing function $t \to [M^{\tau_p} (t)]$. Let $M^{\tau_p}$ be an $L^2$ martingale. Recall that $\tau_p$ is just a localizing sequence for the local martingale $M$.

It is not known whether this increasing function is absolutely continuous.

**Definition 64.0.11** Let $Y \in G$. Then

$$\int_0^t (Y, dM^{\tau_p}) \equiv \lim_{n \to \infty} \int_0^t (Y^n, dM^{\tau_p}) \text{ in } L^2 (\Omega)$$
For example, suppose \( Y \) is a bounded continuous process having values in \( H \). Then you could look at the left step functions

\[
Y^n(t) = \sum_{i=0}^{m_n-1} Y(t_i) \mathcal{X}_{[t_i, t_{i+1})}(t)
\]

The \( Y^n \) would converge to \( Y \) pointwise on \([0, T]\) for each \( \omega \) and these \( Y^n \) are bounded. In fact, in this case, these converge uniformly to \( Y \) on \([0, T]\). Thus this is an example of the situation in the above definition. In this case, the integrand would be bounded by \( C \) for some \( C \) and

\[
E \left( \int_0^T C d[M]^{\tau_p} \right) = E \left( [M]^{\tau_p}(T) \right) = E \left( \|M^{\tau_p}(T)\|^2 \right) < \infty
\]

by assumption. Hence, by the dominated convergence theorem,

\[
\lim_{n \to \infty} E \left( \int_0^T \|Y - Y^n\|_H^2 d[M]^{\tau_p} \right) = 0.
\]

What if \([M]^{\tau_p}\) were bounded and absolutely continuous with respect to Lebesgue measure? This could be the case if you had \( \tau_p \) a stopping time of the form

\[
\tau_p = \inf \{ t : [M](t) > p \}
\]

Then if \( Y \in L^2([0, T] \times \Omega, H) \), and progressively measurable there are left step functions which converge to \( Y \) in \( L^2([0, T] \times \Omega, H) \). Say \( d[M]^{\tau_p} = k(t, \omega) \, dm \) where \( k \) is bounded. Then

\[
E \left( \int_0^T \|Y - Y^n\|_H^2 d[M]^{\tau_p} \right) = E \left( \int_0^T \|Y - Y^n\|_H^2 k dt \right) \to 0
\]

**Lemma 64.0.12** The above definition is well defined. Also, \( \int_0^t (Y, dM^{\tau_p}) \) is a continuous martingale. The inequality

\[
E \left( \left| \int_0^t (Y, dM^{\tau_p}) \right|^2 \right) \leq E \left( \int_0^t \|Y\|_H^2 d[M]^{\tau_p} \right)
\]

is also valid. For any sequence of elementary functions \( \{Y^n\}, \|Y^n(t)\|_H \in L^2(\Omega) \),

\[
\|Y^n - Y\|_{L^2(\Omega; L^2([0,T]; H[d(M^{\tau_p})]))} \to 0
\]

there exists a subsequence, still denoted as \( \{Y^n\} \) of elementary functions for which \( \int_0^t (Y^n, dM^{\tau_p}) \) converges uniformly to \( \int_0^t (Y, dM^{\tau_p}) \) on \([0, T]\) for \( \omega \) off some set of measure zero.
Proof: First of all, why does the limit even exist? From Lemma 64.0.9,
\[ E \left( \int_0^t (Y^n, dM^r) - \int_0^t (Y^m, dM^r) \right)^2 \leq E \left( \int_0^T \|Y^n - Y^m\|_H^2 d[M]^r \right) \]
which converges to 0 as \( n, m \to \infty \) by definition of \( Y \in \mathcal{G} \). This also shows that
the definition is well defined and that the same thing is obtained from any other
sequence converging to \( Y \). \( \left\{ \int_0^t (Y^n, dM^r) \right\} \) is a Cauchy sequence in \( L^2(\Omega) \). Hence
it converges to something \( N(t) \in L^2(\Omega) \). This is a martingale because if \( A \in \mathcal{F}_s, s < t \)
\[ \int_A N(t) dP = \lim_{n \to \infty} \int_A \int_0^t (Y^n, dM^r) dP = \lim_{n \to \infty} \int_A \int_0^s (Y^n, dM^r) dP = \int_A N(s) dP \]
Since \( A \) is arbitrary, this shows that \( E(N(t) | \mathcal{F}_s) = N(s) \). Then
\[ N(t) \equiv \int_0^t (Y^n, dM^r) \]
In fact, this has a continuous version off a set of measure zero.
These are martingales and so actually, by maximal theorems,
\[ P \left( \sup_{t \in [0,T]} \left| \int_0^t (Y^n, dM^r) - \int_0^t (Y^m, dM^r) \right|^2 > \lambda \right) \]
\[ \leq \frac{1}{\lambda} E \left( \left| \int_0^T (Y^n, dM^r) - \int_0^T (Y^m, dM^r) \right|^2 \right) \]
\[ \leq \frac{1}{\lambda} E \left( \int_0^T \|Y^n - Y^m\|_H^2 d[M]^r \right) \]
which converges to 0. Thus there is a subsequence still denoted with index \( k \) such that
\[ P \left( \sup_{t \in [0,T]} \left| \int_0^t (Y^k, dM^r) - \int_0^t (Y^{k+1}, dM^r) \right|^2 > 2^{-k} \right) < 2^{-k} \]
and so there exists a set of measure zero \( N \) such that for \( \omega \notin N \),
\[ \sup_{t \in [0,T]} \left| \int_0^t (Y^k, dM^r) - \int_0^t (Y^{k+1}, dM^r) \right|^2 \leq 2^{-k} \]
for all \( k \) large enough and so for this subsequence, the convergence is uniform.
Hence \( t \to \int_0^t (Y^n, dM^r) \) has a continuous version obtained from the uniform limit
of these.
Finally,
\[
E \left( \left| \int_0^t (Y, dM^\tau) \right|^2 \right) = \lim_{n \to \infty} E \left( \left| \int_0^t (Y^n, dM^\tau) \right|^2 \right) \\
\leq \lim_{n \to \infty} E \left( \int_0^t \|Y^n\|_H^2 d[M]^\tau \right) = E \left( \int_0^t \|Y\|_H^2 d[M]^\tau \right) \]

What is the quadratic variation of the martingale in the above lemma? I am not going to give it exactly but it is easy to give an estimate for it. Recall the following result. It is Theorem 61.6.4.

**Theorem 64.0.13** Let \( H \) be a Hilbert space and suppose \((M, \mathcal{F}_t), t \in [0, T]\) is a uniformly bounded continuous martingale with values in \( H \). Also let \( \{t^n_k\}_{k=1}^{m_n} \) be a sequence of partitions satisfying
\[
\lim_{n \to \infty} \max \left\{ |t^n_i - t^n_{i+1}|, i = 0, \ldots, m_n \right\} = 0, \quad \{t^n_k\}_{k=1}^{m_n} \subseteq \{t^{n+1}_k\}_{k=1}^{m_{n+1}}.
\]
Then
\[
[M](t) = \lim_{n \to \infty} \sum_{k=0}^{m_n-1} |M\left(t \wedge t^n_k\right) - M\left(t \wedge t^n_{k+1}\right)|_H^2
\]
the limit taking place in \( L^2(\Omega) \). In case \( M \) is just a continuous local martingale, the above limit happens in probability.

In the above Lemma, you would find the quadratic variation according to this theorem as follows.
\[
\left[ \int_0^t (Y, dM^\tau) \right](t) = \lim_{n \to \infty} \sum_{k=0}^{m_n-1} \left| \int_{t \wedge t^n_k}^{t \wedge t^n_{k+1}} (Y, dM^\tau) \right|_H^2
\]
where the limit is in probability. Thus
\[
\lim_{n \to \infty} P \left( \left| \int_0^t (Y, dM^\tau) \right| - \sum_{k=0}^{m_n-1} \left| \int_{t \wedge t^n_k}^{t \wedge t^n_{k+1}} (Y, dM^\tau) \right|_H^2 \geq \varepsilon \right) = 0
\]
Then you can obtain from this and the usual appeal to the Borel Cantelli lemma a set of measure zero \( N_t \) and a subsequence still denoted with \( n \) satisfying that for all \( \omega \notin N_t \) and \( n \) large enough,
\[
\left| \int_0^t (Y, dM^\tau) \right| - \sum_{k=0}^{m_n-1} \left| \int_{t \wedge t^n_k}^{t \wedge t^n_{k+1}} (Y, dM^\tau) \right|_H^2 \leq \frac{1}{n}
\]
Hence
\[
\left| \int_0^t (Y, dM^\tau) \right| \leq \frac{1}{n} + \sum_{k=0}^{m_n-1} \left| \int_{t \wedge t^n_k}^{t \wedge t^n_{k+1}} \|Y\|_H^2 d[M]^\tau \right|
\]
Then for that \( t \), you have on taking a limit as \( n \to \infty \),
\[
\left[ \int_0^t (Y, dM^\tau_p) \right] (t) \leq \int_0^t \|Y\|_H^2 \, d[M]^\tau_p
\]
Now take the union of \( N_i \) for \( t \in \mathbb{Q} \cap [0, T] \). Denote this as \( N \). Then if \( \omega \notin N \), the above shows that for such \( t \),
\[
\left[ \int_0^t (Y, dM^\tau_p) \right] (t) \leq \int_0^t \|Y\|_H^2 \, d[M]^\tau_p
\]
But both sides are continuous in \( t \) and so this inequality holds for all \( t \in [0, T] \). Thus the following corollary is obtained.

**Corollary 64.0.14** Let \( M \) be a continuous local martingale and \( \tau_p \) a localizing sequence which makes \( M^\tau_p \) an \( L^2 \) martingale and assume that \( Y \in \mathcal{G} \). Then the quadratic variation of this martingale satisfies
\[
\left[ \int_0^t (Y, dM^\tau_p) \right] (t) \leq \int_0^t \|Y\|_H^2 \, d[M]^\tau_p \leq \int_0^t \|Y\|_H^2 \, d[M]
\]
for \( \omega \) off a set of measure zero.

Does the localization stuff hold for an arbitrary stopping time? Let \( \{t^k_i\} \) denote the \( k^{th} \) partition of a sequence of nested partitions whose maximum length between successive points converges to 0. Let \( \tau \) be a stopping time and let \( \tau_k = t^k_{j+1} \) on \( \tau^{-1}(t^k_j, t^k_{j+1}] \). Then \( \tau_k \) is a stopping time because
\[
[\tau_k \leq t] \in \mathcal{F}_t
\]
Here is why. If \( t \in (t^k_j, t^k_{j+1}] \), then if \( t = t^k_{j+1} \), it would follow that \( \tau_k (\omega) \leq t \) would be the same as saying \( \omega \in [\tau \leq t^k_{j+1}] = [\tau \leq t] \in \mathcal{F}_t \). On the other hand, if \( t < t^k_{j+1} \), then \( [\tau_k \leq t] = [\tau \leq t^k_j] \in \mathcal{F}_{t^k_j} \leq \mathcal{F}_t \) because \( \tau_k \) can only take the values \( t^k_j \).

Let \( Y \) be one of those elementary functions which is in \( \mathcal{G} \), \( \|Y(t)\| M^\ast \in L^2 (\Omega) \).
\[
Y(t) = \sum_{i=0}^{m_k-1} Y_i \chi_{(t^k_i, t^k_{i+1}]}(t)
\]
and consider \( \chi_{[0, \tau_k]}Y \). Here \( Y \) will be always the same for the different partitions. It is just that some of the \( Y_i \) are repeated on smaller and smaller intervals. Does it follow that \( \chi_{[0, \tau_k]}Y \rightarrow \chi_{[0, \tau]}Y \) for each fixed \( \omega \)? This depends only on the indicator function. Let \( \tau (\omega) \in (t^k_j, t^k_{j+1}] \). Fixing \( t \), if \( \chi_{[0, \tau]}(t) = 1 \), then also \( \chi_{[0, \tau_k]}(t) = 1 \) because \( \tau_k \geq \tau \). Therefore, in this case \( \lim_{k \to \infty} \chi_{[0, \tau_k]}(t) = \chi_{[0, \tau]}(t) \). Next suppose
\[ X_{[0, \tau]}(t) = 0 \] so that \( \tau(\omega) < t \). Since the intervals defined by the partition points have lengths which converge to 0, it follows that for all \( k \) large enough, \( \tau_k(\omega) < t \) also and so \( X_{[0, \tau_k]}(t) = 0 \). Therefore,

\[
\lim_{k \to \infty} X_{[0, \tau_k(\omega)]}(t) = X_{[0, \tau(\omega)]}(t).
\]

It follows that \( X_{[0, \tau_k]} Y \to X_{[0, \tau]} Y \). Also it is clear from the dominated convergence theorem,

\[
\|X_{[0, \tau_k]} Y - X_{[0, \tau]} Y\|^2_H \leq 4 \|Y\|_H^2,
\]

that

\[
\lim_{k \to \infty} E \left( \int_0^T \|X_{[0, \tau_k]} Y - X_{[0, \tau]} Y\|^2_H \, d [M^\tau] \right) = 0.
\]

Thus \( X_{[0, \tau]} Y \in G \). By Lemma 64.0.12, there is a subsequence, still denoted as \( X_{[0, \tau_k]} Y \) such that off a set of measure zero,

\[
\int_0^t (X_{[0, \tau_k]} Y, dM^\tau) \to \int_0^t (X_{[0, \tau]} Y, dM^\tau)
\]

uniformly on \([0, T]\). Therefore, from the localization for elementary functions and this uniform convergence,

\[
\int_0^t (X_{[0, \tau]} Y, dM^\tau) = \lim_{n \to \infty} \int_0^t (X_{[0, \tau_n]} Y, dM^\tau) = \lim_{n \to \infty} \int_0^{t \wedge \tau_n} (Y, dM^\tau) = \int_0^{t \wedge \tau} (Y, dM^\tau)
\]

This proves most of the following lemma.

**Lemma 64.0.15** Let \( Y \) be an elementary function. Then if \( \tau \) is any stopping time, then off a set of measure zero,

\[
\int_0^{t \wedge \tau} (Y, dM^\tau) = \int_0^t (X_{[0, \tau]} Y, dM^\tau) = \int_0^t (Y, dM^{t \wedge \tau})
\]

**Proof:** It remains to prove the second equation.

\[
\int_0^{t \wedge \tau} (Y, dM^\tau) = \sum_{i=0}^{m-1} (Y_i, M^{t \wedge \tau_p} (t \wedge t_{i+1} \wedge \tau) - M^{t \wedge \tau_p} (t \wedge t_i \wedge \tau))
\]

\[
= \sum_{i=0}^{m-1} (Y_i, M^{t \wedge \tau_p} (t \wedge t_{i+1}) - M^{t \wedge \tau_p} (t \wedge t_i))
\]

\[
= \int_0^t (Y, dM^{t \wedge \tau_p})
\]

**Lemma 64.0.16** Let \( Y \in G \). Then for any stopping time \( \tau \),

\[
\int_0^{t \wedge \tau} (Y, dM^\tau) = \int_0^t (X_{[0, \tau]} Y, dM^\tau) = \int_0^t (Y, dM^{t \wedge \tau})
\]

for \( \omega \) off some set of measure zero.
CHAPTER 64. THE INTEGRAL \( \int_0^T (Y, dM)_H \)

**Proof:** From Lemma 64.0.14 there exists a sequence of elementary functions \( Y^n \) such that \( t \to \int_0^t (Y^n, dM) \) converges uniformly to \( t \to \int_0^t (Y, dM^\tau) \) on \([0, T]\) for each \( \omega \notin N \), a set of measure zero. Then

\[
\int_0^{t \wedge \tau} (Y, dM^\tau) = \lim_{n \to \infty} \int_0^{t \wedge \tau} (Y^n, dM^\tau) = \lim_{n \to \infty} \int_0^t (X_{[0, \tau]} Y^n, dM^\tau) = \int_0^t (X_{[0, \tau]} Y, dM^\tau)
\]

The last claim needs a little clarification. As shown in the above discussion proving Lemma 64.0.15, while \( X_{[0, \tau]} Y^n \) is no longer obviously an elementary function due to the fact that \( \tau \) has values which are not partition points, it is still the limit of a sequence of elementary functions \( X_{[0, \tau]} Y^n \) and so the integral makes sense. Then from the inequality of Lemma 64.0.12,

\[
E \left( \int_0^T (X_{[0, \tau]} Y^n, dM^\tau) - \int_0^T (X_{[0, \tau]} Y, dM^\tau) \right)^2 \leq E \left( \int_0^T \|Y^n - Y\|_H^2 \ d[M]^\tau \right)
\]

and so by the same Borel Cantelli argument of that lemma, there is a further subsequence for which the convergence is uniform off a set of measure zero as \( n \to \infty \). (Actually, the same subsequence as in the first part of the argument works.) Therefore, the conclusion follows.

What of the second equation? Let \( \{Y^n\} \) be as above where uniform convergence takes place for the stochastic integrals. Then from Lemma 64.0.14

\[
\int_0^t (X_{[0, \tau]} Y^n, dM^\tau) = \int_0^t (Y^n, dM^{\tau \wedge \tau})
\]

Hence

\[
E \left( \int_0^t (X_{[0, \tau]} Y^n, dM^\tau) - \int_0^t (X_{[0, \tau]} Y, dM^\tau) \right)^2 \leq E \left( \int_0^T \|Y^n - Y\|_H^2 \ d[M]^\tau \right)
\]

Now by the usual application of the Borel Canelli lemma, there is a subsequence and a set of measure zero off which \( \int_0^t (Y^n, dM^{\tau \wedge \tau}) \) converges uniformly to \( \int_0^t (Y, dM^{\tau \wedge \tau}) \) on \([0, T]\) and as \( n \to \infty \), and also

\[
\int_0^t (Y^n, dM^{\tau \wedge \tau}) \to \int_0^t (X_{[0, \tau]} Y, dM^\tau)
\]

uniformly on \( t \in [0, T] \). Then from the above,

\[
\int_0^t (Y^n, dM^{\tau \wedge \tau}) \to \int_0^t (X_{[0, \tau]} Y, dM^\tau) = \int_0^{t \wedge \tau} (Y, dM^\tau)
\]

uniformly. Thus \( \int_0^t (Y, dM^{\tau \wedge \tau}) = \int_0^{t \wedge \tau} (Y, dM^\tau) \).
Definition 64.0.17 Let \( \tau_p \) be an increasing sequence of stopping times for which \( \lim_{p \to \infty} \tau_p = \infty \) and such that \( M^{\tau_p} \) is a \( L^2 \) martingale and \( \mathcal{X}_{[0,\tau_p]} Y \in \mathcal{G} \). Then the definition of \( \int_0^t (Y, dM) \) is as follows. For each \( \omega \),

\[
\int_0^t (Y, dM) = \lim_{p \to \infty} \int_0^t (\mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p})
\]

In fact, this is well defined.

Theorem 64.0.18 The above definition is well defined. Also this makes \( \int_0^t (Y, dM) \) a local martingale. In particular,

\[
\int_0^{t \wedge \tau_p} (Y, dM) = \int_0^t (\mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p})
\]

In addition to this, if \( \sigma \) is any stopping time,

\[
\int_0^{t \wedge \sigma} (Y, dM) = \int_0^t (\mathcal{X}_{[0,\sigma]} Y, dM)
\]

In this last formula, \( \mathcal{X}_{[0,\sigma]} \mathcal{X}_{[0,\tau_p]} Y \in \mathcal{G} \). In addition, the following estimate holds for the quadratic variation.

\[
\left[ \int_0^t (Y, dM) \right] (t) \leq \int_0^t \|Y\|^2 d[M]
\]

Proof: Suppose for some \( \omega, t < \tau_p < \tau_q \). Let \( \omega \) be such that both \( \tau_p, \tau_q \) are larger than \( t \). Then for all \( \omega \), and \( \tau \) a stopping time,

\[
\int_0^{t \wedge \tau} (\mathcal{X}_{[0,\tau_q]} Y, dM^{\tau}) = \int_0^t (\mathcal{X}_{[0,\tau_q]} Y, d((M^{\tau_q})^\tau))
\]

In particular, for the given \( \omega \),

\[
\int_0^{t \wedge \tau} (\mathcal{X}_{[0,\tau_q]} Y, d(M^{\tau_q})) = \int_0^t (\mathcal{X}_{[0,\tau_q]} Y, dM^{\tau_q}) = \int_0^{t \wedge \tau_q} (\mathcal{X}_{[0,\tau_q]} Y, dM^{\tau_q})
\]

For the particular \( \omega \), this equals

\[
\int_0^{t \wedge \tau_p} (\mathcal{X}_{[0,\tau_q]} Y, dM^{\tau_q})
\]

Now for all \( \omega \) including the particular one, this equals

\[
\int_0^t (\mathcal{X}_{[0,\tau_q]} Y, d((M^{\tau_q})^{\tau_q})) = \int_0^t (\mathcal{X}_{[0,\tau_q]} Y, dM^{\tau_q})
\]

For the \( \omega \) of interest, this is

\[
\int_0^{t \wedge \tau_p} (\mathcal{X}_{[0,\tau_q]} Y, dM^{\tau_q})
\]
and for all \( \omega \), including the one of interest, the above equals

\[
\int_0^t (\mathcal{X}_{[0,\tau_p]}Y, dM_{\tau_p}) = \int_0^t (\mathcal{X}_{[0,\tau_p]}Y, dM_{\tau_p})
\]

thus for this particular \( \omega \), you get the same for both \( p \) and \( q \). Thus the definition is well defined because for a given \( \omega \), \( \int_0^t (\mathcal{X}_{[0,\tau_p]}Y, dM_{\tau_p}) \) is constant for all \( p \) large enough.

Next consider the claim about this process being a local martingale. Is

\[
\int_0^{t \wedge \tau_p} (Y, dM)
\]

is a martingale? From the definition,

\[
\int_0^{t \wedge \tau_p} (Y, dM) = \lim_{q \to \infty} \int_0^{t \wedge \tau_p} (\mathcal{X}_{[0,\tau_q]}Y, dM_{\tau_q})
\]

\[
= \lim_{q \to \infty} \int_0^t (\mathcal{X}_{[0,\tau_q]}Y, d(M_{\tau_q})_{\tau_p}) = \lim_{q \to \infty} \int_0^t (\mathcal{X}_{[0,\tau_q]}Y, dM_{\tau_p})
\]

\[
= \lim_{q \to \infty} \int_0^t (\mathcal{X}_{[0,\tau_q]}Y_{[0,\tau_p]}Y, dM_{\tau_p}) = \int_0^t (\mathcal{X}_{[0,\tau_p]}Y, dM_{\tau_p})
\] (64.0.6)

which is known to be a martingale since \( \mathcal{X}_{[0,\tau_p]}Y \in \mathcal{G} \). This is what it means to be a local martingale. You localize and get a martingale.

Next consider the claim about an arbitrary stopping time. Why is \( \mathcal{X}_{[0,\sigma]} \hat{Y} \in \mathcal{G} \)? This is part of a more general question. Suppose \( \hat{Y} \in \mathcal{G} \). Then why is \( \mathcal{X}_{[0,\sigma]} \hat{Y} \in \mathcal{G} \). It suffices to show this. Let \( \{Y^n\} \) be the sequence of elementary functions which converge to \( \hat{Y} \) as in the definition. Also let \( \sigma_n \) be the stopping time with discreet values which equals \( t^n_{k+1} \) when \( \sigma \in (t^n_k, t^n_{k+1}) \), \( \{t^n_k\}_{k=0} \) being the partition associated with \( Y^n \). Then, as explained earlier, \( \mathcal{X}_{[0,\sigma_n]}Y^n \) is an acceptable elementary function and also

\[
\left\{ E \left( \int_0^T \| \mathcal{X}_{[0,\sigma]}Y^n - \mathcal{X}_{[0,\sigma]}\hat{Y} \|^2 d[M] \right) \right\}^{1/2} \leq \left\{ E \left( \int_0^T \| \mathcal{X}_{[0,\sigma]}Y^n - \mathcal{X}_{[0,\sigma]}\hat{Y} \|^2 d[M] + \int_0^T \| \mathcal{X}_{[\sigma,\sigma_n]}\hat{Y} \|^2 d[M] \right) \right\}^{1/2}
\]

\[
\leq \left\{ E \left( \int_0^T \| Y^n - \hat{Y} \|^2 d[M] \right) \right\}^{1/2} + \left\{ E \left( \int_0^T \| \mathcal{X}_{[\sigma,\sigma_n]}\hat{Y} \|^2 d[M] \right) \right\}^{1/2}
\]

which converges to 0 from the definition of \( \hat{Y} \in \mathcal{G} \) and the dominated convergence theorem. Thus \( \mathcal{X}_{[0,\sigma]}\hat{Y} \in \mathcal{G} \).
From the above definition, for each $\omega$ off a suitable set of measure zero, from Lemma 64.0.16,

$$\int_0^{t \wedge \sigma} (Y, dM) \equiv \lim_{p \to \infty} \int_0^{t \wedge \sigma} (\mathcal{X}_{[0, \tau_p]}Y, dM^{\tau_p}) = \lim_{p \to \infty} \int_0^t (\mathcal{X}_{[0, \tau_p]} \mathcal{X}_{[0, \sigma]} Y, dM^{\tau_p}) \equiv \int_0^t (\mathcal{X}_{[0, \sigma]} Y, dM)$$

Finally, consider the claim about the quadratic variation. Using 64.0.6,

$$\left[ \left( \int_0^c (Y, dM) \right) \right]^{\tau_p}(t) = \left[ \left( \int_0^c (Y, dM) \right) \right]^{\tau_p}(t) = \left[ \int_0^c (\mathcal{X}_{[0, \tau_p]} Y, dM^{\tau_p}) \right]^{\tau_p}(t) \leq \int_0^t ||\mathcal{X}_{[0, \tau_p]} Y||^2 d[M]^{\tau_p} \leq \int_0^t ||Y||^2 d[M]$$

Now letting $\tau_p \to \infty$,

$$\left[ \left( \int_0^c (Y, dM) \right) \right](t) \leq \int_0^t ||Y||^2 d[M] \blacksquare$$

Next is the case in which $Y$ is continuous in $t$ but not necessarily bounded nor assumed to be in any kind of $L^2$ space either.

**Definition 64.0.19** Let $Y$ be continuous in $t$ and adapted. Let $M$ be a continuous local martingale $M(0) = 0$. Then the definition of a local martingale $\int_0^t (Y, dM)$ is as follows. Let $\tau_p$ be an increasing sequence of stopping times for which $[M]^{\tau_p}$, $||M^{\tau_p}||$, $||\mathcal{X}_{[0, \tau_p]} Y||$ are all bounded by $p$. Then

$$\int_0^t (Y, dM) \equiv \lim_{p \to \infty} \int_0^t (\mathcal{X}_{[0, \tau_p]} Y, dM^{\tau_p})$$

Then it is clear that $\mathcal{X}_{[0, \tau_p]} Y \in \mathcal{G}$. Therefore, the above Theorem yields the following corollary.

**Corollary 64.0.20** The above definition is well defined. Also this makes $\int_0^t (Y, dM)$ a local martingale. In particular,

$$\int_0^{t \wedge \tau_p} (Y, dM) = \int_0^t (\mathcal{X}_{[0, \tau_p]} Y, dM^{\tau_p})$$

In addition to this, if $\sigma$ is any stopping time,

$$\int_0^{t \wedge \sigma} (Y, dM) = \int_0^t (\mathcal{X}_{[0, \sigma]} Y, dM)$$

In this last formula, $\mathcal{X}_{[0, \sigma]} Y$ has the same properties as $Y$, being the pointwise limit on $[0, T]$ of a bounded sequence of elementary functions for each $\omega$. In addition to this, there is an estimate for the quadratic variation

$$\left[ \left( \int_0^c (Y, dM) \right) \right](t) \leq \int_0^t ||Y||^2 d[M]$$
Of course there is no change in anything if $M$ has its values in a Hilbert space $W$ while $Y$ has its values in its dual space. Then one defines $\int_0^t \langle Y, dM \rangle_{W',W}$ by analogy to the above for $Y$ an elementary function, step function which is adapted.

We use the following definition.

**Definition 64.0.21** Let $\tau_p$ be an increasing sequence of stopping times for which $M^{\tau_p}$ is a $L^2$ martingale. If $M$ is already an $L^2$ martingale, simply let $\tau_p \equiv \infty$. Let $G$ denote those functions $Y$ which are adapted and for which there is a sequence of elementary functions $\{Y^n\}$ satisfying $\|Y^n(t)\|_W, M^* \in L^2(\Omega)$ for each $t$ with

$$\lim_{n \to \infty} E \left( \int_0^T \|Y - Y^n\|_W^2, d[M]^{\tau_p} \right) = 0$$

for each $\tau_p$.

Then exactly the same arguments given above yield the following simple generalizations.

**Definition 64.0.22** Let $Y \in G$. Then

$$\int_0^t \langle Y, dM^{\tau_p} \rangle_{W',W} \equiv \lim_{n \to \infty} \int_0^t \langle Y^n, dM^{\tau_p} \rangle_{W',W} \text{ in } L^2(\Omega)$$

**Lemma 64.0.23** The above definition is well defined. Also, $\int_0^t \langle Y, dM^{\tau_p} \rangle_{W',W}$ is a continuous martingale. The inequality

$$E \left( \left\| \int_0^t \langle Y, dM^{\tau_p} \rangle_{W',W} \right\|^2 \right) \leq E \left( \int_0^t \|Y\|^2_W, d[M]^{\tau_p} \right)$$

is also valid. For any sequence of elementary functions $\{Y^n\}, \|Y^n(t)\|_W, M^* \in L^2(\Omega)$,

$$\|Y^n - Y\|_{L^2(\Omega; L^2([0,T];W',d[M^{\tau_p}])}) \to 0$$

there exists a subsequence, still denoted as $\{Y^n\}$ of elementary functions for which $\int_0^t \langle Y^n, dM^{\tau_p} \rangle_{W',W}$ converges uniformly to $\int_0^t \langle Y, dM^{\tau_p} \rangle_{W',W}$ on $[0,T]$ for $\omega$ off some set of measure zero. In addition, the quadratic variation satisfies the following inequality.

$$\left[ \int_0^t \langle Y, dM^{\tau_p} \rangle_{W',W} \right](t) \leq \int_0^t \|Y\|^2_W, d[M]^{\tau_p} \leq \int_0^t \|Y\|^2_W, d[M]$$

As before, you can consider the case where you only know $X_{[0,\tau_p]} Y \in G$. This yields a local martingale as before.
Definition 64.0.24 Let $\tau_p$ be an increasing sequence of stopping times for which $\lim_{p \to \infty} \tau_p = \infty$ and such that $M^{\tau_p}$ is a martingale and $\mathcal{X}_{[0,\tau_p]} Y \in \mathcal{G}$. Then the definition of $\int_0^t \langle Y, dM \rangle_{W', W}$ is as follows. For each $\omega$ off a set of measure zero,

$$\int_0^t \langle Y, dM \rangle_{W', W} \equiv \lim_{p \to \infty} \int_0^t \langle \mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p} \rangle_{W', W}$$

where $\int_0^t \langle \mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p} \rangle_{W', W}$ is a martingale.

In fact, this is well defined.

Theorem 64.0.25 The above definition is well defined. Also this makes $\int_0^t \langle Y, dM \rangle_{W', W}$ a local martingale. In particular,

$$\int_0^{t \wedge \tau_p} \langle Y, dM \rangle_{W', W} = \int_0^t \langle \mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p} \rangle_{W', W}$$

In addition to this, if $\sigma$ is any stopping time,

$$\int_0^{t \wedge \sigma} \langle Y, dM \rangle_{W', W} = \int_0^t \langle \mathcal{X}_{[0,\sigma]} Y, dM \rangle_{W', W}$$

In this last formula, $\mathcal{X}_{[0,\sigma]} \mathcal{X}_{[0,\tau_p]} Y \in \mathcal{G}$. In addition, the following estimate holds for the quadratic variation.

$$\left[ \int_0^t \langle Y, dM \rangle_{W', W} \right](t) \leq \int_0^t \| Y \|_W^2 d[M]$$

Note that from Definition 64.0.24 it is also true that

$$\int_0^t \langle Y, dM \rangle_{W', W} \equiv \lim_{p \to \infty} \int_0^t \langle \mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p} \rangle_{W', W}$$

in probability. In addition, since $\tau_p \to \infty$, it follows that for each $\omega$, eventually $\tau_p > T$. Therefore, $t \to \int_0^t \langle Y, dM \rangle_{W', W}$ is continuous, being equal to $\int_0^t \langle \mathcal{X}_{[0,\tau_p]} Y, dM^{\tau_p} \rangle_{W', W}$ for that $\omega$. 
2348  

CHAPTER 64. THE INTEGRAL $\int_0^T (Y, DM)_H$
Chapter 65

The Easy Ito Formula

First recall where it is shown that for every $\alpha$
\[
E (|W (t) - W (s)|^\alpha) \leq C_\alpha |t - s|^{\alpha/2},
\]
and so by Kolmogorov ˇCentsov continuity theorem
\[
|W (t) - W (s)| \leq C_\gamma |t - s|^{\gamma}
\]
(65.0.1)
for every $\gamma < 1/2$.

65.1 The Situation

The idea is as follows. You have a sufficiently smooth function $F : [0, T] \times H \rightarrow \mathbb{R}$ where $H$ is a separable Hilbert space. You also have the random variable
\[
X (t) = X_0 + \int_0^t \phi (s) ds + \int_0^t \Phi dW
\]
where $\Phi$ is progressively measurable and in $L^2 ([0, T] \times \Omega; L_2 (Q^{1/2} U, H))$ where $Q : U \rightarrow U$ is a positive self adjoint operator. Also assume $X_0$ is $\mathcal{F}_0$ measurable with values in $H$. Recall the descriptive diagram.

Here the Wiener process is in $U_1$ and the filtration with respect to which $\Phi$ is progressively measurable is the usual filtration determined by this Wiener process. Then the Ito formula is about writing the random variable $F (t, X (t))$ in terms of various integrals and derivatives of $F$. 2349
Assume \( F : [0, T] \times H \times \Omega \to \mathbb{R}^1 \) has continuous partial derivatives \( F_t, F_X, \) and \( F_{XX} \) which are uniformly continuous and bounded on bounded subsets of \([0, T] \times H\) independent of \( \omega \in \Omega \). Also assume \( F_{XX} \) is uniformly bounded and that \( F_{XXX} \) exists.

Let \( \phi : [0, T] \times \Omega \to H \) be progressively measurable and Bochner integrable for each \( \omega \). Assume \( \Phi \) is progressively measurable, and is in \( L^2 ([0, T] \times \Omega; L^2 (Q^{1/2}U, H)) \).

Now here is the important lemma which makes the Ito formula possible.

**Lemma 65.2.1** Suppose \( \eta_j \) are real random variables \( E (\eta_j^2) < \infty \), such that \( \eta_k \) is measurable with respect to \( \mathcal{G}_j \) for all \( j > k \) where \( \{\mathcal{G}_k\} \) is increasing. Then

\[
E \left( \sum_{k=0}^{m-1} \eta_k - \sum_{k=0}^{m-1} E (\eta_k|\mathcal{G}_k)^2 \right)
= E \left( \sum_{k=0}^{m-1} \eta_k^2 - E (\eta_k|\mathcal{G}_k)^2 \right)

\]

**Proof:** First consider a mixed term \( i < k \).

\[
E ((\eta_k - E (\eta_k|\mathcal{G}_i)) (\eta_k - E (\eta_k|\mathcal{G}_k)))
\]

This equals

\[
E (\eta_i \eta_k) - E (\eta_i E (\eta_k|\mathcal{G}_k)) - E (\eta_k E (\eta_i|\mathcal{G}_i)) + E (E (\eta_i|\mathcal{G}_i) E (\eta_k|\mathcal{G}_k))
\]

\[
= E (\eta_i \eta_k) - E (E (\eta_i \eta_k|\mathcal{G}_k)) - E (\eta_k E (\eta_i|\mathcal{G}_i)) + E (E (\eta_k E (\eta_i|\mathcal{G}_i)|\mathcal{G}_k))
\]

\[
= E (\eta_i \eta_k) - E (E (\eta_i \eta_k|\mathcal{G}_k)) - E (\eta_k E (\eta_i|\mathcal{G}_i)) + E (\eta_k E (\eta_i|\mathcal{G}_i))
\]

\[
= E (\eta_i \eta_k) - E (\eta_i \eta_k) - E (\eta_k E (\eta_i|\mathcal{G}_i)) + E (\eta_k E (\eta_i|\mathcal{G}_i)) = 0
\]

Thus \( 65.2.2 \) equals

\[
\sum_{k=0}^{m-1} E (\eta_k - E (\eta_k|\mathcal{G}_k))^2
\]
which equals
\[
\sum_{k=0}^{m-1} E(\eta_k^2) - 2E(\eta_k E(\eta_k|G_k)) + E(E(\eta_k|G_k)^2) = \\
\sum_{k=0}^{m-1} E(\eta_k^2) - 2E(\eta_k E(\eta_k|G_k))|G_k) + E(E(\eta_k|G_k)^2) + \\
\sum_{k=0}^{m-1} E(\eta_k^2) - 2E(E(\eta_k|G_k)E(\eta_k|G_k)) + E(E(\eta_k|G_k)^2) = \\
\sum_{k=0}^{m-1} E(\eta_k^2) - E(E(\eta_k|G_k)^2).
\]

65.3 A Special Case

To make it simpler, first consider the situation in which \( \Phi = \Phi_0 \) where \( \Phi_0 \) is \( F_0 \) measurable and has finitely many values in \( \mathcal{L}(U_1, H) \), and \( \phi = \phi_0 \) where \( \phi_0 \) is \( F_0 \) measurable and a simple function with values in \( H \). Thus

\[
X(t) = X_0 + \int_0^t \phi_0 ds + \int_0^t \Phi_0 dW
\]

Now let \( \{t^n_k\}_{k=0}^{m_n} \) denote the \( n^{th} \) partition of \([0, T]\), referred to as \( \mathcal{P}_n \) such that

\[
\lim_{n \to \infty} \left( \max \left\{ |t^n_k - t^n_{k-1}|, k = 0, 1, 2, \cdots, m_n \right\} \right) = \lim_{n \to \infty} ||\mathcal{P}_n|| = 0.
\]

The superscript \( n \) will be suppressed to save notation. Then

\[
F(T, X(T)) - F(0, X_0) = \sum_{k=0}^{m_n-1} (F(t_{k+1}, X(t_{k+1})) - F(t_k, X(t_k))) = \\
\sum_{k=0}^{m_n-1} (F(t_{k+1}, X(t_{k+1})) - F(t_k, X(t_{k+1}))) + \\
\sum_{k=0}^{m_n-1} (F(t_k, X(t_{k+1})) - F(t_k, X(t_k)))
\]

This equals

\[
\sum_{k=0}^{m_n-1} F_t(t_k, X(t_{k+1}))(t_{k+1} - t_k) + o(|t_{k+1} - t_k|) \quad (65.3.3)
\]
\[ + \sum_{k=0}^{m_n-1} F_X(t_k, X(t_k)) (X(t_{k+1}) - X(t_k)) \] (65.3.4)

\[ + \frac{1}{2} \sum_{k=0}^{m_n-1} (F_{XX}(t_k, X(t_k)) (X(t_{k+1}) - X(t_k)) (X(t_{k+1}) - X(t_k)))_H \] (65.3.5)

\[ + \sum_{k=0}^{m_n-1} O(|X(t_{k+1}) - X(t_k)|_H^3) \] (65.3.6)

Recall

\[ X(t) = X_0 + \int_0^t \phi_0 ds + \int_0^t \Phi_0 dW \]

From the properties of the Wiener process in 65.0.1, the term in 65.3.6 converges to 0 as \( n \to \infty \) since these properties of the Wiener process imply \( X \) is Holder continuous with exponent \( 2/5 \).

Now consider the term of 65.3.5. All terms converge to 0 except

\[ \frac{1}{2} \sum_{k=0}^{m_n-1} \left( F_{XX}(t_k, X(t_k)) \int_{t_k}^{t_{k+1}} \Phi_0 dW, \int_{t_k}^{t_{k+1}} \Phi_0 dW \right)_H \] (65.3.7)

Consider one of the terms in 65.3.7. Let \( A \in \mathcal{F}_{t_k} \). By Corollary

\[ \int_A \frac{1}{2} \left( F_{XX}(t_k, X(t_k)) \int_{t_k}^{t_{k+1}} \Phi_0 dW, \int_{t_k}^{t_{k+1}} \Phi_0 dW \right)_H dP \]

\[ = \int_A \frac{1}{2} \left( \int_{t_k}^{t_{k+1}} F_{XX}(t_k, X(t_k)) \Phi_0 dW, \int_{t_k}^{t_{k+1}} \Phi_0 dW \right)_H dP \]

By independence,

\[ = P(A) \frac{1}{2} \int_{t_k}^{t_{k+1}} F_{XX}(t_k, X(t_k)) \Phi_0 dW, \int_{t_k}^{t_{k+1}} \Phi_0 dW \] \( dP \)

By the Ito isometry results presented earlier,

\[ = \int_{t_k}^{t_{k+1}} (F_{XX}(t_k, X(t_k)) \Phi_0, \Phi_0)_{L^2} ds \]

\[ = \int_A \int_{t_k}^{t_{k+1}} (F_{XX}(t_k, X(t_k)) \Phi_0, \Phi_0)_{L^2} ds dP \]

\[ = \int_A \frac{1}{2} (F_{XX}(t_k, X(t_k)) \Phi_0, \Phi_0)_{L^2} (t_{k+1} - t_k) dP \]
A SPECIAL CASE

Since $A \in \mathcal{F}_{t_k}$ was arbitrary,

\[
E \left( \frac{1}{2} \left( F_{XX} (t_k, X (t_k)) \int_{t_k}^{t_{k+1}} \Phi_0 dW \right) \left( \int_{t_k}^{t_{k+1}} \Phi_0 dW \right) | \mathcal{F}_{t_k} \right)
= \frac{1}{2} (F_{XX} (t_k, X (t_k)) \Phi_0, \Phi_0)_{L^2} (t_{k+1} - t_k).
\]

From what was just shown, and Lemma 65.2.1,

\[
E \left( \left[ \frac{1}{2} \sum_{k=0}^{m_n-1} (F_{XX} (t_k, X (t_k)) \Phi_0 \Delta W (t_k), \Phi_0 \Delta W (t_k))_{L^2} \right]^2 \right)
= \frac{1}{4} E \left( \sum_{k=0}^{m_n-1} (F_{XX} (t_k, X (t_k)) \Phi_0 \Delta W (t_k), \Phi_0 \Delta W (t_k))_{L^2}^2 - \sum_{k=0}^{m_n-1} (F_{XX} (t_k, X (t_k)) \Phi_0, \Phi_0)_{L^2}^2 (t_{k+1} - t_k)^2 \right)
\]

Now $F_{XX}$ is bounded and so there exists a constant $M$ independent of $k$ and $n$,

\[
M \geq \|F_{XX} (t_k, X (t_k)) \Phi_0\|, \| (F_{XX} (t_k, X (t_k)) \Phi_0, \Phi_0)_{L^2}\|
\]

Hence the above is dominated by

\[
\leq \frac{1}{4} M^2 \sum_{k=0}^{m_n-1} E \| \Delta W (t_k) \|_{L^1}^4 + \frac{1}{4} M^2 \sum_{k=0}^{m_n-1} (t_{k+1} - t_k)^2
\]

\[
\leq \frac{M^2}{4} \left( \sum_{k=0}^{m_n-1} (C_4 + 1) (t_{k+1} - t_k)^2 \right)
\]

which converges to 0 as $n \to \infty$. Then from (65.3.5) and referring to (65.3.6),

\[
\lim_{n \to \infty} \frac{1}{2} \sum_{k=0}^{m_n-1} (F_{XX} (t_k, X (t_k)) (X (t_{k+1} - X (t_k)) (X (t_{k+1}) - X (t_k)))_{L^2}
= \lim_{n \to \infty} \frac{1}{2} \sum_{k=0}^{m_n-1} \left( F_{XX} (t_k, X (t_k)) \int_{t_k}^{t_{k+1}} \Phi_0 dW \int_{t_k}^{t_{k+1}} \Phi_0 dW \right)_{L^2}
= \lim_{n \to \infty} \frac{1}{2} \sum_{k=0}^{m_n-1} (F_{XX} (t_k, X (t_k)) \Phi_0, \Phi_0)_{L^2} (t_{k+1} - t_k)
\]

(65.3.9)
if this last limit exists in $L^2(\Omega)$. However, since $F_{XX}$ is bounded, this limit certainly exists for a.e. $\omega$ and equals

$$
= \frac{1}{2} \int_0^T (F_{XX}(t,X(t)) \Phi_0 \Phi_0)_{\mathcal{L}_2} dt,
$$

The limit also exists in $L^2(\Omega)$ obviously, since $F_{XX}$ is assumed bounded. Therefore, a subsequence of [65.3.9], still denoted as $n$ must converge for a.e. $\omega$ to the above integral as $n \to \infty$.

Next consider [65.3.4].

$$
\sum_{k=0}^{m_n-1} F_X(t_k, X(t_k)) (X(t_{k+1}) - X(t_k)) = \sum_{k=0}^{m_n-1} F_X(t_k, X(t_k)) \left( \int_{t_k}^{t_{k+1}} \phi_0 ds \right)
$$

$$
+ \sum_{k=0}^{m_n-1} F_X(t_k, X(t_k)) \int_{t_k}^{t_{k+1}} \Phi_0 dW \tag{65.3.10}
$$

Consider the second of these in [65.3.4]. From Corollary [65.3.4], it equals

$$
\int_0^T \left( \sum_{k=0}^{m_n-1} \chi_{(t_k, t_{k+1}]}(t) F_X(t_k, X(t_k)) \right) \Phi_0 dW
$$

which converges as $n \to \infty$ to

$$
\int_0^T F_X(t, X(t)) \Phi_0 dW
$$

because

$$
\lim_{n \to \infty} \left( \sum_{k=0}^{m_n-1} \chi_{(t_k, t_{k+1}]}(t) F_X(t_k, X(t_k)) \right) \Phi_0 = F_X(t, X(t)) \Phi_0
$$

in $L^2([0,T] \times \Omega; \mathcal{L}_2(Q^{1/2}U,H))$. Next consider the first on the right in [65.3.4]. It equals

$$
\sum_{k=0}^{m_n-1} \left( F_X(t_k, X(t_k)) \phi_0 (t_{k+1} - t_k) \right)
$$

and converges to

$$
\int_0^T F_X(t, X(t)) \phi_0 dt.
$$

Finally, it is obviously the case that [65.3.3] converges to

$$
\int_0^T F(t, X(t)) dt
$$
65.4. THE CASE OF ELEMENTARY FUNCTIONS

This has shown

\[ F(T, X(T)) = F(0, X_0) + \int_0^T F_t(t, X(t)) + F_x(t, X(t)) \phi_0 dt + \int_0^T F_X(t, X(t)) \Phi_0 dW + \frac{1}{2} \int_0^T (F_{XX}(t, X(t)) \Phi_0 \Phi_0) \mathcal{L}_2(Q^{1/2}U, H) dt \]

when

\[ X(t) = X_0 + \int_0^t \phi_0 ds + \int_0^t \Phi_0 dW, \]

\( \phi_0, \Phi_0 \mathcal{F}_0 \) measurable as described above. This is the first version of the Ito formula.

65.4 The Case Of Elementary Functions

Of course there was nothing special about the interval \([0, T]\). It follows that for \([a, b] \subseteq [0, T]\), \( \Phi_a \in \mathcal{L}(U_1, U) \) and \( \mathcal{F}_a \) measurable, having finitely many values, \( \phi_a \) a simple function which is \( \mathcal{F}_a \) measurable,

\[ X(t) = X(a) + \int_a^t \phi_a dt + \int_a^t \Phi_a dW \]

\[ F(b, X(b)) = F(a, X(a)) + \int_a^b F_t(t, X(t)) + F_x(t, X(t)) \phi_a dt + \int_a^b F_X(t, X(t)) \Phi_a + \frac{1}{2} \int_a^b (F_{XX}(t, X(t)) \Phi_a \Phi_a) \mathcal{L}_2(Q^{1/2}U, H) dt. \]

Therefore, if \( \Phi \) is any elementary function, being a sum of functions like \( \Phi_a \mathcal{X}_{(a,b]} \), and \( \phi \) a similar sort of elementary function with

\[ X(t) = X_0 + \int_0^t \phi ds + \int_0^t \Phi dW, \]

then

\[ F(T, X(T)) = F(0, X_0) + \int_0^T F_t(t, X(t)) + F_x(t, X(t)) \phi(t) dt + \int_0^T F_X(t, X(t)) \Phi dW + \frac{1}{2} \int_0^T (F_{XX}(t, X(t)) \Phi, \Phi) \mathcal{L}_2(Q^{1/2}U, H) dt \] (65.4.11)

This has proved the following lemma.

**Lemma 65.4.1** Let \( \Phi, \phi \) be elementary functions as described and let

\[ X(t) = X_0 + \int_0^t \phi(s) ds + \int_0^t \Phi dW \]

Then (65.4.11) holds.
65.5 The Integrable Case

Now let \( \Phi \in L^2 \left( [0, T] \times \Omega; \mathcal{L}_2 \left( Q^{1/2} U, H \right) \right) \), \( \phi \in L^1 \left( [0, T] \times \Omega; H \right) \) and be progressively measurable. Let \( \phi \) be as above, and let

\[
X(t) = X_0 + \int_0^t \phi(t) \, dt + \int_0^t \Phi \, dW
\]  

(65.5.12)

Suppose also the additional condition that for some \( M \),

\[
|X(t, \omega)| < M \text{ for all } (t, \omega) \in [0, T] \times N^C, \quad P(N) = 0.
\]

Does it follow that 65.4.11 holds?

There exists a sequence of elementary functions \( \{\Phi_n\} \) converging to \( \Phi \circ J^{-1} \) in \( L^2 \left( [0, T] \times \Omega; \mathcal{L}_2 \left( JQ^{1/2} U, H \right) \right) \). Similarly let \( \{\phi_n\} \) converge to \( \phi \) in \( L^1 \left( [0, T] \times \Omega; H \right) \) where \( \phi_n \) is also an elementary function, \( |\phi_n| \leq |\phi| \) at the mesh points. You could use that theorem about approximating with left and right step functions if desired, Lemma 63.3.1. Let

\[
X_n(t) = X_0 + \int_0^t \phi_n(s) \, ds + \int_0^t \Phi_n \, dW.
\]

Also let \( \tau_n \) be the stopping times

\[
\tau_n \equiv \inf \{ t > 0 : |X_n(t)| > M \}.
\]

Since \( X_n \) is continuous, this is a well defined stopping time. Thus

\[
X_{\tau_n}^n(t) = X_0 + \int_0^{\tau_n} X_{[0, \tau_n]} \phi_n(t) \, dt + \int_0^{\tau_n} X_{[0, \tau_n]} \Phi_n \, dW
\]

and as noted in the discussion of localization for elementary functions, \( X_{[0, \tau_n]} \Phi_n \) is an elementary function.

Claim: \( \lim_{n \to \infty} X_{[0, \tau_n]} = 1 \).

Proof of claim: From maximal estimates as in the construction of the stochastic integral and the Borel Cantelli lemma, it follows that there exists a subsequence still denoted by \( n \) and a set of measure zero \( N \) such that for \( \omega \notin N \),

\[
\int_0^t \Phi_n \, dW \to \int_0^t \Phi \, dW
\]

uniformly on \([0, T]\). Also one can show that off a set of measure zero, there is a subsequence still called \( n \) such that \( \int_0^t \phi_n(s) \, ds \to \int_0^t \phi(s) \, ds \) uniformly on \([0, T]\). Here is why.

\[
E \left( \left| \int_0^t \phi_n(s) \, ds - \int_0^t \phi(s) \, ds \right| \right) \leq \int_{\Omega} \int_0^T |\phi_n - \phi| \, dt \, dP
\]
which is given to converge to 0. Thus

\[
P\left( \max_{t \in [0,T]} \left| \int_0^t \phi_n(s) \, ds - \int_0^t \phi(s) \, ds \right| > \lambda \right) \leq P \left( \int_0^T |\phi_n(s) - \phi(s)| \, ds > \lambda \right)
\]

\[
\leq \frac{1}{\lambda} \int_0^T \int_0^T |\phi_n(s) - \phi(s)| \, ds \, dP
\]

Thus

\[
P\left( \max_{t \in [0,T]} \left| \int_0^t \phi_n(s) \, ds - \int_0^t \phi(s) \, ds \right| > 2^{-k} \right) \leq 2^k \int_0^T \int_0^T |\phi_n(s) - \phi(s)| \, ds \, dP
\]

If \( n > n_k \), the right side is less than \( 2^{-k} \). Use \( \phi_{n_k} \). Then there exists a set of measure zero \( N_2 \) such that for \( \omega \notin N_2 \),

\[
\left| \int_0^t \phi_n(s) \, ds - \int_0^t \phi(s) \, ds \right| \to 0
\]

uniformly. Hence, you can take a couple of subsequences and assert that there exists a subsequence still called \( n \) and a set of measure zero \( N \) such that \( X_n(t) \to X(t) \) uniformly on \([0,T]\) for each \( \omega \notin N \). Since \( |X(t,\omega)| < M \), it follows that for each \( \omega \notin N \), when \( n \) is large enough, \( \tau_n = \infty \) and this proves the claim.

From the claim, it follows that \( X_{[0,\tau_n]}(\Phi_n) \to \Phi \circ J^{-1} \) in \( L^2([0,T] \times \Omega; \mathcal{L}_2(Q^{1/2}U,H)) \) and \( X_{[0,\tau_n]} \Phi_n \to \Phi \) in \( L^1([0,T] \times \Omega; H) \). Thus you can replace \( \Phi_n \) in the above with \( X_{[0,\tau_n]} \Phi_n \) and \( \phi_n \) with \( X_{[0,\tau_n]} \phi_n \). Thus there exists a subsequence, still called \( n \) and a set of measure zero \( N \) such that for \( \omega \notin N \),

\[
\int_0^t X_{[0,\tau_n]} \Phi_n \, dW \to \int_0^t \Phi \, dW
\]

uniformly and

\[
\int_0^t X_{[0,\tau_n]} \phi_n \, ds \to \int_0^t \phi \, ds
\]

uniformly. Hence also \( X_{\tau_n}^n(t) \to X(t) \) uniformly on \([0,T]\) whenever \( \omega \notin N \). Of course \( |X_{\tau_n}^n(t)|_H \) has the advantage of being bounded by \( M \).

From the above,

\[
F(T, X_{\tau_n}^n(T)) = F(0, X_0) + \int_0^T F_1(t, X_{\tau_n}^n(t)) + F_X(t, X_{\tau_n}^n(t)) X_{[0,\tau_n]} \phi_n(t) \, dt
\]

\[
+ \int_0^T F_X(t, X_{\tau_n}^n(t)) \Phi_n \, dW + \frac{1}{2} \int_0^T \left( F_{XX}(t, X_{\tau_n}^n(t)) X_{[0,\tau_n]} \Phi_n, X_{[0,\tau_n]} \Phi_n \right)_{\mathcal{L}_2(Q^{1/2}U,H)} \, dt
\]
Then it is obvious that one can pass to the limit in each of the non stochastic integrals in the above. It is necessary to consider the other one.

From the above claim, \( X_{[0, \tau_n]} \Phi_n \to \Phi \circ J^{-1} \) in \( L^2 ([0, T] \times \Omega; \mathcal{L}_2 (JQ^{1/2} U, H)) \) and also, thanks to the stopping times \( \tau_n \), \( F_X (t, X_{\tau_n} (t)) \) is bounded and converges to \( F_X (t, X (t)) \). Hence the dominated convergence theorem applies, and letting \( n \to \infty \), the following is obtained for a.e. \( \omega \)

\[
F (T, X (T)) = F (0, X_0) + \int_0^T F_t (t, X (t)) dt + \int_0^T F_X (t, X (t)) \phi (t) dW
\]

This is the Ito formula in case that \( \Phi \in L^2 ([0, T] \times \Omega; \mathcal{L}_2 (JQ^{1/2} U, H)) \) and \( |X| \) is bounded above by \( M \).

It is easy to remove this assumption on \( |X| \). Let \( X \) be given in \( 65.5.12 \). Let \( \tau_n \) be the stopping time

\[
\tau_n = \inf \{ t \geq 0 : |X (t, \omega)| > n \}
\]

Then \( 65.5.13 \) holds for the stopped process \( X_{\tau_n} \) and \( \Phi \) and \( \phi \) replaced with \( \Phi X_{[0, \tau_n]} \) and \( \phi X_{[0, \tau_n]} \) respectively. Then let \( n \to \infty \) in this expression, using the continuity of \( X \) and the fact that \( \tau_n \to \infty \) to recover \( 65.5.13 \) without the restriction on \( |X| \).

### 65.6 The General Stochastically Integrable Case

Now suppose that \( \Phi \) is only progressively measurable and stochastically integrable

\[
P \left( \int_0^T \| \Phi \|^2_{\mathcal{L}_2 (Q^{1/2} U, H)} dt < \infty \right) = 1.
\]

Also \( \phi \) is only progressively measurable and Bochner integrable in \( t \). Define a stopping time

\[
\tau (\omega) = \inf \left\{ t \geq 0 : |X (t, \omega)|_H + \int_0^t |\Phi|^2 ds + \int_0^t |\phi| ds > C \right\}
\]

This is just the first hitting time of an open set so it is a stopping time. For \( t \leq \tau \), all of the above quantities must be no larger than \( C \). In particular, \( \mathcal{X}_{[0, \tau]} \Phi \in L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2} U, H)) \). Then

\[
X_{\tau} (t) = X_0 + \int_0^t \mathcal{X}_{[0, \tau]} \phi ds + \int_0^t \mathcal{X}_{[0, \tau]} \Phi dW
\]

and so \( 65.5.13 \) holds with \( X \to X_{\tau} \), \( \Phi \to \mathcal{X}_{[0, \tau]} \Phi \) and \( \phi \to \mathcal{X}_{[0, \tau]} \phi \). Now simply let \( C \to \infty \) and exploit the continuity of \( X \) given by the formula \( 65.5.12 \) to obtain the validity of \( 65.5.13 \) without any reference to the stopping time. Of course arbitrary \( t \) can replace \( T \). This leads to the main result.
Theorem 65.6.1 Let \( \Phi \) be a progressively measurable process having values in \( L_2(Q^{1/2}U, H) \) which is stochastically integrable in \([0,T]\) because

\[
P \left( \int_0^T \|\Phi\|^2_{L_2(Q^{1/2}U, H)}
\right) < \infty = 1
\]

and let \( \phi : [0,T] \times \Omega \rightarrow H \) be progressively measurable and Bochner integrable on \([0,T]\) for a.e. \( \omega \), and let \( X_0 \) be \( \mathcal{F}_0 \) measurable and \( H \) valued. Let

\[
X(t) = X_0 + \int_0^t \phi(s) ds + \int_0^t \Phi dW.
\]

Let \( F : [0,T] \times H \times \Omega \rightarrow \mathbb{R} \) be progressively measurable, have continuous partial derivatives \( F_t, F_X, F_{XX} \) which are uniformly continuous on bounded subsets of \([0,T] \times H \) independent of \( \omega \in \Omega \). Also assume \( F_{XX} \) is bounded and let \( F_{XXX} \) exist and be bounded. Then the following formula holds for a.e. \( \omega \).

\[
F(t, X(t)) = F(0, X_0) + \int_0^t F_X(s, X(s)) \Phi dW + \int_0^t F_t(s, X(s)) + F_X(s, X(s)) \phi(s) ds + \frac{1}{2} \int_0^t (F_{XX}(s, X(s)) \Phi, \Phi)_{L_2(Q^{1/2}U, H)} ds
\]

The dependence of \( F \) on \( \omega \) is suppressed.

That last term is interesting and can be written differently. Let \( \{g_j\} \) be an orthonormal basis for \( Q^{1/2}U \). Then this integrand equals

\[
\sum_{i=1}^L (F_{XX}(s, X(s)) \Phi g_i, \Phi g_i)_{H} = \sum_{i=1}^L (\Phi^* F_{XX}(s, X(s)) \Phi g_i, g_i)_{Q^{1/2}H}
\]

and we write this as

\[
\text{trace}(\Phi^*(s) F_{XX}(s, X(s)) \Phi(s)).
\]

A simple special case is where \( Q = I \) and then \( Q^{1/2}U = U \). Thus it is only required that \( \Phi \) have values in \( L_2(U, H) \).

65.7 Remembering The Formula

I find it almost impossible to remember this formula. Here is a way to do it. Recall that \( |\Delta W|^2 \) is like \( \Delta t \). Therefore, in what follows, neglect all terms which are like \( dW dt, dt^2 \), but keep terms which are \( dW, dt, dW^2 \). Then you start with \( dX = \phi dt + \Phi dW \). Thus for \( F(t, X) \),

\[
dF = F_{t} dt + F_{X} dX + \frac{1}{2} (F_{XX} dX, dX)
\]
other terms from Taylor’s formula are neglected because they involve \(dtdW\) or \(dt^2\). Now the above equals

\[
dF = F_t dt + F_X(\phi dt + \Phi dW) + \frac{1}{2} (F_{XX}\Phi dW, \Phi dW)
\]

Since the \(dW\) occurs twice, in that inner product, you get a \(dt\) out of it. Hence you get

\[
dF = (F_t + F_X \phi) \ dt + \frac{1}{2} (F_{XX} \Phi, \Phi) dt + F_X \Phi dW
\]

Now place an \(\int_0^t\) in front of everything and you have the Ito formula.

### 65.8 An Interesting Formula

Suppose everything is real valued and \(\phi\) is progressively measurable and in

\[
L^2 ([0, T] \times \Omega).
\]

Let

\[
X(t) = \int_0^t \phi dW - \frac{1}{2} \int_0^t \phi^2 ds
\]

and consider \(F(X) = e^X\). Then from the Ito formula,

\[
dF = -\left( e^X \phi \frac{1}{2} \right) dt + \frac{1}{2} e^X \phi^2 dt + e^X \phi dW
\]

and then do an integral

\[
e^{X(t)} - 1 = \int_0^t e^X \phi dW
\]

Thus

\[
e^{X(t)} = 1 + \int_0^t e^{X(s)} \phi dW
\]

That expression on the right is obviously a local martingale and so the expression on the left is also. To see this, you can use a localizing sequence of stopping times which depend on the size of \(X(t)\). This will work fine because \(X(t)\) is continuous.

### 65.9 Some Representation Theorems

In this section is a very interesting representation theorem which comes from the Ito formula. In all of this, \(W\) will be a \(Q\) Wiener process having values in \(\mathbb{R}^n\) for which \(Q = I\). Recall that, letting

\[
\mathcal{G}_t \equiv \sigma (W(s) : s \leq t)
\]
the normal filtration determined by the Wiener process is given by
\[ F_t \equiv \cap_{s > t} G_s \]
where \( G_s \) is the completion of \( G_s \). In this section, the theorems will all feature the smaller filtration \( G_t \), not the filtration \( F_t \). First here are some simple observations which tie this specialized material to what was presented earlier.

When you have an \( G_t \) adapted function in \( L^2(\Omega; \mathbb{R}^n) \), you can consider \( f^T \in L^2([0, T] \times \Omega; L^2(Q^{1/2}\mathbb{R}^n, \mathbb{R})) \) as follows. Letting \( \{g_i\} \) be an orthonormal basis for the subspace \( Q^{1/2}\mathbb{R}^n \) in the norm of \( Q^{1/2}\mathbb{R}^n \),
\[ \|f\|^2_{L^2(Q^{1/2}\mathbb{R}^n, \mathbb{R})} = \sum_i (f^T g_i)^2 < \infty \]
For simplicity, let \( Q = I \). Then you have the simple situation that
\[ \|f^T\|_{L^2(Q^{1/2}\mathbb{R}^n, \mathbb{R})} = \|f\|_{\mathbb{R}^n} \]

In what follows \( W_t \) will be the \( Q \) Wiener process on \( \mathbb{R}^n \) where \( Q = I \). Then the Ito isometry is nothing more than the following lemma.

**Lemma 65.9.1** Let \( f \) be \( F_t \) adapted in the sense that every component is \( F_t \) adapted and \( f \in L^2(\Omega; \mathbb{R}^n) \). Here \( F_t \) is the normal filtration coming from the Wiener process. Then
\[ \left\| \int_0^T f(s)^T dW(s) \right\|_{L^2(\Omega)} = \|f\|_{L^2([0, T], \mathbb{R}^n)}. \]

**Lemma 65.9.2** Let \( X \geq 0 \) and measurable and integrable. Also define a finite measure \( \nu \) on \( \mathcal{B}(\mathbb{R}^p) \) by
\[ \nu(B) \equiv \int \chi_B(Y) dP \]
Then
\[ \int \Omega g(Y) X dP = \int_{\mathbb{R}^p} g(y) d\nu(y) \]
where here \( Y \) is a measurable function with values in \( \mathbb{R}^p \) and \( g \geq 0 \) is Borel measurable. Formally, \( X dP = d\nu \).

**Proof:** First say \( X = \chi_D \) and replace \( g(Y) \) with \( \chi_{Y^{-1}(B)} \). Let
\[ \mu(B) \equiv \int \chi_D \chi_B(Y) dP \]
Then
\[ \int \Omega \chi_D \chi_{Y^{-1}(B)} dP = P(D \cap Y^{-1}(B)) \]
\[
\int_{\mathbb{R}^p} X_B(y) \, d\mu(y) = \mu(B) \equiv \int_{\Omega} X_D X_B(Y) \, dP
\]

\[
= \int_{\Omega} X_D X_{Y^{-1}(B)} \, dP = P(D \cap Y^{-1}(B))
\]

Thus

\[
\int_{\Omega} X_D X_{Y^{-1}(B)} \, dP = \int_{\Omega} X_D X_B(Y) \, dP = \int_{\mathbb{R}^p} X_B(y) \, d\mu(y)
\]

Now let \( s_n(y) \uparrow g(y) \), and let \( s_n(y) = \sum_{k=1}^m c_k X_{B_k}(y) \) where \( B_k \) is a Borel set. Then

\[
\int_{\mathbb{R}^p} s_n(y) \, d\mu(y) = \int_{\mathbb{R}^p} \sum_{k=1}^m c_k X_{B_k}(y) \, d\mu(y) = \sum_{k=1}^m c_k \int_{\mathbb{R}^p} X_{B_k}(y) \, d\mu(y)
\]

\[
= \sum_{k=1}^m c_k P(D \cap Y^{-1}(B_k))
\]

\[
\int_{\Omega} s_n(Y) X_D dP = \sum_{k=1}^m c_k \int_{\Omega} X_D X_{B_k}(Y) \, dP = \sum_{k=1}^m c_k P(D \cap Y^{-1}(B_k))
\]

which is the same thing. Therefore,

\[
\int_{\Omega} s_n(Y) X_D dP = \int_{\mathbb{R}^p} s_n(y) \, d\mu(y)
\]

Now pass to a limit using the monotone convergence theorem to obtain

\[
\int_{\Omega} g(Y) X_D dP = \int_{\mathbb{R}^p} g(y) \, d\mu(y)
\]

Next replace \( X_D \) with \( \sum_{k=1}^m d_k X_{D_k}(\omega) \equiv s_n(\omega) \), a simple function. Then from what was just shown,

\[
\int_{\Omega} g(Y) \sum_{k=1}^m d_k X_{D_k} \, dP = \sum_{k=1}^m d_k \int_{\Omega} g(Y) X_{D_k} \, dP
\]

\[
= \sum_{k=1}^m d_k \int_{\mathbb{R}^p} g(y) \, d\mu_k
\]

where \( \mu_k(B) \equiv \int_{\Omega} X_{D_k} X_B(Y) \, dP \). Now let

\[
\nu_n(B) \equiv \int_{\Omega} \sum_{k=1}^m d_k X_{D_k} X_B(Y) = \int_{\Omega} s_n X_B(Y) \, dP
\]

Then

\[
\nu_n(B) = \sum_{k=1}^m d_k \int_{\Omega} X_{D_k} X_B(Y) \, dP = \sum_{k=1}^m d_k \mu_k(B)
\]
Hence
\[
\int_{\Omega} g(Y) s_n dP = \int_{\Omega} g(Y) \sum_{k=1}^{m} d_k \mathcal{X}_k dP = \sum_{k=1}^{m} d_k \int_{\mathbb{R}^p} g(y) d\mu_k
\]
\[
= \int_{\mathbb{R}^p} g(y) \sum_{k=1}^{m} d_k d\mu_k = \int_{\mathbb{R}^p} g(y) d\nu_n
\]
Then let \( s_n(\omega) \uparrow X(\omega) \). Clearly \( \nu_n \ll \nu \) and so by the Radon Nikodym theorem \( d\nu_n = h_n d\nu \). Then by the monotone convergence theorem, for any \( B \) Borel in \( \mathbb{R}^p \),
\[
\int_B h_n d\nu \equiv \nu_n(B) \equiv \int_{\Omega} s_n(\omega) \chi_B(Y(\omega)) dP \uparrow \int_{\Omega} X(\omega) \chi_B(Y(\omega)) dP \equiv \nu(B)
\]
Thus for each \( B \) Borel, \( 0 \leq h_n \leq 1 \) and
\[
\int_B h_n d\nu \to \nu(B)
\]
and so \( h_n \uparrow 1 \) a.e. Thus, from the above,
\[
\int_{\Omega} g(Y) s_n dP = \int_{\mathbb{R}^p} g(y) d\nu_n = \int_{\mathbb{R}^p} g(y) h_n(y) d\nu
\]
It follows from the monotone convergence theorem that one can pass to a limit in the above and obtain
\[
\int_{\Omega} g(Y) X dP = \int_{\mathbb{R}^p} g(y) d\nu \quad \blacksquare
\]
Note that the same conclusion will hold if the functions are suitably integrable without any restriction on the sign. In particular, this will hold if \( g(y) \) is bounded. One just considers positive and negative parts of real and imaginary parts of \( g \) and applies the above lemma.

Let
\[
\mathcal{G}_t \equiv \sigma(W(s) : s \leq t)
\]
thus the normal filtration for the Wiener process and the Ito integral and so forth is
\[
\mathcal{F}_t = \bigcap_{s \geq t} \mathcal{G}_s
\]

**Lemma 65.9.3** Let \( h \) be a deterministic step function of the form
\[
h = \sum_{i=0}^{m-1} a_i \mathcal{X}_{[t_i, t_{i+1})}, \quad t_m = t
\]
Then for \( h \) of this form, linear combinations of functions of the form
\[
\exp \left( \int_0^t h^T dW - \frac{1}{2} \int_0^t h \cdot h d\tau \right)
\]
are dense in \( L^2(\Omega, \mathcal{G}_t, P) \) for each \( t \).
CHAPTER 65. THE EASY ITO FORMULA

Proof: I will show in the process of the proof that functions of the form \( g \in L^2(\Omega, G_t, P) \) are such that

\[
\int_\Omega g(\omega) \exp \left( \int_0^t h^T dW - \frac{1}{2} \int_0^t h \cdot h d\tau \right) dP = \exp \left( -\frac{1}{2} \int_0^t h \cdot h d\tau \right) \int_\Omega g(\omega) \exp \left( \int_0^t h^T dW \right) dP = 0
\]

for all such \( h \). It is required to show that whenever this happens for all such functions \( \exp \left( \int_0^t h^T dW - \frac{1}{2} \int_0^t h \cdot h d\tau \right) \) then \( g = 0 \).

Letting \( h \) be given as above, \( \int_0^t h^T dW \)

\[
= \sum_{i=0}^{m-1} a_i^T (W(t_{i+1}) - W(t_i)) \quad (65.9.15)
\]

\[
= \sum_{i=0}^{m-1} a_i^T W(t_{i+1}) - \sum_{i=0}^{m-1} a_i^T W(t_i)
\]

\[
= \sum_{i=1}^{m} (a_{i-1}^T - a_i^T) W(t_i) + a_0^T W(t_0) + a_{m-1}^T W(t_n). \quad (65.9.16)
\]

Also \( \exp \left( \int_0^t h^T dW \right) \) is in \( L^2(\Omega, P) \). To see this recall the \( W(t_{i+1}) - W(t_i) \) are independent and the density of \( W(t_{i+1}) - W(t_i) \) is

\[
C(n, \Delta t_i) \exp \left( -\frac{1}{2} \frac{|x|^2}{\Delta t_i} \right), \quad \Delta t_i = t_{i+1} - t_i,
\]

so

\[
\int_\Omega \left( \exp \left( \int_0^t h^T dW \right) \right)^2 dP = \int_\Omega \exp \left( 2 \int_0^t h^T dW \right) dP
\]

\[
= \int_\Omega \exp \left( \sum_{i=0}^{m-1} 2a_i^T (W(t_{i+1}) - W(t_i)) \right) dP
\]

\[
= \prod_{i=0}^{m-1} \exp \left( 2a_i^T (W(t_{i+1}) - W(t_i)) \right) dP
\]

\[
= \prod_{i=0}^{m-1} \int_\Omega \exp \left( 2a_i^T (W(t_{i+1}) - W(t_i)) \right) dP
\]

\[
= \prod_{i=0}^{m-1} \int_{\mathbb{R}^n} C(n, \Delta t_i) \exp \left( -\frac{1}{2} \frac{|x|^2}{\Delta t_i} \right) dx < \infty
\]
Choosing the \( a_i \) appropriately in \( 65.9.16 \), the formula in \( 65.9.16 \) is of the form
\[
\sum_{i=0}^{m} y_i^T W_{t_i}
\]
where \( y_i \) is an arbitrary vector in \( \mathbb{R}^n \). It follows that for all choices of \( y_j \in \mathbb{R}^n \),
\[
\int_{\Omega} g(\omega) \exp \left( \sum_{j=0}^{m} y_j^T W_{t_j}(\omega) \right) dP = 0.
\]
Now the mapping
\[
y = (y_0, \ldots, y_m) \rightarrow \int_{\Omega} g(\omega) \exp \left( \sum_{j=0}^{m} y_j^T W_{t_j}(\omega) \right) dP
\]
is analytic on \( \mathbb{C}^{(m+1)n} \) and equals zero on \( \mathbb{R}^{(m+1)n} \) so from standard complex variable theory, this analytic function must equal zero on \( \mathbb{C}^{(m+1)n} \), not just on \( \mathbb{R}^{(m+1)n} \). In particular, for all \( y = (y_0, \cdots, y_m) \in \mathbb{R}^{n(m+1)} \),
\[
\int_{\Omega} g(\omega) \exp \left( \sum_{j=0}^{m} i y_j^T W_{t_j}(\omega) \right) dP = 0. \tag{65.9.17}
\]
This left side equals
\[
\int_{\Omega} g_+(\omega) \exp \left( \sum_{j=0}^{m} i y_j^T W_{t_j}(\omega) \right) dP - \int_{\Omega} g_-(\omega) \exp \left( \sum_{j=0}^{m} i y_j^T W_{t_j}(\omega) \right) dP
\]
where \( g_+ \) and \( g_- \) are the positive and negative parts of \( g \). By the Lemma \( 65.9.2 \) and the observation at the end, this equals
\[
\int_{\mathbb{R}^{nm}} \exp \left( \sum_{j=0}^{m} i y_j^T x_j \right) \nu_+ - \int_{\mathbb{R}^{nm}} \exp \left( \sum_{j=0}^{m} i y_j^T x_j \right) \nu_-
\]
where \( \nu_+(B) \equiv \int_{\Omega} g_+(\omega) \chi_B(\mathbf{W}_{t_1}(\omega), \cdots, \mathbf{W}_{t_m}(\omega)) dP \) and \( \nu_- \) is defined similarly. Then letting \( \nu \) be the measure \( \nu_+ - \nu_- \), it follows that
\[
0 = \int_{\mathbb{R}^{nm}} \exp \left( \sum_{j=0}^{m} i y_j^T x_j \right) d\nu(y)
\]
and this just says that the inverse Fourier transform of \( \nu \) is 0. It follows that \( \nu = 0 \). Thus
\[
\int_{\Omega} g(\omega) \chi_\mathbf{W}(\mathbf{W}_{t_1}(\omega), \cdots, \mathbf{W}_{t_m}(\omega)) dP
= \int_{\Omega} g(\omega) \chi_{\mathbf{W}_{m}(\mathbf{B}^{-1})}(\omega) dP = 0
\]
for every $B$ Borel in $\mathbb{R}^m$ where 

$$W_m(\omega) \equiv (W_{t_1}(\omega), \ldots, W_{t_m}(\omega))$$

Let $K$ be the $\pi$ system defined as $W_m^{-1}(B)$ for $B$ of the form $\prod_{i=1}^m U_i$ where $U_i$ is open in $\mathbb{R}^n$, this for some $m$ a positive integer. This is indeed a $\pi$ system because it includes $W_m^{-1}(\mathbb{R}^n) = \Omega$ and the empty set. Also it is closed with respect to intersections because, in the situation where each $s_i$ is larger than every $t_i$,

$$(W_{t_1}, \ldots, W_{t_m})^{-1}\left(\prod_{i=1}^{m_1} U_i\right) \cap (W_{s_1}, \ldots, W_{s_{m_2}})^{-1}\left(\prod_{i=1}^{m_2} V_i\right) =$$

$$(W_{t_1}, \ldots, W_{t_m}, W_{s_1}, \ldots, W_{s_{m_2}})^{-1}\left(\prod_{i=1}^{m_1} U_i \times \prod_{k=1}^{m_2} \mathbb{R}^n\right) \cap (W_{t_1}, \ldots, W_{t_m}, W_{s_1}, \ldots, W_{s_{m_2}})^{-1}\left(\prod_{i=1}^{m_1} \mathbb{R}^n \times \prod_{k=1}^{m_2} V_i\right)$$

$$= (W_{t_1}, \ldots, W_{t_m}, W_{s_1}, \ldots, W_{s_{m_2}})^{-1}\left(\prod_{i=1}^{m_1} U_i \times \prod_{k=1}^{m_2} V_k\right)$$

In general, you would just make the obvious modification where you insert a copy of $\mathbb{R}^n$ in the appropriate position after rearranging so that the indices are increasing.

It was just shown that $K \subseteq G$ where

$$G \equiv \left\{ U \in G_t : \int_{\Omega} g X_U dP = 0 \right\}.$$

Now it is clear that $G$ is closed with respect to countable disjoint unions and complements. The case of complements goes as follows. $\Omega \in K$ and so if $U \in G$,

$$\int_{\Omega} g X_U dP + \int_{\Omega} g X_{U^c} dP = \int_{\Omega} g dP$$

The last on the left and the integral on the right are both 0 so it follows that $\int_{\Omega} g X_{U^c} dP = 0$ also. It follows from Dynkin’s lemma that $G \supseteq \sigma(K)$. Now $\sigma(K)$ is $\sigma(W(u) : u \leq t) \equiv G_t$. Hence, $G = G_t$ and so $g$ is in $L^2(\Omega, G_t)$ and for every $U \in G_t$,

$$\int_{\Omega} g X_U dP = 0$$

which requires $g = 0$. Thus functions of the above form are indeed dense in $L^2(\Omega, G_t)$. 

Note that this involves $g$ being $G_t$ measurable, not $\mathcal{F}_t$ measurable. It is not clear to me whether it suffices to assume only that $g$ is $\mathcal{F}_t$ measurable. If true, this above has not proved it. The problem is the argument at the end using Dynkin’s lemma to conclude that $g = 0$. 


Why such a funny lemma? It is because of the following computation which depends on Itô’s formula. Let
\[ X = \int_0^t h^T dW - \frac{1}{2} \int_0^t h \cdot h \, dt \]
and \( g(x) = e^x \) and consider \( g(X) = Y \). Recall the Ito formula. Formally,
\[
dY = g'(X) dX + \frac{1}{2} g''(X) (dX)^2
\]
\[
dY = g(X) \left( h^T dW - \frac{1}{2} |h|^2 dt \right)
+ \frac{1}{2} g(X) \left( h^T dW - \frac{1}{2} |h|^2 dt \right) \left( h^T dW - \frac{1}{2} |h|^2 dt \right)
= Y \left( h^T dW - \frac{1}{2} |h|^2 dt \right) + \frac{1}{2} Y \left[ (h^T dW) (h^T dW) - h^T dW |h|^2 dt + \frac{1}{4} |h|^2 dt^2 \right]
\]
Then neglecting the terms of the form \( dW dt, dt^2 \) and so forth,
\[
dY = Y h^T dW - \frac{1}{2} Y |h|^2 dt + \frac{1}{2} Y (h^T dW) (h^T dW)
\]
Now the \( dW \) occurs twice in the last term so it leads to a \( dt \) and you get
\[
dY = Y h^T dW - \frac{1}{2} Y |h|^2 dt + \frac{1}{2} \left( Y h^T, h^T \right) dt
\]
\[
dY = Y h^T dW - \frac{1}{2} Y |h|^2 dt + \frac{1}{2} Y |h|^2 dt
\]
\[
dY = Y h^T dW
\]
Note that \( \|h^T\|_{L^2(\mathbb{R}^n, \mathbb{R})} = \sum_{k=1}^n (h^T e_k)^2 = |h|^2_{\mathbb{R}^n} \). Place an \( \int_0^t \) in place of both sides to obtain
\[
Y(t) - Y(0) = \int_0^t Y h^T dW
\]
\[
Y(t) = 1 + \int_0^t Y h^T dW \tag{65.9.18}
\]
Now here is the interesting part of this formula.
\[
E \left( \int_0^t Y h^T dW \right) = 0
\]
because the stochastic integral is a martingale and equals 0 at \( t = 0 \).
\[
E \left( \int_0^t Y h^T dW \right) = E \left( E \left( \int_0^t Y h^T dW | \mathcal{F}_0 \right) \right) = 0
\]
Thus

\[ E (Y(t)) = 1 \]

and for \( Y \) one obtains

\[
Y(t) = E(Y(t)) + \int_0^t Y^T dW \\
\equiv E(Y(t)) + \int_0^t f^T dW
\]

where \( f^T \) is adapted and square integrable. It is just \( Y^T \) where \( h \) does not depend on \( \omega \) and \( Y \) is a function of an adapted function.

Does such a function \( f \) exist for all \( F \in L^2(\Omega, G_t, P) \)? The answer is yes and this is the content of the next theorem which is called the Itô representation theorem.

**Theorem 65.9.4** Let \( F \in L^2(\Omega, G_t, P) \). Then there exists a unique \( G_t \) adapted \( f \in L^2(\Omega \times [0,t] : \mathbb{R}^n) \) such that

\[ F = E(F) + \int_0^t f(s,\omega)^T dW. \]

**Proof:** By Lemma 65.9.3, the span of functions of the form

\[
\exp \left( \int_0^t h^T dW - \frac{1}{2} \int_0^t h \cdot h dt \right)
\]

where \( h \) is a vector valued deterministic step function of the sort described in this lemma, are dense in \( L^2(\Omega, G_t, P) \). Given \( F \in L^2(\Omega, G_t, P) \), \( \{G_k\}_{k=1}^\infty \) be functions in the subspace of linear combinations of the above functions which converge to \( F \) in \( L^2(\Omega, G_t, P) \). For each of these functions there exists \( f_k \) an adapted step function such that

\[ G_k = E(G_k) + \int_0^t f_k(s,\omega)^T dW. \]

Then from the Itô isometry, and the observation that \( E(G_k - G_l)^2 \to 0 \) as \( k, l \to \infty \) by the above definition of \( G_k \) in which the \( G_k \) converge to \( F \) in \( L^2(\Omega) \),

\[
0 = \lim_{k,l \to \infty} E ((G_k - G_l)^2) \\
= \lim_{k,l \to \infty} E \left( \left( E(G_k) + \int_0^t f_k(s,\omega)^T dW - \left( E(G_l) + \int_0^t f_l(s,\omega)^T dW \right) \right)^2 \right) \\
= \lim_{k,l \to \infty} \left\{ E(G_k - G_l)^2 + 2E(G_k - G_l) \int_0^t (f_k - f_l)^T dW dP \right. \\
+ \left. \int_0^t \left( \int_0^t (f_k - f_l)^T dW \right)^2 dP \right\}
\]
65.9. SOME REPRESENTATION THEOREMS

\[ \lim_{k,l \to \infty} \left\{ E(G_k - G_l)^2 + \int_{\Omega} \left( \int_0^t (f_k - f_l)^T \, dW \right)^2 \, dP \right\} = \]

\[ \lim_{k,l \to \infty} \int_{\Omega} \left( \int_0^t (f_k - f_l)^T \, dW \right)^2 \, dP = \lim_{k,l \to \infty} ||f_k - f_l||_{L^2(\Omega \times [0,T];\mathbb{R}^n)} \]  

(65.9.19)

Going from the third to the fourth equations, is justified because

\[ \int_{\Omega} \int_0^t (f_k - f_l)^T \, dW \, dP = 0 \]

thanks to the fact that the Ito integral is a martingale which equals 0 at \( t = 0 \).

This shows \( \{f_k\}_{k=1}^{\infty} \) is a Cauchy sequence in \( L^2(\Omega \times [0,t];\mathbb{R}^n,P) \), where \( P \) denotes the progressively measurable sets. It follows there exists a subsequence and \( f \in L^2(\Omega \times [0,t];\mathbb{R}^n) \) such that \( f_k \) converges to \( f \) in \( L^2(\Omega \times [0,t];\mathbb{R}^n,P) \) with \( f \) being progressively measurable. Then by the Itô isometry and the equation

\[ G_k = E(G_k) + \int_0^t f_k(s,\omega)^T \, dW \]

you can pass to the limit as \( k \to \infty \) and obtain

\[ F = E(F) + \int_0^t f(s,\omega)^T \, dW \]

Now \( E(G_k) \to E(F) \). Consider the stochastic integrals. By the maximal estimate, Theorem 60.9.4, and the Itô isometry,

\[ P \left( \sup_{s \in [0,t]} \left| \int_0^s f_k(\cdot,\omega)^T \, dW - \int_0^s f(\cdot,\omega)^T \, dW \right| > \delta \right) \]

\[ < E \left( \left| \int_0^t f_k(\cdot,\omega)^T \, dW - \int_0^t f(\cdot,\omega)^T \, dW \right|^2 \right) \]

\[ = E \left( \int_0^t ||f_k - f||_{\mathbb{R}^n}^2 \, ds \right) \]

From the above convergence result and an application of the Borel Cantelli lemma, there is a set of measure zero \( N \) and a subsequence, still denoted as \( f_k \) such that for \( \omega \notin N \), the convergence of the stochastic integrals for this subsequence is uniform. Thus for \( \omega \notin N \),

\[ F = E(F) + \int_0^t f(s,\omega)^T \, dW \]

This proves the existence part of this theorem.
It remains to consider the uniqueness. Suppose then that
\[ F = E(F) + \int_0^T f(t,\omega)^T dW = E(F) + \int_0^T f_1(t,\omega)^T dW. \]
Then
\[ \int_0^T f(t,\omega)^T dW = \int_0^T f_1(t,\omega)^T dW \]
and so
\[ \int_0^T \left( f(t,\omega)^T - f_1(t,\omega)^T \right) dW = 0 \]
and by the Itô isometry,
\[ 0 = \int_0^T \left( f(t,\omega)^T - f_1(t,\omega)^T \right) dW = \left\| f - f_1 \right\|_{L^2(\Omega \times [0,T];\mathbb{R}^n)} \]
which proves uniqueness.

With the above major result, here is another interesting representation theorem. Recall that if you have an \( \mathcal{F}_t \) adapted function \( f \) and \( f \in L^2(\Omega \times [0,T];\mathbb{R}^n) \), then \( \int_0^t f^T dW \) is a martingale. The next theorem is sort of a converse. It starts with a \( \mathcal{G}_t \) martingale and represents it as an Itô integral. In this theorem, \( \mathcal{G}_t \) continues to be the filtration determined by \( n \) dimensional Wiener process.

**Theorem 65.9.5** Let \( M \) be an \( \mathcal{G}_t \) martingale and suppose \( M(t) \in L^2(\Omega) \) for all \( t \geq 0 \). Then there exists a unique stochastic process, \( g(s,\omega) \) such that \( g \) is \( \mathcal{G}_t \) adapted and in \( L^2(\Omega \times [0,t]) \) for each \( t > 0 \), and for all \( t \geq 0 \),
\[ M(t) = E(M(0)) + \int_0^t g(s,\omega) dW. \]

**Proof:** First suppose \( f \) is an adapted function of the sort that \( g \) is. Then the following claim is the first step in the proof.

**Claim:** Let \( t_1 < t_2 \). Then
\[ E\left( \int_{t_1}^{t_2} f^T dW | \mathcal{G}_{t_1} \right) = 0 \]

**Proof of claim:** This follows from the fact that the Ito integral is a martingale adapted to \( \mathcal{G}_t \). Hence the above reduces to
\[ E\left( \int_0^{t_2} f^T dW - \int_0^{t_1} f^T dW | \mathcal{G}_{t_1} \right) = \int_0^{t_1} f^T dW - \int_0^{t_1} f^T dW = 0. \]

Now to prove the theorem, it follows from Theorem 65.9.4 and the assumption that \( M \) is a martingale that for \( t > 0 \) there exists \( f^t \in L^2(\Omega \times [0,T];\mathbb{R}^n) \) such that
\[ M(t) = E(M(0)) + \int_0^t f^t(s,\omega)^T dW \]
\[ = E(M(0)) + \int_0^t f^t(s,\omega)^T dW. \]
Now let \( t_1 < t_2 \). Then since \( M \) is a martingale and so is the Ito integral,

\[
M ( t_1 ) = E ( M ( t_2 ) | \mathcal{G}_{t_1} ) = E \left( E ( M ( 0 )) + \int_{0}^{t_2} f^{t_2} ( s, \cdot )^T dW | \mathcal{G}_{t_1} \right)
\]

\[
= E ( M ( 0 )) + E \left( \int_{0}^{t_1} f^{t_1} ( s, \cdot )^T dW \right)
\]

Thus

\[
M ( t_1 ) = E ( M ( 0 )) + \int_{0}^{t_1} f^{t_1} ( s, \cdot )^T dW = E ( M ( 0 )) + \int_{0}^{t_1} f^{t_1} ( s, \cdot )^T dW
\]

and so

\[
0 = \int_{0}^{t_1} f^{t_1} ( s, \cdot )^T dW - \int_{0}^{t_1} f^{t_2} ( s, \cdot )^T dW
\]

and so by the Itô isometry,

\[
\| f^{t_1} - f^{t_2} \|_{L^2 ( \Omega \times [0, t_1]; \mathbb{R}^n )} = 0.
\]

Letting \( N \in \mathbb{N} \), it follows that

\[
M ( t ) = E ( M ( 0 )) + \int_{0}^{t} f^N ( s, \cdot )^T dW
\]

for all \( t \leq N \). Let \( g = f^N \) for \( t \in [0, N] \). Then aside from a set of measure zero, this is well defined and for all \( t \geq 0 \)

\[
M ( t ) = E ( M ( 0 )) + \int_{0}^{t} g ( s, \cdot )^T dW
\]

Surely this is an incredible theorem. Note that it implies all the martingales adapted to \( \mathcal{G}_t \) which are in \( L^2 \) for each \( t \) must be continuous a.e. and are obtained from an Ito integral. Also, any such martingale satisfies \( M ( 0 ) = E ( M ( 0 )) \). Isn’t that amazing? Also note that this featured \( \mathbb{R}^n \) as where \( W \) has its values and \( n \) was arbitrary. One could have \( n = 1 \) if desired.

The above theorems can also be obtained from another approach. It involves showing that random variables of the form

\[
\phi ( W ( t_1 ), \cdots , W ( t_k ) )
\]

are dense in \( L^2 ( \Omega , \mathcal{G}_T ) \). This theorem is interesting for its own sake and it involves interesting results discussed earlier. Recall the Doob Dynkin lemma, Lemma 57.3.6 on Page 1913 which is listed here.

**Lemma 65.9.6** Suppose \( X, Y_1, Y_2, \cdots , Y_k \) are random vectors, \( X \) having values in \( \mathbb{R}^n \) and \( Y_j \) having values in \( \mathbb{R}^{p_j} \) and

\[
X, Y_j \in L^1 ( \Omega ).
\]
Suppose \( X \) is \( \sigma(Y_1, \cdots, Y_k) \) measurable. Thus

\[
\{X^{-1}(E) : E \text{ Borel}\} \subseteq \left\{(Y_1, \cdots, Y_k)^{-1}(F) : F \text{ is Borel in } \prod_{j=1}^k \mathbb{R}^{p_j}\right\}
\]

Then there exists a Borel function, \( g : \prod_{j=1}^k \mathbb{R}^{p_j} \to \mathbb{R}^n \) such that

\[ X = g(Y). \]

Recall also the submartingale convergence theorem.

**Theorem 65.9.7 (submartingale convergence theorem)** Let

\[
\{(X_i, S_i)\}_{i=1}^\infty
\]

be a submartingale with \( K \equiv \sup E(|X_n|) < \infty \). Then there exists a random variable \( X \), such that \( E(|X|) \leq K \) and

\[
\lim_{n \to \infty} X_n(\omega) = X(\omega) \text{ a.e.}
\]

Recall

\[ G_t \equiv \sigma(W(u) - W(r) : 0 \leq r < u \leq t) \]

It suffices to consider only \( t \) and \( u, r \) in a countable dense subset of \( \mathbb{R} \) denoted as \( D \).

This follows from continuity of the Wiener process. To see this, let \( 0 \leq r < u \leq t \) \( U \) be open and \( U_n \uparrow U \) where each \( U_n \) is open and \( \overline{U_n} \subseteq U_{n+1}, \cup_n U_n = U \). Then letting \( u_n \uparrow u \) and \( r_n \uparrow r, u_n r_n \) being in the countable dense set,

\[
(W(u) - W(r))^{-1}(U) \subseteq \bigcup_{k=1}^\infty \bigcap_{j \geq k} (W(u_j) - W(r_j))^{-1}(U_n)
\]

and so

\[
(W(u) - W(r))^{-1}(U) = \bigcup_n (W(u) - W(r))^{-1}(U_n)
\]

Now the set in the middle which has two countable unions and a countable intersection is in

\[ \sigma(W(u) - W(r) : 0 \leq r < u \leq t, r, u \in D) \]

Thus in particular, one would get the same filtration from

\[ G_t = \sigma(W(u) - W(r) : 0 \leq r < u \leq t, r, u \in D) \]

Since \( W(0) = 0 \), this is the same as

\[ G_t = \sigma(W(u) : 0 \leq u \leq t, u \in D) \]
Lemma 65.9.8  Random variables of the form  
\[ \phi(W(t_1), \cdots, W(t_k)), \phi \in C_c^\infty(\mathbb{R}^k) \]
are dense in \( L^2(\Omega, \mathcal{G}_T, P) \) where \( t_1 < t_2 < \cdots < t_k \) is a finite increasing sequence of \((\mathbb{Q} \cup \{T\}) \cap [0, T]. \)

**Proof:** Let \( g \in L^2(\Omega, \mathcal{G}_T, P) \). Also let \( \{t_j\}_{j=1}^\infty \) be the points of \((\mathbb{Q} \cup \{T\}) \cap [0, T]\).

Let \( \mathcal{G}_m = \sigma(W(t_k): k \leq m) \)

Thus the \( \mathcal{G}_m \) are increasing but each is generated by finitely many \( W(t_k) \). Also as explained above, 
\[ \mathcal{G}_T = \sigma(W(u): 0 \leq u \leq T, u \in (\mathbb{Q} \cup \{T\}) \cap [0, T]) \]
\[ = \sigma(\mathcal{G}_m, m < \infty). \]

Now consider the martingale, 
\[ \{E(g_M|\mathcal{G}_m)\}_{m=1}^\infty \]
where here 
\[ g_M(\omega) = \begin{cases} 
  g(\omega) & \text{if } g(\omega) \in [-M, M] \\
  M & \text{if } g(\omega) > M \\
  -M & \text{if } g(\omega) < -M 
\end{cases} \]
and \( M \) is chosen large enough that 
\[ ||g - g_M||_{L^2(\Omega)} < \varepsilon/4. \]

(65.9.20)

Now the terms of this martingale are uniformly bounded by \( M \) because 
\[ |E(g_M|\mathcal{G}_m)| \leq E(||g_M|||\mathcal{G}_m) \leq E(M|\mathcal{G}_m) = M. \]

It follows the martingale is certainly bounded in \( L^1 \) and so the martingale convergence theorem stated above can be applied, and so there exists \( f \) measurable in \( \sigma(\mathcal{G}_m, m < \infty) \) such that \( \lim_{m \to \infty} E(g_M|\mathcal{G}_m)(\omega) = f(\omega) \) a.e. Also \( |f(\omega)| \leq M \) a.e. Since all functions are bounded, it follows that this convergence is also in \( L^2(\Omega) \).

Now letting \( A \in \sigma(\mathcal{G}_m, m < \infty) \), it follows from the dominated convergence theorem that 
\[ \int_A f dP = \lim_{m \to \infty} \int_A E(g_M|\mathcal{G}_m) dP = \int_A g_M dP \]

Now \( \mathcal{G}_T = \sigma(W(t_k), t_k \leq T) = \sigma(\mathcal{G}_m, m \geq 1) \) and so the above equation implies that \( f = g_M \) a.e.

By the Doob Dynkin lemma listed above, there exists a Borel measurable \( h: \mathbb{R}^{nm} \to \mathbb{R} \) such that 
\[ E(g_M|\mathcal{G}_m) = h(W_{t_1}, \cdots, W_{t_m}) \text{ a.e.} \]
Of course $h$ is not in $C_c^\infty(\mathbb{R}^{nm})$. Let $m$ be large enough that

$$||g_M - E(g_M|\mathcal{G}_m)||_{L^2} = ||f - E(g_M|\mathcal{G}_m)||_{L^2} < \frac{\varepsilon}{4}. \quad (65.9.21)$$

Let $\lambda((W_{t_1}, \ldots, W_{t_m}))$ be the distribution measure of the random vector $(W_{t_1}, \ldots, W_{t_m})$. Thus $\lambda((W_{t_1}, \ldots, W_{t_m}))$ is a Radon measure and so there exists $\phi \in C_c(\mathbb{R}^{nm})$ such that

$$\left(\int_{\Omega} |E(g_M|\mathcal{G}_m) - \phi(W_{t_1}, \ldots, W_{t_m})|^2 dP\right)^{1/2} = \left(\int_{\Omega} |h(W_{t_1}, \ldots, W_{t_m}) - \phi(W_{t_1}, \ldots, W_{t_m})|^2 dP\right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^{nm}} |h(x_1, \ldots, x_m) - \phi(x_1, \ldots, x_m)|^2 d\lambda((W_{t_1}, \ldots, W_{t_m}))\right)^{1/2} < \varepsilon/4.$$ 

By convolving with a mollifier, one can assume that $\phi \in C_c^\infty(\mathbb{R}^{nm})$ also. It follows from (65.9.24) and (65.9.24) that

$$||g - \phi(W_{t_1}, \ldots, W_{t_m})||_{L^2} \leq ||g - g_M||_{L^2} + ||g_M - E(g_M|\mathcal{G}_m)||_{L^2} + ||E(g_M|\mathcal{G}_m) - \phi(W_{t_1}, \ldots, W_{t_m})||_{L^2}$$

$$\leq 3 \left(\frac{\varepsilon}{4}\right) < \varepsilon \quad \blacksquare$$
Chapter 66

A Different Kind Of Stochastic Integration

For more on this material, see [12] which is what this is based on. Recall the following corollary. It is Corollary 57.20.3 on Page 1984.

Corollary 66.0.9 Let $H$ be a real Hilbert space. Then there exist random variables $W(h)$ for $h \in H$ such that each is normally distributed with mean 0 and for every $h,g,(W(h),W(g))$ is normally distributed and

$$E(W(h)W(g)) = (h,g)_H$$

Furthermore, if $\{e_i\}$ is an orthogonal set of vectors of $H$, then $\{W(e_i)\}$ are independent random variables. Also for any finite set $\{f_1,f_2,\cdots,f_n\}$,

$$(W(f_1),W(f_2),\cdots,W(f_n))$$

is normally distributed.

Here are some simple examples.

Example 66.0.10 Let $H = L^2([0,T])$. For $f \in H$, let

$$W(f) \equiv \int_0^T f(u)\,dW$$

where $W(t)$ is the one dimensional Wiener process.

First of all, note that the integrand is adapted to the usual filtration determined by the Wiener process. This is because $f$ does not depend on $\omega$. That $W(f)$ is normally distributed can be seen from the approximation of the Ito integral with the integral of elementary functions. These are clearly normally distributed because they are just linear combinations of increments of the Wiener process. Recall these

2375
The above implies \( W \) is actually linear.

\[
\begin{align*}
E \left( (W(f + g) - (W(f) + W(g)))^2 \right) &= E \left( (W(f + g))^2 + [W(f)^2 + W(g)^2 + 2W(f)W(g)] \right) \\
&= E \left( (W(f + g))^2 \right) + E \left( W(f)^2 \right) + E \left( W(g)^2 \right) + E(W(f)W(g)) \\
&\quad - 2 \left( E(W(f)W(g)) + E(W(f)W(g)) \right)
\end{align*}
\]

which from the above equals

\[
|f + g|^2 + |f|^2 + |g|^2 + 2(f, g) - 2 [(f + g, f) + (f + g, g)]
\]

\[
= 2|f|^2 + 2|g|^2 + 4(f, g) - 2 \left[ |f|^2 + |g|^2 + 2(f, g) \right] = 0
\]

Thus \( W(f + g) - (W(f) + W(g)) = 0 \). Is it true that

\[
(W(af)) = aW(f)?
\]
66.1. HERMITE POLYNOMIALS

This is easier to show.

\[
E \left( (W(af) - aW(f))^2 \right) = E \left( (W(af))^2 - 2W(af)aW(f) + a^2W(f)^2 \right)
\]

\[
= |af|^2 - 2aE(W(af)W(f)) + a^2E(W(f)^2) = a^2|f|^2 - 2a^2|f|^2 + a^2|f|^2 = 0
\]

Thus \( W \) is indeed linear.

66.1 Hermite Polynomials

Consider

\[
\exp \left( tx - \frac{t^2}{2} \right) = \exp \left( \frac{x^2}{2} - \frac{1}{2} (x - t)^2 \right)
\]

Now the Hermite polynomials are the coefficients of the power series of this function expanded in powers of \( t \). Thus the \( n \)th one of these is

\[
H_n(x) = \exp \left( \frac{x^2}{2} \right) \frac{1}{n!} \frac{d^n}{dt^n} \left( \exp \left( -\frac{1}{2} (x - t)^2 \right) \right) \mid_{t=0} \quad (66.1.1)
\]

and

\[
\exp \left( tx - \frac{t^2}{2} \right) = \sum_{n=0}^{\infty} H_n(x) t^n \quad (66.1.2)
\]

Note that \( H_0(x) = 1 \),

\[
H_1(x) = \exp \left( \frac{x^2}{2} \right) \frac{d}{dt} \left( \exp \left( -\frac{1}{2} (x - t)^2 \right) \right) \mid_{t=0} = -e^{-\frac{1}{2}(t-x)^2} e^{\frac{1}{2}x^2} (t-x) \mid_{t=0} = x
\]

From (66.1.2), differentiating both sides formally with respect to \( x \),

\[
t \exp \left( tx - \frac{t^2}{2} \right) = \sum_{n=1}^{\infty} H'_n(x) t^n
\]

and so

\[
\sum_{n=0}^{\infty} H_n(x) t^n = \exp \left( tx - \frac{t^2}{2} \right) = \sum_{n=1}^{\infty} H'_n(x) t^{n-1} = \sum_{n=0}^{\infty} H'_{n+1}(x) t^n
\]

showing that

\[
H'_n(x) = H_{n-1}(x), \ n \geq 1, \ H_0(x) = 0, \ H_1(x) = x
\]
which could have been obtained with more work from \[66.1.2\]. Also, differentiating both sides of \[66.1.2\] with respect to \(t\),

\[-\exp \left( t x - \frac{t^2}{2} \right) (t - x) = \sum_{n=0}^{\infty} nH_n (x) t^{n-1} \]

Thus

\[(x - t) \sum_{n=0}^{\infty} H_n (x) t^n = \sum_{n=0}^{\infty} nH_n (x) t^{n-1} = \sum_{n=0}^{\infty} (n + 1) H_{n+1} (x) t^n \]

and so

\[\sum_{n=0}^{\infty} xH_n (x) t^n - \sum_{n=0}^{\infty} H_n (x) t^{n+1} = \sum_{n=0}^{\infty} (n + 1) H_{n+1} (x) t^n \]

and so

\[\sum_{n=0}^{\infty} xH_n (x) t^n - \sum_{n=1}^{\infty} H_{n-1} (x) t^n = \sum_{n=0}^{\infty} (n + 1) H_{n+1} (x) t^n \]

Thus for \(n \geq 1\),

\[xH_n (x) - H_{n-1} (x) = (n + 1) H_{n+1} (x) \]

Now also

\[\exp \left( t (-x) - \frac{t^2}{2} \right) = \sum_{n=0}^{\infty} H_n (-x) t^n \]

and taking successive derivatives with respect to \(t\) of the left side and evaluating at \(t = 0\) yields

\[H_n (-x) = (-1)^n H_n (x) . \]

Summarizing these as in \[93\],

\[H'_n (x) = H_{n-1} (x) , n \geq 1 , \quad H_0 (x) = 0 , H_1 (x) = x \]

\[xH_n (x) - H_{n-1} (x) = (n + 1) H_{n+1} (x) , \quad n \geq 1 \]

\[H_n (-x) = (-1)^n H_n (x) \]

Clearly, these relations show that all of these \(H_n\) are polynomials. Also the degree of \(H_n (x)\) is \(n\) and the coefficient of \(x^n\) is \(1/n!\).

**Definition 66.1.1** You can also consider Hermite polynomials which depend on \(\lambda\). These are defined as follows:

\[H_n (x, \lambda) \equiv \left( -\lambda \right)^n e^{\frac{x^2}{2\lambda}} \frac{\partial^n}{\partial x^n} \left( e^{-\frac{x^2}{2\lambda}} \right) \]

You can see clearly that these are polynomials in \(x\). For example, let \(n = 2\). Then you would have from the above definition.

\[H_0 (x, \lambda) = 1 , \quad H_1 (x, \lambda) = \frac{(-\lambda)^1}{1!} e^{\frac{x^2}{2\lambda}} \frac{\partial}{\partial x} \left( e^{-\frac{x^2}{2\lambda}} \right) = x \]
66.1. HERMITE POLYNOMIALS

\[ H_2(x, \lambda) \equiv \frac{(-\lambda)^2}{2!} e^{\frac{x^2}{2\lambda}} \frac{\partial^2}{\partial x^2} \left( e^{-\frac{x^2}{2\lambda}} \right) = \frac{1}{2} x^2 - \frac{1}{2} \lambda \]

The idea is you end up with polynomials of degree \( n \) times \( e^{-x^2/2\lambda} \) in the derivative part and then this cancels with \( e^{x^2/2\lambda} \) to leave you with a polynomial of degree \( n \). Also the leading term will always be \( \frac{x^n}{n!} \) which is easily seen from the above. Then there are some relationships satisfied by these.

Say \( n > 1 \) in what follows.

\[
\frac{\partial}{\partial x} H_n(x, \lambda) = \frac{x (-\lambda)^n}{\lambda} e^{\frac{x^2}{2\lambda}} \frac{\partial^n}{\partial x^n} \left( e^{-\frac{x^2}{2\lambda}} \right) + \frac{(-\lambda)^n}{n!} e^{\frac{x^2}{2\lambda}} \frac{\partial^n}{\partial x^n} \left( \frac{\partial}{\partial x} e^{-\frac{x^2}{2\lambda}} \right)
\]

Now since \( n > 1 \), that last term reduces to

\[
\frac{(-\lambda)^n}{n!} e^{\frac{x^2}{2\lambda}} \left[ -\frac{x}{\lambda} \frac{\partial^n}{\partial x^n} \left( e^{-\frac{x^2}{2\lambda}} \right) + n \frac{\partial}{\partial x} \left( -\frac{x}{\lambda} \right) \frac{\partial^{n-1}}{\partial x^{n-1}} \left( e^{-\frac{x^2}{2\lambda}} \right) \right]
\]

this by Leibniz formula. Thus this cancels with the first term to give

\[
\frac{\partial}{\partial x} H_n(x, \lambda) = \frac{(-\lambda)^n}{n!} \left( -\frac{1}{\lambda} \right) e^{\frac{x^2}{2\lambda}} \frac{\partial^{n-1}}{\partial x^{n-1}} \left( e^{-\frac{x^2}{2\lambda}} \right)
\]

\[
= \frac{(-\lambda)^{n-1}}{(n-1)!} e^{\frac{x^2}{2\lambda}} \frac{\partial^{n-1}}{\partial x^{n-1}} \left( e^{-\frac{x^2}{2\lambda}} \right) \equiv H_{n-1}(x, \lambda)
\]

In case of \( n = 1 \), this appears to also work. \( \frac{\partial}{\partial x} H_1(x, \lambda) = 1 = H_0(x, \lambda) \) from the above computations. This shows that

\[
\frac{\partial}{\partial x} H_n(x, \lambda) = H_{n-1}(x, \lambda)
\]

Next, is the claim that

\[
(n + 1) H_{n+1}(x, \lambda) = x H_n(x, \lambda) - \lambda H_{n-1}(x, \lambda)
\]

If \( n = 1 \), this says that

\[
2H_2(x, \lambda) = x H_1(x, \lambda) - \lambda H_0(x, \lambda)
\]

\[
= x^2 - \lambda
\]

and so the formula does indeed give the correct description of \( H_2(x, \lambda) \) when \( n = 1 \). Thus assume \( n > 1 \) in what follows. The left side equals

\[
\frac{(-\lambda)^{n+1}}{n!} e^{\frac{x^2}{2\lambda}} \frac{\partial^{n+1}}{\partial x^{n+1}} \left( e^{-\frac{x^2}{2\lambda}} \right)
\]
CHAPTER 66. A DIFFERENT KIND OF STOCHASTIC INTEGRATION

This equals

\[
\frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{\lambda} x^2} \partial^n \left( -\frac{x}{\lambda} e^{-\frac{1}{2} x^2} \right)
\]

Now by Liebniz formula,

\[
= \frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{\lambda} x^2} \left[ \frac{-x}{\lambda} \partial^n \frac{1}{\lambda} e^{-\frac{1}{2} x^2} + n \left( \frac{-1}{\lambda} \right) \frac{\partial^{n-1}}{\partial x^{n-1}} \left( e^{-\frac{1}{2} x^2} \right) \right]
\]

\[
= \frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{\lambda} x^2} \left( \frac{-x}{\lambda} \partial^n \left( e^{-\frac{1}{2} x^2} \right) \right) + \frac{(-\lambda)^{n+1}}{n!} e^{\frac{1}{\lambda} x^2} n \left( \frac{-1}{\lambda} \right) \frac{\partial^{n-1}}{\partial x^{n-1}} \left( e^{-\frac{1}{2} x^2} \right)
\]

\[
= xH_n(x, \lambda) - \lambda H_{n-1}(x, \lambda)
\]

which shows the formula is valid for all \( n \geq 1 \).

Next is the claim that

\[ H_n(-x, \lambda) = (-1)^n H_n(x, \lambda) \]

This is easy to see from the observation that

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial (-x)} (-1) \]

Thus if it involves \( n \) derivatives, you end up multiplying by \((-1)^n\).

Finally is the claim that

\[ \frac{\partial}{\partial \lambda} H_n(x, \lambda) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x, \lambda) \]

It is certainly true for \( n = 0, 1, 2 \). So suppose it is true for all \( k \leq n \). Then from earlier claims and induction,

\[
(n + 1) H_{(n+1)\lambda}(x, \lambda) = xH_{n\lambda}(x, \lambda) - H_{(n-1)}(x, \lambda) - \lambda H_{(n-1)\lambda}(x, \lambda)
\]

\[
= x \left( -\frac{1}{2} \right) H_{nxx} - H_{n-1} + \frac{1}{2} H_{(n-1)xx} = x \left( -\frac{1}{2} \right) H_{n-2} - H_{n-1} + \lambda \frac{1}{2} H_{(n-3)}
\]

\[
= -\frac{1}{2} (xH_{n-2} - \lambda H_{n-3} + 2H_{n-1}) = -\frac{1}{2} ((n-1)H_{n-1} + 2H_{n-1}) = -\frac{1}{2} ((n+1)H_{n-1})
\]

comparing the ends,

\[ H_{(n+1)\lambda} = -\frac{1}{2} H_{n-1} = -\frac{1}{2} H_{(n+1)xx} \]

This proves the following theorem.
Theorem 66.1.2 Let $H_n(x, \lambda)$ be defined by

$$H_n(x, \lambda) \equiv \frac{(-\lambda)^n}{n!} e^{\frac{1}{4} \lambda x^2} \frac{\partial^n}{\partial x^n} \left(e^{-\frac{1}{4} \lambda x^2}\right)$$

for $\lambda > 0$. Then the following properties are valid.

$$\frac{\partial}{\partial x} H_n(x, \lambda) = H_{n-1}(x, \lambda) \quad (66.1.4)$$

$$(n + 1) H_{n+1}(x, \lambda) = xH_n(x, \lambda) - \lambda H_{n-1}(x, \lambda) \quad (66.1.5)$$

$$H_n(-x, \lambda) = (-1)^n H_n(x, \lambda) \quad (66.1.6)$$

$$\frac{\partial}{\partial \lambda} H_n(x, \lambda) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_n(x, \lambda) \quad (66.1.7)$$

With this theorem, one can also prove the following.

Theorem 66.1.3 The Hermite polynomials are the coefficients of a certain power series. Specifically,

$$\exp \left( tx - \frac{1}{2} t^2 \lambda \right) = \sum_{n=0}^{\infty} H_n(x, \lambda) t^n$$

Proof: Replace $H_n$ with $K_n$ which really are the coefficients of the power series and then show $K_n = H_n$. Thus

$$\exp \left( tx - \frac{1}{2} t^2 \lambda \right) = \sum_{n=0}^{\infty} K_n(x, \lambda) t^n$$

Then $K_0 = 1 = H_0(x)$. Also $K_1(x) = x = H_1(x)$.

$$\frac{\partial}{\partial t} \left( \exp \left( tx - \frac{1}{2} t^2 \lambda \right) \right) = \exp \left( tx - \frac{1}{2} t^2 \lambda \right) (x - t\lambda)$$

$$= \sum_{n=0}^{\infty} xK_n(x, \lambda) t^n - \sum_{n=0}^{\infty} \lambda K_n(x, \lambda) t^{n+1} = \sum_{n=0}^{\infty} xK_n(x, \lambda) t^n - \sum_{n=1}^{\infty} \lambda K_{n-1}(x, \lambda) t^n$$

Also,

$$\frac{\partial}{\partial t} \left( \exp \left( tx - \frac{1}{2} t^2 \lambda \right) \right) = \sum_{n=1}^{\infty} nK_n(x, \lambda) t^{n-1} = \sum_{n=0}^{\infty} (n + 1) K_{n+1}(x, \lambda) t^n$$

It follows that for $n \geq 1$,

$$(n + 1) K_{n+1}(x, \lambda) = xK_n(x, \lambda) - \lambda K_{n-1}(x, \lambda)$$
Thus the first two $K_0, K_1$ coincide with $H_0$ and $H_1$ respectively. Then since both $K_n$ and $H_n$ satisfy the recursion relation (66.2), it follows that $K_n = H_n$ for all $n$. 

The first version is just letting $\lambda = 1$ in the second version.

There is something very interesting about these Hermite polynomials $H_n(x, \lambda)$. Let $W$ be the real Wiener process. Consider the stochastic process $H_n(W(t), t), n \geq 1$. This ends up being a martingale. Using Ito’s formula, the easy to remember version of it presented above, and the above properties of the Hermite polynomials,

$$dH_n = H_{nx} (W(t), t) dW + H_{nt} (W(t), t) dt + \frac{1}{2} H_{nxx} (W(t), t) dW^2$$

$$= H_{n-1} (W(t), t) dW - \frac{1}{2} H_{nxx} (W(t), t) dt + \frac{1}{2} H_{nxx} (W(t), t) dt$$

Note that if $n < 2$, both of the last two terms are 0. In general, they cancel and so

$$dH_n = H_{n-1} (W(t), t) dW$$

and so

$$H_n (W(t), t) = H_n (W(0), 0) + \int_0^t H_{n-1} (W(t), t) dW$$

Now the constant term in the above equation is $\mathcal{F}_0$ measurable and the stochastic integral is a martingale. Thus this is indeed a martingale assuming everything is suitably integrable. However, this is not hard to see because these $H_n$ are just polynomials. It was shown in Theorem (62.1.3) that $W(t) \in L^q (\Omega)$ for all $q$. Hence there is no integrability issue in doing these things. Actually, $H_n(W(0), 0) = 0$ To see this, note that $E(W(0)^2) = (0, 0)_H = 0$ and so $W(0) = 0$. Now it is not hard to see that $H_n(0, 0) = 0$. Indeed,

$$\exp \left(tx - \frac{1}{2} t^2 \lambda \right) = \sum_{n=0}^{\infty} H_n(x, \lambda) t^n$$

Thus $H_n(x, 0) = \sum_{n=0}^{\infty} H_n(x, 0) t^n = \exp(tx) = \sum_{n=0}^{\infty} \frac{(tx)^n}{n!} = \sum_{n=0}^{\infty} x^n t^n$ and so for all $n \geq 1, H_n(0, 0) = 0$. Thus in fact, for $n \geq 1, t \rightarrow H_n(W(t), t)$ is a martingale which equals 0 when $t = 0$.

### 66.2 A Remarkable Theorem Involving The Hermite Polynomials

**Lemma 66.2.1** Say $(X, Y)$ is generalized normally distributed and $E(X) = E(Y) = 0, E(X^2) = E(Y^2) = 1$. Then for $m, n \geq 0$,

$$E(H_n(X) H_m(Y)) = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{n!} (E(XY))^n & \text{if } n = m \end{cases}$$
**Proof:** By assumption, \( sX + tY \) is normal distributed with mean 0. This follows from Theorem 6.7.16.4.

Also

\[
\sigma^2 \equiv E \left( (sX + tY)^2 \right) = s^2 + t^2 + 2E \left( XY \right) st
\]

and so its characteristic function is

\[
E (\exp (i\lambda (sX + tY))) = \phi_{sX+tY} (\lambda) = e^{-\frac{i}{2} \sigma^2 \lambda^2} = e^{-\frac{i}{2} (s^2 + t^2) \lambda^2} e^{-E(\lambda) st \lambda^2}
\]

So let \( \lambda = -i \). You can do this because both sides are analytic in \( \lambda \in C \) and they are equal for real \( \lambda \), a set with a limit point. This leads to

\[
E (\exp (sX + tY)) = e^{\frac{1}{2} (s^2 + t^2)} e^{E(\lambda) st}
\]

Hence, multiplying both sides by \( e^{-\frac{i}{2} (s^2 + t^2)} \),

\[
e^{-\frac{i}{2} (s^2 + t^2)} E (\exp (sX + tY)) = E \left( \exp \left( sX - \frac{s^2}{2} \right) \exp \left( tY - \frac{t^2}{2} \right) \right)
= \exp (stE(\lambda) )
\]

Now take \( \frac{\partial^{n+m}}{\partial s \partial t^m} \) of both sides. Recall the description of the Hermite polynomials given above

\[
n! H_n (x) = \frac{d^n}{dt^n} \exp \left( tx - \frac{t^2}{2} \right) \big|_{t=0}
\]

Thus

\[
E (n! H_n (X) m! H_m (Y)) = \frac{\partial^{n+m}}{\partial s \partial t^m} \exp (stE(\lambda) ) \big|_{s=t=0}
\]

Consider \( m < n \)

\[
\frac{\partial^{n+m}}{\partial s \partial t^m} \exp (stE(\lambda) ) = \frac{\partial^m}{\partial t^m} \left( (tE(\lambda) )^n \exp (stE(\lambda) ) \right)
\]

You have something like

\[
\frac{\partial^m}{\partial t^m} \left[ t^n \left( (E(\lambda) )^n \exp (stE(\lambda) ) \right) \right]
\]

and \( m < n \) so when you take partial derivatives with respect to \( t \), \( m \) times and set \( s, t = 0 \), you must have 0. Hence, if \( n > m \),

\[
E (n! H_n (X) m! H_m (Y)) = 0
\]

Similarly this equals 0 if \( m > n \). So assume \( m = n \). Then you will go through the same process just described but this time at the end you will have something of the form

\[
n! E (XY)^n + \text{ terms multiplied by } s \text{ or } t
\]
Hence, in this case,

\[ E (n! H_n (X) n! H_n (Y)) = n! E (XY)^n \]

and so

\[ E (H_n (X) H_n (Y)) = \frac{1}{n!} E (XY)^n \]

Let \( W \) be the function defined above, \( W (h) \) is normally distributed with mean 0 and variance \( |h|^2 \) and \( E (W (h) W (g)) = (h, g)_H \). Then from Lemma 66.2.1,

\[ E (H_n (W (h)) H_m (W (g))) = \left\{ \begin{array}{ll} 0 & \text{if } n \neq m \\ \frac{1}{n!} E (W (h) W (g))^n & \text{if } n = m \\ \end{array} \right. \]

This is a really neat result. From definition of \( W \),

\[ E \left( (W (h) W (g))^1 \right) = (h, g)_H \]

Note this is a special case of the above result because \( H_1 (x) = x \). However, we don’t know that \( E ((W (h) W (g))^n) \) is equal to something times \((h, g)^n_H\) but we know that this is true of some \( n^{th} \) degree polynomials in \( W (h) \) and \( W (g) \).

**Definition 66.2.2** Let \( \mathcal{H}_n \equiv \text{span} \{ H_n (W (h)) : h \in H, |h|^2_1 = 1 \} \).

Thus \( \mathcal{H}_n \) is a closed subspace of \( L^2 (\Omega, \mathcal{F}) \). Recall \( \mathcal{F} \equiv \sigma (W (h) : h \in H) \). This subspace \( \mathcal{H}_n \) is called the Wiener chaos of order \( n \).

**Theorem 66.2.3** \( L^2 (\Omega, \mathcal{F}, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \). The symbol denotes the infinite orthogonal sum of the closed subspaces \( \mathcal{H}_n \). That is, if \( f \in L^2 (\Omega) \), there exists \( f_n \in \mathcal{H}_n \) and constants such that \( f = \sum_n c_n f_n \) and if \( f \in \mathcal{H}_n, g \in \mathcal{H}_m \), then \((f, g)_{L^2 (\Omega)} = 0 \).

**Proof:** Clearly each \( \mathcal{H}_n \) is a closed subspace. Also, if \( f \in \mathcal{H}_n \) and \( g \in \mathcal{H}_m \) for \( n \neq m \), what about \((f, g)_{L^2 (\Omega)} \)?

\[
(f, g)_{L^2 (\Omega)} = \lim_{l \to \infty} E \left( \sum_{k=1}^{M_1} a_k^1 H_n (W (h_k^1)) , \sum_{j=1}^{M_2} a_j^1 H_m (W (h_j^1)) \right)
= \lim_{l \to \infty} \sum_{k,j} a_k^1 a_j^1 E \left( H_n (W (h_k^1)) H_m (W (h_j^1)) \right) = 0
\]

Thus these are orthogonal subspaces. Clearly \( L^2 (\Omega) \supseteq \bigoplus \mathcal{H}_n \). Suppose \( X \) is orthogonal to each \( \mathcal{H}_n \). Is \( X = 0 ? \) Each \( x^n \) can be obtained as a linear combination of the \( H_k (x) \) for \( k \leq n \). This is clear because the space of polynomials of degree \( n \) is of dimension \( n + 1 \) and \( \{ H_0 (x), H_1 (x), \cdots, H_n (x) \} \) is independent on \( \mathbb{R} \).
66.2. A REMARKABLE THEOREM INVOLVING THE HERMITE POLYNOMIALS

This is easily seen as follows. Suppose
\[ \sum_{k=0}^{n} c_k H_k(x) = 0 \]
and that not all \( c_k = 0 \). Let \( m \) be the smallest index such that
\[ \sum_{k=0}^{m} c_k H_k(x) = 0 \]
with \( c_m \neq 0 \). Then just differentiate both sides and obtain
\[ \sum_{k=1}^{m} c_k H_{k-1}(x) = 0 \]
contradicting the choice of \( m \).

Therefore, each \( x^n \) is really a unique linear combination of the \( H_k \) as claimed. Say
\[ x^n = \sum_{k=0}^{n} c_k H_k(x) \]
Then for \( |h| = 1 \),
\[ W(h)^n = \sum_{k=0}^{n} c_k H_k(W(h)) \in \mathcal{H}_n \]
Hence \( (X, W(h)^n)_{L^2(\Omega)} = 0 \) whenever \( |h| = 1 \). It follows that for \( h \in H \) arbitrary, and the fact that \( W \) is linear,
\[ (X, W(h)^n)_{L^2} = (X, \left( |h| W \left( \frac{h}{|h|} \right) \right)^n) = |h|^n \left( X, W \left( \frac{h}{|h|} \right)^n \right) = 0 \]
Therefore, \( X \) is perpendicular to \( e^{W(h)} \) for every \( h \in H \) and so from Lemma 62.6.4, \( X = 0 \). Thus \( \oplus \mathcal{H}_n \) is dense in \( L^2(\Omega) \). \( \blacksquare \)

Note that from Lemma 62.6.4, every polynomial in \( W(h) \) is in \( L^p(\Omega) \) for all \( p > 1 \). Now what is next is really tricky.

**Corollary 66.2.4** Let \( P_0^n \) denote all polynomials of the form
\[ p(W(h_1), \ldots, W(h_k)) \], degree of \( p \leq n \), some \( h_1, \ldots, h_k \)
Also let \( \mathcal{P}_n \) denote the closure in \( L^2(\Omega, F, P) \) of \( P_0^n \). Then
\[ \mathcal{P}_n = \oplus_{i=0}^{n} \mathcal{H}_i \]

**Proof:** It is obvious that \( \mathcal{P}_n \supseteq \oplus_{i=0}^{n} \mathcal{H}_i \) because the thing on the right is just the closure of a set of polynomials of degree no more than \( n \), a possibly smaller set than the polynomials used to determine \( P_0^n \) and hence \( \mathcal{P}_n \). If \( \mathcal{P}_n \) is orthogonal to \( \mathcal{H}_m \) for
all $m > n$, then from the above Theorem 66.2.3 you must have $\mathcal{P}_n \subseteq \oplus_{i=0}^n \mathcal{H}_i$. So consider $H_m (W (h))$. Recall that $\mathcal{H}_m$ is the closure of the span of things like this for $|h|_H = 1$. Thus we need to consider

$$E \left( p(W(h_1), \cdots, W(h_k)) H_m(W(h)) \right), \quad |h|_H = 1,$$

and show that this is 0. Now here is the tricky part. Let $\{e_1, \cdots, e_s, h\}$ be an orthonormal basis for

$$\text{span} (h_1, \cdots, h_k, h).$$

Then since $W$ is linear, there is a polynomial $q$ of degree no more than $n$ such that

$$p(W(h_1), \cdots, W(h_k)) = q(W(e_1), \cdots, W(e_s), W(h))$$

Then consider a term of

$$a W(e_1)^{r_1} \cdots W(e_s)^{r_s} W(h)^r H_m(W(h))$$

Now from Corollary 66.2.1 these random variables $\{W(e_1), \cdots, W(e_s), W(h)\}$ are independent due to the fact that the vector $(W(e_1), \cdots, W(e_s), W(h))$ is multivariate normally distributed and the covariance is diagonal. Therefore,

$$E \left(a W(e_1)^{r_1} \cdots W(e_s)^{r_s} W(h)^r H_m(W(h))\right)$$

$$= a E(W(e_1)^{r_1}) \cdots E(W(e_s)^{r_s}) E(W(h)^r) E(H_m(W(h)))$$

Now since $r \leq n$, $W(h)^r = \sum_{k=1}^r c_k H_k(W(h))$ for some choice of scalars $c_k$. By Lemma 66.2.4, this last term,

$$E(W(h)^r H_m(W(h))) = \sum_k c_k E(H_k(W(h)) H_m(W(h))) = 0$$

since each $k < m$. 

Note how remarkable this is. $\mathcal{P}_n^0$ includes all polynomials in $W(h_1), \cdots, W(h_k)$ some $h_1, \cdots, h_k$, of degree no more than $n$, including those which have mixed terms but a typical thing in $\oplus_{i=0}^n \mathcal{H}_i$ is a sum of Hermite polynomials in $W(h_k)$. It is not the case that you would have terms like $W(h_1) W(h_2)$ as could happen in the case of $\mathcal{P}_n$.

Obviously it would be a good idea to obtain an orthonormal basis for $L^2 (\Omega, \mathcal{F}, P)$. This is done next. Let $\Lambda$ be the multiindices, $(a_1, a_2, \cdots)$ each $a_k$ a nonnegative integer. Also in the description of $\Lambda$ assume that $a_k = 0$ for all $k$ large enough. For such a multiindex $a \in \Lambda$,

$$a! \equiv \prod_{i=1}^\infty a_i!, \quad |a| \equiv \sum_i a_i$$

Also for $a \in \Lambda$, define

$$H_a(x) \equiv \prod_{j=1}^\infty H_{a_j} (x)$$
66.2. A REMARKABLE THEOREM INVOLVING THE HERMITE POLYNOMIALS

This is well defined because $H_0(x) = 1$ and all but finitely many terms of this infinite product are therefore equal to 1. Now let $\{e_i\}$ be an orthonormal basis for $H$. For $a \in \Lambda$,

$$\Phi_a = \sqrt{a!} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)) \in L^2(\Omega)$$

Suppose $a, b \in \Lambda$.

$$\int_{\Omega} \Phi_a \Phi_b dP = \sqrt{a!} \sqrt{b!} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)) H_{b_i}(W(e_i)) dP$$

Now recall from Corollary 62.6.1 the random variables $\{W(e_i)\}$ are independent. Therefore, the above equals

$$\sqrt{a!} \sqrt{b!} \prod_{i=1}^{\infty} H_{a_i}(W(e_i)) H_{b_i}(W(e_i)) dP = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

Thus $\{\Phi_a : a \in A\}$ is an orthonormal set in $L^2(\Omega)$.

**Lemma 66.2.5** If $s_k \to h$, then for $n \in \mathbb{N}$, there is a subsequence, still called $s_k$ for which $W(s_k)^n \to W(h)^n$ in $L^2(\Omega)$.

**Proof:** If $s_k \to h$, does $W(s_k)^n \to W(h)^n$ in $L^2(\Omega)$ for some subsequence? First of all,

$$\|W(h) - W(s_k)\|_{L^2(\Omega)}^2 = |s_k - h|^2_H \to 0$$

and so there is a subsequence, still called $k$ such that $W(s_k)(\omega) \to W(h)(\omega)$ for a.e. $\omega$. Consider

$$\int_{\Omega} |W(h)^n - W(s_k)^n|^2 dP \quad (66.2.8)$$

Does this converge to 0? The integrand is bounded by $2 \left(W(h)^{2n} + W(s_k)^{2n}\right)$. Since $W(h), W(s_k)$ are symmetric,

$$\int_{\Omega} \left(2 \left(W(h)^{2n} + W(s_k)^{2n}\right)\right)^2 dP \leq 8 \int_{\Omega} \left(W(h)^{4n} + W(s_k)^{4n}\right) dP$$

$$= 16 \int_{\Omega \cap \{W(h) \geq 0\}} e^{4nW(h)} dP + 16 \int_{\Omega \cap \{W(s_k) \geq 0\}} e^{4nW(s_k)} dP$$

$$\leq 16 \int_{\Omega} e^{4nW(h)} dP + 16 \int_{\Omega} e^{4nW(s_k)} dP$$

$$\leq 16e^{\frac{1}{2}|4nh|} + 16e^{\frac{1}{2}|4ns_k|}$$

which is bounded independent of $k$, the last step following from Lemma 62.6.4. Therefore, the Vitali convergence theorem applies in 66.2.8. ■
Given an \( h \in H \), let \( s_k = \sum_{j=1}^{k} (h, e_j) e_j \), the \( k^{th} \) partial sum in the Fourier series for \( h \).

\[
W(s_k)^m = \left( \sum_{j=1}^{k} (h, e_j) W(e_j) \right)^m = p(W(e_1), \cdots, W(e_k))
\]

where \( p \) is a homogeneous polynomial of degree \( m \). Now this equals

\[
q(H_0(W(e_1)), \cdots, H_0(W(e_k)) \cdots H_m(W(e_1)), \cdots, H_m(W(e_k)))
\]

where \( q \) is a polynomial. This is because each \( W(e_j)^r \) is a linear combination of \( H_s(W(e_j)) \) for \( s \leq r \). Now you look at terms of this polynomial. They are all of the form \( c \Phi_a \) for some constant \( c \) and \( a \in \Lambda \). Therefore, if \( X \in L^2(\Omega) \), there is a subsequence, still denoted as \( \{s_k\} \) such that

\[
E(W(h)^n X) = \lim_{k \to \infty} E(W(s_k)^n X)
\]

Now if \( X \) is orthogonal to each \( \Phi_a \), then for any \( h \) and \( n \), there is a subsequence still denoted with \( k \) such that

\[
E(W(h)^n X) = \lim_{k \to \infty} E(W(s_k)^n X) = 0
\]

It follows from Lemma 62.6.4, the part about the convergence of the partial sums to \( e^{W(h)} \) that \( X \) is orthogonal to \( e^{W(h)} \) for any \( h \). Here are the details. From the lemma, for large \( n \),

\[
\left| E(e^{W(h)}X) - E\left( \sum_{j=0}^{n} \frac{W(h)^j}{j!} X \right) \right| < \varepsilon,
\]

Also for large \( k \),

\[
\left| E\left( \sum_{j=0}^{n} \frac{W(h)^j}{j!} X \right) - E\left( \sum_{j=0}^{n} \frac{W(s_k)^j}{j!} X \right) \right| = \left| E\left( \sum_{j=0}^{n} \frac{W(h)^j}{j!} X \right) \right| < \varepsilon
\]

Therefore,

\[
\left| E(e^{W(h)}X) \right| < 2\varepsilon
\]

Since \( \varepsilon \) is arbitrary, this proves the desired result. By Lemma 62.6.3, \( X = 0 \) and this shows that \( \{\Phi_a : a \in \Lambda\} \) is complete.

**Proposition 66.2.6** \( \{\Phi_a : a \in \Lambda\} \) is a complete orthonormal set for \( L^2(\Omega, F, P) \).
66.3. A MULTIPLE INTEGRAL

66.3 A Multiple Integral

Consider trying to find
\[ \int_0^1 \int_0^1 dB_s dB_t \]

Here \( B_t \) is just one dimensional Wiener process. You would want it to equal
\[ 2 \int_0^1 \int_0^t dB_s dB_t = 2 \int_0^1 B_t dB_t \]

So what should this equal? Let \( F(x) = x^2 \) so \( F'(x) = 2x, F''(x) = 2 \). Consider \( F(B_t) \). Then using the formalism for the Ito formula,
\[
\frac{dF}{dF(B_t)} = 2B_t dB_t + \frac{1}{2} (2) dB_t^2 = 2B_t dB_t + dt
\]

Therefore,
\[ B_t^2 = 2 \int_0^t B_s dB_s + t \]

and letting \( t = 1 \),
\[
\frac{1}{2} B_1^2 - 1 = \int_0^1 B_s dB_s = \int_0^1 \int_0^s dB_r dB_s
\]

and so we would want to have
\[ B_1^2 - 1 = 2 \int_0^1 \int_0^s dB_r dB_s \]

and we want this to equal \( \int_0^1 \int_0^1 dB_s dB_t \) so we need to be defining this in a way such that this will result. Of course, this is just the simplest example of an iterated integral with respect to these one dimensional Wiener processes.

Now partition \([0, 1]\) as \( 0 = t_0 < t_1 < \cdots . t_n = 1 \). Then sum over all \( [t_{i-1}, t_i] \times [t_{j-1}, t_j] \) but leave out those which are on the “diagonal”. These would be of the form \( [t_{i-1}, t_i] \times [t_{i-1}, t_i] \). Here you would have in the sum products of the form \( (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) \). Thus you would have
\[
\sum_{i,j} (B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}}) = \sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2
\]

\[
= (B_1 - B_0)^2 - \sum_{i=1}^n (B_{t_i} - B_{t_{i-1}})^2
\]
Then of course you take a limit as the norm of the partition goes to 0. This yields in the limit

\[ B_1^2 - 1 \]

which is the thing which is wanted. Thus the idea is to consider only functions which are equal to 0 on the “diagonal” and define an integral for these. Then hopefully these will be dense in \( L^2([0,T]^n) \) and the multiple integral can then be defined as some sort of limit.

Now in the above construction, from now on, unless indicated otherwise, \( H = L^2(T) \) where the measure is ordinary Lebesgue measure on \( T = [0,T] \) or \( 0, \infty \) or some other interval of time. However, it could be more general, but for the sake of simplicity let it be Lebesgue measure. Generalities appear to be nothing but identifying that which works in the case of Lebesgue measure. If \( \mu \ll m \) everything would work also. A careful description of what kind of measures work is in [K].

Also, for \( A \) a Borel set having finite Lebesgue measure,

\[ W(A) \equiv W(X_A). \]

This is a random variable, and as explained earlier, since any finite set of these is normally distributed, if all the sets are pairwise disjoint, the random variables are independent because the covariance is a diagonal matrix.

**Definition 66.3.1** Let \( D \equiv \{(t_1, \cdots, t_m) : t_i = t_j \text{ for some } i \neq j\} \). This is called the diagonal set. Here \( D \subseteq T^m \) where \( T \) is an interval \([0,T)\). Assume \( T < \infty \) here. Let

\[ 0 = \tau_0 < \tau_1 < \cdots < \tau_k = T \]

Then this can be used to partition \( T^m \) into sets of the form

\[ [\tau_{i_1 - 1}, \tau_{i_1}) \times \cdots \times [\tau_{i_m - 1}, \tau_{i_m}) \]

such that \( T^m \) is the disjoint union of these. An off diagonal step function \( f \) is one which is of the form

\[ f(t_1, \cdots, t_m) = \sum_{i_1, \cdots, i_m} a_{i_1, \cdots, i_m} X_{[\tau_{i_1 - 1}, \tau_{i_1}) \times \cdots \times [\tau_{i_m - 1}, \tau_{i_m})} (t_1, \cdots, t_m) \]

where \( a_{i_1, \cdots, i_m} = 0 \) if \( i_p = i_q \). This would correspond to a diagonal term because it would result in a repeated half open interval. Thus we assume all these are equal to 0. The collection of all such off diagonal step functions will be denoted as \( E_m \). The \( m \) corresponds to the dimension.

**Definition 66.3.2** Let \( I_m : E_m \rightarrow L^2(\Omega) \) be defined in the obvious way.

\[ I_m \left( \overset{k}{\sum} a_{i_1, \cdots, i_m} X_{[\tau_{i_1 - 1}, \tau_{i_1}) \times \cdots \times [\tau_{i_m - 1}, \tau_{i_m})} (t_1, \cdots, t_m) \right) \]
\[ \equiv \sum_{i_1 \cdots i_m} a_{i_1, \cdots, i_m} \prod_{p=1}^m \left( B_{\tau_p} - B_{\tau_{p-1}} \right) \]

Then \( I_m \) is linear. If you had two different partitions, you could take the union of them both and by letting coefficients be repeated on the smaller boxes, one can assume that a single partition is being used. This is why it is clear that \( I_m \) is linear.

**Definition 66.3.3** Let \( f \in L^2(T^m) \). The symetrization of \( f \) is given by

\[ \tilde{f}(t_1, \cdots, t_m) \equiv \frac{1}{m!} \sum_{\sigma \in S_m} f(t_{\sigma_1}, \cdots, t_{\sigma_m}) \]

**Lemma 66.3.4** The following holds for \( f \in E_m \)

\[ \left\| \tilde{f} \right\|_{L^2(T^m)} \leq \left\| f \right\|_{L^2(T^m)} \]

also

\[ I_m(f) = I_m(\tilde{f}) \]

**Proof:** This follows because, thanks to the properties of Lebesgue measure,

\[ \int_{T^m} |f(t_1, \cdots, t_m)|^2 dt_1 \cdots dt_m = \int_{T^m} |f(t_{\sigma_1}, \cdots, t_{\sigma_m})|^2 dt_1 \cdots dt_m \]

\[ = \int_{T^m} |f(\sigma)|^2 dt_1 \cdots dt_m \]

therefore,

\[ \left\| \tilde{f} \right\|_{L^2(T^m)} \leq \frac{1}{m!} \sum_{\sigma \in S_m} \left\| f(\sigma) \right\|_{L^2(T^m)} = \frac{1}{m!} \sum_{\sigma \in S_m} \left\| f \right\|_{L^2(T^m)} = \left\| f \right\|_{L^2(T^m)} \]

The next claim follows because on the right, the terms making up the sum just happen in a different order for each \( \sigma \).

More generally, here is a lemma about off diagonal things. It uses sets \( A_i \) rather than intervals \([a, b)\).

**Lemma 66.3.5** Let \( \{A_1, \cdots, A_m\} \) be pairwise disjoint sets in \( \mathcal{B}(T) \) each having finite measure. Then the products \( A_{i_1} \times \cdots \times A_{i_n} \) are pairwise disjoint. Also to say that the function

\[ (t_1, \cdots, t_n) \rightarrow \sum_{i} c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}}(t_1, \cdots, t_n) \]

equals 0 whenever some \( t_j = t_i, i \neq j \) is to say that \( c_i = 0 \) whenever there is a repeated index in \( i \).
Proof: Suppose the condition that the $A_k$ are pairwise disjoint holds and consider two of these products, $A_{i_1} \times \cdots \times A_{i_n}$ and $A_{j_1} \times \cdots \times A_{j_n}$. If the two ordered lists $(i_1, \ldots, i_n), (j_1, \ldots, j_n)$ are different, then since the $A_k$ are disjoint the two products have empty intersection because they differ in some position.

Now suppose that $c_1 = 0$ whenever there is a repeated index. Then the sum is taken over all permutations of $n$ things taken from $\{1, \ldots, m\}$ and so if some $t_r = t_s$ for $r \neq s$, all terms of the sum equal zero because $\mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}} \neq 0$ only if $t \in A_{i_1} \times \cdots \times A_{i_n}$ and since $t_r = t_s$ and the sets $\{A_k\}$ are disjoint, there must be the same set in positions $r$ and $s$ so $c_1 = 0$. Hence the function equals 0.

Conversely, suppose the sum $\sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}}$ equals zero whenever some $t_r = t_s$ for $s \neq r$. Does it follow that $c_1 = 0$ whenever some $t_r = t_s$? The value of this function at $t \in A_{i_1} \times \cdots \times A_{i_n}$ is $c_1$ because for any other ordered list of indices, the resulting product has empty intersection with $A_{i_1} \times \cdots \times A_{i_n}$. Thus, since $t_r = t_s$, it is given that this function equals 0 which equals $c_1$.

This says that when you consider such a function $\sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}}$ with the $A_k$ pairwise disjoint, then to say that it equals 0 whenever some $t_i = t_j$ is to say that it is really a sum over all permutations of $n$ indices taken from $\{1, \ldots, m\}$. Thus there are $\binom{m}{n}! = P(m, n)$ possible non zero terms in this sum.

Lemma 66.3.6 Consider the set of all ordered lists of $n$ indices from $\{1, 2, \ldots, m\}$. Thus two lists are the same if they consist of the same numbers in the same positions. We denote by $i$ or $j$ such an index, $i$ from $\{1, \ldots, m\}$ and $j$ from $\{1, \ldots, q\}$. Also let $\{A_1, \ldots, A_m\}, \{B_1, \ldots, B_q\}$ are two lists of pairwise disjoint Borel sets from $T$ having finite Lebesgue measure. Also suppose

$$\sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}} = \sum_j d_j \mathcal{X}_{B_{j_1} \times \cdots \times B_{j_n}}$$

Then

$$\sum_i c_i \prod_{k=1}^{n} W(A_{i_k}) = \sum_j d_j \prod_{k=1}^{n} W(B_{j_k})$$

Proof: Suppose that $n = 1$ first. Then you have

$$\sum_i c_i \mathcal{X}_{A_i} = \sum_j d_j \mathcal{X}_{B_j} \quad (66.3.9)$$

where the sets $\{A_i\}$ and $\{B_j\}$ are disjoint. Clearly

$$A_i \supseteq \cup_j A_i \cap B_j \quad (66.3.10)$$

Consider

$$c_i \mathcal{X}_{A_i}, \sum_j c_i \mathcal{X}_{A_i \cap B_j} \quad (66.3.11)$$

If strict inequality holds in (66.3.11), then you must have a point in $A_i \setminus \cup_j A_i \cap B_j$ where the left side of $\sum_i c_i \mathcal{X}_{A_i}$ equals $c_i$ but the right side would equal 0. Hence
\[ c_i = 0 \text{ and so } \sum_j c_i \chi_{A_i \cap B_j} = 0 \text{ which shows that the two expressions in } 66.3.11 \text{ are equal. If } A_i = \bigcup_j A_i \cap B_j, \text{ it is also true that the two expressions in } 66.3.11 \text{ are equal. Thus}
\]
\[ \sum_i c_i \chi_{A_i} = \sum_i \sum_j c_i \chi_{A_i \cap B_j} \]

Similar considerations apply to the right side. Thus
\[ \sum_i \sum_j c_i \chi_{A_i \cap B_j} = \sum_j \sum_i d_j \chi_{A_i \cap B_j} \]
\[ \sum_{i,j} (c_i - d_j) \chi_{A_i \cap B_j} = 0 \]

hence if \( W(A_i \cap B_j) \neq 0 \), then, since these sets are disjoint, \( c_i - d_j = 0 \). It follows that
\[ \sum_{i,j} (c_i - d_j) W(A_i \cap B_j) = 0 \]

and so
\[ \sum_{i} c_i W(A_i) = \sum_{i} \sum_j c_i W(A_i \cap B_j) = \sum_{j} \sum_i d_j W(A_i \cap B_j) = \sum_{j} d_j W(B_j) \]

This proves the theorem if \( n = 1 \). Consider the general case. Let \( \bar{\mathbf{i}} \) be
\[ (i_1, \cdots, i_{n-1}), i_k \leq m \]
\[ \sum_{i_n=1}^{m} \sum_{t_{n-1}}^{m} c_{(\bar{\mathbf{i}}', \bar{\mathbf{i}}_n)} \chi_{A_{i_n}(t_n)} \chi_{A_{i_1} \times \cdots \times A_{i_{n-1}}} = \sum_i c_i \chi_{A_{i_1} \times \cdots \times A_{i_{n-1}}} \]
\[ = \sum_{j} d_j \chi_{B_{j_1} \times \cdots \times B_{j_n}} = \sum_{j=1}^{m} \sum_{t_{n-1}}^{m} d_{(\bar{\mathbf{j}}', \bar{\mathbf{j}}_n)} \chi_{B_{j_n}(t_n)} \chi_{B_{j_1} \times \cdots \times B_{j_{n-1}}} \]

Now pick \((t_1, \cdots, t_{n-1})\). The above is then
\[ \sum_{i_n=1}^{m} \left( \sum_{\bar{\mathbf{j}}'} c_{(\bar{\mathbf{i}}', \bar{\mathbf{i}}_n)} \chi_{A_{i_1} \times \cdots \times A_{i_{n-1}}}(t_1, \cdots, t_{n-1}) \right) \chi_{A_{i_n}}(t_n) \]
\[ = \sum_{j_n=1}^{m} \left( \sum_{\bar{\mathbf{j}}'} d_{(\bar{\mathbf{j}}', \bar{\mathbf{j}}_n)} \chi_{B_{j_1} \times \cdots \times B_{j_{n-1}}}(t_1, \cdots, t_{n-1}) \right) \chi_{B_{j_n}}(t_n) \]

and by what was just shown for \( n = 1 \), for each such choice,
\[ \sum_{i_n} \left( \sum_{\bar{\mathbf{j}}'} c_{(\bar{\mathbf{i}}', \bar{\mathbf{i}}_n)} \chi_{A_{i_1} \times \cdots \times A_{i_{n-1}}} \right) W(A_{i_n}) \]
\[ = \sum_{j_n} \left( \sum_{\bar{\mathbf{j}}'} d_{(\bar{\mathbf{j}}', \bar{\mathbf{j}}_n)} \chi_{B_{j_1} \times \cdots \times B_{j_{n-1}}} \right) W(B_{j_n}) \]
Then
\[ \sum_i \left( \sum_{t_n} W(A_{i_n}) c_i(v, i_n) \right) X_{A_{i_1} \times \cdots \times A_{i_{n-1}}} = \]
\[ \sum_{j'} \left( \sum_{j_{n}} \int (B_{j_n}) d_{j'(j_n)} \right) X_{B_{j_1} \times \cdots \times B_{j_{n-1}}} \]

Pick \( \omega = \omega_0 \). Then by induction,
\[ \sum_i \left( \sum_{t_n} W(A_{i_n}) (\omega_0) c_i(v, i_n) \right) W(A_{i_1}) \cdots W(A_{i_{n-1}}) = \]
\[ \sum_{j'} \left( \sum_{j_{n}} \int (B_{j_n}) (\omega_0) d_{j'(j_n)} \right) W(B_{j_1}) \cdots W(B_{j_{n-1}}) \]
and this reduces to what was to be shown because \( \omega_0 \) was arbitrary.

In what follows it will be assumed \( c_i = 0 \) if any two of the \( i_k \) are equal. That is
\[ \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} (t_1, \cdots, t_n) = 0 \]
if any \( t_i = t_j \).

**Definition 66.3.7** Let \( E_n \) be functions of the form
\[ f(t_1, \cdots, t_n) = \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} (t_1, \cdots, t_n) \]
where the \( A_k \) come from some list of the form \( \{A_1, A_2, \cdots, A_m\} \) where this list of sets is pairwise disjoint, each \( A_k \neq \emptyset \) and \( c_i = 0 \) whenever two indices are equal. By Lemma 66.3.5 this is the same as saying that \( f = 0 \) if \( t_i = t_j \) for some \( i \neq j \). A function of \( n \) variables \( f \) is symmetric means that for \( \sigma \) a permutation,
\[ f(t_1, \cdots, t_n) = f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) \]

**Lemma 66.3.8** Let \( f(t_1, \cdots, t_n) = \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} (t_1, \cdots, t_n) \). Then \( f \) is symmetric if and only if for all \( \{c_{i_1}, \cdots, i_n\} \)
\[ c_{i_1, \cdots, i_n} = c_{i_{\sigma(1)}, \cdots, i_{\sigma(n)}} \]

**Proof:** First of all, every \( c_i = 0 \) if there are repeated indices so it suffices to consider only the case where all indices are distinct.

Consider all the terms associated with a particular set of indices \( \{i_1, \cdots, i_n\} \). Then, since these sets \( A_{i_k} \) are disjoint, the function \( f \) is symmetric if and only if the part of the sum in the definition of \( f \) associated with each such set of indices is
symmetric. To save on notation, denote such a list by \(\{1, 2, \ldots, n\}\). It suffices then to show that

\[
f(t_1, \ldots, t_n) = \sum_{\sigma \in S_n} c_{\sigma(1) \cdots \sigma(n)} X_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} (t_1, \ldots, t_n)
\]

is symmetric if and only if for all \(\sigma, c_{\sigma(1) \cdots \sigma(n)} = c_{1 \cdots n}\). Suppose then that \(f\) is symmetric. Then

\[
f(t_\beta(1), \ldots, t_\beta(n)) = \sum_{\sigma \in S_n} c_{\sigma(1) \cdots \sigma(n)} X_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} (t_\beta(1), \ldots, t_\beta(n)) = \sum_{\sigma \in S_n} c_{\sigma(1) \cdots \sigma(n)} X_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} (t_1, \ldots, t_n) = f(t_1, \ldots, t_n)
\]

However,

\[
\sum_{\sigma \in S_n} c_{\sigma(1) \cdots \sigma(n)} X_{A_{\beta^{-1} \sigma}(1) \times \cdots \times A_{\beta^{-1} \sigma}(n)} (t_1, \ldots, t_n) = \sum_{\sigma \in S_n} c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)} X_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}} (t_1, \ldots, t_n) \tag{66.3.12}
\]

It is supposed to equal

\[
\sum_{\sigma \in S_n} c_{\sigma(1) \cdots \sigma(n)} X_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} (t_1, \ldots, t_n) = \sum_{\sigma \in S_n} c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)} X_{A_{\beta^{-1} \sigma(1)} \times \cdots \times A_{\beta^{-1} \sigma(n)}} (t_1, \ldots, t_n) \tag{66.3.13}
\]

Thus

\[
\sum_{\sigma \in S_n} c_{\sigma(1) \cdots \sigma(n)} X_{A_{\beta^{-1} \sigma}(1) \times \cdots \times A_{\beta^{-1} \sigma}(n)} (t_1, \ldots, t_n) = \sum_{\sigma \in S_n} c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)} X_{A_{\beta^{-1} \sigma}(1) \times \cdots \times A_{\beta^{-1} \sigma}(n)} (t_1, \ldots, t_n) \tag{66.3.14}
\]

Since the sets \(A_k\) are distinct, as explained above, this requires that

\[X_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} \neq X_{A_{\sigma(1)} \times \cdots \times A_{\alpha(n)}}\]

if \(\alpha \neq \sigma\). Therefore, \(66.3.12\) requires that for all \(\beta\) and each \(\sigma\),

\[
c_{\sigma(1) \cdots \sigma(n)} = c_{\beta^{-1} \sigma(1) \cdots \beta^{-1} \sigma(n)}
\]

In particular, this is true if \(\beta = \sigma\) and so \(c_{\sigma(1) \cdots \sigma(n)} = c_{1 \cdots n}\).
The converse of this is also clear. If \( c_{\sigma(1)\cdots\sigma(n)} = c_{1\cdots n} \) for each \( \sigma \), then

\[
 f (t_{\beta(1)}, \cdots, t_{\beta(n)}) = \sum_{\sigma \in S_n} c_{\sigma(1)\cdots\sigma(n)} X_{A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}} (t_{\beta(1)}, \cdots, t_{\beta(n)}) 
\]

\[
 = \sum_{\sigma \in S_n} c_{\sigma(1)\cdots\sigma(n)} X_{A_{\beta-1_{\sigma(1)}} \times \cdots \times A_{\beta-1_{\sigma(n)}}} (t_1, \cdots, t_n) 
\]

\[
 = \sum_{\sigma \in S_n} c_{1\cdots n} X_{A_{\beta-1_{\sigma(1)}} \times \cdots \times A_{\beta-1_{\sigma(n)}}} (t_1, \cdots, t_n) 
\]

\[
 = \sum_{\sigma \in S_n} c_{1\cdots n} X_{A_{\alpha(1)} \times \cdots \times A_{\alpha(n)}} (t_1, \cdots, t_n) 
\]

\[
 = f (t_1, \cdots, t_n)
\]

Observe that \( \mathcal{E}_n \) is a vector space because if you have two such functions

\[
 \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} (t_1, \cdots, t_n), \sum_i d_i X_{B_{i_1} \times \cdots \times B_{i_n}} (t_1, \cdots, t_n)
\]

where the \( A_{i_k} \) are from \( \{A_1, A_2, \cdots, A_m\} \) and the \( B_{i_k} \) are from \( \{B_1, B_2, \cdots, B_q\} \). Then consider the single list consisting of the sets of the form \( A_k \cap B_j \). You could write each of these functions in terms of indicator functions of products of these disjoint sets. Thus the sum of the two functions can be written in the desired form. Since each are equal to 0 when some \( t_j = t_k \), the same is true of their sum. Thus \( \mathcal{E}_n \) is closed with respect to sums. It is obviously closed with respect to scalar multiplication. Hence it is a subspace of the vector space of all functions and it is therefore, a vector space.

Following [37], for \( f \) one of these elementary functions,

\[
 f (t_1, \cdots, t_n) = \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} (t_1, \cdots, t_n)
\]

where if any two indices are repeated, then \( c_i = 0 \), and the \( A_{i_k} \) are all disjoint,

\[
 I_n (f) = \sum_i c_i W (A_{i_1}) \cdots W (A_{i_n})
\]

**Lemma 66.3.9** \( I_n \) is linear on \( \mathcal{E}_n \). If \( f \in \mathcal{E}_n \) and \( \sigma \) is a permutation of \( (1, \cdots, n) \) and

\[
 f_\sigma (t_1, \cdots, t_n) = f (t_{\sigma(1)}, \cdots, t_{\sigma(n)}),
\]

and \( f \) is symmetric, then

\[
 I_n (f_\sigma) = I_n (f)
\]

For \( f = \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} \), one can conclude that

\[
 I_n (f) = n! \sum_{i_1 < i_2 < \cdots < i_n} c_{i_1, \cdots, i_n} \prod_{i} W (A_{i_n}) \quad (66.3.15)
\]
Also, the following holds for the expectation. For \( f, g \in \mathcal{E}_n, \mathcal{E}_m \) respectively,

\[
E (I_n (f) I_m (g)) = \begin{cases} 
0 & \text{if } n \neq m \\
 n! \left\| \tilde{f} \tilde{g} \right\|_{L^2 (T^n)} & \text{if } n = m
\end{cases}
\]

where \( \tilde{f} \) denotes the symetrization of \( f \) given by

\[
\tilde{f} (t_1, \cdots, t_n) \equiv \frac{1}{n!} \sum_{\sigma \in S_n} f (t_{\sigma(1)}, \cdots, t_{\sigma(n)})
\]

**Proof:** It is clear from the definition being well defined that \( I_n \) is linear. In particular, consider

\[
I_n \left( a \sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}} + b \sum_j d_j \mathcal{X}_{B_{j_1} \times \cdots \times B_{j_n}} \right).
\]

As explained above in the observation that \( \mathcal{E}_n \) is a vector space, it can be assumed that the \( A_{i_k} \) and \( B_{j_k} \) are all from a single set of disjoint Borel sets of \( T \). Then the above is of the form

\[
a \sum_i c_i \prod_k W(A_{i_k}) + b \sum_j d_j \prod_k W(B_{j_k})
\]

\[
= a I_n \left( \sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}} \right) + b I_n \left( \sum_j d_j \mathcal{X}_{B_{j_1} \times \cdots \times B_{j_n}} \right)
\]

Next consider for \( i = (i_1 \cdots i_n) \),

\[
f_\sigma (t_1, \cdots, t_n) = \sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}} (t_{\sigma(1)}, \cdots, t_{\sigma(n)})
\]

\[
= \sum_i c_i \prod_{j=1}^n \mathcal{X}_{A_{i_j}} (t_{\sigma(j)}) = \sum_i c_i \prod_{j=1}^n \mathcal{X}_{A_{i_j-1} (j)} (t_j)
\]

\[
= \sum_i c_i \mathcal{X}_{A_{i_1-1} (1) \times \cdots \times A_{i_n-1} (n)} (t_1, \cdots, t_n)
\]

Thus, it appears that \( f_\sigma \neq f \). However,

\[
I_n (f_\sigma) = \sum_i c_i \prod_{k=1}^n W(A_{i_k-1}) = I_n (f)
\]

(66.3.16)

because one just considers the factors in a different order than the other. The permutation acts on \( (i_1 \cdots i_n) \). Define the symetrization of \( f \) by \( \tilde{f} \) given by

\[
\tilde{f} (t_1, \cdots, t_n) \equiv \frac{1}{n!} \sum_{\sigma} f_\sigma (t_1, \cdots, t_n)
\]
Then \( I_n(\hat{f}) = I_n(f) \) and \( \hat{f}(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) = \hat{f}(t_1, \cdots, t_n) \). If \( f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) = f(t_1, \cdots, t_n) \) for all \( \sigma \) then \( \hat{f} = f \). From the above, \( \hat{f} \) equals

\[
\frac{1}{n!} \sum \sigma \sum_i c_i \mathcal{X}_{A_{i-1(1)} \times \cdots \times A_{i-1(n)}} (t_1, \cdots, t_n)
\]

Note that \((66.3.16)\) implies that 

\[
I_n(f) = I_n(\hat{f}) = \frac{n!}{2} \sum_{i_1 < i_2 < \cdots < i_n} \cdots \cdots \cdot \prod_k W(A_{i_k})
\]

Now consider 

\[
\hat{f} = \sum_i c_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}} \text{ and } \tilde{g} = \sum_i d_i \mathcal{X}_{A_{i_1} \times \cdots \times A_{i_n}}
\]

where without loss of generality, these sets \( A_{i_k} \) come from a single list of disjoint sets. As above, \( I_n(f) = I_n(\hat{f}) \) and so it follows that

\[
E(I_n(f)I_n(g)) = E(I_n(\hat{f})I_n(\tilde{g}))
\]

From the above, \( E(I_n(\hat{f})I_n(\tilde{g})) = \)

\[
E\left(\left(\frac{n!}{2}\right)^2 \sum_{i_1 < \cdots < i_n} \sum_{j_1 < \cdots < j_n} c_{i_1, \cdots, i_n} d_{j_1, \cdots, j_n} \prod_k W(A_{i_k}) \prod_l W(A_{j_l})\right)
\]

\[
= \left(\frac{n!}{2}\right)^2 \sum_{i_1 < \cdots < i_n} \sum_{j_1 < \cdots < j_n} c_{i_1, \cdots, i_n} d_{j_1, \cdots, j_n} E\left(\prod_k W(A_{i_k}) \prod_l W(A_{j_l})\right)
\]

\[
= \left(\frac{n!}{2}\right)^2 \sum_{i_1 < \cdots < i_n} \sum_{j_1 < \cdots < j_n} c_{i_1, \cdots, i_n} d_{j_1, \cdots, j_n} E\left(\prod_k W(A_{i_k}) W(A_{j_k})\right) (66.3.18)
\]

That product is of independent random variables. Recall any collection of the \( W(A_{i_k}) \) are normally distributed and also the covariance is diagonal and so these will all be independent random variables. If any one of them is not repeated, say \( W(A_{i_k}) \), then

\[
E\left(\prod_k W(A_{i_k}) W(A_{j_k})\right) = E(W(A_{i_k})) \text{ (stuff)} = 0
\]

It follows that to get something nonzero out of this, all \( A_{i_k} \) are repeated. That is,
you must have \( j = i \) and \( \text{Lemma 66.3.8} \) reduces to \( E(I_n(f) I_n(g)) = \)

\[
\begin{align*}
(n!)^2 \sum_{i_1 < \cdots < i_n} c_{i_1, \ldots, i_n} d_{i_1, \ldots, i_n} E\left( \prod_k W(A_{i_k})^2 \right) \\
&= (n!)^2 \sum_{i_1 < \cdots < i_n} c_{i_1, \ldots, i_n} d_{i_1, \ldots, i_n} \prod_k E\left( W(A_{i_k})^2 \right) \\
&= (n!)^2 \sum_{i_1 < \cdots < i_n} c_{i_1, \ldots, i_n} d_{i_1, \ldots, i_n} \prod_k m(A_{i_k})
\end{align*}
\]

(66.3.19)

By Lemma 66.3.8, used at the end of the following string of equalities, and the observation that

\[ X_{A_{i_1} \times \cdots \times A_{i_n}} X_{A_{j_1} \times \cdots \times A_{j_m}} = 0 \]

to eliminate mixed terms,

\[
\left( \tilde{f}, \tilde{g} \right)_{L^2(T^n)} =
\]

\[
= \int_0^\infty \cdots \int_0^\infty \left( \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} \right) \left( \sum_i d_i X_{A_{i_1} \times \cdots \times A_{i_n}} \right) dt \cdots dt
\]

\[
= \int_0^\infty \cdots \int_0^\infty \left( \sum_i c_i d_i X_{A_{i_1} \times \cdots \times A_{i_n}} \right) dt \cdots dt
\]

\[
= \sum_i c_i d_i \prod_k m(A_{i_k}) = n! \sum_{i_1 < \cdots < i_n} c_{i_1, \ldots, i_n} d_{i_1, \ldots, i_n} \prod_k m(A_{i_k})
\]

Now it follows from this and \( \text{Lemma 66.3.8} \) that

\[
E(I_n(f) I_n(g)) = n! \left( \tilde{f}, \tilde{g} \right)_{L^2(T^n)}.
\]

What happens if you consider \( E(I_n(f) I_m(g)) \) where \( m < n \)? You would still get \( E(I_n(f) I_m(g)) = E\left( I_n \left( \tilde{f} \right) I_m \left( \tilde{g} \right) \right) \)

\[
= E\left( \left( n! \right) \left( m! \right) \sum_{i_1 < i_2 < \cdots < i_n} \sum_{j_1 < \cdots < j_m} c_{i_1, \ldots, i_n} d_{j_1, \ldots, j_m} W(A_{i_1}) \cdots W(A_{i_n}) \right)
\]

Then at least one of the \( W(A_{i_k}) \) is not repeated. This is because \( n > m \). That product is a product of independent random variables at least one of which is of the form \( W(A_{i_k}) \). Hence when you take the expectation of the product it is of the form \( E(W(A_{i_k})) \) (Other terms) = 0. Thus if \( n \neq m \), the result is 0 as claimed. ■

An integral has now been defined on the functions of the form

\[
f(t_1, \cdots, t_n) \equiv \sum_i c_i X_{A_{i_1} \times \cdots \times A_{i_n}} (t_1, \cdots, t_n)
\]
where $f = 0$ if any $t_i = t_j$ for $i \neq j$. This integral defined on these elementary functions is interesting because for such functions $f, g$

$$E(I_n(f) I_m(g)) = \begin{cases} 0 & \text{if } n \neq m \\ n! \left( \tilde{f}, \tilde{g} \right)_{L^2(T^n)} & \text{if } n = m \end{cases}$$

where $\tilde{f}$ is the symmetrization of $f$. It is desired to extend this integral to $L^2(T^n)$. Simple functions are always dense in $L^2(T^n)$. Also, there is an easy lemma which can be concluded for $L^2(T^n)$.

**Lemma 66.3.10** Let $\mathcal{B}_0(T)$ be the Borel sets having finite measure. Linear combinations of functions of the form

$$\mathcal{X}_{A_1 \times \cdots \times A_n}$$

where $A_i \in \mathcal{B}_0(T)$ are dense in $L^2(T, \mathcal{B}^n)$ where of course $\mathcal{B}^n$ refers to the product $\sigma$-algebra.

**Proof:** If you have $U = A_1 \times \cdots \times A_n$ in $T^n$ one can approximate $\mathcal{X}_{U \cap R_p}$ for $R_p \equiv (-p, p)^n$ in $L^2$ with linear combinations of sets of the desired form. In fact, you just consider $\mathcal{X}_{A_1 \cap \cdots \cap A_n (-p, p)}$ and you get equality. Now let $\mathcal{K}$ denote the $\pi$ system of sets of this sort. Let $G$ denote those Borel sets $G$ such that there exists a sequence of linear combinations of sets of the form $\mathcal{X}_{A_i} A_i = A_1 \times \cdots \times A_n$ which converges to $\mathcal{X}_{G \cap R_p}$ in $L^2(T^n)$. Thus $G \supseteq \mathcal{K}$.

Let $\{G_k\}$ be a disjoint sequence of sets of $G$. Is $G \equiv \bigcup_k G_k \in G$? By monotone convergence theorem,

$$\left\| \mathcal{X}_{G \cap R_p} - \sum_{k=1}^m \mathcal{X}_{G_k \cap R_p} \right\|_{L^2(T^n)} < \varepsilon$$

provided $m$ is large enough. Now by definition of $G$ there exists $L_k$ a linear combination of these special sets such that

$$\left\| \mathcal{X}_{G_k \cap R_p} - L_k \right\|_{L^2(T^n)} < \frac{\varepsilon}{m}$$

It follows that

$$\left\| \mathcal{X}_{G \cap R_p} - \sum_{k=1}^m L_k \right\|_{L^2} \leq \left\| \mathcal{X}_{G \cap R_p} - \sum_{k=1}^m \mathcal{X}_{G_k \cap R_p} \right\|_{L^2} + \left\| \sum_{k=1}^m \mathcal{X}_{G_k \cap R_p} - \sum_{k=1}^m L_k \right\| < \varepsilon + \varepsilon$$

and so, it follows that $G \in G$. If $G \in G$, does it follow that $G^c$ is also?

$$\mathcal{X}_{R_p} = \mathcal{X}_{R_p \cap G} + \mathcal{X}_{R_p \cap G^c}$$
Hence
\[ \mathcal{X}_{R_p} - \mathcal{X}_{R_p \cap G} = \mathcal{X}_{R_p \cap G^c} \]
Both of the functions on the left can be approximated in $L^2$ by the desired kind of functions and so the one on the right can also. It follows from Dynkin’s lemma that $\mathcal{G} = \sigma (\mathcal{K})$ which is the product measurable sets. Thus if $U$ is any set in $\mathcal{B}^n$, it follows that $\mathcal{X}_U$ can be approximated in $L^2 (T^n)$ with linear combinations of sets like $\mathcal{X}_{A_1 \times \cdots \times A_n}$.

Of course nothing is known about whether the sets $A_i$ are disjoint. Also it is not known whether these linear combinations of these functions equals 0 if $t_i = t_j$. Thus there is something which needs to be proved.

**Lemma 66.3.11** The functions in $\mathcal{E}_n$ mentioned above are dense in $L^2 (T^n)$.

**Proof:** From Lemma 66.3.10, it suffices to show that $\mathcal{X}_{A_1 \times \cdots \times A_n}$ can be approximated in $L^2 (T^n)$ with functions in $\mathcal{E}_n$. This is where it will be important that the measure is sufficiently like Lebesgue measure. Let $\{ B_k^i \}_{k=1}^m$ be a partition of $A_i$ such that $m (B_k^i) \leq 2^{m(A_1)}$. Let $\{ B_k \}_{k=1}^p$ denote all these sets so $p = mn$. They are not necessarily disjoint because it is not known that the $A_i$ are disjoint. However, one can say that it is possible to choose $e_i$ equal to either 0 or 1 such that
\[ \mathcal{X}_{A_1 \times \cdots \times A_n} = \sum_{i} e_i \mathcal{X}_{B_1^i \times \cdots \times B_n^i} \]
where we can have $B_i^j \subseteq A_k$. Let $J$ be those indices $i$ which involve a repeated set. That is some $B_j^i = B_k^i$ for some $j \neq k$. How many possibilities are there? There are no more than $C(n, 2) m$ because there are $C(n, 2)$ possibilities for duplicates among the $A_k$ and then there are $m$ sets in the partition of $A_k$.

\[
\int_T \cdots \int_T \left( \sum_{i \in J} e_i \mathcal{X}_{B_1^i \times \cdots \times B_n^i} \right)^2 dt \cdots dt \\
= \int_T \cdots \int_T C(n, 2) m \mathcal{X}_{B_1 \times \cdots \times B_n} dt \cdots dt \\
\leq C(n, 2) m \prod_{k=1}^n m (B_k^i) 
\]
The mixed terms are 0 because for a fixed $k$, $\{ B_k^i \}_{i=1}^m$ are disjoint. Now from the description of these, $m (B_k^i) m < m (A_k)$ and so
\[
\int_T \cdots \int_T \left( \sum_{i \in J} e_i \mathcal{X}_{B_1^i \times \cdots \times B_n^i} \right)^2 dt \cdots dt \\
\leq C(n, 2) m \prod_{k=1}^n \frac{2m (A_k)}{m} = C(n, 2) m \frac{m^n}{m^n} \prod_{k=1}^n m (A_k) 
\]
which clearly converges to 0 as \( m \to \infty \) provided that \( n \geq 2 \). In case \( n = 1 \), all you have to do is approximate \( \mathcal{X}_A \) from something in \( \mathcal{E}_1 \) and of course you just use \( \mathcal{X}_A \).

Let \( f, g \in \mathcal{E}_n \). Then from Lemma 66.3.9,

\[
E \left( (I_n (f-g))^2 \right) = n! \left\| \hat{f} - \hat{g} \right\|_{L^2(T^n)}^2
\]

\[
\left\| \hat{f} \right\|_{L^2(T^n)} = \left( \int_T \cdots \int_T \left| \hat{f}(t) \right|^2 \, dt \right)^{1/2}
\]

\[
= \left( \int_T \cdots \int_T \left| \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) \right|^2 \, dt \right)^{1/2}
\]

\[
\leq \frac{1}{n!} \sum_{\sigma} \left( \int_T \cdots \int_T \left| f(t_{\sigma(1)}, \cdots, t_{\sigma(n)}) \right|^2 \, dt \right)^{1/2}
\]

\[
= \frac{1}{n!} \sum_{\sigma} \|f\|_{L^2(T^n)} = \|f\|_{L^2(T^n)}
\]

Therefore,

\[
E \left( (I_n (f-g))^2 \right) = n! \left\| \hat{f} - \hat{g} \right\|_{L^2(T^n)}^2 \leq n! \|f - g\|_{L^2(T^n)}^2.
\] (66.3.20)

The following theorem comes right away from this and Lemma 66.3.11.

**Theorem 66.3.12** The integral \( I_n \) defined on \( \mathcal{E}_n \) extends uniquely to an integral \( I_n \) defined on \( L^2(T^n) \). This integral satisfies

\[
I_n (f) \in L^2(\Omega)
\]

Also

\[
E (I_n (f) I_n (g)) = n! \left( \hat{f}, \hat{g} \right)_{L^2(T^n)}
\]

**Proof:** This follows right away from the density of \( \mathcal{E}_n \) in \( L^2(T^n) \) and the inequality 66.3.11.

Obviously one wonders whether linear combinations \( \sum c_n I_n (f_n) \) are dense in \( L^2(\Omega) \). It looks like the important thing to notice is that for \( f \in \mathcal{E}_n \), \( I_n (f) \) is a polynomial in \( W(A_{i_k}) \equiv W(X_{A_{i_k}}) \). Recall the corollary above, Corollary 66.2.1.

**Corollary 66.3.13** Let \( \mathcal{P}_n^0 \) denote all polynomials of the form

\[
p(W(h_1), \cdots, W(h_k)), \text{ degree of } p \leq n, \text{ some } h_1, \cdots, h_k
\]

Also let \( \mathcal{P}_n \) denote the closure in \( L^2(\Omega, \mathcal{F}, P) \) of \( \mathcal{P}_n^0 \). Then

\[
\mathcal{P}_n = \bigoplus_{i=0}^n \mathcal{H}_i
\]
Consider $\cup_{p \leq n} \{ I_p (f) : f \in \mathcal{E}_p \}$. This is a subset of $P^0_n$ and so it is a subset of $\bigoplus_{i=0}^n \mathcal{H}_i$. Now for $h \in L^2 (T^n)$, it was shown above that there exists a sequence $g_k \to h$ in $L^2 (T^n)$ where each $h_k \in \mathcal{E}_n$. Then $I_n (g_k) \to I_n (h)$. In particular, if $h \in L^2 (T) \equiv H$, then there is a sequence $g_k \in \mathcal{E}_1$ such that $g_k \to h$ in $L^2 (T)$. Then clearly
\[
E \left( |W (g_k) - W (h)|^2 \right) = E \left( |W (g_k) - h|^2 \right) = \|g_k - h\|^2_{L^2 (T)} \to 0
\]
and so each polynomial $p (W (h_1), \cdots, W (h_k))$ can be approximated in $L^2 (\Omega)$ by one which is of the form $p (W (g_1), \cdots, W (g_k))$ where each $g_j \in \mathcal{E}_1$. Corresponding to each $g_j$ there is a list of disjoint sets. Now consider the union of all the sets just described and let $\{ A_k \}$ be a partition of this union such that the $A_k$ are pairwise disjoint and for each $j$, every set corresponding to $g_j$ is partitioned by a subset of the $\{ A_k \}$. Thus
\[
g_j = \sum_i c_i \chi_{B_i} = \sum_i c_i \sum_{s=1}^{m_i} \chi_{A_{is}^j}
\]
where $B_i$ is partitioned by the $A_{is}^j$. Then consider $p (g_1, \cdots, g_k)$. Then the terms of degree $m$ are of the form
\[
p_m \equiv \sum_i c_i \chi_{A_{1i} \times \cdots \times A_{mi}} \quad (66.3.21)
\]
where the $A_{1i}$ come from the list of disjoint sets $\{ A_k \}$. The terms of degree $m$ in $p (W (g_1), \cdots, W (g_k))$ are also of the form
\[
p_m (W (g_1), \cdots, W (g_k)) \equiv \sum_i c_i \prod_k W (A_{ik})
\]

The problem is that $1 \leq n$ is not in $\mathcal{E}_m$ because it is not known whether $c_1 = 0$ if two indices are repeated. However, as explained in the proof of Lemma 10.3.1, there is a further partition such that the contribution of those terms corresponding to $i$ in which two indices are repeated can be made as small as desired. Therefore, the terms of order $m$ are approximated in $L^2 (T^n)$ by $g_m \in \mathcal{E}_m$. Assume this approximation is good enough that, from the estimates given above in Lemma 10.3.1,
\[
E \left( |I_m (g_m) - p_m (W (g_1), \cdots, W (g_k))|^2 \right)^{1/2} < \frac{\varepsilon}{n + 1}
\]
Thus, taking a succession of partitions if necessary,
\[
E \left( \left| p (W (g_1), \cdots, W (g_k)) - \sum_{m=0}^n I_m (g_m) \right|^2 \right)^{1/2} \leq \sum_{m=1}^n E \left( |I_m (g_m) - p_m (W (g_1), \cdots, W (g_k))|^2 \right)^{1/2} < \sum_{m=1}^n \frac{\varepsilon}{n + 1} < \varepsilon.
\]
This has proved the following lemma.
Lemma 66.3.14 Let \( n \) be given. Then \( \cup_{p \leq n} \{ I_p(f) : f \in \mathcal{E}_p \} \) is dense in \( \mathcal{P}_n = \bigoplus_{i=0}^{n} \mathcal{H}_i \). Consequently, every \( f \in L^2(\Omega, \mathcal{F}) \) may be written as an infinite sum
\[
f = \sum_{k=1}^{\infty} c_k I_k(g_k)
\]
where \( g_k \in \mathcal{E}_k \) and it can also be assumed that \( g_k \) is symmetric.

Proof: It only remains to verify that \( g_k \) can be symmetric. However, this is obvious because if \( g_k \) is replaced with \( \tilde{g}_k \) the integral \( I_k \) is unchanged.

66.4 The Skorokhod Integral

This integral allows for one to obtain a stochastic integral of functions which are not adapted. It is a generalization of the Ito integral. There is also a strange sort of derivative which can be defined and the two are related in a natural way.

66.4.1 The Derivative

Let \( F : \mathbb{R}^n \rightarrow \mathbb{R} \) be smooth and have polynomial growth. Then consider
\[
F(W(h_1), \cdots, W(h_n))
\]
where \( W \) is defined above. Recall that \( h \in H \) a separable real Hilbert space and \( W(h) \in L^2(\Omega, \mathcal{F}, P) \) where \( \mathcal{F} = \sigma(W(h), h \in H) \). Also \( (W(h_1), \cdots, W(h_n)) \) is multivariate normal and \( E(W(g)W(h)) = (h, g)_H \).

Definition 66.4.1 In the above situation,
\[
DF \equiv \sum_{k=1}^{n} D_k F(W(h_1), \cdots, W(h_n)) h_k
\]
Thus from Lemma 66.4.2, \( F, D_k F \) are in \( L^p(\Omega) \) and so \( DF \) is in \( L^p(\Omega; H) \) for every \( p \).

First it is good to consider whether \( DF \) is well defined.

Lemma 66.4.2 The derivative is well defined. Also, if \( F(W(h_1), \cdots, W(h_n)) = 0 \) for \( \{h_1, \cdots, h_n\} \) independent, then for all \( x, F(x) = 0 \).

Proof: Suppose \( F(W(h_1), \cdots, W(h_n)) = 0 \).

Is it true that \( DF = 0 \)? Let \( \lambda \) be the distribution measure of \( (W(h_1), \cdots, W(h_n)) \equiv W(h) \). Then the above requires that for any ball \( B \) in \( \mathbb{R}^n \),
\[
E(\chi_B(W(h))F^2(W(h))) = \int_B F^2(x) d\lambda(x) = 0
\]
66.4. THE SKOROKHOD INTEGRAL

If \( \{h_1, \ldots, h_n\} \) is independent, then \( \lambda \) has a normal density function and \( \lambda \ll m_n \) and so \( F^2(x) = 0 \) for a.e. \( x \). Since \( F \) is smooth, this means that \( F = 0 \) everywhere. Hence \( D_k F = 0 \) and so \( DF = 0 \). Thus the case where the \( h_i \) are independent is easy.

Next suppose without loss of generality that a basis for \( \text{span} \{h_1, \ldots, h_n\} \) is \( \{h_1, \ldots, h_r\} \) where \( r < n \). Say \( h_k = \sum_{i=1}^r c_{ki} h_i \) for \( k > r \). Then

\[
0 = F \left( W(h_1), \ldots, W(h_r), W \left( \sum_{i=1}^r c_{1i} h_i \right), \ldots, W \left( \sum_{i=1}^r c_{ni} h_i \right) \right)
\]

and so in terms of \( \{h_1, \ldots, h_r\} \),

\[
DF = \sum_{i=1}^r (D_i F) h_i + \sum_{i=r+1}^n \left( \sum_{j=1}^r (D_j F) c_{ji} \right) h_j
\]

Now it was just shown that \( G(x) \) is identically 0 and so \( D_j G = 0, \ j \leq r \). So what is \( D_j G \)? From the above, it equals

\[
D_j F + \sum_{i=r+1}^n (D_i F) c_{ji} = 0
\]

Hence \( DF = 0 \). Now if \( F(W(h_1), \ldots, W(h_n)) = G(W(k_1), \ldots, W(k_m)) \), then \( F - G = 0 \) and so from what was just shown, \( D(F - G) = DF - DG = 0 \). Thus the derivative is well defined.

**Lemma 66.4.3** Let \( \mathcal{P} \) denote the set of all polynomials in \( W(h) \) for \( h \in H \). Then \( \mathcal{P} \) is dense in \( L^p(\Omega) \).
CHAPTER 66. A DIFFERENT KIND OF STOCHASTIC INTEGRATION

Proof: Let \( g \in L^p' (\Omega) \) and suppose that for every \( f \in D, \int_\Omega g f dP = 0 \). Does it follow that \( g = 0 \)? If so, then by the Riesz representation theorem, \( \mathcal{P} \) is dense in \( L^p (\Omega) \). From Lemma \ref{lem:62.6.4}, for a given \( h \), there is a sequence of functions of \( \mathcal{P}, \{ f_n \} \) which converges to \( e^{W(h)} \) in \( L^p (\Omega) \). It follows that

\[
\int_\Omega g e^{W(h)} dP = \lim_{n \to \infty} \int_\Omega g f_n dP = 0
\]

Hence by Lemma \ref{lem:62.6.5} it follows that \( g = 0 \). Hence \( \mathcal{P} \) is dense in \( L^p (\Omega) \).

Let \( D^{1,p} \) denote the closure in \( L^p (\Omega) \) of functions in \( \mathcal{P} \) with respect to the seminorm

\[
\| f \|_{1,p} \equiv \left( \| f \|_{L^p(\Omega)}^p + \| Df \|_{L^p(\Omega,H)}^p \right)^{1/p}
\]

By this we mean the following. The above \( \| f \|_{1,p} \) makes perfect sense for every \( f \in \mathcal{P} \) and is algebraically like a norm. Thus it makes \( \mathcal{P} \) into a normed linear space. \( D^{1,p} \) is just the completion of this normed linear space. Then for \( f \in D^{1,p} \), we define \( Df \equiv \lim_{n \to \infty} Df_n \) in \( L^p (\Omega, H) \) where \( f_n \in \mathcal{P} \).

66.4.2 The Integral

The derivative has been defined above. Now here is the definition of the integral defined on functions in \( L^p (\Omega, H) \), possibly not all of them.

Definition 66.4.4 We say a random variable \( F \) is “smooth” if it is of the form \( F(\omega) = F(W(h_1), \cdots W(h_r)) \) where \( x \to F(x) \) is a smooth function of the real variables \( x_i \). It has polynomial growth if

\[
\frac{|F(x)|}{\left(1 + |x|^2\right)^m}
\]

is bounded for some positive integer \( m \). Let \( u \in L^p' (\Omega, H) \). Then \( u \in D (\delta) \) if for all \( F \) smooth having polynomial growth in the \( W(h) \),

\[
|E \langle DF, u \rangle| \leq C(u) \| F \|_{L^p(\Omega)}
\]

Then \( \delta u \in L^p' (\Omega) \) is defined by

\[
E \langle DF, u \rangle = E(F \delta u)
\]

Thus you have \( \delta \) is the adjoint of \( D \).

\[
\begin{align*}
L^p (\Omega) &\overset{D}{\supseteq} D(D) &\overset{\delta}{\supseteq} L^p' (\Omega, H) \\
L^p (\Omega) &\overset{\delta}{\supseteq} L^p' (\Omega, H) &\overset{D}{\supseteq} L^p (\Omega, H)
\end{align*}
\]
Next it is shown that there are functions in $D (\delta)$ by giving examples of them. It turns out that functions of the form $\sum_i F_i h_i$ where $F_i$ is smooth with polynomial growth are in $D (\delta)$. Consider

$$E \langle DG, F (W (h_1), \ldots, W (h_n)) h \rangle$$

where $G = G (W (k_1), \ldots, W (k_p))$ and for simplicity, $\|h\| H = 1$.

Consider the vectors $\{h, h_1, \ldots, h_n, k_1, \ldots, k_p\}$. Starting with the left and moving toward the right, delete vectors which are dependent on the preceding vectors, obtaining a linearly independent set of vectors which includes $h$. Then let $\{h, e_1, \ldots, e_q\}$ be an orthonormal basis having the same span as the original vectors $\{h, h_1, \ldots, h_n, k_1, \ldots, k_p\}$. Then from the fact that $W$ is linear, there are smooth functions having polynomial growth $\hat{G}, \hat{F}$ such that

$$G (W (k_1), \ldots, W (k_p)) = \hat{G} (W (h), W (e_1), \ldots, W (e_q))$$

$$F (W (h_1), \ldots, W (h_n)) = \hat{F} (W (h), W (e_1), \ldots, W (e_q))$$

Note that $h_i = \sum_{j=1}^q (h_i, e_j) e_j + (h, h) h$. Thus

$$F (W (h_1), \ldots, W (h_n)) =$$

$$F \left( W \left( \sum_{j=1}^q (h_1, e_j) e_j + (h_1, h) h \right), \ldots, W \left( \sum_{j=1}^q (h_n, e_j) e_j + (h_n, h) h \right) \right)$$

$$= F \left( \sum_{j=1}^q (h_1, e_j) W (e_j) + (h_1, h) W (h), \ldots, \sum_{j=1}^q (h_n, e_j) W (e_j) + (h_n, h) W (h) \right)$$

and so, $D_1 \hat{F}$ is given by

$$D_1 \hat{F} = \sum_{i=1}^n D_i (F (W (h_1), \ldots, W (h_n))) (h_i, h)$$

Then by Lemma 66.4.2

$$E \langle DG, F (W (h_1), \ldots, W (h_n)) h \rangle = E \langle D \hat{G}, \hat{F} h \rangle$$

$$= E \left( D_1 \hat{G} \right) h + \sum_{k=1}^q D_k \left( \hat{G} \right) e_k \hat{F} h \right) = E \left( D_1 \left( \hat{G} \right) \hat{F} \right)$$

$$= \frac{1}{(\sqrt{2\pi})^{q+1}} \int_{\mathbb{R}^q} \int_{\mathbb{R}^q} D_1 \hat{G} (\mathbf{x}) \hat{F} (\mathbf{x}) e^{-\frac{1}{2} |\mathbf{x}|^2} dx_1 d\mathbf{x}_1$$

$$= \frac{-1}{(\sqrt{2\pi})^{q+1}} \int_{\mathbb{R}^q} \hat{G} (\mathbf{x}) D_1 \left( \hat{F} (\mathbf{x}) e^{-\frac{1}{2} |\mathbf{x}|^2} \right) dx_1 d\mathbf{x}_1$$
Functions of the form $\sum_{j=1}^{m} F_j h_j$ where $F_j$ is a polynomial in variables of the form $W(h)$ dense in $L^p(\Omega, H)$? It was shown earlier in (66.4.22) that polynomial functions $F$ in the $W(h)$ are dense in $L^p(\Omega)$ for any $p$. Let $s(\omega) = \sum_{k=1}^{n} h_k X_{E_k}$ be a simple function. Then $X_{E_k}$ is clearly in $L^p(\Omega)$ and so there exists $F_k$ a polynomial in the $W(h)$ which is as close as desired to $X_{E_k}$ in $L^p$. Hence $\sum_{k=1}^{n} h_k F_k$ is close to $s$ in $L^p(\Omega, H)$ and so since these simple functions are dense, it follows that these kinds of functions are indeed dense in $L^p(\Omega, H)$, this for any $p > 1$. The above discussion is summarized in the following lemma.

**Lemma 66.4.5** Functions of the form $\sum_{k=1}^{n} F_k h_k$ where $F_k$ is a polynomial in the $W(h)$ ($F_j \in \mathcal{P}$) are dense in $L^p(\Omega, H)$ for any $p > 1$. Also each function of this form is in $D\delta$ and

$$\delta \left( \sum_{j=1}^{m} F_j h_j \right) = \sum_{j=1}^{m} \delta (F_j h_j) = \sum_{j=1}^{m} F_j W(h_j) - \langle DF_j, h_j \rangle$$

What does $D$ do to $\delta (Fh)$? It is shown above that $\delta (Fh) = FW(h) - \langle DF, h \rangle$. Say $F = F(W(h_1), \cdots, W(h_n))$. Then when you do $D$ to $\delta (Fh)$, you would get

$$Fh + \sum_{k=1}^{n} D_k(F) W(h) h_k - \sum_{k=1}^{n} \sum_{j=1}^{n} D_j(D_k(F)) h_j(h_k, h)$$
66.4. THE SKOROKHOD INTEGRAL

In other words,

\[ Fh + W(h) D(F) - D\langle DF, h \rangle \]

Recall that \( DG \) is well defined. This means that we can replace \( \{h_1, \cdots, h_n, h\} \) with an orthonormal basis \( \{e_1, \cdots, e_p, h\} \) as in

\[ G(W(h_1), \cdots, W(h_n), W(h)) = \hat{G}(W(e_1), \cdots, W(e_p), W(h)) \]

where we assume \( \|h\| = 1 \) for simplicity. Thus the above equals

\[ D(\delta(F)) = D(\delta(F)) = \hat{F}h + W(h) D(\hat{F}) - D\langle D\hat{F}, h \rangle \]

Now consider \( E(\delta(F)^2) = E(D(\delta(F)), Fh) \). Thus the following must be considered.

\[ E(\langle \hat{F}h + W(h) D(\hat{F}) - D\langle D\hat{F}, h \rangle, \hat{F}h \rangle) \quad (66.4.23) \]

Consider the terms involved. The first term is just

\[ E(\hat{F}h^2) = E\left(\|\hat{F}h\|_H^2\right) = E\left(\|Fh\|_H^2\right) \]

Now consider the third term. It equals

\[ -E(D(D_{p+1}\hat{F})\hat{F}h) = -E(D_{p+1}^2\hat{F}) \]

\[ = \frac{-1}{(\sqrt{2\pi})^{p+1}} \int_{\mathbb{R}^p} \int_{\mathbb{R}} D_{p+1}^2(\hat{F}(x)) \hat{F}(x) e^{-\frac{1}{2}|x|^2} dx_{p+1} d\hat{x}_{p+1} \]

\[ = \frac{1}{(\sqrt{2\pi})^{p+1}} \int_{\mathbb{R}^p} \int_{\mathbb{R}} D_{p+1}(\hat{F}(x)) D_{p+1}(\hat{F}(x)) e^{-\frac{1}{2}|x|^2} dx_{p+1} d\hat{x}_{p+1} \]

\[ - \frac{1}{(\sqrt{2\pi})^{p+1}} \int_{\mathbb{R}^p} \int_{\mathbb{R}} D_{p+1}(\hat{F}(x)) (x_{p+1}\hat{F}(x)) e^{-\frac{1}{2}|x|^2} dx_{p+1} d\hat{x}_{p+1} \]

\[ = E\left(D_{p+1}^2\hat{F}\right) - E(W(h) D_{p+1}(\hat{F}) \hat{F}) \]

\[ = E\left(D_{p+1}\hat{F}\right) - E(W(h) D(\hat{F}), \hat{F}h) \]

Hence \( 66.4.23 \) reduces to

\[ E\left(\|\hat{F}h\|^2\right) + E\left(D_{p+1}\hat{F}\right)^2 = E\left(\|\hat{F}h\|^2\right) + E\left(D(\hat{F}), h^2\right) \]

\[ = E\left(\|Fh\|^2\right) + E\left(D(\hat{F}), h^2\right) \]
This assumed that \( \|h\| = 1 \). For arbitrary nonzero \( h \),

\[
E \left( \delta (Fh)^2 \right) = \|h\|^2 E \left( \delta \left( F \frac{h}{\|h\|} \right)^2 \right) = \|h\|^2 \left( E \left( \left\| F \frac{h}{\|h\|} \right\|^2 \right) + E \left( \left\langle D(F), \frac{h}{\|h\|} \right\rangle^2 \right) \right) = E \left( \|Fh\|^2 \right) + E \left( \langle D(F), h \rangle^2 \right)
\]

Next consider a generalization, \( u = \sum_{j=1}^{m} F_j h_j \) where the \( \{h_j\} \) is an orthonormal set of vectors. Say \( F_j = F_j (W(k_1), \cdots, W(k_n)) \). Let \( \{h_1, \cdots, h_m, e_1, \cdots, e_p\} = \{g_i\}_{i=1}^{m+p} \) be an orthonormal basis for the span of all the \( h_j \) and \( k_i \). Thus \( g_i = h_i \) for \( i \leq m \). Then let

\[
F_j (W(k_1), \cdots, W(k_n)) = \hat{F}_j (W(h_1), \cdots, W(h_m), W(e_1), \cdots, W(e_p))
\]

The computations will be done with respect to this orthonormal set because it will be simpler. Also, the above argument using the density function for the normal distribution will be used without explicitly repeating it.

It is desired to consider \( E \left( \delta (u)^2 \right) \). Recall that

\[
D(\delta (Fh)) = Fh + W(h) D(F) - D(DF, h).
\]

Thus \( E \left( \delta (u)^2 \right) = \)

\[
\sum_{j,k=1}^{m} E \left( \delta (\hat{F}_j h_j) \delta (\hat{F}_k h_k) \right) = \sum_{j,k=1}^{m} E \left( \left\langle D \left( \delta (\hat{F}_j h_j) \right), (\hat{F}_k h_k) \right\rangle \right) = \sum_{j,k=1}^{m} E \left( \left\langle \hat{F}_j h_j + W(h_j) D(\hat{F}_j) - D(D\hat{F}_j, h_j), (\hat{F}_k h_k) \right\rangle \right)
\]

Separating out the first term this is

\[
= E \left( \sum_{k=1}^{m} \left\| \hat{F}_k \right\|^2 \right) + \sum_{k,k} E \left( \left\langle W(h_j) D(\hat{F}_j), \hat{F}_k h_k \right\rangle \right) - \sum_{j,k} E \left( D_k \left( D_j \hat{F}_j \right) F_k \right)
\]

\[
= E \left( \sum_{k=1}^{m} \left\| \hat{F}_k \right\|^2 \right) + \sum_{k,k} E \left( \left\langle W(h_j) D(\hat{F}_j), \hat{F}_k h_k \right\rangle \right) - \sum_{j,k} E \left( D_k \left( D_j \hat{F}_j \right) F_k \right)
\]
66.4. THE SKOROKHOD INTEGRAL

\[ E \left( \sum_{k=1}^{m} \left\| \hat{F}_k \right\|^2 \right) + \sum_{k,k} E \left( W(h_j) D_k \left( \hat{F}_j \right) \hat{F}_k \right) - \sum_{j,k} E \left( D_k \left( D_j \hat{F}_j \right) \hat{F}_k \right) \]

(66.4.24)

By equality of mixed partial derivatives, the third term equals

\[- \sum_{j,k} E \left( D_j \left( D_k \hat{F}_j \right) \hat{F}_k \right) = \sum_{j,k} E \left( D_k \left( D_j \hat{F}_j \right) \hat{F}_k \right) - \sum_{j,k} E \left( D_k \left( \hat{F}_j \right) \hat{F}_k W(h_j) \right) \]

Therefore, (66.4.24) reduces to

\[ E \left( \delta \left( \sum_{j=1}^{m} F_j h_j \right) \right)^2 = E \left( \sum_{k=1}^{m} \left\| \hat{F}_k \right\|^2_H \right) + \sum_{j,k} E \left( \left( D_k \hat{F}_j \right) \left( D_j \hat{F}_k \right) \right) \]

\[ = E \left( \sum_{k=1}^{m} \left\| \hat{F}_k \right\|^2_H \right) + \sum_{j,k} E \left( \left\langle D\hat{F}_j, h_k \right\rangle \left\langle D\hat{F}_k, h_j \right\rangle \right) \]

\[ = E \left( \sum_{k=1}^{m} \left\| F_k \right\|^2_H \right) + \sum_{j,k} E \left( \left\langle D F_j, h_k \right\rangle \left\langle D F_k, h_j \right\rangle \right) \]

because the derivative is well defined. All of this assumes the \( h_k \) form an orthonormal set. Suppose these are just orthogonal but nonzero. Then

\[ E \left( \delta \left( \sum_{j=1}^{m} F_j h_j \right) \right)^2 = E \left( \sum_{j=1}^{m} \delta \left( F_j h_j \right) \right)^2 = E \left( \sum_{j,k} \delta \left( F_j h_j \right) \delta \left( F_k h_k \right) \right) \]

\[ = E \left( \sum_{j,k} \left\| h_j \left\| \left\| h_k \right\| \delta \left( F_j h_j / \left\| h_j \right\| \right) \delta \left( F_k h_k / \left\| h_k \right\| \right) \right) \right) \]

and doing exactly the same steps but keeping the factor \( \left\| h_j \right\| \left\| h_k \right\| \) throughout, this yields

\[ E \left( \sum_{k=1}^{m} \left\| h_k \right\|^2 \left\| F_k \right\|^2_H \right) + \sum_{j,k} E \left( \left\| h_j \right\| \left\| h_k \right\| \left\langle D F_j, h_k / \left\| h_k \right\| \right\rangle \left\langle D F_k, h_j / \left\| h_j \right\| \right\rangle \right) \]

\[ = E \left( \sum_{k=1}^{m} \left\| F_k h_k \right\|^2_H \right) + \sum_{j,k} E \left( \left\langle D F_j, h_k \right\rangle \left\langle D F_k, h_j \right\rangle \right) \]

\[ = E \left( \left\| \sum_{k=1}^{m} F_k h_k \right\|^2_H \right) + \sum_{j,k} E \left( \left\langle D F_j, h_k \right\rangle \left\langle D F_k, h_j \right\rangle \right) \]
It appears from the computations to be correct, but it does not look right. This is because the second term is not clearly nonnegative. It is the expectation of the trace of $A^2$ where $A$ is the matrix whose $jk^{th}$ entry is $\langle DF_j, h_k \rangle$. One wonders whether the end result is nonnegative.

### 66.4.3 The Ito And Skorokhod Integrals

If you let $H = L^2(0, \infty; U)$ where $U$ is a separable Hilbert space, and if $f \in D(\delta)$, it is very natural to ask whether $f \mathcal{X}_{(0,t)} \in D(\delta)$. This is not so. There is a counter example given in [93]. However, this is true if you change the definition of the integral such that in the definition of $\delta$, it is only necessary for

$$|\langle DF, G \rangle| \leq C \|F\|_{L^p(\Omega)}$$

where $F$ is in $\mathcal{P}$. When you see why this is so, it will be clear why it is not so for the definition given above.

**Lemma 66.4.6** Suppose the definition of the Skorokhod integral $\delta$ is changed so that it is only necessary to have

$$|\langle DF, G \rangle| \leq C \|F\|_{L^p(\Omega)}$$

for all $F$ in $\mathcal{P}$. Then let $H \equiv L^2(0, \infty; U)$ or $L^2([0,T]; U)$ where $U$ is a separable real Hilbert space. For this modified definition of the integral, if $f \in D(\delta)$, it follows that $f \mathcal{X}_{(0,t)} \in D(\delta)$.

**Proof:** The case $L^2(0, \infty; U)$ is considered here. The other case is similar. $\delta$ will be defined on some things in $L^2(\Omega, L^2(0, \infty; U), \mathcal{F})$ where, as discussed earlier,

$$\mathcal{F} = \sigma(W(h) : h \in H)$$

Then if you have $f \in D(\delta)$ so $f \in L^2(\Omega, L^2(0, \infty; U))$, does it follow that $f \mathcal{X}_{(0,t)} \in D(\delta)$ also? Let $F$ be one of those polynomial functions of some $W(h)$. Assume first that $a_0$, the constant term is 0 and consider

$$E \langle DF, f \mathcal{X}_{[0,t]} \rangle = E \left( \sum_k D_k(F) h_k, f \mathcal{X}_{[0,t]} \right)$$

Since $h_k \in H = L^2(0, \infty; U)$, so is $h_k \mathcal{X}_{[0,t]}$. Thus the above reduces to

$$= \sum_k E \langle D_k(F) h_k, f \mathcal{X}_{[0,t]} \rangle = \sum_k E \left( \int_0^\infty D_k(F(W(h_1), \cdots, W(h_n))) h_k \mathcal{X}_{[0,t]} f dt \right)$$

Since $F$ is just a polynomial and $W$ is linear and $\mathcal{X}^q_{[0,t]} = \mathcal{X}_{[0,t]}$, this equals

$$\sum_k E \left( \int_0^\infty D_k(F(W(h_1), \cdots, W(h_n))) h_k \mathcal{X}_{[0,t]} f dt \right)$$
Let \( F_t = F(W(\mathcal{X}_{[0,t]} h_1), \ldots W(\mathcal{X}_{[0,t]} h_n)) \) and so the above is nothing more than
\[
E\langle DF, f\mathcal{X}_{[0,t]} \rangle = E\langle DF_t, f \rangle
\]
and since \( f \in D(\delta) \),
\[
|E\langle DF, f\mathcal{X}_{[0,t]} \rangle| = |E\langle DF_t, f \rangle| \leq C(f) \|F_t\|_{L^2(\Omega)}
\]
Also
\[
\|F_t\|_{L^2(\Omega)}^2 = \int_{\Omega} F(\mathcal{X}_{[0,t]} h_1, \ldots W(\mathcal{X}_{[0,t]} h_n))^2 dP
\]
\[
= \int_{\Omega} X_{[0,t]} F(W(h_1), \ldots W(h_n))^2 dP
\]
\[
\leq \int_{\Omega} F(W(h_1), \ldots W(h_n))^2 dP
\]
Thus for such \( F \) which have zero constant term,
\[
|E\langle DF, f\mathcal{X}_{[0,t]} \rangle| \leq C \|F\|_{L^2(\Omega)}
\]
Now what if \( F \) is a constant \( a \)? In this case, \( DF = Da = 0 \)
\[
|E\langle Da, f\mathcal{X}_{[0,t]} \rangle| = 0 \leq \|a\|_{L^2(\Omega)}
\]
It follows that \( \mathcal{X}_{[0,t]} f \in D(\delta) \) whenever \( f \) is. ■

Note how it was essential in this argument to have \( F \) be a polynomial or perhaps more generally an analytic function. However, in the definition of the Skorokhod integral, one must test with functions \( F \) which are smooth and have polynomial growth. In particular, this would include functions which are infinitely differentiable with compact support, none of which have valid power series.

How does the Skorokhod integral relate to the Ito integral? What about elementary functions and so forth? Let \( 0 = t_0 < t_1 < \cdots < t_n = T \). Consider
\[
\sum_{k=0}^{n-1} F_k \mathcal{X}_{(t_k,t_{k+1})}
\]
As shown above, this is one of the things in \( D(\delta) \).
\[
\delta \left( \mathcal{X}_{(0,t)} \sum_{k=0}^{n-1} F_k \mathcal{X}_{(t_k,t_{k+1})} \right) = \delta \left( \sum_{k=0}^{n-1} F_k \mathcal{X}_{[t_k \wedge t,t \wedge t_{k+1}]} \right)
\]
\[
= \sum_{k=0}^{n-1} F_k W(\mathcal{X}_{[t_k \wedge t,t \wedge t_{k+1}]}) - \langle DF_k, \mathcal{X}_{[t_k \wedge t,t \wedge t_{k+1}]} \rangle
\]
\[
= \sum_{k=0}^{n-1} F_k \left( W(\mathcal{X}_{(0,t \wedge t_{k+1})}) - W(\mathcal{X}_{(0,t \wedge t_k)}) \right) - \langle DF_k, \mathcal{X}_{[t_k \wedge t,t \wedge t_{k+1}]} \rangle
\]
CHAPTER 66. A DIFFERENT KIND OF STOCHASTIC INTEGRATION

In terms of the Wiener process, this is of the form
\[
= \sum_{k=0}^{n-1} F_k (W(t \wedge t_{k+1}) - W(t \wedge t_k)) - \langle DF_k, \mathcal{X}_{[0,t \wedge t_{k+1}]} - \mathcal{X}_{[0,t \wedge t_k]} \rangle_H
\]

What if
\[F_k = F_k (W(X_{0,t_k}^h_1), \ldots, W(X_{0,t_k}^h_n))\]?

Let \( F_k \equiv \sigma (W(X_{0,t_k}^h) : h \in H) \). Then this is clearly a filtration. If \( F_k \) is as just described, then \( F_k \) is \( \mathcal{F}_{t_k} \) adapted.

\[
\langle DF_k, \mathcal{X}_{[0,t \wedge t_{k+1}]} - \mathcal{X}_{[0,t \wedge t_k]} \rangle = \int_0^\infty \sum_s D_s (F_k) \mathcal{X}_{(0,t_k)}^h \mathcal{X}_{(t \wedge t_k, t \wedge t_{k+1})} = 0
\]
because the intervals are disjoint. In this case, the troublesome term at the end vanishes and you are left with

\[
\sum_{k=0}^{n-1} F_k (W(t \wedge t_{k+1}) - W(t \wedge t_k)) \tag{66.4.25}
\]

which is similar to the usual definition for the Ito integral.

What if \( F \in L^2 (\Omega \times [0,T]) \) and is progressively measurable. Does it have a Skorokhod integral, and if so, is it the same as the Ito integral? Recall the following useful lemma. It is Lemma 63.3.1 on Page 2290.

Lemma 66.4.7 Let \( \Phi : [0,T] \times \Omega \to E \), be \( \mathcal{B} ([0,T]) \times \mathcal{F} \) measurable and suppose

\[
\Phi \in K \equiv L^p ([0,T] \times \Omega ; E), \ p \geq 1
\]

Then there exists a sequence of nested partitions, \( \mathcal{P}_k \subseteq \mathcal{P}_{k+1} \),

\[
\mathcal{P}_k \equiv \{ t_0^k, \ldots, t_{m_k}^k \}
\]
such that the step functions given by

\[
\Phi_k^t (t) \equiv \sum_{j=1}^{m_k} \Phi (t_j^k) \mathcal{X}_{[t^{k-1}_j, t^k_j]} (t)
\]

\[
\Phi_k^l (t) \equiv \sum_{j=1}^{m_k} \Phi (t_j^k - 1) \mathcal{X}_{[t^{k-1}_j, t^k_j)} (t)
\]

both converge to \( \Phi \) in \( K \) as \( k \to \infty \) and

\[
\lim_{k \to \infty} \max \{ |t_j^k - t_{j+1}^k| : j \in \{0, \ldots, m_k\} \} = 0.
\]
Also, each $\Phi (t^k_j)$ is in $L^p (\Omega; E)$. One can also assume that $\Phi (0) = 0$. The mesh points $\{t^k_j\}_{j=0}^{m_k}$ can be chosen to miss a given set of measure zero. In addition to this, we can assume that

$$|t^k_j - t^k_{j-1}| = 2^{-n_k}$$

except for the case where $j = 1$ or $j = m_{n_k}$ when this is so, you could have $|t^k_j - t^k_{j-1}| < 2^{-n_k}$.

**Theorem 66.4.8** Let $F \in L^2 (\Omega \times [0, T])$ and is progressively measurable. Then it has a Skorokhod integral which coincides with the Ito integral.

**Proof:** From Lemma 66.4.6 there is a sequence of left step functions denoted here as $\{F^l_k\}_{k=1}^\infty$ which converges to $F$ in $L^2 (\Omega \times [0, T])$ where $F^l_k (t^k_j) = F (t^k_j)$. We can take a subsequence if necessary and assume

$$\|F^l_k - F\|_{L^2 ([0, T] \times \Omega)} < 2^{-k}$$

Here the $\{t^k_j\}$ are mesh points corresponding to the $k^{th}$ partition described above. Thus each $F^l_k (t^k_j)$ is in $L^2 (\Omega)$. By Lemma 66.4.6 there exists a random variable $G^l_k (t^k_j)$ which is a polynomial function of some $W (h)$ for $h \in L^2 (0, t^k_j)$ which can approximate $F^l_k (t^k_j)$ as closely as desired in $L^2 (\Omega)$. Then choosing these sufficiently close, it can be assumed that the step functions

$$G^l_k = \sum_{j=0}^{m_k-1} G^l_k (t^k_j) \chi_{(t^k_j, t^k_{j+1})}$$

also converge in $L^2 (\Omega \times [0, T])$ to $F$. Of course, each of these last step functions are in $D (\delta)$.

The idea is to show that $\delta (G^l_k)$ is Cauchy in $L^2 (\Omega)$ as $k \to \infty$ and then use the fact that, since $\delta$ is an adjoint, it must be a closed operator. This will show that $F \in L^2 (\Omega \times [0, T])$, considered as a subspace of $L^2 (\Omega; L^2 (0, \infty, \mathbb{R}))$, is in $D (\delta)$ and $\delta (F)$ is equal to the above limit. Using the fact which comes from the fact that the functions are adapted to the given filtration,

$$\|\delta (\chi_{[0, T]} G^l_k) - \delta (\chi_{[0, T]} G^l_{k+1})\|_{L^2 (\Omega)}^2 = E \left( \sum_{j=0}^{m_{k+1}-1} \left( G^l_k (t^k_{j+1}) - G^l_{k+1} (t^k_{j+1}) \right) \left( W (t^k_{j+1}) - W (t^k_{j+1}) \right) \right)^2$$

Consider a mixed term. To save on space, let $\Delta_j = G^l_k (t^k_{j+1}) - G^l_{k+1} (t^k_{j+1})$ and say $i < j$. Then

$$E \left( (\Delta_j) (\Delta_i) \left( W (t^k_{j+1}) - W (t^k_{j+1}) \right) \left( W (t^k_{i+1}) - W (t^k_{i+1}) \right) \right)$$
CHAPTER 66. A DIFFERENT KIND OF STOCHASTIC INTEGRATION

By independence of the increments for \( W \), this is

\[
E \left( W (t_{j+1}^k) - W (t_j^k) \right) E \left( (\Delta_j)(\Delta_i) \left( W (t_{i+1}^k) - W (t_i^k) \right) \right) = 0
\]

and so the above reduces to

\[
\sum_{j=0}^{m_{k+1}-1} E \left( \Delta_j^2 \left( W (t_{j+1}^k) - W (t_j^k) \right)^2 \right)
\]

\[
= \sum_{j=0}^{m_{k+1}-1} E \left( \Delta_j \right) E \left( \left( W (t_{j+1}^k) - W (t_j^k) \right)^2 \right)
\]

\[
= \sum_{j=0}^{m_{k+1}-1} E \left( (G_{j+1}^k - G_{j+1}^k)^2 \right) \left( t_{j+1}^k - t_j^k \right)
\]

\[
= E \left( \int_0^T \left( G_k - G_{k+1}^d \right)^2 dt \right) \leq 2 \left( E \int_0^T \left( G_k - F \right)^2 dt + E \int_0^T \left( F - G_k \right)^2 dt \right)
\]

which is given to converge to 0 as \( k \to \infty \). It follows that \( X_{[0,T]}G_k \to X_{[0,T]}F \)

in \( L^2(\Omega, L^2(0, \infty, \mathbb{R})) \) by construction and \( \delta (X_{[0,T]}G_k) \) is a Cauchy sequence in \( L^2(\Omega) \). Therefore, it converges to something in \( L^2(\Omega) \) and since \( \delta \) is a closed operator, that which it converges to is \( \delta (F) \).

However, by the definition of the Ito integral, \( \delta (X_{[0,T]}G_k) \) also converges to the Ito integral \( \int_0^T F dW \). \( \blacksquare \)

It follows that the Skorokhod integral is more general than the Ito integral but it gives the Ito integral in the special case where the function is adapted. This also shows that the progressively measurable functions in \( L^2([0, T] \times \Omega) \) are in \( D(\delta) \), but as shown above, there are many other functions which are not progressively measurable but which are still in \( D(\delta) \). Just consider, for example \( \sum_{k=1}^n F h_k \) where \( F \) is just a polynomial in \( W (h) \) for \( h \in L^2(0, \infty; \mathbb{R}) \).
Chapter 67

Gelfand Triples

Let $H$ be a separable real Hilbert space and let $V \subseteq H$ be a separable Banach space which is embedded continuously into $H$ and which is also dense in $H$. Then identifying $H$ and $H'$ you can write

$$V \subseteq H = H' \subseteq V'. $$

This is called a Gelfand triple. If $V$ is reflexive, you could conclude separability of $V$ from the separability of $H$. However, if $V$ is not reflexive, this might not happen. For example, you could take $V = L_\infty (0, 1)$ and $H = L^2 (0, 1)$.

**Proposition 67.0.9** Suppose $V$ is reflexive and a subset of $H$ a separable Hilbert space with the inclusion map continuous. Suppose also that $V$ is dense in $H$. Then identifying $H$ and $H'$, it follows that $H$ is dense in $V'$ and $V$ is separable.

**Proof:** If $H$ is not dense in $V'$, then by the Hahn Banach theorem, there exists $\phi^{**} \in V''$ such that $\phi^{**} (H) = 0$ but $\phi^{**} (\phi^*) \neq 0$ for some $\phi^* \in V' \setminus H$. Since $V$ is reflexive there exists $v \in V$ such that $\phi^{**} = Jv$ for $J$ the standard mapping from $V$ to $V''$. Thus

$$\phi^{**} (h) \equiv \langle h, v \rangle \equiv (v, h)_H = 0$$

for all $h \in H$. Therefore, $v = 0$ and so $Jv = 0 = \phi^{**}$ which contradicts $\phi^{**} (\phi^*) \neq 0$. Therefore, $H$ is dense in $V'$. Now by Theorem which says separability of the dual space implies separability of the space, it follows $V$ is separable as claimed. This proves the proposition.

From now on, it is assumed $V$ and $V'$ are both separable and that $H$ is dense in $V'$. This is summarized in the following definition.

**Definition 67.0.10** $V, H, V'$ will be called a Gelfand triple if $V, V'$ are separable, $V \subseteq H$ with the inclusion map continuous, $H = H'$, and $H = H'$ is dense in $V'$.

What about the Borel sets on $V$ and $H$?
Proposition 67.0.11 Denote by \( \mathcal{B}(X) \) the Borel sets of \( X \) where \( X \) is any separable Banach space. Then

\[
\mathcal{B}(X) = \sigma(X').
\]

Here \( \sigma(X') \) is the smallest \( \sigma \) algebra such that each \( \phi \in X' \) is measurable. Also in the context of the above definition, \( \mathcal{B}(V) = \sigma(i^*H') \) because \( H' \) is dense in \( V' \). Here \( i^* \) is the restriction to \( V \) so that \( i^*h(v) \equiv h(v) \equiv (h,v)_H \) for all \( v \in V \) and \( \sigma(i^*H') \) denotes the smallest \( \sigma \) algebra such that \( i^*h \) is measurable for each \( h \in H' \).

Proof: By Lemma [9.1.4] there exists a countable subset of the unit ball in \( X' \)

\[
\{\phi_n\}_{n=1}^\infty \subseteq D'
\]

such that

\[
||v||_X = \sup \{|\phi(v)| : \phi \in D'\}.
\]

Consider a closed ball \( \overline{B(v_0,r)} \) in \( X \). This equals

\[
\left\{ v \in X : \sup_n |\phi_n(v) - \phi_n(v_0)| \leq r \right\} = \cap_{n=1}^\infty \phi_n^{-1}\left(\overline{B(\phi_n(v_0),r)}\right)
\]

and this last set is in \( \sigma(D') \). Therefore, every closed ball is in \( \sigma(D') \) which implies every open ball is also in \( \sigma(D') \) since open balls are the countable union of closed balls. Since \( X \) is separable, it follows every open set is the countable union of balls and so every open set is in \( \sigma(D') \). It follows \( \mathcal{B}(X) \subseteq \sigma(D') \subseteq \sigma(X') \). On the other hand, every \( \phi \in X' \) is continuous and so it is Borel measurable. Hence \( \sigma(X') \subseteq \mathcal{B}(X) \).

Now consider the last claim. From Lemma [9.1.4] and density of \( H' = H \) in \( V' \), it can be assumed \( D' \subseteq H = H' \). Therefore, from the first part of the argument

\[
\mathcal{B}(V) \subseteq \sigma(D') \subseteq \sigma(i^*H')
\]

Also each \( i^*h \) is continuous on \( V \) so in fact, equality holds in the above because \( \sigma(i^*H') \subseteq \mathcal{B}(V) \). This proves the proposition.

Next I want to verify that \( V \) is in \( \mathcal{B}(H) \). This will be true if \( V \) is reflexive. More generally, here is an interesting result.

Proposition 67.0.12 Let \( X \subseteq Y \), \( X \) dense in \( Y \) and suppose \( X \), \( Y \) are Banach spaces and that \( X \) is reflexive. Then \( X \in \mathcal{B}(Y) \).

Proof: Define the functional

\[
\phi(x) \equiv \begin{cases} 
||x||_X & \text{if } x \in X \\
\infty & \text{if } x \in Y \setminus X
\end{cases}
\]

Then \( \phi \) is lower semicontinuous on \( Y \). Here is why. Suppose \( (x,a) \not\in \text{epi}(\phi) \) so that \( a < \phi(x) \). I need to verify this situation persists for \( (x,b) \) near \( (x,a) \). If this is not so, there exists \( x_n \to x \) and \( a_n \to a \) such that \( a_n \geq \phi(x_n) \). If \( \liminf_{n \to \infty} \phi(x_n) < \infty \), then there exists a subsequence still denoted by \( n \) such that \( ||x_n||_X \) is bounded.
67.1. AN UNNATURAL EXAMPLE

Then by the Eberlein Smulian theorem, there exists a further subsequence such that \( x_n \) converges weakly in \( X \) to some \( z \). Now since \( X \) is dense in \( Y \) it follows \( Y' \) can be considered a subspace of \( X' \) and so for \( f \in Y' \)

\[
f(x_n) \to f(z), \quad f(x_n) \to f(x)
\]

and so \( f(z - x) = 0 \) for all \( f \in Y' \) which requires \( z = x \). Now \( x \to \|x\|_X \) is convex and lower semicontinuous on \( X \) so it follows from Corollary 16.2.12

\[
a = \liminf_{n \to \infty} a_n \geq \liminf_{n \to \infty} \phi(x_n) \geq \phi(x) > a
\]

which is a contradiction. If \( \liminf_{n \to \infty} \phi(x_n) = \infty \), then

\[
\infty > a = \liminf_{n \to \infty} a_n = \infty
\]

another contradiction. Therefore, \( \text{epi}(\phi) \) is closed and so \( \phi \) is lower semicontinuous as claimed. Therefore,

\[
X = Y \setminus \left( \cap_{n=1}^{\infty} \phi^{-1}((n, \infty)) \right)
\]

and since \( \phi \) is lower semicontinuous, each \( \phi^{-1}((n, \infty)) \) is open. Hence \( X \) is a Borel subset of \( Y \). This proves the proposition.

67.1 An Unnatural Example

Recall Gelfand triples are of the form

\[
V \subseteq H \subseteq V'
\]

where \( H \) is a Hilbert space and \( V \) is a Banach space contained in \( H \) and each of the above inclusions is continuous and each space is dense in the next one. The standard example of a Gelfand triple is \( H^1_0(D) \subseteq L^2(D) \subseteq (H^1_0(D))' \) with the convention that \( L^2(D) \) is identified with its dual space. Thus for \( f \in L^2(D) \), \( f \) is considered as something in \( (H^1_0(D))' \) according to the rule

\[
\langle f, \phi \rangle \equiv \langle f, \phi \rangle_{L^2(D)}
\]

This is a very pleasant thing to contemplate and it is natural and transparent. However, there are other ways to come up with a Gelfand triple which are much more perverse. The following is an example of such a thing along with an application. See [MS] and references given there.

First consider the following situation.

\[
X \xrightarrow{\theta} Y
\]

where \( \theta \) is continuous, linear and one to one and \( X \) is a Banach space. Then \( \theta(X) \subseteq Y \) and you could define

\[
\|\theta x\|_{\theta(X)} \equiv \|x\|_X.
\]
Then \( \theta(X) \) can be considered the same thing as \( X \) because \( \theta \) preserves distances and all algebraic properties. Thus people write \( X \subseteq Y \) to save space. In the above simple example, it is obvious what \( \theta \) is. This is because the things in \( H^1_0 \) and things in \( L^2 \) are both functions defined on \( D \) and we can simply take \( \theta \) to be the identity map. However, you might have \( H \) be the dual space of something. Thus it consists of bounded linear transformations defined on some Banach space. Then it becomes necessary to specify the manner in which vectors in \( V \) can be considered as vectors of \( H \).

Let \( \infty > p \geq 2 \). Then letting \( D \) be a bounded open set, \( H^1_0(D) \) embeds continuously into \( L^p(D) \). That is

\[
\|\phi\|_{L^p(D)} \leq C \|\phi\|_{H^1_0}.
\]

(67.1.1)

Here \( \frac{1}{p} + \frac{1}{\frac{1}{p}} = 1 \). Also note that an equivalent inner product on \( H^1_0(D) \) is

\[
(f, g)_{H^1_0} = \int_D \nabla f \cdot \nabla g dx
\]

Then with respect to this inner product, the Riesz map is given by \(-\Delta\).

\(-\Delta : H^1_0(D) \to (H^1_0(D))^\prime\)

Thus a typical vector of \((H^1_0(D))^\prime\) is of the form \(-\Delta \phi \) where \( \phi \in H^1_0(D) \) and the following hold.

\[
(\phi, \psi)_{H^1_0} \equiv \langle -\Delta \phi, \psi \rangle, \quad (-\Delta \phi, -\Delta \psi)_{H^1_0} \equiv (\phi, \psi)_{H^1_0} = \langle -\Delta \psi, \phi \rangle
\]

The following is about the Gelfand triple

\[V = L^p(D) \subseteq (H^1_0)^\prime \subseteq (L^p(D))^\prime\]

**Lemma 67.1.1** It is possible to consider \( L^p(D) \equiv V \) as a dense subspace of \((H^1_0)^\prime \equiv H \) as follows. For \( f \in L^p(D) \) and \( \phi \in H^1_0(D) \),

\[
\langle f, \phi \rangle \equiv \int_D f(x) \phi (x) dx
\]

One can also consider \( H \equiv (H^1_0)^\prime \) as a dense subspace of \((L^p(D))^\prime \equiv V^\prime \) as follows. For \(-\Delta \phi \in H \) and \( f \in L^p(D) \),

\[
\langle -\Delta \phi, f \rangle \equiv (-\Delta \phi, f)_H \equiv \langle f, \phi \rangle
\]

\(-\Delta \) maps \( H^1_0(D) \) to \( H \equiv (H^1_0)^\prime \subseteq V^\prime \). \(-\Delta \) can be extended to yield a map \(-\Delta_1 \) from \( L^p(D) \) to \( V^\prime \).

\[
H^1_0(D) \xrightarrow{-\Delta} (H^1_0)^\prime
\]

\[
L^p(D) = V \xrightarrow{-\Delta_1} V^\prime
\]
67.1. AN UNNATURAL EXAMPLE

**Proof:** First of all, note that by\[ ((f,\phi) \leq \|f\|_{L^p} \|\phi\|_{L^p'} \leq C \|f\|_{L^p} \|\phi\|_{H^1_0} \]
and so it is certainly possible to consider \( L^p \subseteq H \equiv (H^1_0)' \) as just claimed. Now why can \( L^p(D) \) be considered dense in \( H \equiv (H^1_0)' \)? If it isn’t dense, then there exists \( \psi \in H^1_0(D), \psi \neq 0 \) such that
\[ (-\Delta \psi, f)_H = 0 \]
for all \( f \in L^p(D) \). However, the above would say that for all \( f \in L^p \),
\[ (-\Delta \psi, f)_H \equiv \langle f, \psi \rangle \equiv \int_{D} f \psi = 0 \]
But \( \psi \in L^{p'}(D) \) because \( H^1_0(D) \) embeds continuously into \( L^{p'}(D) \) and so the above holding for all \( f \in L^p(D) \) implies by the usual Riesz representation theorem that \( \psi = 0 \) contrary to the way \( \psi \) was chosen.

Now consider the next claim. For \(-\Delta \phi \in H \equiv (H^1_0)' \) and \( f \in L^p(D) \) and from the first part
\[ |\langle -\Delta \phi, f \rangle| \equiv |(-\Delta \phi, f)_H| \equiv |(f, \phi)| \leq C \|f\|_{L^p} \|\phi\|_{H^1_0(D)} \]
Thus \(-\Delta \phi \in H \) can be considered in \( (L^p(D))' \). Why should \( H \) be dense in \( (L^p(D))' \)? If it is not dense, then there exists \( g^* \in (L^p(D))' \) which is not the limit of vectors of \( H \). Then since \( L^p(D) \) is reflexive, an application of the Hahn Banach theorem shows there exists \( f \in L^p(D) \) such that
\[ g^*, f' \equiv (\langle f, \phi \rangle)_{(L^p(D))',L^p(D)} = 0 \]
for all \(-\Delta \phi \in H \). However, it was just shown \( H \) could be considered a subset of \( (L^p(D))' \) in the manner described above. Therefore, the last equation in the above is of the form
\[ 0 = (-\Delta \phi, f)_H = \langle f, \phi \rangle = \int_{D} f \phi dx \]
and since this holds for all \( \phi \in H^1_0(D) \), it follows by density of \( H^1_0(D) \) in \( L^{p'}(D) \), that \( f = 0 \) and now this contradicts the inequality in \( (74.1.1) \).

Now \( \Delta \) is defined on \( H^1_0(D) \) and it delivers something in \( (H^1_0)' \equiv H \). Of course \( H^1_0(D) \) is dense in \( L^{p'}(D) \). Can \( \Delta \) be extended to all of \( L^{p'}(D) \)? The answer is yes and it is more of the same given above. For \( \phi \in H^1_0(D) \), \(-\Delta \phi \in H \subseteq (L^p(D))' \). Then by the above, for \( \phi \in H^1_0(D) \) and \( f \in L^p(D) \),
\[ (-\Delta \phi, f) \equiv \langle f, \phi \rangle \equiv \int_{D} f \phi dx \]
\[ |\langle -\Delta \phi, f \rangle| \equiv |\langle f, \phi \rangle| \equiv \left| \frac{1}{L^p(D)} \right| \leq \|\phi\|_{L^p(D)} \|f\|_{L^p(D)} \]
and so \(-\Delta\) is a continuous linear mapping defined on a dense subspace \(H^1_0(D)\) of \(L^{p'}(D)\) and so this does indeed extend to a continuous linear map defined on all of \(L^{p'}(D)\) given by the formula

\[
\langle -\Delta g, f \rangle = \int_D fg \, dx
\]

This proves the lemma.

Thus letting \(V = L^p(D)\), and \(H = (H^1_0(D))^\prime\), it follows \(V \subseteq H \subseteq V^\prime\) is a Gelfand triple with the understanding of what it means for one space to be included in another described above. To emphasize the above, for \(-\Delta \phi \in H, f \in L^p\),

\[
\langle -\Delta \phi, f \rangle \equiv \langle f, \phi \rangle \equiv \int_D f \phi \, dx
\]

More generally, for \(g \in L^{p'}(D), -\Delta g \in (L^p(D))^\prime\) according to the rule

\[
\langle -\Delta g, f \rangle \equiv \int_D fg \, dx.
\]

With this example of a Gelfand triple, one can define a “porous medium operator” \(A : V \to V^\prime\). Let \(\Psi\) be a real valued function defined on \(\mathbb{R}\) which satisfies

\[
\Psi \text{ is continuous}
\]

\[
(t - s) (\Psi (t) - \Psi (s)) \geq 0
\]

There exists \(p \geq 2, p < \infty\) and \(\alpha \in (0, \infty)\) such that for all \(s \in \mathbb{R}\)

\[
s \Psi (s) \geq \alpha |s|^p - c
\]

There exist \(c_3, c_4 \in (0, \infty)\) such that for all \(s \in \mathbb{R}\)

\[
|\Psi (s)| \leq c_4 + c_3 |s|^{p-1}
\]

Note that Equation 67.1.4 implies that if \(v \in L^p(D)\), then

\[
\int_D |\Psi (v)|^p \, dx \leq C \int_D (1 + |v|^{p(p-1)}) \, dx = C \int_D (1 + |v|^p) \, dx < \infty.
\]

Thus for \(v \in L^p(D)\), \(\Psi (v)\) is something you can do \(\Delta\) to and obtain something in \(V^\prime\). The porous medium operator \(A : V \to V^\prime\) is given as follows.

\[
\langle Av, w \rangle_{V^\prime, V} \equiv \langle \Delta \Psi (v), w \rangle_{V^\prime, V} \equiv -\int_D \Psi (v) \, wd\nu
\]

What are the properties of \(A\)?

\[
\langle A(u + \lambda v), w \rangle \equiv -\int_D \Psi (u + \lambda v) \, wd\nu
\]
67.1. AN UNNATURAL EXAMPLE and this is easily seen to be a continuous function of \( \lambda \). Thus \( A \) is Hemicontinuous.

\[
\langle A(u) - A(v), u - v \rangle \equiv - \int_D \Psi(u)(u - v) \, dx + \int_D \Psi(v)(u - v) \, dx \leq 0
\]

Thus \( -A \) is monotone. Also there is a coercivity estimate which is routine.

\[
\langle A(v), v \rangle \equiv - \int_D \Psi(v) \, v \leq \int_D c - \alpha |v|^p \, dx = C - \alpha \|v\|^p_V
\]

This operator also has a boundedness estimate.

\[
\|A(v)\|_{V'} \equiv \sup_{\|w\|_V \leq 1} \|\langle A(v), w \rangle\| \equiv \sup_{\|w\|_V \leq 1} \left| \int_D \Psi(v) \, w \right|
\]

\[
\leq \sup_{\|w\|_V \leq 1} \left( \int_D \left( c_4 + c_3 |v|^{p-1} \right) w \, dx \right)
\]

\[
\leq \left( \int_D C \left( 1 + |v|^p \right) dx \right)^{1/p'} \leq C + C \left( \int_D |v|^p \, dx \right)^{1/p'}
\]

\[
= C + C \|v|^{p/p'} = C + C \|v\|_{V'}^{p-1}.
\]

Since \( \Psi \) is continuous, it will also follow that \( A \) is \( \mathcal{B}(V) \) measurable. Consider

\[
u \to \langle Au, w \rangle \equiv - \int_D \Psi(u) \, w \, dx
\]

for fixed \( w \in V \). Suppose \( u_n \to u \) in \( V \) and fix \( w \in L^\infty(D) \subseteq V \). Then it follows from an easy argument using the Vitali convergence theorem and the fact that from the estimates above

\[
\Psi(u_n) \, w
\]

is uniformly integrable that

\[
u \to - \int_D \Psi(u) \, w \, dx
\]

is continuous. For general \( w \in L^p(D) \), let \( w_n \to w \) in \( L^p(D) \) where each \( w_n \) is in \( L^\infty(D) \). Then the function

\[
u \to - \int_D \Psi(u) \, w \, dx \equiv \langle Au, w \rangle \tag{67.1.7}
\]

is the limit of the continuous functions

\[
u \to - \int_D \Psi(u) \, w_n \, dx
\]

and so the function \( \text{(67.1.7)} \) is Borel measurable. Now by the Pettis theorem this shows \( A : V \to V' \) is \( \mathcal{B}(V) \) measurable. This shows \( A \) is an example of an operator which satisfies some conditions which will be considered later.
67.2 Standard Techniques In Evolution Equations

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 67.2.1 Suppose \( g \in L^1([a,b];X) \) where \( X \) is a Banach space. Then if \( \int_a^b g(t) \phi(t) \, dt = 0 \) for all \( \phi \in C_c^\infty(a,b) \), then \( g(t) = 0 \) a.e.

**Proof:** Let \( E \) be a measurable subset of \((a,b)\) and let \( K \subseteq E \subseteq V \subseteq (a,b) \) where \( K \) is compact, \( V \) is open and \( m(V \setminus K) < \varepsilon \). Let \( K \prec h \prec V \) as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging \( K \) slightly and convolving with a mollifier, it can be assumed \( h \in C_c^\infty(a,b) \). Then

\[
\left| \int_a^b \mathcal{X}_E(t) g(t) \, dt \right| = \left| \int_a^b (\mathcal{X}_E(t) - h(t)) g(t) \, dt \right| \\
\leq \int_a^b |\mathcal{X}_E(t) - h(t)| \|g(t)\| \, dt \\
\leq \int_{V \setminus K} \|g(t)\| \, dt.
\]

Now let \( K_n \subseteq E \subseteq V_n \) with \( m(V_n \setminus K_n) < 2^{-n} \). Then from the above,

\[
\int_a^b \mathcal{X}_E(t) g(t) \, dt \leq \int_a^b \mathcal{X}_{V_n \setminus K_n}(t) \|g(t)\| \, dt
\]

and the integrand of the last integral converges to 0 a.e. as \( n \to \infty \) because \( \sum_n m(V_n \setminus K_n) < \infty \). By the dominated convergence theorem, this last integral converges to 0. Therefore, whenever \( E \subseteq (a,b) \),

\[
\int_a^b \mathcal{X}_E(t) g(t) \, dt = 0.
\]

Since the endpoints have measure zero, it also follows that for any measurable \( E \), the above equation holds.

Now \( g \in L^1([a,b];X) \) and so it is measurable. Therefore, \( g([a,b]) \) is separable. Let \( D \) be a countable dense subset and let \( E \) denote the set of linear combinations of the form \( \sum a_id_i \) where \( a_i \) is a rational point of \( \mathbb{F} \) and \( d_i \in D \). Thus \( E \) is countable. Denote by \( \bar{Y} \) the closure of \( E \) in \( X \). Thus \( \bar{Y} \) is a separable closed subspace of \( X \) which contains all the values of \( g \).

Now let \( S_n \equiv g^{-1}(B(y_n, \|y_n\|/2)) \) where \( E = \{y_n\}_{n=1}^\infty \). Therefore, \( \cup_n S_n = g^{-1}(X \setminus \{0\}) \). This follows because if \( x \in Y \) and \( x \neq 0 \), then in \( B(x, \|x\|) \) there is a point of \( E \), \( y_n \). Therefore, \( \|y_n\| > \frac{3}{4} \|x\| \) and so \( \frac{\|y_n\|}{2} > \frac{3}{8} \|x\| > \|x\| \) so \( x \in B(y_n, \|y_n\|/2) \). It follows that if each \( S_n \) has measure zero, then \( g(t) = 0 \) for a.e.
For \( t \). Suppose then that for some \( n \), the set, \( S_n \) has positive measure. Then from what was shown above,

\[
\| y_n \| = \left\| \frac{1}{m(S_n)} \int_{S_n} g(t) \, dt - y_n \right\| = \left\| \frac{1}{m(S_n)} \int_{S_n} g(t) \, dt - y_n dt \right\|
\]

\[
\leq \frac{1}{m(S_n)} \int_{S_n} \| g(t) - y_n \| \, dt \leq \frac{1}{m(S_n)} \int_{S_n} \| y_n \| /2 dt = \| y_n \| /2
\]

and so \( y_n = 0 \) which implies \( S_n = \emptyset \), a contradiction to \( m(S_n) > 0 \). This contradiction shows each \( S_n \) has measure zero and so as just explained, \( g(t) = 0 \) a.e. \( \blacksquare \)

**Definition 67.2.2** For \( f \in L^1 (a,b;X) \), define an extension, \( \bar{f} \) defined on

\[
[2a - b, 2b - a] = [a - (b - a), b + (b - a)]
\]

as follows.

\[
\bar{f}(t) = \begin{cases} 
  f(t) & \text{if } t \in [a,b] \\
  f(2a - t) & \text{if } t \in [2a-b,a] \\
  f(2b - t) & \text{if } t \in [b,2b-a] 
\end{cases}
\]

**Definition 67.2.3** Also if \( f \in L^p (a,b;X) \) and \( h > 0 \), define for \( t \in [a,b] \), \( f_h (t) \equiv \bar{f}(t-h) \) for all \( h < b - a \). Thus the map \( f \to f_h \) is continuous and linear on \( L^p (a,b;X) \). It is continuous because

\[
\int_a^b \| f_h(t) \|^p \, dt = \int_a^{a+h} \| f(2a-t+h) \|^p \, dt + \int_{a+h}^b \| f(t) \|^p \, dt
\]

\[
= \int_a^{a+h} \| f(t) \|^p \, dt + \int_{a+h}^b \| f(t) \|^p \, dt \leq 2 \| f \|_p^p.
\]

The following lemma is on continuity of translation in \( L^p (a,b;X) \).

**Lemma 67.2.4** Let \( \bar{f} \) be as defined in Definition 67.2.3. Then for \( f \in L^p (a,b;X) \) for \( p \in [1,\infty) \),

\[
\lim_{\delta \to 0} \int_a^b \| \bar{f}(t-\delta) - f(t) \|_X^p \, dt = 0.
\]

**Proof:** Regarding the measure space as \((a,b)\) with Lebesgue measure, by regularity of the measure, there exists \( g \in C_c (a,b;X) \) such that \( \| f - g \|_p < \varepsilon \). Here the norm is the norm in \( L^p (a,b;X) \). Therefore,

\[
\| f_h - f \|_p \leq \| f_h - g_h \|_p + \| g_h - g \|_p + \| g - f \|_p
\]

\[
\leq (2^{1/p} + 1) \| f - g \|_p + \| g_h - g \|_p
\]

\[
< (2^{1/p} + 1) \varepsilon + \varepsilon
\]

whenever \( h \) is sufficiently small. This is because of the uniform continuity of \( g \). Therefore, since \( \varepsilon > 0 \) is arbitrary, this proves the lemma. \( \blacksquare \)
**Definition 67.2.5** Let \( f \in L^1(a, b; X) \). Then the distributional derivative in the sense of \( X \) valued distributions is given by

\[
f'(\phi) \equiv - \int_a^b f(t) \phi'(t) \, dt
\]

Then \( f' \in L^1(a, b; X) \) if there exists \( h \in L^1(a, b; X) \) such that for all \( \phi \in C_c^\infty(a, b) \),

\[
f'(\phi) = \int_a^b h(t) \phi(t) \, dt.
\]

Then \( f' \) is defined to equal \( h \). Here \( f \) and \( f' \) are considered as vector valued distributions in the same way as was done for scalar valued functions.

**Lemma 67.2.6** The above definition is well defined.

**Proof:** Suppose both \( h \) and \( g \) work in the definition. Then for all \( \phi \in C_c^\infty(a, b) \),

\[
\int_a^b (h(t) - g(t)) \phi(t) \, dt = 0.
\]

Therefore, by Lemma 67.2.1, \( h(t) - g(t) = 0 \) a.e. \( \blacksquare \)

The other thing to notice about this is the following lemma. It follows immediately from the definition.

**Lemma 67.2.7** Suppose \( f, f' \in L^1(a, b; X) \). Then if \([c, d] \subseteq [a, b]\), it follows that \((f|_{[c, d]})' = f'|_{[c, d]}\). This notation means the restriction to \([c, d]\).

Recall that in the case of scalar valued functions, if you had both \( f \) and its weak derivative, \( f' \) in \( L^1(a, b) \), then you were able to conclude that \( f \) is almost everywhere equal to a continuous function, still denoted by \( f \) and

\[
f(t) = f(a) + \int_a^t f'(s) \, ds.
\]

In particular, you can define \( f(a) \) to be the initial value of this continuous function.

It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if \( f' = 0 \) where the derivative is taken in the sense of \( X \) valued distributions, then \( f \) equals a constant.

**Lemma 67.2.8** Suppose \( f \in L^1(a, b; X) \) and for all \( \phi \in C_c^\infty(a, b) \),

\[
\int_a^b f(t) \phi'(t) \, dt = 0.
\]

Then there exists a constant, \( a \in X \) such that \( f(t) = a \) a.e.
67.2. STANDAR D TECHNIQUES IN EVOLUTION EQUATIONS

Proof: Let \( \phi_0 \in C_c^\infty (a, b) \), \( \int_a^b \phi_0 \, dx = 1 \) and define for \( \phi \in C_c^\infty (a, b) \)

\[
\psi_\phi (x) = \int_a^x \left[ \phi (t) - \left( \int_a^t \phi (y) \, dy \right) \phi_0 (t) \right] \, dt
\]

Then \( \psi_\phi \in C_c^\infty (a, b) \) and \( \psi_\phi' = \phi - \left( \int_a^b \phi (y) \, dy \right) \phi_0 \). Then

\[
\int_a^b f (t) (\phi (t)) \, dt = \int_a^b f (t) \left( \psi_\phi' (t) + \left( \int_a^b \phi (y) \, dy \right) \phi_0 (t) \right) \, dt
\]

= 0 by assumption

\[
= \int_a^b f (t) \psi_\phi' (t) \, dt + \left( \int_a^b \phi (y) \, dy \right) \int_a^b f (t) \phi_0 (t) \, dt
\]

\[
= \left( \int_a^b \left( \int_a^b f (t) \phi_0 (t) \, dt \right) \, dy \right). \]

It follows that for all \( \phi \in C_c^\infty (a, b) \),

\[
\int_a^b \left( f (y) - \left( \int_a^b f (t) \phi_0 (t) \, dt \right) \right) \phi (y) \, dy = 0
\]

and so by Lemma 67.2.1,

\[ f (y) - \left( \int_a^b f (t) \phi_0 (t) \, dt \right) = 0 \quad \text{a.e. } y \]

Theorem 67.2.9 Suppose \( f, f' \) both are in \( L^1 (a, b; X) \) where the derivative is taken in the sense of \( X \) valued distributions. Then there exists a unique point of \( X \), denoted by \( f (a) \) such that the following formula holds a.e. \( t \).

\[ f (t) = f (a) + \int_a^t f' (s) \, ds \]

Proof:

\[
\int_a^b \left( f (t) - \int_a^t f' (s) \, ds \right) \phi' (t) \, dt = \int_a^b f (t) \phi' (t) \, dt - \int_a^b \int_a^t f' (s) \phi' (t) \, ds \, dt.
\]

Now consider \( \int_a^b \int_a^t f' (s) \phi' (t) \, ds \, dt \). Let \( \Lambda \in X' \). Then it is routine from approximating \( f' \) with simple functions to verify

\[
\Lambda \left( \int_a^b \int_a^t f' (s) \phi' (t) \, ds \, dt \right) = \int_a^b \int_a^t \Lambda (f' (s)) \phi' (t) \, ds \, dt.
\]
Now the ordinary Fubini theorem can be applied to obtain

\[
= \int_a^b \int_s^b \Lambda \left( f'(s) \phi'(t) \right) dt ds = \Lambda \left( \int_a^b f'(s) \phi'(t) dt ds \right).
\]

Since \( X' \) separates the points of \( X \), it follows

\[
\int_a^b \int_a^t f'(s) \phi'(t) ds dt = \int_a^b \int_s^b f'(s) \phi'(t) dt ds.
\]

Therefore,

\[
\int_a^b \left( f(t) - \int_a^t f'(s) ds \right) \phi'(t) dt
= \int_a^b f(t) \phi'(t) dt - \int_a^b \int_s^b f'(s) \phi'(t) dt ds
= \int_a^b f(t) \phi'(t) dt - \int_a^b f'(s) \int_s^b \phi'(t) dt ds
= \int_a^b f(t) \phi'(t) dt + \int_a^b f'(s) \phi(s) ds = 0.
\]

Therefore, by Lemma, there exists a constant, denoted as \( f(a) \) such that

\[
f(t) - \int_a^t f'(s) ds = f(a).
\]

The integration by parts formula is also important.

**Corollary 67.2.10** Suppose \( f, f' \in L^1(a, b; X) \) and suppose \( \phi \in C^1([a, b]) \). Then the following integration by parts formula holds.

\[
\int_a^b f(t) \phi'(t) dt = f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) dt.
\]
Proof: From Theorem 67.2.9

\[
\int_a^b f(t) \phi'(t) \, dt = \int_a^b \left( f(a) + \int_a^t f'(s) \, ds \right) \phi'(t) \, dt
\]
\[
= f(a) (\phi(b) - \phi(a)) + \int_a^b \int_a^t f'(s) \, ds \phi'(t) \, dt
\]
\[
= f(a) (\phi(b) - \phi(a)) + \int_a^b f'(s) \int_s^b \phi'(t) \, dt \, ds
\]
\[
= f(a) (\phi(b) - \phi(a)) + \int_a^b f'(s) (\phi(b) - \phi(s)) \, ds
\]
\[
= f(a) (\phi(b) - \phi(a)) - \int_a^b f'(s) \phi(s) \, ds + (f(b) - f(a)) \phi(b)
\]
\[
= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(s) \phi(s) \, ds.
\]

The interchange in order of integration is justified as in the proof of Theorem 67.2.9.

With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define \( \mathcal{F} \) as in Definition 67.2.2.

Lemma 67.2.11 Let \( \mathcal{F} \) be given in Definition 67.2.2 and suppose \( f, f' \in L^1(a,b;X) \). Then \( \mathcal{F}, \mathcal{F}' \in L^1(2a - b, 2b - a; X) \) also and

\[
\mathcal{F}'(t) \equiv \begin{cases} 
  f'(t) & \text{if } t \in [a,b] \\
  -f'(2a - t) & \text{if } t \in [2a - b, a] \\
  -f'(2b - t) & \text{if } t \in [b, 2b - a] 
\end{cases}
\] (67.2.8)

Proof: It is clear from the definition of \( \mathcal{F} \) that \( \mathcal{F} \in L^1(2a - b, 2b - a; X) \) and that in fact

\[
\| \mathcal{F} \|_{L^1(2a - b, 2b - a; X)} \leq 3 \| f \|_{L^1(a,b;X)}. 
\] (67.2.9)
Let $\phi \in C_c^\infty (2a - b, 2b - a)$. Then from the integration by parts formula,

\[
\int_{2a-b}^{2b-a} \overline{f}(t) \phi'(t) \, dt
= \int_a^b f(t) \phi'(t) \, dt + \int_b^{2b-a} f(2b - t) \phi'(t) \, dt + \int_{2a-b}^a f(2a - t) \phi'(t) \, dt
= \int_a^b f(t) \phi'(t) \, dt + \int_a^b f(u) \phi'(2b - u) \, du + \int_{2a-b}^b f(u) \phi'(2a - u) \, du
= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) \, dt - f(b) \phi(b) + f(a) \phi(2b - a)
+ \int_a^b f'(u) \phi(2b - u) \, du - f(b) \phi(2a - b)
+ f(a) \phi(a) + \int_a^b f'(u) \phi(2a - u) \, du
= -\int_a^b f'(t) \phi(t) \, dt + \int_a^b f'(u) \phi(2b - u) \, du + \int_a^b f'(u) \phi(2a - u) \, du
= -\int_a^b f'(t) \phi(t) \, dt - \int_b^{2b-a} f'(2b - t) \phi(t) \, dt - \int_{2a-b}^a f'(2a - t) \phi(t) \, dt
= -\int_{2a-b}^{2b-a} \overline{f}(t) \phi'(t) \, dt
\]

where $\overline{f}'(t)$ is given in [72.4.8].

**Definition 67.2.12** Let $V$ be a Banach space and let $H$ be a Hilbert space. (Typically $H = L^2(\Omega)$) Suppose $V \subseteq H$ is dense in $H$ meaning that the closure in $H$ of $V$ gives $H$. Then it is often the case that $H$ is identified with its dual space, and then because of the density of $V$ in $H$, it is possible to write

\[ V \subseteq H = H' \subseteq V' \]

When this is done, $H$ is called a pivot space. Another notation which is often used is $\langle f, g \rangle$ to denote $f(g)$ for $f \in V'$ and $g \in V$. This may also be written as $\langle f, g \rangle_{V', V}$. Another term is that $V \subseteq H = H' \subseteq V'$ is called a Gelfand triple.

The next theorem is an example of a trace theorem. In this theorem, $f \in L^p(0, T; V)$ while $f' \in L^p(0, T; V')$. It makes no sense to consider the initial values of $f$ in $V$ because it is not even continuous with values in $V$. However, because of the derivative of $f$ it will turn out that $f$ is continuous with values in a larger space and so it makes sense to consider initial values of $f$ in this other space. This other space is called a trace space.

**Theorem 67.2.13** Let $V$ and $H$ be a Banach space and Hilbert space as described in Definition 67.2.12. Suppose $f \in L^p(0, T; V)$ and $f' \in L^p(0, T; V')$. Then $f$ is
a.e. equal to a continuous function mapping \([0, T]\) to \(H\). Furthermore, there exists \(f(0) \in H\) such that
\[
\frac{1}{2} |f(t)|_H^2 - \frac{1}{2} |f(0)|_H^2 = \int_0^t \langle f'(s), f(s) \rangle \, ds,
\]
and for all \(t \in [0, T]\),
\[
\int_0^t f'(s) \, ds \in H,
\]
and for a.e. \(t \in [0, T]\),
\[
f(t) = f(0) + \int_0^t f'(s) \, ds \quad \text{in} \quad H,
\]
Here \(f'\) is being taken in the sense of \(V'\) valued distributions and \(\frac{1}{p} + \frac{1}{p'} = 1\) and \(p \geq 2\).

**Proof:** Let \(\Psi \in C_c^\infty (-T, 2T)\) satisfy \(\Psi (t) = 1\) if \(t \in [-T/2, 3T/2]\) and \(\Psi (t) \geq 0\).

For \(t \in \mathbb{R}\), define
\[
\widehat{f}(t) = \begin{cases} 
\overline{f}(t) \Psi(t) & \text{if } t \in [-T, 2T] \\
0 & \text{if } t \notin [-T, 2T]
\end{cases}
\]
and
\[
f_n(t) = \int_{-1/n}^{1/n} \widehat{f}(t-s) \phi_n(s) \, ds
\]
where \(\phi_n\) is a mollifier having support in \((-1/n, 1/n)\). Then by Minkowski's inequality
\[
\|f_n - \overline{f}\|_{L^p(\mathbb{R}; V)} = \left( \int_{\mathbb{R}} \left\| \int_{-1/n}^{1/n} \widehat{f}(t-s) \phi_n(s) \, ds \right\|_V^p \, dt \right)^{1/p}
\]
\[
= \left( \int_{\mathbb{R}} \left( \int_{-1/n}^{1/n} \left| \widehat{f}(t) - \widehat{f}(t-s) \right| \phi_n(s) \, ds \right)_V^p \, dt \right)^{1/p}
\]
\[
\leq \left( \int_{\mathbb{R}} \left( \int_{-1/n}^{1/n} \left| \widehat{f}(t) - \widehat{f}(t-s) \right| \phi_n(s) \, ds \right)_V^p \, dt \right)^{1/p}
\]
\[
\leq \int_{-1/n}^{1/n} \phi_n(s) \left( \int_{\mathbb{R}} \left| \widehat{f}(t) - \widehat{f}(t-s) \right|_V^p \, dt \right)^{1/p} \, ds
\]
\[
\leq \int_{-1/n}^{1/n} \phi_n(s) \varepsilon \, ds = \varepsilon
\]
provided \(n\) is large enough. This follows from continuity of translation in \(L^p\) with Lebesgue measure. Since \(\varepsilon > 0\) is arbitrary, it follows \(f_n \to \overline{f}\) in \(L^p(\mathbb{R}; V)\). Similarly,
\(f_n \rightarrow f\) in \(L^2(\mathbb{R}; H)\). This follows because \(p \geq 2\) and the norm in \(V\) and norm in \(H\) are related by \(|x|_H \leq C||x||_V\) for some constant, \(C\). Now

\[
\tilde{f}(t) = \begin{cases} 
\Psi(t)f(t) & \text{if } t \in [0,T], \\
\Psi(t)(2T-t) & \text{if } t \in [T,2T], \\
\Psi(t)f(-t) & \text{if } t \in [0,T], \\
0 & \text{if } t \notin [-T,2T]. 
\end{cases}
\]

An easy modification of the argument of Lemma \[\text{(???)}\] yields

\[
\tilde{f}'(t) = \begin{cases} 
\Psi'(t)f(t) + \Psi(y)f'(t) & \text{if } t \in [0,T], \\
\Psi'(t)(2T-t) - \Psi(t)f'(2T-t) & \text{if } t \in [T,2T], \\
\Psi'(t)f(-t) - \Psi(t)f'(t) & \text{if } t \in [-T,0], \\
0 & \text{if } t \notin [-T,2T]. 
\end{cases}
\]

Recall

\[
f_n(t) = \int_{-1/n}^{1/n} \tilde{f}(t-s)\phi_n(s)\,ds = \int_{\mathbb{R}} \tilde{f}(t-s)\phi_n(s)\,ds = \int_{\mathbb{R}} \tilde{f}(s)\phi_n(t-s)\,ds.
\]

Therefore,

\[
f'_n(t) = \int_{\mathbb{R}} \tilde{f}'(s)\phi_n'(t-s)\,ds = \int_{-T-h}^{2T+h} \tilde{f}(s)\phi_n'(t-s)\,ds
\]

\[
= \int_{-T-h}^{2T+h} \tilde{f}'(s)\phi_n(t-s)\,ds = \int_{\mathbb{R}} \tilde{f}'(s)\phi_n(t-s)\,ds
\]

\[
= \int_{\mathbb{R}} \tilde{f}'(t-s)\phi_n(s)\,ds = \int_{-1/n}^{1/n} \tilde{f}'(t-s)\phi_n(s)\,ds
\]

and it follows from the first line above that \(f'_n\) is continuous with values in \(V\) for all \(t \in \mathbb{R}\). Also note that both \(f'_n\) and \(f_n\) equal zero if \(t \notin [-T,2T]\) whenever \(n\) is large enough. Exactly similar reasoning to the above shows that \(f'_n \to \tilde{f}'\) in \(L^p'(\mathbb{R}; V')\).

Now let \(\phi \in C_0^\infty(0,T)\).

\[
\int_{\mathbb{R}} |f_n(t)|_H^2 \phi'(t)\,dt = \int_{\mathbb{R}} (f_n(t), f_n(t))_H \phi'(t)\,dt \quad (67.2.14)
\]

\[
= -\int_{\mathbb{R}} 2(f'_n(t), f_n(t))_H \phi(t)\,dt = -\int_{\mathbb{R}} 2(f'_n(t), f_n(t))_H \phi(t)\,dt
\]

Now

\[
\int_{\mathbb{R}} |(f'_n(t), f_n(t))_H \phi(t)\,dt - \int_{\mathbb{R}} (f'(t), f(t))_H \phi(t)\,dt
\]

\[
\leq \int_{\mathbb{R}} |(|f'_n(t) - f'(t), f_n(t))| + |f'(t), f_n(t) - f(t))|\phi(t)\,dt.
\]
From the first part of this proof which showed that \( f_n \to \hat{f} \) in \( L^p(\mathbb{R}; V) \) and \( f_n' \to \hat{f}' \) in \( L^{p'}(\mathbb{R}; V') \), an application of Holder’s inequality shows the above converges to 0 as \( n \to \infty \). Therefore, passing to the limit as \( n \to \infty \) in the

\[
\int_{\mathbb{R}} |\hat{f}(t)|^2_H \phi'(t) dt = - \int_{\mathbb{R}} 2 \left< \hat{f}'(t), \hat{f}(t) \right> \phi(t) dt
\]

which shows \( t \to |\hat{f}(t)|^2_H \) equals a continuous function a.e. and it also has a weak derivative equal to \( 2 \left< \hat{f}', \hat{f} \right> \).

It remains to verify that \( \hat{f} \) is continuous on \([0,T]\). Of course \( \hat{f} = f \) on this interval. Let \( N \) be large enough that \( f_n(\pm T) = 0 \) for all \( n > N \). Then for \( m, n > N \) and \( t \in [-T, 2T] \)

\[
|f_n(t) - f_m(t)|^2_H = 2 \int_{-T}^t (f'_n(s) - f'_m(s), f_n(s) - f_m(s)) ds
\]

\[
= 2 \int_{-T}^t (f'_n(s) - f'_m(s), f_n(s) - f_m(s))_{V',V} ds
\]

\[
\leq 2 \int \|f'_n(s) - f'_m(s)\|_{V'} \|f_n(s) - f_m(s)\|_{V} ds
\]

\[
\leq 2 \|f_n - f_m\|_{L^p(\mathbb{R};V')} \|f_n - f_m\|_{L^p(\mathbb{R};V')}
\]

which shows from the above that \( \{f_n\} \) is uniformly Cauchy on \([-T,2T]\) with values in \( H \). Therefore, there exists \( g \) a continuous function defined on \([-T,2T]\) having values in \( H \) such that

\[
\lim_{n \to \infty} \max \{ |f_n(t) - g(t)|_H ; t \in [-T,2T] \} = 0.
\]

However, \( g = \hat{f} \) a.e. because \( f_n \) converges to \( f \) in \( L^p(0,T;V) \). Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that \( V \subseteq H = H' \subseteq V' \) and Theorem 67.2.9, there exists \( f(0) \in V' \) such that for a.e. \( t \),

\[
f(t) = f(0) + \int_0^t f'(s) ds \text{ in } V'
\]

Now \( g = f \) a.e. and \( g \) is continuous with values in \( H \) hence continuous with values in \( V' \) and so

\[
g(t) = f(0) + \int_0^t f'(s) ds \text{ in } V'
\]

for all \( t \). Since \( g \) is continuous with values in \( H \) it is continuous with values in \( V' \). Taking the limit as \( t \downarrow 0 \) in the above, \( g(a) = \lim_{t \to 0+} g(t) = f(0) \), showing that \( f(0) \in H \). Therefore, for a.e. \( t \),

\[
f(t) = f(0) + \int_0^t f'(s) ds \text{ in } H, \int_0^t f'(s) ds \in H. \]

\[\square\]
Note that if \( f \in L^p(0, T; V) \) and \( f' \in L^{p'}(0, T; V') \), then you can consider the initial value of \( f \) and it will be in \( H \). What if you start with something in \( H \)? Is it an initial condition for a function \( f \in L^p(0, T; V) \) such that \( f' \in L^{p'}(0, T; V') \)? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if \( V \) is a closed subspace of the Sobolev space, \( W^{1,p}(\Omega) \) which contains \( W^{1,0}(\Omega) \) for \( p \geq 2 \) and \( H = L^2(\Omega) \) the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

### 67.3 An Important Formula

It is not necessary to have \( p > 2 \) in order to do the sort of thing just described. Here is a major result which will have a much more difficult stochastic version presented later. First is an approximation theorem of Doob. See Lemma 63.3.1.

**Lemma 67.3.1** Let \( Y : [0, T] \to E \), be \( B([0, T]) \) measurable and suppose

\[
Y \in L^p(0, T; E) \equiv K, \quad p \geq 1
\]

Then there exists a sequence of nested partitions, \( \mathcal{P}_k \subseteq \mathcal{P}_{k+1} \),

\[
\mathcal{P}_k = \{i_0^k, \ldots, i_{m_k}^k\}
\]

such that the step functions given by

\[
Y_k^r(t) = \sum_{j=1}^{m_k} Y(t_j^k) \chi_{[t_j^k, t_{j+1}^k)}(t)
\]

\[
Y_k^l(t) = \sum_{j=1}^{m_k} Y(t_{j-1}^k) \chi_{(t_{j-1}^k, t_j^k]}(t)
\]

both converge to \( Y \) in \( K \) as \( k \to \infty \) and

\[
\lim_{k \to \infty} \max \{|t_j^k - t_{j+1}^k| : j \in \{0, \ldots, m_k\}\} = 0.
\]

Also, each \( Y(t_j^k), Y(t_{j-1}^k) \) is in \( E \). One can also assume that \( Y(0) = 0 \). The mesh points \( \{t_j^k\}_{j=0}^{m_k} \) can be chosen to miss a given set of measure zero. In addition to this, we can assume that

\[
|t_j^k - t_{j-1}^k| = 2^{-n_k}
\]

except for the case where \( j = 1 \) or \( j = m_{n_k} \) when this might not be so. In the case of the last subinterval defined by the partition, we can assume

\[
|t_m^k - t_{m-1}^k| = |T - t_{m-1}^k| \geq 2^{-(n_k+1)}
\]

Theorem 67.3.2 Let $V \subseteq H = H' \subseteq V'$ be a Gelfand triple and suppose $Y \in L^{p'} (0, T; V') \equiv K'$ and

$$X(t) = X_0 + \int_0^t Y(s) \, ds \text{ in } V'$$  \hfill (67.3.15)

where $X_0 \in H$, and it is known that $X \in L^p (0, T, V) \equiv K$ for $p > 1$. Then $t \rightarrow X(t)$ is in $C([0, T], H)$ and also

$$\frac{1}{2} |X(t)|_H^2 = \frac{1}{2} |X_0|_H^2 + \int_0^t \langle Y(s), X(s) \rangle \, ds$$

Proof: By Lemma 67.3.1 there exists a sequence of uniform partitions $(t_k^n)_{k=0}^{m_n} = \mathcal{P}_n, \mathcal{P}_n \subseteq \mathcal{P}_{n+1}$, of $[0, T]$ such that the step functions

$$\sum_{k=0}^{m_n-1} X(t_k^n) X_{(t_k^n, t_{k+1}^n)}(t) \equiv X^l(t)$$

$$\sum_{k=0}^{m_n-1} X(t_{k+1}^n) X_{(t_k^n, t_{k+1}^n)}(t) \equiv X^r(t)$$

converge to $X$ in $K$ and in $L^2([0, T], H)$.

Lemma 67.3.3 Let $s < t$. Then for $X, Y$ satisfying 67.3.15

$$|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du - |X(t) - X(s)|^2$$  \hfill (67.3.16)

Proof: It follows from the following computations

$$X(t) - X(s) = \int_s^t Y(u) \, du$$

$$- |X(t) - X(s)|^2 = - |X(t)|^2 + 2 \langle X(t), X(s) \rangle - |X(s)|^2$$

$$= - |X(t)|^2 + 2 \left( X(t), X(t) - \int_s^t Y(u) \, du \right) - |X(s)|^2$$

$$= - |X(t)|^2 + 2 |X(t)|^2 - 2 \left( \int_s^t Y(u) \, du, X(t) \right) - |X(s)|^2$$

Hence

$$|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du - |X(t) - X(s)|^2$$

$\blacksquare$
Lemma 67.3.4 In the above situation,
\[ \sup_{t \in [0, T]} |X(t)|_H \leq C(\|Y\|_{K'}, \|X\|_K) \]

Also, \( t \to X(t) \) is weakly continuous with values in \( H \).

Proof: From the above formula applied to the \( k^{th} \) partition of \([0, T]\) described above,
\[
|X(t_m)|^2 - |X_0|^2 = \sum_{j=0}^{m-1} |X(t_{j+1})|^2 - |X(t_j)|^2
\]
\[
= \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) \rangle \, du - |X(t_{j+1}) - X(t_j)|^2_H
\]
\[
= \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X_k^e(u) \rangle \, du - |X(t_{j+1}) - X(t_j)|^2_H
\]

Thus, discarding the negative terms and denoting by \( P_k \) the \( k^{th} \) of these partitions,
\[
\sup_{t_j \in P_k} |X(t_j)|^2_H \leq |X_0|^2 + 2 \int_0^T |\langle Y(u), X_k^e(u) \rangle| \, du
\]
\[
\leq |X_0|^2 + 2 \int_0^T \|Y(u)\|_{V'} \|X_k^e(u)\|_V \, du
\]
\[
\leq |X_0|^2 + 2 \left( \int_0^T \|Y(u)\|^{p'}_{V'} \, du \right)^{1/p'} \left( \int_0^T \|X_k^e(u)\|^p_V \, du \right)^{1/p} \leq C(\|Y\|_{K'}, \|X\|_K)
\]
because these partitions are chosen such that
\[
\lim_{k \to \infty} \left( \int_0^T \|X_k^e(u)\|^p_V \, du \right)^{1/p} = \left( \int_0^T \|X(u)\|^p_V \, du \right)^{1/p}
\]
and so these are bounded. This has shown that for the dense subset of \([0, T]\),
\[
D \equiv \bigcup_k P_k,
\]
\[
\sup_{t \in D} |X(t)| < C(\|Y\|_{K'}, \|X\|_K)
\]

Now let \( \{g_k\}_{k=1}^\infty \) be linearly independent vectors of \( V \) whose span is dense in \( V \).
This is possible because \( V \) is separable. Then let \( \{e_j\}_{j=1}^\infty \) be an orthonormal basis
for \( H \) such that \( e_k \in \text{span}(g_1, \ldots, g_k) \) and each \( g_k \in \text{span}(e_1, \ldots, e_k) \).
This is done with the Gram Schmidt process. Then it follows that span \( \{e_k\}_{k=1}^\infty \) is dense in \( V \).
I claim
\[
|y|^2_H = \sum_{j=1}^\infty |\langle y, e_j \rangle|^2.
\]
67.3. AN IMPORTANT FORMULA

This is certainly true if \( y \in H \) because
\[
\langle y, e_j \rangle = \langle y, e_j \rangle_H
\]
If \( y \notin H \), then the series must diverge since otherwise, you could consider the infinite sum
\[
\sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \in H
\]
because
\[
\left| \sum_{j=p}^{q} \langle y, e_j \rangle e_j \right|^2 = \sum_{j=p}^{q} \left| \langle y, e_j \rangle \right|^2 \to 0 \text{ as } p, q \to \infty.
\]
Letting \( z = \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \), it follows that \( \langle y, e_j \rangle \) is the \( j^{th} \) Fourier coefficient of \( z \) and that
\[
\langle z - y, v \rangle = 0
\]
for all \( v \in \text{span} \{e_k\}_{k=1}^{\infty} \) which is dense in \( V \). Therefore, \( z = y \) in \( V' \) and so \( y \in H \).

It follows
\[
|X(t)|^2 = \sup_n \sum_{j=1}^{n} |\langle X(t), e_j \rangle|^2
\]
which is just the sup of continuous functions of \( t \). Therefore, \( t \to |X(t)|^2 \) is lower semicontinuous. It follows that for any \( t \), letting \( t_j \to t \) for \( t_j \in D \),
\[
|X(t)|^2 \leq \lim \inf_{j \to \infty} |X(t_j)|^2 \leq C (\|Y\|_{K'}, \|X\|_K)
\]
This proves the first claim of the lemma.

Consider now the claim that \( t \to X(t) \) is weakly continuous. Letting \( v \in V \),
\[
\lim_{t \to s} \langle X(t), v \rangle = \lim_{t \to s} \langle X(s), v \rangle = \langle X(s), v \rangle = \langle X(s), v \rangle
\]
Since it was shown that \( |X(t)| \) is bounded independent of \( t \), and since \( V \) is dense in \( H \), the claim follows. ■

Now
\[
- \sum_{j=0}^{m-1} |X(t_{j+1}) - X(t_j)|_H^2 = |X(t_m)|^2 - |X_0|^2 - \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X_k^+(u) \rangle \, du
\]
\[
= |X(t_m)|^2 - |X_0|^2 - 2 \int_0^{t_m} \langle Y(u), X_k^+(u) \rangle \, du
\]
Thus, since the partitions are nested, eventually \( |X(t_m)|^2 \) is constant for all \( k \) large enough and the integral term converges to
\[
\int_0^{t_m} \langle Y(u), X(u) \rangle \, du
\]
CHAPTER 67. GELFAND TRIPLES

It follows that the term on the left does converge to something. It just remains to consider what it does converge to. However, from the equation solved by \( X \),

\[
X(t_{j+1}) - X(t_j) = \int_{t_j}^{t_{j+1}} Y(u) \, du
\]

Therefore, this term is dominated by an expression of the form

\[
\sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y(u) \, du, X(t_{j+1}) - X(t_j) \right)
\]

\[
= \sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y(u) \, du, X(t_{j+1}) - X(t_j) \right)
\]

\[
= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} (Y(u), X(t_{j+1}) - X(t_j)) \, du
\]

\[
= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) - X(t_j) \rangle \, du
\]

\[
= \int_0^T \langle Y(u), X^r(u) \rangle \, du - \int_0^T \langle Y(u), X^l(u) \rangle \, du
\]

However, both \( X^r \) and \( X^l \) converge to \( X \) in \( K = L^p(0, T, V) \). Therefore, this term must converge to 0. Passing to a limit, it follows that for all \( t \in D \), the desired formula holds. Thus, for such \( t \),

\[
|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(u), X(u) \rangle \, du
\]

It remains to verify that this holds for all \( t \). Let \( t \notin D \) and let \( t(k) \in P_k \) be the largest point of \( P_k \) which is less than \( t \). Suppose \( t(m) \leq t(k) \) so that \( m \leq k \). Then

\[
X(t(m)) = X_0 + \int_0^{t(m)} Y(s) \, ds,
\]

a similar formula for \( X(t(k)) \). Thus for \( t > t(m) \),

\[
X(t) - X(t(m)) = \int_{t(m)}^t Y(s) \, ds
\]

which is the same sort of thing already looked at except that it starts at \( t(m) \) rather than at 0 and \( X_0 = 0 \). Therefore,

\[
|X(t(k)) - X(t(m))|^2 = 2 \int_{t(m)}^{t(k)} \langle Y(s), X(s) - X(t(m)) \rangle \, ds
\]
Thus, for \( m \leq k \)

\[
\lim_{m,k \to \infty} |X(t(k)) - X(t(m))|^2 = 0
\]

Hence \( \{X(t(k))\}_{k=1}^{\infty} \) is a convergent sequence in \( H \). Does it converge to \( X(t) \)? Let \( \xi(t) \in H \) be what it does converge to. Let \( v \in V \). Then

\[
(\xi(t), v) = \lim_{k \to \infty} (X(t(k)), v) = \lim_{k \to \infty} \langle X(t(k)), v \rangle = \langle X(t), v \rangle = (X(t), v)
\]

because it is known that \( t \to X(t) \) is continuous into \( V' \) and it is also known that \( X(t) \in H \) and that the \( X(t) \) for \( t \in [0, T] \) are uniformly bounded. Therefore, since \( V \) is dense in \( H \), it follows that \( \xi(t) = X(t) \).

Now for every \( t \in D \), it was shown above that

\[
|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle ds
\]

Thus, using what was just shown, if \( t \notin D \) and \( t_k \to t \),

\[
|X(t)|^2 = \lim_{k \to \infty} |X(t_k)|^2 = \lim_{k \to \infty} \left( |X_0|^2 + 2 \int_0^{t_k} \langle Y(s), X(s) \rangle ds \right)
\]

\[
= |X_0|^2 + 2 \int_0^t \langle Y(s), X(s) \rangle ds
\]

which proves the desired formula. From this it follows right away that \( t \to X(t) \) is continuous into \( H \) because it was just shown that \( t \to |X(t)| \) is continuous and \( t \to X(t) \) is weakly continuous. Since Hilbert space is uniformly convex, this implies the \( t \to X(t) \) is continuous. To see this in the special case of Hilbert space,

\[
|X(t) - X(s)|^2 = |X(t)|^2 - 2 \langle X(s), X(t) \rangle + |X(s)|^2
\]

Then \( \lim_{t \to s} \left( |X(t)|^2 - 2 \langle X(s), X(t) \rangle + |X(s)|^2 \right) = 0 \) by weak convergence of \( X(t) \) to \( X(s) \) and the convergence of \( |X(t)|^2 \) to \( |X(s)|^2 \).

### 67.4 The Implicit Case

The above theorem can be generalized to the case where the formula is of the form

\[
BX(t) = BX_0 + \int_0^t Y(s) ds
\]

This involves an operator \( B \in \mathcal{L}(W, W') \) and \( B \) satisfies

\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle
\]

for

\[
V \subseteq W, W' \subseteq V'
\]

Where \( V \) is dense in the Hilbert space \( W \). Before giving the theorem, here is a technical lemma.
Lemma 67.4.1 Suppose $V, W$ are separable Banach spaces, $W$ also a Hilbert space such that $V$ is dense in $W$ and $B \in \mathcal{L}(W, W')$ satisfies

$$\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle, \quad B \neq 0.$$ 

Then there exists a countable set $\{e_i\}$ of vectors in $V$ such that

$$\langle Be_i, e_j \rangle = \delta_{ij}$$

and for each $x \in W,$

$$\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,$$

and also

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i,$$

the series converging in $W'$.

Proof: Let $\{g_k\}_{k=1}^{\infty}$ be linearly independent vectors of $V$ whose span is dense in $V$. This is possible because $V$ is separable. Thus, their span is also dense in $W$. Let $n_1$ be the first index such that $\langle Bg_{n_1}, g_{n_1} \rangle \neq 0$.

Claim: If there is no such index, then $B = 0$.

Proof of claim: First note that if there is no such first index, then if $x = \sum_{i=1}^{k} a_i g_i$

$$|\langle Bx, x \rangle| = \left| \sum_{i \neq j} a_i a_j \langle Bg_i, g_j \rangle \right| \leq \sum_{i \neq j} |a_i| |a_j| |\langle Bg_i, g_j \rangle|$$

$$\leq \sum_{i \neq j} |a_i| |a_j| \langle Bg_i, g_i \rangle^{1/2} \langle Bg_j, g_j \rangle^{1/2} = 0$$

Therefore, if $x$ is given, you could take $x_k$ in the span of $\{g_1, \cdots, g_k\}$ such that $\|x - x_k\|_W \to 0$. Then

$$|\langle Bx, y \rangle| = \lim_{k \to \infty} |\langle Bx_k, y \rangle| \leq \lim_{k \to \infty} \langle Bx_k, x_k \rangle^{1/2} \langle By, y \rangle^{1/2} = 0$$

because $\langle Bx_k, x_k \rangle$ is zero by what was just shown.

Thus assume there is such a first index. Let

$$e_1 = \frac{g_{n_1}}{\langle Bg_{n_1}, g_{n_1} \rangle^{1/2}}$$

Then $\langle Be_1, e_1 \rangle = 1$. Now if you have constructed $e_j$ for $j \leq k$,

$$e_j \in \text{span} \{g_{n_1}, \cdots, g_{n_k}\}, \quad \langle Be_i, e_j \rangle = \delta_{ij},$$
67.4. THE IMPLICIT CASE

$g_{n_{j+1}}$ being the first for which

\[ \left\langle Bg_{n_{j+1}} - \sum_{i=1}^{j} \langle Bg_{n_{j+1}}, e_i \rangle Be_i, g_{n_{j+1}} - \sum_{i=1}^{j} \langle Bg_{n_{j}}, e_i \rangle e_i \right\rangle \neq 0, \]

and

\[ \text{span} \left( g_{n_1}, \ldots, g_{n_k} \right) = \text{span} \left( e_1, \ldots, e_k \right), \]

let $g_{n_{k+1}}$ be such that $g_{n_{k+1}}$ is the first in the list \{ $g_{n_k}$ \} such that

\[ \left\langle Bg_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle Be_i, g_{n_{k+1}} - \sum_{i=1}^{k} \langle Bg_{n_{k+1}}, e_i \rangle e_i \right\rangle \neq 0 \]

Note the difference between this and the Gram Schmidt process. Here you don’t necessarily use all of the $g_k$ due to the possible degeneracy of $B$.

**Claim:** If there is no such first $g_{n_{k+1}}$, then $B \left( \text{span} \left( e_1, \ldots, e_k \right) \right) = BW$ so in this case, $\{ Be_i \}_{i=1}^{k}$ is actually a basis for $BW$.

**Proof:** Let $x \in W$. Let $x_r \in \text{span} \left( g_1, \ldots, g_r \right)$, $r > n$ such that $\lim_{r \to \infty} x_r = x$ in $W$. Then

\[ x_r = \sum_{i=1}^{k} c_i \epsilon_i + \sum_{i \notin \{ n_1, \ldots, n_k \}}^{r} d_i g_i \equiv y_r + z_r \quad (67.4.17) \]

If $l \notin \{ n_1, \ldots, n_k \}$, then by the construction and the above assumption, for some $j \leq k$

\[ \left\langle B g_l - \sum_{i=1}^{j} \langle B g_l, e_i \rangle B e_i, g_l - \sum_{i=1}^{j} \langle B g_l, e_i \rangle e_i \right\rangle = 0 \quad (67.4.18) \]

If $l < n_k$, this follows from the construction. If the above is nonzero all $j \leq k$, then $l$ would have been chosen but it wasn’t. Thus

\[ B g_l = \sum_{i=1}^{j} \langle B g_l, e_i \rangle B e_i \]

If $l > n_k$, then by assumption, $67.4.18$ holds for $j = k$. Thus, in any case, it follows that for each $l \notin \{ n_1, \ldots, n_k \}$,

\[ B g_l \in B \left( \text{span} \left( e_1, \ldots, e_k \right) \right). \]

Now it follows from $67.4.17$ that

\[ B x_r = \sum_{i=1}^{k} c_i \epsilon_i + \sum_{i \notin \{ n_1, \ldots, n_k \}}^{r} d_i g_i \]

\[ = \sum_{i=1}^{k} c_i \epsilon_i + \sum_{i \notin \{ n_1, \ldots, n_k \}}^{r} d_i \sum_{j=1}^{k} c_j \epsilon_j \]
and so \( Bx_r \in B(\text{span}(e_1, \ldots, e_k)) \). Then \( Bx = \lim_{r \to \infty} Bx_r = \lim_{r \to \infty} By_r \) where \( y_r \in \text{span}(e_1, \ldots, e_k) \). Say

\[
Bx_r = \sum_{i=1}^{k} a_i^r Be_i
\]

It follows easily that \( \langle Bx_r, e_j \rangle = a_j^r \). (Act on \( e_j \) by both sides and use \( \langle Be_i, e_j \rangle = \delta_{ij} \).) Now since \( x_r \) is bounded, it follows that these \( a_j^r \) are also bounded. Hence, defining \( y_r = \sum_{i=1}^{k} a_i^r e_i \), it follows that \( y_r \) is bounded in \( \text{span}(e_1, \ldots, e_k) \) and so, there exists a subsequence, still denoted by \( r \) such that \( y_r \to y \in \text{span}(e_1, \ldots, e_k) \). Therefore, \( Bx = \lim_{r \to \infty} By_r = By \). In other words, \( BW = B(\text{span}(e_1, \ldots, e_k)) \) as claimed. This proves the claim.

If this happens, the process being described stops. You have found what is desired which has only finitely many vectors involved.

As long as the process does not stop, let

\[
e_{k+1} = \frac{g_{nk+1} - \sum_{i=1}^{k} \langle Bg_{nk+1}, e_i \rangle e_i}{\langle B \left( g_{nk+1} - \sum_{i=1}^{k} \langle Bg_{nk+1}, e_i \rangle e_i \right), g_{nk+1} - \sum_{i=1}^{k} \langle Bg_{nk+1}, e_i \rangle e_i \rangle^{1/2}}
\]

Thus, as in the usual argument for the Gram Schmidt process, \( \langle Be_i, e_j \rangle = \delta_{ij} \) for \( i, j \leq k + 1 \). This is already known for \( i, j \leq k \). Letting \( l \leq k \), and using the orthogonality already shown,

\[
\langle Be_{k+1}, e_l \rangle = C \left( \langle Bg_{k+1}, e_l \rangle - \langle Bg_{nk+1}, e_l \rangle \right) = 0
\]

Consider

\[
\langle Bg_p - B \left( \sum_{i=1}^{k} \langle Bg_p, e_i \rangle e_i \right), g_p - \sum_{i=1}^{k} \langle Bg_p, e_i \rangle e_i \rangle
\]

Either this equals 0 because \( p \) is never one of the \( n_k \) or eventually it equals 0 for some \( k \) because \( g_p = g_{nk} \) for some \( n_k \) and so, from the construction, \( g_{nk} = g_p \in \text{span}(e_1, \ldots, e_k) \) and therefore,

\[
g_p = \sum_{j=1}^{k} a_j e_j
\]

which requires easily that

\[
Bg_p = \sum_{i=1}^{k} \langle Bg_p, e_i \rangle Be_i,
\]
the above holding for all $k$ large enough. It follows that for any $x \in \text{span} (\{g_k\}_{k=1}^{\infty})$,
(finite linear combination of vectors in $\{g_k\}_{k=1}^{\infty}$)

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \quad (67.4.19)$$

because for all $k$ large enough,

$$Bx = \sum_{i=1}^{k} \langle Bx, e_i \rangle Be_i$$

Also note that for such $x \in \text{span} (\{g_k\}_{k=1}^{\infty})$,

$$\langle Bx, x \rangle = \left( \sum_{i=1}^{k} \langle Bx, e_i \rangle Be_i, x \right) = \sum_{i=1}^{k} \langle Bx, e_i \rangle \langle Bx, e_i \rangle$$

$$= \sum_{i=1}^{k} |\langle Bx, e_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2$$

Now for $x$ arbitrary, let $x_k \to x$ in $W$ where $x_k \in \text{span} (\{g_k\}_{k=1}^{\infty})$. Then by Fatou’s lemma,

$$\sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \leq \liminf_{k \to \infty} \sum_{i=1}^{\infty} |\langle Bx_k, e_i \rangle|^2$$
$$= \liminf_{k \to \infty} \langle Bx_k, x_k \rangle = \langle Bx, x \rangle \quad (67.4.20)$$

$$\leq \|Bx\|_W, \|x\|_W \leq \|B\| \|x\|_W$$

Thus the series on the left converges. Then also, from the above inequality,

$$\left| \sum_{i=p}^{q} \langle Bx, e_i \rangle Be_i, y \right| \leq \sum_{i=p}^{q} |\langle Bx, e_i \rangle| |\langle Be_i, y \rangle|$$

$$\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=p}^{q} |\langle By, e_i \rangle|^2 \right)^{1/2}$$

$$\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \sum_{i=1}^{\infty} |\langle By, e_i \rangle|^2 \right)^{1/2}$$

By [THEorem 6.11.6],

$$\leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \left( \|B\| \|y\|_W \right)^{1/2} \leq \left( \sum_{i=p}^{q} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \|B\|^{1/2} \|y\|_W$$
CHAPTER 67. GELFAND TRIPLES

It follows that

\[ \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \]

(67.4.21)

converges in \( W' \) because it was just shown that

\[ \sum_{i=p}^{q} \langle Bx, e_i \rangle \cdot \| W' \| \leq \left( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \| B \|^{1/2} \]

and it was shown above that \( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 < \infty \), so the partial sums of the series are a Cauchy sequence in \( W' \). Also, the above estimate shows that for \( \| y \| = 1 \),

\[ \sum_{i=1}^{\infty} \langle Bx, e_i \rangle, y \rangle \leq \left( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \| B \|^{1/2} \]

and so

\[ \sum_{i=1}^{\infty} \langle Bx, e_i \rangle \cdot \| W' \| \leq \left( \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \right)^{1/2} \| B \|^{1/2} \]

(67.4.22)

Now for \( x \) arbitrary, let \( x_k \in \text{span} \left\{ g_j \right\}_{j=1}^{\infty} \) and \( x_k \to x \) in \( W \). Then for a fixed \( k \) large enough,

\[ \| Bx - \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \| \leq \| Bx - Bx_k \| \]

\[ + \| Bx_k - \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle Be_i \| + \| \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle Be_i - \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \| \]

\[ \leq \varepsilon + \sum_{i=1}^{\infty} \langle B(x_k - x), e_i \rangle Be_i \]

the term

\[ \| Bx_k - \sum_{i=1}^{\infty} \langle Bx_k, e_i \rangle Be_i \| \]

equaling 0 by 67.4.19. From 67.4.20 and 67.4.21,

\[ \leq \varepsilon + \| B \|^{1/2} \left( \sum_{i=1}^{\infty} |\langle B(x_k - x), e_i \rangle|^2 \right)^{1/2} \]

\[ \leq \varepsilon + \| B \|^{1/2} \langle B(x_k - x), x_k - x \rangle^{1/2} < 2\varepsilon \]
whenever \( k \) is large enough. Therefore,

\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i
\]

in \( W' \). It follows that

\[
\langle Bx, x \rangle = \lim_{k \to \infty} \left( \sum_{i=1}^{k} \langle Bx, e_i \rangle Be_i, x \right) = \lim_{k \to \infty} \sum_{i=1}^{k} |\langle Bx, e_i \rangle|^2 \equiv \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2 \quad \blacksquare
\]

**Theorem 67.4.2** Let \( V \subseteq W, W' \subseteq V' \) be separable Banach spaces, \( W \) a separable Hilbert space, and let \( Y \in L^p' (0,T;V') \equiv K' \) and

\[
BX(t) = BX_0 + \int_{0}^{t} Y(s) \, ds \quad \text{in} \quad V'
\]

where \( X_0 \in W \), and it is known that \( X \in L^p (0,T,V) \equiv K \) for \( p > 1 \). Then \( t \to BX(t) \) is in \( C ([0,T], W') \) and also

\[
\frac{1}{2} \langle BX(t),X(t) \rangle = \frac{1}{2} \langle BX_0,X_0 \rangle + \int_{0}^{t} \langle Y(s),X(s) \rangle \, ds
\]

**Proof:** By Lemma 67.4.1, there exists a sequence of uniform partitions \( \{ t^n_k \}_{k=0}^{m_n} = \mathcal{P}_n, \mathcal{P}_n \subseteq \mathcal{P}_{n+1}, \) of \([0,T]\) such that the step functions

\[
\sum_{k=0}^{m_n-1} X(t^n_k) \chi_{(t^n_k,t^n_{k+1}]}(t) \equiv X^l(t)
\]

\[
\sum_{k=0}^{m_n-1} X(t^n_{k+1}) \chi_{(t^n_k,t^n_{k+1}]}(t) \equiv X^r(t)
\]

converge to \( X \) in \( K \) and also \( BX^l, BX^r \to BX \) in \( L^2 ([0,T], W') \).

**Lemma 67.4.3** Let \( s < t \). Then for \( X,Y \) satisfying 67.4.23

\[
\langle BX(t),X(t) \rangle = \langle BX(s),X(s) \rangle + 2 \int_{s}^{t} \langle Y(u),X(t) \rangle \, du - \langle B(X(t) - X(s)),(X(t) - X(s)) \rangle
\]

**Proof:** It follows from the following computations

\[
B(X(t) - X(s)) = \int_{s}^{t} Y(u) \, du
\]

and so

\[
2 \int_{s}^{t} \langle Y(u),X(t) \rangle \, du - \langle B(X(t) - X(s)),(X(t) - X(s)) \rangle
\]
\[= 2 \langle BX(t) - X(s) \rangle, X(t) - \langle BX(t) - X(s) \rangle, (X(t) - X(s)) \]

\[= 2 \langle BX(t), X(t) \rangle - 2 \langle BX(s), X(t) \rangle - \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle + 2 \langle BX(s), X(t) \rangle - \langle BX(s), X(s) \rangle \]

\[= \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle \]

Thus

\[= 2 \int_s^t \langle Y(u), X(t) \rangle du - \langle B(X(t) - X(s)), (X(t) - X(s)) \rangle \]

**Lemma 67.4.4** In the above situation,

\[\sup_{t \in [0,T]} \langle BX(t), X(t) \rangle \leq C(\|Y\|_{K'}, \|X\|_K)\]

Also, \( t \to BX(t) \) is weakly continuous with values in \( W' \).

**Proof:** From the above formula applied to the \( k^{th} \) partition of \([0,T]\) described above,

\[\langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle = \sum_{j=0}^{m-1} \langle BX(t_{j+1}), X(t_{j+1}) \rangle - \langle BX(t_j), X(t_j) \rangle \]

\[= \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X(t_{j+1}) \rangle du - \langle B(X(t_{j+1}) - X(t_j)), X(t_{j+1}) - X(t_j) \rangle \]

\[= \sum_{j=0}^{m-1} 2 \int_{t_j}^{t_{j+1}} \langle Y(u), X^*_k(u) \rangle du - \langle B(X(t_{j+1}) - X(t_j)), X(t_{j+1}) - X(t_j) \rangle \]

Thus, discarding the negative terms and denoting by \( P_k \) the \( k^{th} \) of these partitions,

\[\sup_{t_j \in P_k} \langle BX(t_j), X(t_j) \rangle \leq \langle BX_0, X_0 \rangle + 2 \int_0^T \|Y(u)\|_{V'} \|X^*_k(u)\|_V \]

\[\leq \langle BX_0, X_0 \rangle + 2 \int_0^T \|Y(u)\|_{V'}^p \|X^*_k(u)\|_V^p \]

\[\leq \langle BX_0, X_0 \rangle + 2 \left( \int_0^T \|Y(u)\|_{V'}^p \right) \left( \int_0^T \|X^*_k(u)\|_V^p \right) \leq C(\|Y\|_{K'}, \|X\|_K) \]

because these partitions are chosen such that

\[\lim_{k \to \infty} \left( \int_0^T \|X^*_k(u)\|_V^p \right)^{1/p} = \left( \int_0^T \|X(u)\|_{V'}^p \right)^{1/p} \]
and so these are bounded. This has shown that for the dense subset of \([0,T]\),
\[ D \equiv \cup \mathcal{P}_k, \]
\[ \sup_{t \in D} \langle BX(t), X(t) \rangle < C (\|Y\|_{K'}, \|X\|_K) \]

From Lemma 67.4.1 above, there exists \(\{e_i\} \subseteq V\) such that \(\langle Be_i, e_j \rangle = \delta_{ij}\) and
\[ \langle BX(t), X(t) \rangle = \sum_{k=1}^{\infty} |\langle BX(t), e_i \rangle|^2 = \sup_{m} \sum_{k=1}^{m} |\langle BX(t), e_i \rangle|^2 \]
Since each \(e_i \in V\), and since \(t \to BX(t)\) is continuous into \(V'\) thanks to the formula \(BX \in BV\), it follows that \(t \to \sum_{k=1}^{m} |\langle BX(t), e_i \rangle|\) is continuous and so \(t \to \langle BX(t), X(t) \rangle\) is the sup of continuous functions. Therefore, this function of \(t\) is lower semicontinuous. Since \(D\) is dense in \([0,T]\), it follows that for all \(t\),
\[ \langle BX(t), X(t) \rangle \leq C (\|Y\|_{K'}, \|X\|_K) \]
It only remains to verify the claim about weak continuity.
Consider now the claim that \(t \to BX(t)\) is weakly continuous. Letting \(v \in V\),
\[ \lim_{t \to s} \langle BX(t), v \rangle = \langle BX(s), v \rangle = \langle BX(s), v \rangle \quad (67.4.25) \]
The limit follows from the formula \(BX \in BV\) which implies \(t \to BX(t)\) is continuous into \(V'\). Now
\[ \|BX(t)\| = \sup_{\|v\|_1} |\langle BX(t), v \rangle| \leq \langle Bv, v \rangle \langle BX(t), X(t) \rangle \]
which was shown to be bounded for \(t \in [0,T]\). Now let \(w \in W\). Then
\[ |\langle BX(t), w \rangle - \langle BX(s), w \rangle| \leq |\langle BX(t) - BX(s), w - v \rangle| + |\langle BX(t) - BX(s), v \rangle| \]
Then the first term is less than \(\varepsilon\) if \(v\) is close enough to \(w\) and the second converges to 0 so \(67.4.25\) holds for all \(v \in W\) and so this shows the weak continuity. □

Now pick \(t \in D\), the union of all the mesh points. Then for all \(k\) large enough,
\(t \in \mathcal{P}_k\). Say \(t = t_m\). From Lemma 67.4.3
\[ - \sum_{j=0}^{m-1} \langle B(X(t_{j+1}) - X(t_j)), (X(t_{j+1}) - X(t_j)) \rangle = \]
\[ \langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle - 2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X_k(u) \rangle \, du \]
Thus, \(\langle BX(t_m), X(t_m) \rangle\) is constant for all \(k\) large enough and the integral term converges to
\[ \int_0^{t_m} \langle Y(u), X(u) \rangle \, du \]
It follows that the term on the left does converge to something as \( k \to \infty \). It just remains to consider what it does converge to. However, from the equation solved by \( X \),

\[
BX (t_{j+1}) - BX (t_j) = \int_{t_j}^{t_{j+1}} Y (u) \, du
\]

Therefore, this term is dominated by an expression of the form

\[
\sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y (u) \, du, X (t_{j+1}) - X (t_j) \right)
\]

\[
= \sum_{j=0}^{m_k-1} \left( \int_{t_j}^{t_{j+1}} Y (u) \, du, X (t_{j+1}) - X (t_j) \right)
\]

\[
= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} (Y (u), X (t_{j+1}) - X (t_j)) \, du
\]

\[
= \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} (Y (u), X (t_{j+1}) - X (t_j)) \, du
\]

\[
= \int_0^T \langle Y (u), X (t) \rangle \, du - \int_0^T \langle Y (u), X (t) \rangle \, du
\]

However, both \( X^r \) and \( X^l \) converge to \( X \) in \( K = L^p (0, T, V) \). Therefore, this term must converge to 0. Passing to a limit, it follows that for all \( t \in D \), the desired formula holds. Thus, for such \( t \in D \),

\[
\langle BX (t), X (t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y (u), X (t) \rangle \, du
\]

It remains to verify that this holds for all \( t \). Let \( t \notin D \) and let \( t (k) \in P_k \) be the largest point of \( P_k \) which is less than \( t \). Suppose \( t (m) \leq t (k) \) so that \( m \leq k \). Then

\[
BX (t (m)) = BX_0 + \int_0^{t (m)} Y (s) \, ds,
\]

a similar formula for \( X (t (k)) \). Thus for \( t > t (m) \),

\[
BX (t) - BX (t (m)) = \int_{t (m)}^t Y (s) \, ds
\]

which is the same sort of thing already looked at except that it starts at \( t (m) \) rather than at 0 and \( X_0 = 0 \). Therefore,

\[
\langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle = 2 \int_{t (m)}^{t (k)} \langle Y (s), X (s) - X (t (m)) \rangle \, ds
\]
Then which follows from the above, this is no larger than that provided

Then using the lower semicontinuity of

From the above, this is dominated by an expression of the form

Then the second term converges to 0. The first equals

Also it was just shown that

Therefore, since it is known that

Hence \( \{ BX(t(k)) \} \) is a convergent sequence in \( W' \) because

because it is known that \( t \to BX(t) \) is continuous into \( V' \). It is also known that \( BX(t) \in W' \subseteq V' \) and that the \( BX(t) \) for \( t \in [0,T] \) are uniformly bounded in \( W' \). Therefore, since \( V \) is dense in \( W \), it follows that \( \xi(t) = BX(t) \).

Now for every \( (m,k) \), it was shown above that

Also it was just shown that \( BX(t(k)) \to BX(t) \). Then

Then the second term converges to 0. The first equals

From the above, this is dominated by an expression of the form

Then using the lower semicontinuity of \( t \to \langle B(X(t(k)) - X(t)), X(t(k)) - X(t) \rangle \) which follows from the above, this is no larger than

provided \( k \) is large enough. This follows from \( \text{pre.} \). Since \( \varepsilon \) is arbitrary, it follows that

Thus, for \( m \leq k \)

\[
\lim_{m,k \to \infty} \langle B(X(t(k)) - X(t(m))), X(t(k)) - X(t(m)) \rangle = 0 \quad (67.4.26)
\]
Then from the formula,

\[ \langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y(u), X(u) \rangle \, du \]

valid for \( t \in D \), it follows that the same formula holds for all \( t \). This formula implies \( t \to \langle BX(t), X(t) \rangle \) is continuous. Also recall that \( t \to BX(t) \) was shown to be weakly continuous into \( W' \). Then

\[ \langle B(X(t) - X(s)), X(t) - X(s) \rangle = \langle BX(t), X(t) \rangle - 2 \langle BX(t), X(s) \rangle + \langle BX(s), X(s) \rangle \]

From this, it follows that \( t \to BX(t) \) is continuous into \( W' \) because \( \lim_{t \to s} \) of the right side gives 0 and so the same is true of the left. Hence,

\[ \|B(X(t) - X(s)), y\| \leq \|B\|^{1/2} \langle B(X(t) - X(s)), X(t) - X(s)\rangle^{1/2} \|y\| \]

so

\[ \|B(X(t) - X(s))\|_{W'} \leq \|B\|^{1/2} \langle B(X(t) - X(s)), X(t) - X(s)\rangle^{1/2} \]

which converges to 0 as \( t \to s \). ■

### 67.5 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling’s lemma.

**Definition 67.5.1** Let \( E, W \) be Banach spaces such that \( E \subseteq W \) and the injection map from \( E \) into \( W \) is continuous. The injection map is said to be compact if every bounded set in \( E \) has compact closure in \( W \). In other words, if a sequence is bounded in \( E \) it has a convergent subsequence converging in \( W \). This is also referred to by saying that bounded sets in \( E \) are precompact in \( W \).

**Theorem 67.5.2** Let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Then for every \( \varepsilon > 0 \) there exists a constant, \( C_\varepsilon \) such that for all \( u \in E \),

\[ \|u\|_W \leq \varepsilon \|u\|_E + C_\varepsilon \|u\|_X \]

**Proof:** Suppose not. Then there exists \( \varepsilon > 0 \) and for each \( n \in \mathbb{N} \), \( u_n \) such that

\[ \|u_n\|_W > \varepsilon \|u_n\|_E + n \|u_n\|_X \]

Now let \( v_n = u_n / \|u_n\|_E \). Therefore, \( \|v_n\|_E = 1 \) and

\[ \|v_n\|_W > \varepsilon + n \|v_n\|_X \]
67.5. SOME IMBEDDING THEOREMS

It follows there exists a subsequence, still denoted by \( v_n \) such that \( v_n \) converges to \( v \) in \( W \). However, the above inequality shows that \( \|v_n\|_X \to 0 \). Therefore, \( v = 0 \). But then the above inequality would imply that \( \|v_n\| > \varepsilon \) and passing to the limit yields \( 0 > \varepsilon \), a contradiction. ■

**Definition 67.5.3** Define \( C([a, b]; X) \) the space of functions continuous at every point of \([a, b]\) having values in \( X \).

You should verify that this is a Banach space with norm

\[
\|u\|_{\infty, X} = \max \left\{ \|u_{n_k}(t) - u(t)\| : t \in [a, b] \right\}.
\]

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem. It is like a well known result due to Simon. It is the appropriate generalization to stochastic problems in which you do not have weak derivatives. See Theorem [2.12.1] on the Holder continuity of the stochastic integral.

**Theorem 67.5.4** Let \( q > 1 \) and let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Let \( S \) be defined by

\[
\left\{ u \text{ such that } \|u(t)\|_E \leq R \text{ for all } t \in [a, b], \text{ and } \|u(s) - u(t)\|_X \leq R|t - s|^{1/q} \right\}.
\]

Thus \( S \) is bounded in \( L^\infty(0, T, E) \) and in addition, the functions are uniformly Holder continuous into \( X \). Then \( S \subseteq C([a, b]; W) \) and if \( \{u_n\} \subseteq S \), there exists a subsequence, \( \{u_{n_k}\} \) which converges to a function \( u \in C([a, b]; W) \) in the following way.

\[
\lim_{k \to \infty} \|u_{n_k} - u\|_{\infty, W} = 0.
\]

**Proof:** First consider the issue of \( S \) being a subset of \( C([a, b]; W) \). Let \( \varepsilon > 0 \) be given. Then by Theorem [67.5.2] there exists a constant, \( C_\varepsilon \) such that for all \( u \in W \)

\[
\|u\|_W \leq \frac{\varepsilon}{4R} \|u\|_E + C_\varepsilon \|u\|_X.
\]

Therefore, for all \( u \in S \),

\[
\|u(t) - u(s)\|_W \leq \frac{\varepsilon}{6R} \|u(t) - u(s)\|_E + C_\varepsilon \|u(t) - u(s)\|_X
\leq \frac{\varepsilon}{6R} (\|u(t)\|_E + \|u(s)\|_E) + C_\varepsilon \|u(t) - u(s)\|_X
\leq \frac{\varepsilon}{3} + C_\varepsilon |t - s|^{1/q}.
\]

(67.5.27)

Since \( \varepsilon \) is arbitrary, it follows \( u \in C([a, b]; W) \).

Let \( D = \mathbb{Q} \cap [a, b] \) so \( D \) is a countable dense subset of \([a, b]\). Let \( D = \{t_n\}_{n=1}^\infty \). By compactness of the embedding of \( E \) into \( W \), there exists a subsequence \( u_{n(n, 1)} \) such that as \( n \to \infty \), \( u_{n(n, 1)}(t_1) \) converges to a point in \( W \). Now take a subsequence of this, called \( (n, 2) \) such that as \( n \to \infty \), \( u_{n(n, 2)}(t_2) \) converges to a point in \( W \).
It follows that \( u_{(n,2)}(t) \) also converges to a point of \( W \). Continue this way. Now consider the diagonal sequence, \( u_k \equiv u_{(k,k)} \) This sequence is a subsequence of \( u_{(n,t)} \) whenever \( k > l \) Therefore, \( u_k(t_j) \) converges for all \( t_j \in D \).

**Claim:** Let \( \{u_k\} \) be as just defined, converging at every point of \( D \equiv [a,b] \cap \mathbb{Q} \). Then \( \{u_k\} \) converges at every point of \( [a,b] \).

**Proof of claim:** Let \( \varepsilon > 0 \) be given. Let \( t \in [a,b] \). Pick \( t_m \in D \cap [a,b] \) such that in (67.5.28) \( C \varepsilon R |t - t_m| < \varepsilon / 3 \). Then there exists \( N \) such that if \( l, n > N \), then \( ||u_l(t_m) - u_n(t_m)||_X < \varepsilon / 3 \). It follows that for \( l, n > N \),

\[
||u_l(t) - u_n(t)||_W \leq ||u_l(t) - u_l(t_m)||_W + ||u_l(t_m) - u_n(t_m)||_W \\
+ ||u_n(t_m) - u_n(t)||_W \\
\leq \frac{2\varepsilon}{3} + \frac{2\varepsilon}{3} < 2\varepsilon
\]

Since \( \varepsilon \) was arbitrary, this shows \( \{u_k(t)\}_{k=1}^{\infty} \) is a Cauchy sequence. Since \( W \) is complete, this shows this sequence converges.

Now for \( t \in [a,b] \), it was just shown that if \( \varepsilon > 0 \) there exists \( N_t \) such that if \( n, m > N_t \), then

\[
||u_n(t) - u_m(t)||_W < \frac{\varepsilon}{3}.
\]

Now let \( s \neq t \). Then

\[
||u_n(s) - u_m(s)||_W \leq ||u_n(s) - u_n(t)||_W + ||u_n(t) - u_m(t)||_W + ||u_m(t) - u_m(s)||_W
\]

From (67.5.28)

\[
||u_n(s) - u_m(s)||_W \leq 2\left(\frac{\varepsilon}{3} + C \varepsilon |t - s|^{1/2}\right) + ||u_n(t) - u_m(t)||_W
\]

and so it follows that if \( \delta \) is sufficiently small and \( s \in B(t, \delta) \), then when \( n, m > N_t \)

\[
||u_n(s) - u_m(s)|| < \varepsilon.
\]

Since \([a,b]\) is compact, there are finitely many of these balls, \( \{B(t_i, \delta)\}_{i=1}^{N_t} \), such that for \( s \in B(t_i, \delta) \) and \( n, m > N_i \), the above inequality holds. Let \( N > \max \{N_{t_1}, \ldots, N_{t_p}\} \). Then if \( m,n > N \) and \( s \in [a,b] \) is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

**Claim:** Let \( \varepsilon > 0 \) be given. Then there exists \( N \) such that if \( m,n > N \), then

\[
||u_n - u_m||_{\infty,W} < \varepsilon.
\]

Now let \( u(t) = \lim_{k \to \infty} u_k(t) \).

\[
||u(t) - u(s)||_W \leq ||u(t) - u_n(t)||_W + ||u_n(t) - u_n(s)||_W + ||u_n(s) - u(s)||_W
\]

Let \( N \) be in the above claim and fix \( n > N \). Then

\[
||u(t) - u_n(t)||_W = \lim_{m \to \infty} ||u_m(t) - u_n(t)||_W \leq \varepsilon
\]
The next theorem is a well known result probably due to Lions.
Theorem 67.5.6 Let $E \subseteq W \subseteq X$ where the injection map is continuous from $W$ to $X$ and compact from $E$ to $W$. Let $p \geq 1$, let $q > 1$, and define

$$S \equiv \{ u \in L^p ([a, b]; E) : \text{for some } C, \| u(t) - u(s) \|_X \leq C |t - s|^{1/q}$$

and $\| u \|_{L^p([a,b];E)} \leq R$. Thus $S$ is bounded in $L^p ([a, b]; E)$ and Holder continuous into $X$. Then $S$ is pre-compact in $L^p ([a, b]; W)$. This means that if $\{ u_n \}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\{ u_{n_k} \}$ which converges in $L^p ([a, b]; W)$.

Proof: By Proposition 6.7.2 on Page 146 it suffices to show $S$ has an $\eta$ net in $L^p ([a, b]; W)$ for each $\eta > 0$.

If not, there exists $\eta > 0$ and a sequence $\{ u_n \} \subseteq S$, such that

$$\| u_n - u_m \| \geq \eta \quad (67.5.29)$$

for all $n \neq m$ and the norm refers to $L^p ([a, b]; W)$. Let

$$a = t_0 < t_1 < \cdots < t_k = b, \ t_i - t_{i-1} = (b - a)/k.$$ 

Now define

$$\overline{u}_n(t) = \sum_{i=1}^{k} \overline{u}_{n,i} \chi_{[t_{i-1}, t_i)}(t), \quad \overline{u}_{n,i} = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(s) \, ds.$$ 

The idea is to show that $\overline{u}_n$ approximates $u_n$ well and then to argue that a subsequence of the $\{ \overline{u}_n \}$ is a Cauchy sequence yielding a contradiction to Theorem 6.24.

Therefore,

$$u_n(t) - \overline{u}_n(t) = \sum_{i=1}^{k} u_n(t) \chi_{[t_{i-1}, t_i)}(t) - \sum_{i=1}^{k} \overline{u}_{n,i} \chi_{[t_{i-1}, t_i)}(t)$$

$$= \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(t) \, ds \chi_{[t_{i-1}, t_i)}(t) - \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(s) \, ds \chi_{[t_{i-1}, t_i)}(t)$$

$$= \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \chi_{[t_{i-1}, t_i)}(t).$$

It follows from Jensen’s inequality that

$$\| u_n(t) - \overline{u}_n(t) \|_W^p$$

$$= \sum_{i=1}^{k} \left( \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) \, ds \right)_{\chi_{[t_{i-1}, t_i)}(t)}^p$$

$$\leq \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n(t) - u_n(s) \|_W^p \, ds \chi_{[t_{i-1}, t_i)}(t)$$
and so
\[
\int_a^b \| (u_n(t) - u_n(s)) \|_{W^1}^p \, ds \\
\leq \int_a^b \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \| u_n(t) - u_n(s) \|_{W^1}^p \, ds \chi_{[t_{i-1}, t_i]}(t) \, dt \\
= \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \| u_n(t) - u_n(s) \|_{W^1}^p \, dsdt. \tag{67.5.30}
\]

From Theorem 67.5.2 if \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that
\[
\| u_n(t) - u_n(s) \|_{W^1}^p \leq \varepsilon \| u_n(t) - u_n(s) \|_{E^p}^p + C_\varepsilon \| u_n(t) - u_n(s) \|_X^p \\
\leq 2^{p-1} \varepsilon (\| u_n(t) \|_E^p + \| u_n(s) \|_E^p) + C_\varepsilon |t - s|^{p/q}
\]
This is substituted in to 67.5.30 to obtain
\[
\int_a^b \| (u_n(t) - \bar{u}_n(s)) \|_{W}^p \, ds \leq \\
\sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \left( 2^{p-1} \varepsilon (\| u_n(t) \|_E^p + \| u_n(s) \|_E^p) + C_\varepsilon |t - s|^{p/q} \right) dsdt \\
= \sum_{i=1}^k 2^p \varepsilon \int_{t_{i-1}}^{t_i} \| u_n(t) \|_{W}^p \, dt + \frac{C_\varepsilon}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} |t - s|^{p/q} dsdt \\
\leq 2^p \varepsilon \int_a^b \| u_n(t) \|_E^p \, dt + C_\varepsilon \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})^{p/q}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} dsdt \\
= 2^p \varepsilon \int_a^b \| u_n(t) \|_E^p \, dt + C_\varepsilon \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})^{1+p/q}} \leq 2^p \varepsilon R^p + C_\varepsilon k \left( \frac{b - a}{k} \right)^{1+p/q}. 
\]

Taking \( \varepsilon \) so small that \( 2^p \varepsilon R^p < \eta^p / 8p \) and then choosing \( k \) sufficiently large, it follows
\[
\| u_n - \bar{u}_n \|_{L^p([a,b]; W)} < \frac{\eta}{4}.
\]

Thus \( k \) is fixed and \( \bar{u}_n \) at a step function with \( k \) steps having values in \( E \). Now use compactness of the embedding of \( E \) into \( W \) to obtain a subsequence such that \( \{ \bar{u}_n \} \) is Cauchy in \( L^p(a, b; W) \) and use this to contradict 67.5.24. The details follow.

Suppose \( \bar{u}_n(t) = \sum_{i=1}^k u_{II}^p \chi_{[t_{i-1}, t_i]}(t) \). Thus
\[
\| \bar{u}_n(t) \|_E = \sum_{i=1}^k \| u_{II}^p \|_E \chi_{[t_{i-1}, t_i]}(t).
\]
and so
\[ R \geq \int_a^b \|\pi_n(t)\|^p_E \, dt = \frac{T}{k} \sum_{i=1}^k \|u^n_i\|^p_E. \]

Therefore, the \( \{u^n_i\} \) are all bounded. It follows that after taking subsequences \( k \) times there exists a subsequence \( \{u_{n_k}\} \) such that \( u_{n_k} \) is a Cauchy sequence in \( L^p(a,b;W) \). You simply get a subsequence such that \( u^{n_k}_i \) is a Cauchy sequence in \( W \) for each \( i \). Then denoting this subsequence by \( n_k \),
\[
\|u_n - u_m\|_{L^p(a,b;W)} \leq \|u_n - \pi_n\|_{L^p(a,b;W)} + \|\pi_n - \pi_m\|_{L^p(a,b;W)} + \|\pi_m - u_m\|_{L^p(a,b;W)}
\]
\[
\leq \frac{\eta}{4} + \|\pi_n - \pi_m\|_{L^p(a,b;W)} + \frac{\eta}{4} < \eta
\]
provided \( m, n \) are large enough, contradicting \( \text{Proposition 67.5.29} \). \( \square \)
Chapter 68

Measurability Without Uniqueness

With the Ito formula which holds for a single space, it is time to consider stochastic ordinary differential equations. First is a general theory which allows one to consider measurable solutions to stochastic equations in which there is no uniqueness available. Unfortunately, it does not include obtaining adapted solutions. Instead, it includes measurability of functions with respect to a single \( \sigma \) algebra. Then when path uniqueness is available, one can include the concept of adapted solutions rather easily and this will be done for ordinary differential equations. First is a general result about multifunctions.

68.1 Multifunctions And Their Measurability

Let \( X \) be a separable complete metric space and let \( (\Omega, \mathcal{C}, \mu) \) be a set, a \( \sigma \) algebra of subsets of \( \Omega \), and a measure \( \mu \) such that this is a complete \( \sigma \) finite measure space. Also let \( \Gamma : \Omega \to P_F(X) \), the closed subsets of \( X \).

Definition 68.1.1 We define \( \Gamma^- (S) \equiv \{ \omega \in \Omega : \Gamma(\omega) \cap S \neq \emptyset \} \)

We will consider a theory of measurability of set valued functions. The following theorem is the main result in the subject. In this theorem the third condition is what we will refer to as measurable. The second condition is called strongly measurable. More can be said than what we will prove here.

Theorem 68.1.2 In the following, 1. \( \Rightarrow \) 2. \( \Rightarrow \) 3. \( \Rightarrow \) 4.

1. For all \( B \) a Borel set in \( X \), \( \Gamma^- (B) \in \mathcal{C} \).
2. For all \( F \) closed in \( X \), \( \Gamma^- (F) \in \mathcal{C} \).
3. For all \( U \) open in \( X \), \( \Gamma^- (U) \in \mathcal{C} \).
4. There exists a sequence, \( \{\sigma_n\} \) of measurable functions satisfying \( \sigma_n(\omega) \in \Gamma(\omega) \) such that for all \( \omega \in \Omega \),

\[
\Gamma(\omega) = \{\sigma_n(\omega) : n \in \mathbb{N}\}
\]

These functions are called measurable selections.

Also 4.\(\Rightarrow\) 3. If \( \Gamma(\omega) \) is compact for each \( \omega \), then also 3.\(\Rightarrow\) 2.

**Proof:** It is obvious that 1.) \(\Rightarrow\) 2.). To see that 2.) \(\Rightarrow\) 3.) note that \( \Gamma^-(\bigcup_{i=1}^\infty F_i) = \bigcup_{i=1}^\infty \Gamma^-(F_i) \). Since any open set in \( X \) can be obtained as a countable union of closed sets, this implies 2.) \(\Rightarrow\) 3.).

Now we verify that 3.) \(\Rightarrow\) 4.). Let \( \{x_n\}_{n=1}^\infty \) be a countable dense subset of \( X \). For \( \omega \in \Omega \), let \( \psi_1(\omega) = x_n \) where \( n \) is the smallest integer such that \( \Gamma(\omega) \cap B(x_n, 1) \neq \emptyset \).

Therefore, \( \psi_1(\omega) \) has countably many values, \( x_{n_1}, x_{n_2}, \ldots \) where \( n_1 < n_2 < \cdots \).

Now

\[
\{\omega : \psi_1 = x_n\} = \{\omega : \Gamma(\omega) \cap B(x_n, 1) \neq \emptyset\} \cap [\Omega \setminus \bigcup_{k<n} \{\omega : \Gamma(\omega) \cap B(x_k, 1) \neq \emptyset\}] \in C.
\]

Thus we see that \( \psi_1 \) is measurable and \( \text{dist}(\psi_1(\omega), \Gamma(\omega)) < 1 \). Let

\[
\Omega_n \equiv \{\omega \in \Omega : \psi_1(\omega) = x_n\}.
\]

Then \( \Omega_n \in C \) and \( \Omega_n \cap \Omega_m = \emptyset \) for \( n \neq m \) and \( \bigcup_{n=1}^\infty \Omega_n = \Omega \). Let \( D_n \equiv \{x_k : x_k \in B(x_n, 1)\} \). Now for each \( n \), and \( \omega \in \Omega_n \), let \( \psi_2(\omega) = x_k \) where \( k \) is the smallest index such that \( x_k \in D_n \) and \( B(x_k, \frac{1}{2}) \cap \Gamma(\omega) \neq \emptyset \). Thus \( \text{dist}(\psi_2(\omega), \Gamma(\omega)) < \frac{1}{2} \) and \( d(\psi_2(\omega), \psi_1(\omega)) < 1 \). Continue this way obtaining \( \psi_k \) a measurable function such that

\[
\text{dist}(\psi_k(\omega), \Gamma(\omega)) < \frac{1}{2^{k-1}}, \quad d(\psi_k(\omega), \psi_{k+1}(\omega)) < \frac{1}{2^k}.
\]

Then for each \( \omega, \{\psi_k(\omega)\} \) is a Cauchy sequence converging to a point, \( \sigma(\omega) \in \Gamma(\omega) \).

This has shown that if \( \Gamma \) is measurable there exists a measurable selection, \( \sigma(\omega) \in \Gamma(\omega) \). It remains to show there exists a sequence of these measurable selections, \( \sigma_n \) such that the conclusion of 4.) holds. To do this we define

\[
\Gamma_{ni}(\omega) = \begin{cases} 
\Gamma(\omega) \cap B(x_n, 2^{-i}) & \text{if } \Gamma(\omega) \cap B(x_n, 2^{-i}) \neq \emptyset \\
\Gamma(\omega) & \text{otherwise.}
\end{cases}
\]

First we show that \( \Gamma_{ni} \) is measurable. Let \( U \) be open. Then

\[
\{\omega : \Gamma_{ni}(\omega) \cap U \neq \emptyset\} = \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \cap U \neq \emptyset\} \cup
\]

\[
\left[\{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) = \emptyset\} \cap \{\omega : \Gamma(\omega) \cap U \neq \emptyset\}\right]
\]

\[
= \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \cap U \neq \emptyset\} \cup
\]

\[
\left[(\Omega \setminus \{\omega : \Gamma(\omega) \cap B(x_n, 2^{-i}) \neq \emptyset\}) \cap \{\omega : \Gamma(\omega) \cap U \neq \emptyset\}\right],
\]
68.2. A MEASURABLE SELECTION

a measurable set. By what was just shown there exists \( \sigma_{n_1} \), a measurable function such that \( \sigma_{n_1}(\omega) \in \Gamma_{n_1}(\omega) \subseteq \Gamma(\omega) \) for all \( \omega \in \Omega \). If \( x \in \Gamma(\omega) \), then \( x \in B(x_n, 2^{-i}) \) whenever \( x_n \) is close enough to \( x \). Therefore, \( |\sigma_{n_1}(\omega) - x| < 2^{-i} \). And it follows that condition 4.) holds.

Now we verify that 4.) \( \Rightarrow \) 3.). Suppose there exist measurable selections \( \sigma_n(\omega) \in \Gamma(\omega) \) satisfying condition 4.). Let \( U \) be open. Then

\[
\{ \omega : \Gamma(\omega) \cap U \neq \emptyset \} = \bigcup_{n=1}^{\infty} \sigma_n^{-1}(U) \in \mathcal{C}.
\]

Now suppose \( \Gamma(\omega) \) is compact for every \( \omega \) and that \( \Gamma^{-}(U) \in \mathcal{C} \) for every \( U \) open. Then let \( F \) be a closed set and let \( \{U_n\} \) be a decreasing sequence of open sets whose intersection equals \( F \) such that also, for all \( n \), \( U_n \supseteq \overline{U}_{n+1} \). Then

\[
\Gamma(\omega) \cap F = \cap_n \Gamma(\omega) \cap U_n = \cap_n \Gamma(\omega) \cap U_n.
\]

Now because of compactness, the set on the left is nonempty if and only if each set on the right is also nonempty. Thus \( \Gamma^{-}(F) = \cap_n \Gamma^{-1}(U_n) \in \mathcal{C} \).

Actually these are all equivalent in the case of complete measure spaces but we do not need this and it is much harder to show.

68.2 A Measurable Selection

This section deals with the problem of getting product measurable functions in a context of no uniqueness. The following is the main result. It is stated in great generality because it has fairly wide application although it will be used first in finite dimensions.

**Theorem 68.2.1** Let \( V \) be a reflexive separable Banach space and \( V' \) its dual and \( \frac{1}{p} + \frac{1}{p'} = 1 \) where \( p > 1 \) as usual. For \( n \in \mathbb{N} \) let the functions \( t \to u_n(t, \omega) \) be in \( L^{p'}([0, T]; V') \) and \( (t, \omega) \to u_n(t, \omega) \) be \( B([0, T]) \times \mathcal{P} \) measurable into \( V' \). Suppose there is a set of measure zero \( N \) such that if \( \omega \notin N \), then for all \( n \),

\[
\sup_{t \in [0, T]} \|u_n(t, \omega)\|_{V'} \leq C(\omega).
\]

Also suppose for each \( \omega \notin N \), each subsequence of \( \{u_n\} \) has a further subsequence which converges weakly in \( L^{p'}([0, T]; V') \) to \( u(\cdot, \omega) \in L^{p'}([0, T]; V') \) such that \( t \to u(t, \omega) \) is weakly continuous into \( V' \). Then there exists \( u \) product measurable, with \( t \to u(t, \omega) \) being weakly continuous into \( V' \). Moreover, there exists, for each \( \omega \notin N \), a subsequence \( u_n(\omega) \) such that \( u_n(\omega)(\cdot, \omega) \to u(\cdot, \omega) \) weakly in \( L^{p'}([0, T]; V') \).

Note that the exceptional set is given. It could be the empty set with no change in the conclusion of the theorem.

Let \( X = \prod_{k=1}^{\infty} C([0, T]) \) with the product topology. One can consider this as a metric space using the metric

\[
d(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|f_k - g_k\|}{1 + \|f_k - g_k\|},
\]
where the norm is the maximum norm in \( C([0, T]) \). With this metric, \( X \) is complete and separable.

**Lemma 68.2.2** Let \( \{ f_n \} \) be a sequence in \( X \) and suppose that the \( k \)-th components \( f_{nk} \) are bounded in \( C^{0,1}([0, T]) \). (This refers to the Hölder space with \( \gamma = 1 \).) Then there exists a subsequence converging to some \( f \in X \). Thus if \( \{ f_n \} \) has each component bounded in \( C^{0,1}([0, T]) \), then \( \{ f_n \} \) is pre-compact in \( X \).

**Proof:** By the Ascoli–Arzelà theorem, there exists a subsequence \( n_1 \) such that the first component \( f_{n_1,1} \) converges in \( C([0, T]) \). Then taking a subsequence, one can obtain \( n_2 \) a subsequence of \( n_1 \) such that both the first and second components of \( f_{n_2} \) converge. Continuing this way one obtains a sequence of subsequences, each a subsequence of the previous one such that \( f_{n_j} \) has the first \( j \) components converging to functions in \( C([0, T]) \). Therefore, the diagonal subsequence has the property that it has every component converging to a function in \( C([0, T]) \). The resulting function in \( \prod_k C([0, T]) \) is \( f \). \( \blacksquare \)

Now for \( m \in \mathbb{N} \) and \( \phi \in V' \), define \( l_m(t) = \max(0, t - (1/m)) \) and \( \psi_{m,\phi} : L^p([0, T]; V') \rightarrow C([0, T]) \) as follows

\[
\psi_{m,\phi}(u)(t) = \int_0^T \langle m\phi, \chi_{[l_m(t),t]}(s), u(s) \rangle_{V,V}, \, ds = m \int_{l_m(t)}^t \langle \phi(u(s))_{V,V}, ds.
\]

Let \( D = \{ \phi_r \}_{r=1}^\infty \) denote a countable dense subset of \( V \). Then the pairs \( (\phi, m) \) for \( \phi \in D \) and \( m \in \mathbb{N} \) yield a countable set. Let \( (m_k, \phi_{r_k}) \) denote an enumeration of these pairs \( (m, \phi) \in \mathbb{N} \times D \). To save notation, we denote

\[
f_k(u)(t) \equiv \psi_{m_k,\phi_{r_k}}(u)(t) = m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u(s) \rangle_{V,V}, \, ds
\]

For fixed \( \omega \notin N \) and \( k \), the functions \( \{ t \rightarrow f_k(u_j(\cdot, \omega))(t) \} \) are uniformly bounded and equicontinuous because they are in \( C^{0,1}([0, T]) \). Indeed,

\[
|f_k(u_j(\cdot, \omega))(t)| = \left| m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u_j(s, \omega) \rangle_{V,V}, \, ds \right| \leq C(\omega) \| \phi_{r_k} \|_{V'},
\]

and for \( t \leq t' \)

\[
|f_k(u_j(\cdot, \omega))(t) - f_k(u_j(\cdot, \omega))(t')| \leq \left| m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u_j(s, \omega) \rangle_{V,V}, ds - m_k \int_{l_{m_k}(t')}^t \langle \phi_{r_k}, u_j(s, \omega) \rangle_{V,V}, ds \right| \\
\leq 2m_k |t' - t| \| \phi_{r_k} \|_{V'}, C(\omega).
\]

By Lemma 68.2.2, the set of functions \( \{ f(u_j(\cdot, \omega)) \}_{j \geq n} \) is pre-compact in \( X = \prod_k C([0, T]) \). Then define a set valued map \( \Gamma^n : \Omega \rightarrow X \) as follows.

\[
\Gamma^n(\omega) = \bigcup_{j \geq n} \{ f(u_j(\cdot, \omega)) \},
\]
where the closure is taken in $X$. Then $\Gamma^n(\omega)$ is the closure of a pre-compact set in $\prod_k C([0,T])$ and so $\Gamma^n(\omega)$ is compact in $\prod_k C([0,T])$. From the definition, a function $f$ is in $\Gamma^n(\omega)$ if and only if $d(f, f(w_l)) \to 0$ as $l \to \infty$, where each $w_l$ is one of the $u_j(\cdot, \omega)$ for $j \geq n$. From the topology on $X$ this happens if and only if for every $k$,

$$f_k(t) = \lim_{l \to \infty} m_k \int_t^l \langle \phi_{r_k}, w_l(s,\omega) \rangle_{V,V}, ds,$$

where the limit is the uniform limit in $t$.

Note that in the case of a filtration, instead of a single $\sigma$-algebra $\mathcal{F}$ where each $u_j$ is progressively measurable, if the sequence $w_l$ does not have the index $l$ dependent on $\omega$, then if such a limit holds for each $\omega$, it follows that $(t, \omega) \to f_k(t, \omega)$ will inherit progressive measurability from the $w_l$. This situation will be typical when dealing with stochastic equations with path uniqueness known. Thus this is a reasonable way to attempt to consider measurability and the more difficult question of whether a process is adapted.

**Lemma 68.2.3** $\omega \to \Gamma^n(\omega)$ is a $\mathcal{F}$ measurable set valued map with values in $X$. If $\sigma$ is a measurable selection, $(\sigma(\omega) \in \Gamma^n(\omega))$ so $\sigma(\cdot, \omega)$ a continuous function. To have this measurable would mean that $\sigma^{-1}_k$ (open) $\in \mathcal{F}$ where the open set is in $C([0,T])$. Then for each $t$, $\omega \to \sigma(t, \omega)$ is $\mathcal{F}$ measurable and $(t, \omega) \to \sigma(t, \omega)$ is $B([0,T]) \times \mathcal{F} \equiv \mathcal{P}$ measurable.

**Proof:** Let $O$ be a basic open set in $X$. Thus $O = \prod_{k=1}^\infty O_k$ where $O_k$ is a proper open set of $C([0,T])$ only for $k \in \{k_1, \cdots, k_r\}$. We need to consider whether

$$\Gamma^n(\omega) \equiv \{\omega : \Gamma^n(\omega) \cap O \neq \emptyset\} \in \mathcal{F}.$$

Now $\Gamma^n(\omega)$ equals

$$\bigcap_{i=1}^n \{\omega : \Gamma^n(\omega)_{k_i} \cap O_{k_i} \neq \emptyset\}$$

Thus we consider whether

$$\{\omega : \Gamma^n(\omega)_{k_i} \cap O_{k_i} \neq \emptyset\} \in \mathcal{F} \quad (68.2.1)$$

From the definition of $\Gamma^n(\omega)$, this is equivalent to the condition that for some $j \geq n$,

$$f_{k_i}(u_j(\cdot, \omega)) = (f(u_j(\cdot, \omega)))_{k_i} \in O_{k_i}$$

and so the above set in $\mathcal{F}$ is of the form

$$\bigcup_{j=n}^{\infty} \{\omega : (f(u_j(\cdot, \omega)))_{k_i} \in O_{k_i}\}$$

Now $\omega \to (f(u_j(\cdot, \omega)))_{k_i}$ is $\mathcal{F}$ measurable into $C([0,T])$ and so the above set is in $\mathcal{F}$. To see this, let $g \in C([0,T])$ and consider the inverse image of the ball $B(g,r)$,

$$\{\omega : \|(f(u_j(\cdot,\omega)))_{k_i} - g\|_{C([0,T])} < r\}.$$
By continuity considerations,
\[ \left\| \left( f(u_j(\cdot,\omega)) \right)_{k_i} - g \right\|_{C([0,T])} = \sup_{t \in Q \cap [0,T]} \left( f(u_j(t,\omega)) \right)_{k_i} - g(t) \]
which is the sup of countably many $\mathcal{F}$ measurable functions. Thus it is $\mathcal{F}$ measurable. Since every open set is the countable union of such balls, it follows that the claim about $\mathcal{F}$ measurability is valid. Thus $\Gamma^n(O)$ is $\mathcal{F}$ measurable whenever $O$ is a basic open set.

Now $X$ is a separable metric space and so every open set is a countable union of these basic sets. Let $U$ be an open set in $X$ and let $U = \bigcup_{l=1}^{\infty} O_l$ where $O_l$ is a basic open set as above. Then
\[ \Gamma^n(U) = \bigcup_{l=1}^{\infty} \Gamma^n(O_l) \in \mathcal{F}. \]
That there exists a measurable selection follows from the standard theory of measurable multi-functions [9, 63]. This is proved in Theorem 68.1.2 above. For $\sigma$ one of these measurable selections, the evaluation at $t$ is $\mathcal{F}$ measurable. Thus $\omega \rightarrow \sigma(t,\omega)$ is $\mathcal{F}$ measurable with values in $\mathbb{R}^\infty$. Also $t \rightarrow \sigma(t,\omega)$ is continuous, and so it follows that in fact $\sigma$ is product measurable as claimed.

**Definition 68.2.4** Let $\Gamma(\omega) \equiv \cap_{n=1}^{\infty} \Gamma^n(\omega)$.

**Lemma 68.2.5** $\Gamma$ is a nonempty $\mathcal{F}$ measurable set valued function having values in the compact sub-sets of $X$. There exists a measurable selection $\gamma$. For $\gamma$ a $\mathcal{F}$ measurable selection, $(t,\omega) \rightarrow \gamma(t,\omega)$ is $\mathcal{P}$ measurable. Also, for each $\omega$, there exists a subsequence, $u_n(\omega)(\cdot,\omega)$ such that for each $k$,
\[ \gamma_k(t,\omega) = \lim_{n(\omega) \to \infty} f(u_n(\omega)(t,\omega)) = \lim_{n(\omega) \to \infty} m_k \int_{	au_k(t)}^{t} \langle \phi_{\tau_k}, u_n(\omega)(s,\omega) \rangle_{V,V'} \, ds \]

**Proof:** Consider $\Gamma(\omega) = \cap_{n=1}^{\infty} \Gamma^n(\omega)$. Then $\omega \rightarrow \Gamma(\omega)$ is a compact set valued map in $X$. It is nonempty because each $\Gamma^n(\omega)$ is nonempty and compact, and these sets are nested. Is it $\mathcal{F}$ measurable? Each $\Gamma^n$ is compact valued and $\mathcal{F}$ measurable. Hence if $F$ is closed,
\[ \Gamma(\omega) \cap F = \cap_{n=1}^{\infty} \Gamma^n(\omega) \cap F \]
and the left is non empty if and only if each $\Gamma^n(\omega) \cap F \neq \emptyset$. Hence for $F$ closed,
\[ \{ \omega : \Gamma(\omega) \cap F \neq \emptyset \} = \cap_n \{ \omega : \Gamma^n(\omega) \cap F \neq \emptyset \} \]
and so
\[ \Gamma^-(F) = \cap_n \Gamma^{n-}(F) \in \mathcal{F} \]
The last claim follows from the theory of multi-functions Theorem 68.1.2, [9, 63]. Since $\Gamma^n(\omega)$ is compact, the measurability of $\Gamma^n$, that $\Gamma^{n-}(U) \in \mathcal{F}$ for $U$ open implies the strong measurability of $\Gamma^n$, that $\Gamma^{n-}(F) \in \mathcal{F}$. Thus $\omega \rightarrow \Gamma(\omega)$ is non empty compact valued in $X$ and $\mathcal{F}$ measurable.
68.2. A MEASURABLE SELECTION

From standard theory of measurable multi-functions, Theorem 64.2.1, there exists a $\mathcal{F}$ measurable selection $\omega \to \gamma (\omega)$ with $\gamma (\omega) \in \Gamma (\omega)$ for each $\omega$. Now it follows that $t \to \gamma_k (t, \omega)$ is continuous. This is what it means for $\gamma (\omega) \in X$. What of the product measurability of $\gamma_k$? We know that $\omega \to \gamma_k (\omega)$ is $\mathcal{F}$ measurable into $C ([0,T])$ and so since pointwise evaluation is continuous, $\omega \to \gamma_k (t, \omega)$ is $\mathcal{F}$ measurable. Then since $t \to \gamma_k (t, \omega)$ is continuous, it follows that $\gamma_k$ is a $\mathcal{P}$ measurable real valued function and that $\gamma$ is a $\mathcal{P}$ measurable selection.

Since $\gamma (\omega) \in \Gamma (\omega)$, it follows that for each $n, \gamma (\omega) \in \Gamma^n (\omega)$. Therefore, there exists $j_n \geq n$ such that for each $\omega$,

$$d (f (u_{jn} (\cdot, \omega)), \gamma (\omega)) < 2^{-n}$$

It follows that, taking a suitable subsequence, denoted as $\{u_n (\omega) (\cdot, \omega)\}$,

$$\gamma (\omega) = \lim_{n(\omega) \to \infty} f (u_n (\omega) (\cdot, \omega))$$

for each $\omega$. In particular, for each $k$

$$\gamma_k (t, \omega) = \lim_{n(\omega) \to \infty} f (u_n (\omega) (t, \omega)) = \lim_{n(\omega) \to \infty} m_k \int_{l_m (t)}^{t} \langle \phi_{r_k}, u_n (\omega) (s, \omega) \rangle_{V, V}, ds$$

(68.2.2)

for each $t$. ■

Note that it is not clear that $(t, \omega) \to f (u_n (\omega) (t, \omega))$ is $\mathcal{P}$ measurable although $(t, \omega) \to \gamma (t, \omega)$ is $\mathcal{P}$ measurable.

**Proof of the theorem:** By assumption, there exists a further subsequence still denoted by $n (\omega)$ such that, in addition to 64.2.2 above, the weak limit

$$\lim_{n(\omega) \to \infty} u_n (\omega) (\cdot, \omega) = u (\cdot, \omega)$$

exists in $L^{p'} ([0,T]; V')$ such that $t \to u (t, \omega)$ is weakly continuous into $V'$. Then the above equation 64.2.2 continues to hold for this further subsequence and in addition to this,

$$m_k \int_{l_m (t)}^{t} \langle \phi_{r_k}, u (s, \omega) \rangle_{V, V}, ds = \lim_{n(\omega) \to \infty} m_k \int_{l_m (t)}^{t} \langle \phi_{r_k}, u_n (\omega) (s, \omega) \rangle_{V, V}, ds = \gamma_k (t, \omega)$$

Letting $\phi \in \mathcal{D}$ given, there exists a sub-sequence denoted by $k$ such that $m_k \to \infty$ and $\phi_{r_k} = \phi$ for all $k$. Then passing to a limit and using the assumed continuity of $s \to u (s, \omega)$, the left side of this equation converges to $\langle \phi, u (t, \omega) \rangle_{V, V}$, and so the right side, $\gamma_k (t, \omega)$ must also converge, this for each $\omega$. Since the right side is a product measurable function of $(t, \omega)$, it follows that the pointwise limit is also product measurable. Hence $(t, \omega) \to \langle \phi, u (t, \omega) \rangle_{V, V}$ is product measurable, this for each $\phi \in \mathcal{D}$. Since $\mathcal{D}$ is a dense set, it follows that $(t, \omega) \to \langle \phi, u (t, \omega) \rangle_{V, V}$ is $\mathcal{P}$ measurable for all $\phi \in V$ and so by the Pettis theorem, 64.17, $(t, \omega) \to u (t, \omega)$ is $\mathcal{P}$ measurable into $V'$. ■

One can say more about the measurability of the approximating sequence. In fact, we can obtain one for which $\omega \to u_n (\omega) (t, \omega)$ is also $\mathcal{F}$ measurable.
Lemma 68.2.6 Suppose, \( u_{n(\omega)} \to u \) weakly in \( L^p([0,T];V') \) where \( u \) is product measurable measurable and \( \{u_{n(\omega)}\} \) is a subsequence of \( \{u_n\} \) where

\[
\sup_{t \in [0,T]} \|u_n(t,\omega)\|_{V'} < C(\omega), \text{ for } \omega \notin N \text{ a set of measure zero,}
\]

Then for each \( \omega \notin N \), there exists a subsequence of \( \{u_n\} \) denoted as \( \{u_{k(\omega)}\} \) such that \( u_{k(\omega)} \to u \) weakly in \( L^p([0,T];V') \), \( \omega \to k(\omega) \) is \( F \) measurable, and \( \omega \to u_{k(\omega)}(t,\omega) \) is also \( F \) measurable, the last assertions holding for all \( \omega \notin N \).

**Proof:** For \( f,g \in L^p([0,T];V') \equiv V', \ L^p([0,T];V) \equiv V \), let \( \{\phi_k\} \) be a countable dense subset of \( L^p([0,T];V) \) with the weak topology can be considered a complete metric space using the following metric.

\[
d(f,g) \equiv \sum_{j=1}^{\infty} 2^{-j} \frac{|\langle \phi_k, f-g \rangle_{V',V}|}{1 + |\langle \phi_k, f-g \rangle_{V',V}|}
\]

Now let \( k(\omega) \) be the first index from the indices of \( \{u_n\} \) at least as large as \( k \) such that

\[
d(u_{k(\omega)},u) \leq 2^{-k}
\]

Such an index exists because there exists a convergent sequence \( u_{n(\omega)} \) which does converge weakly to \( u \). This is just picking another one which happens to also retain measurability. In fact,

\[
\{\omega : k(\omega) = l\} = \{\omega : d(\omega, u) \leq 2^{-k}\} \cap \cap_{j=1}^{k-1} \{\omega : d(\omega, u) > 2^{-k}\}
\]

Since \( u \) is product measurable and each \( u_t \) is also product measurable, these are all measurable sets with respect to \( F \) and so \( \omega \to k(\omega) \) is \( F \) measurable. Now we have \( u_{k(\omega)} \to u \) weakly in \( L^p([0,T];V') \) for each \( \omega \) with each function being \( F \) measurable because

\[
u_{k(\omega)}(t,\omega) = \sum_{j=1}^{\infty} \mathcal{X}_{[k(\omega)=j]} u_j(t,\omega)
\]

and every term in the sum is \( F \) measurable. \( \blacksquare \)

The following obvious corollary shows the significance of this lemma.

Corollary 68.2.7 Let \( V \) be a reflexive separable Banach space and \( V' \) its dual and \( \frac{1}{p} + \frac{1}{p'} = 1 \) where \( p > 1 \) as usual. Let the functions \( t \to u_{n(\omega)}(t,\omega) \) be in \( L^p([0,T];V') \) and \( (t,\omega) \to u_{n(\omega)}(t,\omega) \) be \( B([0,T]) \times F \equiv \mathcal{P} \) measurable into \( V' \). Here \( \{u_{n(\omega)}\}_{n=1}^{\infty} \) is a sequence, one for each \( \omega \). Suppose there is a set of measure zero \( N \) such that if \( \omega \notin N \), then for all \( u \),

\[
\sup_{t \in [0,T]} \|u_{n(\omega)}(t,\omega)\|_{V'} \leq C(\omega).
\]

Also suppose for each \( \omega \notin N \), each subsequence of \( \{u_{n(\omega)}\} \) has a further subsequence which converges weakly in \( L^p([0,T];V') \) to \( u(\cdot,\omega) \in L^p([0,T];V') \) such that \( t \to
68.3. Measurability In Finite Dimensional Problems

Let \( u(t, \omega) \) be weakly continuous into \( V' \). Then there exists a product measurable, with \( t \to u(t, \omega) \) being weakly continuous into \( V' \). Moreover, there exists, for each \( \omega \notin N \), a subsequence \( u_{n(\omega)} \) such that \( u_{n(\omega)}(\cdot, \omega) \to u(\cdot, \omega) \) weakly in \( L^p([0,T]; V') \).

**Proof:** It suffices to consider the functions \( v_n(t, \omega) \equiv u_{n(\omega)}(t, \omega) \) and use the result of Theorem 68.2.1.

Of course when you have all functions having values in \( H \) a separable Hilbert space, there is no change in the argument to obtain the following theorem.

**Theorem 68.2.8** Let \( H \) be a real separable Hilbert space. For \( n \in \mathbb{N} \) let the functions \( t \to u_n(t, \omega) \) be in \( L^2([0,T]; H) \) and \( (t, \omega) \to u_n(t, \omega) \) be \( B([0,T]) \times \mathcal{F} \equiv \mathcal{P} \) measurable into \( H \). Suppose there is a set of measure zero \( N \) such that if \( \omega \notin N \), then for all \( n \),

\[
\sup_{t \in [0,T]} |u_n(t, \omega)|_H \leq C(\omega).
\]

Also suppose for each \( \omega \notin N \), each subsequence of \( \{u_n\} \) has a further subsequence which converges weakly in \( L^2([0,T]; H) \) to \( u(\cdot, \omega) \in L^2([0,T]; H) \) such that \( t \to u(t, \omega) \) is weakly continuous into \( H \). Then there exists a product measurable, with \( t \to u(t, \omega) \) being weakly continuous into \( H \). Moreover, there exists, for each \( \omega \notin N \), a subsequence \( u_{n(\omega)} \) such that \( u_{n(\omega)}(\cdot, \omega) \to u(\cdot, \omega) \) weakly in \( L^2([0,T]; H) \).

**68.3 Measurability In Finite Dimensional Problems**

What follows is like the Peano existence theorem from ordinary differential equations except that it provides a solution which retains product measurability. It is a nice example of the above theory. It will be used in the next section in the Galerkin method.

**Lemma 68.3.1** Suppose \( N(t, u, v, w, \omega) \in \mathbb{R}^d \) for \( u, v, w \in \mathbb{R}^d, t \in [0,T] \) and \( (t, u, v, w, \omega) \to N(t, u, v, w, \omega) \) is progressively measurable relative to the filtration consisting of the single \( \sigma \) algebra \( \mathcal{F} \). Also suppose \( (t, u, v, w) \to N(t, u, v, w, \omega) \) is continuous and that \( N(t, u, v, w, \omega) \) is uniformly bounded in \( (t, u, v, w) \) by \( M(\omega) \).

Let \( f \) be \( \mathcal{P} \) measurable and \( f(\cdot, \omega) \in L^2([0,T]; \mathbb{R}^d) \). Then for \( h > 0 \), there exists a \( \mathcal{P} \) measurable solution \( u \) to the integral equation

\[
u(t, \omega) - u_0(\omega) + \int_0^t N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) \, ds = \int_0^t f(s, \omega) \, ds.
\]

Here \( u_0 \) has values in \( \mathbb{R}^d \) and is \( \mathcal{F} \) measurable, \( u(s-h, \omega) \equiv u_0(\omega) \) if \( s-h < 0 \) and for \( w_0 \) a given \( \mathcal{F} \) measurable function,

\[
w(t, \omega) \equiv w_0(\omega) + \int_0^t u(s, \omega) \, ds.
\]
CHAPTER 68. MEASURABILITY WITHOUT UNIQUENESS

**Proof:** Let \( u_n \) be the solution to the following equation:

\[
\begin{align*}
    u_n(t, \omega) - u_0(\omega) &= \int_0^t N\left(s, \tau_{1/n}u_n(s, \omega), u_n(s - h, \omega), \tau_{1/n}w_n(s, \omega), \omega\right) ds \\
    &= \int_0^t f(s, \omega) ds.
\end{align*}
\]

where here \( \tau_{1/n} \) is defined as follows. For \( \delta > 0, \)

\[
\tau_\delta u(s) = \begin{cases} 
    u(s - \delta) & \text{if } s > \delta \\
    0 & \text{if } s - \delta \leq 0
\end{cases}
\]

It follows that \( (t, \omega) \rightarrow u_n(t, \omega) \) is \( \mathcal{P} \) measurable. From the assumptions on \( N, \) it follows that for fixed \( \omega, \{u_n(\cdot, \omega)\} \) is uniformly bounded:

\[
\sup_{t \in [0,T]} |u_n(t, \omega)| \leq |u_0(\omega)| + \int_0^T M(\omega) ds + \int_0^T |f(s, \omega)| ds =: C(\omega),
\]

and is also equicontinuous because for \( s < t, \)

\[
|u_n(t, \omega) - u_n(s, \omega)| \\
\leq \int_s^t |N(r, \tau_{1/n}u_n(r, \omega), u_n(r - h, \omega), \tau_{1/n}w_n(r, \omega), \omega)| dr
\]

\[+ \int_s^t |f(r, \omega)| dr \leq C(\omega, f) |t - s|^{1/2}.\]

Therefore, by the Ascoli–Arzelà theorem, for each \( \omega, \) there exists a subsequence \( \tilde{n}(\omega) \) depending on \( \omega \) and a function \( \tilde{u}(t, \omega) \) such that

\[
    u_{\tilde{n}(\omega)}(t, \omega) \rightarrow \tilde{u}(t, \omega) \text{ uniformly in } C\left([0,T]; \mathbb{R}^d\right).
\]

This verifies the assumptions of Theorem 68.2.28.

It follows that there exists \( \tilde{u} \) product measurable and a subsequence \( \{u_{n(\omega)}\} \) for each \( \omega \) such that

\[
    \lim_{n(\omega) \rightarrow \infty} u_{n(\omega)}(\cdot, \omega) = \tilde{u}(\cdot, \omega) \text{ weakly in } L^2\left([0,T]; \mathbb{R}^d\right)
\]

and that \( t \rightarrow \tilde{u}(t, \omega) \) is continuous. (Note that weak continuity is the same as continuity in \( \mathbb{R}^d. \) The same argument given above applied to the \( u_{n(\omega)} \) for a fixed \( \omega \) yields a further subsequence, denoted as \( \{u_{\tilde{n}(\omega)}(\cdot, \omega)\} \) which converges uniformly to a function \( u(\cdot, \omega) \) on \([0,T].\) So \( \tilde{u}(t, \omega) = u(t, \omega) \) in \( L^2\left([0,T]; \mathbb{R}^d\right).\)

Since both of these functions are continuous in \( t, \) they must be equal for all \( t. \) Hence, \( (t, \omega) \rightarrow u(t, \omega) \) is product measurable. Passing to the limit in the equation solved by \( \{u_{n(\omega)}(\cdot, \omega)\} \) using the dominated convergence theorem, we obtain

\[
    u(t, \omega) - u_0(\omega) + \int_0^t N\left(s, u(s, \omega), u(s - h, \omega), w(s, \omega), \omega\right) ds = \int_0^t f(s, \omega) ds.
\]
Thus $t \to u(t, \omega)$ is a product measurable solution to the integral equation. ■

This lemma gives the existence of the approximate solutions in the following theorem in which the assumption that the integrand is bounded is replaced with an estimate. The following elementary consideration will be used whenever convenient. Note that it holds for all $\omega$.

**Remark 68.3.2** When $w(t) \equiv w_0(\omega) + \int_0^t u(s, \omega) ds$,

$$v(t) = \begin{cases} u(t - h) & \text{if } t \geq h \\ u_0 & \text{if } t < h \end{cases}$$

and when the estimate

$$(N(t, u, v, w, \omega), u) \geq -C(t, \omega) - \mu \left( |u|^2 + |v|^2 + |w|^2 \right)$$

holds, it follows that

$$\int_0^t (N(t, u, v, w, \omega), u) ds \geq -C \left( C(\omega) + \int_0^t |u|^2 ds \right)$$

for some constant $C$ depending on the initial data but not on $u$.

To see this,

$$\int_0^t |u(s - h)|^2 ds = \int_0^h |u_0|^2 ds + \int_h^t |u(s - h)|^2 ds = |u_0|^2 h + \int_h^{t-h} |u(s)|^2 ds \leq |u_0|^2 h + \int_0^t |u(s)|^2 ds$$

if $t \geq h$ and if $s < h$, this is dominated by

$$|u_0|^2 t \leq |u_0|^2 h \leq |u_0|^2 h + \int_0^t |u(s)|^2 ds$$

As to the terms from $w$,

$$\int_0^t |w(s)|^2 ds \leq \int_0^t \left( |w_0| + \int_0^s |u(r)| dr \right)^2 ds \leq \left( |w_0| + \int_0^t |u(r)| dr \right)^2 ds \leq T |w_0|^2 + T \left( \int_0^t |u(r)| dr \right)^2 ds \leq 2T |w_0|^2 + 2 \int_0^t \left( \int_0^s |u(r)| dr \right)^2 ds + \int_0^t \left( \int_0^s |u(r)| dr \right)^2 ds \leq 2T |w_0|^2 + 2T \int_0^t \left( \int_0^s |u(r)| dr \right)^2 ds \leq 2T |w_0|^2 + 2T \int_0^t \left( \int_0^s |u(r)|^2 dr \right) ds \leq 2T |w_0|^2 + 2T \int_0^t \int_0^s |u(r)|^2 dr ds \leq 2T |w_0|^2 + 2T^2 \int_0^t |u(r)|^2 dr ds \leq 2T |w_0|^2 + 2T^2 \int_0^t |u(r)|^2 dr.$$
From this, the claimed result follows.

**Theorem 68.3.3** Suppose \( N(t, u, v, w, \omega) \in \mathbb{R}^d \) for \( u, v, w \in \mathbb{R}^d, t \in [0, T] \) and \( (t, u, v, w, \omega) \rightarrow N(t, u, v, w, \omega) \) is progressively measurable with respect to a constant filtration \( \mathcal{F}_t = \mathcal{F} \). Also suppose \( (t, u, v, w) \rightarrow N(t, u, v, w, \omega) \) is continuous and satisfies the following conditions for \( C(\cdot, \omega) \geq 0 \) in \( L^1([0, T]) \) and some \( \mu > 0 \):

\[
(N(t, u, v, w, \omega), u) \geq -C(t, \omega) - \mu \left( |u|^2 + |v|^2 + |w|^2 \right).
\]

Also let \( f \) be product measurable and \( f(\cdot, \omega) \in L^2([0, T]; \mathbb{R}^d) \). Then for \( h > 0 \), there exists a product measurable solution \( u \) to the integral equation

\[
u(t, \omega) = u_0(\omega) + \int_0^t N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) ds = \int_0^t f(s, \omega) ds,
\]

where \( u_0 \) has values in \( \mathbb{R}^d \) and is \( \mathcal{F} \) measurable. Here \( u(s-h, \omega) \equiv u_0(\omega) \) for all \( s-h \leq 0 \) and for \( w_0 \) a given \( \mathcal{F} \) measurable function,

\[
w(t, \omega) \equiv w_0(\omega) + \int_0^t u(s, \omega) ds
\]

**Proof:** Let \( P_m \) denote the projection onto the closed ball \( \overline{B(0, 9^m)} \). Then from the above lemma, there exists a product measurable solution \( u_m \) to the integral equation

\[
u_m(t, \omega) = u_0(\omega) + \int_0^t N(s, P_m u_m(s, \omega), P_m u_m(s-h, \omega), P_m w_m(s, \omega), \omega) ds
\]

Define a stopping time

\[
\tau_m(\omega) \equiv \inf \left\{ t \in [0, T] : |u_m(t, \omega)|^2 + |w_m(t, \omega)|^2 > 2^m \right\},
\]

where \( \inf \emptyset \equiv T \). Localizing with the stopping time,

\[
u_{m}^\tau(t, \omega) = u_0(\omega) + \int_0^t \mathcal{X}_{[0, \tau_m]} N(s, u_{m}^\tau(s, \omega), u_{m}^\tau(s-h, \omega), w_{m}^\tau(s, \omega), \omega) ds
\]

Note how the stopping time allowed the elimination of the projection map in the equation. Then we get

\[
\frac{1}{2} |u_{m}^\tau(t, \omega)|^2 - \frac{1}{2} |u_0(\omega)|^2 + \int_0^t \mathcal{X}_{[0, \tau_m]} [N(s, u_{m}^\tau(s, \omega), u_{m}^\tau(s-h, \omega), w_{m}^\tau(s, \omega), \omega, u_{m}^\tau(s, \omega)) ds
\]

\[
= \int_0^t \mathcal{X}_{[0, \tau_m]} [f(s, \omega), u_{m}^\tau(s, \omega)) ds.
\]
68.3. MEASURABILITY IN FINITE DIMENSIONAL PROBLEMS

From the estimate,
\[
\frac{1}{2} |u_m^\tau_m(t, \omega)|^2 - \frac{1}{2} |u_0(\omega)|^2 \leq \int_0^t \left( \mu \left( |u_m^\tau_m(s, \omega)|^2 + |u_m^\tau_m(s - h, \omega)|^2 + |w_m^\tau_m(s, \omega)|^2 \right) \\
+ C(s, \omega) + \frac{1}{2} |f(s, \omega)|^2 \right) ds + \frac{1}{2} \int_0^t |u_m^\tau_m(s, \omega)|^2 ds.
\]

Note that
\[
|u_0|^2 h + \int_0^t |u_n^\tau_n(s)|^2 ds \geq \int_0^t |u_n^\tau_n(s - h, \omega)|^2 ds
\]
and
\[
\begin{align*}
\int_0^t |w_n^\tau_n(s, \omega)|^2 ds &= \int_0^t |w_0 + \int_0^s X_{[0, \tau_n]} u_n(r) dr |^2 ds \\
&= \int_0^t |w_0 + \int_0^s X_{[0, \tau_n]} u_n^\tau_n(r) dr |^2 ds \\
&\leq C(w_0(\omega)) + CT \int_0^t |u_n^\tau_n|^2 ds.
\end{align*}
\]

By Gronwall’s inequality,
\[
|u_m^\tau_m(t, \omega)|^2 \leq C \left( u_0(\omega), w_0(\omega), \mu, ||C(\cdot, \omega)||_{L^1([0, T]; \mathbb{R}^d)}, T; ||f(\cdot, \omega)||_{L^2([0, T]; \mathbb{R}^d)} \right) \\
\equiv C(\omega).
\]

Thus, for a.e. \( \omega, \tau_m = T \) for all \( m \) large enough, say for \( m \geq M(\omega) \) where
\[
C(\omega) \leq 2^{M(\omega)}.
\]

Then define the functions
\[
y_n(t, \omega) \equiv u_n^\tau_n(t, \omega).\]

These are product measurable and
\[
y_n(t, \omega) - u_0(\omega) + \\
\int_0^t X_{[0, \tau_n]} N \left( s, y_n(s, \omega), y_n(s - h, \omega), w_0(\omega) + \int_0^s y_n(r, \omega) dr, \omega \right) ds
\]
\[
= \int_0^t X_{[0, \tau_n]} f(s, \omega) ds.
\]

So each is continuous in \( t \). For large enough \( n, \tau_n = T \) and hence
\[
y_n(t, \omega) - u_0(\omega) + \int_0^t N \left( s, y_n(s, \omega), y_n(s - h, \omega), w_0(\omega) + \int_0^s y_n(r, \omega) dr, \omega \right) ds
\]
\[
= \int_0^t f(s, \omega) ds.
\]
Also these satisfy the inequality
\[
\sup_{t \in [0,T]} |y_n(t,\omega)|^2 \leq C(\omega) \leq 2^M(\omega) < 9M(\omega), \tag{68.3.4}
\]
the constant on the right not depending on \(n\). Thus for fixed \(\omega\), we can regard \(N\) as bounded and the same reasoning used in the above lemma involving the Ascoli–Arzelà theorem implies that every subsequence has a further subsequence which converges to a solution to the integral equation for that \(\omega\). Thus it is continuous into \(\mathbb{R}^d\). It follows from the measurable selection theorem above that there exists \(u\) product measurable and continuous in \(t\) such that \(u(\cdot,\omega) = \lim_{n(\omega) \to \infty} y_n(\omega) (\cdot,\omega)\) in \(L^2([0,T];\mathbb{R}^d)\). By the reasoning of the above lemma, there is a further subsequence, denoted the same way, for which \(\lim_{n \to \infty} y_n(\omega)\) in \(C([0,T];\mathbb{R}^d)\) solves the integral equation for a fixed \(\omega\). Thus \(u\) is a product measurable solution to the integral equation as claimed. \(\blacksquare\)

We made use of an estimate in order to get the conclusion of this theorem. However, all that is really needed is the following.

**Corollary 68.3.4** Suppose \(N(t, u, v, w, \omega) \in \mathbb{R}^d\) for \(u, v, w \in \mathbb{R}^d, t \in [0,T]\) and \((t, u, v, w, \omega) \to N(t, u, v, w, \omega)\) is progressively measurable with respect to a constant filtration \(\mathcal{F}_t = \mathcal{F}\). Also suppose \((t, u, v, w) \to N(t, u, v, w, \omega)\) is continuous. Suppose for each \(\omega\), there exists an estimate for any solution \(u(\cdot,\omega)\) to the integral equation

\[
u(t,\omega) - u_0(\omega) + \int_0^t N(s, u(s,\omega), u(s-h,\omega), w(s,\omega), \omega) \, ds = \int_0^t f(s,\omega) \, ds,
\]
which is of the form
\[
\sup_{t \in [0,T]} |u(t,\omega)| \leq C(\omega) < \infty
\]
Also let \(f\) be product measurable and \(f(\cdot,\omega) \in L^1([0,T];\mathbb{R}^d)\). Here \(u_0\) has values in \(\mathbb{R}^d\) and is \(\mathcal{F}\) measurable and \(u(s-h,\omega) \equiv u_0(\omega)\) whenever \(s-h \leq 0\) and

\[
w(t,\omega) \equiv w_0(\omega) + \int_0^t u(s,\omega) \, ds
\]
where \(w_0\) is a given \(\mathcal{F}\) measurable function. Then for \(h > 0\), there exists a product measurable solution \(u\) to the integral equation. \(\blacksquare\)

Of course the same conclusions apply when there is no dependence in the integral equation on \(u(s-h,\omega)\) or the integral \(w(t,\omega)\). Note that these theorems hold for all \(\omega\).

### 68.4 The Navier–Stokes Equations

In this section, we study the stochastic Navier–Stokes equations of arbitrary dimension. We prove there exists a global solution which is product measurable. The
main result is Theorem 68.4.6. We use the Galerkin method and Theorem 68.3.3 to get product measurable approximate solutions. Then we take weak limits and get path solutions. After this, we apply Theorem 68.2.8 to get product measurable global solutions.

As in [14], an important part of our argument is the theorem in Lions [82] which follows. See Theorem 67.5.6.

Theorem 68.4.1 Let $W$, $H$, and $V'$ be separable Banach spaces. Suppose $W \subseteq H \subseteq V'$ where the injection map is continuous from $H$ to $V'$ and compact from $W$ to $H$. Let $q_1 \geq 1$, $q_2 > 1$, and define

$$S \equiv \{ u \in L^{q_1}([a,b];W) : u' \in L^{q_2}([a,b];V')$$

and $||u||_{L^{q_1}([a,b];W)} + ||u'||_{L^{q_2}([a,b];V')} \leq R$. Then $S$ is pre-compact in $L^{q_1}([a,b];H)$. This means that if $\{u_n\}_{n=1}^\infty \subseteq S$, it has a subsequence $\{u_{n_k}\}$ which converges in $L^{q_1}([a,b];H)$.

A proof of a generalization of this theorem is found on Page 2454. Let $U$ be a bounded open set in $\mathbb{R}^d$ and let $S$ denote the functions which are infinitely differentiable having zero divergence and also having compact support in $U$. We have in mind $d=3$, but the approach is not limited by dimension. We use the same Galerkin method found in [14], the details being included in slightly abbreviated form for convenience of the reader. The difference is that we switch the roles of $V$ and $W$ along with a few other minor modifications. This is the part of the argument which gives a solution for each $\omega$ and it is standard material. Define

$$V \equiv \overline{S} \text{ in } \left( H^{d^*} (U) \right)^d, \ W \equiv \overline{S} \text{ in } \left( H^1 (U) \right)^d, \text{ and } H \equiv \overline{S} \text{ in } \left( L^2 (U) \right)^d,$$

where $d^*$ is such that for $w \in \left( H^{d^*} (U) \right)^d$ then $\|Dw\|_{L^\infty (U)} < \infty$. For example, you could take $d^* = 3$ for $d = 3$. In [14], they take $d^* = 8$ which is large enough to work for all dimensions of interest.

Let $A : W \to W'$ and $N : W \to V'$ be defined by

$$\langle Au, v \rangle \equiv \int_U \nabla u_i \cdot \nabla v_i \, dx, \quad \langle Nu, v \rangle \equiv -\int_U u_i u_j v_{j,i} \, dx.$$ 

Then $N$ is a continuous function. Indeed, pick $v \in V$ and suppose $u_n \to u$ in $W$, then

$$|\langle Nu - Nu_n, v \rangle| \leq \int_U \sum_{i,j} (u_{ni} u_{nj} - u_i u_j) v_{j,i} \, dx \leq C \|v\|_V \int_U (|u_n| + |u|) (|u_n - u|) \, dx$$

$$\leq C \|v\|_V \left( \int_U |u_n|^2 + |u|^2 \, dx \right)^{1/2} \left( \int_U |u_n - u|^2 \, dx \right)^{1/2},$$
where what multiplies $\|v\|_V$ clearly converges to 0.

An abstract form for the incompressible Navier–Stokes equations is

$$u' + \nu A u + N u = f, \quad u(0) = u_0,$$

where $f \in L^2([0, T]; W')$, for some fixed $T > 0$. As in [14], we will let $\nu = 1$ to simplify the presentation. A stochastic version of this would be the integral equation

$$u(t, \omega) - u_0(\omega) + \int_0^t A(u(s, \omega)) \, ds + \int_0^t N(u(s, \omega)) \, ds = \int_0^t f(s, \omega) \, ds + q(t, \omega),$$

where $q(\cdot, \omega)$ will be continuous into $V$, $(t, \omega) \rightarrow q(t, \omega)$ will be product measurable having values in $V$, and $q(0, \omega) = 0$. So $q$ here is a fixed stochastic process, which serves as the random source. Also $(t, \omega) \rightarrow f(t, \omega)$ will be product measurable into $W'$ as well as having $t \rightarrow f(t, \omega)$ in $L^2([0, T]; W')$. Our problem is to show the existence of a product measurable solution.

Let $T$ be any fixed positive number and let $q$ be any fixed process satisfying the above.

**Definition 68.4.2** A global solution to the above integral equation is a process $u(t, \omega)$, for which $\omega \rightarrow u(t, \omega)$ is $F$ measurable and satisfies for each $\omega$ outside a set of measure zero and all $t \in [0, T]$,

$$u(t, \omega) - u_0(\omega) + \int_0^t A(u(s, \omega)) \, ds + \int_0^t N(u(s, \omega)) \, ds = \int_0^t f(s, \omega) \, ds + q(t, \omega).$$

In order to apply the earlier result, let $w(t, \omega) = u(t, \omega) - q(t, \omega)$ and write the equation in terms of $w$,

$$w(t, \omega) - u_0(\omega) + \int_0^t A(w(s, \omega) + q(s, \omega)) \, ds + \int_0^t N(w(s, \omega) + q(s, \omega)) \, ds = \int_0^t f(s, \omega) \, ds.$$

It turns out that it is convenient to define

$$\langle B(u, v), w \rangle \equiv -\int_U u_i v_j w_{j,i} \, dx,$$

and write the equation in the following form:

$$w(t, \omega) - u_0(\omega) + \int_0^t A(w(s, \omega)) \, ds + \int_0^t \hat{N}(w(s, \omega)) \, ds = \int_0^t \hat{f}(s, \omega) \, ds,$$

where

$$\hat{N}(w(t, \omega)) \equiv N(w(t, \omega)) + B(w(t, \omega), q(t, \omega)) + B(q(t, \omega), w(t, \omega)),$$

$$\hat{f}(t, \omega) \equiv f(t, \omega) - A(q(t, \omega)) - N(q(t, \omega)).$$

This is an equation in $V'$. Moreover, we have the following:
Lemma 68.4.3 For fixed \( \omega \in \Omega \), \( \tilde{f} \in L^2 ([0, T] ; W') \), and

\[
(t, \mathbf{w}) \to B(\mathbf{w}, \mathbf{q}(t, \omega)), \ (t, \mathbf{w}) \to B(\mathbf{q}(t, \omega), \mathbf{w})
\]

are continuous functions having values in \( W' \). For fixed \( \mathbf{w} \in W \),

\[
(t, \omega) \to B(\mathbf{w}, \mathbf{q}(t, \omega)), \ (t, \omega) \to B(\mathbf{q}(t, \omega), \mathbf{w})
\]

are product measurable. In addition to this, if \( \mathbf{z} \in W \),

\[
|\langle B(\mathbf{w}, \mathbf{q}(t, \omega)), \mathbf{z} \rangle| \leq C \|\mathbf{q}(t, \omega)\|_V \|\mathbf{w}\|_H \|\mathbf{z}\|_H,
\]

Proof: The first claim is straightforward to prove from the definition of \( A \) and \( N \). Consider the next claim about continuity. Let \( \mathbf{z} \in W \) be given. Then from the fact that all the functions are divergence free,

\[
|\langle B(\mathbf{w}, \mathbf{q}(t)) - B(\tilde{\mathbf{w}}, \mathbf{q}(s)), \mathbf{z} \rangle| = \left| \int_U (w_i q_j(t) - \tilde{w}_i q_j(s)) z_{j,i} dx \right| \leq C \left( \|\mathbf{q}(t)\|_V \int_U |\mathbf{w} - \tilde{\mathbf{w}}| |\mathbf{z}| dx + \|\mathbf{q}(t) - \mathbf{q}(s)\|_V \int_U |\tilde{\mathbf{w}}| |\mathbf{z}| dx \right) \leq C \left( \|\mathbf{q}(t)\|_V \left\{ |\mathbf{w} - \tilde{\mathbf{w}}|_H + \|\mathbf{q}(t) - \mathbf{q}(s)\|_V |\tilde{\mathbf{w}}|_H \right\} |\mathbf{z}|_H \right),
\]

where we have suppressed the dependence of \( \mathbf{q} \) on \( \omega \) to simplify the notation. The other function is similar.

As to the claim about product measurability, this follows from the above definition and assumptions about \( \mathbf{q} \) being product measurable. For the estimates,

\[
|\langle B(\mathbf{w}, \mathbf{q}), \mathbf{z} \rangle| = \left| \int_U w_i q_j z_{j,i} \right| = \left| \int_U w_i q_j z_j \right| \leq C \|\mathbf{q}\|_V \int_U |\mathbf{w}| |\mathbf{z}| dx,
\]

and apply Hölder’s inequality. The other estimate is similar. 

This has shown that it suffices to verify that there exists a global solution \( \mathbf{u} \) to the equation

\[
\mathbf{u}(t, \omega) - \mathbf{u}_0(\omega) + \int_0^t A(\mathbf{u}(s, \omega)) ds + \int_0^t \tilde{N}(\mathbf{u}(s, \omega)) ds = \int_0^t \mathbf{f}(s, \omega) ds,
\]

where \( \mathbf{f}(\cdot, \omega) \in L^2 ([0, T] ; W') \) for each \( \omega \in \Omega \).

Let \( R \) be the Riesz map from \( V \) to \( V' \), so \( \langle R \mathbf{v}_1, \mathbf{v}_2 \rangle_{V', V} = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle_V \) for any \( \mathbf{v}_1, \mathbf{v}_2 \in V \). The compactness of the embeddings imply that \( R^{-1} \) is a compact self adjoint operator on \( H \) and so there is a complete orthonormal basis \( \{ w_k \} \) for \( H \) such that \( R^{-1} w_k = \mu_k w_k \), where \( \{ \mu_k \} \) is a decreasing sequence of positive numbers.
By Theorem in Chapter 68.8, and from the above lemma and that all functions are divergence free, we obtain

Recall that \( \hat{\psi} \) is the orthogonal projection of \( u \) onto \( \mathbb{R}^n \). Thus \( R_y \in \text{span}(w_1, \ldots, w_M) \) and in fact, \( R \) maps \( V_n \) onto \( V_n \) and so this shows that \( y \) is perpendicular to \( \text{span}(w_1, \ldots, w_M) \) for each \( M \) so \( y = 0 \) and \( \phi = 0 \) after all. Thus \( \cup \mathbb{M} V_n \) is also dense in \( V \) and hence it is also dense in \( W \).

Let \( u_n(t, \omega) = \sum_{k=1}^n x_k(t, \omega) w_k \), where \( x(t, \omega) = (x_1(t, \omega), \ldots, x_n(t, \omega))^T \in \mathbb{R}^n \). We consider the problem of finding \( x(t, \omega) \) such that for all \( w_k, k \leq n \),

\[
(u_n(t, \omega), w_k)_H = (u_{0n}(\omega), w_k)_H + \int_0^t \langle A(u_n(s, \omega)), w_k \rangle \, ds \\
+ \int_0^t \langle \tilde{N}(u_n(s, \omega)), w_k \rangle \, ds = \int_0^t \langle f(s, \omega), w_k \rangle \, ds, 
\]

(68.4.6)

where \( u_{0n} \) is the orthogonal projection of \( u_0 \) onto \( V_n \).

By the continuity of the operators described above, and the orthogonality of the \( w_k \), this is nothing but an ordinary differential equation for the vector \( x(t, \omega) \). By Theorem [NSW76] there exists a product measurable solution \( x \) and therefore, \( u_n(t, \omega) \) is also product measurable in \( H \).

Take the derivative, multiply by \( x_k(t, \omega) \), add, and integrate again in the usual way to obtain

\[
\frac{1}{2} |u_n(t, \omega)|_H^2 - \frac{1}{2} |u_{0n}(\omega)|_H^2 \\
= - \int_0^t \langle A(u_n(s, \omega)), u_n(s, \omega) \rangle \, ds - \int_0^t \langle \tilde{N}(u_n(s, \omega)), u_n(s, \omega) \rangle \, ds \\
+ \int_0^t \langle f(s, \omega), u_n(s, \omega) \rangle \, ds.
\]

Recall that \( \tilde{N}(u(t, \omega)) = N(u(t, \omega)) + B(u(t, \omega), q(s, \omega)) + B(q(t, \omega), u(t, \omega)) \).

From the above lemma and that all functions are divergence free, we obtain

\[
\int_0^t \langle B(q(s, \omega), u_n(s, \omega)), u_n(s, \omega) \rangle \, ds = 0,
\]

and

\[
\left| \int_0^t \langle B(u_n(s, \omega), q(s, \omega)), u_n(s, \omega) \rangle \, ds \right| \leq C \int_0^t \|q(s, \omega)\|_V |u_n(s, \omega)|_H^2 \, ds.
\]
Then one can obtain an inequality of the following form

\[
\frac{1}{2} \| u_n(t, \omega) \|^2_H + \int_0^t \| u_n(s, \omega) \|_W^2 \, ds \\
\leq \frac{1}{2} \| u_{0n}(\omega) \|^2_H + C \int_0^t \| q(s, \omega) \|_V \| u_n(s, \omega) \|^2_H \, ds \\
+ C \int_0^t \| f(s, \omega) \|^2_W \, ds + \frac{1}{2} \int_0^t \| u_n(s, \omega) \|^2_W \, ds.
\]

Since \( t \to \| q(t, \omega) \|_V \) is continuous, it follows from Gronwall’s inequality that there is an estimate of the form

\[
\| u_n(t, \omega) \|^2_H + \int_0^t \| u_n(s, \omega) \|_W^2 \, ds \leq C (u_0, f, q, T, \omega).
\]

(68.4.7)

The next task is to estimate \( \| u_n'(\omega) \|_{L^2([0,T];V')} \) for each fixed \( \omega \in \Omega \). We will suppress the dependence on \( \omega \) of all functions whenever it is appropriate. With 68.4.6, the fundamental theorem of calculus implies that for each \( w \in V_n \),

\[
\langle u_n'(t), w \rangle_{V', V} + \langle A(u_n(t)), w \rangle_{V', V} + \langle \acute{N}(u_n(t)), w \rangle = \langle f(t), w \rangle.
\]

In terms of inner products in \( V \),

\[
\left( R^{-1} u_n'(t) + R^{-1} A(u_n(t)) + R^{-1} \acute{N}(u_n(t)) - R^{-1} f(t), w \right)_V = 0
\]

for all \( w \in V_n \). This is equivalent to saying that for \( P_n \) the orthogonal projection in \( V \) onto \( V_n \),

\[
\left( R^{-1} u_n'(t) + R^{-1} A(u_n(t)) + R^{-1} \acute{N}(u_n(t)) - R^{-1} f(t), P_n w \right)_V = 0
\]

for all \( w \in V \). This is to say that

\[
R^{-1} u_n'(t) + P_n R^{-1} A(u_n(t)) + P_n R^{-1} \acute{N}(u_n(t)) = P_n R^{-1} f(t).
\]

Now the projection map decreases norms and \( R^{-1} \) preserves norms. Hence

\[
\| u_n'(t) \|_{V'} = \| R^{-1} u_n'(t) \|_V \leq \| A(u_n(t)) \|_{V'} + \| \acute{N}(u_n(t)) \|_{V'} + \| f(t) \|_{V'},
\]

from which it follows that \( u_n' \) is bounded in \( L^2([0,T];V') \). Indeed, this is the case because \( A(u_n) \) and \( \acute{N}(u_n) \) are both bounded in \( L^2([0,T];V') \). The term \( \| \acute{N}(u_n(t)) \|_{V'} \) can be split further into terms involving \( \| N(u_n) \|, \| B(u_n, q) \| \), and \( \| B(q, u_n) \| \). For example, consider \( N(u_n) \) which is the least obvious. Let \( w \in L^2([0,T];V) \). From the definitions,

\[
\langle N(u_n), w \rangle_{L^2([0,T];V)} = \left| \int_0^T \int_U u_n_i u_n_j w_{i,j} \, dxdt \right|
\]
that there is a subsequence, still
decomposition shows then that a solution to the evolution equation is obtained. Then, when this is done, we will apply the measurable selection result to obtain a product measurable solution.

This condition holds for all \( \omega \). Now for each \( \omega \), one can take a subsequence such that a strongly convergent subsequence exists. Since \( A \) is linear, we can also assume that

\[
Ax_n \to Ax \quad \text{weakly in } L^2([0,T];H), \quad (68.4.9)
\]

\[
A^*x_n' \to A^*x' \quad \text{weakly in } L^2([0,T];V'), \quad (68.4.10)
\]

This last convergence follows from Theorem 68.3.1. The sequence is bounded in \( L^2([0,T];W) \) and the derivative is bounded in \( L^2([0,T];V') \) so such a strongly convergent subsequence exists. Since \( A \) is linear, we can also assume that

\[
Ax_n \to Ax \quad \text{weakly in } L^2([0,T];W'). \quad (68.4.11)
\]

What happens with the nonlinear operator \( \tilde{N} \)? Let \( w \in L^\infty([0,T];V) \). A computation shows then that

\[
\left| \int_0^T \langle Nu_n(t) - Nu(t), w(t) \rangle \, dt \right| 
\]

\[
= \left| \int_0^T \int_U (u_{nj}(t)u_j(t) - u_i(t)u_i(t)) \, w, dxdt \right| 
\]

\[
\leq \|w\|_{L^\infty([0,T],V)} \int_0^T \int_U \left( |u_n(t)| + |u(t)| \right) ||u_n(t) - u(t)|| \, dxdt 
\]

\[
\leq \|w\|_{L^\infty([0,T],V)} \left( \int_0^T \int_U |u_n| + |u|^2 \, dxdt \right)^{1/2} \left( \int_0^T \int_U (u - u_n)^2 \, dxdt \right)^{1/2}. 
\]

This converges to 0 thanks to the estimates and the strong convergence. Similar converging holds for the other nonlinear terms \( B(u_n(t), q), B(q, u_n(t)) \).

We have shown that for any \( n \geq m \), and \( w \in V_m \),

\[
\langle u_n'(t), w \rangle_{V', V} + \langle A(u_n(t)), w \rangle_{V', V} + \langle \tilde{N}(u_n(t)), w \rangle = \langle f(t), w \rangle. \quad (68.4.12)
\]
68.4. THE NAVIER–STOKES EQUATIONS

Let \( \zeta \in C^\infty ([0, T]) \) be such that \( \zeta (T) = 0 \). Then

\[
\langle u_n'(t), w\zeta(t) \rangle_{V', V} + \langle A(u_n(t), w\zeta(t)) \rangle_{V', V} + \langle \hat{N}(u_n(t)), w\zeta(t) \rangle = \langle f(t), w\zeta(t) \rangle.
\]

Integrating this equation from 0 to \( T \) we obtain

\[
- \langle u_{0m}(\omega), w\zeta(0) \rangle_H - \int_0^T \zeta'(s) \langle u_n(s, \omega), w \rangle_H \ ds
= - \int_0^T \langle A(u_n(s, \omega)), w\zeta(s) \rangle \ ds - \int_0^T \langle \hat{N}(u_n(s, \omega)), w\zeta(s) \rangle \ ds
+ \int_0^T \langle f(s, \omega), w\zeta(s) \rangle \ ds.
\]

Now letting \( n \to \infty \), from the above list of convergent sequences,

\[
- \langle u_0(\omega), w\zeta(0) \rangle_H - \int_0^T \zeta'(s) \langle u(s, \omega), w \rangle_H \ ds
= - \int_0^T \langle A(u(s, \omega)), w\zeta(s) \rangle \ ds - \int_0^T \langle \hat{N}(u(s, \omega)), w\zeta(s) \rangle \ ds
+ \int_0^T \langle f(s, \omega), w\zeta(s) \rangle \ ds.
\]

It follows that in the sense of \( V' \) valued distributions,

\[
u'(\omega) + A(u(\omega)) + \hat{N}(u(\omega)) = f(\omega) \tag{68.4.13}
\]

along with the initial condition

\[
u(0) = u_0. \tag{68.4.14}
\]

This has proved most of the following lemma:

**Lemma 68.4.4** Let \( u_0 \) have values in \( H \) and be \( F \) measurable, and let \( u_n \) be a solution to \(68.4.7\). Then for each \( \omega \), the estimate \(68.4.8\) holds. Also there is a subsequence, still called \( u_n \) such that the convergence for \(68.4.9 - 68.4.13\) are valid. For all \( \omega \), the function \( u(\cdot, \omega) \) is a solution to \(68.4.13 - 68.4.14\) and satisfies

\[
u(\cdot, \omega) \in L^\infty ([0, T]; H) \cap L^2 ([0, T]; W), u' \in L^2 ([0, T]; V').
\]

This solution is also weakly continuous into \( H \) for each \( \omega \).

**Proof:** All that remains to show is the last claim about weak continuity into \( H \). The equation \(68.4.7\) shows that \( u(\cdot, \omega) \) is continuous into \( V' \). However, the weak convergence and the estimate \(68.4.8\) show that \( u(\cdot, \omega) \) is bounded in \( H \). It follows from density of \( V \) in \( H \) that \( t \to u(t, \omega) \) is weakly continuous into \( H \).

From \(68.4.13 - 68.4.14\), the following integral equation for a path solution holds:

\[
u(t, \omega) - u_0(\omega) + \int_0^t A(u(s, \omega)) \ ds + \int_0^t \hat{N}(u(s, \omega)) \ ds = \int_0^t f(s, \omega) \ ds.
\]

We apply Theorem \(68.4.8\) to prove the above solution could be taken product measurable.
Chapter 68. Measurability Without Uniqueness

Theorem 68.4.5 Let \( f(t, \omega), q(t, \omega) \) be product measurable and \( u_0 \) be measurable, such that for each \( \omega \in \Omega \), \( f(\cdot, \omega) \in L^2([0, T]; W') \), \( q(\cdot, \omega) \in C([0, T]; V) \) with \( q(0) = 0 \), and \( u_0(\omega) \in H \). Then there exists a global solution to the integral equation

\[
    u(t, \omega) - u_0(\omega) + \int_0^t A(u(s, \omega)) \, ds + \int_0^t N(u(s, \omega)) \, ds = \int_0^t f(s, \omega) \, ds.
\]

Proof: Letting \( u_n \) be a solution to (68.4.8) for \( u_n \), we verify the conditions of Theorem 68.4.5 for \( u_n \).

The assumption in this theorem that the \( u_n \) are bounded follows from the above estimate (68.4.8). Then it was shown in the above lemma that whenever a sequence satisfies the estimate (68.4.6), it has a subsequence which converges as in (68.4.6) to a weakly continuous \( u(\cdot, \omega) \). Therefore, by Theorem 68.4.5 there is a subsequence \( u_{n_k}(\cdot, \omega) \) converging weakly to \( u(\cdot, \omega) \), such that \( (t, \omega) \mapsto u(t, \omega) \) is a product measurable function into \( H \). Then a further subsequence converges to a path solution to the above integral equation, which must be the same function because when a sequence converges, all subsequences converge to the same thing. In addition to this, \( u \) is also product measurable into \( W \). This follows from the above estimate (68.4.8). For \( \phi \in H, (t, \omega) \mapsto (\phi, u(t, \omega)) \) is product measurable. However, \( H \) is dense in \( W \) and so if \( \psi \in W \), there is a sequence \( \{ \phi_n \} \) in \( H \) such that \( \phi_n \to \psi \).

Then

\[
    \langle \psi, u \rangle = \lim_{n \to \infty} \langle \phi_n, u \rangle,
\]

so by the Pettis theorem (68.4.10), \( u \) is product measurable into \( W \) also.\]

This shows much of the following theorem which is the main result.

Theorem 68.4.6 Let \( f(t, \omega), q(t, \omega) \) be product measurable and \( u_0 \) be measurable, such that for each \( \omega \in \Omega \), \( f(\cdot, \omega) \in L^2([0, T]; W') \), \( q(\cdot, \omega) \in C([0, T]; V) \) with \( q(0) = 0 \), and \( u_0(\omega) \in H \). Then there exists a global solution to the integral equation

\[
    u(t, \omega) - u_0(\omega) + \int_0^t A(u(s, \omega)) \, ds + \int_0^t N(u(s, \omega)) \, ds = \int_0^t f(s, \omega) \, ds + q(t, \omega).
\]

In addition to this, \( t \mapsto u(t, \omega) \) is continuous into \( H \) and satisfies

\[
    u(\cdot, \omega) \in L^\infty([0, T]; H) \cap L^2([0, T]; W).
\]

If, in addition to the above, \( u_0 \in L^2(\Omega; H) \) and \( f \in L^2([0, T] \times \Omega; W') \) and \( q \in L^2([0, T] \times \Omega; V) \), then the solution \( u \) is in \( L^2([0, T] \times \Omega; H) \cap L^2([0, T] \times \Omega; W) \).

Proof: The last claim follows from the estimates used in the Galerkin method, taking expectations and passing to a limit. To verify the continuity into \( H \), one can observe that from the integral equation, \( u \) is continuous into \( W' \). One has

\[
    |u(t)|_H = \sum_{k=1}^\infty \langle u(t), w_k \rangle^2 = \sum_{k=1}^\infty \langle u(t), w_k \rangle^2,
\]
and so $t \to |u(t)|^2_H$ is lower semi-continuous. Since it is in $L^\infty$, this implies this function is bounded. Hence the continuity into $V'$ and density of $V$ in $H$ implies that $u(t)$ is weakly continuous into $H$. Then one can use the formulation in Theorem 68.4.5 to verify $t \to |u(t)|_H$ is continuous and apply uniform convexity of the Hilbert space $H$. ■

One can replace $q(t,\omega)$ with $q(t,\omega,u)$ and $f(t,\omega)$ with $f(t,\omega,u)$ in the above with no change in the argument, provided it is assumed that $(t,\omega,u) \to q(t,\omega,u)$, $f(t,\omega,u)$ are product measurable, continuous in $(t,u)$ and bounded.

68.5 A Friction contact problem

In this section we will consider a friction contact problem which has a coefficient of friction which is dependent on the slip speed.

$$\ddot{u} = \sigma_{ij,j}(u,\dot{u}) + f_i \text{ for } (t,x) \in (0,T) \times U, \quad (68.5.15)$$

$$u(0,x) = u_0(x), \quad (68.5.16)$$

$$\dot{u}(0,x) = v_0(x), \quad (68.5.17)$$

where $U$ is a bounded open subset of $\mathbb{R}^3$ having Lipschitz boundary, along with some boundary conditions which pertain to a part of the boundary of $U$, $\Gamma_C$. For $x \in \Gamma_C$,

$$\sigma_n = -p((u_n - g)_+)C_n, \quad (68.5.18)$$

This is the normal compliance boundary condition.

$$|\sigma_T| \leq F((u_n - g)_+)\mu \left(\left|\dot{u}_T - \dot{U}_T\right|\right), \quad (68.5.19)$$

$$|\sigma_T| < F((u_n - g)_+)\mu \left(\left|\dot{u}_T - \dot{U}_T\right|\right) \text{ implies } \dot{u}_T - \dot{U}_T = 0, \quad (68.5.20)$$

$$|\sigma_T| = F((u_n - g)_+)\mu \left(\left|\dot{u}_T - \dot{U}_T\right|\right) \text{ implies } \dot{u}_T - \dot{U}_T = -\lambda \sigma_T(u,\dot{u}). \quad (68.5.21)$$

Here $C_n$ is a positive function in $L^\infty(\Gamma_C)$, which we take equal to 1 to simplify notation, $\dot{U}_T$ is the velocity of the foundation, $\lambda$ is non negative, and $\mu$ is a bounded positive function having a bounded continuous derivative. We could also let $\mu$ depend on $x \in \Gamma_C$ to model the roughness of the contact surface but we will suppress this dependence in the interest of simpler notation. Also, $n$ is the unit outward normal to $\partial U$ and $u_n, u_T, \sigma_T,$ and $\sigma_n$ are defined by the following.

$$u_n = u \cdot n$$

$$u_T = u - (u \cdot n)n$$

$$\sigma_n = \sigma_{ij}n_jn_i$$
\[ \sigma_{Ti} = \sigma_{ij}n_j - \sigma_n n_i, \]

written more simply,

\[ \sigma_T = \sigma_n - \sigma_n n \]

Systems like the above model dynamic friction contact problems \cite{24}, \cite{48} \cite{43}. The function \( g \) represents the gap between the contact surface of \( U, \Gamma_C \), and a foundation which is sliding tangent to \( \Gamma_C \) with tangential velocity \( \dot{U}_T \).

The new ingredient in this paper is that we allow \( g = g(t, x, \omega) \) where \( \omega \in (\Omega, \mathcal{F}) \) and we assume \((t, x, \omega) \rightarrow g(x, \omega) \) is \( B([0, T] \times \Gamma_C) \times \mathcal{F} \) measurable. Also, we make the reasonable assumption that \( 0 \leq g(t, x, \omega) \leq l < \infty \) for all \((t, x, \omega)\). We also assume that the given motion of the foundation \( \dot{U}_T \) is a stochastic process \( \dot{U}_T = \dot{U}_T (t, x, \omega) \) and is \( B([0, T] \times \Gamma_C) \times \mathcal{F} \) measurable. Here \( B([0, T] \times \Gamma_C) \) denotes the Borel sets of \([0, T] \times \Gamma_C\). We make the reasonable assumption that \( \dot{U}_T (t, x, \omega) \) is uniformly bounded. In the interest of notation, we will often suppress the dependence on \( t, x, \) and \( \omega \).

The condition \( 68.5.18 \) is the contact condition. It says the normal component of the traction force density is dependent on the normal penetration of the body into the foundation surface. Conditions \( 68.5.19 - 68.5.21 \) model friction. They say that the tangential part of the traction force density is bounded by a function determined by the normal force or penetration. No sliding takes place until \(|\dot{T}| \) reaches this bound, \( F((u_n - g)_+)\mu (0) \), \( 68.5.20 \). When this occurs, the tangential force density has a direction opposite the relative tangential velocity \( \dot{u}_T - \dot{U}_T \). The dependence of the friction coefficient on the magnitude of the slip velocity, \( |\dot{u}_T - \dot{U}_T| \) may be experimentally verified and so it has been included. The new feature in this model is the assumption that the gap is a random variable for each \( x \in \Gamma_C \) and we want to consider measurability of the solutions. Thus for a fixed \( \omega \), we have a standard friction problem and it is the measurability which is of interest here.

In this paper, we assume the following on \( p \) and \( F \). The functions \( p \) and \( F \) are increasing and

\[ \delta^2 r - K \leq p(r) \leq K (1 + r), \quad r \geq 0, \quad (68.5.22) \]
\[ p(r) = 0, \quad r < 0, \]
\[ F(r) \leq K (1 + r), \quad r \geq 0, \quad (68.5.23) \]
\[ F(r) = 0 \text{ if } r < 0, \]
\[ |\mu (r_1) - \mu (r_2)| \leq \text{Lip} (\mu) |r_1 - r_2|, \quad ||\mu||_{\infty} \leq C, \quad (68.5.24) \]
68.5. A FRICTION CONTACT PROBLEM

and for \( a = F, p, \) and \( r_1, r_2 \geq 0, \)

\[
|a(r_1) - a(r_2)| \leq K|r_1 - r_2|.
\]  

(68.5.25)

One can consider more general growth conditions than this, but we are keeping this part simple to emphasize the new stochastic considerations.

It will be assumed that

\[
\sigma_{ij} = A_{ijkl}u_k + C_{ijkl}\dot{u}_k,
\]  

(68.5.26)

where \( A \) and \( C \) are in \( \mathbb{L}^\infty(U) \) and for \( B = A \) or \( C \), we have the following symmetries.

\[
B_{ijkl} = B_{ijlk}, B_{jikl} = B_{ijkl}, B_{ijkl} = B_{klij},
\]  

(68.5.27)

and we also assume for \( B = A \) or \( C \) that

\[
B_{ijkl}H_{ij}H_{kl} \geq \varepsilon H_{rs}H_{rs}
\]  

(68.5.28)

for all symmetric \( H \).

Throughout the paper, \( V \) will be a closed subspace of \((H^1(U))^3\) containing the test functions \((C_0^\infty(U))^3\), \( \rightharpoonup \) will denote weak or weak * convergence while \( \rightarrow \) will mean strong convergence. \( \gamma \) will denote the trace map from \( W^{1,2}(U) \) into \( L^2(\partial U) \). \( H \) will denote \((L^2(U))^3\) and we will always identify \( H \) and \( H' \) to write

\[
V \subseteq H = H' \subseteq V'
\]

We define

\[
\mathcal{V} = L^2(0,T;V), \mathcal{H} = L^2(0,T,H), V' = L^2(0,T;V')
\]

68.5.1 The Abstract Problem

We shall use two theorems found in Lions [72] and Simon [106] respectively. These theorems apply for fixed \( \omega \). Proofs of generalizations of these theorems begin on Page 2451.

**Theorem 68.5.1** If \( p \geq 1, q > 1 \), and \( W \subseteq U \subseteq Y \) where the inclusion map of \( W \) into \( U \) is compact and the inclusion map of \( U \) into \( Y \) is continuous, let

\[
S = \{ u \in L^p(0,T;W) : u' \in L^q(0,T;Y) \text{ and } ||u||_{L^p(0,T;W)} + ||u'||_{L^q(0,T;Y)} < R \}
\]

Then \( S \) is pre compact in \( L^p(0,T;U) \).

**Theorem 68.5.2** Let \( W, U, \) and \( Y \) be as in Theorem 68.5.1 and let

\[
S = \{ u : ||u(t)||_W + ||u'||_{L^q(0,T;Y)} \leq R \text{ for } t \in [0,T] \}
\]

for \( q > 1 \). Then \( S \) is pre compact in \( C(0,T;U) \).
Now we give an abstract formulation of the problem described roughly in 68.5.15 - 68.5.21. We begin by defining several operators. Let \( M, A : V \to V' \) be given by
\[
\langle Mu, v \rangle = \int_U C_{ijkl} u_{k,l} v_{i,j} \, dx, \quad (68.5.29)
\]
\[
\langle Au, v \rangle = \int_U A_{ijkl} u_{k,l} v_{i,j} \, dx. \quad (68.5.30)
\]
Also let the operator \( v \to P(u) \) map \( V \) to \( V' \) be given by
\[
\langle P(u), w \rangle = \int_0^T \int_{\Gamma_C} p((u_n - g)_+) w_n \, d\alpha dt, \quad (68.5.31)
\]
where
\[
u(t) = \nu_0 + \int_0^t v(s) \, ds \quad (68.5.32)
\]
for \( \nu_0 \in V_0 \). (Technically, \( P \) depends on \( \nu_0 \) but we suppress this in favor of simpler notation ). Let
\[
\gamma^*_T : L^2 \left( 0, T; L^2 \left( \Gamma_C \right) \right) \to V'
\]
is defined as
\[
\langle \gamma^*_T \xi, w \rangle = \int_0^T \int_{\Gamma_C} \xi \cdot w \, d\alpha dt.
\]
Now the abstract form of the problem, denoted by \( \mathcal{P} \), is the following.
\[
v' + Mu + Au + Pu + \gamma^*_T \xi = f \quad \text{in} \quad V', \quad (68.5.33)
\]
\[
v(0) = \nu_0 \in H, \quad (68.5.34)
\]
where
\[
u(t) = \nu_0 + \int_0^t v(s) \, ds, \quad \nu_0 \in V_0, \quad (68.5.35)
\]
and for all \( w \in V \),
\[
\langle \gamma^*_T \xi, w \rangle \leq \int_0^T \int_{\Gamma_C} F((u_n - g)_+) \mu \left( \left| \nu_T - \dot{\nu}_T \right| \right) \cdot \left[ \nu_T - \dot{\nu}_T + w_T \right] \, d\alpha dt. \quad (68.5.36)
\]
Also \( f \in L^2(0, T; V') \) so \( f \) can include the body force as well as traction forces on various parts of \( \partial U \). If \( v \) solves the above abstract problem, then \( u \) can be considered a weak solution to 68.5.15 - 68.5.21 along with other variational and stable boundary conditions depending on the choice of \( W \) and \( f \in L^2(0, T; V') \).

In order to carry out our existence and uniqueness proofs, we assume \( M \) and \( A \) satisfy the following for some \( \delta > 0, \lambda \geq 0 \).
\[
\langle Bu, u \rangle \geq \delta \|u\|^2_{V'} - \lambda \|u\|^2_H, \quad \langle Bu, u \rangle \geq 0, \quad \langle Bu, v \rangle = \langle Bv, u \rangle, \quad (68.5.37)
\]
for \( B = M \) or \( A \). This is the assumption that we use, and we note that 68.5.37 is a consequence of 68.5.20 - 68.5.28 and Korn's inequality 19.
68.5.2 An Approximate Problem

We will use the Galerkin method. To do this, we will first regularize that subgradient material. Let

\[ \psi_\varepsilon (r) = \sqrt{|r|^2 + \varepsilon} \]

Then this is a convex, Lipschitz continuous function having bounded derivative which converges uniformly to \( \psi(r) = |r| \) on \( \mathbb{R} \). Also

\[ |\psi_\varepsilon (x) - \psi_\varepsilon (y)| \leq |x - y| \]
\[ |\psi_\varepsilon' (t)| \leq 1 \]

And finally, \( \psi_\varepsilon' \) is Lipschitz continuous with a Lipschitz constant \( C/\sqrt{\varepsilon} \). Here \( \psi_\varepsilon' \) denotes the gradient or Frechet derivative of the scalar valued function.

Our approximate problem for which we will apply the Galerkin method will be

\[ (68.5.38) \quad v' + Mv + Au + \gamma_T F ((u_n - g)_+) \mu \left( \begin{vmatrix} v_T - \dot{U}_T \end{vmatrix} \right) \psi_\varepsilon' \left( v_T - \dot{U}_T \right) = f \text{ in } V', \]

\[ v(0) = v_0 \in H, \quad (68.5.39) \]

where

\[ u(t) = u_0 + \int_0^t v(s)ds, \quad u_0 \in V, \quad (68.5.40) \]

Here the long operator on the left is defined in the following manner.

\[ \left\langle \gamma_T^* F ((u_n - g)_+) \mu \left( \begin{vmatrix} v_T - \dot{U}_T \end{vmatrix} \right) \psi_\varepsilon' \left( v_T - \dot{U}_T \right), w \right\rangle = \int_{\Gamma_C} F ((u_n - g)_+) \mu \left( \begin{vmatrix} v_T - \dot{U}_T \end{vmatrix} \right) \psi_\varepsilon' \left( v_T - \dot{U}_T \right) \cdot w_T dS \]

Let \( R \) denote the Riesz map from \( V \) to \( V' \) defined by \( \langle Ru, v \rangle = (u, v)_V \). Then \( R^{-1} : H \to V \) is a compact self adjoint operator and so there exists a complete orthonormal basis for \( H \), \( \{e_k\} \subset V \) such that

\[ Re_k = \lambda_k e_k \]

where \( \lambda_k \to \infty \). Let \( V_n = \text{span} \{e_1, \ldots, e_n\} \). Thus \( \cup_n V_n \) is dense in \( H \). In addition \( \cup_n V_n \) is dense in \( V \) and \( \{e_k\} \) is also orthogonal in \( V \). To see first that \( \{e_k\} \) is orthogonal in \( V \),

\[ 0 = (e_k, e_l)_H = \frac{1}{\lambda_k} \langle Re_k, e_l \rangle_H = \frac{1}{\lambda_k} \langle e_l, e_k \rangle_V \]

Next consider why \( \cup_n V_n \) is dense in \( V \). If this is not so, then there exists \( f \in V', \ f \neq 0 \) such that \( \cup_n V_n \) is in \( \ker (f) \). But \( f = Ru \) and so

\[ 0 = \langle Ru, e_k \rangle = \langle Re_k, u \rangle = \lambda_k \langle e_k, u \rangle_H \]

for all $e_k$ and so $u = 0$ by density of $\cup_n V_n$ in $H$. Hence $R \mathbf{u} = 0 = f$ after all, a contradiction. Hence $\cup_n V_n$ is dense in $V$ as claimed.

Now we set up the Galerkin method for Problem $P_\varepsilon$. Let

$$v_k(t, \omega) = \sum_{j=1}^k x_j(t, \omega) e_j, \quad u_k(t) = u_0 + \int_0^t v_k(s)ds$$

and let $v_k$ be the solution to the following integral equation for each $\omega$ and $j \leq k$.

The dependence on $\omega$ is suppressed in most terms in order to save space.

$$\langle v_k(t) - v_0, v_0 + \int_0^t Mv_k + Au_k + P u_k + \gamma_T^* F ((u_{kn} - g(\omega))_+ \cdot \mu (v_{kT} - \dot{U}_T), \psi_{\varepsilon} (v_{kT} - \dot{U}_T) \rangle \cdot e_j \rangle = \int_0^t \langle f, e_j \rangle ds$$

Here $v_0 \to v_0 \in H$ and the equation holds for each $e_j$ for each $j \leq k$. Then this integral equation reduces to a system of ordinary differential equations for the vector $x(t, \omega)$ whose $j$th component is $x_j(t, \omega)$ mentioned above. Differentiate, multiply by $x_j$ and add. Then integrate. This will yield some terms which need to be estimated. Here is the one which comes from the long term.

$$\int_0^t \int_{\Gamma_C} F ((u_{kn} - g(\omega))_+ \mu (v_{kT} - \dot{U}_T) \psi_{\varepsilon} (v_{kT} - \dot{U}_T) \cdot v_{kT} dSds$$

The first of these is nonnegative and the second is bounded below by an expression of the form

$$-C \int_0^t \int_{\Gamma_C} (1 + |u_{kn}|) |\dot{U}_T| dSds \geq -C \int_0^t \|u_k\|_W \|\dot{U}_T\|_{L^2(\Gamma_C)}^3 ds - C$$

Where $W$ embeds compactly into $V$ and the trace map from $W$ to $L^2(\Gamma_C)^3$ is continuous. In the above, $C$ is independent of $\varepsilon, \omega$ and $k$. To estimate the term from $P$ one exploits the linear growth condition of $P$ in $H^1$ to obtain a suitable estimate.
It follows from equivalence of norms in finite dimensional spaces, the assumed estimates on $M$, $A$, and $P$ and standard manipulations depending on compact embeddings that there exists an estimate suitable to apply Theorem 68.3.3 to obtain the existence of a solution such that $(t, \omega) \to x(t, \omega)$ is measurable into $\mathbb{R}^k$ which implies that $(t, \omega) \to v_k(t, \omega)$ is product measurable into $V$ and $H$. This yields the measurable Galerkin approximation.

Also, the estimates and compact embedding results for Sobolev spaces imply an inequality of the form

$$|v_k(t)|^2_H + \int_0^T \|v_k\|_V^2 \, ds + \|u_k(t)\|_V^2 \leq C$$

where in fact $C$ does not depend on $\varepsilon, \omega$ or $k$. Everything would work if $C$ depended on $\omega$ but because of our simplifying assumptions, we can get a single $C$ as above.

Next we need to estimate the time derivative in $V'$. The integral equation implies that for all $w \in V_k$,

$$(v'_k(t), w)_{V', V} + \left( Mv_k + Au_k + Pu_k + \gamma^*_T F ((u_n - g(\omega))^+) \cdot \mu \left(\begin{vmatrix} v_k T - \dot{U}_T \end{vmatrix} \psi'_\varepsilon \left( v_k T - \dot{U}_T \right) \right), w \right)_{V', V}$$

$$= (f, w)$$

where the dependence on $t$ and $\omega$ is suppressed in most terms. In terms of inner products in $V$ this reduces to

$$(R^{-1}v'_k(t), w)_{V'} + \left( R^{-1} \left( Mv_k + Au_k + Pu_k + \gamma^*_T F ((u_n - g(\omega))^+) \cdot \mu \left(\begin{vmatrix} v_k T - \dot{U}_T \end{vmatrix} \psi'_\varepsilon \left( v_k T - \dot{U}_T \right) \right), w \right)_{V}$$

$$= (R^{-1}f, w)_V$$

In terms of $P_k$ the orthogonal projection in $V$ onto $V_k$, this takes the form

$$(R^{-1}v'_k(t), P_k w)_{V'} + \left( R^{-1} \left( Mv_k + Au_k + Pu_k + \gamma^*_T F ((u_n - g(\omega))^+) \cdot \mu \left(\begin{vmatrix} v_k T - \dot{U}_T \end{vmatrix} \psi'_\varepsilon \left( v_k T - \dot{U}_T \right) \right), P_k w \right)_{V}$$

$$= (R^{-1}f, P_k w)_V$$

for all $w \in V$. Now $v'_k(t) \in V_k$ and so the first term can be simplified and we can write

$$(R^{-1}v'_k(t), w)_{V'} + \left( R^{-1} \left( Mv_k + Au_k + Pu_k + \gamma^*_T F ((u_n - g(\omega))^+) \cdot \mu \left(\begin{vmatrix} v_k T - \dot{U}_T \end{vmatrix} \psi'_\varepsilon \left( v_k T - \dot{U}_T \right) \right), P_k w \right)_{V}$$

$$= (R^{-1}f, P_k w)_V$$
for all $w \in V$. Then it follows that for all $w \in V$,

$$(R^{-1}v_k'(t), w)_V +$$

$$\left( P_kR^{-1} \left( \begin{array}{c} Mv_k + Au_k + Pu_k + \gamma_T^*F \left( (u_n - g(\omega))_+ \right) \cdot \\ \mu \left( |v_{kT} - \hat{U}_T| \right) \psi_\varepsilon \left( v_{kT} - \hat{U}_T \right) \end{array} \right), w \right)_V$$

$$= (P_kR^{-1}f, w)_V$$

Thus in $V$ we have

$$R^{-1}v_k'(t) + P_kR^{-1} \left( \begin{array}{c} Mv_k + Au_k + Pu_k + \gamma_T^*F \left( (u_n - g(\omega))_+ \right) \cdot \\ \mu \left( |v_{kT} - \hat{U}_T| \right) \psi_\varepsilon \left( v_{kT} - \hat{U}_T \right) \end{array} \right) = P_kR^{-1}f$$

and $R^{-1}$ preserves norms while $P_k$ decreases them. Hence the estimate implies that $\|v_k'\|_V$ is also bounded independent of $\varepsilon, \omega$, and $k$. Then summarizing this yields

$$|v_k(t, \omega)|_H + \|v_k(\cdot, \omega)\|_V + \|v_k'(\cdot, \omega)\|_{V'} + \|u_k(t, \omega)\|_V \leq C(\omega) \quad (68.5.44)$$

where $C$ is some constant which does not depend on $\varepsilon, \omega$, and $k$. Also, integrating it follows that

$$i_k^* \left( v_k(t) - v_0 + \int_0^t Mv_k ds + \int_0^t Au_k ds + \int_0^t Pu_k ds + \int_0^t \gamma_T^*F \left( (u_{kn} - g(\omega))_+ \right) \mu \left( |v_{kT} - \hat{U}_T| \right) \psi_\varepsilon \left( v_{kT} - \hat{U}_T \right) ds \right) = i_k^*\int_0^t f ds \quad (68.5.45)$$

Where $i_k^*$ is the dual map to the inclusion map $i_k : V_k \rightarrow V$.

Let

$$V \subseteq W, \ V \text{ dense in } W,$$

where the embedding is compact and the trace map onto the boundary of $U$ is continuous. Using Theorem and it follows that for a fixed $\omega$, there exist the following convergences valid for a suitable subsequence, still denoted as $\{v_k\}$ which may depend on $\omega$.

$$v_k \rightarrow v \text{ in } V \quad (68.5.46)$$

$$v_k' \rightarrow v' \text{ in } V' \quad (68.5.47)$$

$$v_k \rightarrow v \text{ strongly in } C([0, T], W') \quad (68.5.48)$$

$$v_k \rightarrow v \text{ strongly in } L^2([0, T]; W) \quad (68.5.49)$$

$$v_k(t) \rightarrow v(t) \text{ in } W \text{ for a.e. } t \quad (68.5.50)$$

$$u_k \rightarrow u \text{ strongly in } C([0, T]; W) \quad (68.5.51)$$
68.5. A FRICTION CONTACT PROBLEM

\[ Au_k \to Au \text{ in } V' \]  
\[ Mv_k \to Mv \text{ in } V' \]  

(68.5.52)  
(68.5.53)

Now from these convergences and the density of \( \cup_n V_n \), it follows on passing to a limit and using dominated convergence theorem and the strong convergences above in the nonlinear terms, we obtain the following equation which holds in \( V' \).

\[ v(t) - v_0 + \int_0^t Mv ds + \int_0^t Au ds + \int_0^t Pu ds + \]
\[ \int_0^t \gamma_T^* F \left( \left( u_{n_k} - g(\omega) \right)_+ \right) \mu \left( \left| v_T - \bar{U}_T \right| \right) \psi_{1/k} \left( v_T - \bar{U}_T \right) ds = \int_0^t f ds \]  

(68.5.54)

Thus \( t \to v(t, \omega) \) is continuous into \( V' \). This along with the estimate \( \text{NS.5.1.2} \) implies that the conditions of Theorem \( \text{NS.5.1.3} \) are satisfied. It follows that there is a function \( \bar{v} \) which is product measurable into \( V' \) and weakly continuous in \( t \) and for each \( \omega \), a subsequence \( v_{k(\omega)} \) such that \( v_{k(\omega)}(\cdot, \omega) \to \bar{v}(\cdot, \omega) \) in \( V' \). Then by a repeat of the above argument, for each \( \omega \), there exists a further subsequence still denoted as \( v_{k(\omega)} \) which converges in \( V' \) to \( v(\cdot, \omega) \) which is a solution to the above integral equation which is continuous into \( V' \). Hence, \( \bar{v}(\cdot, \omega) = v(\cdot, \omega) \) and since these are both weakly continuous into \( V' \) they must be the same function. Hence, there is a product measurable solution \( v \).

Next we pass to a limit as \( \varepsilon \to 0 \). Denoting the product measurable solution to the above integral equation as \( \psi_{1/k} \), where \( \varepsilon = 1/k \). The estimate \( \text{NS.5.1.4} \) is obtained as before. Then we get a subsequence, still denoted as \( v_k \) which has the same convergences as in \( \text{NS.5.1.1} \) \( - \) \( \text{NS.5.1.3} \). Thus we obtain these convergences along with the fact that \( v_k \) is product measurable and for each \( \omega \), it is a solution of

\[ v_k(t) = v_0 + \int_0^t Mv_k ds + \int_0^t Au_k ds + \int_0^t Pu_k ds + \]
\[ \int_0^t \gamma_T^* F \left( \left( u_{kn} - g(\omega) \right)_+ \right) \mu \left( \left| v_{kT} - \bar{U}_T \right| \right) \psi_{1/k} \left( v_{kT} - \bar{U}_T \right) ds = \int_0^t f ds \]  

Now in addition to these convergences, we can also obtain

\[ \psi_{1/k} \left( v_{kT} - \bar{U}_T \right) \to \xi \text{ in } L^\infty \left( [0, T]; L^\infty (\Gamma_c)^3 \right) \]

(68.5.55)

We have also

\[ \psi_{1/k} \left( v_{kT} - \bar{U}_T \right) \cdot w_T \leq \psi_{1/k} \left( v_{kT} - \bar{U}_T + w_T \right) - \psi_{1/k} \left( v_{kT} - \bar{U}_T \right) \]

and so, passing to a limit, using the strong convergence of \( v_{kT} \) to \( v_T \) in \( L^2 ([0, T]; W) \), uniform convergence of \( \psi_{1/k} \) to \( \| \cdot \| \), and pointwise convergence in \( W \), we obtain using the dominated convergence theorem that for \( w \in V \),

\[ \int_0^t \int_{\Gamma_c} F \left( \left( u_{kn} - g(\omega) \right)_+ \right) \mu \left( \left| v_{kT} - \bar{U}_T \right| \right) \psi_{1/k} \left( v_{kT} - \bar{U}_T \right) \cdot w_T dx ds \]
\[ \to \int_0^t \int_{\Gamma_c} F \left( \left( u_n - g(\omega) \right)_+ \right) \mu \left( \left| v_T - \bar{U}_T \right| \right) \xi \cdot w_T dx ds \]
where

\[ \int_0^t \int_{\Gamma_C} \xi \cdot w_T \, d\alpha \, ds \leq \int_0^t \int_{\Gamma_C} \left| v_{kT} - \mathbf{U}_T + w_T \right| - \left| v_{kT} - \mathbf{U}_T \right| \, d\alpha \, ds \]  

(68.5.56)

Then passing to the limit in the integral equation, we obtain that \( v \) is a solution for each \( \omega \) to the integral equation

\[
v(t) - v_0 + \int_0^t Mv \, ds + \int_0^t Au \, ds + \int_0^t Pu \, ds + \int_0^t \gamma T F \left( (u_n - g(\omega))_+ \right) \mu \left( \left| v_T - \mathbf{U}_T \right| \right) \xi \, ds = \int_0^t f \, ds \]  

(68.5.57)

where \( \xi \) satisfies the inequality. In particular, \( v \) is continuous into \( V' \) and now, the conclusion of the measurable selection theorem applies and yields the existence of a measurable solution to the integral equation just displayed for each \( \omega \). Taking a weak derivative, it follows that we have obtained a measurable solution to the system

In this case of Lipschitz \( \mu \) one can show that the solution for each \( \omega \) to the above integral equation is unique although this it is not an obvious theorem. This follows standard procedures involving Gronwall’s inequality and estimates. Therefore, it is possible to obtain the measurability using more elementary methods. In addition, it becomes possible to include a stochastic integral of the form \( \int_0^t \Phi \, dW \). In this case one must consider a filtration and obtain solutions which are adapted to the filtration. In the next section we consider the case of discontinuous friction coefficient and in this case it is not clear whether there is uniqueness but we have still obtained a measurable solution.

### 68.5.3 Discontinuous coefficient of friction

In this section we consider the case where the coefficient of friction is a discontinuous function of the slip speed. This is the case described in elementary physics courses which state that the coefficient of sliding friction is less than the coefficient of static friction. Specifically, we assume the function \( \mu_s \), has a jump discontinuity at 0, becoming smaller when the speed is positive.
Fig. 2. The graph of $\mu$ vs. the slip rate $|v_\ast|$, and $\nu$.

We assume the function $\mu_s$ of the picture is Lipschitz continuous and decreasing just as shown. The new function $\nu$ is extended for $r < 0$ as shown and is just $\mu_s(r) + \eta$ for $r > 0$.

Let

$$h_\varepsilon (r) \equiv (\eta^2 r^2 + \varepsilon)^{1/2}$$

$$\mu_\varepsilon (r) = \nu (r) - h_\varepsilon (r)$$

Thus $\mu_\varepsilon$ is bounded, Lipschitz continuous and as $\varepsilon \to 0$, $\mu_\varepsilon (r) \to \mu (r)$ for $r > 0$.

Thus, for each $\varepsilon = 1/k$, there exists a measurable solution to the integral equation

$$v_k (t) - v_0 + \int_0^t M v_k ds + \int_0^t A u_k ds + \int_0^t P u_k ds + \int_0^t \gamma F ((u_{kn} - g (\omega))_+) \mu_{1/k} \left( |v_k T - \hat{U}_T| \right) \xi_k ds = \int_0^t f ds$$

(68.5.58)

where

$$\int_0^t \int_{\Gamma_C} \xi_k \cdot w_T d\alpha ds \leq \int_0^t \int_{\Gamma_C} |v_k T - \hat{U}_T + w_T| - |v_k T - \hat{U}_T| d\alpha ds$$

(68.5.59)

Now for a given $\omega$, the same estimate obtained earlier is available. Thus

$$|v_k (t)|^2_H + \int_0^T \|v_k\|^2_V ds + \|u_k (t)\|^2_V \leq C$$

where $C$ is not dependent on $k$. Recall also that $\xi_k$ is bounded. Hence from (68.5.42) - (68.5.53), this estimate, it also follows that $v_k'$ is bounded in $V'$. Thus

$$|v_k (t)|^2_H + \int_0^T \|v_k\|^2_V ds + \|u_k (t)\|^2_V + \|v_k'\|_{V'} \leq C$$

As earlier, we can take $C$ independent of $k$ and $\omega$ although we do not need this constant to be independent of $\omega$. Now for fixed $\omega$, there exists a subsequence, still denoted as $\{v_k\}$ such that the convergences obtained earlier all hold, that is (68.5.46) - (68.5.53). Taking a further subsequence, we may assume also that

$$\psi - h'_{1/k} \left( |v_k T - \hat{U}_T| \right) \to 0 \text{ in } L^\infty \left( [0, T], L^\infty (\Gamma_C) \right),$$

$$\xi_k \rightharpoonup \xi \text{ weak } \ast \text{ in } L^\infty \left( [0, T], L^\infty (\Gamma_C) \right).$$

That is, $h'_{1/k} \left( |v_{(1/k)T} - \hat{U}_T| \right)$ converges weak $\ast$ in $L^\infty \left( [0, T], L^\infty (\Gamma_C) \right)$ to some $\psi$. This is because

$$h'_{\varepsilon} (r) = \frac{\eta^2 r}{\sqrt{r^2 \eta^2 + \varepsilon}}$$
and this is bounded. Letting \( w \in L^1 ([0,T]; L^1 (\Gamma_C)) \),
\[
\int_0^T \int_{\Gamma_C} h'_{(1/k)} \left( |v_{kT} - \hat{U}_T| \right) w \, d\alpha \, dt \\
\leq \int_0^T \int_{\Gamma_C} h_{(1/k)} \left( |v_{kT} - \hat{U}_T| + w \right) - h_{(1/k)} \left( |v_{kT} - \hat{U}_T| \right) d\alpha \, dt
\]

Thanks to the strong convergences and the uniform convergence of \( h_{(1/k)} (r) \) to \(|\eta r|\),
\[
\int_0^T \int_{\Gamma_C} \psi \, d\alpha \, dt \leq \int_0^T \int_{\Gamma_C} \eta \left( |v_T - \hat{U}_T| + w \right) - \eta |v_T - \hat{U}_T| d\alpha \, dt
\]

Therefore, for \( a.e. t, \psi (t,x,\omega) \) is in the subgradient of the function \( \phi \eta (r) = |\eta r| \) for \( a.e. x \in \Gamma_C \) at the point \( v_{kT} - \hat{U}_T \).

In particular, \( \psi = \eta \) and \( \nu \left( |v_{kT} - \hat{U}_T| \right) - \psi \) reduces to \( \mu \left( |v_{kT} - \hat{U}_T| \right) \).

Thus
\[
\left( |v_{kT} - \hat{U}_T| , \nu \left( |v_{kT} - \hat{U}_T| \right) - \psi \right)
\]
is in the graph of \( \mu \) a.e. Similar reasoning based on strong convergence and \( 68.5.59 \) implies that for \( a.e.t, \xi \in \partial \gamma \) where \( \gamma (y) = |y| \) at the point \( v_{kT} - \hat{U}_T \) for \( a.e. x \in \Gamma_C \).

Consider the friction terms in \( 68.5.60 \). Letting \( w \in V \) and recalling that \( \mu_{(1/k)} (r) = \nu (r) - h'_{(1/k)} (r) \),
\[
\int_0^T \int_{\Gamma_C} F \left( (u_n - g)_+ \right) \mu_{(1/k)} \left( |v_{kT} - \hat{U}_T| \right) \xi_k \cdot w \, d\alpha \, dt \\
= \int_0^T \int_{\Gamma_C} F \left( (u_n - g)_+ \right) \left( \nu \left( |v_{kT} - \hat{U}_T| \right) - h'_{(1/k)} \left( |v_{kT} - \hat{U}_T| \right) \right) \xi_k \cdot w \, d\alpha \, dt \\
= \int_0^T \int_{\Gamma_C} F \left( (u_n - g)_+ \right) \left( \nu \left( |v_{kT} - \hat{U}_T| \right) - \psi \right) \xi_k \cdot w \, d\alpha \, dt \\
+ \int_0^T \int_{\Gamma_C} F \left( (u_n - g)_+ \right) \left( \psi - h'_{(1/k)} \left( |v_{kT} - \hat{U}_T| \right) \right) \xi_k \cdot w \, d\alpha \, dt
\]

Now consider the first integral. The strong convergence yields that this integral in \( 68.5.60 \) converges to
\[
\int_0^T \int_{\Gamma_C} F \left( (u_n - g)_+ \right) \left( \nu \left( |v_T - \hat{U}_T| \right) - \psi \right) \xi \cdot w \, d\alpha \, dt
\]

where \( \nu \left( |v_T - \hat{U}_T| \right) - \psi \) is in the graph of \( \mu \) a.e.
Consider the second integral in (68.5.61)

$$\int_0^T \int_{\Gamma_C} F \left( u_{kn} - g \right)_+ \left( \psi - h'(1/k) \left( |v_{kT} - \dot{\mathbf{U}}_T| \right) \right) \xi_k \cdot \mathbf{w}_T \, d\alpha \, dt$$

$$\leq \int_0^T \int_{\Gamma_C} F \left( u_{kn} - g \right)_+ \left( \psi - h'(1/k) \left( |v_{kT} - \dot{\mathbf{U}}_T| \right) \right) \cdot \left( |v_{kT} - \dot{\mathbf{U}}_T + \mathbf{w}_T| - |v_{kT} - \dot{\mathbf{U}}_T| \right) \, d\alpha \, dt$$

Similarly,

$$- \int_0^T \int_{\Gamma_C} F \left( u_{kn} - g \right)_+ \left( \psi - h'(1/k) \left( |v_{kT} - \dot{\mathbf{U}}_T| \right) \right) \xi_k \cdot \mathbf{w}_T \, d\alpha \, dt$$

$$\leq \int_0^T \int_{\Gamma_C} F \left( u_{kn} - g \right)_+ \left( \psi - h'(1/k) \left( |v_{kT} - \dot{\mathbf{U}}_T| \right) \right) \cdot \left( |v_{kT} - \dot{\mathbf{U}}_T - \mathbf{w}_T| - |v_{kT} - \dot{\mathbf{U}}_T| \right) \, d\alpha \, dt$$

Each of these integrals on the right side converge to 0 because, from the strong convergence results,

$$F \left( u_{kn} - g \right)_+ \left( |v_{kT} - \dot{\mathbf{U}}_T| \pm \mathbf{w}_T \right)$$

converges in $L^1([0,T], L^1(\Gamma_C))$ and so the weak $\ast$ convergence to 0 of

$$\psi - h'(1/k) \left( |v_{kT} - \dot{\mathbf{U}}_T| \right)$$

implies that these integrals converge to 0. Thus the integral in (68.5.61)

$$\int_0^T \int_{\Gamma_C} F \left( u_{kn} - g \right)_+ \left( \psi - h'(1/k) \left( |v_{kT} - \dot{\mathbf{U}}_T| \right) \right) \xi_k \cdot \mathbf{w}_T \, d\alpha \, dt$$

is between two sequences each of which converges to 0 so it also converges to 0.

To save space, denote by

$$\hat{\mu} = \nu \left( |v_T - \dot{\mathbf{U}}_T| \right) - \psi$$

Then passing to the limit in this subsequence, we obtain for fixed $\omega$ the existence of a solution to the following integral equation.

$$\mathbf{v}(t) - \mathbf{v}_0 + \int_0^t \mathbf{M} \mathbf{v} \, ds + \int_0^t \mathbf{A} \mathbf{u} \, ds + \int_0^t \mathbf{P} \mathbf{u} \, ds + \int_0^t \gamma^*_T F \left( (u_n - g)_+ \right) \hat{\mu} \xi \, ds = \int_0^t f \, ds$$

(68.5.62)
where 
\[ u(t) = u_0 + \int_0^t v(s) \, ds \quad (68.5.63) \]
and \( \left( |v_k - \dot{U}_T|, \dot{\mu} \right) \) is contained in the graph of \( \mu \) a.e. Also for each \( w \in V \),
\[ \int_0^T \int_{\Gamma_C} \xi \cdot w \, d\alpha \, ds \leq \int_0^T \int_{\Gamma_C} \left| v_k T - \dot{U}_T + w \right| - \left| v_k T - \dot{U}_T \right| \, d\alpha \, ds \quad (68.5.64) \]

The remaining issue concerns the existence of a measurable solution. However, this follows in the same way as before from the measurable selection theorem, Theorem 68.2.1. From the above reasoning, for fixed \( \omega \) any sequence has a subsequence which leads to a solution to the integral equation 68.5.62 - 68.5.64 which is continuous into \( V' \). There is also an estimate of the right sort for all of the \( v_k \). Therefore, from this theorem, there is a function \( v(t, \omega) \) in \( V' \) which is weakly continuous into \( V' \) and a sequence \( v_{k(\omega)}(\cdot, \omega) \) converging to \( v(\cdot, \omega) \). Then from the above argument, a subsequence converges to a solution to the integral equation and since both are weakly continuous into \( V' \), it follows that the solution to the integral equation equals this measurable function for all \( t \), this for each \( \omega \). Thus there is a measurable solution to the stochastic friction problem. The result is stated in the following theorem.

**Theorem 68.5.3** For each \( \omega \) let \( u_0(\omega) \in V, v_0(\omega) \in H \). Let \( f \in V' \). Also assume the gap \( g \) and sliding velocity \( \dot{U}_T \) are \( F \) measurable. Then there exists a solution \( v \), to the problem summarized in 68.5.62 - 68.5.64 for each \( \omega \). This solution \( (t, \omega) \rightarrow v(t, \omega) \) is measurable into \( V', H' \) and \( V' \).

It only remains to check the last claim about measurability into the other spaces. By density of \( V \) into \( H \), it follows that \( H' \) is dense in \( V' \) and so a simple Pettis theorem argument implies right away that \( \omega \rightarrow v(t, \omega) \) is \( F \) measurable into both \( V \) and \( H \).
Chapter 69

Stochastic O.D.E. One Space

69.1 Adapted Solutions With Uniqueness

Instead of a single $\sigma$ algebra $\mathcal{F}$, one can generalize to the case of a normal filtration $\mathcal{F}_t$ and obtain adapted solutions to finite dimensional theorems, provided one also knows path uniqueness of the solutions. Recall that a filtration is normal includes the following condition which is what we will use.

$$\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s \quad (69.1.1)$$

**Theorem 69.1.1** Suppose $N(t,u,v,w,\omega) \in \mathbb{R}^d$ for $u,v,w \in \mathbb{R}^d, t \in [0,T]$ and $(t,u,v,w,\omega) \rightarrow N(t,u,v,w,\omega)$ is progressively measurable with respect to a normal filtration or more generally one which satisfies (69.1.1). Also suppose $(t,u,v,w) \rightarrow N(t,u,v,w,\omega)$ is continuous. Suppose for each $\omega$, there exists an estimate for any solution $u(\cdot,\omega)$ to the integral equation

$$u(t,\omega) - u_0(\omega) + \int_0^t N(s,u(s,\omega),u(s-h,\omega),w(s,\omega),\omega) \, ds = \int_0^t f(s,\omega) \, ds,$$

which is of the form

$$\sup_{t \in [0,T]} |u(t,\omega)| \leq C(\omega) < \infty \quad (69.1.2)$$

Also let $f$ be progressively measurable and $f(\cdot,\omega) \in L^1([0,T];\mathbb{R}^d)$. Here $u_0$ has values in $\mathbb{R}^d$ and is $\mathcal{F}_0$ measurable and $u(s-h,\omega) \equiv u_0(\omega)$ whenever $s-h \leq 0$ and

$$w(t,\omega) \equiv w_0(\omega) + \int_0^t u(s,\omega) \, ds$$

where $w_0$ is a given $\mathcal{F}_0$ measurable function. Also assume that for each $\omega$ there is at most one solution to the integral equation (69.1.2). Then for $h > 0$, there exists a progressively measurable solution $u$ to the integral equation (69.1.2).
**Proof:** Let \(0 = t_0 < t_1 < \cdots < t_n = T\). From Theorem 68.3.3, there exists a solution to the integral equation \(u\) which has the property that \(u(t \wedge t_j)\) is \(\mathcal{F}_{t_j}\) measurable. One simply applies this theorem to the succession of intervals determined by the given partition. Now suppose \(\mathcal{P}^n\) consists of the points \(k2^{-n}T = t^n_j\) so that these satisfy \(\mathcal{P}^n \subseteq \mathcal{P}^{n+1}\) and the lengths of the sub-intervals decreases to 0 with increasing \(n\). Let \(u_n\) denote the solution just described corresponding to \(\mathcal{P}_n\) such that \(u_n(t \wedge t^n_j)\) is \(\mathcal{F}_{t^n_j}\) measurable. As before, using the estimate, these \(u_n(\cdot, \omega)\) for a fixed \(\omega\) are uniformly bounded and equicontinuous. This is because it is a solution to the integral equation for each \(\omega\) and so by assumption, there is an estimate. Therefore, for fixed \(\omega\), there exists \(u(\cdot, \omega)\) and a subsequence, denoted as \(u_n(\cdot, \omega)\) which converges uniformly to \(u(\cdot, \omega)\) on \([0, T]\). Therefore, \(u(\cdot, \omega)\) will be a solution to the integral equation for that \(\omega\). It follows from the uniqueness assumption, that it is not necessary to take a subsequence. Thus

\[
u(t, \omega) = \lim_{n \to \infty} u_n(t, \omega)
\]

For \(t \in (t^n_{j-1}, t^n_j]\), it follows that \(\omega \to u(t, \omega)\) is \(\mathcal{F}_{t^n_j}\) measurable. Since this is true for each \(n\) and the filtration is assumed to be a normal filtration, we conclude that \(\omega \to u(t, \omega)\) is \(\mathcal{F}_t\) measurable. ■

Why can’t this be generalized to the situation where no uniqueness is known? We have been unable to do this. It appears that the difficulty is related to the need to use theorems about measurable selections and these theorems pertain to a single \(\sigma\) algebra. Attempts to use the \(\sigma\)-algebra of progressively measurable sets have not been successful either.

### 69.2 Including Stochastic Integrals

It is not surprising that Theorem 69.1.1 is sufficient to allow the inclusion of a stochastic integral. Thus, with the same descriptions of the symbols used in that theorem, one could consider the following integral equation.

\[
u(t, \omega) - \nu_0(\omega) + \int_0^t N(s, \nu(s, \omega), \nu(s-h, \omega), w(s, \omega), \omega) ds
\]

\[= \int_0^t f(s, \omega) ds + \int_0^t \Phi dW\]

where, as usual \(\Phi \in L^2([0, T] \times \Omega; L_2(Q^{1/2}U, \mathbb{R}^d))\) where \(U\) is a Hilbert space. It could be \(\mathbb{R}^d\) of course. To include a stochastic integral, you define a new variable.

\[
u(t) = \nu(t) - \int_0^t \Phi dW
\]

Then in terms of this new variable, the integral equation is

\[
u(t, \omega) - \nu_0(\omega) + \int_0^t N(s, \nu(s, \omega) + \int_0^s \Phi dW, \nu(s-h, \omega) + \int_0^{s-h} \Phi dW, \omega) ds
\]
69.2. INCLUDING STOCHASTIC INTEGRALS

\[
\int_0^t \left( \hat{u}(r) + \int_0^r \Phi dW \right) dr, \omega \right) ds = \int_0^t f(s, \omega) ds
\]

This is in the situation of Theorem 69.1 provided \( N \) is progressively measurable with respect to the normal filtration \( \mathcal{F}_t \) determined by the Wiener process and there exists an estimate of the sort in this theorem and for a given \( \omega \) there is at most one solution \( t \to \hat{u}(t, \omega) \) to the above integral equation.

**Theorem 69.2.1** Suppose \( N(t, u, v, w, \omega) \in \mathbb{R}^d \) for \( u, v, w \in \mathbb{R}^d, t \in [0, T] \) and \( (t, u, v, w, \omega) \to N(t, u, v, w, \omega) \) is progressively measurable with respect to the normal filtration \( \mathcal{F}_t \) determined by a given Wiener process \( W(t) \). Also suppose

\[
(t, u, v, w) \to N(t, u, v, w, \omega)
\]

is continuous and satisfies the following conditions for \( C(\cdot, \omega) \geq 0 \) in \( L^1([0, T]) \) and some \( \mu > 0 \):

\[
(N(t, u, v, w, \omega), u) \geq -C(t, \omega) - \mu \left( |u|^2 + |v|^2 + |w|^2 \right).
\]

(69.2.3)

Also let \( f \) be progressively measurable and \( f(\cdot, \omega) \in L^2([0, T] ; \mathbb{R}^d) \). Let

\[
\Phi \in L^2 \left( [0, T] \times \Omega ; \mathcal{L}_2 \left( Q^{1/2}U \right) \right)
\]

where \( U \) is some Hilbert space, \( \mathbb{R}^d \), for example. Also suppose path uniqueness.

That is, for each \( \omega \), there is at most one solution to the integral equation

\[
u(t, \omega) - u_0(\omega) + \int_0^t N(s, u(s, \omega), u(s - h, \omega), w(s, \omega), \omega) ds
\]

\[
= \int_0^t f(s, \omega) ds + \int_0^t \Phi dW,
\]

(69.2.4)

Then for \( h > 0 \), there exists a unique progressively measurable solution \( u \) to the integral equation [69.2.4] where \( u_0 \) has values in \( \mathbb{R}^d \) and is \( \mathcal{F}_0 \) measurable. Here

\[
u(s - h, \omega) \equiv u_0(\omega) \quad \text{for all } s - h \leq 0 \quad \text{and for } w_0 \text{ a given } \mathcal{F}_0 \text{ measurable function,}
\]

\[
u(t, \omega) \equiv w_0(\omega) + \int_0^t u(s, \omega) ds
\]

**Proof:** The only thing left is to observe that the given estimate is sufficient to obtain an estimate for the solutions to the integral equation for \( \hat{u} \) defined above.

Then from Theorem 69.1 there exists a unique progressively measurable solution for \( \hat{u} \) and hence for \( u \).

Note that the integral equation holds for all \( t \) for each \( \omega \). There is no exceptional set of measure zero which might depend on the initial condition needed.

What is a sufficient condition for path uniqueness? Suppose the following weak monotonicity condition for \( \mu = \mu(\omega) \).

\[
(N(t, u_1, v_1, w_1, \omega) - N(t, u_2, v_2, w_2, \omega), u_1 - u_2)
\]

\[
\geq -\mu \left( |u_1 - u_2|^2 + |v_1 - v_2|^2 + |w_1 - w_2|^2 \right)
\]

(69.2.5)
Then path uniqueness will hold. This follows from subtracting the two integral equations, one for \( u_1 \) and one for \( u_2 \), using the estimate and then applying Gronwall’s inequality.

Recall the Ito formula

\[
 u(t) - u_0 + \int_0^t N ds = \int_0^t f ds + \int_0^t \Phi dW
\]

where \( u(t) \in H \) a Hilbert space. Consider \( F(u) = \frac{1}{2} \| u \|^2 \). Also let \( R \) denote the Riesz map from \( H \to H' \) such that \( \langle Rx, y \rangle \equiv (x, y)_H \). Then proceeding formally, to see what the Ito formula says,

\[
dF = DF(u) du + \frac{1}{2} D^2 F(u) (du, du) + O(du^3)
\]

Recall then that

\[
du = -N dt + f dt + \Phi dW
\]

and so recalling \( (dW, dW) = dt \),

\[
 R(u)(-N dt + f dt + \Phi dW) + \frac{1}{2} \| \Phi \|^2 dt
\]

Hence

\[
\frac{1}{2} |u(t)|_H^2 - \frac{1}{2} |u_0|^2 + \int_0^t (N, u)_H ds - \frac{1}{2} \int_0^t \| \Phi \|^2 \| \| ds = \int_0^t (f, u) ds + \int_0^t R(u)(\Phi) dW
\]

This is all that is of importance in what follows. Therefore, this martingale may be simply denoted as \( M(t) \) in what follows.

Under the assumption \( 69.2.5 \) you can include instead of the term \( \int_0^t \Phi dW \), the more general term \( \int_0^t \sigma(s, u, \omega) dW \). This will be shown by doing the argument and indicating what extra assumptions are needed as this is done. Let \( z \) be progressively measurable and in \( L^2(\Omega; C([0, T]; \mathbb{R}^n)) \). Also assume that \( \sigma \) has linear growth. That is

\[
\| \sigma (s, u, \omega) \|_{L^2} \leq a + b \| u \|_{\mathbb{R}^n}
\]

Then from the above theorem, there exists a unique progressively measurable solution \( u \) to

\[
u(t, \omega) - u_0(\omega) + \int_0^t N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) ds
\]

\[
= \int_0^t f(s, \omega) ds + \int_0^t \sigma(s, z) dW;
\]

This holds for all \( \omega \). There is no exceptional set needed. Now assume

\[
u_0 \in L^2(\Omega)
\]
and also a Lipschitz condition

\[ \| \sigma (s, u, \omega) - \sigma (s, \tilde{u}, \omega) \|_{L^2} \leq K |u - \tilde{u}| \]  

(69.2.9)

Then let \( u \) coincide with \( z \) and \( \tilde{u} \) come from \( \tilde{z} \). Then applying the Ito formula, one can obtain the following for a constant \( C \) which does not depend on \( u, \tilde{u} \).

\[
\frac{1}{2} |u(t) - \tilde{u}(t)|^2 - C \int_0^t |u(s) - \tilde{u}(s)|^2 ds - K \int_0^t |u(s) - \tilde{u}(s)|^2 ds = M(t)
\]

where \( M(t) \) is a local martingale whose quadratic variation satisfies

\[
[M](t) = \int_0^t \| \sigma (s, z, \omega) - \sigma (s, \tilde{z}, \omega) \|^2_{L^2} |u - \tilde{u}|^2 ds
\]

Thus, simplifying the constants,

\[
\sup_{s \in [0, t]} |u(s) - \tilde{u}(s)|^2 \leq C \int_0^t |u(s) - \tilde{u}(s)|^2 ds + M^*(t)
\]

where \( M^*(t) = \sup_{s \in [0, t]} |M(s)| \). Then by Gronwall’s inequality,

\[
\sup_{s \in [0, t]} |u(s) - \tilde{u}(s)|^2 \leq CM^*(t)
\]

Then take the expectation of both sides. Using the Burkholder Davis Gundy inequality,

\[
E \left( \sup_{s \in [0, t]} |u(s) - \tilde{u}(s)|^2 \right) \leq CE \left( \left( \int_0^t K |z - \tilde{z}|^2 |u - \tilde{u}|^2 ds \right)^{1/2} \right)
\]

Then adjusting the constant again,

\[
\leq \frac{1}{2} E \left( \sup_{s \in [0, t]} |u(s) - \tilde{u}(s)|^2 \right) + CE \left( \int_0^t K |z - \tilde{z}|^2 ds \right)
\]

and so,

\[
E \left( \sup_{s \in [0, t]} |u(s) - \tilde{u}(s)|^2 \right) \leq C \int_0^t E \left( \sup_{r \in [0, s]} |z(r) - \tilde{z}(r)|^2 \right) ds
\]

Letting \( Tz = u \) where \( u \) is defined from \( z \) in the integral equation 69.2.7, the above inequality implies that

\[
E \left( \sup_{s \in [0, t]} |T^n z_1 (s) - T^n z_2 (s)|^2 \right) \leq C \int_0^t E \left( \sup_{r \in [0, s]} |T^{n-1} z_1 (r) - T^{n-1} z_2 (r)|^2 \right) ds
\]
\[
\leq C^2 \int_0^t \int_0^s E \left( \sup_{r_1 \in [0,r]} |T^{n-2} z_1 (r_1) - T^{n-2} z_2 (r_1)|^2 \right) \, dr \, ds
\]

One can iterate this, eventually finding that

\[
E \left( \sup_{s \in [0,t]} |T^n z_1 (s) - T^n z_2 (s)|^2_{H^n} \right)
\leq C^n \int_0^t \int_0^1 \cdots \int_0^{s_{n-1}} \cdots \, dt \, dE \left( \sup_{s \in [0,t]} |z_1 (s) - z_2 (s)|^2_{H^n} \right)
= \frac{C^n T^n}{(n!)} E \left( \sup_{s \in [0,t]} |z_1 (s) - z_2 (s)|^2_{H^n} \right)
\]

In particular, this holds for \( t = T \) and so, letting \( z \in L^2 (\Omega, C ([0,T], \mathbb{R}^n)) \), \( \{T^n z\} \) is a Cauchy sequence in this space because a high enough power is a contraction map, so it converges to a unique fixed point \( u \). Each \( T^n z \) is progressively measurable and so the fixed point is also. In \( L^2 (\Omega, C ([0,T], \mathbb{R}^n)) \), you get the integral equation

\[
u \ (t, \omega) - \nu_0 (\omega) + \int_0^t \nu (s, \nu (s, \omega), \nu (s - h, \omega), \nu (s, \omega), \omega) \, ds
= \int_0^t f (s, \omega) \, ds + \int_0^t \sigma (s, \nu) \, dW,
\]  

(69.2.10)

Thus off a set of measure zero, the equation holds for all \( t \) and \( \nu \) is progressively measurable. The place where \( \nu_0 \in L^2 (\Omega) \) is needed is in having \( T \nu \in L^2 (\Omega, C ([0,T]; \mathbb{R}^n)) \). One uses a similar procedure involving the Ito formula, the growth condition

\[
\langle \nu (t, \nu, v, w, \omega), \nu \rangle \geq -C (t, \omega) - \mu (|\nu|^2 + |v|^2 + |w|^2) .
\]

and the Burkholder Davis Gundy inequality to verify this. Since \( T \) depends on \( \nu_0 \), it appears that the set of measure zero, off which the integral equation holds, will also depend on \( \nu_0 \). It appears that this ultimately results from the need to take an expectation in order to deal with the stochastic integral. If this integral could be generalized in such a way that it made sense for each \( \omega \) as in the usual Riemann Stieltjes integral, then likely this restriction could be removed. It is a problem because the Wiener process is not of bounded variation.

**Theorem 69.2.2** Suppose the weak monotonicity condition and the growth estimate. Also assume \( \nu (t, \nu, v, w, \omega) \in \mathbb{R}^d \) for \( \nu, v, w \in \mathbb{R}^d, t \in [0,T] \) and \( (t, \nu, v, w, \omega) \rightarrow \nu (t, \nu, v, w, \omega) \) is progressively measurable with respect to the normal filtration \( \mathcal{F}_t \) determined by a given Wiener process \( W (t) \). Also suppose \( (t, \nu, v, w) \rightarrow \nu (t, \nu, v, w, \omega) \) is continuous. Let \( f \in L^2 (\Omega, C ([0,T], \mathbb{R}^n)) \) and
69.3. **Stochastic Differential Equations In A Hilbert Space**

$(t, u, \omega) \rightarrow \sigma(t, u, \omega)$ is progressively measurable and satisfies the linear growth condition \( \| \sigma(t, u, \omega) \| \leq a \) and the Lipschitz condition \( \| \sigma(t, u, \omega) - \sigma(t, \hat{u}, \omega) \| \leq b \langle u, \hat{u} \rangle_H \). Also suppose $u_0$ is $F_0$ measurable and in $L^2(\Omega, \mathbb{R}^n)$. Then there exists a progressively measurable solution $u$ to \( \mathcal{G}_1 \). If $\hat{u}$ is another such solution, then there is a set of measure zero $N$ such that for $\omega \notin N$, $\hat{u}(t) = u(t)$ for all $t$.

**Proof:** It only remains to verify the uniqueness assertion. This happens because the fixed point is unique in $L^2(\Omega, C([0, T], \mathbb{R}^n))$. Therefore, off a set of measure zero the two solutions are equal for all $t$. ■

### 69.3 Stochastic Differential Equations In A Hilbert Space

In this section, ordinary differential equations in Hilbert space which are of the form

$$ du + N(u) dt = f dt + \sigma(u) dW $$

are considered under Lipschitz assumptions on $N$ and $\sigma$. A very satisfactory theorem can be proved.

The assumptions made are as follows.

\[ \| \sigma(t, u, \omega) - \sigma(t, \hat{u}, \omega) \|_{\mathcal{L}_2(Q^{1/2}; H)} \leq K \| u - \hat{u} \|_H, \]  \hspace{1cm} (69.3.11)

\[ |N(t, u_1, v_1, w_1, \omega) - N(t, u_2, v_2, w_2, \omega)| \leq K (|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) \]  \hspace{1cm} (69.3.12)

where the norms $|\cdot|$ refer here to the Hilbert space $H$. Assume $N, \sigma$ are both progressively measurable. From the Lipschitz condition given above,

\[ |N(t, u, v, w, \omega) - N(t, 0, 0, 0, \omega)| \leq K (|u| + |v| + |w|) \]

and it is assumed that

$$ t \rightarrow N(t, 0, 0, 0, \omega) $$  \hspace{1cm} (69.3.13)

is in $L^2(\Omega, C([0, T]; H))$. Also consider the growth condition which is implied by the above condition and the Lipschitz assumption.

\[ (N(t, u, v, w, \omega), u) \geq - C(t, \omega) - \mu \left( |u|^2 + |v|^2 + |w|^2 \right) \] \hspace{1cm} (69.3.14)

where $C \in L^1([0, T] \times \Omega)$ and the linear growth condition for $\sigma$,

\[ \| \sigma(t, u, \omega) \| \leq a + b \| u \|_H \] \hspace{1cm} (69.3.15)
69.3.1 The Lipschitz Case

Theorem 69.3.1 Suppose \( w(t) = w_0 + \int_0^t u(s) \, ds, \ w_0 \in L^2(\Omega), \ w_0 \) is \( \mathcal{F}_0 \) measurable.

Then there exists a unique progressively measurable solution \( u \) to the integral equation
\[
\begin{align*}
    u(t, \omega) - u_0(\omega) + \int_0^t N(s, u(s, \omega), u(s - h, \omega), w(s, \omega), \omega) \, ds &= \int_0^t f(s, \omega) \, ds + \int_0^t \sigma(s, u, \omega) \, dW.
\end{align*}
\]

where \( u \in L^2(\Omega; C([0, T]; H)), \ u_0 \in L^2(\Omega), \ u_0 \) is \( \mathcal{F}_0 \) measurable, \( f \) is progressively measurable and in \( L^2([0, T] \times \Omega; H) \). Here there is a set of measure zero such that if \( \omega \) is not in this set, then \( u(\cdot, \omega) \) solves the above integral equation and furthermore, if \( \hat{u}(\cdot, \omega) \) is another solution to it, then \( u(t, \omega) = \hat{u}(t, \omega) \) for all \( t \) if \( \omega \) is off some set of measure zero.

Proof: Let \( v \in L^2(\Omega; C([0, T]; H)) \) where \( v \) is also progressively measurable. Then let \( u \) be given by
\[
\begin{align*}
    u(t, \omega) - u_0(\omega) + \int_0^t N(s, v(s, \omega), v(s - h, \omega), w(s, \omega), \omega) \, ds &= \int_0^t f(s, \omega) \, ds + \int_0^t \sigma(s, v, \omega) \, dW.
\end{align*}
\]

The Lipschitz condition, the assumption \( \mathcal{F}_0 \) measurable, and the linear growth assertion imply that \( u \) is also in \( L^2(\Omega; C([0, T]; H)) \). The proof of this involves the same arguments about to be given in order to show that this determines a mapping which has a sufficiently high power a contraction map. They are also the same arguments to be used in the following theorem to establish estimates which imply a stopping time is eventually infinity.

Let \( v_1, v_2 \) be two given functions of this sort and let the corresponding \( u \) be denoted by \( u_1, u_2 \) respectively. Then
\[
\begin{align*}
    u_1(t) - u_2(t) + \int_0^t N(s, v_1(s), v_1(s - h), w_1(s)) - N(s, v_2(s), v_2(s - h), w_2(s)) \, ds &= \int_0^t \sigma(s, v_1, \omega) - \sigma(s, v_2, \omega) \, dW
\end{align*}
\]

Use the Itô formula and the Lipschitz condition on \( N \) to obtain an expression of the form
\[
\frac{1}{2} \left| u_1(t) - u_2(t) \right|^2 - C \int_0^t |v_1 - v_2|^2 \, ds - C \int_0^t |u_1 - u_2|^2 \, ds
\]
69.3. **STOCHASTIC DIFFERENTIAL EQUATIONS IN A HILBERT SPACE**

\[
- \frac{1}{2} \int_0^t \| \sigma(s, v_1, \omega) - \sigma(s, v_2, \omega) \|^2 ds \leq |M(t)|
\]

where \( M(t) \) is a martingale whose quadratic variation is dominated by

\[
C \int_0^t \| \sigma(s, v_1, \omega) - \sigma(s, v_2, \omega) \|^2 |u_1 - u_2|^2 ds
\]

Therefore, using the Lipschitz condition on \( \sigma \) and the Burkholder-Davis-Gundy inequality, the above implies

\[
E \left( \sup_{s \in [0,t]} |u_1(s) - u_2(s)|^2 \right) \leq C E \int_0^t \sup_{r \in [0,s]} |u_1(r) - u_2(r)|^2 ds
\]

\[
+ CE \int_0^t \sup_{r \in [0,s]} |v_1(r) - v_2(r)|^2 ds
\]

\[
+ CE \left( \left( \int_0^t \| \sigma(s, v_1, \omega) - \sigma(s, v_2, \omega) \|^2 |u_1 - u_2|^2 ds \right)^{1/2} \right)
\]

Then a use of Gronwall’s inequality allows this to be simplified to an expression of the form

\[
E \left( \sup_{s \in [0,t]} |u_1(s) - u_2(s)|^2 \right) \leq C E \int_0^t \sup_{r \in [0,s]} |v_1(r) - v_2(r)|^2 ds
\]

\[
+ CE \left( \left( \int_0^t \| \sigma(s, v_1, \omega) - \sigma(s, v_2, \omega) \|^2 |u_1 - u_2|^2 ds \right)^{1/2} \right)
\]

\[
\leq C \int_0^t E \left( \sup_{r \in [0,s]} |v_1(r) - v_2(r)|^2 \right) ds + \frac{1}{2} E \left( \sup_{s \in [0,t]} |u_1(s) - u_2(s)|^2 \right)
\]

\[
+ CE \left( \int_0^t \| \sigma(s, v_1, \omega) - \sigma(s, v_2, \omega) \|^2 ds \right)
\]

Now using the Lipschitz condition on \( \sigma \), this simplifies further to give an inequality of the form

\[
E \left( \sup_{s \in [0,t]} |u_1(s) - u_2(s)|^2 \right) \leq C \int_0^t E \left( \sup_{r \in [0,s]} |v_1(r) - v_2(r)|^2 \right) ds
\]

Letting \( T v = u \) where \( u \) is defined from \( v \) in the integral equation 69.3.17, the above inequality implies that

\[
E \left( \sup_{s \in [0,t]} |T^n v_1(s) - T^n v_2(s)|^2 \right) \leq C \int_0^t E \left( \sup_{r \in [0,s]} |T^{n-1} v_1(r) - T^{n-1} v_2(r)|^2 \right) ds
\]
\[ \leq C^2 \int_0^t \int_0^s E \left( \sup_{r_1 \in [0,r]} |T^{r_2}v_1(r_1) - T^{r_2}v_2(r_1)|^2 \right) dr ds \]

One can iterate this, eventually finding that

\[ E \left( \sup_{s \in [0,t]} |T^n v_1(s) - T^n v_2(s)|_H^2 \right) \]

\[ \leq C^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_{n-1} \cdots dt E \left( \sup_{s \in [0,t]} |v_1(s) - v_2(s)|_H^2 \right) \]

\[ = \frac{C^n T^n}{(n!)} E \left( \sup_{s \in [0,t]} |v_1(s) - v_2(s)|_H^2 \right) \]

In particular, one could take \( t = T \). This shows that for all \( n \) large enough, \( T^n \) is a contraction map on \( L^2(\Omega; C([0,T]; H)) \). Therefore, picking \( v \in L^2(\Omega; C([0,T]; H)) \), such that \( v \) is also progressively measurable, \( \{T^n v\}_{n=1}^{\infty} \) converges in \( L^2(\Omega; C([0,T]; H)) \) to the unique fixed point of \( T \) denoted as \( u \). Thus \( Tu = u \) in \( L^2(\Omega; C([0,T]; H)) \).

That is,

\[ \int \sup_t |Tu - u|^2 dP = 0 \]

It follows that there is a set of measure zero such that for \( \omega \) not in this set,

\[ u(t,\omega) - u_0(\omega) + \int_0^t N(s,u(s,\omega),u(s-h,\omega),w(s,\omega),\omega) ds \]

\[ = \int_0^t f(s,\omega) ds + \int_0^t \sigma(s,u,\omega) dW. \quad (69.3.18) \]

The function \( u \) is progressively measurable because each \( T^n v \) is progressively measurable and there exists a subsequence still indexed with \( n \) such that for \( \omega \) off a set of measure zero, \( T^n \cdot (\cdot,\omega) \to u(\cdot,\omega) \) in \( C([0,T]; H) \).

Note that the fixed point of \( T \) is unique in the space \( L^2(\Omega; C([0,T]; H)) \) and so any solution to the integral equation in this space must equal this one. Hence, there exists a set of measure zero such that for \( \omega \) off this set, the two solutions are equal for all \( t \). \[ \blacksquare \]

### 69.3.2 The Locally Lipschitz Case

Now replace the Lipschitz assumption with the locally Lipschitz assumption which says that if \( \max(|u|,|v|,|w|) < R \), then there is a constant \( K_R \) such that

\[ |N(t,u_1,v_1,w_1,\omega) - N(t,u_2,v_2,w_2,\omega)| \leq K(R)(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|) \]

(69.3.19)

Also assume the growth condition

\[ (N(t, u, v, w, \omega), u) \geq -C(t, \omega) - \mu (|u|^2 + |v|^2 + |w|^2) \]

(69.3.20)
69.3. Stochastic Differential Equations in a Hilbert Space

and the linear growth condition on $\sigma$

$$\|\sigma(t, u, \omega)\| \leq a + b|u|_H$$

and the Lipschitz condition on $\sigma$. This can likely be relaxed as in the case of the Lipschitz condition for $N$ but for simplicity, we keep it.

**Theorem 69.3.2** Suppose $69.3.11$, $69.3.14$, $69.3.15$, $69.3.13$, $69.3.19$ and let

$$w(t) = w_0 + \int_0^t u(s) \, ds, \quad w_0 \in L^2(\Omega), \quad w_0 \text{ is } F_0 \text{ measurable.}$$

Then there exists a unique progressively measurable solution $u$ to the integral equation

$$u(t, \omega) = u_0(\omega) + \int_0^t N(s, u(s, \omega), u(s-h, \omega), w(s, \omega), \omega) \, ds$$

$$= \int_0^t f(s, \omega) \, ds + \int_0^t \sigma(s, u, \omega) \, dW. \quad (69.3.21)$$

where $u \in L^2(\Omega, C([0, T]; H))$, $u_0 \in L^2(\Omega, H)$, $u_0$ is $F_0$ measurable, $f$ is progressively measurable and in $L^2([0, T] \times \Omega; H)$. Here there is a set of measure zero such that if $\omega$ is not in this set, then $u(\cdot, \omega)$ solves the above integral equation $69.3.11$ and furthermore, if $\hat{u}(\cdot, \omega)$ is another solution to it, then $u(t, \omega) = \hat{u}(t, \omega)$ for all $t$ if $\omega$ is off some set of measure zero.

**Proof:** Let $u_n$ be the unique solution to the integral equation

$$u_n(t, \omega) = u_0(\omega) + \int_0^t N(s, P_n u_n(s, \omega), P_n u_n(s-h, \omega), P_n w_n(s, \omega), \omega) \, ds$$

$$= \int_0^t f(s, \omega) \, ds + \int_0^t \sigma(s, u_n, \omega) \, dW. \quad (69.3.22)$$

where $P_n$ is the projection onto $B(0, 9^n)$. Thus the modified problem is in the situation of Theorem 69.3.1 so there exists such a solution. Then let

$$\tau_n = \inf \{t : |u_n(t)| + |w_n(t)| > 2^n\}$$

Then stopping the equation with this stopping time, we can write

$$u_{\tau_n}^n(t, \omega) = u_0(\omega) + \int_0^{\tau_n} N(s, u_{\tau_n}^n(s, \omega), u_{\tau_n}^n(s-h, \omega), w_{\tau_n}^n(s, \omega), \omega) \, ds$$

$$= \int_0^{\tau_n} f(s, \omega) \, ds + \int_0^{\tau_n} \sigma(s, u_{\tau_n}^n, \omega) \, dW. \quad (69.3.23)$$

Then using the growth condition $69.3.20$ and the Itô formula,

$$\frac{1}{2} \|u_{\tau_n}^n(t)\|_H^2 \leq C(u_0, w_0, f) + C \int_0^t \|u_{\tau_n}^n\|_H^2 \, ds + \sup_{s \in [0, t]} |M(t)|$$
where $M(t)$ is a martingale whose quadratic variation is dominated by
\[ \int_0^t \| \sigma(s, u_n^s) \|^2 |u_n^s|^2 ds \]

Then it follows by the Burkholder-Davis-Gundy inequality
\[ E \left( \sup_{s \in [0,t]} |u_n^s(s)|^2_H \right) \leq E(C(u_0, w_0, f)) + C \int_0^t E \left( \sup_{r \in [0,s]} |u_n^r(r)|^2 dr \right) ds \]
\[ + CE \left( \left( \int_0^t \| \sigma(s, u_n^s) \|^2 |u_n^s|^2 ds \right)^{1/2} \right) \]

Now apply Gronwall’s inequality and modify the constants so that
\[ E \left( \sup_{s \in [0,t]} |u_n^s(s)|^2_H \right) \leq E(C(u_0, w_0, f)) + CE \left( \left( \int_0^t \| \sigma(s, u_n^s) \|^2 |u_n^s|^2 ds \right)^{1/2} \right) \]
\[ \leq E(C(u_0, w_0, f)) + \frac{1}{2} E \left( \sup_{s \in [0,t]} |u_n^s(s)|^2_H \right) + CE \left( \int_0^t \| \sigma(s, u_n^s) \|^2 ds \right) \]

Then, using the linear growth condition on $\sigma$, it follows on modification of the constants again that
\[ E \left( \sup_{s \in [0,t]} |u_n^s(s)|^2_H \right) \leq E(C(u_0, w_0, f)) + CE \left( \int_0^t |u_n^s|^2 ds \right) \]
\[ \leq E(C(u_0, w_0, f)) + CE \left( \int_0^t \sup_{r \in [0,s]} |u_n^r|^2 dr \right) \]

and so, another application of Gronwall’s inequality implies that
\[ E \left( \sup_{s \in [0,T]} |u_n^s(s)|^2_H \right) \leq E(C(u_0, w_0, f)) < \infty \]

Then
\[ P \left( \sup_{s \in [0,T]} |u_n^s(s)|^2_H > \left( \frac{3}{2} \right)^n \right) \leq E(C(u_0, w_0, f)) \left( \frac{2}{3} \right)^n \]

Now an application of the Borel Cantelli lemma shows that there exists a set of measure zero $\bar{N}$ such that for $\omega \notin \bar{N}$, it follows that for all $n$ large enough,
\[ \sup_{s \in [0,T]} |u_n^s(s)|^2_H < (3/2)^n \]

and so $\tau_n = \infty$ for all $n$ large enough.
Claim: For \( m < n \), there is a set of measure zero \( N_{mn} \) such that if \( \omega \not\in N_{mn} \), then \( u^m_n (s) = u^n_m (s) \) on \([0, T \wedge \tau_m] \).

Proof of the claim: Note that \( \tau_m \leq \tau_n \). Therefore, these are both progressively measurable solutions to the integral equation

\[
u (t \wedge \tau_m, \omega) - u_0 (\omega) + \int_0^t \mathcal{X}_{[0, \tau_m]} N (s, u(s, \omega), u(s-h, \omega), w_\nu (s, \omega), \omega) ds
= \int_0^t \mathcal{X}_{[0, \tau_m]} f (s, \omega) ds + \int_0^t \mathcal{X}_{[0, \tau_m]} \sigma (s, u, \omega) dW.
\]  

(69.3.24)

where

\[
w_\nu (t) = w_0 + \int_0^t u (s) ds.
\]

To save notation, refer to these functions as \( u, v \) and let \( \tau_m = \tau \). Subtract and use the Ito formula to obtain

\[
\frac{1}{2} \| u (t \wedge \tau) - v (t \wedge \tau) \|_H^2 \leq \int_0^t \mathcal{X}_{[0, \tau]} \| \sigma (s, u, \omega) - \sigma (s, v, \omega) \|^2 \| u - v \|^2 ds
+ \sup_{s \in [0, t]} | M (s) |
\]

where the quadratic variation of the martingale \( M (t) \) is dominated by

\[
\int_0^t \mathcal{X}_{[0, \tau]} \| \sigma (s, u, \omega) - \sigma (s, v, \omega) \|^2 \| u - v \|^2 ds
\]

Then from the assumption that \( N \) is locally Lipschitz and routine manipulations,

\[
\frac{1}{2} \| u (t \wedge \tau) - v (t \wedge \tau) \|_H^2 \leq C_m \int_0^t \mathcal{X}_{[0, \tau]} | u - v |^2 ds + \sup_{s \in [0, t]} | M (s) |
\]

and so, adjusting the constants yields

\[
\sup_{s \in [0, t]} | u (s \wedge \tau) - v (s \wedge \tau) |_H^2
\leq C_m \int_0^t \mathcal{X}_{[0, \tau]} \sup_{r \in [0, s]} | u (r \wedge \tau) - v (r \wedge \tau) |^2 ds + \sup_{s \in [0, t]} | M (s) |
\]

and so, by Gronwall’s inequality followed by the Burkholder-Davis-Gundy inequality,

\[
E \left( \sup_{s \in [0, t]} | u (s \wedge \tau) - v (s \wedge \tau) |_H^2 \right) \leq \]
CHAPTER 69. STOCHASTIC O.D.E. ONE SPACE

\[
CE \left( \left( \int_0^t X_{[0,\tau]} \| \sigma(s, u, \omega) - \sigma(s, v, \omega) \|^2 |u(s \land \tau) - v(s \land \tau)|^2 \, ds \right)^{1/2} \right)
\]

\[
\leq \frac{1}{2} CE \left( \sup_{s \in [0,t]} |u(s \land \tau) - v(s \land \tau)|^2 \right) + CE \left( \int_0^t X_{[0,\tau]} \| \sigma(s, u) - \sigma(s, v) \|^2 \, ds \right)
\]

and so, adjusting the constant again,

\[
E \left( \sup_{s \in [0,t]} |u(s \land \tau) - v(s \land \tau)|^2 \right)
\]

\[
\leq CE \left( \int_0^t X_{[0,\tau]} \| \sigma(s, u \land \tau) - \sigma(s, v \land \tau) \|^2 \, ds \right)
\]

\[
\leq CE \left( \int_0^t X_{[0,\tau]} K |u(s \land \tau) - v(s \land \tau)|^2 \, ds \right)
\]

\[
\leq C \int_0^t E \left( \sup_{r \in [0,s]} |u(r \land \tau) - v(r \land \tau)|^2 \right) \, ds
\]

and so, Gronwall’s inequality shows that for every \( t \),

\[
E \left( \sup_{s \in [0,t]} |u(s \land \tau) - v(s \land \tau)|^2 \right) = 0
\]

In particular, for \( t = T \) this holds. Hence

\[
E \left( \sup_{s \in [0,T]} |u(s \land \tau) - v(s \land \tau)|^2 \right) = 0
\]

It follows that

\[
E \left( \sup_{s \in [0,\tau \land T]} |u(s) - v(s)|^2 \right) = 0
\]

so that off a set of measure zero, \( u(s) = v(s) \) for all \( s \in [0, \tau] \). This proves the claim.

Now let the set of measure zero \( N \) be given by \( N \equiv \cup_{m<n} N_{mn} \cup \hat{N} \) where \( \hat{N} \) is the set of measure zero off which \( \tau_m = \infty \) for all \( m \) large enough. Then for \( \omega \notin N \), it follows that \( u_n^\tau(s) = u_m^\tau(s) \) on \([0, \tau_m \land T]\) and, for all \( m \) large enough, \( \tau_m = \infty \). Hence for all \( m \) large enough, and such \( \omega \), \( u_n(s, \omega) = u_m(s, \omega) \) for all \( s \in [0, T] \). Thus, for \( \omega \) off \( N \), it follows that \( \lim_{m \to \infty} u_m^\tau(s, \omega) = u(s, \omega) \) exists, this for each \( s \in [0, T] \) and \( \omega \) off a fixed set of measure zero. In fact, this convergence is uniform on \([0, T]\) because for all \( n \) sufficiently large and for such a fixed \( \omega \notin N \), there is no change in increasing \( m \). Hence, \( u \) is progressively measurable and satisfies the integral equation 69.3.21.
It remains to verify uniqueness. Suppose there are two solutions $u, v$ each progressively measurable solutions of the given integral equation. Then let $\tau_n$ be a stopping time
$$\tau_n = \inf \{ t : |u(t)| + |v(t)| > 2^n \}$$
Then a repeat of the arguments given in the above claim shows that on $[0, \tau_n \wedge T]$ the two functions $u^{\tau_n}, v^{\tau_n}$ are equal on $[0, \tau_n \wedge T]$ off a set of measure zero $N_n$. Let $N$ be the union of the exceptional sets. Then for $\omega \notin N$, $u(t, \omega) = v(t, \omega)$ for all $t \in [0, \tau_n \wedge T]$. However, $\tau_n(\omega) = \infty$ for all $n$ large enough because each of these functions is continuous. Hence, the two functions are equal on $[0, T]$ for such $\omega$. This shows uniqueness. ■
Chapter 70

The Hard Ito Formula

Recall the following definition of stochastically continuous.

$X$ is stochastically continuous at $t_0 \in I$ means: for all $\varepsilon > 0$ and $\delta > 0$ there exists $\rho > 0$ such that

$$P (||X(t) - X(t_0)|| \geq \varepsilon) \leq \delta$$

whenever $|t - t_0| < \rho, t \in I$.

Note the above condition says that for each $\varepsilon > 0$,

$$\lim_{t \to t_0} P (||X(t) - X(t_0)|| \geq \varepsilon) = 0.$$

70.1 Predictable And Stochastic Continuity

**Definition 70.1.1** Let $F_t$ be a filtration. The predictable sets consists of those sets which are in the smallest $\sigma$-algebra which contains the sets $E \times \{0\}$ for $E \in F_0$ and $E \times (a,b]$ where $E \in F_a$. Thus every predictable set is a progressively measurable set.

First of all, here is an important observation.

**Proposition 70.1.2** Let $X(t)$ be a stochastic process having values in $E$ a complete metric space and let it be $F_t$ adapted and left continuous where $F_t$ is a normal filtration. Then it is predictable. If $t \to X(t, \omega)$ is continuous for all $\omega \notin N, P(N) = 0$, then $(t, \omega) \to X(t, \omega)X_{NC}(\omega)$ is predictable. Also, if $X(t)$ is stochastically continuous and adapted on $[0,T]$, then it has a predictable version. If $X \in C([0,T]; L^p(\Omega; F)), p \geq 1$ for $F$ a Banach space, then $X$ is stochastically continuous.

**Proof:** First suppose $X$ is continuous for all $\omega \in \Omega$. Define

$$I_{m,k} \equiv ((k - 1)2^{-m}T, k2^{-m}T]$$

2509
if \( k \geq 1 \) and \( I_{m,0} = \{0\} \) if \( k = 1 \). Then define
\[
X_m(t) \equiv \sum_{k=1}^{2^m} X(T(k-1)2^{-m}) X_{((k-1)2^{-m}T,k2^{-m}T]}(t) \\
+ X(0) X_{[0,0]}(t)
\]
Here the sum means that \( X_m(t) \) has value \( X(T(k-1)2^{-m}) \) on the interval \( ((k-1)2^{-m}T,k2^{-m}T] \).
Thus \( X_m \) is predictable because each term in the formal sum is. Thus
\[
X_{m-1}(U) = \bigcup_{k=1}^{2^m} (X(T(k-1)2^{-m}) X_{((k-1)2^{-m}T,k2^{-m}T]}(U))^{-1} \bigcup_{k=1}^{2^m}((k-1)2^{-m}T,k2^{-m}T] \times (X(T(k-1)2^{-m}))^{-1} (U),
\]
a finite union of predictable sets. Since \( X \) is left continuous,
\[
X(t,\omega) = \lim_{m \to \infty} X_m(t,\omega)
\]
and so \( X \) is predictable.

Now suppose that for \( \omega \notin N, P(N) = 0, t \to X(t,\omega) \) is continuous. Then applying the above argument to \( X(t) X_{N^C} \) it follows \( X(t) X_{N^C} \) is predictable by completeness of \( \mathcal{F}_t \), \( X(t) X_{N^C} \) is \( \mathcal{F}_t \) measurable.

Next consider the other claim. Since \( X \) is stochastically continuous on \([0,T]\) it is uniformly stochastically continuous on this interval by Lemma 70.1.1. Therefore, there exists a sequence of partitions of \([0,T]\), the \( m \)-th being
\[
0 = t_{m,0} < t_{m,1} < \cdots < t_{m,n(m)} = T
\]
such that for \( X_m \) defined as above, then for each \( t \)
\[
P \left( \left[ d(X_m(t),X(t)) \geq 2^{-m} \right] \right) \leq 2^{-m} \tag{70.1.1}
\]
Then as above, \( X_m \) is predictable. Let \( A \) denote those points of \( \mathcal{P}_T \) at which \( X_m(t,\omega) \) converges. Thus \( A \) is a predictable set because it is just the set where \( X_m(t,\omega) \) is a Cauchy sequence. Now define the predictable function \( Y \)
\[
Y(t,\omega) \equiv \begin{cases} \lim_{m \to \infty} X_m(t,\omega) & \text{if } (t,\omega) \in A \\ 0 & \text{if } (t,\omega) \notin A \end{cases}
\]
From Lemma 70.1.1 it follows from the Borel Cantelli lemma that for fixed \( t \), the set of \( \omega \) which are in infinitely many of the sets,
\[
[d(X_m(t),X(t)) \geq 2^{-m}]
\]
has measure zero. Therefore, for each \( t \), there exists a set of measure zero, \( N(t) \) such that for \( \omega \notin N(t) \) and all \( m \) large enough
\[
[d(X_m(t,\omega),X(t,\omega)) < 2^{-m}]
\]
Hence for $\omega \notin N(t)$, $(t, \omega) \in A$ and so $X_m(t, \omega) \to Y(t, \omega)$ which shows
\[ d(Y(t, \omega), X(t, \omega)) = 0 \text{ if } \omega \notin N(t). \]
The predictable version of $X(t)$ is $Y(t)$.

Finally consider the claim about the specific example where $X \in C([0, T] ; L^p(\Omega; F))$. 

\[ P(\||X(t) - X(s)||_{F} \geq \varepsilon \leq \int_{\Omega} ||X(t) - X(s)||_{F}^p dP \leq \varepsilon^p \delta \]
provided $|s - t|$ sufficiently small. Thus
\[ P(\||X(t) - X(s)||_{F} \geq \varepsilon) < \delta \]
when $|s - t|$ is small enough. 

### 70.2 Approximating With Step Functions

This Ito formula seems to be the fundamental idea which allows one to obtain solutions to stochastic partial differential equations using a variational point of view. I am following the treatment found in [98]. The following lemma is fundamental to the presentation. It approximates a function with a sequence of two step functions $X_{r_k}, X_{l_k}$ where $X_{r_k}$ has the value of $X$ at the right end of each interval and $X_{l_k}$ gives the value $X$ at the left end of the interval. The lemma is very interesting for its own sake. You can obviously do this sort of thing for a continuous function but here the function is not continuous and in addition, it is a stochastic process depending on $\omega$ also. This lemma was proved earlier Lemma [63.3.1].

**Lemma 70.2.1** Let $\Phi : [0, T] \times \Omega \to V$, be $B([0, T]) \times F$ measurable and suppose $\Phi \in K \equiv L^p([0, T] \times \Omega; E), p \geq 1$

Then there exists a sequence of nested partitions, $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$,
\[ \mathcal{P}_k \equiv \{t_0^k, \ldots, t_{m_k}^k\} \]
such that the step functions given by
\[ \Phi^r_k(t) = \sum_{j=1}^{m_k} \Phi(t_j^k) X(t_{j-1}^k, t_j^k)(t) \]
\[ \Phi^l_k(t) = \sum_{j=1}^{m_k} \Phi(t_j^k-1) X(t_{j-1}^k, t_j^k)(t) \]
both converge to $\Phi$ in $K$ as $k \to \infty$ and
\[ \lim_{k \to \infty} \max \{|t_j^k - t_{j+1}^k| : j \in \{0, \ldots, m_k\}\} = 0. \]
Also, each \( \Phi(t^k_j), \Phi(t^k_{j-1}) \) is in \( L^p(\Omega; E) \). One can also assume that \( \Phi(0) = 0 \). The mesh points \( \{t^k_j\}_{j=0}^{m^n_k} \) can be chosen to miss a given set of measure zero. In addition to this, we can assume that \( \Phi(0) = 0 \).

The mesh points \( \{t^k_j\}_{m^n_k}^{m^n_k-1} \) can be chosen to miss a given set of measure zero. In addition to this, we can assume that

\[
|t^k_j - t^k_{j-1}| = 2^{-n_k}
\]

except for the case where \( j = 1 \) or \( j = m^n_k \) when this might not be so. In the case of the last subinterval defined by the partition, we can assume

\[
|t^k_m - t^k_{m-1}| = |T - t^k_{m-1}| \geq 2^{-(n_k+1)}
\]

The following lemma is convenient.

**Lemma 70.2.2** Let \( f_n \to f \) in \( L^p([0,T] \times \Omega, E) \). Then there exists a subsequence \( n_{k} \) and a set of measure zero \( N \) such that if \( \omega \notin N \), then

\[
f_{n_k}(\cdot, \omega) \to f(\cdot, \omega)
\]

in \( L^p([0,T], E) \) and for a.e. \( t \).

**Proof:** We have

\[
P \left( \left\| f_n - f \right\|_{L^p([0,T], E)} > \lambda \right) \leq \frac{1}{\lambda} \int \Omega \left\| f_n - f \right\|_{L^p([0,T], E)} dP \leq \frac{1}{\lambda} \left\| f_n - f \right\|_{L^p([0,T] \times \Omega, E)}
\]

Hence there exists a subsequence \( n_{k} \) such that

\[
P \left( \left\| f_{n_k} - f \right\|_{L^p([0,T], E)} \geq 2^{-k} \right) \leq 2^{-k}
\]

Then by the Borel Cantelli lemma, it follows that there exists a set of measure zero \( N \) such that for all \( k \) large enough and \( \omega \notin N \),

\[
\left\| f_{n_k} - f \right\|_{L^p([0,T], E)} \leq 2^{-k}
\]

Because of this lemma, it can also be assumed that for a.e. \( \omega \) pointwise convergence is obtained on \([0,T]\) as well as convergence in \( L^p([0,T]) \). This kind of assumption will be tacitly made whenever convenient in the context of the above lemma.

Also recall the diagram for the definition of the integral.

\[
\begin{align*}
&U_1 \supseteq JJ^{1/2}U \\
&\downarrow \quad \downarrow J^{1/2}
\end{align*}
\]

The idea was to get \( \int_t^t \Phi dW \) where \( \Phi \in L^2([0,T] \times \Omega; L_2(J^{1/2}U, H)) \). Here \( W(t) \) was a cylindrical Wiener process. This meant that it was a \( Q_1 \) Wiener process on \( U_1 \) for \( Q_1 = JJ^* \) and \( J \) was a Hilbert Schmidt operator mapping \( Q^{1/2}U \) to \( U_1 \).
70.3. THE SITUATION

Now consider the following situation.

**Situation 70.3.1** Let $X$ satisfy the following.

$$X(t) = X_0 + \int_0^t Y(s) \, ds + \int_0^t Z(s) \, dW(s),$$  \hspace{1cm} (70.3.2)

$X_0 \in L^2(\Omega; H)$ and is $\mathcal{F}_0$ measurable, where $Z$ is $L_2(Q^{1/2}U, H)$ progressively measurable and

$$\int_0^T \int_\Omega \| Z(s) \|_{L_2(Q^{1/2}U, H)}^2 \, dP \, dt < \infty$$

so that the stochastic integral makes sense. Also $X$ has a measurable representative $\bar{X}$ which has values in $V$. (For a.e. $t$, $\bar{X}(t) = X(t)$ for $P$ a.e. $\omega$). This representative satisfies

$$\bar{X} \in L^2([0, T] \times \Omega, \mathcal{B}([0, T] \times \mathcal{F}, H)) \cap L^p([0, T] \times \Omega, \mathcal{B}([0, T] \times \mathcal{F}, V)$$

Assume $Y(s)$ satisfies

$$Y \in K' = L^{p'}([0, T] \times \Omega; V')$$

where $1/p' + 1/p = 1$ and $Y$ is $V'$ progressively measurable. The situation in which the equation holds is as follows. For a.e. $t$, $\bar{X}(t) = X(t)$ for $P$ a.e. $\omega$. Thus it follows that $X(t)$ is automatically progressively measurable into $V'$ from Proposition 70.1.2. Also $W(t)$ is a Wiener process on $U_1$ in the above diagram. Thus $X$ is continuous into $V'$ off a set of measure zero, and it is also $V'$ predictable.

The goal is to prove the following Itô formula.

$$|X(t)|^2 = |X_0|^2 + \int_0^t \left( 2 \langle Y(s), \bar{X}(s) \rangle + \| Z(s) \|_{L_2(Q^{1/2}U, H)}^2 \right) \, ds$$

$$+ 2 \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \bar{X}(s) \right) \circ JdW(s)$$  \hspace{1cm} (70.3.3)

where $\mathcal{R}$ is the Riesz map which takes $U_1$ to $U_1'$. The main thing is that the last term above be a local martingale.

In all that follows, the mesh points $t_j$ will be points where $\bar{X}(t_j) = X(t_j)$ a.e. $\omega$.

**Lemma 70.3.2** Let $X$ be as in Situation 70.3.1 and let $X^i_k$ be as in Lemma 70.2.1 corresponding to $\bar{X}$ above. Say

$$X^i_k(t) = \sum_{j=0}^{m_k} \bar{X}(t_j) X_{[t_j, t_{j+1})}(t), \ X^i_k(0) \equiv 0.$$

Then each term in the above sum for which $t_j > 0$ is predictable into $H$. As mentioned earlier, we can take $X(0) \equiv 0$ in the definition of the “left step function”. Since, at the mesh points, $\bar{X} = X$ a.e., it makes no difference off a set of measure zero whether we use $\bar{X}(t_j)$ or $X(t_j)$ at the left end point.
Proof: This is a step function and a typical term is of the form $X(a) \mathcal{X}_{[a,b]}(t)$. I will try and show this is predictable. Let $a_n$ be an increasing sequence converging to $a$ and let $b_n$ be an increasing sequence converging to $b$. Then for a.e. $\omega,$

$$X(a_n) \mathcal{X}_{[a_n,b_n]}(t) \to X(a) \mathcal{X}_{[a,b]}(t)$$

in $V'$ due to the fact that $t \to X(t)$ is continuous into $V'$ for a.e. $\omega$. Therefore, letting $v \in V$ be given, it follows that for a.e. $\omega$

$$\langle X(a_n) \mathcal{X}_{[a_n,b_n]}(t), v \rangle \to \langle X(a) \mathcal{X}_{[a,b]}(t), v \rangle,$$

and since the filtration is a normal filtration in which all sets of measure zero from $\mathcal{F}_T$ are in $\mathcal{F}_0$, this shows

$$(t, \omega) \to \langle X(a) \mathcal{X}_{[a,b]}(t), v \rangle$$

is real predictable because it is the pointwise limit of real predictable functions, those in the sequence being real predictable because of the continuity of $X(t)$ into $V'$ and Proposition 70.3.3. Now since $H \subseteq V'$ it follows that for all $v \in V,$

$$(t, \omega) \to \langle X(a) \mathcal{X}_{[a,b]}(t), v \rangle$$

is real predictable. This holds for $h \in H$ replacing $v$ in the above because $V$ is dense in $H$. By the Pettis theorem, this proves the lemma. $\blacksquare$

Lemma 70.3.3 In Situation $\{70.3.\}$ the following formula holds for a.e. $\omega$ for $0 < s < t$ where $M(t) \equiv \int_0^t Z(u) dW(u)$. Here and elsewhere, $|\cdot|$ denotes the norm in $H$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V, V'$. Also $X = \bar{X}$ for a.e. $\omega$ at $t, s$ so that it makes no difference off a set of measure zero whether we write $\langle Y(u), X(t) \rangle$ or $\langle Y(u), \bar{X}(t) \rangle$

$$|X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du + 2 \langle X(s), M(t) - M(s) \rangle + |M(t) - M(s)|^2 - |X(t) - X(s) - (M(t) - M(s))|^2 \quad (70.3.4)$$

Also for $t > 0$

$$|X(t)|^2 = |X_0|^2 + 2 \int_0^t \langle Y(u), X(t) \rangle \, du + 2 \langle X_0, M(t) \rangle + |M(t)|^2 - |X(t) - X_0 - M(t)|^2 \quad (70.3.5)$$

Proof: The formula is a straight forward computation which holds a.e. $\omega.$

$$|M(t) - M(s)|^2 - |X(t) - X(s) - (M(t) - M(s))|^2 + 2 \langle X(s), M(t) - M(s) \rangle$$

$$= |M(t) - M(s)|^2 - |X(t) - X(s)|^2 - |M(t) - M(s)|^2 + 2 \langle X(t) - X(s), M(t) - M(s) \rangle + 2 \langle X(s), M(t) - M(s) \rangle$$
70.4. The Main Estimate

The following phenomenal estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if $y \in L^p(0,T;V)$ and $y' \in L^{p'}(0,T;V')$, then $y$ is actually continuous with values in $H$. Later, this continuity result is strengthened further to give strong continuity.
Lemma 70.4.1 In the Situation [70.X.7],

\[ E \left( \sup_{t \in [0,T]} |X(t)|_H^2 \right) < C \left( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|X_0\|_{L^2(\Omega,H)} \right) < \infty. \]

where

\[ J = L^2 \left( [0,T] \times \Omega; \mathcal{L}_2 \left( Q^{1/2}U; H \right) \right), \quad K_2 = L^p (\Omega; V), \quad K' = L^p (\Omega; V'). \]

Also, \( C \) is a continuous function of its arguments and \( C (0,0,0,0) = 0 \). Thus for a.e. \( \omega \),

\[ \sup_{t \in [0,T]} |X(t,\omega)|_H \leq C(\omega) < \infty. \]

Also for a.e. \( \omega, t \rightarrow X(t,\omega) \) is weakly continuous with values in \( H \).

**Proof:** Consider the formula in Lemma [70.X.3]:

\[ |X(t)|^2 = |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle \, du + 2 \langle X(s), M(t) - M(s) \rangle \]

\[ + |M(t) - M(s)|^2 - |X(t) - X(s)|^2 + |X(t)|^2 - |X_0|^2 \quad (70.4.6) \]

Now let \( t_j \) denote a point of \( \mathcal{P}_k \) from Lemma [70.X.1]. Then for \( t_j > 0 \), \( X(t_k) \) is just the value of \( X \) at \( t_k \) but when \( t = 0 \), the definition of \( X(0) \) in this step function is \( X(0) \equiv 0 \). Thus

\[ |X(t_m)|^2 - |X_0|^2 = \sum_{j=1}^{m-1} |X(t_{j+1})|^2 - |X(t_j)|^2 + |X(t_1)|^2 - |X_0|^2 \]

Using the formula in Lemma [70.X.3], for \( t = t_m \) this yields

\[ |X(t_m)|^2 - |X_0|^2 = 2 \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X_k(u) \rangle \, du + \]

\[ + 2 \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} Z(u) \, dW, X(t_j) \right) \] + \( \sum_{j=1}^{m-1} |M(t_{j+1}) - M(t_j)|^2 \]

\[ - \sum_{j=1}^{m-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \]

\[ + 2 \int_0^{t_1} \langle Y(u), X(t_1) \rangle \, du + 2 \left( X_0, \int_0^{t_1} Z(u) \, dW \right) + |M(t_1)|^2 \]

\[ - |X(t_1) - X_0 - M(t_1)|^2 \quad (70.4.7) \]
Of course
\[ 2 \int_0^{t_1} \langle Y (u), X (t_1) \rangle \, du + 2 \left( X_0, \int_0^{t_1} Z (u) \, dW \right) + |M (t_1)|^2 \]
converges to 0 for a.e. \( \omega \) as \( k \to \infty \) because the norms of the partitions converge to 0 and the stochastic integral is continuous off a set of measure zero. Actually this is not completely clear for the first of the above terms. This term is dominated by
\[
\int_0^{t_1} \| Y (u) \|^p' \, du \quad \left( \int_0^T \| X_k^r (u) \|^p \, du \right)^{1/p} \leq C (\omega) \int_0^{t_1} \| Y (u) \|^p' \, du
\]
Hence this converges to 0 for a.e. \( \omega \). At this time, not much is known about the last term in (70.4.7), but it is negative and is about to be neglected anyway. The Ito isometry implies the other two terms converge to 0 in \( L^1 (\Omega) \) also, in addition to converging for a.e. \( \omega \). At this time, not much is known about the last term in (70.4.7), but it is negative and is about to be neglected anyway.

The term involving the stochastic integral equals
\[ 2 \sum_{j=1}^{m-1} \left( \int_{t_j}^{t_{j+1}} Z (u) \, dW, X (t_j) \right)_H \]
By Theorem (63.14.1), this equals
\[ 2 \int_{t_1}^{t_m} \mathcal{R} \left( (Z (u) \circ J^{-1})^* X_k^l (u) \right) \circ JdW, \]
t \to \int_t^T \mathcal{R} \left( (Z (u) \circ J^{-1})^* X_k^l (u) \right) \circ JdW being a local martingale. Therefore, (70.4.7) equals
\[ |X (t_m)|^2 - |X_0|^2 = 2 \int_0^{t_m} \langle Y (u), X_k^r (u) \rangle \, du + e(k) \]
\[ 2 \int_{t_1}^{t_m} \mathcal{R} \left( (Z (u) \circ J^{-1})^* X_k^l (u) \right) \circ JdW + \sum_{j=1}^{m-1} |M (t_{j+1}) - M (t_j)|^2 \]
\[ - \sum_{j=1}^{m-1} \left| X (t_{j+1}) - X (t_j) - (M (t_{j+1}) - M (t_j)) \right|^2 - \left| X (t_1) - X_0 - M (t_1) \right|^2 \]
where \( e(k) \) converges to 0 in \( L^1 (\Omega) \) and for a.e. \( \omega \). Note that \( X_k^l (u) = 0 \) on \([0, t_1)\) and so that stochastic integral equals
\[ \int_0^{t_m} \mathcal{R} \left( (Z (u) \circ J^{-1})^* X_k^l (u) \right) \circ JdW. \]
Therefore, from the above,

\[ |X(t_m)|^2 - |X_0|^2 = 2 \int_0^{t_m} \langle Y(u), X_k(u) \rangle \, du + e(k) \]

\[
2 \int_0^{t_m} R \left( (Z(u) \circ J^{-1})^* X_k(u) \right) \circ JdW + \sum_{j=0}^{m-1} |M(t_{j+1}) - M(t_j)|^2 - |M(t_1)|^2
- \sum_{j=1}^{m-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 - |X(t_1) - X_0 - M(t_1)|^2
\]

Then since \(|M(t_1)|^2\) converges to 0 in \(L^1(\Omega)\) and for a.e. \(\omega\), as discussed above,

\[
|X(t_m)|^2 - |X_0|^2 = 2 \int_0^{t_m} \langle Y(u), X_k(u) \rangle \, du + e(k)
+ 2 \int_0^{t_m} R \left( (Z(u) \circ J^{-1})^* X_k(u) \right) \circ JdW + \sum_{j=0}^{m-1} |M(t_{j+1}) - M(t_j)|^2
- |X(t_1) - X_0 - M(t_1)|^2
- \sum_{j=1}^{m-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2
\]

(70.4.8)

where \(e(k) \to 0\) for a.e. \(\omega\) and also in \(L^1(\Omega)\).

Now it follows on discarding the negative terms,

\[
\sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 \leq |X_0|^2 + 2 \int_0^T \langle Y(u), X_k(u) \rangle \, du
+ 2 \sup_{t \in [0,T]} \left| \int_0^t R \left( (Z(u) \circ J^{-1})^* X_k(u) \right) \, \circ JdW \right| + \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} Z(u) \, dW \right|^2
\]

where there are \(m_k + 1\) points in \(\mathcal{P}_k\).

Do \(\int_\Omega\) to both sides. Using the Ito isometry, this yields

\[
\int_\Omega \left( \sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 \right) \, dP \leq E \left( |X_0|^2 \right) + 2 \|Y\|_{\mathcal{K}} \cdot \|X_k\|_{\mathcal{K}}
+ \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \int_\Omega ||Z(u)||^2 \, dP \, du
+ 2 \int_\Omega \left( \sup_{t \in [0,T]} \left| \int_0^T R \left( (Z(u) \circ J^{-1})^* X_k(u) \right) \, \circ JdW \right| \right) \, dP + E \left( |e(k)| \right)
\]
70.4. THE MAIN ESTIMATE

\[ \leq C + \int_0^T \int_{\Omega} \|Z(u)\|^2 dP \, du + \]

\[ + 2 \int_{\Omega} \left( \sup_{t \in [0,T]} \left| \int_0^T \mathcal{R} \left( (Z(u) \circ J^{-1})^* X_k^1(u) \right) \circ J \, dW \right| \right) \, dP \]

\[ \leq C + 2 \int_{\Omega} \left( \sup_{t \in [0,T]} \left| \int_0^T \mathcal{R} \left( (Z(u) \circ J^{-1})^* X_k^1(u) \right) \circ J \, dW \right| \right) \, dP \]

where the result of Lemma 70.2.1 that \( X_k^r \) converges to \( \bar{X} \) in \( K \) shows the term 2 \( \|Y\|_{K'} \|X_k^r\|_K \) is bounded. Note that the constant \( C \) is a continuous function of \( \|Y\|_{K'}, \|\bar{X}\|_K, \|Z\|_J, \|X_0\|_{L^2(\Omega,H)} \)

which equals zero when all are equal to zero. The term involving the stochastic integral is next.

Applying the Burkholder Davis Gundy inequality, Theorem 61.4.4 for \( F(r) = r \) along with the description of the quadratic variation of the Ito integral found in Corollary 63.11.1

\[ \int_{\Omega} \sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 dP \]

\[ \leq C + C \int_{\Omega} \left( \int_0^T \|\mathcal{R} \left( (Z(u) \circ J^{-1})^* X_k^1(u) \right) \circ J \|^2 \, du \right)^{1/2} \, dP \]

\[ \leq C + C \int_{\Omega} \left( \int_0^T \|Z(u)\|^2 \|X_k^1(u)\|^2 \, du \right)^{1/2} \, dP \]

Now for each \( \omega \), there are only finitely many values of \( X_k^1(u) \) and they equal \( X(t_j) \) for \( t_j \in \mathcal{P}_k \) with the convention that \( X(0) = 0 \). Therefore, the above is dominated by

\[ C + C \int_{\Omega} \left( \sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 \right)^{1/2} \left( \int_0^T \|Z(u)\|^2 \, du \right)^{1/2} \, dP \]

\[ \leq C + \frac{1}{2} \int_{\Omega} \sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 + C \int_{\Omega} \int_0^T \|Z(u)\|^2_{Z^2(Q^{1/2}U,H)} \, du \, dP \]

and so

\[ \frac{1}{2} \int_{\Omega} \sup_{t_j \in \mathcal{P}_k} |X(t_k)|^2 \, dP \leq C \]

for some constant \( C \) independent of \( \mathcal{P}_k \) dependent on \( \int_{\Omega} \int_0^T \|Z(u)\|^2_{Z^2(Q^{1/2}U,H)} \, du \, dP \).

This constant is dependent on \( \|Y\|_{K'}, \|\bar{X}\|_K, \|Z\|_J, \|X_0\|_{L^2(\Omega,H)} \) and equals zero when all of these quantities equal 0.
Let $D$ denote the union of all the $\mathcal{P}_k$. Thus $D$ is a dense subset of $[0, T]$ and it has just been shown that for a constant $C$ independent of $\mathcal{P}_k$,

$$E \left( \sup_{t \in D} |X(t)|^2 \right) \leq C.$$ 

Let $\{e_j\}$ be an orthonormal basis for $H$ which is also contained in $V$ and has the property that $\text{span} \{e_k\}_{k=1}^{\infty}$ is dense in $V$. I claim that for $y \in V'$

$$|y|_H^2 = \sup_n \sum_{j=1}^{n} |\langle y, e_j \rangle|^2$$

This is certainly true if $y \in H$ because in this case

$$\langle y, e_j \rangle = \langle y, e_j \rangle$$

If $y \not\in H$, then the series must diverge. If not, you could consider the infinite sum

$$z = \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \in H$$

and argue that $(z - y, v) = 0$ for all $v \in \text{span} \{e_k\}_{k=1}^{\infty}$ which would also imply that this is true for all $v \in V$. Then since $z = y$ in $V'$, it follows that $y \in H$ contrary to the assumption that $y \not\in H$.

It follows

$$|X(t)|^2 = \sup_n \sum_{j=1}^{n} |\langle X(t), e_j \rangle|^2$$

and for a.e. $\omega$, this is just the sup of continuous functions of $t$. Therefore, for given $\omega$ off a set of measure zero,

$$t \to |X(t)|^2$$

is lower semicontinuous. Hence letting $t \in [0, T]$ and $t_j \to t$ where $t_j \in D$,

$$|X(t)|^2 \leq \liminf_{j \to \infty} |X(t_j)|^2$$

so it follows for a.e. $\omega$

$$\sup_{t \in [0, T]} |X(t)|^2 \leq \sup_{t \in D} |X(t)|^2 \leq \sup_{t \in [0, T]} |X(t)|^2$$

Hence

$$E \left( \sup_{t \in [0, T]} |X(t)|^2 \right) \leq C \left( |Y|_{K'}, |X|_{K'}, |Z|_J, \|X_0\|_{L^2(\Omega, H)} \right) \tag{70.4.9}$$
Note the above shows that for a.e. $\omega$, $\sup_{t \in [0, T]} |X(t)|_H < \infty$ so that for such $\omega$, $X(t)$ has values in $H$. Note that we began by assuming it had a representative with values in $H$ although the equation only held in $V'$. Say

$$|X(t, \omega)| \leq C(\omega).$$

Hence if $v \in V$, then for a.e. $\omega$

$$\lim_{t \to s} (X(t), v) = \lim_{t \to s} (X(t), v) = \langle X(s), v \rangle = (X(s), v)$$

Therefore, since for such $\omega$, $|X(t, \omega)|$ is bounded, the above holds for all $h \in H$ in place of $v$ as well. Therefore, for a.e. $\omega$, $t \to X(t, \omega)$ is weakly continuous with values in $H$. \[ \Box \]

Eventually, it is shown that in fact, the function $t \to X(t, \omega)$ is continuous with values in $H$.

This lemma also provides a way to simplify one of the formulas derived earlier in the case that $X_0 \in L^p(\Omega, V)$. Refer to (70.4.10). One term there is

$$|X(t_1) - X_0 - M(t_1)|^2 \leq 2|X(t_1) - X_0|^2 + 2|M(t_1)|^2$$

It was shown above that $2|M(t_1)|^2 \to 0$ a.e. and also in $L^1(\Omega)$ as $k \to \infty$. Apply the above lemma to $|X(t) - X_0|^2$ using $[0, t_1]$ instead of $[0, T]$. The new $X_0$ equals 0. Then from the estimate (70.4.10), it follows that

$$E\left(|X(t_1) - X_0|^2\right) \to 0$$

as $k \to \infty$. Taking a subsequence, we could also assume that $|X(t_1) - X_0|^2 \to 0$ a.e. $\omega$ as $k \to \infty$. Then, using this subsequence, it would follow from (70.4.10) that

$$|X(t_m)|^2 - |X_0|^2 = 2 \int_0^{t_m} \langle Y(u), X_k(u) \rangle du + e(k)$$

$$+ 2 \int_0^{t_m} R\left(\left(Z(u) \circ J^{-1}\right)^* X_k^1(u)\right) \circ JdW + \sum_{j=0}^{m-1} |M(t_{j+1}) - M(t_j)|^2$$

$$- \sum_{j=1}^{m-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \quad (70.4.10)$$

where $e(k) \to 0$ in $L^1(\Omega)$ and a.e. $\omega$.

Can you obtain something similar even in case $X_0$ is not assumed to be in $L^p(\Omega, V)$? Let $Z_{0k} \in L^p(\Omega, V) \cap L^2(\Omega, H), Z_{0k} \to X_0$ in $L^2(\Omega, H)$. Then

$$|X(t_1) - X_0| \leq |X(t_1) - Z_{0k}| + |Z_{0k} - X_0|$$
Also, restoring the superscript to identify the partition,

\[ X(t^k_1) - Z_{0k} = X_0 - Z_{0k} + \int_0^{t^k_1} Y(s) \, ds + \int_0^{t^k_1} Z(s) \, dW. \]

Of course \( \|\bar{X} - Z_{0k}\|_K \) is not bounded but for each \( k \) it is at least finite. There is a sequence of partitions \( P_k, \|P_k\| \to 0 \) such that all the above holds. In the definitions of \( K, K', J \) replace \([0, T]\) with \([0, t]\) and let the resulting spaces be denoted by \( K_t, K'_t, J_t \). Let \( n_k \) denote a subsequence of \( \{k\} \) such that

\[ \|\bar{X} - Z_{0k}\|_{K'_1 n_k} < 1/k. \]

Then from the above lemma,

\[
E\left( \sup_{t \in [0, t^{n_k}_1]} \left| X(t^{n_k}_1) - Z_{0k} \right|^2_H \right)
\leq C \left( \|Y\|_{K'_1 n_k}, \|\bar{X} - Z_{0k}\|_{K'_1 n_k}, \|Z\|_{J_1 n_k}, \|X_0 - Z_{0k}\|_{L^2(\Omega, H)} \right)
\leq C \left( \|Y\|_{K'_1 n_k}, \frac{1}{k}, \|Z\|_{J_1 n_k}, \|X_0 - Z_{0k}\|_{L^2(\Omega, H)} \right)
\]

Hence

\[
E\left( |X(t^{n_k}_1) - X_0|^2 \right) \leq 2E\left( |X(t^{n_k}_1) - Z_{0k}|^2_H \right) + 2E\left( |Z_{0k} - X_0|^2_H \right)
\leq 2C \left( \|Y\|_{K'_1 n_k}, \frac{1}{k}, \|Z\|_{J_1 n_k}, \|X_0 - Z_{0k}\|_{L^2(\Omega, H)} \right) + 2\|Z_{0k} - X_0\|^2
\]

which converges to 0 as \( k \to \infty \). It follows that there exists a suitable subsequence such that \( 70.4.10 \) holds even in the case that \( X_0 \) is only known to be in \( L^2(\Omega, H) \). From now on, assume this subsequence for the partitions \( P_k \). Thus \( k \) will really be \( n_k \).

## 70.5 Converging In Probability

I am working toward the Ito formula \( 70.3.3 \). In order to get this, there is a technical result which will be needed.

**Lemma 70.5.1** Let \( X(s) - X_s^k (s) \equiv \Delta_k (s) \). Then the following limit occurs.

\[
\lim_{k \to \infty} P \left( \sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \Delta_k (s) \right) \circ JdW(s) \geq \varepsilon \right) = 0. \quad (70.5.11)
\]

That is,

\[
\sup_{t \in [0, T]} \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* (X(s) - X_s^k (s)) \right) \circ JdW(s)
\]
70.6. THE ITO FORMULA

converges to 0 in probability. Also the stochastic integral makes sense because \(X\) is \(H\) predictable.

Proof: First note that from Lemma 70.4.1, for a.e. \(\omega\), \(X(t)\) has values in \(H\) for \(t \in [0,T]\) and so it makes sense to consider it in the stochastic integral provided it is \(H\) progressively measurable. However, as noted in Situation 70.3.1, this function is automatically \(V'\) predictable. Therefore,

\[
\langle X(t), v \rangle = (X(t), v)
\]

is real predictable for every \(v \in V\). Now if \(h \in H\), let \(v_n \to h\) in \(H\) and so for each \(\omega\),

\[
(X(t, \omega), v_n) \to (X(t, \omega), h)
\]

By the Pettis theorem, \(X\) is \(H\) predictable, hence progressively measurable. Also it was shown above that \(t \to X(t)\) is weakly continuous into \(H\). Therefore, the desired result follows from Lemma 63.14.3 on Page 2326.

70.6 The Ito Formula

Now at long last, here is the first version of the Ito formula.

Lemma 70.6.1 In Situation 70.3.4, let \(D\) be as above, the union of all the positive mesh points for all the \(P_k\). Also assume \(X_0 \in L^2(\Omega; H)\). Then for every \(t \in D\),

\[
|X(t)|^2 = |X_0|^2 + \int_0^t \left( 2 \langle Y(s), \bar{X}(s) \rangle + ||Z(s)||_{L^2(Q^{1/2}U,H)}^2 \right) ds
+ 2 \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* X(s) \right) \circ JdW(s) \tag{70.6.12}
\]

Note that it was shown above that \(X(t, \omega)\) has values in \(H\) for a.e. \(\omega\).

Proof: Let \(t \in D\). Then \(t \in P_k\) for all \(k\) large enough. Consider the

\[
|X(t)|^2 - |X_0|^2 = 2 \int_0^t \langle Y(u), X_k^t(u) \rangle du
+ 2 \int_0^t \mathcal{R} \left( (Z(u) \circ J^{-1})^* X_k^t(u) \right) \circ JdW + \sum_{j=0}^{q_k-1} |M(t_{j+1}) - M(t_j)|^2
\]

\[
- \sum_{j=1}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 + c(k) \tag{70.6.13}
\]

where \(t_{q_k} = t\). By Lemma 70.5.1 the second term on the right, the stochastic integral, converges to

\[
2 \int_0^t \mathcal{R} \left( (Z(u) \circ J^{-1})^* \bar{X}(u) \right) \circ JdW
\]
in probability. The first term on the right converges to
\[ 2 \int_0^t \langle Y(u), \bar{X}(u) \rangle \, du \]
in \( L^1(\Omega) \) because \( X_k \to X \) in \( K \). Therefore, this also happens in probability. Consider the next term.

\[ E \left( \sum_{j=0}^{q_k-1} |M(t_{j+1}) - M(t_j)|^2 \right) \]

It is known from the theory of the quadratic variation that this term converges in probability to \( [M](t) = \int_0^t ||Z(s)||^2 \, ds \). See Theorem 61.6.4 on Page 2200 and the description of the quadratic variation in Corollary 63.11.1.

Thus all the terms in 70.6.13 converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

\[ |X(t)|^2 - |X_0|^2 = 2 \int_0^t \langle Y(u), \bar{X}(u) \rangle \, du \]

\[ + 2 \int_0^t \mathcal{R} \left( (Z(u) \circ J^{-1})^* \bar{X}(u) \right) \circ JdW + \int_0^t ||Z(s)||^2 \, ds - a \]

where \( a \) is the limit in probability of the term

\[ \sum_{j=1}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \]

Let \( P_n \) be the projection onto span \( \{ e_1, \ldots, e_n \} \) as before where \( \{ e_k \} \) is an orthonormal basis for \( H \) with each \( e_k \in V \). Then using

\[ X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)) = \int_{t_j}^{t_{j+1}} Y(s) \, ds \]

the troublesome term above is of the form

\[ \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), \bar{X}(t_{j+1}) - X(t_j) - P_n (M(t_{j+1}) - M(t_j)) \rangle \, ds \]

\[ - \sum_{j=1}^{q_k-1} (X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))) \, (I - P_n) (M(t_{j+1}) - M(t_j)) \]

The sum in 70.6.14 is dominated by

\[ \left( \sum_{j=1}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \right)^{1/2} \cdot \]
Now it is known that $\sum_{j=1}^{n} \left| X(t_{j+1}) - X(t_{j}) - (M(t_{j+1}) - M(t_{j})) \right|^2$ converges in probability to $a$. If you take the expectation of the other factor it is

$$E \left( \sum_{j=1}^{q_{k}-1} \left| (I - P_{n}) \int_{t_{j}}^{t_{j+1}} Z(s) dW(s) \right|^2 \right)$$

$$= \sum_{j=1}^{q_{k}-1} E \left( \left| \int_{t_{j}}^{t_{j+1}} (I - P_{n}) Z(s) dW(s) \right|^2 \right)$$

$$= \sum_{j=1}^{q_{k}-1} E \left( \left| (I - P_{n}) Z(s) \right|^2 \right) ds$$

$$\leq E \left( \left| (I - P_{n}) Z(s) \right|^2 \right) ds$$

$$= \int_{\Omega} \int_{0}^{T} \sum_{i=n+1}^{\infty} (Z(s) , e_{i})^2 ds dP$$

The integrand converges to 0 as $n \to \infty$ and is dominated by $\sum_{i=1}^{\infty} (Z(s) , e_{i})^2$ which is given to be in $L^{1}(\Omega)$. Therefore, it converges to 0.

Thus the expression in (70.6.16) is of the form $f_{k}g_{nk}$ where $f_{k}$ converges in probability to $a$ as $k \to \infty$ and $g_{nk}$ converges in probability to 0 as $n \to \infty$ independently of $k$. Now this implies $f_{k}g_{nk}$ converges in probability to 0. Here is why.

$$P(\left| f_{k}g_{nk} \right| > \varepsilon) \leq P(2\delta \left| g_{nk} \right| > \varepsilon) + P(2C_{\delta} \left| g_{nk} \right| > \varepsilon)$$

$$\leq P(2\delta \left| f_{k} - a \right| + 2\delta \left| a \right| > \varepsilon) + P(2C_{\delta} \left| g_{nk} \right| > \varepsilon)$$

where $\delta \left| f_{k} \right| + C_{\delta} \left| g_{nk} \right| > \left| f_{k}g_{nk} \right|$ and $\lim_{\delta \to 0} C_{\delta} = \infty$. Pick $\delta$ small enough that $\varepsilon - 2\delta \left| a \right| > \varepsilon/2$. Then this is dominated by

$$\leq P(2\delta \left| f_{k} - a \right| > \varepsilon/2) + P(2C_{\delta} \left| g_{nk} \right| > \varepsilon)$$

Fix $n$ large enough that the second term is less than $\eta$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in (70.6.16) and consequently in (70.6.15) converges to 0 in probability.

Now consider the other term, (70.6.15) using the $n$ just determined. This term is of the form

$$\sum_{j=1}^{q_{k}-1} \int_{t_{j}}^{t_{j+1}} \langle Y(s) , X_{k}^{r}(s) - X_{k}^{l}(s) - P_{n}(M_{k}^{r}(s) - M_{k}^{l}(s)) \rangle ds$$

$$= \int_{t_{1}}^{t} \langle Y(s) , X_{k}^{r}(s) - X_{k}^{l}(s) - P_{n}(M_{k}^{r}(s) - M_{k}^{l}(s)) \rangle ds$$
where \( M^r_k \) denotes the step function
\[
M^r_k(t) = \sum_{i=0}^{m_k-1} M(t_{i+1}) \mathcal{X}_{(t_i,t_{i+1}]}(t)
\]
and \( M^l_k \) is defined similarly. The term
\[
\int_{t_1}^t \langle Y(s), P_n (M^r_k(s) - M^l_k(s)) \rangle \, ds
\]
converges to 0 for a.e. \( \omega \) as \( k \to \infty \). This is because the integrand converges to 0 thanks to the continuity of \( M(t) \) and also since this is a projection onto a finite dimensional subspace of \( V \). Therefore, for each \( \omega \) off a set of measure zero,
\[
\int_{t_1}^t \| Y(s) \|_{V'} \| P_n (M^r_k(s) - M^l_k(s)) \|_V \, ds
\]
and this last integral converges to 0 as \( k \to \infty \) because \( P_n(M(s)) \) is uniformly bounded in \( V \) so there is no problem getting a dominating function for the dominated convergence theorem. Let
\[
A_k = \left\{ \left. \int_{t_1}^t \| Y(s) \|_{V'} \| P_n (M^r_k(s) - M^l_k(s)) \|_V \, ds \right| > \varepsilon \right\}
\]
Then since the partitions are increasing, these sets are decreasing as \( k \) increases and their intersection has measure zero. Hence \( P(A_k) \to 0 \). It follows that
\[
\lim_{k \to \infty} P \left( \left| \int_{t_1}^t \langle Y(s), P_n (M^r_k(s) - M^l_k(s)) \rangle \, ds \right| > \varepsilon \right) \leq 0
\]
Now consider
\[
\int_{t_1}^t \langle Y(s), X^r_k(s) - X^l_k(s) \rangle \, ds
\]
This converges to 0 in \( L^1(\Omega) \) because it is of the form
\[
\int_{t_1}^t \langle Y(s), X^r_k(s) \rangle \, ds - \int_{t_1}^t \langle Y(s), X^l_k(s) \rangle \, ds
\]
and both \( X^l_k \) and \( X^r_k \) converge to \( X \) in \( K \). Therefore, the expression
\[
\sum_{j=1}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2
\]
converges to 0 in probability. ■

In fact, the formula \ref{ito-formula} is valid for all \( t \in [0,T] \).
70.6. THE ITO FORMULA

**Theorem 70.6.2** In Situation (70.5.7), off a set of measure zero, for every \( t \in [0, T] \),

\[
|X(t)|^2 = |X_0|^2 + \int_0^t \left( 2 \langle Y(s), \dot{X}(s) \rangle + ||Z(s)||_{L_2(Q^{1/2}U,H)}^2 \right) ds \\
+ 2 \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* X(s) \right) \circ JdW(s) \\
\tag{70.6.17}
\]

Furthermore, for \( t \in [0,T] \), \( t \to X(t) \) is continuous as a map into \( H \) for \( a.e. \ \omega \). In addition to this,

\[
E \left( |X(t)|^2 \right) = E \left( |X_0|^2 \right) + E \left( \int_0^t \left( 2 \langle Y(s), \dot{X}(s) \rangle + ||Z(s)||_{L_2(Q^{1/2}U,H)}^2 \right) ds \right) \\
\tag{70.6.18}
\]

**Proof:** Let \( t \notin D \). For \( t > 0 \), let \( t(k) \) denote the largest point of \( \mathcal{P}_k \) which is

less than \( t \). Suppose \( t(m) < t(k) \). Hence \( m \leq k \). Then

\[ X(t(m)) = X_0 + \int_0^{t(m)} Y(s) ds + \int_0^{t(m)} Z(s) dW(s), \]

a similar formula holding for \( X(t(k)) \). Thus for \( t > t(m) \),

\[ X(t) - X(t(m)) = \int_{t(m)}^t Y(s) ds + \int_{t(m)}^t Z(s) dW(s) \]

which is the same sort of thing studied so far except that it starts at \( t(m) \) rather than at 0 and \( X_0 = 0 \). Therefore, from Lemma 70.5.4 it follows

\[
|X(t(k)) - X(t(m))|^2 = \int_{t(m)}^{t(k)} \left( 2 \langle Y(s), X(s) - X(t(m)) \rangle + ||Z(s)||^2 \right) ds \\
+ 2 \int_{t(m)}^{t(k)} \mathcal{R} \left( (Z(s) \circ J^{-1})^* (X(s) - X(t(m))) \right) \circ JdW(s) \\
\tag{70.6.19}
\]

Consider that last term. It equals

\[
2 \int_{t(m)}^{t(k)} \mathcal{R} \left( (Z(s) \circ J^{-1})^* (X(s) - X_{t(m)}^l) \right) \circ JdW(s) \\
\tag{70.6.20}
\]

This is dominated by

\[
2 \int_0^{t(k)} \mathcal{R} \left( (Z(s) \circ J^{-1})^* (X(s) - X_{t(m)}^l) \right) \circ JdW(s) \\
- 2 \int_0^{t(m)} \mathcal{R} \left( (Z(s) \circ J^{-1})^* (X(s) - X_{t(m)}^l) \right) \circ JdW(s)
\]
In Lemma 70.5.1, the above expression was shown to converge to 0 in probability. Therefore, by the usual appeal to the Borel Cantelli lemma, there is a subsequence still referred to as \( \{m\} \), such that it converges to 0 pointwise in \( \omega \) for all \( \omega \) off some set of measure 0 as \( m \to \infty \). It follows there is a set of measure 0 such that for \( \omega \) not in that set, \( \{m\} \) converges to 0 in \( \mathbb{R} \). Note that \( t > 0 \) is arbitrary. Similar reasoning shows the first term in the non stochastic integral of 70.6.19 is dominated by an expression of the form

\[
4 \int_0^T \langle Y(s), \bar{X}(s) - X_m^l(s) \rangle \, ds
\]

which clearly converges to 0 for \( \omega \) not in some set of measure zero because \( X_m^l \) converges in \( K \) to \( \bar{X} \). Finally, it is obvious that

\[
\lim_{m \to \infty} \int_0^T ||Z(s)||^2 \, ds = 0 \quad \text{for a.e. } \omega
\]

due to the assumptions on \( Z \).

This shows that for \( \omega \) off a set of measure 0

\[
\lim_{m,k \to \infty} |X(t(k)) - X(t(m))|^2 = 0
\]

and so \( \{X(t(k))\}_{k=1}^{\infty} \) is a convergent sequence in \( H \). Does it converge to \( X(t) \)? Let \( \xi(t) \in H \) be what it converges to. Let \( v \in V \) then

\[
\langle \xi(t), v \rangle = \lim_{k \to \infty} \langle X(t(k)), v \rangle = \lim_{k \to \infty} \langle X(t(k)), v \rangle = \langle X(t), v \rangle = \langle X(t), v \rangle
\]

and now, since \( V \) is dense in \( H \), this implies \( \xi(t) = X(t) \).

Now for every \( t \in D \),

\[
|X(t)|^2 = |X_0|^2 + \int_0^t \left( 2 \langle Y(s), \bar{X}(s) \rangle + ||Z(s)||^2 \right) \, ds
\]

\[
+ 2 \int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* \bar{X}(s) \right) \circ JdW(s)
\]

and so, using what was just shown along with the obvious continuity of the functions of \( t \) on the right of the equal sign, it follows the above holds for all \( t \in [0, T] \) off a set of measure zero.
70.6. THE ITO FORMULA

It only remains to verify \( t \to X(t) \) is continuous with values in \( H \). However, the above shows \( t \to |X(t)|^2 \) is continuous and it was shown in Lemma 70.4.1 that \( t \to X(t) \) is weakly continuous into \( H \). Therefore, from the uniform convexity of the norm in \( H \) it follows \( t \to X(t) \) is continuous. This is very easy to see in Hilbert space. Say \( a_n \to a \) and \( |a_n| \to |a| \). From the parallelogram identity,

\[
|a_n - a|^2 + |a_n + a|^2 = 2|a_n|^2 + 2|a|^2
\]

so

\[
|a_n - a|^2 = 2|a_n|^2 + 2|a|^2 - \left( |a_n|^2 + 2 \langle a_n, a \rangle + |a|^2 \right)
\]

Then taking \( \limsup \) both sides,

\[
0 \leq \limsup_{n \to \infty} |a_n - a|^2 \leq 2|a|^2 + 2|a|^2 - \left( |a|^2 + 2 \langle a, a \rangle + |a|^2 \right) = 0.
\]

Of course this fact also holds in any uniformly convex Banach space.

Now consider the last claim. If the last term in \( 70.6.17 \) were a martingale, then there would be nothing to prove. This is because if \( M(t) \) is a martingale which equals 0 when \( t = 0 \), then

\[
E(M(t)) = E(E(M(t) | \mathcal{F}_0)) = E(M(0)) = 0.
\]

However, that last term is unfortunately only a local martingale. One can obtain a localizing sequence as follows.

\[
\tau_n(\omega) \equiv \inf \{ t : |X(t, \omega)| > n \}
\]

where as usual \( \inf(\emptyset) \equiv \infty \). This is all right because it was shown above that \( t \to X(t, \omega) \) is continuous into \( H \) for a.e. \( \omega \). Then stopping both processes on the two sides of \( 70.6.17 \) with \( \tau_n \),

\[
|X(t \wedge \tau_n)|^2 = |X_0|^2 + \int_0^{t \wedge \tau_n} \left( 2 \langle Y(s), X(s) \rangle + ||Z(s)||_{L^2(Q^{1/2}U,H)}^2 \right) ds
\]

\[
+ 2 \int_0^{t \wedge \tau_n} \mathcal{R} \left( (Z(s) \circ J^{-1})^* X(s) \right) \circ J dW(s)
\]

Now from Lemma 70.6.14

\[
|X(t \wedge \tau_n)|^2 = |X_0|^2 + \int_0^t \mathcal{X}_{[0,\tau_n]}(s) \left( 2 \langle Y(s), X(s) \rangle + ||Z(s)||_{L^2(Q^{1/2}U,H)}^2 \right) ds
\]

\[
+ 2 \int_0^t \mathcal{X}_{[0,\tau_n]}(s) \mathcal{R} \left( (Z(s) \circ J^{-1})^* X(s) \right) \circ J dW(s)
\]

That last term is now a martingale and so you can take the expectation of both sides. This gives

\[
E \left( |X(t \wedge \tau_n)|^2 \right) = E \left( |X_0|^2 \right)
\]
\[ +E \left( \int_0^t \chi_{[0,\tau_n]}(s) \left( 2 \langle Y(s), \tilde{X}(s) \rangle + \| Z(s) \|^2_{L^2(Q^{1/2}; L^2)} \right) ds \right) \]

Letting \( n \to \infty \) and using the dominated convergence theorem and \( \tau_n \to \infty \) yields the desired result. \( \blacksquare \)

**Notation 70.6.3** The stochastic integrals are unpleasant to look at.

\[
\int_0^t \mathcal{R} \left( (Z(s) \circ J^{-1})^* X(s) \right) \circ JdW(s) \\
\equiv \int_0^t \langle X(s), Z(s) dW(s) \rangle .
\]
Chapter 71

The Hard Ito Formula, Implicit Case

71.1 Approximating With Step Functions

This Ito formula seems to be the fundamental idea which allows one to obtain solutions to stochastic partial differential equations using a variational point of view. I am following the treatment found in [18]. The following lemma is fundamental to the presentation. It approximates a function with a sequence of two step functions $X^r, X^l$ where $X^r$ has the value of $X$ at the right end of each interval and $X^l$ gives the value $X$ at the left end of the interval. The lemma is very interesting for its own sake. You can obviously do this sort of thing for a continuous function but here the function is not continuous and in addition, it is a stochastic process depending on $\omega$ also. This lemma was proved earlier, Lemma 63.3.1.

**Lemma 71.1.1** Let $\Phi : [0,T] \times \Omega \to V$, be $B([0,T]) \times F$ measurable and suppose $\Phi \in K \equiv L^p([0,T] \times \Omega; E), p \geq 1$

Then there exists a sequence of nested partitions, $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$,

$$\mathcal{P}_k \equiv \{ t_{0}^k, \ldots, t_{m_k}^k \}$$

such that the step functions given by

$$\Phi_k^r (t) \equiv \sum_{j=1}^{m_k} \Phi(t_j^k) \cdot X_{(t_{j-1}^k, t_j^k]} (t)$$

$$\Phi_k^l (t) \equiv \sum_{j=1}^{m_k} \Phi(t_{j-1}^k) \cdot X_{(t_{j-1}^k, t_j^k)} (t)$$

both converge to $\Phi$ in $K$ as $k \to \infty$ and

$$\lim_{k \to \infty} \max \{ |t_j^k - t_{j+1}^k| : j \in \{0, \ldots, m_k\} \} = 0.$$
Also, each \( \Phi(t^k_j), \Phi(t^k_{j-1}) \) is in \( L^p(\Omega;E) \). One can also assume that \( \Phi(0) = 0 \).

The mesh points \( \{t^k_j\}^m_{j=0} \) can be chosen to miss a given set of measure zero. In addition to this, we can assume that

\[
|t^k_j - t^k_{j-1}| = 2^{-n_k}
\]

except for the case where \( j = 1 \) or \( j = m_{n_k} \) when this might not be so. In the case of the last subinterval defined by the partition, we can assume

\[
|t^k_m - t^k_{m-1}| = |T - t^k_{m-1}| \geq 2^{-(n_k+1)}
\]

The following lemma is convenient.

**Lemma 71.1.2** Let \( f_n \to f \) in \( L^p([0,T] \times \Omega, E) \). Then there exists a subsequence \( n_k \) and a set of measure zero \( N \) such that if \( \omega \notin N \), then

\[
f_{n_k} (\cdot, \omega) \to f (\cdot, \omega)
\]

in \( L^p([0,T], E) \) and for a.e. \( t \).

**Proof:** We have

\[
P \left( \left\| f_n - f \right\|_{L^p([0,T], E)} > \lambda \right) \leq \frac{1}{\lambda} \int_{\Omega} \left\| f_n - f \right\|_{L^p([0,T], E)} dP \leq \frac{1}{\lambda} \left\| f_n - f \right\|_{L^p([0,T] \times \Omega, E)}
\]

Hence there exists a subsequence \( n_k \) such that

\[
P \left( \left\| f_{n_k} - f \right\|_{L^p([0,T], E)} > 2^{-k} \right) \leq 2^{-k}
\]

Then by the Borel Cantelli lemma, it follows that there exists a set of measure zero \( N \) such that for all \( k \) large enough and \( \omega \notin N \),

\[
\left\| f_{n_k} - f \right\|_{L^p([0,T], E)} \leq 2^{-k}
\]

Now by the usual arguments used in proving completeness, \( f_{n_k}(t) \to f(t) \) for a.e. \( t \).

Because of this lemma, it can also be assumed that for a.e. \( \omega \), pointwise convergence is obtained on \([0,T]\) as well as convergence in \( L^p([0,T]) \). This kind of assumption will be tacitly made whenever convenient.

Also recall the diagram for the definition of the integral which has values in a Hilbert space \( W \).

\[
\begin{array}{c}
U \\
\downarrow \\
JQ^{1/2}U \subseteq \prod_{i=1}^J Q^{1/2}U \\
\downarrow \\
Z_n \searrow \downarrow Z \\
\searrow W
\end{array}
\]
71.2. THE SITUATION

The idea was to get \( \int_0^t Z \, dW \) where \( Z \in L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2} U, W)) \). Here \( W (t) \) was a cylindrical Wiener process. This meant that it was a \( Q_1 \) Wiener process on \( U_1 \) for \( Q_1 = JJ^* \) and \( J \) was a Hilbert Schmidt operator mapping \( Q^{1/2} U \) to \( U_1 \). To get \( \int_0^t Z \, dW \), \( Z \circ J^{-1} \) was approximated by a sequence of elementary functions having values in \( \mathcal{L} (U_1, W) \). Then

\[
\int_0^t Z \, dW \equiv \lim_{n \to \infty} \int_0^t Z_n \, dW
\]

and this limit existed in \( L^2 (\Omega, W) \).

### 71.2 The Situation

Now consider the following situation. There are real separable Banach spaces \( V, W \) such that \( W \) is a Hilbert space and

\[
V \subseteq W, \quad W' \subseteq V'
\]

where \( V \) is dense in \( W \). Also let \( B \in \mathcal{L} (W, W') \) satisfy

\[
\langle Bw, w \rangle \geq 0, \quad \langle Bu, v \rangle = \langle Bv, u \rangle
\]

Note that \( B \) does not need to be one to one. Also allowed is the case where \( B \) is the Riesz map. It could also happen that \( V = W \).

**Situation 71.2.1** Let \( X \) have values in \( V \) and satisfy the following

\[
BX (t) = BX_0 + \int_0^t Y (s) \, ds + B \int_0^t Z (s) \, dW (s), \quad (71.2.1)
\]

where \( X_0 \in L^2 (\Omega; W) \) and is \( \mathcal{F}_0 \) measurable, where \( Z \) is \( \mathcal{L}_2 (Q^{1/2} U, W) \) progressively measurable and

\[
\|Z\|_{L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2} U, W))} < \infty.
\]

This is what is needed to define the stochastic integral in the above formula.

Assume \( X, Y \) satisfy

\[
BX, Y \in K' \equiv L^{p'} ([0, T] \times \Omega; V'),
\]

the \( \sigma \) algebra of measurable sets defining \( K' \) will be the progressively measurable sets. Here \( 1/p' + 1/p = 1, \ p > 1 \).

Also the sense in which the equation holds is as follows. For a.e. \( \omega \), the equation holds in \( V' \) for all \( t \in [0, T] \). Thus we are considering a particular representative \( X \) of \( K \) for which this happens. **Also it is only assumed that** \( BX (t) = B (X (t)) \) **for a.e.** \( t \). **Thus BX is the name of a function having values in** \( V' \) **for which** \( BX (t) = B (X (t)) \) **for a.e.** \( t \). Assume that \( X \) is progressively measurable also and

\[
X \in L^p ([0, T] \times \Omega, V)
\]
Also \( W(t) \) is a \( JJ^* \) Wiener process on \( U_1 \) in the above diagram.

The goal is to prove the following Itô formula valid for a.e. \( t \) for each \( \omega \) off a set of measure zero.

\[
\langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t \left( 2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle \right) ds \\
+ \int_0^t (Z \circ J^{-1})^* BX \circ JdW
\] (71.2.2)

The most significant feature of the last term is that it is a local martingale. The term \( \langle BZ, Z \rangle \mathbb{L}^2 \) will be discussed later, as will the meaning of the stochastic integral.

The idea is that \((Z \circ J^{-1})^* BX \circ J\) has values in \( L^2(Q^{1/2}U, \mathbb{R}) \) and so it makes sense to consider this stochastic integral. To see this, \( BX \in W' \) and \((Z \circ J^{-1})^* \in L^2(W', (JQ^{1/2}U)^\prime) \) and so \((Z \circ J^{-1})^* BX \in (JQ^{1/2}U)^\prime\).

The main item of interest relative to this stochastic integral will be a statement about its quadratic variation. It appears to depend on \( J \) but this is not the case because the other terms in the formula do not.

### 71.3 Preliminary Results

Here are discussed some preliminary results which will be needed. From the integral equation, if \( \phi \in L^q(\Omega; V) \) and \( \psi \in C^\infty_c(0, T) \) for \( q = \max(p, 2) \),

\[
\int_0^T \int_0^T \left( (BX)(t) - B \int_0^t Z(s) dW(s) - BX_0 \right) \phi(t) dtdP \\
= \int_0^T \int_0^T \int_0^t Y(s) \psi'(t) dsd\phi(\omega) dP
\]

Then the term on the right equals

\[
\int_0^T \int_0^T Y(s) \psi'(t) dtd\phi(\omega) dP = \int_0^T \left( - \int_0^T Y(s) \psi(s) ds \right) \phi(\omega) dP
\]

It follows that, since \( \phi \) is arbitrary,

\[
\int_0^T \left( (BX)(t) - B \int_0^t Z(s) dW(s) - BX_0 \right) \psi'(t) dt = - \int_0^T Y(s) \psi(s) ds
\]
in $L^q (\Omega; V')$ and so the weak time derivative of
\[ t \to (BX) (t) - B \int_0^t Z (s) dW (s) - BX_0 \]
equals $Y$ in $L^q \left( [0, T]; L^q (\Omega; V') \right)$. Thus, by Theorem 31.2.9, for a.e. $t$, say $t \notin \mathcal{N} \subset [0, T]$, $m \left( \mathcal{N} \right) = 0$,
\[ B \left( X (t) - \int_0^t Z (s) dW (s) \right) = BX_0 + \int_0^t Y (s) ds \text{ in } L^q (\Omega; V'). \]
That is,
\[ (BX) (t) = BX_0 + \int_0^t Y (s) ds + B \int_0^t Z (s) dW (s) \]
holds in $L^q (\Omega; V')$ where $(BX) (t) = B (X (t))$ a.e. $t$, in addition to holding for all $t$ for each $\omega$. Now let $\{ t_k^n \}_{k=1}^{m_n} \geq 1$ be partition sets for which, from Lemma 31.2.3, there are left and right step functions $X_k^n, X_k^n$, which converge in $L^p \left( [0, T] \times \Omega; V \right)$ to $X$ and such that each $\{ t_k^n \}_{k=1}^{m_n} \geq 1$ has empty intersection with the set of measure zero $\mathcal{N}$ where, in $L^q (\Omega; V')$, $(BX) (t) \neq B (X (t))$ in $L^q (\Omega; V')$. Thus for $t_k$ a generic partition point, \[ BX (t_k) = B (X (t_k)) \text{ in } L^q (\Omega; V') \]
Hence there is an exceptional set of measure zero, $N (t_k) \subset \Omega$ such that for $\omega \notin N (t_k), BX (t_k) (\omega) = B (X (t_k, \omega))$. We define an exceptional set $N \subset \Omega$ to be the union of all these $N (t_k)$. There are countably many and so $N$ is also a set of measure zero. Then for $\omega \notin N$, and $t_k$ any mesh point at all, $BX (t_k) (\omega) = B (X (t_k, \omega))$. This will be important in what follows. In addition to this, from the integral equation, for each of these $\omega \notin N, BX (t) (\omega) = B (X (t, \omega))$ for all $t \notin N_\omega \subset [0, T]$ where $N_\omega$ is a set of Lebesgue measure zero. Thus the $t_k$ from the various partitions are always in $N_\omega$. By Lemma 31.2.3, there exists a countable set $\{ e_i \}$ of vectors in $V$ such that \[ \langle Be_i, e_j \rangle = \delta_{ij} \]
and for each $x \in W$,
\[ \langle Bx, x \rangle = \sum_{i=0}^{\infty} \| Bx, e_i \|^2, \quad Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i \]
Thus the conclusion of the above discussion is that at the mesh points, it is valid to write
\[ \langle (BX) (t_k), X (t_k) \rangle = \langle B (X (t_k)), X (t_k) \rangle \]
\[ = \sum_i \langle (BX) (t_k), e_i \rangle^2 = \sum_i \langle B (X (t_k)), e_i \rangle^2 \]
just as would be the case if \((BX)(t) = B(X(t))\) for every \(t\). In all which follows, the mesh points will be like this and an appropriate set of measure zero which may be replaced with a larger set of measure zero finitely many times is being neglected. Obviously, one can take a subsequence of the sequence of partitions described above without disturbing the above observations. We will denote these partitions as \(P_k\).

As a case of this, we obtain the following interesting lemma.

**Lemma 71.3.1** In the above situation, there exists a set of measure zero \(N \subseteq \Omega\) and a dense subset of \([0,T]\), \(D\) such that for \(\omega \notin N\), \(BX(t,\omega) = B(X(t,\omega))\) for all \(t \in D\).

**Theorem 71.3.2** Let \(Z\) be progressively measurable and in

\[
L^2\left([0,T] \times \Omega, \mathcal{L}_2\left(Q^{1/2}U,W\right)\right).
\]

Also suppose \(X\) is progressively measurable and in \(L^2\left([0,T] \times \Omega,W\right)\). Let \(\left\{t^n_j\right\}_{j=0}^{m_n}\) be a sequence of partitions of the sort in Lemma 71.1.1 such that if \(X_n(t) \equiv m_n-1 \sum_{j=0}^{k} X(t^n_j) X(t^n_j,t^n_{j+1}) (t) \equiv X^n_l(t)\)

then \(X_n \to X\) in \(L^p\left([0,T] \times \Omega,W\right)\). Also, it can be assumed that none of these mesh points are in the exceptional set off which \(BX(t,\omega) = B(X(t,\omega))\). (Thus it will make no difference whether we write \(BX(t,\omega)\) or \(B(X(t,\omega))\) in what follows for all one of these mesh points.) Then the expression

\[
\sum_{j=0}^{m_n-1} \left\langle B \int_{t^n_j \wedge t}^{t^n_{j+1} \wedge t} ZdW, X^n_j (t) \right\rangle = \sum_{j=0}^{m_n-1} \left\langle BX(t^n_j), \int_{t^n_j \wedge t}^{t^n_{j+1} \wedge t} ZdW \right\rangle \quad (71.3.3)
\]

is a local martingale which can be written in the form

\[
\int_0^t (Z \circ J^{-1})^* BX^n_l \circ JdW
\]

where

\[
X^n_l(t) = \sum_{k=0}^{m_n-1} X(t^n_k) \mathcal{X}_{[t^n_k,t^n_{k+1}]} (t)
\]

**Proof:** First suppose that \(\langle BX(t^n_k), X(t^n_k) \rangle \in L^\infty\left([0,T] \times \Omega,W^\prime\right)\). Then

\[
\left\langle BX(t^n_j), \int_{t^n_j \wedge t}^{t^n_{j+1} \wedge t} ZdW \right\rangle
\]

is in \(L^1(\Omega)\) for each \(t\) since both entries are in \(L^2(\Omega)\). Why is this a martingale?

\[
E \left( \left\langle BX(t^n_j), \int_{t^n_j \wedge t}^{t^n_{j+1} \wedge t} ZdW \right\rangle \right) = E \left( E \left( \left\langle BX(t^n_j), \int_{t^n_j \wedge t}^{t^n_{j+1} \wedge t} ZdW \right\rangle | \mathcal{F}_{t^n_j} \right) \right)
\]
71.3. PRELIMINARY RESULTS

\[ E \left( \langle BX (t^n), E \left( \int_{t^n}^{t^{n+1} \land t} ZdW | \mathcal{F}_{t^n} \right) \rangle \right) = E \left( \langle BX (t^n), 0 \rangle \right) = 0 \]

because the stochastic integral is a martingale. Now let \( \sigma \) be a bounded stopping time.

\[ E \left( \langle BX (t^n), E \left( \int_{t^n}^{t^{n+1} \land \sigma} ZdW \right) \rangle \right) = E \left( \langle BX (t^n), E \left( \int_{t^n}^{t^{n+1} \land \sigma} ZdW | \mathcal{F}_{t^n} \right) \rangle \right) \]

\[ = E \left( \langle BX (t^n), \left( E \int_{t^n}^{t^{n+1} \land \sigma} ZdW - \int_{0}^{t^n \land \sigma} ZdW \right) | \mathcal{F}_{t^n} \rangle \right) \]

\[ = E \left( \langle BX (t^n), 0 \rangle \right) = 0 \]

and so this is a martingale. I want to write the formula in 71.3.3 as a stochastic integral. First note that \( W \) has values in \( U_1 \).

Consider one of the terms of the sum more simply as

\[ \left\langle B \int_{a}^{b} ZdW, X (a) \right\rangle, \ a = t^n \land t, \ b = t^{n+1} \land t. \]

Then from the definition of the integral, let \( Z_n \) be a sequence of elementary functions converging to \( Z \circ J^{-1} \) in \( L^2 ([a, b] \times \Omega, \mathcal{L}_2 (JQ^{1/2} U, W)) \) and

\[ \left\| \int_{a}^{t} ZdW - \int_{a}^{t} Z_n dW \right\|_{L^2(\Omega, W)} \rightarrow 0 \]

Using a maximal inequality and the fact that the two integrals are martingales along with the Borel Cantelli lemma, there exists a set of measure 0 \( N \) such that for \( \omega \notin N \), the convergence of a suitable subsequence of these integrals, still denoted by \( n \), is uniform for \( t \in [a, b] \). It follows that for such \( \omega \),

\[ \left\langle B \int_{a}^{t} ZdW, X (a) \right\rangle = \lim_{n \rightarrow \infty} \left\langle B \int_{a}^{t} Z_n dW, X (a) \right\rangle. \] (71.3.4)

Say

\[ Z_n (u) = \sum_{k=0}^{m_n-1} Z^n_k X_{[t^k, t^{k+1})} (u) \]

where \( Z^n_k \) has finitely many values in \( \mathcal{L} (U_1, W) \), the restrictions of maps in \( \mathcal{L} (U_1, W) \) to \( JQ^{1/2} U \), and the \( t^n_k \) refer to a partition of \( [a, b] \). Then the product on the right in 71.3.4 is of the form

\[ \sum_{k=0}^{m_n-1} \left\langle BZ^n_k (W (t \land t^k_{k+1}) - W (t \land t^k_k)), X (a) \right\rangle_{W'.W} \]
Note that it makes sense because $Z^n_k$ is the restriction to $JQ^{1/2}U$ of a map from $U_1$ to $W$ and so $BZ^n_k$ is a map from $U_1$ to $W'$. Then the Wiener process has values in $U_1$ so when you apply $BZ^n_k$ to $W \left(t \wedge t^\nu_{k+1}\right) - W \left(t \wedge t^\nu_k\right)$, you get something in $W'$ and so the duality pairing is between $W'$ and $W$ as shown. Also, $Z^n_k \left(W \left(t \wedge t^\nu_{k+1}\right) - W \left(t \wedge t^\nu_k\right)\right)$ gives something in $W$ because the Wiener process has values in $U_1$ and $Z^n_k$ acts on these things to give something in $W$. Thus the above equals

\[
\sum_{k=0}^{m-1} \langle BX(a), Z^n_k \left(W \left(t \wedge t^\nu_{k+1}\right) - W \left(t \wedge t^\nu_k\right)\right) \rangle_{W',W}
\]

\[
= \sum_{k=0}^{m-1} \langle (Z^n_k)^* BX(a), (W \left(t \wedge t^\nu_{k+1}\right) - W \left(t \wedge t^\nu_k\right)\rangle_{U_1'} U_1
\]

\[
= \sum_{k=0}^{m-1} (Z^n_k)^* BX(a) (W \left(t \wedge t^\nu_{k+1}\right) - W \left(t \wedge t^\nu_k\right)\)
\]

\[
= \int_a^t Z^n_k BX(a) dW
\]

Note that the restriction of $(Z_n)^* BX(a)$ is in

\[
L(U_1, \mathbb{R}) \subseteq \mathcal{L}_2 \left( JQ^{1/2}U, \mathbb{R} \right).
\]

Recall also that the space on the left is dense in the one on the right. Now let $\{g_i\}$ be an orthonormal basis for $Q^{1/2}U$, so that $\{Jg_i\}$ is an orthonormal basis for $JQ^{1/2}U$. Then

\[
\sum_{i=1}^{\infty} \left| \langle (Z_n)^* BX(a) - (Z \circ J^{-1})^* BX(a) \rangle (Jg_i) \right|^2
\]

\[
= \sum_{i=1}^{\infty} \left| \langle BX(a), (Z_n - Z \circ J^{-1}) (Jg_i) \rangle \right|^2
\]

\[
\leq \langle BX(a), X(a) \rangle \sum_{i=1}^{\infty} \langle B \left(Z_n - Z \circ J^{-1}\right) (Jg_i), (Z_n - Z \circ J^{-1}) (Jg_i) \rangle
\]

\[
\leq \langle BX(a), X(a) \rangle \|B\| \sum_{i=1}^{\infty} \left\| (Z_n - Z \circ J^{-1}) (Jg_i) \right\|^2_{W}
\]

\[
= \langle BX(a), X(a) \rangle \|B\| \left\| Z_n - Z \circ J^{-1} \right\|^2_{\mathcal{L}_2(JQ^{1/2}U,W)}
\]

When integrated over $[a, b] \times \Omega$, it is given that this converges to 0, assuming that $\langle BX(a), X(a)\rangle \in L^\infty(\Omega)$, which is assumed for now.
It follows that, with this assumption,
\[ Z_n^* BX (a) \to (Z \circ J^{-1})^* BX (a) \]
in \( L^2 ([a, b] \times \Omega, \mathcal{L}_2 (JQ^{1/2} U, \mathbb{R})) \). Writing this differently, it says
\[ Z_n^* BX (a) \to \left((Z \circ J^{-1})^* BX (a) \circ J\right) \circ J^{-1} \text{ in } L^2 ([a, b] \times \Omega, \mathcal{L}_2 (JQ^{1/2} U, \mathbb{R})) \]
It follows from the definition of the integral that the Ito integrals converge. Therefore,
\[ \langle B \int_a^t Z W, X (a) \rangle = \int_a^t (Z \circ J^{-1})^* BX (a) \circ JdW \]
The term on the right is a martingale because the one on the left is.

Next it is necessary to drop the assumption that \( \langle BX (a), X (a) \rangle \in L^\infty (\Omega) \). Note that \( X^l_n \) is right continuous and \( BX^l_n \) progressively measurable. Thus,
\[ \langle BX^l_n (t), X^l_n (t) \rangle = \sum_i \langle BX^l_i (t), \varepsilon_i \rangle^2 \]
where \( \{ \varepsilon_i \} \) is the set defined in Lemma 71.3.1 each in \( V \). Thus \( \langle BX^l_n, X^l_n \rangle \) is also progressively measurable and right continuous, and one can define the stopping time
\[ \sigma^n_q \equiv \inf \{ t : \langle BX^l_n (t), X^l_n (t) \rangle > q \} , \quad (71.3.5) \]
the first hitting time of an open set. Also, for each \( \omega \), there are only finitely many values for \( \langle BX^l_n (t), X^l_n (t) \rangle \) and so \( \sigma^n_q = \infty \) for all \( q \) large enough.

From localization,
\[ \langle B \int_{a \wedge \sigma^n_q}^{t \wedge \sigma^n_q} Z W, X (a) \rangle = \langle B \int_a^t X_{[0, \sigma^n_q]} Z W, X (a) \rangle \]
\[ = \int_a^t \left(X_{[0, \sigma^n_q]} \circ J^{-1}\right)^* BX (a) \circ JdW \]
\[ = \int_{a \wedge \sigma^n_q}^{t \wedge \sigma^n_q} \left(Z \circ J^{-1}\right)^* BX (a) \circ JdW \]
Then it follows that, using the stopping time,
\[ \sum_{j=0}^{m_n-1} \left(B \int_{t^j_n \wedge \sigma^n_q}^{t^{j+1}_n \wedge \sigma^n_q} Z W, X (t^j_n) \right) = \int_0^{t \wedge \sigma^n_q} \left(Z \circ J^{-1}\right)^* BX^l_n \circ JdW \]
where \( X^l_n \) is the step function
\[ X^l_n (t) = \sum_{k=0}^{m_n-1} X (t^j_k) X_{[t^j_k, t^{j+1}_k]} (t) . \]
Thus the given sum equals the local martingale
\[ \int_0^t (Z \circ J^{-1})^* BX_n \circ J dW. \]

Note that the sum does not depend on \( J \) or on \( U_1 \) so the same must be true of what it equals although it does not look that way. The question of convergence as \( n \to \infty \) is considered later.

What follows is the main estimate and discrete formulas.

### 71.4 The Main Estimate

The argument will be based on a formula which follows in the next lemma.

**Lemma 71.4.1** In Situation (71.2.1) the following formula holds for a.e. \( \omega \) for \( 0 < s < t \) where \( M(t) \equiv \int_0^t Z(u) dW(u) \) which has values in \( W \). In the following, \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( V, V' \).

\[
\langle BX(t), X(t) \rangle = \langle BX(s), X(s) \rangle + 
2 \int_s^t \langle Y(u), X(t) \rangle du + \langle B(M(t) - M(s)), M(t) - M(s) \rangle
- \langle BX(t) - BX(s) - (M(t) - M(s)), X(t) - X(s) - (M(t) - M(s)) \rangle
+ 2 \langle BX(s), M(t) - M(s) \rangle \tag{71.4.6}
\]

Also for \( t > 0 \)

\[
\langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y(u), X(t) \rangle du + 2 \langle BX_0, M(t) \rangle + 
\langle BM(t), M(t) \rangle - \langle BX(t) - BX_0 - BM(t), X(t) - X_0 - M(t) \rangle \tag{71.4.7}
\]

**Proof:** From the formula which is assumed to hold,

\[
BX(t) = BX_0 + \int_0^t Y(u) du + BM(t)
\]

\[
BX(s) = BX_0 + \int_0^s Y(u) du + BM(s)
\]

Then

\[
BM(t) - BM(s) + \int_s^t Y(u) du = BX(t) - BX(s)
\]

It follows that

\[
\langle B(M(t) - M(s)), M(t) - M(s) \rangle - 
\langle BX(t) - BX(s) - (M(t) - M(s)), X(t) - X(s) - (M(t) - M(s)) \rangle
\]
71.4. THE MAIN ESTIMATE

\[ +2 \langle BX (s), M (t) - M (s) \rangle \]

\[ = \langle B (M (t) - M (s)), M (t) - M (s) \rangle - \langle BX (t) - BX (s), (X (t) - X (s)) \rangle \]

\[ + 2 \langle BX (t) - BX (s), M (t) - M (s) \rangle \]

\[ - \langle B (M (t) - M (s)), M (t) - M (s) \rangle + 2 \langle BX (s), M (t) - M (s) \rangle \]

Some terms cancel and this yields

\[ = - \langle BX (t) - BX (s), (X (t) - X (s)) \rangle + 2 \langle BX (s), M (t) - M (s) \rangle \]

\[ = - \langle BX (t) - BX (s), (X (t) - X (s)) \rangle + 2 \langle B (M (t) - M (s)), X (t) \rangle \]

\[ = - \langle BX (t), X (t) \rangle - \langle BX (s), X (s) \rangle + 2 \langle BX (s), X (t) \rangle \]

\[ + 2 \langle BX (t), X (s) \rangle + 2 \langle BX (t), X (t) \rangle \]

\[ - 2 \langle BX (s), X (t) \rangle - 2 \int_s^t \langle Y (u), X (t) \rangle du \]

\[ = \langle BX (t), X (t) \rangle - \langle BX (s), X (s) \rangle - 2 \int_s^t \langle Y (u), X (t) \rangle du \]

Therefore,

\[ \langle BX (t), X (t) \rangle - \langle BX (s), X (s) \rangle \]

\[ = 2 \int_s^t \langle Y (u), X (t) \rangle du + \langle B (M (t) - M (s)), M (t) - M (s) \rangle \]

\[ - \langle BX (t) - BX (s) - (M (t) - M (s)), X (t) - X (s) - (M (t) - M (s)) \rangle \]

\[ + 2 \langle BX (s), M (t) - M (s) \rangle \]

The case with \( X_0 \) is similar. \( \blacksquare \)

The following phenomenal estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if \( y \in L^p (0, T; V) \) and \( y' \in L^p' (0, T; V') \), then \( y \) is actually continuous a.e. with values in \( H \), for \( V, H, V' \) a Gelfand triple. Later, this continuity result is strengthened further to give strong continuity. In all of this, \( X^t_k \) and \( X^t_k \) are as described above, converging in \( K \) to \( X \).
Lemma 71.4.2 In the Situation [71.2.4], the following holds. For a.e. \( t \)
\[
E\left(\langle BX(t), X(t) \rangle\right) < C \left( \|Y\|_{K'}, \|X\|_{K}, \|Z\|_{J}, \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right) < \infty.
\]
(71.4.8)
where \( K, K' \) were defined earlier and \( J = L^2 \left( [0,T] \times \Omega; L^2 \left( Q^{1/2} U; W \right) \right) \).

In fact,
\[
E \left( \sup_{t \in [0,T]} \sum_i \langle BX(t), e_i \rangle^2 \right) \leq C \left( \|Y\|_{K'}, \|X\|_{K}, \|Z\|_{J}, \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right)
\]
Also, \( C \) is a continuous function of its arguments, increasing in each one, and \( C(0,0,0,0) = 0 \). Thus for a.e. \( \omega \),
\[
\sup_{t \in N^C} \langle BX(t, \omega), X(t, \omega) \rangle \leq C(\omega) < \infty.
\]
Also for \( \omega \) off a set of measure zero described earlier, \( t \to BX(t)(\omega) \) is weakly continuous with values in \( W' \) on \([0,T]\). Also \( t \to \langle BX(t), X(t) \rangle \) is lower semi-continuous on \( N^C \).

Proof: Consider the formula in Lemma [71.3.1]
\[
\langle BX(t), X(t) \rangle = \langle BX(s), X(s) \rangle + 2 \int_s^t \langle Y(u), X(t) \rangle \, du + \langle B(M(t) - M(s)), M(t) - M(s) \rangle
\]
\[
- \langle B(X(t) - X(s) - (M(t) - M(s))), X(t) - X(s) - (M(t) - M(s)) \rangle + 2 \langle BX(s), M(t) - M(s) \rangle.
\]
(71.4.9)
Now let \( t_j \) denote a point of \( P_k \) from Lemma [71.3.1]. Then for \( t_j > 0, X(t_j) \) is just the value of \( X \) at \( t_j \) but when \( t = 0 \), the definition of \( X(0) \) in this step function is \( X(0) \equiv 0 \). Thus
\[
\sum_{j=1}^{m-1} \langle BX(t_{j+1}), X(t_{j+1}) \rangle - \langle BX(t_j), X(t_j) \rangle
\]
\[
+ \langle BX(t_1), X(t_1) \rangle - \langle BX_0, X_0 \rangle
\]
\[
= \langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle
\]
Using the formula in Lemma [71.3.1], for \( t = t_m \) this yields
\[
\langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle = 2 \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X_k^r(u) \rangle \, du + \ldots
\]
71.4. THE MAIN ESTIMATE

\[ + 2 \sum_{j=1}^{m-1} \left\langle B \int_{t_j}^{t_{j+1}} Z(u) \, dW, X(t_j) \right\rangle \]

\[ + \sum_{j=1}^{m-1} \left( B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \right) \]

\[ - \sum_{j=1}^{m-1} \left( B(X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))), X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)) \right) \]

\[ + 2 \int_0^{t_1} \langle Y(u), X(t_1) \rangle \, du + 2 \left\langle BX_0, \int_0^{t_1} Z(u) \, dW \right\rangle + \langle BM(t_1), M(t_1) \rangle \]

\[ - \langle B(X(t_1) - X_0 - M(t_1)), X(t_1) - X_0 - M(t_1) \rangle \]

(71.4.10)

First consider

\[ 2 \int_0^{t_1} \langle Y(u), X(t_1) \rangle \, du + 2 \left\langle BX_0, \int_0^{t_1} Z(u) \, dW \right\rangle + \langle BM(t_1), M(t_1) \rangle . \]

Each term of the above converges to 0 for a.e. \( \omega \) as \( k \to \infty \) and in \( L^1(\Omega) \). This follows right away for the second two terms from the Ito isometry and continuity properties of the stochastic integral. Consider the first term. This term is dominated by

\[ \left( \int_0^{t_1} \|Y(u)\|^p' \, du \right)^{1/p'} \left( \int_0^T \|X_k(u)\|^p \, du \right)^{1/p} \leq C(\omega) \left( \int_0^{t_1} \|Y(u)\|^p' \, du \right)^{1/p'} \left( \int_\Omega C(\omega)^p \, dP \right)^{1/p} < \infty \]

Hence this converges to 0 for a.e. \( \omega \) and also converges to 0 in \( L^1(\Omega) \).

At this time, not much is known about the last term in (71.4.10), but it is negative and is about to be neglected anyway.

The term involving the stochastic integral equals

\[ 2 \sum_{j=1}^{m-1} \left\langle B \int_{t_j}^{t_{j+1}} Z(u) \, dW, X(t_j) \right\rangle \]

By Theorem 71.3.2, this equals

\[ 2 \int_{t_1}^{t_m} (Z \circ J^{-1})^* BX_k^l \circ JdW \]
CHAPTER 71. THE HARD ITO FORMULA, IMPLICIT CASE

Also note that since \( \langle BM (t_1), M (t_1) \rangle \) converges to 0 in \( L^1 (\Omega) \) and for a.e. \( \omega \), the sum involving

\[
\langle B (M (t_{j+1}) - M (t_j)), M (t_{j+1}) - M (t_j) \rangle
\]

can be started at 0 rather than 1 at the expense of adding in a term which converges to 0 a.e. and in \( L^1 (\Omega) \). Thus (71.4.10) is of the form

\[
\langle BX (t_m), X (t_m) \rangle - \langle BX_0, X_0 \rangle = e (k) + 2 \int_0^{t_m} \langle Y (u), X_r (u) \rangle \, du +
\]

\[
+ 2 \int_0^{t_m} (Z \circ J^{-1})^* BX_k^l \circ JdW + \sum_{j=0}^{m-1} \langle B (M (t_{j+1}) - M (t_j)), M (t_{j+1}) - M (t_j) \rangle
\]

\[
- \sum_{j=1}^{m-1} \langle B (X (t_{j+1}) - X (t_j) - (M (t_{j+1}) - M (t_j))), X (t_{j+1}) - X (t_j) - (M (t_{j+1}) - M (t_j)) \rangle
\]

\[
- \langle B (X (t_1) - X_0 - M (t_1)), X (t_1) - X_0 - M (t_1) \rangle
\]

(71.4.11)

where \( e (k) \to 0 \) for a.e. \( \omega \) and also in \( L^1 (\Omega) \).

By definition, \( M (t_{j+1}) - M (t_j) = \int_{t_j}^{t_{j+1}} ZdW \). Now it follows, on discarding the negative terms,

\[
\langle BX (t_m), X (t_m) \rangle - \langle BX_0, X_0 \rangle \leq e (k) + 2 \int_0^{t_m} \langle Y (u), X_r (u) \rangle \, du +
\]

\[
+ 2 \int_0^{t_m} (Z \circ J^{-1})^* BX_k^l \circ JdW + \sum_{j=0}^{m-1} \left\langle B \int_{t_j}^{t_{j+1}} ZdW, \int_{t_j}^{t_{j+1}} ZdW \right\rangle
\]

Therefore,

\[
\sup_{t_m \in P_k} \langle BX (t_m), X (t_m) \rangle \leq \langle BX_0, X_0 \rangle + e (k) + 2 \int_0^T |\langle Y (u), X_r (u) \rangle| \, du +
\]

\[
+ 2 \sup_{t_m \in P_k} \left| \int_0^{t_m} (Z \circ J^{-1})^* BX_k^l \circ JdW \right|
\]

\[
+ \sum_{j=0}^{m_k-1} \left\langle B \left( \int_{t_j}^{t_{j+1}} Z (u) \, dW \right), \int_{t_j}^{t_{j+1}} Z (u) \, dW \right\rangle
\]

where there are \( m_k + 1 \) points in \( P_k \).
The next task is to somehow take the expectation of both sides. However, this is problematic because the stochastic integral is only a local martingale. Let

\[ \tau_p = \inf \{ t : \langle BX_k^l (t), X_k^l (t) \rangle > p \} \]

By right continuity this is a well defined stopping time. Then you obtain the above inequality for \( (X_k^l)^{\tau_p} \) in place of \( X_k^l \). Take the expectation and use the Ito isometry to obtain

\[
\int_{\Omega} \left( \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} (t), (X_k^l)^{\tau_p} (t_m) \right\rangle \right) dP \\
\leq E \left( \langle BX_0, X_0 \rangle \right) + 2 \|Y\|_{K'} \|X_k^l\|_K \\
+ \|B\| \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \int_{\Omega} \|Z(u)\|^2 dP \, du
\]

\[
+ 2 \int_{\Omega} \left( \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B (X_k^l)^{\tau_p} \circ J \, dW \right| \right) dP + E (|e(k)|)
\]

\[
\leq C + \|B\| \int_0^T \int_{\Omega} \|Z(u)\|^2 dP \, du + E (|e(k)|)
\]

\[
+ 2 \int_{\Omega} \left( \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B (X_k^l)^{\tau_p} \circ J \, dW \right| \right) dP \leq
\]

\[
C + E (|e(k)|) + 2 \int_{\Omega} \left( \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B (X_k^l)^{\tau_p} \circ J \, dW \right| \right) dP \quad (71.4.12)
\]

where the convergence of \( X_k^l \) to \( X \) in \( K \) shows the term \( 2 \|Y\|_{K'} \|X_k^l\|_K \) is bounded. Thus the constant \( C \) can be assumed to be a continuous function of

\[
\|Y\|_{K'}, \|X\|_K, \|Z\|_{J'}, \|BX_0, X_0\|_{L^1(\Omega)}
\]

which equals zero when all are equal to zero and is increasing in each. The term involving the stochastic integral is next.

Let \( M(t) = \int_0^t (Z \circ J^{-1})^* B (X_k^l)^{\tau_p} \circ J \, dW \). Then thanks to Corollary [63.11.1]

\[
d [M] = \left\| (Z \circ J^{-1})^* B (X_k^l)^{\tau_p} \circ J \right\|^2 \, ds \]

Applying the Burkholder Davis Gundy inequality, Theorem [61.4.4] for \( F(r) = r \) in that stochastic integral,

\[
2 \int_{\Omega} \left( \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B (X_k^l)^{\tau_p} \circ J \, dW \right| \right) dP
\]
\[
\leq C \int_\Omega \left( \int_0^T \left\| (Z \circ J^{-1})^* B \left( X_k^l \right)^{\tau_p} \circ J \right\|^2_{L^2(Q^{1/2}U, R)} \, ds \right)^{1/2} \, dP \quad (71.4.13)
\]

So let \( \{g_i\} \) be an orthonormal basis for \( Q^{1/2}U \) and consider the integrand in the above. It equals

\[
\sum_{i=1}^\infty \left( \left\langle B \left( X_k^l \right)^{\tau_p} \circ J \left( X_{l_k}^l \tau_p \circ J \right) \right\rangle \right) ^2 = \sum_{i=1}^\infty \left\langle B \left( X_k^l \right)^{\tau_p}, Z \left( g_i \right) \right\rangle^2
\]

\[
\leq \sum_{i=1}^\infty \left\langle B \left( X_k^l \right)^{\tau_p}, \left( X_k^l \right)^{\tau_p} \right\rangle \left\langle BZ \left( g_i \right), Z \left( g_i \right) \right\rangle
\]

\[
\leq \left( \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) \| B \| \| Z \|_{L^2}^2.
\]

It follows that the integral in (71.4.13) is dominated by

\[
C \int_\Omega \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle^{1/2} \| B \|^{1/2} \left( \int_0^T \left\| Z \|_{L^2}^2 \, ds \right) \right)^{1/2} \, dP
\]

Now return to (71.4.12). From what was just shown,

\[
E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) 
\leq C + E (|e(k)|) + 2 \int_\Omega \left( \sup_{t \in [0,T]} \left| \int_0^T \left( Z \circ J^{-1})^* B \left( X_k^l \right)^{\tau_p} \circ JdW \right) \right| \, dP
\]

\[
\leq C + C \int_\Omega \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle^{1/2} \| B \|^{1/2} \left( \int_0^T \left\| Z \|_{L^2}^2 \, ds \right) \right)^{1/2} \, dP + E (|e(k)|)
\]

\[
\leq C + \frac{1}{2} E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) 
+ C \| Z \|_{L^2([0,T] \times \Omega, L^2)}^2 + E (|e(k)|).
\]

It follows that

\[
\frac{1}{2} E \left( \sup_{t_m \in \mathcal{P}_k} \left\langle B \left( X_k^l \right)^{\tau_p} \left( t_m \right), \left( X_k^l \right)^{\tau_p} \left( t_m \right) \right\rangle \right) \leq C + E (|e(k)|)
\]
Now let $p \to \infty$ and use the monotone convergence theorem to obtain

$$\mathbb{E} \left( \sup_{t \in \mathcal{P}_k} \langle BX^t (t_m), X^t (t_m) \rangle \right) = \mathbb{E} \left( \sup_{t \in \mathcal{P}_k} \langle BX (t_m), X (t_m) \rangle \right) \leq C + \mathbb{E} (|e (k)|)$$

(71.4.14)

As mentioned above, this constant $C$ is a continuous function of

$$||Y||_{K'}, ||X||_K, ||Z||_J, ||BX_0, X_0||_{L^1(\Omega, H)}$$

and equals zero when all of these quantities equal 0 and is increasing with respect to each of the above quantities. Also, for each $\varepsilon > 0$,

$$\mathbb{E} \left( \sup_{t \in \mathcal{P}_k} \langle BX (t_m), X (t_m) \rangle \right) \leq C + \varepsilon$$

whenever $k$ is large enough.

Let $D$ denote the union of all the $\mathcal{P}_k$. Thus $D$ is a dense subset of $[0, T]$ and it has just been shown, since the $\mathcal{P}_k$ are nested, that for a constant $C$ dependent only on the above quantities which is independent of $\mathcal{P}_k$,

$$\mathbb{E} \left( \sup_{t \in D} \langle BX (t), X (t) \rangle \right) \leq C + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\mathbb{E} \left( \sup_{t \in D} \langle BX (t), X (t) \rangle \right) \leq C \quad (71.4.15)$$

Thus, enlarging $N$, for $\omega \notin N,$

$$\sup_{t \in D} \langle BX (t), X (t) \rangle = C (\omega) < \infty \quad (71.4.16)$$

where $\int_{\Omega} C (\omega) dP < \infty$. By Lemma 71.4.1, there exists a countable set $\{e_i\}$ of vectors in $V$ such that

$$\langle Be_i, e_j \rangle = \delta_{ij}$$

and for each $x \in W$,

$$\langle Bx, x \rangle = \sum_{i=0}^{\infty} \langle Bx, e_i \rangle^2, \quad Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i$$

Thus for $t$ not in a set of measure zero off which $BX (t) = B (X (t))$,

$$\langle BX (t), X (t) \rangle = \sum_{i=0}^{\infty} \langle BX (t), e_i \rangle^2 = \sup_{m} \sum_{k=1}^{m} \langle BX (t), e_i \rangle^2$$
Now from the formula for $BX(t)$, it follows that $BX$ is continuous into $V'$. For any $t \notin \hat{N}$ so that $(BX)(t) = B(X(t))$ in $L^q(\Omega; V')$ and letting $t_k \to t$ where $t_k \in D$, Fatou's lemma implies

$$E(\langle BX(t), X(t) \rangle) = \sum_i E(\langle BX(t), e_i \rangle^2) = \sum_i \lim_{k \to \infty} E(\langle BX(t_k), e_i \rangle^2) \leq \lim_{k \to \infty} \inf \sum_i E(\langle BX(t_k), e_i \rangle^2) = \lim_{k \to \infty} \inf E(\langle BX(t_k), X(t_k) \rangle) \leq C \left( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right)$$

In addition to this, for arbitrary $t \in [0,T]$, and $t_k \to t$ from $D$,

$$\sum_i \langle BX(t), e_i \rangle^2 \leq \lim inf_{k \to \infty} \sum_i \langle BX(t_k), e_i \rangle^2 \leq \sup_{s \in D} \langle BX(s), X(s) \rangle$$

Hence

$$\sup_{t \in [0,T]} \sum_i \langle BX(t), e_i \rangle^2 \leq \sup_{s \in D} \langle BX(s), X(s) \rangle = \sup_{s \in D} \sum_i \langle BX(s), e_i \rangle^2 \leq \sup_{t \in [0,T]} \sum_i \langle BX(t), e_i \rangle^2$$

It follows that $\sup_{t \in [0,T]} \sum_i \langle BX(t), e_i \rangle^2$ is measurable and

$$E \left( \sup_{t \in [0,T]} \sum_i \langle BX(t), e_i \rangle^2 \right) \leq E \left( \sup_{s \in D} \langle BX(s), X(s) \rangle \right) \leq C \left( \|Y\|_{K'}, \|X\|_K, \|Z\|_J, \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right)$$

And so, for $\omega$ off a set of measure zero, $\sup_{t \in [0,T]} \sum_i \langle BX(t), e_i \rangle^2$ is bounded above. Also for $t \notin \hat{N}_\omega$ and a given $\omega \notin \hat{N}$, letting $t_k \to t$ for $t_k \in D$,

$$\langle BX(t), X(t) \rangle = \sum_i \langle BX(t), e_i \rangle^2 \leq \lim_{k \to \infty} \inf \sum_i \langle BX(t_k), e_i \rangle^2 \leq \lim_{k \to \infty} \inf \langle BX(t_k), X(t_k) \rangle \leq \sup_{t \in D} \langle BX(t), X(t) \rangle$$

and so

$$\sup_{t \notin \hat{N}_\omega} \langle BX(t), X(t) \rangle \leq \sup_{t \in D} \langle BX(t), X(t) \rangle \leq \sup_{t \notin \hat{N}_\omega} \langle BX(t), X(t) \rangle$$

From \textit{Fatou}

$$\sup_{t \notin \hat{N}_\omega} \langle BX(t), X(t) \rangle = C(\omega) \ a.e. \omega$$
where \( \int_\Omega C (\omega) \, dP < \infty \). In particular, \( \sup_{t \in N_\omega} |\langle BX (t) , X (t) \rangle| \) is bounded for a.e. \( \omega \) say for \( \omega \notin N \) where \( N \) includes the earlier sets of measure zero. This shows that \( BX (t) \) is bounded in \( W' \) for \( t \in N_\omega^c \).

If \( v \in V \), then for \( \omega \notin N \),

\[
\lim_{t \to s} \langle BX (t) , v \rangle = \langle BX (s) , v \rangle, \quad t, s
\]

Therefore, since for such \( \omega \), \( \|BX (t)\|_{W'} \) is bounded for \( t \notin N_\omega \), the above holds for all \( v \in W \) also. Therefore, for a.e. \( \omega \), \( t \to BX (t, \omega) \) is weakly continuous with values in \( W' \) for \( t \notin N_\omega \).

Note also that

\[
\int_0^T \int_\Omega \|BX (t)\|^2 
\]

\[
\|t^2 \| \leq \int_\Omega \int_0^T |B|^2 \langle BX (t) , X (t) \rangle 
\]

\[
\leq C \left( |Y|_{K'} + |X|_{K} + |Z|_{J} + \|\langle BX_0 , X_0 \rangle\|_{L^1 (\Omega)} \right) \|B\|^{1/2} T \quad (71.4.17)
\]

Eventually, it is shown that in fact, the function \( t \to BX (t, \omega) \) is continuous with values in \( W' \). The above shows that \( BX \in L^2 ([0, T] \times \Omega, W') \).

Finally consider the claim of weak continuity of \( BX \) into \( W' \). From the integral equation, \( BX \) is continuous into \( V' \). Also \( BX \) is bounded on \( N_\omega^C \). Let \( s \in [0, T] \) be arbitrary. I claim that if \( t_n \to s, t_n \notin D \), it follows that \( BX (t_n) \to BX (s) \) weakly in \( W' \). If not, then there is a subsequence, still denoted as \( t_n \) such that \( BX (t_n) \to Y \) weakly in \( W' \) but \( Y \neq BX (s) \). However, the continuity into \( V' \) means that for all \( v \in V \),

\[
\langle Y, v \rangle = \lim_{n \to \infty} \langle BX (t_n) , v \rangle = \langle BX (s) , v \rangle
\]

which is a contradiction since \( V \) is dense in \( W \). This establishes the claim. Also this shows that \( BX (s) \) is bounded in \( W' \).

\[
|\langle BX (s) , w \rangle| = \lim_{n \to \infty} |\langle BX (t_n) , w \rangle| \leq \lim \inf_{n \to \infty} \|BX (t_n)\|_{W'}, \|w\|_{W'} \leq C (\omega) \|w\|_W
\]

Now a repeat of the above argument shows that \( s \to BX (s) \) is weakly continuous into \( W' \).

### 71.5 A Simplification Of The Formula

This lemma also provides a way to simplify one of the formulas derived earlier in the case that \( X_0 \in L^p (\Omega, V) \) so that \( X - X_0 \in L^p ([0, T] \times \Omega, V) \). Refer to 71.4.17. One term there is

\[
\langle B (X (t_1) - X_0 - M (t_1)) , X (t_1) - X_0 - M (t_1) \rangle
\]

Also,

\[
\langle B (X (t_1) - X_0 - M (t_1)) , X (t_1) - X_0 - M (t_1) \rangle
\]
It was observed above that \( 2 \langle BM(t_1), M(t_1) \rangle \to 0 \) a.e. and also in \( L^1(\Omega) \) as \( k \to \infty \). Apply the above lemma to \( \langle B(X(t_1) - X_0), X(t_1) - X_0 \rangle \) using \([0, t_1]\) instead of \([0, T]\). The new \( X_0 \) equals 0. Then from the estimate 14.4.8, it follows that
\[
E(\langle B(X(t_1) - X_0), X(t_1) - X_0 \rangle) \to 0
\]
as \( k \to \infty \). Taking a subsequence, we could also assume that
\[
\langle B(X(t_1) - X_0), X(t_1) - X_0 \rangle \to 0
\]
a.e. \( \omega \) as \( k \to \infty \). Then, using this subsequence, it would follow from 14.4.11,
\[
\langle BX_{(m)}, X_{(m)} \rangle - \langle BX_0, X_0 \rangle = e(k) + 2 \int_0^{t_m} \langle Y(u), X'_k(u) \rangle \, du +
\]
\[
+ 2 \int_0^{t_m} (Z \circ J^{-1})^* BX'_k \circ JdW
\]
\[
+ \sum_{j=0}^{m-1} \langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle
\]
\[
- \sum_{j=1}^{m-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), \Delta X(t_j) - \Delta M(t_j) \rangle \tag{71.5.18}
\]
where \( e(k) \to 0 \) in \( L^1(\Omega) \) and a.e. \( \omega \) and

\( \Delta X(t_j) \equiv X(t_{j+1}) - X(t_j) \)

\( \Delta M(t_j) \) being defined similarly. Note how this eliminated the need to consider the term
\[
\langle B(X(t_1) - X_0 - M(t_1)), X(t_1) - X_0 - M(t_1) \rangle
\]
in passing to a limit. This is a very desirable thing to be able to conclude.

Can you obtain something similar even in case \( X_0 \) is not assumed to be in \( L^p(\Omega, V) \)? Let \( Z_{0k} \in L^p(\Omega, V) \cap L^2(\Omega, W) \), \( Z_{0k} \to X_0 \) in \( L^2(\Omega, W) \). Then from the usual arguments involving the Cauchy Schwarz inequality,
\[
\langle B(X(t_1) - X_0), X(t_1) - X_0 \rangle^{1/2} \leq \langle B(X(t_1) - Z_{0k}), X(t_1) - Z_{0k} \rangle^{1/2}
\]
\[
+ \langle B(Z_{0k} - X_0), Z_{0k} - X_0 \rangle^{1/2}
\]
Also, restoring the superscript to identify the partition,
\[
B(X(t_1^k) - Z_{0k}) = B(X_0 - Z_{0k}) + \int_0^{t_1^k} Y(s) \, ds + B \int_0^{t_1^k} Z(s) \, dW.
\]
Of course \( \|X - Z_{0k}\|_K \) is not bounded, but for each \( k \) it is finite. There is a sequence of partitions \( P_k, \|P_k\| \to 0 \) such that all the above holds. In the definitions
of $K, K', J$ replace $[0, T]$ with $[0, t]$ and let the resulting spaces be denoted by $K_t, K'_t, J_t$. Let $n_k$ denote a subsequence of $\{k\}$ such that

$$\|X - Z_{0k}\|_{K'_{t_{n_k}}} < 1/k.$$ 

Then from the above lemma,

$$E \left( \langle B \left( X \left( t_{1}^{n_k} \right) - Z_{0k} \right), X \left( t_{1}^{n_k} \right) - Z_{0k} \rangle \right)$$

$$\leq C \left( \|Y\|_{K'_{t_{1}^{n_k}}}, \|X - Z_{0k}\|_{K'_{t_{1}^{n_k}}}, \|Z\|_{J_{t_{1}^{n_k}}}, \langle B \left( X_0 - Z_{0k} \right), X_0 - Z_{0k} \rangle_{L^1(\Omega)} \right)$$

$$\leq C \left( \|Y\|_{K'_{t_{1}^{n_k}}}, \frac{1}{k}, \|Z\|_{J_{t_{1}^{n_k}}}, \langle B \left( X_0 - Z_{0k} \right), X_0 - Z_{0k} \rangle_{L^1(\Omega)} \right)$$

Hence

$$E \left( \langle B \left( X \left( t_{1}^{n_k} \right) - X_0 \right), X \left( t_{1}^{n_k} \right) - X_0 \rangle \right)$$

$$\leq 2E \left( \langle B \left( X \left( t_{1}^{n_k} \right) - Z_{0k} \right), X \left( t_{1}^{n_k} \right) - Z_{0k} \rangle \right) + 2E \left( \langle B \left( Z_{0k} - X_0 \right), Z_{0k} - X_0 \rangle \right)$$

$$\leq \quad 2C \left( \|Y\|_{K'_{t_{1}^{n_k}}}, \frac{1}{k}, \|Z\|_{J_{t_{1}^{n_k}}}, \langle B \left( X_0 - Z_{0k} \right), X_0 - Z_{0k} \rangle_{L^1(\Omega)} \right)$$

$$+ 2\|B\| \|Z_{0k} - X_0\|_{L^2(\Omega, W)}^2$$

which converges to 0 as $k \to \infty$. It follows that there exists a suitable subsequence such that holds even in the case that $X_0$ is only known to be in $L^2(\Omega, W)$. From now on, assume this subsequence for the partitions $P_k$. Thus $k$ will really be $n_k$ and it suffices to consider the limit as $k \to \infty$ of the equation of 71.5.18. To emphasize this point again, the reason for the above observations is to argue that, even when $X_0$ is only in $L^2(\Omega, W)$, one can neglect

$$\langle B \left( X \left( t_1 \right) - X_0 - M \left( t_1 \right) \right), X \left( t_1 \right) - X_0 - M \left( t_1 \right) \rangle$$

in passing to the limit as $k \to \infty$ provided a suitable subsequence is used.

### 71.6 Convergence

The question is whether the above stochastic integral $\int_0^t (Z \circ J^{-1})^* BX \circ J dW$ converges as $n \to \infty$ in some sense to

$$(71.6.19) \quad \int_0^t (Z \circ J^{-1})^* BX \circ J dW$$

and whether the above is also a local martingale. Maybe it is well to pause and consider the integral and and what it means. $Z \circ J^{-1}$ maps $Q^{1/2}U$ to $W$ and so $(Z \circ J^{-1})^*$ maps $W'$ to $(Q^{1/2}U)'$. Thus

$$(Z \circ J^{-1})^* BX \in \left( Q^{1/2}U \right)' \quad \text{and} \quad (Z \circ J^{-1})^* BX \circ J \in Q^{1/2}(U)' \quad \text{is} \quad L_2 \left( Q^{1/2}U, \mathbb{R} \right)$$

Thus it has the right values.

Does the stochastic integral just written even make sense? The integrand is Hilbert Schmidt and has values in $\mathbb{R}$ so it seems like we ought to be able to define an integral. The problem is that the integrand is not in $L^2 ([0,T] \times \Omega; L^2 (Q^{1/2} U, \mathbb{R}))$.

By assumption, $t \to BX (t)$ is continuous into $V'$ thanks to the integral equation solved, and also $BX (t) = B (X (t))$ for $t \notin N_\omega$ a set of measure zero. For such $t$, it follows from Lemma 67.4.1,

$$\langle BX (t), X (t) \rangle = \sum_i \langle BX (t), e_i \rangle^2_{V', V} \text{ a.e.}\omega$$

and so $t \to \sum_i \langle BX (t), e_i \rangle^2_{V', V}$ is lower semicontinuous and as just explained, it equals $\langle BX (t), X (t) \rangle$ for a.e. $t$, this for each $\omega \notin N$, a single set of measure zero. Also, $t \to \sum_i \langle BX (t), e_i \rangle^2_{V', V}$ is progressively measurable and lower semicontinuous in $t$ so by Proposition 60.7.3, one can define a stopping time

$$\tau_p \equiv \inf \left\{ t : \sum_i \langle BX (t), e_i \rangle^2_{V', V} > p \right\}, \tau_0 \equiv 0 \quad (71.6.20)$$

Instead of referring to this Proposition, you could consider

$$\tau_p^m \equiv \inf \left\{ t : \sum_{i=1}^m \langle BX (t), e_i \rangle^2_{V', V} > p \right\}$$

which is clearly a stopping time because $t \to \sum_{i=1}^m \langle BX (t), e_i \rangle^2_{V', V}$ is a continuous process. Then observe that $\tau_p = \sup_m \tau_p^m$. Then

$$[\tau_p \leq t] = \bigcup_m [\tau_p^m \leq t] \in \mathcal{F}_t.$$

Is it the case that $\tau_p = \infty$ for all $p$ large enough? Yes, this follows from Lemma 61.4.4.

**Lemma 71.6.1** Suppose $\tau_p = \infty$ for all $p$ large enough off a set of measure zero, then

$$P \left( \int_0^T \left| (Z \circ J^{-1})^* BX \circ J \right|^2 \, dt < \infty \right) = 1$$

Also $\int_0^t (Z \circ J^{-1})^* BX \circ J dW$ can be defined as a local martingale.

**Proof:** Let

$$A \equiv \left\{ \omega : \int_0^T \left| (Z \circ J^{-1})^* BX \circ J \right|^2 \, dt = \infty \right\}$$

Then from the assumption that $\tau_p = \infty$ for all $p$ large enough, it follows that

$$A = \bigcup_{m=1}^\infty A \cap (\tau_m = \infty) \setminus (\tau_{m-1} < \infty).$$
Now
\[ P (A \cap [\tau_m = \infty]) \leq P \left( \omega : \int_0^T \mathcal{X}_{[0, \tau_m]} \left| (Z \circ J^{-1})^* BX \circ J \right|^2 dt = \infty \right) \] (71.6.21)

Look at the integrand. What is the meaning of \( (Z \circ J^{-1})^* BX \circ J \)? You have \( (Z \circ J^{-1})^* \in L_2 \left( W', J \left( Q^{1/2} U \right) \right) \) while \( BX \in W' \) and so \( (Z \circ J^{-1})^* BX \in L_2 \left( J \left( Q^{1/2} U \right), \mathbb{R} \right) \) which is just \( (J \left( Q^{1/2} U \right))' \). Thus \( (Z \circ J^{-1})^* BX \circ J \) would be in \( (Q^{1/2} U)' \) and to get the \( L_2 \) norm, you would take an orthonormal basis in \( Q^{1/2} U \) denoted as \( \{g_i\} \) and the square of this norm is just
\[
\sum_i \left[ (Z \circ J^{-1})^* BX \circ J (g_i) \right]^2 = \sum_i \left[ (Z \circ J^{-1})^* BX (Jg_i) \right]^2
\]
\[
= \sum_i [BX (Z \circ J^{-1}(Jg_i))]^2
\]
\[
= \sum_i [(BX)(Zg_i)]^2
\]
\[
\leq \sum_i \|BX\|^2 \|Zg_i\|^2_{W}
\]

Now incorporating the stopping time, you know that for a.e. \( t, \langle BX, X \rangle(t) = \langle BX(t), X(t) \rangle \leq m \) and so \( \|BX(t)\| \) can be estimated in terms of \( m \) as follows.
\[
|\langle BX(t), w \rangle| \leq \langle BX(t), X(t) \rangle^{1/2} \|B\|^{1/2} \|w\|_W
\]
\[
= \left( \sum_i \langle BX(t), e_i \rangle_{V_i, V}^2 \right)^{1/2} \|B\|^{1/2} \|w\|_W
\]
\[
\leq \sqrt{m} \|B\|^{1/2} \|w\|_W, \text{ so } \|BX(t)\| \leq m \|B\|^{1/2}
\]

Thus the integrand satisfies for a.e. \( t \)
\[
\mathcal{X}_{[0, \tau_m]} \left| (Z \circ J^{-1})^* BX \circ J \right|^2 \leq m \|B\| \|Z\|_{L_2}^2
\]

Hence, from (71.6.21), \( P (A \cap [\tau_m = \infty]) \)
\[
\leq P \left( \omega : \int_0^T \|Z\|_{L_2}^2 m \|B\| dt = \infty \right)
\]

However,
\[
\int_\Omega \int_0^T \|Z\|_{L_2}^2 m \|B\| dtdP < \infty
\]
by the assumptions on \( Z \). Therefore, \( P(A \cap [\tau_m = \infty]) = 0 \). It follows that

\[
P(A) = \sum_m P(A \cap ([\tau_m = \infty] \setminus [\tau_m-1 < \infty])) = \sum_m 0 = 0
\]

It follows that \( P\left( \int_0^T \left| (Z \circ J^{-1})^* BX \circ J \right|^2 \, dt < \infty \right) = 1 \) and so from Definition 71.4.2, one can define \( \int_0^T (Z \circ J^{-1})^* BX \circ J \, dW \) as a local martingale. \( \blacksquare \)

Convergence will be shown for a subsequence and from now on every sequence will be a subsequence of this one. As part of Lemma 71.6.2, see 71.4.1. It was shown that \( BX \in L^2([0,T], \Omega) \). Therefore, there exist partitions of \([0,T]\) like the above such that

\[
BX_{n_k}^l, BX_{n_k}^r \to BX \text{ in } L^2([0,T], \Omega) \]

in addition to the convergence of \( X_{n_k}^l, X_{n_k}^r \) to \( X \) in \( K \). From now on, the argument will involve a subsequence of these.

**Lemma 71.6.2** There exists a subsequence still denoted with the subscript \( k \) and an enlarged set of measure zero \( N \) including the earlier one such that \( BX_{n_k}^l(t), BX_{n_k}^r(t) \) also converges pointwise a.e. \( t \) to \( BX(t) \) in \( W' \) and \( X_{n_k}^l(t), X_{n_k}^r(t) \) converge pointwise a.e. in \( V \) to \( X(t) \) for \( \omega \notin N \) as well as having convergence of \( X_{n_k}^l(\cdot, \omega) \) to \( X(\cdot, \omega) \) in \( L^p([0,T]; V) \) and \( BX_{n_k}^l(\cdot, \omega) \) to \( BX(\cdot, \omega) \) in \( L^2([0,T]; W) \).

**Proof:** To see that such a sequence exists, let \( n_k \) be such that

\[
\int_\Omega \int_0^T \| BX_{n_k}^l(t) - BX(t) \|_{W'}^2 \, dt \, dP + \int_\Omega \int_0^T \| X_{n_k}^r(t) - X(t) \|_V^p \, dt \, dP + \\
\int_\Omega \int_0^T \| BX_{n_k}^r(t) - BX(t) \|_{W'}^2 \, dt \, dP + \int_\Omega \int_0^T \| X_{n_k}^l(t) - X(t) \|_V^p \, dt \, dP < 4^{-k}.
\]

Then

\[
P\left( \int_0^T \| BX_{n_k}^l(t) - BX(t) \|_{W'}^2 \, dt + \int_0^T \| X_{n_k}^r(t) - X(t) \|_V^p \, dt > 2^{-k} \right)
\]

\[
\leq 2^k (4^{-k}) = 2^{-k}
\]

and so by Borel Cantelli lemma, there is a set of measure zero \( N \) such that if \( \omega \notin N \),

\[
\int_0^T \| BX_{n_k}^l(t) - BX(t) \|_{W'}^2 \, dt + \int_0^T \| X_{n_k}^r(t) - X(t) \|_V^p \, dt + \\
\int_0^T \| BX_{n_k}^r(t) - BX(t) \|_{W'}^2 \, dt + \int_0^T \| X_{n_k}^l(t) - X(t) \|_V^p \, dt \leq 2^{-k}
\]
for all $k$ large enough. By the usual proof of completeness of $L^p$, it follows that $X_n(t) \to X(t)$ for a.e. $t$, this for each $\omega \notin N$, a similar assertion holding for $X_{n_k}^r$.

We denote these subsequences as $\{X^r_k\}_{k=1}^\infty$, $\{X^l_k\}_{k=1}^\infty$.

Now with this preparation, it is possible to show the desired convergence.

**Lemma 71.6.3** In the above context, let $X(s) - X_k^l(s) \equiv \Delta_k^l(s)$. Then the integral

$$\int_0^t (Z \circ J^{-1})^* BX \circ JdW$$

exists as a local martingale and the following limit is valid for the subsequence of Lemma 71.6.2

$$\lim_{k \to \infty} P \left( \sup_{t \in [0,T]} \int_0^t (Z \circ J^{-1})^* B\Delta_k \circ JdW \geq \varepsilon \right) = 0.$$

That is,

$$\sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B\Delta_k \circ JdW \right|$$

converges to 0 in probability.

**Proof:** In the argument $\tau_m$ will be defined in 71.6.20. Let

$$A_k \equiv \left\{ \omega : \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B\Delta_k \circ JdW \right| \geq \varepsilon \right\}$$

then

$$A_k \cap \{ \omega : \tau_m = \infty \} \subseteq \left\{ \omega : \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B\Delta_{\tau_m}^k \circ JdW \right| \geq \varepsilon \right\}$$

By Burkholder Davis Gundy inequality,

$$P( A_k \cap \{ \omega : \tau_m = \infty \}) \leq \frac{C}{\varepsilon} \int_{\Omega} \left( \int_0^T \|Z\|_{L^2}^2 \|B\Delta_{\tau_m}^k\|^2 dt \right)^{1/2} dP$$

Recall that if $\langle Bx, x \rangle \leq m$, then $\|Bx\|_{W^r} \leq m^{1/2} \|B\|^{1/2}$. Then the integrand is bounded for a.e. $t$ by $\|Z\|_{L^2}^2 4m \|B\|$. Next use the result of Lemma 71.6.2 and
the dominated convergence theorem to conclude that the above converges to 0 as 
$k \to \infty$. Then from the assumption that $\tau_m = \infty$ for all $m$ large enough,

$$P(A_k) = \sum_{m=1}^{\infty} P(A_k \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty]))$$

Now $\sum_m P([\tau_m = \infty] \setminus [\tau_{m-1} < \infty]) = 1$ and so, one can apply the dominated convergence theorem to conclude that

$$\lim_{k \to \infty} P(A_k) = \sum_{m=1}^{\infty} \lim_{k \to \infty} P(A_k \cap ([\tau_m = \infty] \setminus [\tau_{m-1} < \infty])) = 0$$

**Lemma 71.6.4** Let $X$ be as in Situation [71.2.2] and let $X^l_k$ be as in Lemma [71.1.1] corresponding to $X$ above. Let $X^l_k$ and $X^r_k$ both converge to $X$ in $K$ and also $BX^l_k, BX^r_k \to BX$ in $L^2([0, T] \times \Omega, W')$

Say

$$X^l_k(t) = \sum_{j=0}^{m_k} X(t_j) X_{[t_j, t_{j+1})}(t), \quad (71.6.22)$$

$$BX^l_k(t) = \sum_{j=0}^{m_k} BX(t_j) X_{[t_j, t_{j+1})}(t) \quad (71.6.23)$$

Then the sum in (71.6.23) is progressively measurable into $W'$. As mentioned earlier, we can take $X(0) \equiv 0$ in the definition of the “left step function”.

**Proof:** This follows right away from the definition of progressively measurable.

One can take a further subsequence such that uniform convergence is obtained.

**Lemma 71.6.5** Let $X(s) - X^l_k(s) \equiv \Delta_k(s)$. Then the following limit occurs.

$$\lim_{k \to \infty} P\left(\sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B\Delta_k \circ JdW \right| \geq \varepsilon \right) = 0$$

The stochastic integral

$$\int_0^t (Z \circ J^{-1})^* BX_k \circ JdW$$

makes sense because $BX$ is $W'$ progressively measurable and is in $L^2([0, T] \times \Omega; W')$. Also, there exists a further subsequence, still denoted as $k$ such that

$$\int_0^t (Z \circ J^{-1})^* BX_k \circ JdW \to \int_0^t (Z \circ J^{-1})^* BX \circ JdW$$

uniformly on $[0, T]$ for a.e. $\omega$. 

71.7. THE ITO FORMULA

Proof: This follows from Lemma 71.6.3. The last conclusion follows from the usual use of the Borel Cantelli lemma. There exists a further subsequence, still denoted with subscript $k$ such that

$$P \left( \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B\Delta_k \circ JdW \right| \geq \frac{1}{k} \right) < 2^{-k}$$

Then by the Borel Cantelli lemma, one can enlarge the set of measure zero such that for $\omega \notin N$,

$$\sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B\Delta_k \circ JdW \right| < \frac{1}{k}$$

for all $k$ large enough. That is, the claimed uniform convergence holds. $\blacksquare$

From now on, the sequence will either be this subsequence or a further subsequence.

71.7 The Ito Formula

Now at long last, here is the first version of the Ito formula valid on the partition points.

Lemma 71.7.1 In Situation 71.2.1, let $D$ be as above, the union of all the positive mesh points for all the $P_k$. Also assume $X_0 \in L^2(\Omega; W)$. Then for $\omega \notin N$ the exceptional set of measure zero in $\Omega$ and every $t \in D$,

$$\langle BX (t), X (t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t (2 \langle Y (s), X (s) \rangle + \langle BZ, Z \rangle_{L^2}) \, ds$$

$$+ 2 \int_0^t (Z \circ J^{-1})^* BX \circ JdW$$

(71.7.24)

where, in the above formula,

$$\langle BZ, Z \rangle_{L^2} \equiv (R^{-1} BZ, Z)_{L^2(Q^{1/2}U, W)}$$

for $R$ the Riesz map from $W$ to $W'$.

Note first that for $\{g_i\}$ an orthonormal basis for $Q^{1/2}(U)$,

$$(R^{-1} BZ, Z)_{L^2} \equiv \sum_i (R^{-1} BZ (g_i), Z (g_i))_W = \sum_i \langle BZ (g_i), Z (g_i) \rangle_{W'W} \geq 0$$

Proof: Let $t \in D$. Then $t \in P_k$ for all $k$ large enough. Consider $\{X^{(k)}\}$

$$\langle BX (t), X (t) \rangle - \langle BX_0, X_0 \rangle = \varepsilon (k) + 2 \int_0^t \langle Y (u), X^k (u) \rangle \, du$$
\[ +2 \int_0^t (Z \circ J^{-1})^* BX_k^t \circ JdW + \sum_{j=0}^{q_k-1} \langle B (M (t_{j+1}) - M (t_j)), M (t_{j+1}) - M (t_j) \rangle \]
\[ - \sum_{j=0}^{q_k-1} \langle B (\Delta X (t_j) - \Delta M (t_j)), \Delta X (t_j) - \Delta M (t_j) \rangle \]
\[ (71.7.25) \]

where \( t_{q_k} = t \), \( \Delta X (t_j) = X (t_{j+1}) - X (t_j) \) and \( e (k) \to 0 \) in probability. By Lemma 71.6.5 the stochastic integral on the right converges uniformly for \( t \in [0, T] \) to
\[ 2 \int_0^t (Z \circ J^{-1})^* BX \circ JdW \]
for \( \omega \) off a set of measure zero. The deterministic integral on the right converges uniformly for \( t \in [0, T] \) to
\[ 2 \int_0^t \langle Y (u), X (u) \rangle \, du \]
thanks to Lemma 71.6.2.

\[ \left| \int_0^t \langle Y (u), X (u) \rangle \, du - \int_0^t \langle Y (u), X_k^t (u) \rangle \, du \right| \leq \int_0^T \| Y (u) \|_{V'} \| X (u) - X_k^t (u) \|_V \]
\[ \leq \| Y \|_{L^{p'} ([0, T])} 2^{-k} \]

for all \( k \) large enough. Consider the fourth term. It equals
\[ \sum_{j=0}^{q_k-1} \left( R^{-1} B (M (t_{j+1}) - M (t_j)), M (t_{j+1}) - M (t_j) \right)_W \]
\[ (71.7.26) \]

where \( R^{-1} \) is the Riesz map from \( W \) to \( W' \). This equals
\[ \frac{1}{4} \left( \sum_{j=0}^{q_k-1} \| R^{-1} BM (t_{j+1}) + M (t_{j+1}) - (R^{-1} BM (t_j) + M (t_j)) \|^2 \right) \]
\[ - \sum_{j=0}^{q_k-1} \| R^{-1} BM (t_{j+1}) - M (t_{j+1}) - (R^{-1} BM (t_j) - M (t_j)) \|^2 \]

From Theorem 71.6.4, as \( k \to \infty \), the above converges in probability to \( (t_{q_k} = t) \)
\[ \frac{1}{4} \left( [R^{-1} BM + M] (t) - [R^{-1} BM - M] (t) \right) \]

However, from the description of the quadratic variation of \( M \), the above equals
\[ \frac{1}{4} \left( \int_0^t \| R^{-1} BZ + Z \|^2_{L_2} ds - \int_0^t \| R^{-1} BZ - Z \|^2_{L_2} ds \right) \]
which equals
\[ \int_0^t (R^{-1}BZ, Z)_{\mathcal{L}_2} ds \equiv \int_0^t (BZ, Z)_{\mathcal{L}_2} ds \]

This is what was desired.

Note that in the case of a Gelfand triple, when \( W = H = H' \), the term \( (BZ, Z)_{\mathcal{L}_2} \) will end up reducing to nothing more than \( \|Z\|_{\mathcal{L}_2}^2 \).

Thus all the terms in (71.7.25) converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

\[
(BX(t), X(t)) - (BX_0, X_0) = 2 \int_0^t (Y(u), X(u)) du \\
+ 2 \int_0^t (Z \circ J^{-1})^* BX \circ J dW + \int_0^t (BZ, Z)_{\mathcal{L}_2} ds - a
\]

where \( a \) is the limit in probability of the term

\[
\sum_{j=1}^{q_k-1} (B \Delta X(t_j) - \Delta M(t_j), \Delta X(t_j) - \Delta M(t_j)) \quad (71.7.27)
\]

Let \( P_n \) be the projection onto span \((e_1, \cdots, e_n)\) where \( \{e_k\} \) is an orthonormal basis for \( W \) with each \( e_k \in V \). Then using

\[
BX(t_{j+1}) - BX(t_j) - (BM(t_{j+1}) - BM(t_j)) = \int_{t_j}^{t_{j+1}} Y(s) ds
\]

the troublesome term of (71.7.25) above is of the form

\[
\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} (Y(s), \Delta X(t_j) - \Delta M(t_j)) ds \\
= \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} (Y(s), \Delta X(t_j) - P_n \Delta M(t_j)) ds \\
+ \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} (Y(s), -(I - P_n) \Delta M(t_j)) ds
\]

which equals

\[
\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} (Y(s), X(t_{j+1}) - X(t_j) - P_n (M(t_{j+1}) - M(t_j))) ds \quad (71.7.28)
\]
CHAPTER 71. THE HARD ITO FORMULA, IMPLICIT CASE

+ \sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), -(I - P_n)(M(t_{j+1}) - M(t_j)) \rangle \quad (71.7.29)

The reason for the $P_n$ is to get $P_n(M(t_{j+1}) - M(t_j))$ in $V$. The sum in (71.7.29) is dominated by

\begin{align*}
\left( \sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), (\Delta X(t_j) - \Delta M(t_j)) \rangle \right)^{1/2} \\
\left( \sum_{j=1}^{q_k-1} |\langle B(I - P_n)\Delta M(t_j), (I - P_n)\Delta M(t_j) \rangle|^2 \right)^{1/2}
\end{align*}

(71.7.30)

Now it is known from the above that $\sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), (\Delta X(t_j) - \Delta M(t_j)) \rangle$ converges in probability to $a \geq 0$. If you take the expectation of the square of the other factor, it is no larger than

$\|B\| E \left( \sum_{j=1}^{q_k-1} \| (I - P_n) \Delta M(t_j) \|_W^2 \right)$

$= \|B\| \left( \sum_{j=1}^{q_k-1} \left\| \int_{t_j}^{t_{j+1}} (I - P_n) Z(s) dW(s) \right\|_W^2 \right)$

$= \|B\| \sum_{j=1}^{q_k-1} E \left( \left\| \int_{t_j}^{t_{j+1}} (I - P_n) Z(s) dW(s) \right\|_W^2 \right)$

$= \|B\| \sum_{j=1}^{q_k-1} E \left( \int_{t_j}^{t_{j+1}} \left\| (I - P_n) Z(s) \right\|_{L_2(Q^{1/2}U,W)}^2 d\tau \right)$

$\leq \|B\| E \left( \int_0^T \left\| (I - P_n) Z(s) \right\|_{L_2(Q^{1/2}U,H)}^2 d\tau \right)$

Now letting $\{g_i\}$ be an orthonormal basis for $Q^{1/2}U$,

$= \|B\| \int_0^T \sum_{i=1}^\infty \left\| (I - P_n) Z(s) (g_i) \right\|_W^2 d\tau dP \quad (71.7.31)$

The integrand $\sum_{i=1}^\infty \left\| (I - P_n) Z(s) (g_i) \right\|_W^2$ converges to 0. Also, it is dominated by

$\sum_{i=1}^\infty \|Z(s) (g_i)\|_W^2 \equiv \|Z\|_{L_2(Q^{1/2}U,W)}^2$
which is given to be in $L^1 ([0, T] \times \Omega)$. Therefore, from the dominated convergence theorem, the expression in (71.7.31) converges to 0 as $n \to \infty$.

Thus the expression in (71.7.30) is of the form $f_k g_{nk}$ where $f_k$ converges in probability to $a^{1/2}$ as $k \to \infty$ and $g_{nk}$ converges in probability to 0 as $n \to \infty$ independently of $k$. Now this implies $f_k g_{nk}$ converges in probability to 0. Here is why.

\[
P (|f_k g_{nk}| > \varepsilon) \leq P (2\delta |f_k| > \varepsilon) + P (2C_{\delta} |g_{nk}| > \varepsilon)
\leq P \left( 2\delta \left| f_k - a^{1/2} \right| + 2\delta \left| a^{1/2} \right| > \varepsilon \right) + P (2C_{\delta} |g_{nk}| > \varepsilon)
\]

where $\delta |f_k| + C_{\delta} |g_{nk}| > |f_k g_{nk}|$ and $\lim_{\delta \to 0} C_{\delta} = \infty$. Pick $\delta$ small enough that $\varepsilon - 2\delta a^{1/2} > \varepsilon/2$. Then this is dominated by

\[
P \left( 2\delta \left| f_k - a^{1/2} \right| > \varepsilon/2 \right) + P (2C_{\delta} |g_{nk}| > \varepsilon)
\]

Fix $n$ large enough that the second term is less than $\eta$ for all $k$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in (71.7.30) and consequently in (71.7.29) converges to 0 in probability.

Now consider the other term (71.7.28) using the $n$ just determined. This term is of the form

\[
\sum_{j=1}^{q-1} \int_{t_j}^{t_{j+1}} \langle Y (s), X (t_{j+1}) - X (t_j) - P_n (M (t_{j+1}) - M (t_j)) \rangle \, ds =
\]

\[
\sum_{j=1}^{q-1} \int_{t_j}^{t_{j+1}} \langle Y (s), X_k^r (s) - X_k^l (s) - P_n (M^r_k (s) - M^l_k (s)) \rangle \, ds
\]

\[
= \int_{t_1}^{t} \langle Y (s), X_k^r (s) - X_k^l (s) - P_n (M^r_k (s) - M^l_k (s)) \rangle \, ds
\]

where $M^r_k$ denotes the step function

\[
M^r_k (t) = \sum_{i=0}^{m_k-1} M (t_{i+1}) \chi_{(t_i, t_{i+1}]} (t)
\]

and $M^l_k$ is defined similarly. The term

\[
\int_{t_1}^{t} \langle Y (s), P_n (M^r_k (s) - M^l_k (s)) \rangle \, ds
\]

converges to 0 for a.e. $\omega$ as $k \to \infty$ thanks to continuity of $t \to M (t)$. However, more is needed than this. Define the stopping time

\[
\tau_p = \inf \{ t > 0 : \|M (t)\|_W > p \}.
\]
Then $\tau_p = \infty$ for all $p$ large enough, this for a.e. $\omega$. Let

$$A_k = \left[ \left[ \int_{t_1}^t \langle Y(s), P_n (M^r_k(s) - M^l_k(s)) \rangle ds \right] > \varepsilon \right]$$

$$P(A_k) = \sum_{p=0}^{\infty} P(A_k \cap (\{\tau_p = \infty\} \setminus \{\tau_{p-1} < \infty\})) \quad (71.7.32)$$

Now

$$P(A_k \cap (\{\tau_p = \infty\} \setminus \{\tau_{p-1} < \infty\})) \leq P \left( \left[ \int_{t_1}^t \langle Y(s), P_n ((M^r)^i_k(s) - (M^l)^i_k(s)) \rangle ds \right] > \varepsilon \right)$$

This is so because if $\tau_p = \infty$, then it has no effect but also it could happen that the defining inequality may hold even if $\tau_p < \infty$ hence the inequality. This is no larger than an expression of the form

$$\frac{C_n}{\varepsilon} \int_{\Omega} \int_0^T \|Y(s)\|_{V^r} \| (M^r)^i_k(s) - (M^l)^i_k(s) \|_W \, ds \, dP \quad (71.7.33)$$

The inside integral converges to 0 by continuity of $M$. Also, thanks to the stopping time, the inside integral is dominated by an expression of the form

$$\int_0^T \|Y(s)\|_{V^r} \, 2pds$$

and this is a function in $L^1(\Omega)$ by assumption on $Y$. It follows that the integral in $71.7.33$ converges to 0 as $k \to \infty$ by the dominated convergence theorem. Hence

$$\lim_{k \to \infty} P(A_k \cap (\{\tau_p = \infty\})) = 0.$$ 

Since the sets $[\tau_p = \infty] \setminus [\tau_{p-1} < \infty]$ are disjoint, the sum of their probabilities is finite. Hence there is a dominating function in $71.7.32$ and so, by the dominated convergence theorem applied to the sum,

$$\lim_{k \to \infty} P(A_k) = \sum_{p=0}^{\infty} \lim_{k \to \infty} P(A_k \cap (\{\tau_p = \infty\} \setminus \{\tau_{p-1} < \infty\})) = 0$$

Thus $\int_{t_1}^t \langle Y(s), P_n (M^r_k(s) - M^l_k(s)) \rangle ds$ converges to 0 in probability as $k \to \infty$. Now consider

$$\left| \int_{t_1}^t \langle Y(s), X^r_k(s) - X^l_k(s) \rangle ds \right| \leq \int_0^T |\langle Y(s), X^r_k(s) - X(s) \rangle| ds$$

$$+ \int_0^T |\langle Y(s), X^l_k(s) - X(s) \rangle| ds$$
71.7. THE ITO FORMULA

\[ \leq 2 \|Y(\cdot, \omega)\|_{L^p([0,T])} 2^{-k}, \]

for all \( k \) large enough, this by Lemma 71.6.2. Therefore,

\[ \sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), \Delta X(t_j) - \Delta M(t_j) \rangle \]

converges to 0 in probability. This establishes the desired formula for \( t \in D \). \( \blacksquare \)

In fact, the formula \( 71.7.24 \) is valid for all \( t \in N^C_\omega \).

**Theorem 71.7.2** In Situation 71.2.1, for \( \omega \) off a set of measure zero, for every \( t \notin N_\omega \),

\[ \langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t \left( 2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle_{\mathcal{L}_2} \right) ds \]

\[ + 2 \int_0^t (Z \circ J^{-1})^* B X \circ J dW \quad (71.7.34) \]

Also, there exists a unique continuous, progressively measurable function denoted as \( \langle BX, X \rangle \) such that it equals \( \langle BX(t), X(t) \rangle \) for a.e. \( t \) and \( \langle BX, X \rangle(t) \) equals the right side of the above for all \( t \). In addition to this,

\[ E(\langle BX, X \rangle(t)) = \]

\[ E(\langle BX_0, X_0 \rangle) + E \left( \int_0^t \left( 2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle_{\mathcal{L}_2} \right) ds \right) \quad (71.7.35) \]

Also the quadratic variation of the stochastic integral in \( 71.7.34 \) is dominated by

\[ C \int_0^t \|Z\|_{\mathcal{L}_2}^2 \|BX\|_{W'}^2, ds \quad (71.7.36) \]

for a suitable constant \( C \). Also \( t \to BX(t) \) is continuous with values in \( W' \) for \( t \in N^C_\omega \).

**Proof:** Let \( t \in N^C_\omega \setminus D \). For \( t > 0 \), let \( t(k) \) denote the largest point of \( \mathcal{P}_k \) which is less than \( t \). Suppose \( t(m) < t(k) \). Hence \( m \leq k \). Then

\[ BX(t(m)) = BX_0 + \int_0^{t(m)} Y(s) ds + B \int_0^{t(m)} Z(s) dW(s), \]

a similar formula holding for \( X(t(k)) \). Thus for \( t > t(m), t \notin N_\omega \),

\[ B(X(t) - X(t(m))) = \int_{t(m)}^t Y(s) ds + B \int_{t(m)}^t Z(s) dW(s) \]
which is the same sort of thing studied so far except that it starts at $t(m)$ rather
than at 0 and $BX_0 = 0$. Therefore, from Lemma 71.7.1 it follows
\[
\langle B (X (t (k)) - X (t (m))) , X (t (k)) - X (t (m)) \rangle \\
= \int_{t(m)}^{t(k)} (2 \langle Y (s) , X (s) - X (t (m)) \rangle + \langle BZ , Z \rangle_{\mathcal{L}_2}) \, ds
\]
\[
+ 2 \int_{t(m)}^{t(k)} (Z \circ J^{-1})^* B (X (s) - X (t (m))) \circ JdW \tag{71.7.37}
\]
Consider that last term. It equals
\[
2 \int_{t(m)}^{t(k)} (Z \circ J^{-1})^* B (X (s) - X_m^l (s)) \circ JdW \tag{71.7.38}
\]
This is dominated by
\[
2 \int_0^{t(k)} (Z \circ J^{-1})^* B (X (s) - X_m^l (s)) \circ JdW \\
- \int_0^{t(m)} (Z \circ J^{-1})^* B (X (s) - X_m^l (s)) \circ JdW
\]
\[
\leq 4 \sup_{t \in [0,T]} \left| \int_0^t (Z \circ J^{-1})^* B (X (s) - X_m^l (s)) \circ JdW \right|
\]
In Lemma 71.6.5 the above expression was shown to converge to 0 in probability.
Therefore, by the usual appeal to the Borel Cantelli lemma, there is a subsequence
still referred to as $\{m\}$, such that it converges to 0 pointwise in $\omega$ for all $\omega$ off some
set of measure 0 as $m \to \infty$. It follows there is a set of measure 0 including the
earlier one such that for $\omega$ not in that set, \text{71.7.38} converges to 0 in $\mathbb{R}$. Similar
reasoning shows the first term on the right in the non stochastic integral of \text{71.7.37}
is dominated by an expression of the form
\[
4 \int_0^T \left| \langle Y (s) , X (s) - X_m^l (s) \rangle \right| \, ds
\]
which clearly converges to 0 thanks to Lemma 71.6.4. Finally, it is obvious that
\[
\lim_{m \to \infty} \int_{t(m)}^{t(k)} \langle BZ , Z \rangle_{\mathcal{L}_2} \, ds = 0 \quad \text{for a.e. } \omega
\]
due to the assumptions on $Z$. For $\{g_i\}$ an orthonormal basis of $Q^{1/2} (U)$,
\[
\langle BZ , Z \rangle_{\mathcal{L}_2} \equiv \sum_i (R^{-1} BZ (g_i) , Z (g_i)) = \sum_i \langle BZ (g_i) , Z (g_i) \rangle \\
\leq \|B\| \sum_i \|Z (g_i)\|^2_{L^1 (0,T)} \in L^1 (0,T) \quad \text{a.e.}
\]
This shows that for $\omega$ off a set of measure 0
\[
\lim_{m,k \to \infty} \langle B (X (t (k))) - X (t (m)), X (t (k)) - X (t (m)) \rangle = 0
\]
Then for $x \in W$,
\[
|\langle B (X (t (k))) - X (t (m)), x \rangle| \\
\leq \langle B (X (t (k))) - X (t (m)), X (t (k)) - X (t (m)) \rangle^{1/2} \langle B x, x \rangle^{1/2} \\
\leq \langle B (X (t (k))) - X (t (m)), X (t (k)) - X (t (m)) \rangle^{1/2} \|B\|^{1/2} \|x\|_W
\]
and so
\[
\lim_{m,k \to \infty} \|B (X (t (k))) - B X (t (m))\|_W = 0
\]
Recall $t$ was arbitrary in $N^C_\omega$ and $\{t (k)\}$ is a sequence converging to $t$. Then the above has shown that $\{B X (t (k))\}_{k=1}^\infty$ is a convergent sequence in $W'$. Does it converge to $B X (t)$? Let $\xi (t) \in W'$ be what it converges to. Letting $v \in V$ then, since the integral equation shows that $t \to B X (t)$ is continuous into $V'$,
\[
\langle \xi (t), v \rangle = \lim_{k \to \infty} \langle B X (t (k)), v \rangle = \langle B X (t), v \rangle,
\]
and now, since $V$ is dense in $W$, this implies $\xi (t) = B X (t) = B (X (t))$. Recall also that it was shown earlier that $B X$ is weakly continuous into $W'$ hence the strong convergence of $\{B X (t (k))\}_{k=1}^\infty$ in $W'$ implies that it converges to $B X (t)$, this for any $t \in N^C_\omega$.

For every $t \in D$ and for $\omega$ off the exceptional set of measure zero described earlier,
\[
\langle B (X (t)), X (t) \rangle = \langle B X_0, X_0 \rangle + \int_0^t (2 \langle Y (s), X (s) \rangle + \langle B Z, Z \rangle \_2 ds) \ ds \\
+ 2 \int_0^t (Z \circ J^{-1})^* B X \circ J dW
\] (71.7.39)
Does this formula hold for all $t \in [0, T]$? Maybe not. However, it will hold for $t \notin N_\omega$. So let $t \notin N_\omega$.
\[
|\langle B X (t (k)), X (t (k)) \rangle - \langle B X (t), X (t) \rangle| \\
\leq |\langle B X (t (k)), X (t (k)) \rangle - \langle B X (t), X (t (k)) \rangle| \\
+ |\langle B X (t), X (t (k)) \rangle - \langle B X (t), X (t) \rangle| \\
= |\langle B (X (t (k)) - X (t)), X (t (k)) \rangle| + |\langle B (X (t (k)) - X (t)), X (t) \rangle| \\
\]
Then using the Cauchy Schwarz inequality on each term,
\[
\leq \langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle^{1/2} \\
\times \left( (\langle B X (t (k)), X (t (k)) \rangle^{1/2} + \langle B X (t), X (t) \rangle^{1/2} \right)^2
\]
As before, one can use the lower semicontinuity of
\[ t \to \langle B(X(t(k)) - X(t)), X(t(k)) - X(t) \rangle \]
on $N_C$ along with the boundedness of $\langle BX(t), X(t) \rangle$ also shown earlier off $N_\omega$ to conclude
\[
\left| \langle BX(t(k)) - X(t(k)) \rangle - \langle BX(t), X(t) \rangle \right|
\leq C \left( \langle B(X(t(k)) - X(t)), X(t(k)) - X(t) \rangle \right)^{1/2}
\leq C \lim_{m \to \infty} \inf \left( \langle B(X(t(k)) - X(t(m)), X(t(k)) - X(t(m)) \rangle \right)^{1/2} < \varepsilon
\]
provided $k$ is sufficiently large. Since $\varepsilon$ is arbitrary,
\[
\lim_{k \to \infty} \langle BX(t(k)), X(t(k)) \rangle = \langle BX(t), X(t) \rangle.
\]
It follows that the formula \[(71.7.39)\] is valid for all $t \notin N_\omega$. Now define the function $\langle BX, X \rangle(t)$ as
\[
\langle BX, X \rangle(t) = \begin{cases} 
\langle B(X(t)), X(t) \rangle, & t \notin N_\omega \\
\text{The right side of } (71.7.39) & \text{if } t \in N_\omega
\end{cases}
\]
Then in short, $\langle BX, X \rangle(t)$ equals the right side of \[(71.7.39)\] for all $t \in [0, T]$ and is consequently progressively measurable and continuous. Furthermore, for a.e. $t$, this function equals $\langle B(X(t)), X(t) \rangle$. Since it is known on a dense subset, it must be unique.

This implies that $t \to BX(t)$ is continuous with values in $W'$ for $t \notin N_\omega$. Here is why. The fact that the formula \[(71.7.39)\] holds for all $t \notin N_\omega$ implies that $t \to \langle BX(t), X(t) \rangle$ is continuous on $N_C$. Then for $x \in W$,
\[
\left| \langle BX(t) - BX(s), x \rangle \right| \leq \left( \langle B(X(t) - X(s)), X(t) - X(s) \rangle \right)^{1/2} \|B\|^{1/2} \|x\|_W.
\]
\[
(71.7.40)
\]
Also
\[
\langle B(X(t) - X(s)), X(t) - X(s) \rangle = \langle BX(t), X(t) \rangle + \langle BX(s), X(s) \rangle - 2 \langle BX(t), X(s) \rangle
\]
By weak continuity of $t \to BX(t)$ shown earlier,
\[
\lim_{t \to s} \langle BX(t), X(s) \rangle = \langle BX(s), X(s) \rangle.
\]
Therefore,
\[
\lim_{t \to s} \langle B(X(t) - X(s)), X(t) - X(s) \rangle = 0
\]
and so the inequality \[(71.7.40)\] implies the continuity of $t \to BX(t)$ into $W'$ for $t \notin N_\omega$. Note that by assumption this function is continuous into $V'$ for all $t$. 

It was also shown that it is weakly continuous into $W'$ on $[0,T]$ and hence it is bounded in $W'$.

Now consider the claim about the expectation. Since the stochastic integral
\[
2 \int_0^t (Z \circ J^{-1})^* BX \circ JdW
\]
is only a local martingale, it is necessary to employ a stopping time. We use the function $\langle BX, X \rangle$ to define this stopping time as
\[
\tau_p \equiv \inf \{ t > 0 : \langle BX, X \rangle (t) > p \}
\]
This is the first hitting time of a continuous process and so it is a valid stopping time. Using this, leads to
\[
\langle BX, X \rangle_{\tau_p} (t) = \langle BX_0, X_0 \rangle + \int_0^t \mathcal{X}_{[0,\tau_p]} (s) \left( 2 \langle Y (s), X (s) \rangle + \langle BZ, Z \rangle_{L^2} ds \right) ds
\]
\[
+ 2 \int_0^t \mathcal{X}_{[0,\tau_p]} (s) \left( Z \circ J^{-1} \right)^* BX_{\tau_p} \circ JdW
\]
By continuity of $\langle BX, X \rangle$, $\tau_p = \infty$ for all $p$ large enough. Take expectation of both sides of the above. In the integrand of the last term, $BX$ refers to the function $BX (t,\omega) \equiv B (X (t,\omega))$ and so it is progressively measurable because $X$ is assumed to be so. Hence $BX_{\tau_p}$ is also progressively measurable and for a.e. $s$, $\|BX (s \wedge \tau_p)\|_{W'} \leq \sqrt{p} \sqrt{\|B\|}$. Therefore, one can take expectations and get
\[
E (\langle BX, X \rangle_{\tau_p} (t)) = E (\langle BX_0, X_0 \rangle) + E \left( \int_0^t \mathcal{X}_{[0,\tau_p]} (s) \left( 2 \langle Y (s), X (s) \rangle + \langle BZ, Z \rangle_{L^2} ds \right) ds \right)
\]
Now let $p \to \infty$ and use the monotone convergence theorem on the left and the dominated convergence theorem on the right to obtain the desired result.

The claim about the quadratic variation follows from Corollary 63.11.1. ■
Chapter 72

A More Attractive Version

The following lemma is convenient.

Lemma 72.0.3 Let \( f_n \to f \) in \( L^p ([0, T] \times \Omega, E) \). Then there exists a subsequence \( n_k \) and a set of measure zero \( N \) such that if \( \omega \notin N \), then

\[
f_{n_k} (\cdot, \omega) \to f (\cdot, \omega)
\]

in \( L^p ([0, T], E) \) and for a.e. \( t \).

Proof: We have

\[
P \left( \left\| f_n - f \right\|_{L^p([0, T], E)} > \lambda \right) \leq \frac{1}{\lambda} \int_{\Omega} \left\| f_n - f \right\|_{L^p([0, T], E)} dP
\]

\[
\leq \frac{1}{\lambda} \| f_n - f \|_{L^p([0, T] \times \Omega, E)}
\]

Hence there exists a subsequence \( n_k \) such that

\[
P \left( \left\| f_{n_k} - f \right\|_{L^p([0, T], E)} > 2^{-k} \right) \leq 2^{-k}
\]

Then by the Borel Cantelli lemma, it follows that there exists a set of measure zero \( N \) such that for all \( k \) large enough and \( \omega \notin N \),

\[
\| f_{n_k} - f \|_{L^p([0, T], E)} \leq 2^{-k}
\]

Now by the usual arguments used in proving completeness, \( f_{n_k} (t) \to f (t) \) for a.e.t.

Also, we have the approximation lemma proved earlier, Lemma 63.3.1

Lemma 72.0.4 Let \( \Phi : [0, T] \times \Omega \to V \), be \( B ([0, T]) \times F \) measurable and suppose

\[
\Phi \in K = L^p ([0, T] \times \Omega; E), \ p \geq 1
\]
Then there exists a sequence of nested partitions, \( \mathcal{P}_k \subseteq \mathcal{P}_{k+1} \),
\[
\mathcal{P}_k \equiv \{ t^{k}_0, \ldots, t^{k}_{m_k} \}
\]
such that the step functions given by
\[
\Phi^+_k (t) \equiv \sum_{j=1}^{m_k} \Phi (t^{k}_j) X_{(t^{k}_{j-1}, t^{k}_j)} (t)
\]
\[
\Phi^-_k (t) \equiv \sum_{j=1}^{m_k} \Phi (t^{k}_{j-1}) X_{(t^{k}_{j-1}, t^{k}_j)} (t)
\]
both converge to \( \Phi \) in \( K \) as \( k \to \infty \) and
\[
\lim_{k \to \infty} \max \{ |t^{k}_j - t^{k}_{j+1}| : j \in \{0, \ldots, m_k \} \} = 0.
\]

Also, each \( \Phi (t^{k}_j) \) is in \( L^p (\Omega; E) \). One can also assume that \( \Phi (0) = 0 \).
The mesh points \( \{ t^{k}_j \}_{j=0}^{m_k} \) can be chosen to miss a given set of measure zero. In addition to this, we can assume that
\[
|t^{k}_j - t^{k}_{j-1}| = 2^{-n_k}
\]
except for the case where \( j = 1 \) or \( j = m_k \) when this might not be so. In the case of the last subinterval defined by the partition, we can assume
\[
|t^{k}_m - t^{k}_{m-1}| = |T - t^{k}_{m-1}| \geq 2^{-(n_k+1)}
\]

### 72.1 The Situation

Now consider the following situation. There are real separable Banach spaces \( V, W \) such that \( W \) is a Hilbert space and
\[
V \subseteq W, \ W' \subseteq V'
\]
where \( V \) is dense in \( W \). Also let \( B \in L (W, W') \) satisfy
\[
\langle Bw, w \rangle \geq 0, \ \langle Bu, v \rangle = \langle Bv, u \rangle
\]
Note that \( B \) does not need to be one to one. Also allowed is the case where \( B \) is the Riesz map. It could also happen that \( V = W \). Assume that \( B = B (\omega) \) where \( B \) is \( \mathcal{F}_0 \) measurable into \( L (W, W') \). This dependence on \( \omega \) will be suppressed in the interest of simpler notation. For convenience, assume \( \|B (\omega)\| \) is bounded. This is assumed mainly so that an estimate can be made on \( \langle BX_0, X_0 \rangle \) for \( X_0 \) given in \( L^2 (\Omega) \). It probably suffices to simply give an estimate on \( \|\langle BX_0, X_0 \rangle\|_{L^1 (\Omega)} \) along with something else on the Ito integral. However, it seems at this time like this is more trouble than it is worth.
Situation 72.1.1 Let $X$ have values in $V$ and satisfy the following

$$BX(t) = BX_0 + \int_0^t Y(s) \, ds + BM(t),$$  \hspace{1cm} (72.1.1)

$X_0 \in L^2(\Omega; W)$ and is $\mathcal{F}_0$ measurable. Here $M(t)$ is a continuous $L^2$ martingale having values in $W$. By this is meant that $\lim_{t \to 0^+} \|M(t)\|_{L^2(\Omega)} = 0$ and for each $\omega, \lim_{t \to 0^+} M(t) = 0$, $\|M\|_{L^2([0, T] \times \Omega)}^2 \in L^2(\Omega)$. Assume that $d[M] = kdm$ for $k \in L^1([0, T] \times \Omega)$, that is, the measure determined by the quadratic variation for the martingale is absolutely continuous with respect to Lebesgue measure as just described.

Assume $Y$ satisfies

$$Y \in K' \equiv L^p'([0, T] \times \Omega; V'),$$

the $\sigma$ algebra of measurable sets defining $K'$ will be the progressively measurable sets. Here $1/p' + 1/p = 1$, $p > 1$.

Also the sense in which the equation holds is as follows. For a.e. $\omega$, the equation holds in $V'$ for all $t \in [0, T]$. Thus we are considering a particular representative $X$ for which this happens. Also it is only assumed that $BX(t) = B(X(t))$ for a.e. $t$. Thus $BX$ is the name of a function having values in $V'$ for which $BX(t) = B(X(t))$ for a.e. $t$. Assume that $X$ is progressively measurable also and $X \in L^p([0, T] \times \Omega, V)$.

The goal is to prove the following Itô formula valid for a.e. $t$ for each $\omega$ off a set of measure zero.

$$\langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t (2 \langle Y(s), X(s) \rangle) \, ds$$

$$+ [R^{-1}BM, M](t) + 2 \int_0^t \langle BX, dM \rangle$$  \hspace{1cm} (72.1.2)

where $R$ is the Riesz map from $W$ to $W'$. The most significant feature of the last term is that it is a local martingale. The third term on the right is the covariation of the two martingales $R^{-1}BM$ and $M$. It will follow from the argument that this will be nonnegative.

Note that the assumptions on $M$ imply that $[M] \in L^1([0, T] \times \Omega)$.

72.2 Preliminary Results

Here are discussed some preliminary results which will be needed. From the integral equation, if $\phi \in L^q(\Omega; V)$ and $\psi \in C_c^\infty(0, T)$ for $q = \max(p, 2)$,

$$\int_\Omega \int_0^T ((BX)(t) - BM(t) - BX_0) \psi' \phi \, dt \, dP$$

$$= \int_\Omega \int_0^T \int_0^t Y(s) \psi'(t) \, ds \phi \, dt \, dP$$
Then the term on the right equals

\[
\int_{\Omega} \int_{0}^{T} \int_{s}^{T} Y(s) \psi'(t) \, dt \, ds \, d\phi(\omega) \, dP = \int_{\Omega} \left( - \int_{0}^{T} Y(s) \psi(s) \, ds \right) \, d\phi(\omega) \, dP
\]

It follows that, since \( \phi \) is arbitrary,

\[
\int_{0}^{T} ((BX)(t) - BM(t) - BX_{0}) \psi'(t) \, dt = - \int_{0}^{T} Y(s) \psi(s) \, ds
\]

in \( L^{q'}(\Omega; V') \) and so the weak time derivative of

\[
t \to (BX)(t) - BM(t) - BX_{0}
\]

equals \( Y \) in \( L^{q'}([0,T]; L^{q'}(\Omega, V')) \). Thus, by Theorem 22.4.1, for a.e. \( t \),

\[
B(X(t) - M(t)) = BX_{0} + \int_{0}^{t} Y(s) \, ds \text{ in } L^{q'}(\Omega, V').
\]

That is,

\[
(BX)(t) = BX_{0} + \int_{0}^{t} Y(s) \, ds + BM(t), \quad t \notin \hat{N}, \quad m(\hat{N}) = 0
\]

holds in \( L^{q'}(\Omega, V') \) where \( (BX)(t) = B(X(t)) \) a.e. \( t \) in this space, for all \( t \notin \hat{N}, \) a set of Lebesgue measure zero, in addition to holding for all \( t \) for each \( \omega \). Now let \( \{t_{k}^{n}\}_{k=1}^{\infty} \) be partitions for which, from Lemma 22.4.2, there are left and right step functions \( X_{k}, X_{k}' \), which converge in \( L^{p}([0,T] \times \Omega; V) \) to \( X \) and such that each \( \{t_{k}^{n}\}_{k=1}^{\infty} \) has empty intersection with the set of measure zero \( \hat{N} \) where, in \( L^{q'}(\Omega; V') \),

\[
(BX)(t) \neq B(X(t)) \text{ in } L^{q'}(\Omega, V').
\]

Thus for \( t_{k} \) a generic partition point,

\[
BX(t_{k}) = B(X(t_{k})) \text{ in } L^{q'}(\Omega, V')
\]

Hence there is an exceptional set of measure zero, \( N(t_{k}) \subseteq \Omega \) such that for \( \omega \notin N(t_{k}), BX(t_{k})(\omega) = B(X(t_{k},\omega)). \) Define an exceptional set \( N \subseteq \Omega \) to be the union of all these \( N(t_{k}). \) There are countably many and so \( N \) is also a set of measure zero. Then for \( \omega \notin N \), and \( t_{k} \) any mesh point at all, \( BX(t_{k})(\omega) = B(X(t_{k},\omega)). \) This will be important in what follows. In addition to this, from the integral equation, for each of these \( \omega \notin N, BX(t)(\omega) = B(X(t,\omega)) \) for all \( t \notin N_{\omega} \subseteq [0,T] \) where \( N_{\omega} \) is a set of Lebesgue measure zero. Thus the \( t_{k} \) from the various partitions are always in \( N_{\omega}. \) By Lemma 22.4.3, there exists a countable set \( \{e_{i}\} \) of vectors in \( V \) such that

\[
\langle Be_{i}, e_{j} \rangle = \delta_{ij}
\]

and for each \( x \in W, \)

\[
\langle Bx, x \rangle = \sum_{i=0}^{\infty} |\langle Bx, e_{i} \rangle|^{2}, \quad Bx = \sum_{i=1}^{\infty} \langle Bx, e_{i} \rangle Be_{i}
\]
By this lemma, if \( B = B(\omega) \) where \( B \) is \( \mathcal{F}_0 \) measurable into \( \mathcal{L}(W,W') \), then the \( e_i \) are also \( \mathcal{F}_0 \) measurable into \( V \). Thus the conclusion of the above discussion is that at the mesh points, it is valid to write

\[
\langle (BX) (t_k), X (t_k) \rangle = \langle B (X (t_k)), X (t_k) \rangle = \sum_i \langle (BX) (t_k), e_i \rangle^2 = \sum_i \langle B (X (t_k)), e_i \rangle^2
\]

just as would be the case if \( (BX) (t) = B (X (t)) \) for every \( t \). In all which follows, the mesh points will be like this and an appropriate set of measure zero which may be replaced with a larger set of measure zero finitely many times is being neglected. Obviously, one can take a subsequence of the sequence of partitions described above without disturbing the above observations. We will denote these partitions as \( P_k \).

Thus we obtain the following interesting lemma.

**Lemma 72.2.1** In the above situation, there exists a set of measure zero \( N \subseteq \Omega \) and a dense subset of \([0, T]\), \( D \) such that for \( \omega \notin N \), \( BX (t, \omega) = B (X (t, \omega)) \) for all \( t \in D \). This set \( D \) is the union of nested partitions \( \{ P_k \} = \{ t_k^i \}_{j=1,k=1}^{m, \infty} \) such that the left and right step functions \( \{ X_k^i \}, \{ X_k^r \} \) converge to \( X \) in \( L^p ([0, T] \times \Omega; V) \). There is also a set of Lebesgue measure zero \( \hat{N} \subseteq [0, T] \) such that \( BX (t) = B (X (t)) \) in \( L^q (\Omega; V') \) for all \( t \notin \hat{N} \). Thus for such \( t \), \( BX (t) (\omega) = B (X (t), \omega) \) for a.e.\( \omega \). In particular, for such \( t \notin \hat{N} \),

\[
\langle BX (t) (\omega), X (t, \omega) \rangle = \sum_i \langle B (X (t)), e_i \rangle^2 \text{ a.e.}\omega.
\]

\( D \) has empty intersection with \( \hat{N} \). There is also a set of Lebesgue measure zero \( N_\omega \) for each \( \omega \notin N \) defined by \( BX (t, \omega) = B (X (t, \omega)) \) for all \( t \notin N_\omega \).

Now define a stopping time.

\[
\sigma_q^n = \inf \left\{ t : \langle BX^n_t (t), X^n_t (t) \rangle > q \right\}, \tag{72.2.3}
\]

Thus this pertains to the \( n^{th} \) partition. Since \( X^n_t \) is right continuous, this will be a well defined stopping time. Thus, for \( t \) one of the partition points,

\[
\langle BX^{\sigma_q^n} (t, \omega), X^{\sigma_q^n} (t, \omega) \rangle \leq q \tag{72.2.4}
\]

From the definition of \( X^n_t \) and the observation that these partitions are nested,

\[
\lim_{n \to \infty} \sigma_q^n = \sigma_q
\]

exists because this is a decreasing sequence. There are more available times to consider as \( n \) gets larger and so when the \( \inf \) is taken, it can only get smaller. Thus

\[
[\sigma_q \leq t] = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \left[ \sigma_q^n \leq t + \frac{1}{m} \right] \in \bigcap_{m=1}^{\infty} \mathcal{F}_{t+(1/m)} = \mathcal{F}_t
\]

since it is assumed that the filtration is normal. Thus this appears to be a stopping time. However, I don’t know how to use this.
Theorem 72.2.2 Let \( \{t^n_j\}_{j=0}^{m_n} \) be the above sequence of partitions of the sort in Lemma 72.0.4 such that if
\[
X^l_n(t) = \sum_{j=0}^{m_n-1} X(t^n_j) X_{(t^n_j, t^n_{j+1})}(t)
\]
then \( X_n \to X \) in \( L^p([0,T] \times \Omega, V) \) with the other conditions holding which were discussed above. In particular, \( BX(t) = B(X(t)) \) for \( t \) one of these mesh points. Then the expression
\[
\sum_{j=0}^{m_n-1} \langle B\left(M(t_{j+1}^n \land t) - M(t_j^n \land t)\right), X(t_j^n) \rangle
\]
is a local martingale
\[
\int_0^t \langle BX^l_k, dM \rangle
\]
with \( \{\sigma^n_q\}_{q=1}^\infty \) being a localizing sequence.

Proof: This follows from Lemma 64.0.23. This can be seen because, thanks to the fact that \( BX^l_k \) is bounded, the function \( BX^l_k \) is in the set \( G \) described there. This is a place where we use that \( d[M] = kdt \).

72.3 The Main Estimate

The argument will be based on a formula which follows in the next lemma.

Lemma 72.3.1 In Situation 72.1.1 the following formula holds for a.e. \( \omega \) for \( 0 < s < t \). In the following, \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( V, V' \).
\[
\langle BX(t), X(t) \rangle = \langle BX(s), X(s) \rangle +
+2 \int_s^t \langle Y(u), X(t) \rangle \, du + \langle B(M(t) - M(s)), M(t) - M(s) \rangle
- \langle BX(t) - BX(s), X(t) - X(s) - (M(t) - M(s)) \rangle
+ 2 \langle BX(s), M(t) - M(s) \rangle
\]
Also for \( t > 0 \)
\[
\langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + 2 \int_0^t \langle Y(u), X(t) \rangle \, du + 2 \langle BX_0, M(t) \rangle +
\langle BM(t), M(t) \rangle - \langle BX(t) - BX_0 - BM(t), X(t) - X_0 - M(t) \rangle
\]
72.3. THE MAIN ESTIMATE

**Proof:** From the formula which is assumed to hold,

\[ BX(t) = BX_0 + \int_0^t Y(u) \, du + BM(t) \]

\[ BX(s) = BX_0 + \int_0^s Y(u) \, du + BM(s) \]

Then

\[ BM(t) - BM(s) + \int_s^t Y(u) \, du = BX(t) - BX(s) \]

It follows that

\[
\langle B(M(t) - M(s)), M(t) - M(s) \rangle - \\
\langle BX(t) - BX(s), (M(t) - M(s)) \rangle, X(t) - X(s) - (M(t) - M(s)) \rangle \\
+ 2 \langle BX(s), M(t) - M(s) \rangle \\
- (B(M(t) - M(s)), M(t) - M(s)) + 2 \langle BX(s), M(t) - M(s) \rangle
\]

Some terms cancel and this yields

\[ = - \langle BX(t) - BX(s), X(t) - X(s) \rangle + 2 \langle BX(t), M(t) - M(s) \rangle \]

\[ = - \langle BX(t) - BX(s), X(t) - X(s) \rangle + 2 \langle B(M(t) - M(s)), X(t) \rangle \]

\[ = - \langle BX(t) - BX(s), X(t) - X(s) \rangle + 2 \langle B(X(t) - X(s)), X(t) \rangle \\
+ 2 \langle BX(t) - BX(s), \int_s^t Y(u) \, du, X(t) \rangle \]

\[ = - \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle \\
+ 2 \langle BX(t), X(s) \rangle + 2 \langle BX(t), X(t) \rangle \\
- 2 \langle BX(s), X(s) \rangle + 2 \int_s^t \langle Y(u), X(t) \rangle \, du \]

\[ = \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle - 2 \int_s^t \langle Y(u), X(t) \rangle \, du \]

Therefore,

\[ \langle BX(t), X(t) \rangle - \langle BX(s), X(s) \rangle \\
= 2 \int_s^t \langle Y(u), X(t) \rangle \, du + \langle B(M(t) - M(s)), M(t) - M(s) \rangle \]
The following phenomenal estimate holds and it is this estimate which is the main idea in proving the Ito formula. The last assertion about continuity is like the well known result that if \( y \in L^p (0,T;V) \) and \( y' \in L^p (0,T;V') \), then \( y \) is actually continuous a.e. with values in \( H \), for \( V,H,V' \) a Gelfand triple. Later, this continuity result is strengthened further to give strong continuity.

**Lemma 72.3.2** In the Situation \([72.1.1,72.3.8]\), the following holds for all \( t \notin \mathbb{N} \):

\[
E \left( \langle BX (t), X (t) \rangle \right) < C \left( \|Y\|_{K'}, \|X\|_K, E \left( \|M\| (T) \right), \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right) < \infty. \tag{72.3.8}
\]

where \( K, K' \) were defined earlier. In fact,

\[
E \left( \sup_{t \in [0,T]} \sum_i \langle BX (t), c_i \rangle^2 \right) \leq C \left( \|Y\|_{K'}, \|X\|_K, E \left( \|M\| (T) \right), \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right)
\]

Also, \( C \) is a continuous function of its arguments, increasing in each one, and \( C (0,0,0,0) = 0 \). Thus for a.e. \( \omega \),

\[
\sup_{t \notin \mathbb{N}} \langle BX (t, \omega), X (t, \omega) \rangle \leq C (\omega) < \infty.
\]

Also for \( \omega \) off a set of measure zero described earlier, \( t \rightarrow BX (t) (\omega) \) is weakly continuous with values in \( W' \) on \( 0,T \). Also \( t \rightarrow \langle BX (t), X (t) \rangle \) is lower semi-continuous on \( \mathbb{N} \).

**Proof:** Consider the formula in Lemma \([72.3.1]\)

\[
\langle BX (t), X (t) \rangle = \langle BX (s), X (s) \rangle + 2 \int_s^t \langle Y (u), X (t) \rangle \, du + \langle B (M (t) - M (s)), M (t) - M (s) \rangle
\]

\[
- \langle B (X (t) - X (s) - (M (t) - M (s))), X (t) - X (s) - (M (t) - M (s)) \rangle
\]

\[
+ 2 \langle BX (s), M (t) - M (s) \rangle \tag{72.3.9}
\]

Now let \( t_j \) denote a point of \( \mathcal{P}_k \) from Lemma \([72.1.3]\). Then for \( t_j > 0 \), \( X (t_j) \) is just the value of \( X \) at \( t_j \) but when \( t = 0 \), the definition of \( X (0) \) in this step function is \( X (0) = 0 \). Thus

\[
\sum_{j=1}^{m-1} \langle BX (t_{j+1}), X (t_{j+1}) \rangle - \langle BX (t_j), X (t_j) \rangle + \langle BX (t_1), X (t_1) \rangle - \langle BX_0, X_0 \rangle
\]
72.3. THE MAIN ESTIMATE

\[ \langle BX (t_m), X (t_m) \rangle - \langle BX_0, X_0 \rangle \]

Using the formula in Lemma 72.3.1, for \( t = t_m \) this yields

\[ \langle BX (t_m), X (t_m) \rangle - \langle BX_0, X_0 \rangle = 2 \sum_{j=1}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y (u), X'_k (u) \rangle \, du + \]

\[ + 2 \sum_{j=1}^{m-1} \langle BX (t_j), M (t_{j+1}) - M (t_j) \rangle \]

\[ + \sum_{j=1}^{m-1} \left( B (M (t_{j+1}) - M (t_j)) \right) (t_{j+1} - t_j) \]

\[ - \sum_{j=1}^{m-1} \left( B (X (t_{j+1}) - X (t_j)) - (M (t_{j+1}) - M (t_j)) \right) \]

\[ + 2 \int_0^{t_1} \langle Y (u), X (t_1) \rangle \, du + 2 \langle BX_0, M (t_1) \rangle + \langle BM (t_1), M (t_1) \rangle \]

\[ - \langle B (X (t_1) - X_0 - M (t_1)) \rangle + (X (t_1) - X_0 - M (t_1)) \]  

(72.3.10)

First consider

\[ 2 \int_0^{t_1} \langle Y (u), X (t_1) \rangle \, du + 2 \langle BX_0, M (t_1) \rangle + \langle BM (t_1), M (t_1) \rangle . \]

Each term of the above converges to 0 for a.e. \( \omega \) as \( k \to \infty \) and in \( L^1 (\Omega) \). This follows right away for the second two terms from the assumptions on \( M \) given in the situation. Recall it was assumed that \( \| B (\omega) \| \) is bounded. This is where it is convenient to make this assumption. Consider the first term. This term is dominated by

\[ \left( \int_0^{t_1} \| Y (u) \|^{p'} \, du \right)^{1/p'} \left( \int_0^{t_1} \| X'_k (u) \|^{p} \, du \right)^{1/p} \]

\[ \leq C (\omega) \left( \int_0^{t_1} \| Y (u) \|^{p'} \, du \right)^{1/p'} \left( \int_\Omega C (\omega)^p \, dP \right)^{1/p} < \infty \]

Hence this converges to 0 for a.e. \( \omega \) and also converges to 0 in \( L^1 (\Omega) \).

At this time, not much is known about the last term in (72.3.11), but it is negative and is about to be neglected anyway.

The second term on the right equals

\[ 2 \int_{t_1}^{t_m} \langle BX'_k, dM \rangle = 2 \int_0^{t_m} \langle BX'_k, dM \rangle + e (k) \]
where \( e(k) \to 0 \) for a.e. \( \omega \) and in \( L^1(\Omega) \). Also note that since \( \langle BM(t_1), M(t_1) \rangle \) converges to 0 in \( L^1(\Omega) \) and for a.e. \( \omega \), the sum involving

\[
\langle B (M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle
\]

can be started at 0 rather than 1 at the expense of adding in a term which converges to 0 a.e. and in \( L^1(\Omega) \). Thus \( 72.3.10 \) is of the form

\[
\langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle = e(k) + 2 \int_0^{t_m} \langle Y(u), X^r_k(u) \rangle \, du + 2 \int_0^{t_m} \langle BX^l_k, dM \rangle + \sum_{j=0}^{m-1} \langle B (M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle - \sum_{j=1}^{m-1} \langle B (X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))), X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)) \rangle
\]

\[
- \langle B (X(t_1) - X_0 - (M(t_1))), X(t_1) - X_0 - M(t_1) \rangle
\]

(72.3.11)

where \( e(k) \to 0 \) for a.e. \( \omega \) and also in \( L^1(\Omega) \).

Now it follows, on discarding the negative terms,

\[
\langle BX(t_m), X(t_m) \rangle - \langle BX_0, X_0 \rangle \leq e(k) + 2 \int_0^{t_m} \langle Y(u), X^r_k(u) \rangle \, du + 2 \int_0^{t_m} \langle BX^l_k, dM \rangle + \sum_{j=0}^{m-1} \langle B (M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle
\]

Therefore,

\[
\sup_{t_m \in \mathcal{P}_k} \langle BX(t_m), X(t_m) \rangle \leq \langle BX_0, X_0 \rangle + e(k) + 2 \int_0^T \langle Y(u), X^r_k(u) \rangle \, du + 2 \sup_{t_m \in \mathcal{P}_k} \left| \int_0^{t_m} \langle BX^l_k, dM \rangle \right| + \sum_{j=0}^{m-1} \langle B (M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle
\]

(72.3.12)
where there are \( m_k + 1 \) points in \( P_k \). Consider that last term. It is no larger than

\[
\|B\| \sum_{j=0}^{m-1} \|M(t_{j+1}) - M(t_j)\|^2
\]

Say the last point in the partition is \( t_p = T \) and consider the sum

\[
\sum_{j=0}^{p-1} \|M(t_{j+1} \wedge t) - M(t_j \wedge t)\|^2 = \sum_{j=0}^{p-1} \|M^{t_{j+1}} - M^{t_j}\|^2(t)
\]

\[
= \sum_{j=0}^{p-1} [M^{t_{j+1}} - M^{t_j}](t) + N_j(t) = \sum_{j=0}^{p-1} [M^{t_{j+1}}(t) - [M]^{t_j}(t) + N_j(t)
\]

for \( N_j \) a martingale which equals 0 for \( t \leq t_j \). Now when you put in \( t = t_m \), this becomes

\[
\sum_{j=0}^{m-1} [M^{t_{j+1}}(t_m) - [M]^{t_j}(t_m) + N_j(t_m)
\]

Thus the expectation of that last term is no larger than

\[
\|B\| E \left( \sum_{j=0}^{m-1} [M^{t_{j+1}}(t_m) - [M]^{t_j}(t_m)\right) = \|B\| E ([M] (t_m))
\]

The next task is to take the expectation of both sides of \( 72.3.12 \). Of course there is a small problem with things not being in \( L^1 \). Hence it is appropriate to localize with the stopping time \( \sigma_k \) defined in \( 72.2.3 \). That is, we obtain all of the above with \( X \) replaced with \( X^{\sigma_k} \), stopping the original integral equation by introducing \( X_{[0, \sigma_k]} \) in the integrals. Then carry out the following argument and pass to a limit as \( q \to \infty \). In fact \( \sigma_k = \infty \) if \( q \) is large enough. Then carry out everything with \( X^{\sigma_k} \). We don’t write it, but this is what is being done in the following argument.

\[
E \left( \sup_{t_m \in P_k} \langle BX(t_m), X(t_m) \rangle \right) \leq E (\langle BX_0, X_0 \rangle) + E (|e (k)|) + 2 \|Y\|_{K'} \|X_{\epsilon}\|_K
\]

\[
+ 2E \left( \sup_{t_m \in P_k} \left| \int_0^{t_m} \langle BX^k, dM \rangle \right| \right) + \|B\| E ([M] (T))
\]

Now using the Burkholder Davis Gundy inequality and the inequality for the quadratic variation of that funny integral involving \( \langle BX^k, dM \rangle \),

\[
\leq E (\langle BX_0, X_0 \rangle) + E (|e (k)|) + 2 \|Y\|_{K'} \|X_{\epsilon}\|_K
\]

\[
+ CE \left( \left( \int_0^T \|BX^k\|^2 d[M] \right)^{1/2} \right) + \|B\| E ([M] (T))
\]
Now \( \|Bv\|^2 \leq \|B\| \langle Bv, v \rangle \). Hence the above reduces to the following after adjusting the constant \( C \),
\[
\leq E (\langle BX_0, X_0 \rangle) + E (|e(k)|) + 2 \|Y\|_{K'} \|X'_k\|_K
+ CE \left( \left( \int_0^T \langle BX'_k, X'_k \rangle d[M] \right)^{1/2} \right) + \|B\| E ([M](T))
\]
\[
\leq \frac{1}{2} \sup_{t_m \in P_k} \langle BX(t_m), X(t_m) \rangle + (C + \|B\|) E ([M](T))
+ C (E (\langle BX_0, X_0 \rangle), \|Y\|_{K'}, \|X'_k\|_K) + E (|e(k)|)
\]

It follows on subtracting the first term on the right and adjusting constants again,
\[
E \left( \sup_{t_m \in P_k} \langle BX(t_m), X(t_m) \rangle \right)
\leq (C + \|B\|) E ([M](T)) + C (E (\langle BX_0, X_0 \rangle), \|Y\|_{K'}, \|X'_k\|_K) + E (|e(k)|)
\]

Now let \( q \to \infty \) and use the monotone convergence theorem which yields the above for un-modified \( X \).

Observe that these partitions are nested and that the constant \( C (\cdots) \) is continuous and increasing in each argument with \( C (0) = 0 \), \( C (\cdots) \) not depending on \( T \). Thus the left side is increasing and for given \( \varepsilon > 0 \), there exists \( N \) such that \( k \geq N \) implies the right side is no larger than
\[
C (E (\langle BX_0, X_0 \rangle), \|Y\|_{K'}, \|X'_k\|_K) + E (|e(k)|) + \varepsilon \quad (72.3.13)
\]

Now let \( D \) denote the union of these nested partitions. Then from the monotone convergence theorem,
\[
E \left( \sup_{t \in D} \langle BX(t), X(t) \rangle \right)
\]
is no larger than the right side of (72.3.13). Since this is true for all \( \varepsilon > 0 \), it follows
\[
E \left( \sup_{t \in D} \langle BX(t), X(t) \rangle \right) \leq C (E (\langle BX_0, X_0 \rangle), \|Y\|_{K'}, \|X'_k\|_K, E ([M](T)))
\]

where \( C (\cdots) \) is increasing in each argument, continuous, and \( C (0) = 0 \). Thus, enlarging \( N \), for \( \omega \notin N \),
\[
\sup_{t \in D} \langle BX(t), X(t) \rangle = C (\omega) < \infty \quad (72.3.15)
\]

where \( \int_\Omega C (\omega) dP < \infty \). By Lemma 76.3.1, there exists a countable set \( \{e_i\} \) of vectors in \( V \) such that
\[
\langle Be_i, e_j \rangle = \delta_{ij}
\]
and for each $x \in W$, 
\[
\langle Bx, x \rangle = \sum_{i=0}^{\infty} \langle Bx, e_i \rangle^2, \quad Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i
\]
Thus for $t$ not in a set of measure zero off which $BX (t) = B (X (t))$, 
\[
\langle BX (t), X (t) \rangle = \sum_{i=0}^{\infty} \langle BX (t), e_i \rangle^2 = \sup_{m} \sum_{k=1}^{m} \langle BX (t), e_i \rangle^2
\]
Now from the formula for $BX (t)$, it follows that $BX$ is continuous into $V'$. For any $t \notin \mathcal{N}$ so that $(BX) (t) = B (X (t))$ in $L^2 (\Omega; V')$ and letting $t_k \to t$ where $t_k \in D$, Fatou's lemma implies
\[
E (\langle BX (t), X (t) \rangle) = \sum_{i} E (\langle BX (t), e_i \rangle^2) = \sum_{i} \lim_{k \to \infty} E (\langle BX (t_k), e_i \rangle^2)
\]
\[
\leq \lim_{k \to \infty} \inf_i \sum_{i} E (\langle BX (t_k), e_i \rangle^2) = \lim_{k \to \infty} E (\langle BX (t_k), X (t_k) \rangle)
\]
\[
\leq C \left( \|Y\|_{K'}, \|X\|_{K'}, \|Z\|_J, \|BX_0, X_0\|_{L^1 (\Omega)} \right)
\]
In addition to this, for arbitrary $t \in [0, T]$, and $t_k \to t$ from $D$,
\[
\sum_{i} \langle BX (t), e_i \rangle^2 \leq \inf_{k \to \infty} \sum_{i} \langle BX (t_k), e_i \rangle^2 \leq \sup_{s \in D} \langle BX (s), X (s) \rangle
\]
Hence
\[
\sup_{t \in [0, T]} \sum_{i} \langle BX (t), e_i \rangle^2 \leq \sup_{s \in D} \langle BX (s), X (s) \rangle
\]
\[
= \sup_{s \in D} \sum_{i} \langle BX (s), e_i \rangle^2 \leq \sup_{t \in [0, T]} \sum_{i} \langle BX (t), e_i \rangle^2
\]
It follows that $\sup_{t \in [0, T]} \sum_{i} \langle BX (t), e_i \rangle^2$ is measurable and
\[
E \left( \sup_{t \in [0, T]} \sum_{i} \langle BX (t), e_i \rangle^2 \right) \leq E \left( \sup_{s \in D} \langle BX (s), X (s) \rangle \right)
\]
\[
\leq C \left( \|Y\|_{K'}, \|X\|_{K'}, \|Z\|_J, \|BX_0, X_0\|_{L^1 (\Omega)} \right)
\]
And so, for $\omega$ off a set of measure zero, $\sup_{t \in [0, T]} \sum_{i} \langle BX (t), e_i \rangle^2$ is bounded above. Include this exceptional set in $N$. 
Also for $t \notin \mathcal{N}_\omega$ and a given $\omega \notin N$, letting $t_k \to t$ for $t_k \in D$, 
\[
\langle BX (t), X (t) \rangle = \sum_{i} \langle BX (t), e_i \rangle^2 \leq \lim_{k \to \infty} \sum_{i} \langle BX (t_k), e_i \rangle^2
\]
\[
= \lim_{k \to \infty} \inf_{k \to \infty} \langle BX (t_k), X (t_k) \rangle \leq \sup_{t \in D} \langle BX (t), X (t) \rangle
\]
and so
\[
\sup_{t \notin N_\omega} \langle BX(t), X(t) \rangle \leq \sup_{t \in D} \langle BX(t), X(t) \rangle \leq \sup_{t \notin N_\omega} \langle BX(t), X(t) \rangle
\]
From (72.3.16)
\[
\sup_{t \notin N_\omega} \langle BX(t), X(t) \rangle = C(\omega) \text{ a.e.}\omega
\]
where \(\int_\Omega C(\omega) \, dP < \infty\). In particular, \(\sup_{t \notin N_\omega} \langle BX(t), X(t) \rangle\) is bounded for a.e. \(\omega\) say for \(\omega \notin N\) where \(N\) includes the earlier sets of measure zero. This shows that \(BX(t)\) is bounded in \(W'\) for \(t \in N'_\omega\).

If \(v \in V\), then for \(\omega \notin N\),
\[
\lim_{t \to s} \langle BX(t), v \rangle = \langle BX(s), v \rangle, \ t, s
\]
Therefore, since for such \(\omega\), \(\|BX(t)\|_{W'}\) is bounded for \(t \notin N_\omega\), the above holds for all \(v \in W\) also. Therefore, for a.e. \(\omega\), \(t \to BX(t, \omega)\) is weakly continuous with values in \(W'\) for \(t \notin N_\omega\).

Note also that
\[
\int_0^T \int_\Omega \|BX(t)\|^2 \, dP dt \leq \int_0^T \|B\|^{1/2} \langle BX(t), X(t) \rangle \, dtdP
\]
\[
\leq C \left( \|Y\|_{K'}, \|X\|_{K'}, \|Z\|_J, \|\langle BX_0, X_0 \rangle\|_{L^1(\Omega)} \right) \|B\|^{1/2} T
\]
(72.3.16)
Eventually, it is shown that in fact, the function \(t \to BX(t, \omega)\) is continuous with values in \(W'\). The above shows that \(BX \in L^2([0, T] \times \Omega, W')\).

Finally consider the claim of weak continuity of \(BX\) into \(W'\). From the integral equation, \(BX\) is continuous into \(V'\). Also \(t \to BX(t)\) is bounded in \(W'\) on \(N'_\omega\). Let \(s \in [0, T]\) be arbitrary. I claim that if \(t_n \to s, t_n \in D\), it follows that \(BX(t_n) \to BX(s)\) weakly in \(W'\). If not, then there is a subsequence, still denoted as \(t_n\), such that \(BX(t_n) \to Y\) weakly in \(W'\) but \(Y \neq BX(s)\). However, the continuity into \(V'\) means that for all \(v \in V\),
\[
\langle Y, v \rangle = \lim_{n \to \infty} \langle BX(t_n), v \rangle = \langle BX(s), v \rangle
\]
which is a contradiction since \(V\) is dense in \(W\). This establishes the claim. Also this shows that \(BX(s)\) is bounded in \(W'\).
\[
|\langle BX(s), w \rangle| = \lim_{n \to \infty} |\langle BX(t_n), w \rangle| \leq \lim_{n \to \infty} \inf \|BX(t_n)\|_{W'}, \|w\|_{W'} \leq C(\omega) \|w\|_{W'}
\]
Now a repeat of the above argument shows that \(s \to BX(s)\) is weakly continuous into \(W'\).
72.4 A Simplification Of The Formula

This estimate in Lemma 72.3.2 also provides a way to simplify one of the formulas derived earlier in the case that \( X_0 \in L^p (\Omega, V) \) so that \( X - X_0 \in L^p ([0, T] \times \Omega, V) \). Refer to 72.3.11 One term there is

\[
\langle B (X (t_1) - X_0 - M (t_1)) , X (t_1) - X_0 - M (t_1) \rangle
\]

Also,

\[
\langle B (X (t_1) - X_0 - M (t_1)) , X (t_1) - X_0 - M (t_1) \rangle \leq 2 \langle B (X (t_1) - X_0) , X (t_1) - X_0 \rangle + 2 \langle BM (t_1), M (t_1) \rangle
\]

It was observed above that \( 2 \langle BM (t_1), M (t_1) \rangle \to 0 \) a.e. and also in \( L^1 (\Omega) \) as \( k \to \infty \). Apply the above lemma to \( \langle B (X (t_1) - X_0), X (t_1) - X_0 \rangle \) using \([0, t_1]\) instead of \([0, T]\). The new \( X_0 \) equals 0. Then from the estimate 72.3.2 it follows that

\[
E (\langle B (X (t_1) - X_0), X (t_1) - X_0 \rangle) \to 0
\]

as \( k \to \infty \). Taking a subsequence, we could also assume that

\[
\langle B (X (t_1) - X_0), X (t_1) - X_0 \rangle \to 0
\]

a.e. \( \omega \) as \( k \to \infty \). Then, using this subsequence, it would follow from 72.3.11.

\[
\langle BX (t_m), X (t_m) \rangle - \langle BX_0, X_0 \rangle = e (k) + 2 \int_0^{t_m} \langle Y (u), X_k^1 (u) \rangle du + 2 \int_0^{t_m} \langle BX_k^1, dM \rangle + \sum_{j=0}^{m-1} \langle B (M (t_{j+1}) - M (t_j)) , M (t_{j+1}) - M (t_j) \rangle - \sum_{j=1}^{m-1} \langle B (\Delta X (t_j) - \Delta M (t_j)) , \Delta X (t_j) - \Delta M (t_j) \rangle
\]

(72.4.17)

where \( e (k) \to 0 \) in \( L^1 (\Omega) \) and a.e. \( \omega \) and

\[
\Delta X (t_j) \equiv X (t_{j+1}) - X (t_j)
\]

\( \Delta M (t_j) \) being defined similarly. Note how this eliminated the need to consider the term

\[
\langle B (X (t_1) - X_0 - M (t_1)) , X (t_1) - X_0 - M (t_1) \rangle
\]

in passing to a limit. This is a very desirable thing to be able to conclude.

Can you obtain something similar even in case \( X_0 \) is not assumed to be in \( L^p (\Omega, V) \)? Let \( X_{0k} \in L^p (\Omega, V) \cap L^2 (\Omega, W) \), \( X_{0k} \to X_0 \) in \( L^2 (\Omega, W) \). Then from the usual arguments involving the Cauchy Schwarz inequality,

\[
\langle B (X (t_1) - X_0), X (t_1) - X_0 \rangle^{1/2} \leq \langle B (X (t_1) - X_{0k}), X (t_1) - X_{0k} \rangle^{1/2} + \langle B (X_{0k} - X_0), X_{0k} - X_0 \rangle^{1/2}
\]
Also, restoring the superscript to identify the partition,

\[ B \left( X \left( t_1^k \right) - X_{0k} \right) = B \left( X_0 - X_{0k} \right) + \int_0^{t_1^k} Y(s) \, ds + BM \left( t_1^k \right). \]

Of course \( \|X - X_{0k}\|_K \) is not bounded, but for each \( k \) it is finite. There is a sequence of partitions \( \mathcal{P}_k, \|\mathcal{P}_k\| \to 0 \) such that all the above holds. In the definitions of \( K, K', E \left( [M] (T) \right) \) replace \([0, T]\) with \([0, t]\) and let the resulting spaces be denoted by \( K_t, K'_t \). Let \( n_k \) denote a subsequence of \( \{k\} \) such that

\[ \|X - X_{0k}\|_{K'_t} < 1/k. \]

Then from the above lemma,

\[
E \left( \langle B (X (t_1^k) - X_0), X (t_1^k) - X_{0k} \rangle \right) \\
\leq C \left( \langle B (X_0 - X_{0k}), X_0 - X_{0k} \rangle_{L^1(\Omega)} ; \|Y\|_{K'_t}, \|X - X_{0k}\|_{K'_t}, E \left( [M] (t_1^k) \right) \right) \\
\leq C \left( \langle B (X_0 - X_{0k}), X_0 - X_{0k} \rangle_{L^1(\Omega)} ; \|Y\|_{K'_t} \frac{1}{K}, E \left( [M] (t_1^k) \right) \right).
\]

Hence

\[
E \left( \langle B (X (t_1^k) - X_0), X (t_1^k) - X_0 \rangle \right) \\
\leq 2E \left( \langle B (X (t_1^k) - X_{0k}), X (t_1^k) - X_{0k} \rangle \right) + 2E \left( \langle B (X_0 - X_{0k}), X_0 - X_{0k} \rangle \right) \\
\leq 2C \left( \langle B (X_0 - X_{0k}), X_0 - X_{0k} \rangle_{L^1(\Omega)} ; \|Y\|_{K'_t} \frac{1}{K}, E \left( [M] (t_1^k) \right) \right) \\
+ 2 \|B\| \|X_0 - X_0\|^2_{L^2(\Omega, W)}
\]

which converges to 0 as \( k \to \infty \). It follows that there exists a suitable subsequence such that \( 72.4.18 \) holds even in the case that \( X_0 \) is only known to be in \( L^2(\Omega, W) \). From now on, assume this subsequence for the partitions \( \mathcal{P}_k \). Thus \( k \) will really be \( n_k \) and it suffices to consider the limit as \( k \to \infty \) of the equation of \( 72.4.17 \). To emphasize this point again, the reason for the above observations is to argue that, even when \( X_0 \) is only in \( L^2(\Omega, W) \), one can neglect

\[ \langle B (X (t_1) - X_0 - M (t_1)), X (t_1) - X_0 - M (t_1) \rangle \]

in passing to the limit as \( k \to \infty \) provided a suitable subsequence is used.

### 72.5 Convergence

Convergence will be shown for a subsequence and from now on every sequence will be a subsequence of this one. Since \( BX \in L^2([0, T] \times \Omega; W') \) which was shown
above, there exists a sequence of partitions of the sort described above such that

\[ BX^t_k \to BX, BX^r_k \to BX \]

in \( L^2 ([0, T] \times \Omega, W') \). Then the next lemma improves on this.

**Lemma 72.5.1** There exists a subsequence still denoted with the subscript \( k \) and an
enlarged set of measure zero \( N \) including the earlier one such that \( BX_k^t (t), BX_k^r (t) \)
also converges pointwise a.e. \( \tau \) to \( BX (t) \) in \( W' \) and \( X_k^r (t), X_k^t (t) \) converge pointwise
a.e. in \( V \) to \( X (t) \) for \( \omega \notin N \) as well as having convergence of \( X_k^t (\cdot, \omega) \) to \( X (\cdot, \omega) \)
in \( L^p ([0, T] \setminus \Omega) \) and \( BX_k^t (\cdot, \omega) \) to \( BX (\cdot, \omega) \) in \( L^2 ([0, T] \setminus \Omega, W') \).

**Proof:** To see that such a sequence exists, let \( n_k \) be such that

\[
\int_\Omega \int_0^T \| BX^r_{n_k} (t) - BX (t) \|^2_{W'} dt \, dP + \int_\Omega \int_0^T \| X^r_{n_k} (t) - X (t) \|^p_{V} dt \, dP + \\
\int_\Omega \int_0^T \| BX^l_{n_k} (t) - BX (t) \|^2_{W'} dt \, dP + \int_\Omega \int_0^T \| X^l_{n_k} (t) - X (t) \|^p_{V} dt \, dP < 4^{-k}.
\]

Then

\[
P \left( \int_0^T \| BX^l_{n_k} (t) - BX (t) \|^2_{W'} dt > 0, \int_0^T \| X^l_{n_k} (t) - X (t) \|^p_{V} dt > 0 \right) \leq 2^k (4^{-k}) = 2^{-k}
\]

and so by Borel Cantelli lemma, there is a set of measure zero \( N \) such that if \( \omega \notin N \),

\[
\int_0^T \| BX^l_{n_k} (t) - BX (t) \|^2_{W'} dt + \int_0^T \| X^l_{n_k} (t) - X (t) \|^p_{V} dt + \\
\int_0^T \| BX^r_{n_k} (t) - BX (t) \|^2_{W'} dt + \int_0^T \| X^r_{n_k} (t) - X (t) \|^p_{V} dt \leq 2^{-k}
\]

for all \( k \) large enough. By the usual proof of completeness of \( L^p \), it follows that
\( X^l_{n_k} (t) \to X (t) \) for a.e. \( t \), this for each \( \omega \notin N \), a similar assertion holding for \( X^r_{n_k} \).
Also \( BX^l_{n_k} (t) \to BX (t) \) for a.e. \( t \), similar for \( BX^r_{n_k} (t) \). We denote these
subsequences as \( \{ X^l_k \}_{k=1}^{\infty}, \{ X^r_k \}_{k=1}^{\infty} \).

Define the following stopping time.

\[
\tau_p \equiv \inf \left\{ t : \sum_i \langle BX (t), e_i \rangle^2 > p \right\}
\]

By Lemma [72.3.2], \( \tau_p = \infty \) for all \( p \) large enough off some set of measure zero. Also, \( BX (\omega) = B (X (t), \omega) \) for a.e. \( t \) and so for a.e.t, \( \langle BX (t), X (t) \rangle = \sum_i \langle BX (t), e_i \rangle \)
and so \( \| BX^r (t) \|_{W'} \leq \| B \| \sqrt{p} \) for a.e.t. Hence \( BX^r \in L^\infty ([0, T] \times \Omega, W') \).
Lemma 72.5.2 The process \( \int_0^t \langle BX_k, dM \rangle \) converges in probability as \( k \to \infty \) to \( \int_0^t \langle BX, dM \rangle \) which is a local martingale. Also, there is a subsequence and an enlarged set of measure zero \( N \) such that for \( \omega \) not in this set, the convergence is uniform on \( [0, T] \).

**Proof:** By assumption, \( d[M] = kdt \) for some \( k \in L^1 ([0, T] \times \Omega) \) and so \( BX^{\tau_p} \in \mathcal{G} \) where \( \mathcal{G} \) was the class of functions for which one can write \( \int_0^t \langle BX, dM \rangle \). By the Burkholder Davis Gundy inequality,

\[
P\left( \sup_t \left| \int_0^{t \wedge \tau_p} \langle B (X_k^l) - BX, dM \rangle \right| > \varepsilon \right) = P\left( \sup_t \left| \int_0^{\tau_p} X([t, \tau_p]) \langle B (X_k^l) - BX, dM \rangle \right| > \varepsilon \right) \leq \frac{C}{\varepsilon} \int \Omega \left( \int_0^T \| B (X_k^l) - BX \|_W^2 kdt \right)^{1/2} dP = \frac{C}{\varepsilon} \int \Omega \left( \int_0^T X([0, \tau_p]) \| B (X_k^l) - BX \|_W^2 kdt \right)^{1/2} dP \tag{72.5.19}
\]

Let

\[ A_k = \left[ \sup_t \left| \int_0^t \langle BX_k^l - BX, dM \rangle \right| > \varepsilon \right] \]

Then, since \( \tau_p = \infty \) for all \( p \) large enough,

\[ A_k = \bigcup_{p=0}^\infty A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} \neq \infty]) \]

Consider \( BX_k^{\tau_p} \). If \( t > \tau_p \), what of the values of \( BX_k^{\tau_p} \)? It equals \( BX(s) \) where \( s \) is one of the mesh points \( s \leq \tau_p \) because this is a left step function. Therefore,

\[
\langle BX_k^{\tau_p} (s), X_k^{\tau_p} (s) \rangle = \langle B (X_k^{\tau_p} (s)), X_k^{\tau_p} (s) \rangle = \sum_i \langle BX (s), e_i \rangle \}
\]

As to \( X([0, \tau_p])BX \), it follows that for all \( t \leq \tau_p \) you have \( \sum_i \langle BX (t), e_i \rangle \} \leq p \) and so, since this equals \( \langle B (X (t)), X (t) \rangle \) a.e. \( t \), it follows that \( \| X([0, \tau_p])BX (t) \|_W \) is bounded by a constant depending on \( p \) for a.e.\( t \). It follows that \( BX \) and \( BX_k^l \) are bounded. Now by Lemma 72.5.1, \( BX_k^l (t) \to BX (t) \) a.e. \( t \) and the term \( \| B (X_k^l) - BX \|_W^2 \) is essentially bounded. Therefore, in 72.5.19, the integral converges to 0. From this formula,

\[
P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} \neq \infty])) \leq P(A_k \cap ([\tau_p = \infty])) \leq C \int \Omega \left( \int_0^{T \wedge \tau_p} \| B (X_k^l) - BX \|_W^2 kdt \right)^{1/2} dP
\]
Thus
\[ \lim_{k \to \infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} \neq \infty])) = 0 \]
Then
\[ P(A_k) = \sum_{p=1}^{\infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} \neq \infty])) \]
and taking limits using the dominated convergence theorem on the sum on the right,
\[ \lim_{k \to \infty} P(A_k) = \sum_{p=1}^{\infty} \lim_{k \to \infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} \neq \infty])) = 0 \]
This proves convergence in probability.

\[ \lim_{k \to \infty} P \left( \sup_t \left| \int_0^t \langle B(X_k^t), X(t) \rangle - B X, dM \right| > \varepsilon \right) = 0 \]
Then selecting a subsequence, still denoted with \(k\), we can obtain
\[ P \left( \sup_t \left| \int_0^t \langle B(X_k^t), X(t) \rangle - B X, dM \right| > \frac{1}{k} \right) < 2^{-k} \]
and so, by the Borel Cantelli lemma, there is a set of measure zero \(N\) such that for this subsequence, for all \(\omega \notin N\),
\[ \sup_t \left| \int_0^t \langle B(X_k^t), X(t) \rangle - B X, dM \right| \leq \frac{1}{k} \]
for all \(k\) large enough. Thus convergence is uniform. \(\blacksquare\)

From now on, include \(N\) in the exceptional set and every subsequence will be a subsequence of this one.

### 72.6 The Ito Formula

Now at long last, here is the first version of the Ito formula valid on the partition points.

**Lemma 72.6.1** In Situation [72.1.1], let \(D\) be as above, the union of all the positive mesh points for all the \(\mathcal{P}_k\). Also assume \(X_0 \in L^2(\Omega; W)\). Then for \(\omega \notin N\) the exceptional set of measure zero in \(\Omega\) and every \(t \in D\),
\[ \langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t 2 \langle Y(s), X(s) \rangle \, ds \]
\[ + \left[ R^{-1}BM, M \right](t) + 2 \int_0^t \langle BX, dM \rangle \] (72.6.20)
for \(R\) the Riesz map from \(W\) to \(W'\). The covariation term \([R^{-1}BM, M](t)\) is nonnegative.
Proof: Let $t \in D$. Then $t \in \mathcal{P}_k$ for all $k$ large enough. Consider

$$\langle BX(t), X(t) \rangle - \langle BX_0, X_0 \rangle = e(k) + 2 \int_0^t \langle Y(u), X_k^r(u) \rangle \, du$$

$$+ 2 \int_0^t \langle BX_k^r, dM \rangle + \sum_{j=0}^{q_k-1} \langle B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \rangle$$

$$- \sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), \Delta X(t_j) - \Delta M(t_j) \rangle$$

(72.6.21)

where $t_{q_k} = t$, $\Delta X(t_j) = X(t_{j+1}) - X(t_j)$ and $e(k) \to 0$ in probability. By Lemma 72.5.2 the stochastic integral on the right converges uniformly for $t \in [0, T]$ to

$$2 \int_0^t \langle BX, dM \rangle$$

for $\omega$ off a set of measure zero. The deterministic integral on the right converges uniformly for $t \in [0, T]$ to

$$2 \int_0^t \langle Y(u), X(u) \rangle \, du$$

Thanks to Lemma 72.5.1.

$$\left| \int_0^t \langle Y(u), X(u) \rangle \, du - \int_0^t \langle Y(u), X_k^r(u) \rangle \, du \right|$$

$$\leq \int_0^T \| Y(u) \|_{W'} \| X(u) - X_k^r(u) \|_V$$

$$\leq \| Y \|_{L^p([0, T])} (2^{-k})^{1/p}$$

for all $k$ large enough. Consider the fourth term. It equals

$$\sum_{j=0}^{q_k-1} \left( R^{-1} B(M(t_{j+1}) - M(t_j)), M(t_{j+1}) - M(t_j) \right)_W$$

(72.6.22)

where $R^{-1}$ is the Riesz map from $W$ to $W'$. This equals

$$\frac{1}{4} \left( \sum_{j=0}^{q_k-1} \left\| R^{-1} BM(t_{j+1}) + M(t_{j+1}) - (R^{-1} BM(t_j) + M(t_j)) \right\|^2$$

$$- \sum_{j=0}^{q_k-1} \left\| R^{-1} BM(t_{j+1}) - M(t_{j+1}) - (R^{-1} BM(t_j) - M(t_j)) \right\|^2 \right)$$

From Theorem 61.6.4, as $k \to \infty$, the above converges in probability to $(t_{q_k} = t)$

$$\frac{1}{4} \left( [R^{-1} BM + M](t) - [R^{-1} BM - M](t) \right) \equiv [R^{-1} BM, M](t)$$
Also note that from \( 72.6.22 \), this term must be nonnegative since it is a limit of nonnegative quantities. This is what was desired.

Thus all the terms in \( 72.6.21 \) converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

\[
\langle BX (t), X (t) \rangle - \langle BX_0, X_0 \rangle = 2 \int_0^t \langle Y (u), X (u) \rangle \, du \\
+ 2 \int_0^t \langle BX, dM \rangle + [R^{-1} BM, M] (t) - a
\]

where \( a \) is the limit in probability of the term

\[
\sum_{j=1}^{q_k-1} \langle B (\Delta X (t_j) - \Delta M (t_j)), \Delta X (t_j) - \Delta M (t_j) \rangle
\]

(72.6.23)

Let \( P_n \) be the projection onto \( \text{span} (e_1, \cdots, e_n) \) where \( \{e_k\} \) is an orthonormal basis for \( W \) with each \( e_k \in V \). Then using

\[
BX (t_{j+1}) - BX (t_j) - (BM (t_{j+1}) - BM (t_j)) = \int_{t_j}^{t_{j+1}} Y (s) \, ds
\]

the troublesome term of \( 72.6.23 \) above is of the form

\[
\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y (s), \Delta X (t_j) - \Delta M (t_j) \rangle \, ds
\]

\[
= \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y (s), \Delta X (t_j) - P_n \Delta M (t_j) \rangle \, ds
\]

\[
+ \sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y (s), -(I - P_n) \Delta M (t_j) \rangle \, ds
\]

which equals

\[
\sum_{j=1}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y (s), X (t_{j+1}) - X (t_j) - P_n (M (t_{j+1}) - M (t_j)) \rangle \, ds
\]

(72.6.24)

\[
+ \sum_{j=1}^{q_k-1} \langle B (\Delta X (t_j) - \Delta M (t_j)), -(I - P_n) (M (t_{j+1}) - M (t_j)) \rangle
\]

(72.6.25)

The reason for the \( P_n \) is to get \( P_n (M (t_{j+1}) - M (t_j)) \) in \( V \). The sum in \( 72.6.25 \) is dominated by

\[
\left( \sum_{j=1}^{q_k-1} \langle B (\Delta X (t_j) - \Delta M (t_j)), (\Delta X (t_j) - \Delta M (t_j)) \rangle \right)^{1/2}.
\]
$\left( \sum_{j=1}^{q_k-1} |\langle B ( I - P_n) \Delta M (t_j) , ( I - P_n) \Delta M (t_j) \rangle|^2 \right)^{1/2}$ \hfill (72.6.26)

Now it is known from the above that

$$\sum_{j=1}^{q_k-1} \langle B (\Delta X (t_j) - \Delta M (t_j)), (\Delta X (t_j) - \Delta M (t_j)) \rangle$$

converges in probability to $a \geq 0$. If you take the expectation of the square of the other factor, it is no larger than

$$\|B\| E \left( \sum_{j=1}^{q_k-1} \| (I - P_n) \Delta M (t_j) \|^2_W \right)$$

$$= \|B\| E \left( \sum_{j=1}^{q_k-1} \| (I - P_n) (M (t_{j+1}) - M (t_j)) \|^2_W \right)$$

$$= \|B\| \sum_{j=1}^{q_k-1} E \left( \| (I - P_n) (M (t_{j+1}) - M (t_j)) \|^2_W \right)$$

Then

$$\| (I - P_n) (M (t_{j+1} \wedge t) - M (t_j \wedge t)) \|^2_W = \left[ [(1 - P_n) M^{t_{j+1}} - (1 - P_n) M^{t_j}] (t) + N (t) \right]$$

$$\left[ [(1 - P_n) M^{t_{j+1}} (t) - [(1 - P_n) M^{t_j} (t)] + N (t) \right]$$

for $N (t)$ a martingale. In particular, taking $t = t_{q_k}$, the above reduces to

$$\|B\| \sum_{j=1}^{q_k-1} E \left( \| (I - P_n) (M (t_{j+1}) - M (t_j)) \|^2_W \right)$$

$$= \|B\| \sum_{j=1}^{q_k-1} E \left( [(1 - P_n) M] (t_{j+1}) - [(1 - P_n) M] (t_j) \right)$$

$$= \|B\| E \left( [(1 - P_n) M] (t_{q_k}) \right) = \|B\| E \left( \| (1 - P_n) M (t_{q_k}) \|^2_W \right)$$

From maximal theorems, Theorem 60.9.4,

$$\|B\| E \left( \sup_{t_{q_k}} \| (1 - P_n) M (t_{q_k}) \|^2_W \right) \leq 2 \|B\| E \left( \| (1 - P_n) M (T) \|^2_W \right)$$

and this on the right converges to zero as $n \to \infty$ by assumption that $M (t)$ is in $L^2$ and the dominated convergence theorem. In particular, this shows that

$$\left( \sum_{j=1}^{q_k-1} |\langle B ( I - P_n) \Delta M (t_j) , ( I - P_n) \Delta M (t_j) \rangle|^2 \right)^{1/2}$$
converges to 0 in $L^2(\Omega)$ independent of $k$ as $n \to \infty$.

Thus the expression in (24.6.25) is of the form $f_k g_{nk}$ where $f_k$ converges in probability to $a^{1/2}$ as $k \to \infty$ and $g_{nk}$ converges in probability to 0 as $n \to \infty$ independent of $k$. Now this implies $f_k g_{nk}$ converges in probability to 0. Here is why.

$$P ([|f_k g_{nk}| > \varepsilon]) \leq P (2\delta |f_k| > \varepsilon) + P (2C_\delta |g_{nk}| > \varepsilon) \leq P \left( 2\delta \left| f_k - a^{1/2} \right| + 2\delta \left| a^{1/2} \right| > \varepsilon \right) + P (2C_\delta |g_{nk}| > \varepsilon)$$

where $\delta |f_k| + C_\delta |g_{nk}| > |f_k g_{nk}|$ and $\lim_{\delta \to 0} C_\delta = \infty$. Pick $\delta$ small enough that $\varepsilon - 2\delta a^{1/2} > \varepsilon/2$. Then this is dominated by

$$\leq P \left( 2\delta \left| f_k - a^{1/2} \right| > \varepsilon/2 \right) + P (2C_\delta |g_{nk}| > \varepsilon)$$

Fix $n$ large enough that the second term is less than $\eta$ for all $k$. Now taking $k$ large enough, the above is less than $\eta$. It follows the expression in (24.6.26) and consequently in (24.6.24) converges to 0 in probability.

Now consider the other term (24.6.23) using the $n$ just determined. This term is of the form

$$\sum_{j=1}^{q_k-1} \int_{t_j}^{t_j+1} \langle Y (s) , X (t_{j+1}) - X (t_j) - P_n (M (t_{j+1}) - M (t_j)) \rangle ds =$$

$$\sum_{j=1}^{q_k-1} \int_{t_j}^{t_j+1} \langle Y (s) , X^r_k (s) - X^l_k (s) - P_n (M^r_k (s) - M^l_k (s)) \rangle ds$$

$$= \int_{t_1}^{t} \langle Y (s) , X^r_k (s) - X^l_k (s) - P_n (M^r_k (s) - M^l_k (s)) \rangle ds$$

where $M^r_k$ denotes the step function

$$M^r_k (t) = \sum_{i=0}^{m_k-1} M (t_{i+1}) \mathcal{X}_{(t_i, t_{i+1}]} (t)$$

and $M^l_k$ is defined similarly. The term

$$\int_{t_1}^{t} \langle Y (s) , P_n (M^r_k (s) - M^l_k (s)) \rangle ds$$

converges to 0 for a.e. $\omega$ as $k \to \infty$ thanks to continuity of $t \to M (t)$. However, more is needed than this. Define the stopping time

$$\tau_p = \inf \{ t > 0 : ||M (t)||_W > p \} .$$

Then $\tau_p = \infty$ for all $p$ large enough, this for a.e. $\omega$. Let

$$A_k = \left[ \int_{t_1}^{t} \langle Y (s) , P_n (M^r_k (s) - M^l_k (s)) \rangle ds \right] > \varepsilon$$
CHAPTER 72. A MORE ATTRACTION VERSION

\[ P(A_k) = \sum_{p=0}^{\infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) \quad (72.6.27) \]

Now
\[ P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) \leq P \left( \left\| \int_{t_1}^{t} \langle Y(s), P_n \left( (M^{\tau_p})_k(s) - (M^{\tau_p})_k^l(s) \right) \right\|_W ds \right\| > \varepsilon \right) \]

This is so because if \( \tau_p = \infty \), then it has no effect but also it could happen that the defining inequality may hold even if \( \tau_p < \infty \) hence the inequality. This is no larger than an expression of the form

\[ \frac{C_n}{\varepsilon} \int_{\Omega} \int_{0}^{T} \|Y(s)\|_{V'} \left\| (M^{\tau_p})_k^l(s) - (M^{\tau_p})_k^l(s) \right\|_{W} ds dP \quad (72.6.28) \]

The inside integral converges to 0 by continuity of \( M \). Also, thanks to the stopping time, the inside integral is dominated by an expression of the form

\[ \int_{0}^{T} \|Y(s)\|_{V'} 2pds \]

and this is a function in \( L^1(\Omega) \) by assumption on \( Y \). It follows that the integral in \( 72.6.28 \) converges to 0 as \( k \to \infty \) by the dominated convergence theorem. Hence

\[ \lim_{k \to \infty} P(A_k \cap ([\tau_p = \infty])) = 0. \]

Since the sets \( [\tau_p = \infty] \setminus [\tau_{p-1} < \infty] \) are disjoint, the sum of their probabilities is finite. Hence there is a dominating function in \( 72.6.27 \) and so, by the dominated convergence theorem applied to the sum,

\[ \lim_{k \to \infty} P(A_k) = \sum_{p=0}^{\infty} \lim_{k \to \infty} P(A_k \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) = 0 \]

Thus \( \int_{t_1}^{t} \langle Y(s), P_n (M^p_k(s) - M^l_k(s)) \rangle ds \) converges to 0 in probability as \( k \to \infty \).

Now consider
\[
\left| \int_{t_1}^{t} \langle Y(s), X^p_k(s) - X^l_k(s) \rangle ds \right| \leq \int_{0}^{T} |\langle Y(s), X^p_k(s) - X(s) \rangle| ds \\
+ \int_{0}^{T} |\langle Y(s), X^l_k(s) - X(s) \rangle| ds \\
\leq 2 \|Y(\cdot, \omega)\|_{L^{p'}(0,T)} (2^{-k})^{1/p}
\]

for all \( k \) large enough, this by Lemma 72.5.1. Therefore,

\[ \sum_{j=1}^{q_k-1} \langle B(\Delta X(t_j) - \Delta M(t_j)), \Delta X(t_j) - \Delta M(t_j) \rangle \]

converges to 0 in probability. This establishes the desired formula for \( t \in D. \]

In fact, the formula \( 72.6.20 \) is valid for all \( t \in N^C_{\omega}. \)
72.6. THE ITO FORMULA

Theorem 72.6.2 In Situation \([2.1.1]\), for \(\omega\) off a set of measure zero, it follows that for every \(t \in N^C\),

\[
\langle BX (t) , X (t) \rangle = \langle BX_0 , X_0 \rangle + \int_0^t 2 \langle Y (s) , X (s) \rangle \, ds
\]

\[
[R^{-1} BM , M] (t) + 2 \int_0^t \langle BX , dM \rangle
\]

(72.6.29)

Also, there exists a unique continuous, progressively measurable function denoted as \(\langle BX , X \rangle\) such that it equals \(\langle BX (t) , X (t) \rangle\) for a.e. \(t\) and \(\langle BX , X \rangle (t)\) equals the right side of the above for all \(t\). In addition to this,

\[
E\left( \langle BX , X \rangle (t) \right) = E\left( \langle BX_0 , X_0 \rangle \right) + E \left( \int_0^t 2 \langle Y (s) , X (s) \rangle \, ds + [R^{-1} BM , M] (t) \right)
\]

(72.6.30)

Also the quadratic variation of the stochastic integral in (72.6.29) is dominated by

\[
\int_0^t \|BX\|_{\mathcal{W}}^2 , d[M]
\]

(72.6.31)

Also \(t \to BX (t)\) is continuous with values in \(W^t\) for \(t \in N^C\).

**Proof:** Let \(t \in N^C \setminus D\). For \(t > 0\), let \(t (k)\) denote the largest point of \(\mathcal{P}_k\) which is less than \(t\). Suppose \(t (m) < t (k)\). Hence \(m \leq k\). Then

\[
BX (t (m)) = BX_0 + \int_0^{t (m)} Y (s) \, ds + BM (t (m))
\]

a similar formula holding for \(X (t (k))\). Thus for \(t > t (m)\), \(t \in N^C\),

\[
B (X (t) - X (t (m))) = \int_{t (m)}^t Y (s) \, ds + B (M (t) - M (t (m)))
\]

which is the same sort of thing studied so far except that it starts at \(t (m)\) rather than at 0 and \(BX_0 = 0\). Therefore, from Lemma (72.6.1) it follows

\[
\langle B (X (t (k)) - X (t (m))) , X (t (k)) - X (t (m)) \rangle
\]

\[
= \int_{t (m)}^{t (k)} 2 \langle Y (s) , X (s) - X (t (m)) \rangle \, ds
\]

\[
+ [R^{-1} BM , M] (t (k)) - [R^{-1} BM , M] (t (m))
\]

\[
+ 2 \int_{t (m)}^{t (k)} \langle B (X - X (t (m))) , dM \rangle
\]

(72.6.32)
Consider that last term. It equals
\[ 2 \int_{t(m)}^{t(k)} \langle B (X - X_m^t), dM \rangle \]  
(72.6.33)

This is dominated by
\[ 2 \left| \int_{0}^{t(k)} \langle B (X - X_m^t), dM \rangle - \int_{0}^{t(m)} \langle B (X - X_m^t), dM \rangle \right| \]
\[ \leq 4 \sup_{t \in [0, T]} \left| \int_{0}^{t} \langle B (X - X_m^t), dM \rangle \right| \]

In Lemma 72.5.2, the above expression converges to 0. It follows there is a set of measure 0 including the earlier one such that for \( \omega \) not in that set, 72.6.33 converges to 0 in \( \mathbb{R} \). Similar reasoning shows the first term on the right in the non stochastic integral of 72.6.32 is dominated by an expression of the form
\[ 4 \int_{0}^{T} \left| \langle Y (s), X (s) - X_m^t (s) \rangle \right| ds \]
which clearly converges to 0 thanks to Lemma 72.5.1. Finally, it is obvious that
\[ \lim_{m, k \to \infty} \left[ R^{-1}BM, M \right] (t (k)) - \left[ R^{-1}BM, M \right] (t (m)) = 0 \text{ for a.e. } \omega \]
due to the continuity of the quadratic variation.

This shows that for \( \omega \) off a set of measure 0
\[ \lim_{m, k \to \infty} \langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle = 0 \]

Then for \( x \in W, \)
\[ \left| \langle B (X (t (k)) - X (t (m))), x \rangle \right| \]
\[ \leq \langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle^{1/2} \langle Bx, x \rangle^{1/2} \]
\[ \leq \langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle^{1/2} \|B\|^{1/2} \|x\|_W \]
and so
\[ \lim_{m, k \to \infty} \|BX (t (k)) - BX (t (m))\|_W, = 0 \]

Recall \( t \) was arbitrary and \( \{t (k)\} \) is a sequence converging to \( t \). Then the above has shown that \( \{BX (t (k))\}_{k=1}^{\infty} \) is a convergent sequence in \( W' \). Does it converge to \( BX (t) \)? Let \( \xi (t) \in W' \) be what it converges to. Letting \( v \in V \) then, since the integral equation shows that \( t \to BX (t) \) is continuous into \( V' \),
\[ \langle \xi (t), v \rangle = \lim_{k \to \infty} \langle BX (t (k)), v \rangle = \langle BX (t), v \rangle, \]
and now, since \( V \) is dense in \( W \), this implies \( \xi (t) = BX (t) = B (X (t)) \) since \( t \notin N_\omega \).

Recall also that it was shown earlier that \( BX \) is weakly continuous into \( W' \) on \([0, T]\).

hence the strong convergence of \( \{ BX (t (k)) \}_{k=1}^\infty \) in \( W' \) implies that it converges to \( BX (t) \), this for any \( t \in N^C_\omega \).

For every \( t \in D \) and for \( \omega \) off the exceptional set of measure zero described earlier,

\[
\langle B (X (t)), X (t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t 2 \langle Y (s), X (s) \rangle \, ds + [R^{-1}BM, M] (t) + 2 \int_0^t \langle BX, dM \rangle
\]

(72.6.34)

Does this formula hold for all \( t \in [0, T] \)? Maybe not. However, it will hold for \( t \notin N_\omega \).

So let \( t \notin N_\omega \).

\[
\| \langle BX (t (k)), X (t (k)) \rangle - \langle BX (t), X (t) \rangle \|
\]

\[
\leq |(\langle BX (t (k)), X (t (k)) \rangle - \langle BX (t), X (t (k)) \rangle) + (\langle BX (t), X (t (k)) \rangle - \langle BX (t), X (t) \rangle)|
\]

\[
= |\langle B (X (t (k)) - X (t)), X (t (k)) \rangle + \langle B (X (t (k)) - X (t)), X (t) \rangle|
\]

Then using the Cauchy–Schwarz inequality on each term,

\[
\leq \langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle^{1/2} \cdot \left( (\langle BX (t (k)), X (t (k)) \rangle)^{1/2} + \langle BX (t), X (t) \rangle^{1/2} \right)
\]

As before, one can use the lower semicontinuity of

\[
t \to \langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle
\]

on \( N^C_\omega \) along with the boundedness of \( \langle BX (t), X (t) \rangle \) also shown earlier off \( N_\omega \) to conclude

\[
\| \langle BX (t (k)), X (t (k)) \rangle - \langle BX (t), X (t) \rangle \|
\]

\[
\leq C \langle B (X (t (k)) - X (t)), X (t (k)) - X (t) \rangle^{1/2}
\]

\[
\leq C \lim_{m \to \infty} \inf_{m \to \infty} \langle B (X (t (k)) - X (t (m))), X (t (k)) - X (t (m)) \rangle^{1/2} < \varepsilon
\]

provided \( k \) is sufficiently large. Since \( \varepsilon \) is arbitrary,

\[
l \lim_{k \to \infty} \langle BX (t (k)), X (t (k)) \rangle = \langle BX (t), X (t) \rangle.
\]

It follows that the formula (72.6.34) is valid for all \( t \in N^C_\omega \). Now define the function \( \langle BX, X \rangle (t) \) as

\[
\langle BX, X \rangle (t) \equiv \left\{ \begin{array}{ll}
\langle B (X (t)), X (t) \rangle, t \notin N_\omega \\
\text{The right side of (72.6.34) if } t \in N_\omega
\end{array} \right.
\]
Then in short, \( \langle BX, X \rangle (t) \) equals the right side of (72.6.34) for all \( t \in [0, T] \) and is consequently progressively measurable and continuous. Furthermore, for a.e. \( t \), this function equals \( \langle B(X(t)), X(t) \rangle \). Since it is known on a dense subset, it must be unique.

This implies that \( t \to BX(t) \) is continuous with values in \( W' \) for \( t \in \mathcal{N}_\omega^C \). Here is why. The fact that the formula (72.6.34) holds for all \( t \in \mathcal{N}_\omega^C \) implies that \( t \to \langle BX(t), X(t) \rangle \) is continuous on \( \mathcal{N}_\omega^C \). Then for \( x \in W, t, s \in \mathcal{N}_\omega \)

\[
|\langle BX(t) - BX(s), x \rangle| \leq \langle B(X(t) - X(s)), X(t) - X(s) \rangle^{1/2} \|B\|^{1/2} \|x\|_W.
\]

(72.6.35)

Also

\[
\langle B(X(t) - X(s)), X(t) - X(s) \rangle
= \langle BX(t), X(t) \rangle + \langle BX(s), X(s) \rangle - 2 \langle BX(t), X(s) \rangle
\]

By weak continuity of \( t \to BX(t) \) shown earlier,

\[
\lim_{t \to s} \langle BX(t), X(s) \rangle = \langle BX(s), X(s) \rangle.
\]

Therefore,

\[
\lim_{t \to s} \langle B(X(t) - X(s)), X(t) - X(s) \rangle = 0
\]

and so the inequality (72.6.35) implies the continuity of \( t \to BX(t) \) into \( W' \) for \( t \notin \mathcal{N}_\omega \). Note that by assumption this function is continuous into \( V' \) for all \( t \).

Now consider the claim about the expectation. Use the function \( \langle BX, X \rangle \) to define a stopping time as

\[
\tau_p \equiv \inf \{ t > 0 : \langle BX, X \rangle (t) > p \}
\]

This is the first hitting time of a continuous process and so it is a valid stopping time. Using this, leads to

\[
\langle BX, X \rangle^{\tau_p} (t) = \langle BX_0, X_0 \rangle + \int_0^t X_{[0, \tau_p]} 2 \langle Y(s), X(s) \rangle \, ds + [R^{-1} BM, M]^{\tau_p} (t) + 2 \int_0^t X_{[0, \tau_p]} \langle BX, dM \rangle
\]

(72.6.36)

The term at the end is now a martingale because \( X_{[0, \tau_p]} BX \) is bounded. Hence the expectation of the martingale at the end equals 0. Thus you obtain

\[
E(\langle BX, X \rangle^{\tau_p} (t)) = E(\langle BX_0, X_0 \rangle)
+ E \left( \int_0^t X_{[0, \tau_p]} 2 \langle Y(s), X(s) \rangle \, ds \right) + E \left( [R^{-1} BM, M]^{\tau_p} (t) \right)
\]

Now use the monotone convergence theorem and the dominated convergence theorem to pass to a limit as \( p \to \infty \) and obtain (72.6.30). The claim about the quadratic variation follows from Theorem (64.0.25).
72.6. THE ITO FORMULA

What of the special case where $W = H = H'$ and you are in the context of a Gelfand triple

$$V \subseteq H = H' \subseteq V'$$

and $B$ is simply the identity. Then we obtain the following theorem as a special case.

**Theorem 72.6.3** In Situation 72.1.1 in which $W = H = H'$ and $B = I$, it follows that off a set of measure zero, for every $t \in [0, T]$, there is a set of measure zero $N$ such that for $\omega \notin N$, there is a continuous function $\langle X, X \rangle$ which equals $|X(t)|_H^2$ for a.e. $t$ such that

$$\langle X, X \rangle(t) = |X_0|_H^2 + \int_0^t 2 \langle Y(s), X(s) \rangle \, ds$$

$$+ [M](t) + 2 \int_0^t (X, dM)$$

(72.6.37)

Furthermore, off a set of measure zero, $t \to X(t)$ is continuous as a map into $H$ for a.e. $\omega$. In addition to this,

$$E(\langle X, X \rangle(t)) =$$

$$E(|X_0|^2) + E\left(\int_0^t 2 \langle Y(s), X(s) \rangle \, ds \right) + E([M](t))$$

(72.6.38)

The quadratic variation of the stochastic integral satisfies

$$\left[\int_0^t (X, dM) \right](t) \leq \int_0^t ||X||_H^2 \, d[M]$$

It is more attractive to write $|X(t)|_H^2$ in place of $\langle X, X \rangle(t)$. However, I guess this is not strictly right although the discrepancy is only on a set of measure zero so it seems fairly harmless to indulge in this sloppiness. However, for $t \notin N_\omega$,

$$|X(t)|_H^2 = \sum_i \langle X(t), e_i \rangle^2$$

where the orthonormal basis $\{e_i\}$ is in $V$. Then for $s \in N_\omega$, you can get the following. Let $t_n \to s$ where $t_n \in N_\omega$. Then in the above notation,

$$\sum_i \langle X(s), e_i \rangle^2 \leq \lim_{n \to \infty} \inf \sum_i \langle X(t_n), e_i \rangle_H^2 = \lim_{n \to \infty} \inf \|X(t_n)\|_H^2 \leq C(\omega)$$

It follows that in fact $X(s) \in H$ and you can take $X(s) = \sum_i \langle X(s), e_i \rangle e_i \in H$ because $\sum_i \langle X(s), e_i \rangle^2 < \infty$. Hence

$$|X(s)|^2 = \sum_i \langle X(s), e_i \rangle^2 \leq \lim_{n \to \infty} \inf \|X(t_n)\|_H^2$$

so $X$ has values in $H$ and is lower semicontinuous on $[0, T]$. 
Chapter 73

Some Nonlinear Operators

In this chapter is a description and properties of some standard nonlinear maps.

73.1 An Assortment Of Nonlinear Operators

Definition 73.1.1 For $V$ a real Banach space, $A : V \to V'$ is a pseudomonotone map if whenever

$$u_n \rightharpoonup u$$

and

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0$$

it follows that for all $v \in V$,

$$\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle.$$  

The half arrows denote weak convergence.

Definition 73.1.2 $A : V \to V'$ is monotone if for all $v, u \in V$,

$$\langle Au - Av, u - v \rangle \geq 0,$$

and $A$ is Hemicontinuous if for all $v, u \in V$,

$$\lim_{t \to 0^+} \langle A (u + t (v - u)), u - v \rangle = \langle Au, u - v \rangle.$$

Theorem 73.1.3 Let $V$ be a Banach space and let $A : V \to V'$ be monotone and hemicontinuous. Then $A$ is pseudomonotone.

Proof: Let $A$ be monotone and Hemicontinuous. First here is a claim.

Claim: If $\text{[Claim 1]}$ and $\text{[Claim 2]}$ hold, then $\lim_{n \to \infty} \langle Au_n, u_n - u \rangle = 0$.

Proof of the claim: Since $A$ is monotone,

$$\langle Au_n - Au, u_n - u \rangle \geq 0$$
so
\[ \langle Au_n, u_n - u \rangle \geq \langle Au, u_n - u \rangle. \]

Therefore,
\[ 0 = \liminf_{n \to \infty} \langle Au, u_n - u \rangle \leq \liminf_{n \to \infty} \langle Au_n, u_n - u \rangle \leq \limsup_{n \to \infty} \langle Au_n, u_n - u \rangle \leq 0. \]

Now using that \( A \) is monotone again, then letting \( t > 0 \),
\[ \langle Au_n - A(u + t(v - u)), u_n - u + t(u - v) \rangle \geq 0 \]
and so
\[ \langle Au_n, u_n - u + t(u - v) \rangle \geq \langle A(u + t(v - u)), u_n - u + t(u - v) \rangle. \]

Taking the \( \liminf \) on both sides and using the claim and \( t > 0 \),
\[ t \liminf_{n \to \infty} \langle Au_n, u - v \rangle \geq t \langle A(u + t(v - u)), (u - v) \rangle. \]
Next divide by \( t \) and use the Hemicontinuity of \( A \) to conclude that
\[ \liminf_{n \to \infty} \langle Au_n, u - v \rangle \geq \langle Au, u - v \rangle. \]

From the claim,
\[ \liminf_{n \to \infty} \langle Au_n, u - v \rangle = \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle + \langle Au_n, u_n - u \rangle \]
\[ = \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle. \]

Monotonicity is very important in the above proof. The next example shows that even if the operator is linear and bounded, it is not necessarily pseudomonotone.

**Example 73.1.4** Let \( H \) be any Hilbert space and let \( A : H \to H' \) be given by
\[ \langle Ax, y \rangle \equiv \langle -x, y \rangle_H. \]

Then \( A \) fails to be pseudomonotone.

**Proof:** Let \( \{x_n\}_{n=1}^\infty \) be an orthonormal set of vectors in \( H \). Then Parsevall’s inequality implies
\[ \|x\|^2 \geq \sum_{n=1}^\infty |(x_n, x)|^2 \]
and so for any \( x \in H, \lim_{n \to \infty} (x_n, x) = 0 \). Thus \( x_n \to 0 \equiv x \). Also
\[ \limsup_{n \to \infty} \langle Ax_n, x_n - x \rangle = \limsup_{n \to \infty} \langle Ax_n, x_n \rangle = \limsup_{n \to \infty} \left(-\|x_n\|^2\right) = -1 \leq 0. \]
If $A$ were pseudomonotone, we would need to be able to conclude that for all $y \in H$,

$$\liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle \geq \langle Ax, x - y \rangle = 0.$$ 

However,

$$\liminf_{n \to \infty} \langle Ax_n, x_n - 0 \rangle = -1 < 0 = \langle A0, 0 - 0 \rangle.$$ 

Now the following proposition is useful.

**Proposition 73.1.5** Suppose $A : V \to V'$ is pseudomonotone and bounded where $V$ is separable. Then it must be demicontinuous. This means that if $u_n \to u$, then $Au_n \rightharpoonup Au$.

**Proof:** Since $u_n \to u$ is strong convergence and since $Au_n$ is bounded, it follows

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle = \lim_{n \to \infty} \langle Au_n, u_n - u \rangle = 0.$$ 

Suppose this is not so that $Au_n$ converges weakly to $Au$. Since $A$ is bounded, there exists a subsequence, still denoted by $n$ such that $Au_n \to \xi$ weakly. I need to verify $\xi = Au$. From the above, it follows that for all $v \in V$

$$\langle Au, u - v \rangle \leq \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle = \liminf_{n \to \infty} \langle Au_n, u - v \rangle = \langle \xi, u - v \rangle$$

Hence $\xi = Au$. ☐

There is another type of operator which is more general than pseudomonotone.

**Definition 73.1.6** Let $A : V \to V'$ be an operator. Then $A$ is called type $M$ if whenever $u_n \to u$ and $Au_n \rightharpoonup \xi$, and

$$\limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle$$

it follows that $Au = \xi$.

**Proposition 73.1.7** If $A$ is pseudomonotone, then $A$ is type $M$.

**Proof:** Suppose $A$ is pseudomonotone and $u_n \to u$ and $Au_n \to \xi$, and

$$\limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle$$

Then

$$\limsup_{n \to \infty} \langle Au_n, u_n - u \rangle = \limsup_{n \to \infty} \langle Au_n, u_n \rangle - \langle \xi, u \rangle \leq 0$$

Hence

$$\liminf_{n \to \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle$$
for all \( v \in V \). Consequently, for all \( v \in V \),
\[
\langle Au, u - v \rangle \leq \liminf_{n \to \infty} \langle Au_n, u_n - v \rangle
= \liminf_{n \to \infty} (\langle Au_n, u - v \rangle + \langle Au_n, u_n - v \rangle)
= \langle \xi, u - v \rangle + \liminf_{n \to \infty} \langle Au_n, u_n - u \rangle \leq \langle \xi, u - v \rangle
\]
and so \( Au = \xi \). □

An interesting result is the following which states that a monotone linear function added to a type M is also type M.

**Proposition 73.1.8** Suppose \( A : V \to V' \) is type M and suppose \( L : V \to V' \) is monotone, bounded and linear. Then \( L + A \) is type M. Let \( V \) be separable or reflexive so that the weak convergences in the following argument are valid.

**Proof:** Suppose \( u_n \rightharpoonup u \) and \( Au_n + Lu_n \rightharpoonup \xi \) and also that
\[
\limsup_{n \to \infty} \langle Au_n + Lu_n, u_n \rangle \leq \langle \xi, u \rangle
\]
Does it follow that \( \xi = Au + Lu \)? Suppose not. By assumption, \( u_n - u_0 \rightharpoonup u - u_0 \) and so, since \( L \) is bounded, there is a further subsequence, still called \( n \) such that
\[
\langle Lu_n, u_n \rangle \geq \langle Lu_n, u \rangle + \langle L(u), u_n - u \rangle
\]
Hence with this further subsequence, the lim sup is no larger and so
\[
\limsup_{n \to \infty} \langle Au_n, u_n \rangle + \lim_{n \to \infty} (\langle Lu_n, u \rangle + \langle L(u), u_n - u \rangle) \leq \langle \xi, u \rangle
\]
and so
\[
\limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi - Lu, u \rangle
\]
It follows since \( A \) is type M that \( Au = \xi - Lu \), which contradicts the assumption that \( \xi \neq Au + Lu \). □

There is also the following useful generalization of the above proposition.

**Corollary 73.1.9** Suppose \( A : V \to V' \) is type M and suppose \( L : V \to V' \) is monotone, bounded and linear. Then for \( u_0 \in V \) define \( M(u) \equiv L(u - u_0) \). Then \( M + A \) is type M. Let \( V \) be separable or reflexive so that the weak convergences in the following argument are valid.

**Proof:** Suppose \( u_n \rightharpoonup u \) and \( Au_n + Mu_n \rightharpoonup \xi \) and also that
\[
\limsup_{n \to \infty} \langle Au_n + Mu_n, u_n \rangle \leq \langle \xi, u \rangle
\]
Does it follow that \( \xi = Au + Mu \)? Suppose not. By assumption, \( u_n - u_0 \rightharpoonup u - u_0 \) and so, since \( L \) is bounded, there is a further subsequence, still called \( n \) such that
\[
Mu_n = L(u_n - u_0) \rightharpoonup L(u - u_0) = Mu.
\]
73.1. AN ASSORTMENT OF NONLINEAR OPERATORS

Since $M$ is monotone, 
\[ \langle Mu_n - Mu, u_n - u \rangle \geq 0 \]

Thus 
\[ \langle Mu_n, u_n \rangle - \langle Mu_n, u \rangle - \langle Mu, u_n \rangle + \langle Mu, u \rangle \geq 0 \]

and so 
\[ \langle Mu_n, u_n \rangle \geq \langle Mu_n, u \rangle + (Mu, u_n - u) \]

Hence with this further subsequence, the lim sup is no larger and so 
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle + \lim_{n \to \infty} (\langle Mu_n, u \rangle + \langle M(u), u_n - u \rangle) \leq \langle \xi, u \rangle \]

and so 
\[ \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi - Mu, u \rangle \]

It follows since $A$ is type $M$ that 
\[ Au = \xi - Mu, \]

which contradicts the assumption that $\xi \neq Au + Mu$. \[ \blacksquare \]

The following is Browder’s lemma. It is a very interesting application of the Brouwer fixed point theorem.

**Lemma 73.1.10 (Browder)** Let $K$ be a convex closed and bounded set in $\mathbb{R}^n$ and let $A : K \to \mathbb{R}^n$ be continuous and $f \in \mathbb{R}^n$. Then there exists $x \in K$ such that for all $y \in K$,
\[ (f - Ax, y - x) \leq 0 \]

**Proof:** Let $P_K$ denote the projection onto $K$. Thus $P_K$ is Lipschitz continuous.

\[ x \to P_K (f - Ax + x) \]

is a continuous map from $K$ to $K$. By the Brouwer fixed point theorem, it has a fixed point $x \in K$. Therefore, for all $y \in K$,
\[ (f - Ax + x - x, y - x) = (f - Ax, y - x) \leq 0 \] \[ \blacksquare \]

From this lemma, there is an interesting theorem on surjectivity.

**Proposition 73.1.11** Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and coercive,
\[ \lim_{|x| \to \infty} \frac{(A(x + x_0), x)}{|x|} = \infty \]
for some $x_0$. Then for all $f \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $Ax = f$.

**Proof:** Define the closed convex sets $B_n \equiv B(x_0, n)$. By Browder’s lemma, there exists $x_n$ such that 
\[ (f - Ax_n, y - x_n) \leq 0 \]
for all $y \in B_n$. Then taking $y = x_0$, it follows from the coercivity condition that the $x_n - x_0$ are bounded. It follows that for large $n$, $x_n$ is an interior point of $B_n$. Therefore, 
\[ (f - Ax_n, z) \leq 0 \]
for all $z$ in some open ball centered at $x_0$. Hence $f = Ax_n$. \[ \blacksquare \]
**Lemma 73.1.12** Let $A : V \to V'$ be type $M$ and bounded and suppose $V$ is reflexive or $V$ is separable. Then $A$ is demicontinuous.

**Proof:** Suppose $u_n \to u$ and $Au_n$ fails to converge weakly to $Au$. Then there is a further subsequence, still denoted as $u_n$ such that $Au_n \rightharpoonup \zeta \neq Au$. Then thanks to the strong convergence, you have

$$\limsup_{n \to \infty} \langle Au_n, u_n \rangle = \langle \zeta, u_n \rangle$$

which implies $\zeta = Au$ after all. ■

With these lemmas and the above proposition, there is a very interesting surjectivity result.

**Theorem 73.1.13** Let $A : V \to V'$ be type $M$, bounded, and coercive

$$\lim_{\|u\| \to \infty} \frac{\langle A(u + u_0), u \rangle}{\|u\|} = \infty, \quad (73.1.13)$$

for some $u_0$, where $V$ is a separable reflexive Banach space. Then $A$ is surjective.

**Proof:** Since $V$ is separable, there exists an increasing sequence of finite dimensional subspaces $\{ V_n \}$ such that $\bigcup_n V_n = V$. Say span $(v_1, \cdots, v_n) = V_n$. Then consider the following diagram.

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{\theta} & V' \\
\mathbb{R}^n & \xrightarrow{i} & V \\
\end{array}$$

Here the map $\theta$ is the one which does the following.

$$\theta(x) = \sum_{i=1}^{n} x_i v_i.$$ 

The map $i$ is the inclusion map. Consider the map $\theta^* i^* Ai\theta$. By Lemma this map is continuous. The map $\theta$ is continuous, one to one, and onto. Thus its inverse is also continuous. Let $x_0$ correspond to $u_0$. Then for some constant $C$,

$$\frac{(\theta^* i^* Ai\theta (x + x_0), x)}{|x|} \geq \frac{(Ai\theta (x + x_0), i\theta x)}{C \|i\theta x\|_V},$$

and to say $|x| \to \infty$ is the same as saying that $\|i\theta x\|_V \to \infty$. Hence $\theta^* i^* Ai\theta$ is coercive. Let $f \in V'$. Then from there exists $x_n$ such that

$$\theta^* i^* Ai\theta x_n = \theta^* i^* f$$

Thus, $i^* Ai\theta x_n = i^* f$ and this implies that for $v_n = \theta x_n$,

$$i^* Av_n = i^* f$$
In other words,
\[ \langle Av_n, y \rangle = \langle f, y \rangle \]
for all \( y \in V_n \). Then from the coercivity condition \[73.1.4\], the \( v_n \) are bounded independent of \( n \). Since \( V \) is reflexive, there is a subsequence, still called \( \{v_n\} \) which converges weakly to \( v \in V \). Since \( A \) is bounded, it can also be assumed that \( Av_n \to \zeta \in V' \). Then
\[
\limsup_{n \to \infty} \langle Av_n, v_n \rangle = \limsup_{n \to \infty} \langle f, v_n \rangle = \langle f, v \rangle
\]
Also, passing to the limit in \[73.1.5\],
\[
\langle \zeta, y \rangle = \langle f, y \rangle
\]
for any \( y \in V_n \), this for any \( n \). Since the union of these \( V_n \) is dense, it follows that the above equation holds for all \( y \in V \). Therefore, \( f = \zeta \) and so
\[
\limsup_{n \to \infty} \langle Av_n, v_n \rangle = \limsup_{n \to \infty} \langle f, v_n \rangle = \langle f, v \rangle = \langle \zeta, v \rangle
\]
Since \( A \) is type \( M \),
\[
Av = \zeta = f \tag*{■}
\]

### 73.2 Duality Maps

The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space.

**Definition 73.2.1** A Banach space is said to be strictly convex if whenever \( ||x|| = ||y|| \) and \( x \neq y \), then
\[
\left|\left|\frac{x + y}{2}\right|\right| < ||x||.
\]

\( F : X \to X' \) is said to be a duality map if it satisfies the following: a.) \( ||F(x)|| = ||x||^{p-1} \). b.) \( F(x)(x) = ||x||^p \), where \( p > 1 \).

Duality maps exist. Here is why. Let
\[
F(x) \equiv \left\{ x^* : ||x^*|| \leq ||x||^{p-1} \text{ and } x^*(x) = ||x||^p \right\}
\]
Then \( F(x) \) is not empty because you can let \( f(\alpha x) = \alpha ||x||^p \). Then \( f \) is linear and defined on a subspace of \( X \). Also
\[
\sup_{||\alpha x|| \leq 1} |f(\alpha x)| = \sup_{||\alpha x|| \leq 1} |\alpha||x||^p \leq ||x||^{p-1}
\]
Also from the definition,
\[
f(x) = ||x||^p
\]
and so, letting $x^*$ be a Hahn Banach extension, it follows $x^* \in F(x)$. Also, $F(x)$ is closed and convex. It is clearly closed because if $x^*_n \to x^*$, the condition on the norm clearly holds and also the other one does too. It is convex because

$$||x^* + (1 - \lambda) y^*|| \leq \lambda ||x^*|| + (1 - \lambda) ||y^*|| \leq \lambda ||x||^{p-1} + (1 - \lambda) ||x||^{p-1}$$

If the conditions hold for $x^*$, then we can show that in fact $||x^*|| = ||x||^{p-1}$.

This is because

$$||x^*|| \geq \left| x^* \left( \frac{x}{||x||} \right) \right| = \frac{1}{||x||} |x^*(x)| = ||x||^{p-1}.$$

Now how many things are in $F(x)$ assuming the norm on $X'$ is strictly convex? Suppose $x_1^*$ and $x_2^*$ are two things in $F(x)$. Then by convexity, so is $(x_1^* + x_2^*)/2$. Hence by strict convexity, if the two are different, then

$$\left| \frac{x_1^* + x_2^*}{2} \right| = ||x||^{p-1} < \frac{1}{2} ||x_1^*|| + \frac{1}{2} ||x_2^*|| = ||x||^{p-1}$$

which is a contradiction. Therefore, $F$ is an actual mapping.

What are some of its properties? First is one which is similar to the Cauchy Schwarz inequality. Since $p-1 = {p}/p'$,

$$\sup_{||y|| \leq 1} |\langle Fx, y \rangle| = ||x||^{p/p'}$$

and so for arbitrary $y \neq 0$,

$$|\langle Fx, y \rangle| = ||y|| \left| \left\langle Fx, \frac{y}{||y||} \right\rangle \right| \leq ||y|| ||x||^{p/p'}$$

$$\quad = |\langle Fy, x \rangle|^{1/p} |\langle Fx, x \rangle|^{1/p'}$$

Next we can show that $F$ is monotone.

$$\langle Fx - Fy, x - y \rangle = \langle Fx, x \rangle - \langle Fx, y \rangle - \langle Fy, x \rangle + \langle Fy, y \rangle \geq ||x||^p + ||y||^p - ||y|| ||x||^{p/p'} - ||y||^{p/p'} ||x||$$

$$\quad \geq ||x||^p + ||y||^p - \left( \frac{||y||^p}{p} + \frac{||x||^p}{p'} \right) - \left( \frac{||y||^{p/p'}}{p} + \frac{||x||^{p/p'}}{p'} \right) = 0$$

Next it can be shown that $F$ is hemi-continuous. By the construction, $F(x + ty)$ is bounded as $t \to 0$. Let $t \to 0$ be a subsequence such that

$$F(x + ty) \to x^*$$

Then we ask: Does $x^*$ do what it needs to do in order to be $F(x)$? The answer is yes. First of all $||F(x + ty)|| = ||x + ty||^{p-1} \to ||x||^{p-1}$. The set

$$\left\{ x^*: ||x^*|| \leq ||x||^{p-1} + \epsilon \right\}$$
is closed and convex and so it is weak ∗ closed as well. For all small enough \( t \), it follows \( F(x + ty) \) is in this set. Therefore, the weak limit is also in this set and it follows \( ||\xi|| \leq ||x||^{p-1} + \varepsilon \). Since \( \varepsilon \) is arbitrary, it follows \( ||\xi|| \leq ||x||^{p-1} \). Is \( \xi(x) = ||x||^p \)? We have

\[
||x||^p = \lim_{t \to 0} ||x + ty||^p = \lim_{t \to 0} \langle F(x + ty), x + ty \rangle
\]

and so, \( \xi \) does what it needs to do to be \( F(x) \). This would be clear if \( ||\xi|| = ||x||^{p-1} \).

However, \( ||(\xi,x)|| = ||x||^p \) and so \( ||\xi|| \geq \left( \langle \xi, \frac{x}{||x||} \rangle \right) = ||x||^{p-1} \). Thus \( ||\xi|| = ||x||^{p-1} \), which shows \( \xi \) does everything it needs to do to equal \( F(x) \) and so it is \( F(x) \).

Since this conclusion follows for any convergent sequence, it follows that \( F(x + ty) \) converges to \( F(x) \) weakly as \( t \to 0 \). This is what it means to be hemicontinuous. This proves the following theorem. One can show also that \( F \) is demicontinuous which means strongly convergent sequences go to weakly convergent sequences. Here is a proof for the case where \( p = 2 \). You can clearly do the same thing for arbitrary \( p \).

**Lemma 73.2.2** Let \( F \) be a duality map for \( p = 2 \), where \( X, X' \) are reflexive and have strictly convex norms. (If \( X \) is reflexive, there is always an equivalent strictly convex norm \( \Box \).) Then \( F \) is demicontinuous.

**Proof:** Say \( x_n \to x \). Then does it follow that \( Fx_n \rightharpoonup Fx \)? Suppose not. Then there is a subsequence, still denoted as \( x_n \) such that \( x_n \to x \) but \( Fx_n \rightharpoonup y \neq Fx \) where here \( \rightharpoonup \) denotes weak convergence. This follows from the Eberlein Smulian theorem. Then

\[
\langle y, x \rangle = \lim_{n \to \infty} \langle Fx_n, x_n \rangle = \lim_{n \to \infty} ||x_n||^2 = ||x||^2
\]

Also, there exists \( z, ||z|| = 1 \) and \( \langle y, z \rangle \geq ||y|| - \varepsilon \). Then

\[
||y|| - \varepsilon \leq \langle y, z \rangle = \lim_{n \to \infty} \langle Fx_n, z \rangle \leq \lim \inf_{n \to \infty} ||Fx_n|| = \lim \inf_{n \to \infty} ||x_n|| = ||x||
\]

and since \( \varepsilon \) is arbitrary, \( ||y|| \leq ||x|| \). It follows from the above construction of \( Fx \), that \( y = Fx \) after all, a contradiction.

**Theorem 73.2.3** Let \( X \) be a reflexive Banach space with \( X' \) having strictly convex norm\(^\Box\). Then for \( p > 1 \), there exists a mapping \( F : X \to X' \) which is bounded, monotone, hemicontinuous, coercive in the sense that \( \lim_{||x|| \to \infty} \langle Fx, x \rangle / ||x|| = \infty \), which also satisfies the inequalities

\[
||\langle Fx, y \rangle|| \leq ||\langle Fx, x \rangle||^{1/p} ||\langle Fy, y \rangle||^{1/p}
\]

\(^1\)It is known that if the space is reflexive, then there is an equivalent norm which is strictly convex. However, in most examples, this strict convexity is obvious.
CHAPTER 73. SOME NONLINEAR OPERATORS

Note that these conclusions about duality maps show that they map onto the dual space.

The duality map was onto and it was monotone. This was shown above. Consider the form of a duality map for the $L^p$ spaces. Let $F : L^p \to (L^p)'$ be the one which satisfies

$$||Ff|| = ||f||^{p-1}, \quad \langle Ff, f \rangle = ||f||^p$$

Then in this case,

$$Ff = |f|^{p-2}f$$

This is because it does what it needs to do.

$$||Ff||_{L^p'} = \left( \int_\Omega \left( |f|^{p-1} \right)^{p'} d\mu \right)^{1/p'} = \left( \frac{\int_\Omega |f|^p d\mu}{\left( \int_\Omega |f|^{p/p'} d\mu \right)^{1/p}} \right)^{p-1} = ||f||^{p-1}_{L^p}$$

while it is obvious that

$$\langle Ff, f \rangle = \int_\Omega |f|^p d\mu = ||f||^p_{L^p(\Omega)}.$$ 

Now here is an interesting inequality which I will only consider in the case where the quantities are real valued.

**Lemma 73.2.4** Let $p > 2$. Then for $a, b$ real numbers, \( |a|^{p-2}a - |b|^{p-2}b \) \((a - b) \geq C|a - b|^p \) for some constant $C$ independent of $a, b$.

**Proof:** There is nothing to show if $a = b$. Without loss of generality, assume $a > b$. Also assume $p \geq 2$. There is nothing to show if $p = 2$. I want to show that there exists a constant $C$ such that for $a > b$,

$$\frac{|a|^{p-2}a - |b|^{p-2}b}{|a - b|^{p-1}} \geq C \quad (73.2.6)$$

First assume also that $b \geq 0$. Now it is clear that as $a \to \infty$, the quotient above converges to 1. Take the derivative of this quotient. This yields

$$(p - 1)|a - b|^{p-2|a|^{p-2}a - |b|^{p-2}b - |a|^{p-2}a - |b|^{p-2}b}{|a - b|^{2p-2}}$$

Now remember $a > b$. Then the above reduces to

$$(p - 1)|a - b|^{p-2}b|b|^{p-2} - |a|^{p-2}{|a - b|^{2p-2}}$$

Since $b \geq 0$, this is negative and so 1 would be a lower bound. Now suppose $b < 0$. Then the above derivative is negative for $b < a \leq -b$ and then it is positive for
73.2. DUALITY MAPS

$a > -b$. It equals 0 when $a = -b$. Therefore the quotient in the expression achieves its minimum value when $a = -b$. This value is

$$\frac{|b|^{p-2} (-b) - |b|^{p-2} b}{|b| - b} = |b|^{p-2} \frac{-2b}{2b} = |b|^{p-2} \frac{1}{2b} = \frac{1}{2^{p-2}}.$$ 

Therefore, the conclusion holds whenever $p \geq 2$. That is

$$\langle Fu - Fv, u - v \rangle \geq \frac{1}{2^{p-2}} |a - b|^p.$$ 

This proves the lemma.

This holds for $p > 1$ also, but I don’t remember how to show this at this time.

However, in the context of strictly convex norms on the reflexive Banach space $X$, the following important result holds. I will give it for the case where $p = 2$ since this is the case of most interest.

**Theorem 73.2.5** Let $X$ be a reflexive Banach space and $X, X'$ have strictly convex norms as discussed above. Let $F$ be the duality map with $p = 2$. Then $F$ is strictly monotone. This means

$$\langle Fu - Fv, u - v \rangle \geq 0$$

and it equals 0 if and only if $u = v$.

**Proof:** First why is it monotone? By definition of $F$, $\langle F(u), u \rangle = \|u\|^2$ and $\|F(u)\| = \|u\|$. Then

$$|\langle Fu, v \rangle| = \left| \left\langle Fu, \frac{v}{\|v\|} \right\rangle \right| \|v\| \leq \|Fu\| \|v\| = \|u\| \|v\|$$

Hence

$$\langle Fu - Fv, u - v \rangle = \|u\|^2 + \|v\|^2 - \langle Fu, v \rangle - \langle Fv, u \rangle \geq \|u\|^2 + \|v\|^2 - 2 \|u\| \|v\| \geq 0.$$ 

Now suppose $\|x\| = \|y\| = 1$ but $x \neq y$. Then

$$\left\langle Fx, \frac{x + y}{2} \right\rangle \leq \left\| \frac{x + y}{2} \right\| < \frac{\|x\| + \|y\|}{2} = 1.$$ 

It follows that

$$\frac{1}{2} \langle Fx, x \rangle + \frac{1}{2} \langle Fx, y \rangle = \frac{1}{2} + \frac{1}{2} \langle Fx, y \rangle < 1$$

and so

$$\langle Fx, y \rangle < 1.$$ 

For arbitrary $x, y, x/ \|x\| \neq y/ \|y\|$,

$$\langle Fx, y \rangle = \|x\| \|y\| \left\langle F \left( \frac{x}{\|x\|} \right), y/ \|y\| \right\rangle.$$
It is easy to check that $F(\alpha x) = \alpha F(x)$. Therefore,
\[ |\langle Fx, y \rangle| = \|x\| \|y\| \left( \frac{x}{\|x\|} , \frac{y}{\|y\|} \right) < \|x\| \|y\| \]

Now say that $x \neq y$ and consider
\[ \langle Fx - Fy, x - y \rangle \]

First suppose $x = \alpha y$. Then the above is
\[ \langle F(\alpha y) - Fy, (\alpha - 1)y \rangle = (\alpha - 1) \left( \langle F(\alpha y), y \rangle - \|y\|^2 \right) \]
\[ = (\alpha - 1) \left( \langle \alpha F(y), y \rangle - \|y\|^2 \right) \]
\[ = (\alpha - 1)^2 \|y\|^2 > 0 \]

The other case is that $x/\|x\| \neq y/\|y\|$ and in this case,
\[ \langle Fx - Fy, x - y \rangle = \|x\|^2 + \|y\|^2 - \langle Fx, y \rangle - \langle Fy, x \rangle \]
\[ > \|x\|^2 + \|y\|^2 - 2\|x\| \|y\| \geq 0 \]

Thus $F$ is strictly monotone as claimed. ■

Another useful observation about duality maps for $p = 2$ is that \( \|F^{-1}y^*\|_{V'} = \|y^*\|_{V'} \). This is because
\[ \|y^*\|_{V'} = \|FF^{-1}y^*\|_{V'} = \|F^{-1}y^*\|_{V'} \]
also from similar reasoning,
\[ \langle y^*, F^{-1}y^* \rangle = \langle FF^{-1}y^*, F^{-1}y^* \rangle = \|F^{-1}y^*\|^2_{V'} = \|y^*\|^2_{V'} \]
Chapter 74

Implicit Stochastic Equations

74.1 Introduction

In this chapter, implicit evolution equations are considered. These are of the form

\[ Bu(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u(t, \omega), \omega) \, ds = \int_0^t f(s) \, ds + B \int_0^t \Phi dW \]

the term on the end being a stochastic integral. The novelty is in allowing \( B \) to be an operator which could vanish or have other interesting features. Thus the integral equation could degenerate to a non stochastic elliptic equation. This generalization of evolution equations has proven useful in the study of deterministic evolution equations and we give some interesting examples which indicate that this may be true in the case of stochastic equations also. In any case, it is an interesting generalization and equations of the usual form are recovered by using a Gelfand triple in which \( B = I \).

Like deterministic equations, there are many ways to consider stochastic equations. Here it is based on an approach due to Bardos and Brezis [13] which avoids the consideration of finite dimensional problems. A generalized Ito formula is summarized in the next section. It is Theorem 74.2.3.

74.2 Preliminary Results

Let \( X \) have values in \( W \) and satisfy the following

\[ BX(t) = BX_0 + \int_0^t Y(s) \, ds + B \int_0^t Z(s) \, dW(s), \quad (74.2.1) \]

\( X_0 \in L^2(\Omega; W) \) and is \( \mathcal{F}_0 \) measurable, where \( Z \) is \( L^2(Q^{1/2}U, W) \) progressively measurable and

\[ \| Z \|_{L^2([0,T] \times \Omega, L^2(Q^{1/2}U, W))} < \infty. \]
This is what is needed to define the stochastic integral in the above formula. Here $Q$ is a nonnegative self adjoint operator defined on a separable real Hilbert space $U$. In what follows, $J$ will denote a one to one Hilbert Schmidt operator mapping $Q^{1/2}U$ into another separable Hilbert space $U_1$. For more explanation on this situation see $[98]$. Assume $X, Y$ satisfy

$$X \in K \equiv L^p \left( [0, T] \times \Omega; V \right), \quad Y \in K' = L^{p'} \left( [0, T] \times \Omega; V' \right)$$

where $1/p' + 1/p = 1, p > 1$, and $X, Y$ are progressively measurable into $V$ and $V'$ respectively.

The sense in which the equation holds is as follows. For a.e. $\omega$, the equation holds in $V'$ for all $t \in [0, T]$. Assume that

$$X \in L^2 \left( [0, T] \times \Omega, W \right),$$

$$BX \in L^2 \left( [0, T] \times \Omega, B \left( [0, T] \times \mathcal{F}, W' \right) \right), \quad X \in L^p \left( [0, T] \times \Omega, B \left( [0, T] \times \mathcal{F}, V \right) \right)$$

Note that, since $X$ is progressively measurable into $V$, this implies that $BX$ is progressively measurable into $W'$. Also $W(t)$ is a $JJ^*$ Wiener process on $U_1$ in the following diagram. ($W$ is a cylindrical Wiener process.)

\[
\begin{array}{c}
U \\
\downarrow \Phi \\
Q^{1/2}U \\
\downarrow JQ^{1/2}U \\
U_1 \supseteq JQ^{1/2}U \\
\downarrow \\
W
\end{array}
\]

We will also make use of the following generalization of familiar concepts from Hilbert space.

**Lemma 74.2.1** Suppose $V, W$ are separable Banach spaces, $W$ also a Hilbert space such that $V$ is dense in $W$ and $B \in \mathcal{L}(W, W')$ satisfies

$$\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle, B \neq 0.$$ 

Then there exists a countable set $\{e_i\}$ of vectors in $V$ such that

$$\langle Be_i, e_j \rangle = \delta_{ij}$$

and for each $x \in W$,

$$\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,$$

and also

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i,$$

the series converging in $W'$. 
74.2. PRELIMINARY RESULTS

Then in the above situation, we have the following fundamental estimate.

Lemma 74.2.2 In the above situation where, off a set of measure zero, \( \text{[74.2.1]} \) holds for all \( t \in [0, T] \), and \( X \) is progressively measurable into \( V \),

\[
E \left( \sup_{t \in [0, T]} \langle BX, X \rangle (t) \right) < C \left( ||Y||_{K'}, ||X||_{K}, ||Z||_{J}, ||\langle BX_0, X_0 \rangle||_{L^1(\Omega)} \right) < \infty.
\]

where \( \langle BX, X \rangle (t) = \langle B (X (t)), X (t) \rangle \) a.e. and \( \langle BX, X \rangle \) is progressively measurable and continuous in \( t \).

\[
J = L^2 \left[ [0, T] \times \Omega; L_2 \left( Q^{1/2} U; W \right) \right], K \equiv L^p \left( [0, T] \times \Omega; V \right), \\
K' \equiv L^{p'} \left( [0, T] \times \Omega; V' \right).
\]

Also, \( C \) is a continuous function of its arguments and \( C (0, 0, 0, 0) = 0 \). Thus for a.e. \( \omega \),

\[
\sup_{t \in [0, T]} \langle BX, X \rangle (t) \leq C (\omega) < \infty.
\]

For a.e. \( \omega \), \( t \to BX (t, \omega) \) is weakly continuous with values in \( W' \) for \( t \) off a set of measure zero. Also \( t \to \langle BX (t), X (t) \rangle \) is lower semicontinuous off a set of measure zero.

Then from this fundamental lemma, the following Ito formula is valid. The proof of this theorem follows the same methods used for a similar result in [25].

Theorem 74.2.3 Off a set of measure zero, for every \( t \in [0, T] \),

\[
\langle BX, X \rangle (t) = \langle BX_0, X_0 \rangle + \int_0^t \left( 2 \langle Y (s), X (s) \rangle + \langle BZ, Z \rangle_{L_2} \right) ds \\
+ 2 \int_0^t (Z \circ J^{-1})^* BX \circ JdW
\] (74.2.2)

Also

\[
E (\langle BX, X \rangle (t)) = \\
E (\langle BX_0, X_0 \rangle) + E \left( \int_0^t \left( 2 \langle Y (s), X (s) \rangle + \langle BZ, Z \rangle_{L_2} \right) ds \right)
\] (74.2.3)

The quadratic variation of the stochastic integral is dominated by

\[
C \int_0^t \|Z\|_{L_2}^2 \|BX\|_{W'}^2 ds
\] (74.2.4)

for a suitable constant \( C \). Also \( t \to BX (t) \) is continuous with values in \( W' \) for \( t \in N_C^\omega \).
We will often abuse the notation and write $\langle BX(t), X(t) \rangle$ instead of the more precise $\langle BX, X \rangle(t)$. No harm is done because these two are equal a.e.

In addition to the above, we will use the following basic theorems about nonlinear operators. This is Proposition 73.1.8 above.

**Proposition 74.2.4** Suppose $A : V \to V'$ is type $M$, see [82], and suppose $L : V \to V'$ is monotone, bounded and linear. Here $V$ is a separable reflexive Banach space. Then $L + A$ is type $M$.

As an important example, we give the following definition.

**Definition 74.2.5** Let $f : [0, T] \times \Omega \to V$

$$\tau_h f(t, \omega) \equiv \begin{cases} f(t-h, \omega) & \text{if } t \geq h \\ 0 & \text{if } t < h \end{cases}$$

Then letting $B$ be a monotone nonnegative, self adjoint operator, $B : W \to W'$ for $W$ a separable Hilbert space, consider the linear operator $L : L^2(0, T, W) \equiv W \to L^2(0, T, W') \equiv W'$ given as

$$Lu = \left( I - \frac{\tau_h}{h} \right) Bu.$$ 

Is it the case that $L$ is monotone? Clearly it is linear and so it suffices to consider $\langle Lu, u \rangle_{W', W}$ which equals

$$\frac{1}{h} \int_0^T \langle Bu(t), u(t) \rangle \, dt - \frac{1}{h} \int_h^T \langle Bu(t-h), u(t) \rangle \, dt$$

$$= \frac{1}{h} \int_0^T \langle Bu(t), u(t) \rangle \, dt - \frac{1}{h} \int_0^{T-h} \langle Bu(t), u(t+h) \rangle \, dt$$

$$\geq \frac{1}{h} \int_0^T \langle Bu(t), u(t) \rangle \, dt$$

$$- \frac{1}{h} \int_0^{T-h} \left( \frac{1}{2} \langle Bu(t), u(t) \rangle + \frac{1}{2} \langle Bu(t+h), u(t+h) \rangle \right) \, dt$$

$$= \frac{1}{2h} \int_0^{T-h} \langle Bu(t), u(t) \rangle \, dt + \frac{1}{h} \int_{T-h}^T \langle Bu(t), u(t) \rangle \, dt$$

$$- \frac{1}{2h} \int_0^{T-h} \langle Bu(t+h), u(t+h) \rangle \, dt$$

$$= \frac{1}{2h} \int_0^{T-h} \langle Bu(t), u(t) \rangle \, dt + \frac{1}{h} \int_{T-h}^T \langle Bu(t), u(t) \rangle \, dt$$

$$- \frac{1}{2h} \int_{T-h}^T \langle Bu(t), u(t) \rangle \, dt$$

$$- \frac{1}{2h} \int_{T-h}^T \langle Bu(t+h), u(t+h) \rangle \, dt$$
\[ \begin{align*}
&= \frac{1}{2h} \int_{T-h}^{T} \langle Bu(t), u(t) \rangle \, dt + \frac{1}{2h} \int_{T}^{h} \langle Bu(t), u(t) \rangle \, dt \\
&+ \frac{1}{h} \int_{T-h}^{T} \langle Bu(t), u(t) \rangle \, dt - \frac{1}{2h} \int_{h}^{T} \langle Bu(t), u(t) \rangle \, dt \\
&= \frac{1}{2h} \int_{h}^{T-h} \langle Bu(t), u(t) \rangle \, dt + \frac{1}{h} \int_{T-h}^{T} \langle Bu(t), u(t) \rangle \, dt \\
&- \frac{1}{2h} \int_{T}^{h} \langle Bu(t), u(t) \rangle \, dt \\
&+ \frac{1}{2h} \int_{0}^{h} \langle Bu(t), u(t) \rangle \, dt - \frac{1}{2h} \int_{h}^{T-h} \langle Bu(t), u(t) \rangle \, dt \\
&= \frac{1}{2h} \int_{T-h}^{T} \langle Bu(t), u(t) \rangle \, dt + \frac{1}{2h} \int_{h}^{T} \langle Bu(t), u(t) \rangle \, dt \geq 0 \quad (74.2.5)
\end{align*} \]

The following is a restatement of Theorem 73.1.13

**Theorem 74.2.6** Let \( A : V \to V' \) be type \( M \), bounded, and coercive

\[
\lim_{\|u\| \to \infty} \frac{\langle A(u + u_0), u \rangle}{\|u\|} = \infty, \quad (74.2.6)
\]

for some \( u_0 \in V \), where \( V \) is a separable reflexive Banach space. Then \( A \) is surjective.

In addition, there is a fundamental definition and theorem about weak derivatives which will be used.

**Definition 74.2.7** Let \( f \in L^1(a,b,V') \) where \( V' \) is the dual of a Banach space \( V \). Let \( D^* (a,b) \) linear mappings from \( C_\infty^c (a,b) \) to \( V' \). Then we can consider \( f \in D^* (a,b) \), the linear transformations defined on \( C_\infty^c (a,b) \) as follows.

\[
f(\phi) \equiv \int_{a}^{b} f \phi ds
\]

This is well defined due to regularity considerations for Lebesgue measure. Then define \( Df \in D^* (a,b) \) by

\[
Df(\phi) \equiv - \int_{a}^{b} f \phi' ds
\]

To say that \( Df \in L^1(a,b,V') \) is to say that there exists \( g \in L^1(a,b,V') \) such that

\[
Df(\phi) \equiv - \int_{a}^{b} f \phi' ds = \int_{a}^{b} g \phi ds
\]

for all \( \phi \in C_\infty^c (a,b) \). Note that regularity considerations imply that \( g \) is unique if it exists.
The following is Theorem 67.2.9.

**Theorem 74.2.8** Suppose that $f$ and $Df$ are both in $L^1(a,b,V')$. Then $f$ is equal to a continuous function a.e., still denoted by $f$ and

$$f(x) = f(a) + \int_a^x Df(t)\,dt.$$  

In the next section are theorems about how shifts in time relate to progressive measurability.

### 74.3 The Existence Of Approximate Solutions

The situation is as follows. There are spaces $V \subseteq W$ where $V$ is a reflexive separable Banach space and $W$ is a separable Hilbert space. It is assumed that $V$ is dense in $W$. Define the spaces

$$V \equiv L^p([0,T] \times \Omega, V), \quad W \equiv L^2([0,T] \times \Omega, W)$$

where in each case, the $\sigma$ algebra of measurable sets will be the progressively measurable sets. Thus, from the Riesz representation theorem,

$$V' = L^{p'}([0,T] \times \Omega, V'), \quad W' = L^2([0,T] \times \Omega, W')$$

It will be assumed for the sake of convenience that $p \geq 2$. It follows that

$$V \subseteq W, \quad W' \subseteq V'$$

The entire presentation will be based on the following lemma.

**Lemma 74.3.1** Let $V \equiv L^p([0,T] \times \Omega, V)$ where $V$ is a separable Banach space and the $\sigma$ algebra of measurable sets consists of those which are progressively measurable. Then for $h \in (0,T)$, $\tau_h : V \to V$.

**Proof:** First consider $Q$ which is a progressively measurable set. Is it the case that $\tau_hX_Q$ is also progressively measurable? Define $Q + h$ as

$$Q + h \equiv \{(t + h, \omega) : (t, \omega) \in Q\}$$

Then

$$\tau_hX_Q(t,\omega) = \begin{cases} X_{Q+h}(t,\omega) & \text{if } t \geq h \\ 0 & \text{if } t < h \end{cases}$$

Is this function progressively measurable? For $(s,\omega) \in [0,t] \times \Omega$, we have the following

$$0 < \alpha \leq 1, \{(s,\omega) : \tau_hX_Q(s,\omega) \geq \alpha\} = [h,t] \times \Omega \cap (Q + h)$$

$$\alpha > 1, \{(s,\omega) : \tau_hX_Q(s,\omega) \geq \alpha\} = \emptyset \in \mathcal{B}([0,t]) \times \mathcal{F}_t$$
74.3. THE EXISTENCE OF APPROXIMATE SOLUTIONS

$$\alpha \leq 0, \left[ (s, \omega) : \tau_h x_Q(s, \omega) \geq \alpha \right] = [0, t] \times \Omega \in \mathcal{B}([0, t]) \times \mathcal{F}_t$$

It suffices to show that for $$t \geq h, [h, t] \times \Omega \cap (Q + h)$$ is $$\mathcal{B}([0, t]) \times \mathcal{F}_t$$ measurable. It is known that $$[0, t] \times \Omega \cap Q$$ is $$\mathcal{B}([0, t]) \times \mathcal{F}_t$$ measurable and also that $$[0, t-h] \times \Omega \cap Q$$ is $$\mathcal{B}([0, t-h]) \times \mathcal{F}_{t-h}$$ measurable. Let

$$\mathcal{G} \equiv \{ Q \in \mathcal{B}([0, t-h]) \times \mathcal{F}_{t-h} : [h, t] \times \Omega \cap Q + h \in \mathcal{B}([0, t]) \times \mathcal{F}_t \}$$

First consider $$I \times B$$ where $$I$$ is an interval in $$\mathcal{B}([0, t-h])$$ and $$B \in \mathcal{F}_{t-h}$$. Then

$$[h, t] \times \Omega \cap (I + h) \times B = I' \times B$$

where $$I'$$ is in $$\mathcal{B}([0, t])$$ and of course $$B \in \mathcal{F}_{t-h} \subseteq \mathcal{F}_t$$. Thus the sets of this form, are in $$\mathcal{G}$$. Next suppose $$Q \in \mathcal{G}$$. Is $$Q^C \in \mathcal{G}$$?

$$\left( [h, t] \times \Omega \cap (Q^C + h) \right) \cup [h, t] \times \Omega \cap (Q + h) \cup [0, h] \times \Omega = [0, t] \times \Omega$$

Then all of these disjoint sets but the first are in $$\mathcal{B}([0, t]) \times \mathcal{F}_t$$. It follows that the first is also in $$\mathcal{B}([0, t]) \times \mathcal{F}_t$$. It is clear that $$\mathcal{G}$$ is also closed with respect to countable disjoint unions. Therefore, $$\mathcal{G}$$ contains the $$\pi$$ system of sets of the form $$I \times B$$ just described. It follows that $$\mathcal{G} = \mathcal{B}([0, t-h]) \times \mathcal{F}_{t-h}$$.

Now if $$Q$$ is progressively measurable, then $$[0, t-h] \times \Omega \cap Q$$ is $$\mathcal{B}([0, t-h]) \times \mathcal{F}_{t-h}$$ measurable and so from what was just shown, $$[h, t] \times \Omega \cap Q + h \in \mathcal{B}([0, t]) \times \mathcal{F}_t$$. Thus $$\tau_h x_Q$$ is progressively measurable. It follows that if $$f \in \mathcal{V}$$, you could consider $$\phi(f)$$ for $$\phi \in \mathcal{V}'$$ and the positive and negative parts of this function. Each of these is the limit of a sequence of simple functions involving combinations of indicator functions of the form $$x_Q$$. Thus $$\tau_h \phi(f) = \phi(\tau_h f)$$ is the limit of simple functions involving combinations of functions $$\tau_h x_Q$$ and, as just shown, these simple functions are progressively measurable. Thus $$\tau_h f$$ is also progressively measurable by the Pettis theorem. ■

This Lemma states that you can do $$\tau_h$$ to progressively measurable functions and end up with one which is progressively measurable. Let

$$B \in L(W, W')$$

satisfy

$$\langle Bx, y \rangle = \langle By, x \rangle, \quad \langle Bx, x \rangle \geq 0$$

(74.3.7)

Also suppose that

$$A$$ is monotone and hemiuniform from $$\mathcal{V}$$ to $$\mathcal{V}'$$

(74.3.8)

This means the operator is monotone:

$$\langle Au - Au, u - v \rangle_{\mathcal{V}' , \mathcal{V}} \geq 0$$

and hemiuniform:

$$\lim_{t \to 0} \langle A(u + tv), w \rangle_{\mathcal{V}' , \mathcal{V}} = \langle Au, w \rangle_{\mathcal{V}' , \mathcal{V}}$$
Also we assume that \( A \) is bounded and takes the form

\[
Au (t, \omega) = A (t, u (t, \omega), \omega)
\]

for \( u \in V \). Such an operator is type \( M \) and this is what we use. Such an operator is defined by:

If \( u_n \to u \) weakly in \( V \), and \( Au_n \to \xi \) weakly in \( V' \) and \( \limsup_{n \to \infty} \langle Au_n, u_n \rangle \leq \langle \xi, u \rangle \)

Then the above implies

\[
Au = \xi.
\]

We define \( V_\omega \) as \( L^p (0, T, V) \) with the definition of \( V'_\omega \) similar, the subscript denoting that \( \omega \) is fixed, the \( \sigma \)-algebra of measurable sets being the Borel sets, \( B ([0, T]) \). Also, \((t, u, \omega) \to A (t, u, \omega) \) (74.3.9) is progressively measurable.

Suppose \( A (\omega) \) is monotone and hemicontinuous and bounded from \( V_\omega \) to \( V'_\omega \). Thus

\[
A (\omega) \text{ is type } M \text{ from } V_\omega \text{ to } V'_\omega
\]

where

\[
A (\omega) u \equiv A (t, u, \omega)
\]

We assume the estimates found in the next lemma.

**Lemma 74.3.2** If \( p \geq 2 \) and

\[
\langle A (t, u, \omega), u \rangle_V \geq \delta \| u \|_V^p - c (t, \omega)
\]

\[
\| A (t, u, \omega) \|_{V'} \leq k \| u \|_V^{p-1} + c^{1/p'} (t, \omega)
\]

where \( c \geq 0, c \in L^1 ([0, T] \times \Omega) \), then if \((t, \omega) \to q (t, \omega) \) is in \( V_\omega \), it follows that for a.e. \( \omega \), similar inequalities hold for \( \bar{A} \) given by

\[
\bar{A} (t, u, \omega) \equiv A (t, u + q (t, \omega), \omega)
\]

**Proof:** Letting \( q \) be progressively measurable, \( q (t, \omega) \in V \) only consider \( \omega \) such that \( t \to q (t, \omega) \) is in \( L^p (0, T, V) \).

\[
\langle \bar{A} (t, u, \omega), u \rangle =
\]

\[
\langle A (t, u + q (t, \omega), \omega), u \rangle = \langle A (t, u + q (t, \omega), \omega), u + q (t, \omega) \rangle - \langle A (t, u + q (t, \omega), \omega), q (t, \omega) \rangle
\]

\[
\geq \delta \| u + q (t, \omega) \|_V^p - k \| u + q (t, \omega) \|_V^{p-1} \| q (t, \omega) \|_V - c^{1/p'} (t, \omega) \| q (t, \omega) \|_V - c (t, \omega)
\]

\[
\geq \delta \| u + q (t, \omega) \|_V^p - k \| u + q (t, \omega) \|_V^{p-1} \| q (t, \omega) \|_V - \| q (t, \omega) \|_V^{p} - 2c (t, \omega)
\]
\[ \geq \frac{\delta}{2} \| u + q(t, \omega) \|_V^p - C(k, \delta, T) \| q(t, \omega) \|_V^p - 2c(t, \omega) \]

Now
\[ \| u + q(t, \omega) \| \geq \| u \| - \| q(t, \omega) \| \]

and so by convexity,
\[ \| u + q(t, \omega) \| \geq \| u \| - \| q(t, \omega) \| \]
\[ \| u + q(t, \omega) \| \geq (\| u \| - \| q(t, \omega) \|) \]
\[ \| u + q(t, \omega) \| \geq 2 \]}

This implies
\[ \| u + q(t, \omega) \| \geq 2 \left( \frac{\| u \|_V^p - \| q(t, \omega) \|_V^p}{2} \right) \]

Therefore,
\[ \langle \bar{A}(t, u, \omega) \rangle V \geq \delta \| u \|_V^p - c(t, \omega) \]}

where \( c' \in L^1([0, T] \times \Omega) \).

Consider the other inequality. Let \( \| z \|_V \leq 1 \).
\[ |\langle A(t, u + q(t, \omega), \omega) \rangle z| \leq k \| u + q(t, \omega) \|^{p-1} + c^{1/p'}(t, \omega) \]

Since \( p \geq 2 \), a convexity argument shows that
\[ \langle A(t, u + q(t, \omega), \omega) \rangle z \leq k \left( 2^{p-2} \| u \|^{p-1} + 2^{p-2} \| q(t, \omega) \|^{p-1} \right) + c^{1/p'}(t, \omega) \]
\[ = 2^{p-2}k \| u \|^{p-1} + (c(t, \omega))^{1/p'} \]

where \( c \in L^1([0, T] \times \Omega) \). Thus the same two inequalities continue to hold.

In what follows, \( c \geq 0 \) and is in \( L^1([0, T] \times \Omega) \), the \( \sigma \) algebra being \( \mathcal{B}(\{0, T\}) \times \mathcal{F}_T \).
 fittings.

\[ \langle A(t, u, \omega) \rangle V \geq \delta \| u \|_V^p - c(t, \omega) \]}
\[ \| A(t, u, \omega) \|_V \leq k \| u \|^{p-1} + c^{1/p'}(t, \omega) \]

Letting \( \bar{A} \) be defined above in \( \mathcal{F}_T \),
\[ \bar{A}(t, u, \omega) \equiv A(t, u + q, \omega) \equiv \bar{A}(t, u) \]

Assume the following pathwise uniqueness condition which is the hypothesis of the following lemma.
Lemma 74.3.3  Suppose it is true that whenever $u, v \in V_\omega$ and

$$Bu(t) - Bv(t) + \int_0^t A(u) - A(v) = 0 \quad (74.3.16)$$

it follows that $u = v$. Then if

$$(Bu)' + \bar{A}(\omega) u = f \text{ in } V'_\omega, \quad Bu(0) = Bu_0$$

$$(Bv)' + \bar{A}(\omega) v = f \text{ in } V'_\omega, \quad Bv(0) = Bu_0$$

(74.3.17)

it follows that $u = v$ in $V_\omega$. Here $u_0 \in W$.

Proof: If $(Bu)' + \bar{A}(\omega) u = f$ and $(Bv)' + \bar{A}(\omega) v = f$, then

$$Bu(t) - Bv(t) + \int_0^t A(u) - A(v) ds = 0$$

Hence

$$B(u(t) + q(t)) - B(v(t) + q(t)) + \int_0^t A(u) - A(v) ds = 0$$

and so $u + q = v + q$ showing that $u = v$. \blacksquare

We give the following measurability lemma.

Lemma 74.3.4  Suppose $f_n$ is progressively measurable and converges weakly to $\bar{f}$ in

$$L^\alpha([0,T] \times \Omega, X, \mathcal{B}([0,T]) \times \mathcal{F}_T), \quad \alpha > 1$$

where $X$ is a reflexive separable Banach space. Also suppose that for each $\omega \notin N$ a set of measure zero,

$$f_n(\cdot, \omega) \to f(\cdot, \omega) \text{ weakly in } L^\alpha(0,T,X)$$

Then there is an enlarged set of measure zero, still denoted as $N$ such that for $\omega \notin N$,

$$\bar{f}(\cdot, \omega) = f(\cdot, \omega) \text{ in } L^\alpha(0,T,X).$$

Also $\bar{f}$ is progressively measurable.

Proof: By the Pettis theorem, $\bar{f}$ is progressively measurable. Letting $\phi \in L^{\alpha'}([0,T] \times \Omega, X', \mathcal{B}([0,T]) \times \mathcal{F}_T)$, it is known that for a.e. $\omega$,

$$\int_0^T \langle \phi(t,\omega), f_n(t,\omega) \rangle dt \to \int_0^T \langle \phi(t,\omega), f(t,\omega) \rangle dt$$

and so $u + q = v + q$ showing that $u = v$. \blacksquare
Therefore, the function of $\omega$ on the right is at least $F$ measurable. Now let $g \in L^\infty(\Omega, X')$, and let $\psi \in C([0,T])$. Then for $1 < p \leq \alpha$,

$$
\int_\Omega \int_0^T \langle g(\omega) \psi(t), f_n(t,\omega) \rangle dt \, dP 
\leq C(T) \int_\Omega \|g\|_{L^\infty(\Omega,X')} \int_0^T |\psi(t)|^p \|f_n(t,\omega)\|_{X}^p \, dtdP 
\leq C(T,g,\psi) \int_\Omega \int_0^T \|f_n(t,\omega)\|_{X}^p \, dtdP \leq C < \infty
$$

for some $C$. Since $\int_0^T \langle g(\omega) \psi(t), f_n(t,\omega) \rangle dt$ is bounded in $L^p(\Omega)$ independent of $n$ because $\int_\Omega \int_0^T \|f_n(t,\omega)\|_{X}^p \, dtdP$ is given to be bounded, it follows that the functions

$$
\omega \to \int_0^T \langle g(\omega) \psi(t), f_n(t,\omega) \rangle dt
$$

are uniformly integrable and so it follows from the Vitali convergence theorem that

$$
\int_\Omega \int_0^T \langle g(\omega) \psi(t), f_n(t,\omega) \rangle dt \, dP \to \int_\Omega \int_0^T \langle g(\omega) \psi(t), f(t,\omega) \rangle dt \, dP
$$

But also from the assumed weak convergence to $\bar{f}$

$$
\int_\Omega \int_0^T \langle g(\omega) \psi(t), f_n(t,\omega) \rangle dt \, dP \to \int_\Omega \int_0^T \langle g(\omega) \psi(t), \bar{f}(t,\omega) \rangle dt \, dP
$$

It follows that

$$
\int_\Omega \left( g(\omega), \int_0^T (f - \bar{f}) \psi(t) dt \right) \, dP = 0
$$

This is true for every such $g \in L^\infty(\Omega, X')$, and so for a fixed $\psi \in C([0,T])$ and the Riesz representation theorem,

$$
\int_\Omega \left\| \int_0^T (f - \bar{f}) \psi(t) dt \right\|_X \, dP = 0
$$

Therefore, there exists $N_\psi$ such that if $\omega \notin N_\psi$, then

$$
\int_0^T (f - \bar{f}) \psi(t) dt = 0
$$

Enlarge $N$, the exceptional set to also include $\cup_{\psi \in \mathcal{D}} N_\psi$, where $\mathcal{D}$ is a countable dense subset of $C([0,T])$. Therefore, if $\omega \notin N$, then the above holds for all $\psi \in C([0,T])$. It follows that for such $\omega$, $f(t,\omega) = \bar{f}(t,\omega)$ for a.e. $t$. Therefore, $f(\cdot,\omega) = \bar{f}(\cdot,\omega)$ in $L^\alpha(0,T,X)$ for all $\omega \notin N$. 

Then one can obtain the following existence theorem using a technique of Bardos and Brezis.
Lemma 74.3.5 Let $q \in \mathcal{V}$ and let the conditions (74.3.13) - (74.3.14) be valid. Let $f \in \mathcal{V}'$ be given. Then for each $\omega$ off a set of measure zero, there exists $u(\cdot, \omega) \in \mathcal{V}_\omega$ such that $(Bu)'(\cdot, \omega) \in \mathcal{V}_\omega'$ and

$$Bu(0, \omega) = 0$$

and also the following equation holds in $\mathcal{V}_\omega'$ for a.e. $\omega$

$$(Bu)'(\cdot, \omega) + \bar{A}(\omega)(\cdot, u(\cdot, \omega)) = f(\cdot, \omega)$$

In addition to this, it can be assumed that $(t, \omega) \to u(t, \omega)$ is progressively measurable into $\mathcal{V}$. That is, for each $\omega$ off a set of measure zero, $t \to u(t, \omega)$ can be modified on a set of measure zero in $[0, T]$ such that the resulting $u$ is progressively measurable.

Proof: Consider the equation

$$L_h Bu + \bar{A}u = \frac{1}{h} (I - \tau_h)(Bu) + \bar{A}u = f \text{ in } \mathcal{V}'$$  \hspace{1cm} (74.3.18)$$

By Proposition 74.2.4 and Theorem 74.2.6, there exists a solution to the above equation if the left side is coercive. However, it was shown above in the computations leading to (74.2.5) that $L_h \circ B$ is monotone. Hence the coercivity follows right away from Lemma 74.3.2.

Thus (74.3.18) holds in $\mathcal{V}'$. It follows that, indexing the solution by $h$,

$$\int_\Omega \int_0^T \left\| \frac{1}{h} (I - \tau_h)(Bu_h) + \bar{A}u_h - f \right\|_{\mathcal{V}'}^p dt dP = 0$$

and so there exists a set of measure zero $N_h$ such that for $\omega \notin N_h$, the following equation holds in $\mathcal{V}_\omega'$

$$\frac{1}{h} (I - \tau_h)(Bu_h(\cdot, \omega)) + \bar{A}(\omega)(u_h(\cdot, \omega)) = f(\cdot, \omega)$$

Let $h$ denote a sequence converging to 0 and let $N$ be a set of measure zero which includes $\cup_h N_h$.

Letting $u_h \in \mathcal{V}$ be the above solution to (74.3.18), it also follows from the above estimates (74.3.13) - (74.3.14) that for $\omega$ off $N$, $\|u_h(\cdot, \omega)\|_{\mathcal{V}_\omega'}$ is bounded independent of $h$. Thus, for such $\omega$ off this set, there exists a subsequence still called $u_h$ such that the following convergences hold.

$$u_h \rightharpoonup u \text{ in } \mathcal{V}_\omega$$

$$\bar{A}(\omega) u_h \rightharpoonup \xi \text{ in } \mathcal{V}_\omega'$$

$$\frac{1}{h} (I - \tau_h)(Bu_h) \rightharpoonup \zeta \text{ in } \mathcal{V}_\omega'$$
First we need to identify \( \zeta \). Let \( \phi \in C^\infty ([0,T]) \) where \( \phi = 0 \) near \( T \) and let \( w \in V \). Then

\[
\left\langle \int_0^T \zeta \phi, w \right\rangle = \lim_{h \to 0} \left\langle \int_0^T \frac{1}{h} (I - \tau_h) (Bu_h), w \phi \right\rangle
\]

\[
= \lim_{h \to 0} \left\langle \int_0^T \frac{Bu_h(t)}{h} \phi(t) - \int_h^T \frac{Bu_h(t-h)}{h} \phi(t), w \right\rangle
\]

\[
= \lim_{h \to 0} \left\langle \int_0^T \frac{Bu_h(t)}{h} \phi(t) - \int_0^{T-h} \frac{Bu_h(t)}{h} \phi(t+h), w \right\rangle
\]

\[
= \lim_{h \to 0} \left( \left\langle \int_0^{T-h} \frac{Bu_h(t)}{h} \phi(t) - \phi(t+h), w \right\rangle + \int_0^T \frac{Bu_h(t)}{h} \phi(t) \right)
\]

\[
= \left\langle \int_0^T Bu(t) \phi'(t), w \right\rangle
\]

Since this holds for all \( \phi \in C^\infty_c (0,T) \), it follows that \( \zeta = (Bu)' \). Hence letting \( \phi \) be an arbitrary function in \( C^\infty ([0,T]) \) which equals zero near \( T \), this implies from the above that

\[
\left\langle - \int_0^T \zeta \phi, w \right\rangle = \left\langle \int_0^T Bu(t) \phi'(t), w \right\rangle
\]

\[
= \left\langle \int_0^T \left( Bu(0) + \int_0^t (Bu)'(s) ds \right) \phi'(t), w \right\rangle
\]

\[
= \int_0^T \left( Bu(0), w \right) \phi'(t) dt + \left\langle \int_0^T \int_0^t (Bu)'(s) ds \phi'(t), w \right\rangle
\]

\[
= - \left( Bu(0), w \right) \phi(0) + \left\langle \int_0^T (Bu)'(s) \int_s^T \phi'(t) dt ds, w \right\rangle
\]

\[
= - \left( Bu(0), w \right) \phi(0) - \left\langle \int_0^T (Bu)'(s) \phi(s) ds, w \right\rangle
\]

Hence, since \( \zeta = (Bu)' \),

\[
0 = - \left( Bu(0), w \right) \phi(0)
\]

Then it follows that, \( \left( Bu(0), w \right) = 0 \). Since \( w \) was arbitrary, \( Bu(0) = 0 \) and \( \zeta = (Bu)' \).

Thus, passing to a limit in 74.3.18, \( (Bu)' + \xi = f \) in \( V'_\omega \), \( Bu(0) = 0 \)
It is desired to identify $\xi$ with $\bar{A}(\omega) u$. First let

$$L_h \equiv \frac{I - \tau_h}{h}$$

Then

$$L_h (Bu) (t) = \begin{cases} \frac{1}{h} \int_{t-h}^{t} (Bu)' ds & \text{if } t \geq h \\ \frac{1}{h} \int_{0}^{t} (Bu)' ds & \text{if } t < h \end{cases}$$

Then from standard considerations involving approximate identities,

$$\lim_{h \to 0} L_h (Bu) = (Bu)' \text{ strongly in } V''_\omega \quad (74.3.19)$$

Thus

$$\langle L_h (Bu_h) - (Bu)', u_h - u \rangle =$$

$$\langle L_h (Bu_h) - L_h (Bu), u_h - u \rangle + \langle L_h (Bu) - (Bu)', u_h - u \rangle$$

and the above strong convergence implies that this converges to 0. Therefore, from

$$L_h Bu_h + \bar{A} u_h = f \text{ in } V''_\omega$$

and so

$$\langle L_h Bu_h, u_h - u \rangle + \langle \bar{A} u_h, u_h - u \rangle = \langle f, u_h - u \rangle$$

From the above,

$$\langle (Bu)', u_h - u \rangle + \langle L_h (Bu) - (Bu)', u_h - u \rangle + \langle \bar{A} u_h, u_h - u \rangle \leq \langle f, u_h - u \rangle$$

and so, taking $\limsup_{h \to 0}$ of both sides, it follows from that

$$\limsup_{h \to 0} \langle \bar{A} u_h, u_h - u \rangle \leq 0, \quad \limsup_{h \to 0} \langle \bar{A} u_h, u_h \rangle \leq \langle \xi, u \rangle$$

Since $A$ is monotone and hemi-continuous, the same is true of $\bar{A}$ and so

$$\bar{A} u = \xi$$

Thus

$$((Bu)' (\cdot, \omega)) + \bar{A} (\omega) u (\cdot, \omega) = f (\cdot, \omega) \text{ in } V''_\omega, \quad Bu (0, \omega) = 0 \quad (74.3.20)$$

It follows from the uniqueness assumption that for each $\omega$ off a set of measure zero, there exists a unique solution to

$$\begin{align*}
(Bu)' (\cdot, \omega) + \bar{A} (\cdot, u (\cdot, \omega), \omega) &= f (\cdot, \omega) \text{ in } V''_\omega, \\
B (u (\cdot, \omega)) (0) &= 0
\end{align*}$$
74.3. THE EXISTENCE OF APPROXIMATE SOLUTIONS

You can consider the function of two variables \( u(t, \omega) \). Is this function progressively measurable? Right now, this is not clear because we have done nothing more than solve a problem for each \( \omega \).

However, we can at least say that \( u_h \) is progressively measurable because \( u_h \in V \). Recall also that
\[
\frac{1}{h} (I - \tau_h) Bu_h + \bar{A}(\omega) u_h = f, \quad u_h \in V
\]

Next we show that because of uniqueness, one can assume that \( u \) is progressively measurable. To do this, we show that the sequence for which convergence holds in the above can be chosen independent of \( \omega \).

**Claim:** A single sequence \( h \to 0 \) works for all \( \omega \) off a set of measure zero.

**Proof:** Since there is only one solution to the above initial value problem for \( \omega \not\in N \), then letting \( h \to 0 \) be a single sequence, one can conclude that \( u_h (\cdot, \omega) \to u(\cdot, \omega) \) in \( V_\omega = L^p(0,T,V) \). Otherwise, from the above argument, one could obtain another subsequence which converges to a solution different than \( u(\cdot, \omega) \) which would violate uniqueness.

From the coercivity condition, it follows that there exists a constant \( C(f) \) depending on \( f \) such that for all \( h \),
\[
\|u_h\|_V \leq C(f)
\]

Therefore, there is a further subsequence still denoted by \( h \) such that
\[
u_h \to \bar{u} \text{ in } L^p([0,T] \times \Omega; V)\]

where the measurable sets are just the product measurable sets \( B([0,T]) \times \mathcal{F}_T \). Then it follows from Lemma 74.3.4 that \( u(\cdot, \omega) = \bar{u}(\cdot, \omega) \) in \( V_\omega \) for all \( \omega \) off a set of measure zero. It follows that in all of the above, we could substitute \( \bar{u} \) for \( u \) at least for \( \omega \) off a single set of measure zero. Thus \( u \) can be assumed progressively measurable.

Note the importance of path uniqueness in obtaining the result on progressive measurability of the solutions.

We will write \( u \) rather than \( \bar{u} \) to save notation. Now with this lemma, it is easy to obtain the following proposition.

**Proposition 74.3.6** Let \( q \in V \) such that \( t \to q(t, \omega) \) is continuous and \( q(0, \omega) = 0 \), and let the conditions 74.3.12 - 74.3.17 be valid. Also let \( u_0 \in L^2(\Omega, V) \) such that \( u_0 \) is \( F_0 \) measurable. Let \( f \in V' \) be given. Then for each \( \omega \) off a set of measure zero, there exists \( u(\cdot, \omega) \in V_\omega \) such that \( (Bu)(\cdot, \omega) \in V'_\omega \) and
\[
Bu(0, \omega) = Bu_0
\]

and also the following equation holds in \( V'_\omega \) for a.e. \( \omega \)
\[
(Bu - Bq)'(\cdot, \omega) + A(\cdot, u(\cdot, \omega), \omega) = f(\cdot, \omega)
\]

In addition to this, it can be assumed that \( (t, \omega) \to u(t, \omega) \) is progressively measurable into \( V \). That is, for each \( \omega \) off a set of measure zero, \( t \to u(t, \omega) \) can be
modified on a set of measure zero in $[0, T]$ such that the resulting $u$ is progressively measurable. Then one also obtains that $u$ is the unique solution to the integral equation which holds for a.e. $\omega$

$$Bu(t, \omega) - Bu_0 + \int_0^t A(s, u(s, \omega), \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bq(t, \omega) \quad (74.3.22)$$

**Proof:**
Recall
$$\bar{A}(\omega) (t, u) \equiv A(t, u + q(t, \omega), \omega)$$
where $q$ was in $V$. Therefore, replace this definition of $\bar{A}$ with
$$\bar{A}(\omega) (t, u) \equiv A(t, u + q(t, \omega) + u_0, \omega)$$
Then from Lemma 74.3.5, there exists $w \in V$ such that
$$(Bw)'(\cdot, \omega) + A(\cdot, w(\cdot, \omega) + q(\cdot, \omega) + u_0(\omega), \omega) = f(\cdot, \omega), \quad Bw(0) = 0$$
Let $u(t, \omega) = w(t, \omega) + q(t, \omega) + u_0(\omega)$. Then for fixed $\omega$, $Bu(0) = Bw(0) + Bu_0 = Bu_0$. Also
$$(Bu - q)' + A(\cdot, u(\cdot, \omega)) = f(\cdot, \omega), \quad Bu(0) = Bu_0$$
Then an integration yields

Corollary 74.3.7 Suppose the situation of the above proposition but that all that is known is that $\lambda B + A$ is monotone and hemicontinuous on $V_\omega$ and $V$ for all $\lambda$ sufficiently large. Then defining

$$\lambda (Bu, u) + \langle A(\omega)(u(\cdot, \omega)) \rangle \geq \delta \|u\|^p_{V'} - c(t, \omega) \quad (74.3.23)$$

for all $\lambda$ large enough.

$$\|A(t, u, \omega)\|_{V'} \leq k \|u\|_{V'}^{p-1} + c^{1/p'}(t, \omega) \quad (74.3.24)$$

where $c \in L^1([0, T] \times \Omega), c \geq 0$. Then the conclusion of Proposition 74.3.6 is still valid. There exists a unique $u \in V$ such that for a.e. $\omega$,

$$(Bu - Bq)'(\cdot, \omega) + A(\omega)(u(\cdot, \omega)) = f(\cdot, \omega), \quad B(u - q)(0) = Bu_0 \quad (74.3.25)$$

**Proof:**
That $\lambda B + A_\lambda$ is monotone and hemicontinuous follows from the definition. Also, from the above estimates,

$$\lambda (Bu, u) + \langle A_\lambda(\cdot, u, \omega) \rangle \geq e^{-2\lambda t} \left( \lambda \langle B(e^{\lambda t}u), e^{\lambda t}u \rangle + \langle A(t, e^{\lambda t}u, \omega) \rangle, e^{\lambda t}u \rangle \right)$$
74.3. THE EXISTENCE OF APPROXIMATE SOLUTIONS

\[ \geq e^{-2\lambda t} \left( \delta \left\| e^{\lambda t} u \right\|^p_V - e^{-\lambda t} c(t, \omega) \right) \geq e^{-2\lambda t} \left( \delta \left\| e^{\lambda t} u \right\|^p_V - e^{\lambda pt} e^{-\lambda pt} c(t, \omega) \right) \]

\[ \geq e^{-2\lambda t} e^{\lambda p t} \left( \delta \left\| u \right\|^p_V - e^{-\lambda pt} c(t, \omega) \right) \geq \delta \left\| u \right\|^p_V - e^{-\lambda pt} c(t, \omega) \]

which is of the right form.

Similarly

\[ \left\| \lambda Bw + A\lambda (t, w, \omega) \right\|_V \leq \left\| \lambda Bw \right\|_V + \left\| e^{-\lambda t} A(t, e^{\lambda t} w, \omega) \right\|_V \]

\[ \leq \lambda \left\| B \right\| \left\| w \right\|_V + e^{-\lambda t} \left\| A(t, e^{\lambda t} w, \omega) \right\|_V \]

\[ \leq \lambda \left\| B \right\| \left\| w \right\|_V + e^{-\lambda t} k \left\| e^{\lambda t} w \right\|^{p-1} + e^{-\lambda t} c^{1/p'} (t, \omega) \]

Since \( p \geq 2 \), this is no larger than

\[ \leq (\lambda \left\| B \right\|)^{p/(p-p')} + \left\| w \right\|^{p-1}_V + e^{(p-1)\lambda t} e^{-\lambda t} k \left\| w \right\|^{p-1}_V + e^{-\lambda t} c^{1/p'} (t, \omega) \]

\[ \leq \left( e^{(p-2)\lambda T} k + 1 \right) \left\| w \right\|^{p-1}_V + e^{-\lambda t} c^{1/p'} (t, \omega) + (\lambda \left\| B \right\|)^{p/(p-p')} \]

\[ \equiv \bar{k} \left\| w \right\|^{p-1}_V + \bar{c} (t, \omega)^{1/p'} \]

Now note that if \( w \) is a solution to

\[ B \left( w - e^{-\lambda (\cdot)} q \right)' + \lambda Bw + e^{-\lambda (\cdot)} A(t, e^{\lambda (\cdot)} w, \omega) = e^{-\lambda (\cdot)} f (\cdot, \omega) + \lambda Be^{-\lambda (\cdot)} q (\cdot, \omega) \text{ in } V_\omega \]

\[ B \left( w - e^{-\lambda (\cdot)} q \right)(0) = Bu_0 \]

if and only if \( u(t) \equiv e^{\lambda t} w(t) \) is a solution to

\[ (B (u - q))' + A(t, u, \omega) = f (\cdot, \omega) \text{, } B (u - q)(0) = Bu_0 \]

Thus the necessary uniqueness condition holds for the initial value problem for \( w \) and hence it follows from Proposition 74.3.6 that there exists a unique progressively measurable solution to the initial value problem for \( w \) and hence a unique progressively measurable solution to the above one for \( u \).

Now suppose the situation of the above corollary and let \( E \) be a separable Hilbert space which is dense in \( V \) and let

\[ \Phi \in L^2 \left( [0, T] \times \Omega, L^2 \left( Q^{1/2} U, E \right) \right) \text{, } \Phi \text{ being progressively measurable,} \]

so that one can consider the stochastic integral \( \int_0^t \Phi dW \). Let

\[ \tau_n \equiv \inf \left\{ t : \left\| \int_0^t \Phi dW \right\|_E > 2^n \right\} \]
Thus
\[ \left\| \int_0^{t \land \tau_n} \Phi dW \right\| _E \leq 2^n \]
Then you could pick \( u_0 \in L^p(\Omega, V) \), \( u_0 \) being \( \mathcal{F}_0 \) measurable, and let
\[ q(t, \omega) = \int_0^{t \land \tau_n} \Phi dW. \]
The result is clearly in \( V \) and is continuous in \( t \). Therefore, from Corollary 74.3.7, there exists a unique solution \( u \in V \) to the initial value problem
\[
\left( Bu - B \int_0^{t \land \tau_n} \Phi dW \right)'(\cdot, \omega) + A(\omega)(u(\cdot, \omega)) = f(\cdot, \omega), \quad Bu(0) = Bu_0.
\]
Integrating, one obtains a unique solution \( u_n \in V \) to the integral equation
\[
Bu_n(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u_n, \omega) ds = \int_0^t f(s, \omega) ds + B \int_0^{t \land \tau_n} \Phi dW.
\]
This holds in \( V'_\omega \) and is so for all \( \omega \) off a set of measure zero \( N_n \). Let \( N = \cup_n N_n \).
For \( \omega \notin N \), \( t \to \int_0^t \Phi dW \) is continuous and so for all \( n \) large enough, \( \tau_n = \infty \). Thus for a fixed \( \omega \), it follows that for all \( n \) large enough \( \tau_n = \infty \) and so one obtains
\[
Bu_n(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u_n, \omega) ds = \int_0^t f(s, \omega) ds + B \int_0^{t \land \tau_n} \Phi dW
\]
Then for \( k \) some other index sufficiently large, the same holds for \( u_k \). By the uniqueness assumption 74.3.17, \( u_k(t, \omega) = u_n(t, \omega) \) and so it follows that \( \lim_{n \to \infty} u_n(t, \omega) \) exists because for each \( \omega \) off a set of measure zero, there is eventually no change in \( u_n \). Defining \( u(t, \omega) = \lim_{n \to \infty} u_n(t, \omega) = u_n(t, \omega) \) for all \( n \) large enough, it follows that \( u \) is progressively measurable since it is the pointwise limit of progressively measurable functions and
\[
Bu(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u, \omega) ds = \int_0^t f(s, \omega) ds + B \int_0^{t \land \tau_n} \Phi dW.
\]
This has shown the following lemma.

**Lemma 74.3.8** Let \( (t, u, \omega) \to A(t, u, \omega) \) be progressively measurable into \( V' \) and suppose for some \( \lambda \),
\[
\lambda B + A(\omega) : V_{\omega} \to V'_{\omega},
\]
\[
\lambda B + A : V \to V'
\]
are both monotone bounded and hemicontinuous. Also suppose the two estimates giving boundedness and coercivity 74.3.23 - 74.3.24 of Corollary 74.3.7 above. Here \( V, W \) are as described above \( V \subseteq W, W' \subseteq V' \), \( W \) is a separable Hilbert space and
74.4. THE GENERAL CASE 2629

$V$ is a separable reflexive Banach space. $B : W \rightarrow W'$ is nonnegative and self adjoint. Let $f \in V'$ and let $u_0 \in L^p (\Omega, V)$ where $u_0$ is $\mathcal{F}_0$ measurable. Then if $\Phi \in L^2 ([0, T] \times \Omega, \mathcal{L}_2 (Q^{1/2} U, E))$, $\Phi$ being progressively measurable, into $E$, where $E$ is a Hilbert space dense in $V$ with $\|u\|_E \geq \|u\|_V$, then there exists a unique solution to the integral equation

$$Bu(t) - Bu_0 + \int_0^t A(s, u, \omega) \, ds = \int_0^t f(s, \omega) \, ds + B \int_0^t \Phi \, dW$$

in the sense that $u$ is in $V$ and there exists a set of measure zero $N$ such that if $\omega \notin N$, then the above integral equation holds for all $t$.

74.4 The General Case

Suppose $\lambda B + A(\omega), \lambda B + A$ are both monotone bounded and hemicontinuous on $V_\omega$ and $V$ respectively for $\lambda$ sufficiently large. Also suppose the two estimates giving boundedness and coercivity of Corollary 74.3.7 above. We strengthen the assumption that $\lambda B + A(\omega)$ is monotone as follows. In the usual case where $B$ is the identity, this conclusion is obvious, but here we need to assume it.

$$(\lambda B + A(\omega)) (u) - (\lambda B + A(\omega)) (v), u - v \geq \delta \|u - v\|_U^\alpha, \alpha \geq 1 \quad (74.4.26)$$

where here $U$ is a reflexive Banach space such that $V \subseteq U$ and the inclusion map is continuous, $V$ being dense in $U$. In regards to this monotonicity condition, here is a simple lemma which will be used later.

**Lemma 74.4.1** Suppose $u_n \rightarrow w$ weakly in $V_\omega$ and that for a.e.t, $u_n(t) \rightarrow u(t)$ in $U$. Then $w(t) = u(t)$ a.e.

**Proof:** You know that $\|u_n\|_{L^p([0, T], V)}$ is bounded. Now consider $\phi \in U'$ and $\psi \in C([0, T])$. Then the weak convergence implies

$$\lim_{n \rightarrow \infty} \int_0^T \langle \phi, u_n \rangle_{U', U} \psi \, dt = \int_0^T \langle \phi, w \rangle_{U', U} \psi \, dt$$

because it is also the case that $u_n \rightarrow w$ weakly in $L^p([0, T], U)$. However, the fact that $\|u_n\|_{L^p([0, T], V)}$ is bounded means that, by the assumed pointwise convergence,

$$\lim_{n \rightarrow \infty} \int_0^T \langle \phi, u_n \rangle_{U', U} \psi \, dt = \int_0^T \langle \phi, u \rangle_{U', U} \psi \, dt$$

It follows that

$$\int_0^T \langle \phi, u - w \rangle \psi \, dt = 0$$
CHAPTER 74. IMPLICIT STOCHASTIC EQUATIONS

Since this is true for all \( \psi \in C ([0, T]) \), there exists a set of measure zero \( Q_\phi \) such that for \( t \notin Q_\phi \),
\[
\langle \phi, u (t) - w (t) \rangle = 0
\]
Letting \( Q = \cup_{\phi \in D} Q_\phi \), where \( D \) is a countable dense subset of \( U' \), it follows that for \( t \notin Q \), the above holds for all \( \phi \in U' \). Hence \( u (t) = w (t) \) for \( t \notin Q \) and \( m (Q) = 0 \).

Typically \( \alpha = 2 \) and \( U = W \).

Recall that
\[
V \subseteq W, \quad W' \subseteq V'
\]
each space dense in the one to its right and the inclusion maps are continuous.

Assume only
\[
\Phi \in L^2 \left( [0, T] \times \Omega, \mathcal{L}_2 \left( Q^{1/2} U, W \right) \right).
\]

By density of \( E \) into \( W \), there exists a sequence
\[
\Phi_n \in L^2 \left( [0, T] \times \Omega, \mathcal{L}_2 \left( Q^{1/2} U, E \right) \right)
\]
such that
\[
\| \Phi_n - \Phi \|_{L^2([0, T] \times \Omega, \mathcal{L}_2(Q^{1/2} U, W))} \to 0,
\]
\[
\| \Phi_n \|_{\mathcal{L}_2(Q^{1/2} U, W)} \leq \| \Phi \|_{\mathcal{L}_2(Q^{1/2} U, W)}.
\]

Also let \( u_{0n} \in L^p (\Omega, V) \) where \( u_{0n} \) is \( F_0 \) measurable and such that \( u_{0n} \in L^p (\Omega, V) \) and
\[
\| u_{0n} (\omega) - u_0 (\omega) \|_W \to 0, \quad \langle Bu_{0n}, u_{0n} \rangle \leq 2 \langle Bu_0, u_0 \rangle
\]
for each \( \omega \). The existence of such an approximating sequence follows from density considerations of \( E \) into \( V \) and of \( V \) into \( W \).

By Lemma there is a solution \( u_n \) to the integral equation
\[
Bu_n (t) - Bu_{0n} + \int_0^t A (s, u_n, \omega) ds = \int_0^t f (s, \omega) ds + B \int_0^t \Phi_n dW \quad (74.4.27)
\]

Then by the Implicit Ito formula there is a set of measure zero such that for all \( n, m \)
\[
\frac{1}{2} \langle B (u_n - u_m), u_n - u_m \rangle (t) - \frac{1}{2} \langle Bu_{0n} - Bu_{0m}, u_{0n} - u_{0m} \rangle \\
+ \delta \int_0^t \| u_n - u_m \|_U^2 ds
\]
\[
\leq \lambda \int_0^t \langle B (u_n - u_m), u_n - u_m \rangle (s) ds \\
+ \frac{1}{2} \int_0^t \langle B (\Phi_n - \Phi_m), \Phi_n - \Phi_m \rangle_{L^2} ds + M_{mn} (t)
\]
Also the last term is a martingale whose quadratic variation satisfies
\[
\left[ M_{mn}(t) \right] = C \int_0^t \left\| \Phi_n - \Phi_m \right\|_{L_2(Q^{1/2} U, W)}^2 \left\| B(u_n - u_m) \right\|_{W}^2 \, ds
\]
\[
\leq C \int_0^t \left\| \Phi_n - \Phi_m \right\|_{L_2(Q^{1/2} U, W)}^2 \left\langle B(u_n - u_m), u_n - u_m \right\rangle \, ds
\]
Then from Gronwall’s inequality, and adjusting the constants,
\[
\left\langle B(u_n - u_m), u_n - u_m \right\rangle + \int_0^t \left\| u_n - u_m \right\|_{U}^\alpha \, ds
\leq C (u_0 - u_0 + \Phi_n - \Phi_m) + C(T) M_{mn}^*(t)
\]
where the expectation of the first constant on the right converges to 0 as \( m,n \to \infty \).
Here
\[
M_{mn}^*(t) = \sup_{s \in [0,t]} |M_{mn}(t)|
\]
Since \( M^* \) is increasing, this implies that after adjusting constants,
\[
\sup_{s \in [0,t]} \left( \left\langle B(u_n - u_m), u_n - u_m \right\rangle(s) + \int_0^t \left\| u_n - u_m \right\|_{U}^\alpha \, ds \right) \leq C (u_0 - u_0 + \Phi_n - \Phi_m) + C(T) M_{mn}^*(t)
\]
Then taking expectations and using the Burkholder Davis Gundy inequality,
\[
E \left( \sup_{s \in [0,t]} \left( \left\langle B(u_n - u_m), u_n - u_m \right\rangle(s) + \int_0^t \left\| u_n - u_m \right\|_{U}^\alpha \, ds \right) \right) \leq C (u_0 - u_0 + \Phi_n - \Phi_m) +
\]
\[
C(T) \int_\Omega \left( \int_0^t \left\| \Phi_n - \Phi_m \right\|_{L_2(Q^{1/2} U, W)}^2 \left\langle B(u_n - u_m), u_n - u_m \right\rangle \, ds \right)^{1/2} \, dP
\leq C_{n,m} + 2C \int_\Omega \sup_{s \in [0,t]} \left\langle B(u_n - u_m), u_n - u_m \right\rangle^{1/2}(s) \cdot \left( \int_0^t \left\| \Phi_n - \Phi_m \right\|_{L_2(Q^{1/2} U, W)}^2 \right)^{1/2} \, dP
\]
Then adjusting the constants,
\[
E \left( \sup_{s \in [0,t]} \left( \left\langle B(u_n - u_m), u_n - u_m \right\rangle(s) + \int_0^t \left\| u_n - u_m \right\|_{U}^\alpha \, ds \right) \right) \leq C_{n,m} + C \int_0^T \int_\Omega \left\| \Phi_n - \Phi_m \right\|_{L_2(Q^{1/2} U, W)}^2 \, dt \, dP \equiv C_{n,m}
where \( C_{n,m} \to 0 \) as \( n, m \to \infty \). In particular, it is true for \( t = T \)

\[
E \left( \sup_{s \in [0,T]} \langle B (u_n - u_m), u_n - u_m \rangle (s) + \int_0^T \|u_n - u_m\|^2_U \, ds \right) \leq C_{n,m}
\]

Then

\[
P \left( \sup_{s \in [0,T]} \langle B (u_n - u_m), u_n - u_m \rangle (s) + \int_0^T \|u_n - u_m\|^2_U \, ds \geq \lambda \right) \leq \frac{C_{n,m}}{\lambda}
\]

Now take a subsequence such that if \( m > n_k, C_{n_k,m} < 4^{-k} \). Then the above inequality implies that

\[
P \left( \sup_{s \in [0,T]} \langle B (u_n - u_m), u_n - u_m \rangle (s) + \int_0^T \|u_n - u_{n_k+1}\|^2_U \, ds \geq 2^{-k} \right) \leq \frac{4^{-k}}{2^{-k}} = 2^{-k}
\]

and so, by the Borel Cantelli lemma, there is a set of measure zero \( N \) including all earlier exceptional sets of measure zero such that for \( \omega \notin N \),

\[
\sup_{s \in [0,T]} \langle B (u_n - u_m), u_n - u_m \rangle (s) + \int_0^T \|u_{n_k} - u_{n_k+1}\|^2_U \, ds < 2^{-k}
\]

for all \( k \) large enough. We will denote this new subsequence by \( \{u_n\} \). Thus for such \( \omega \), it follows that \( Bu_n \) is a Cauchy sequence in \( C (N^C, W') \) for \( N^C \) an exceptional set of measure zero where \( B (u_n - u_m) (t) \neq B (u_n (t) - u_m (t)) \) and also \( \{u_n\} \) is a Cauchy sequence in \( L^a (0, T, U) \). It follows

\[
Bu_n \to z \text{ strongly in } C (N^C, W') \text{ with uniform norm} \quad (74.4.28)
\]

\[
\lim_{m,n \to \infty} \sup_{s \in [0,T]} \langle B (u_n - u_m), u_n - u_m \rangle (s) = 0 \quad (74.4.29)
\]

There exists \( u \in L^a (0, T, U) \) such that for \( \omega \notin N \),

\[
\|u_n - u\|_{L^a (0,T,U)} \to 0, \ u_n (t, \omega) \to u (t, \omega) \text{ for a.e.t in } U \quad (74.4.30)
\]

Of course a technical issue is the fact that \( B \) is a degenerate operator which might not be invertible. In the above limit, we do not know that \( z = Bu \) for some \( u \). We resolve this issue by obtaining pointwise estimates for a given \( \omega \) and then pass to a limit. After this, a time integration will give the desired result. There are easier ways to do this if \( B \) is not degenerate.

From now on, this or a subsequence of this one will be the sequence of interest. Return to (74.4.25) and use the Ito formula again. Thus using the estimates,

\[
\frac{1}{2} \langle Bu_n, u_n \rangle (t) - \frac{1}{2} \langle Bu_{0n}, u_{0n} \rangle + \delta \int_0^t \|u_n\|^p_V \, ds - \lambda \int_0^t \langle Bu_n, u_n \rangle \, ds
\]

\[
= \frac{1}{2} \int_0^t \langle B\Phi_n, \Phi_n \rangle \, ds + \int_0^t c (s, \omega) \, ds + \int_0^t \langle f, u_n \rangle \, ds + M_n (t)
\]
where \( M_n(t) \) is a local martingale whose quadratic variation satisfies

\[
[M_n](t) \leq C \int_0^t \| \Phi_n \|_{L_2}^2 \| Bu_n \|_W^2 \, ds
\]

Then adjusting the constants,

\[
\langle Bu_n, u_n \rangle(t) + \int_0^t \| u_n \|_V^p \, ds \leq C (u_{0n}, \Phi_n, f, c) + CM_n^* (t)
\]

where the expectation of the first constant on the right is no larger than a constant \( C \) which is independent of \( n \). Since the right term is increasing in \( t \),

\[
\sup_{s \in [0, t]} \langle Bu_n, u_n \rangle(s) + \int_0^t \| u_n \|_V^p \, ds \leq C (u_{0n}, \Phi_n, f, c) + CM_n^* (t) \tag{74.4.31}
\]

Now using the Burkholder Davis Gundy inequality as before and taking the expectation,

\[
E \left( \sup_{s \in [0, t]} \langle Bu_n, u_n \rangle(s) \right) + E \int_0^t \| u_n \|_V^p \, ds \leq C + C \int_{\Omega} \left( \int_0^t \| \Phi_n \|_{L_2}^2 \| Bu_n \|_W^2 \, ds \right)^{1/2} \, dP
\]

\[
\leq C + C \int_{\Omega} \left( \int_0^t \| \Phi_n \|_{L_2}^2 \langle Bu_n, u_n \rangle \, ds \right)^{1/2} \, dP
\]

\[
\leq C + C \int_{\Omega} \sup_{s \in [0, t]} \langle Bu_n, u_n \rangle(s)^{1/2} \left( \int_0^t \| \Phi_n \|_{L_2}^2 \, ds \right)^{1/2} \, dP
\]

Then adjusting the constants and using the approximation properties of \( \Phi_n \) given above, there is a constant \( C \) independent of \( n, t \leq T \) such that

\[
E \left( \sup_{s \in [0, t]} \langle Bu_n, u_n \rangle(s) \right) + E \int_0^t \| u_n \|_V^p \, ds \leq C
\]

In particular

\[
E \left( \sup_{s \in [0, T]} \langle Bu_n, u_n \rangle(s) \right) + E \int_0^T \| u_n \|_V^p \, ds \leq C \tag{74.4.32}
\]

Next use monotonicity to obtain

\[
\frac{1}{2} \langle Bu_r - Bu_q, u_r - u_q \rangle(t) \leq \frac{1}{2} \int_0^t \left( (\Phi_r - \Phi_q) \circ J^{-1} \right)^* B (u_r - u_q) \circ J dW + C \lambda \int_0^t \langle Bu_r - Bu_q, u_r - u_q \rangle \, ds + \int_0^t \| \Phi_r - \Phi_q \|^2 \, ds
\]
and so, from Gronwall’s inequality, there is a constant $C$ which is independent of $r, q$ such that

$$\langle Bu_r - Bu_q, u_r - u_q \rangle (t) \leq CM_{rq} (t) \leq CM_{rq}^* (T) + C \int_0^t \| \Phi_r - \Phi_q \|^2 ds$$

where $M_{rq}$ refers to that local martingale on the right. Thus also

$$\sup_{t \in [0, T]} \langle Bu_r - Bu_q, u_r - u_q \rangle (t) \leq CM_{rq} (t) \leq CM_{rq}^* (T) + C \int_0^T \| \Phi_r - \Phi_q \|^2 ds$$

Taking the expectation and using the Burkholder Davis Gundy inequality again, and similar estimates to the above, using appropriate stopping times as needed, we obtain

$$E \left( \sup_{t \in [0, T]} \langle Bu_r - Bu_q, u_r - u_q \rangle (t) \right) \leq C \int_\Omega \int_0^T \| \Phi_r - \Phi_q \|^2 dtdP$$

Now the right side converges to 0 as $r, q \to \infty$ and so there is a subsequence, denoted with the index $k$ such that if $p > k$,

$$E \left( \sup_{t \in [0, T]} \langle Bu_k - Bu_p, u_k - u_p \rangle (t) \right) \leq \frac{1}{2^k} \tag{74.4.34}$$

Then consider the earlier local martingales. One of these is of the form

$$M_k = \int_0^t (\Phi_k \circ J^{-1})^* Bu_k \circ J dW$$

Then by the Burkholder Davis Gundy inequality and modifying constants as appropriate,

$$E \left( \left( M_k - M_{k+1} \right)^* \right)$$

$$\leq C \int_\Omega \left( \int_0^T \left\| (\Phi_k \circ J^{-1})^* Bu_k - (\Phi_{k+1} \circ J^{-1})^* Bu_{k+1} \right\|^2 dt \right)^{1/2} dP$$

$$\leq C \int_\Omega \left( \int_0^T \left\| \Phi_k - \Phi_{k+1} \right\|^2 \langle Bu_k, u_k \rangle + \left\| \Phi_{k+1} \right\|^2 \langle Bu_k - Bu_{k+1, u_k - u_{k+1}} \rangle dt \right)^{1/2} dP$$

$$\leq C \int_\Omega \left( \int_0^T \left\| \Phi_k - \Phi_{k+1} \right\|^2 \langle Bu_k, u_k \rangle dt \right)^{1/2} + C \int_\Omega \left( \int_0^T \left\| \Phi_{k+1} \right\|^2 \langle Bu_k - Bu_{k+1, u_k - u_{k+1}} \rangle dt \right)^{1/2} dP$$
74.4. THE GENERAL CASE

\[\leq C \int_{\Omega} \sup_t \langle Bu_k, u_k \rangle^{1/2} \left( \int_0^T \| \Phi_k - \Phi_{k+1} \|^2 \, dt \right)^{1/2} \, dP \]

\[+ C \int_{\Omega} \sup_t \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle^{1/2} \left( \int_0^T \| \Phi_{k+1} \|^2 \, dt \right)^{1/2} \, dP \]

\[\leq C \left( \int_{\Omega} \sup_t \langle Bu_k, u_k \rangle \, dP \right)^{1/2} \left( \int_{\Omega} \int_0^T \| \Phi_k - \Phi_{k+1} \|^2 \, dt dP \right)^{1/2} \]

\[+ C \left( \int_{\Omega} \sup_t \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle \, dP \right)^{1/2} \left( \int_{\Omega} \int_0^T \| \Phi_{k+1} \|^2 \, dtdP \right)^{1/2} \]

From the above inequalities, after adjusting the constants, the above is no larger than an expression of the form

\[C \left( \frac{1}{2} \right)^{k/2} \text{ which is a summable sequence. Then} \]

\[\sum_k \int_{\Omega} \sup_{t \in [0, T]} |M_k(t) - M_{k+1}(t)| \, dP < \infty \]

Then \(\{M_k\}\) is a Cauchy sequence in \(M^1_T\) and so there is a continuous martingale \(M\) such that

\[\lim_{k \to \infty} E \left( \sup_t |M_k(t) - M(t)| \right) = 0 \quad (74.4.35)\]

Taking a further subsequence if needed, one can also have

\[P \left( \sup_t |M_k(t) - M(t)| > \frac{1}{k} \right) \leq \frac{1}{2^k} \]

and so by the Borel Cantelli lemma, there is a set of measure zero such that off this set, \(\sup_t |M_k(t) - M(t)|\) converges to 0. Hence for such \(\omega\), \(M_k^\omega(T)\) is bounded independent of \(k\). Thus for \(\omega\) off a set of measure zero, \(\sup_{t \in [0, T]} \langle Bu_n, u_n \rangle(s) + \int_0^T \| u_r(s) \|_V^p \, ds \leq C(\omega)\)

where \(C(\omega)\) does not depend on the index \(r\), this for the subsequence just described which will be the sequence of interest in what follows. Using the boundedness assumption for \(A\), one also obtains an estimate of the form

\[\sup_{s \in [0, T]} \langle Bu_r, u_r \rangle(s) + \int_0^T \| u_r(s) \|_V^p \, ds + \int_0^T \| z_r \|_V^p \, ds \leq C(\omega) \quad (74.4.36)\]

**Lemma 74.4.2** There is a subsequence, still indexed by \(n\) and a set of measure zero \(N\), containing all the preceding sets of measure zero such that for \(\omega \notin N\),

\[\sup_{s \in [0, T]} \langle Bu_n, u_n \rangle(s) + \int_0^T \| u_n \|_V^p \, ds \leq C(\omega) < \infty\]
CHAPTER 74. IMPLICIT STOCHASTIC EQUATIONS

From the theory of the stochastic integral, there is a further subsequence of the above such that

\[
\int_0^t \Phi_n dW \to \int_0^t \Phi dW \text{ strongly in } C(0, T, W)
\]
for all \( \omega \) off a set of measure zero.Enlarge the exceptional set \( N \) and only use subsequence of this one so that both the above estimate in the lemma and the above convergence hold for \( \omega \notin N \). Recall the integral equation solved.

\[ Bu_n(t) - Bu_0 + \int_0^t A(s, u_n, \omega) ds = \int_0^t f(s, \omega) ds + B \int_0^t \Phi_n dW \quad (74.4.37) \]

Thus

\[ \left( Bu_n - B \int_0^t \Phi_n dW - Bu_0 \right)' + Au_n = f \]

Then for \( \omega \notin N \), a subsequence of the one for which the above lemma holds, still denoted as \( \{ u_n \} \) yields the following convergences,

\[ u_n \to u \text{ weakly in } V_\omega \quad (74.4.38) \]

\[ Au_n \rightharpoonup \xi \text{ weakly in } V'_\omega \quad (74.4.39) \]

\[ \left( Bu_n - B \int_0^t \Phi_n dW - Bu_0 \right) \rightharpoonup \zeta \text{ weakly in } V'_\omega \quad (74.4.40) \]

By the earlier convergence \([4.30.34]\), this \( u \) is the same as the one in \([4.30.34]\).

Consider \( \zeta \). Let \( \psi \) be infinitely differentiable and equal to 0 near \( T \) and let \( g \in V \). Then since \( Bu_n(0) = Bu_0 \),

\[
\int_0^T \langle \zeta, \psi g \rangle dt = \lim_{n \to \infty} \int_0^T \left\langle \left( Bu_n - B \int_0^t \Phi_n dW - Bu_0 \right)', \psi g \right\rangle dt
\]

\[
= - \lim_{n \to \infty} \int_0^T \left\langle \left( Bu_n - B \int_0^t \Phi_n dW - Bu_0 \right)', \psi' g \right\rangle dt
\]

\[
= - \int_0^T \left\langle \psi'Bg, \left( u - \int_0^t \Phi dW - u_0 \right) \right\rangle dt
\]

\[
= - \int_0^T \left\langle B \left( u - \int_0^t \Phi dW - u_0 \right), \psi' g \right\rangle dt
\]

which shows that

\[ \zeta = \left( B \left( u - \int_0^t \Phi dW - u_0 \right) \right)' \]
in the sense of $V'$ valued distributions. Also from the above,
\[
\int_0^T \langle \zeta, \psi g \rangle \, dt = \langle Bu(0) - Bu_0, \psi(0) g \rangle + \int_0^T \langle (Bu - B\int_0^t \Phi dW - Bu_0), \psi' g \rangle \, dt
\]

\[
= \langle Bu(0) - Bu_0, \psi(0) g \rangle + \int_0^T \langle \zeta, \psi g \rangle \, dt
\]

Hence $B(u(0, \omega)) = Bu_0$. Thus this has shown that
\[
\left( B \left( u - \int_0^t \Phi dW - u_0 \right) \right)' + \xi(\cdot, \omega) = f(\cdot, \omega) \text{ in } V_\omega', \; Bu(0) = Bu_0.
\]

Thus integrating this, we get
\[
Bu(t, \omega) - Bu_0(\omega) + \int_0^t \xi(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + B \int_0^t \Phi dW \tag{74.4.41}
\]

**Lemma 74.4.3** The above sequence does not depend on $\omega \notin N$. In fact, it is not necessary to take a further subsequence.

**Proof:** In fact, it is not necessary to take a subsequence to get the convergences in (74.4.38) - (74.4.40). This is because of the pointwise convergence of $\xi$ and Lemma (74.4.1). If the original sequence did not converge, then there would be two subsequences converging weakly to two different functions in $V_\omega,\; v, w$ which is impossible because of (74.4.30) and this lemma since it would require $v(t) = w(t)$ a.e. \(\blacksquare\)

The question at this point is whether $u$ is progressively measurable. From the assumed estimates, the Itô formula, and (74.4.27), the same kind of estimates used earlier show that there exists an estimate of the form
\[
\|u_n\|_{V} \leq C
\]

Therefore, there exists a further subsequence such that
\[
u_n \rightarrow \bar{u} \text{ weakly in } V
\]

It follows from Lemma (74.3.4) that off an enlarged exceptional set of measure zero, still denoted as $N$,
\[
\bar{u}(\cdot, \omega) = u(\cdot, \omega) \text{ in } V_\omega
\]

Hence we can assume that $u$ is progressively measurable into $V$. It follows that $Bu$ is progressively measurable into $W'$.

Thus also
\[
(t, \omega) \rightarrow \int_0^t \xi(s, \omega) \, ds
\]
is progressively measurable into $V'$. Of course the next task is to identify $\xi$. This is always a problem even in the non stochastic case. Here it is especially difficult because in order to identify $\xi$ we need to use the implicit Ito formula which only holds if $\xi$ is sufficiently measurable. However, we have obtained $\xi$ as a weak limit for fixed $\omega$. Therefore, this is a significant issue. In stochastic evolution problems where $B = I$ this is not as difficult because one gets $\xi$ as a weak limit in $V$ and then $\xi$ is progressively measurable. We cannot do it in this way and still get the best results in which there is a solution to the integral equation which holds for all $t$ off a set of measure zero because of the degenerate nature of the operator $B$. However, $\xi$ is only an equivalence class of functions. We show in the next lemma that there exists a representative of this equivalence class for each $\omega$ off an exceptional set of measure zero such that the resulting $\xi$ is progressively measurable. This will enable us to use the implicit Ito formula and identify $\xi$.

The following lemma will allow the use of the Ito formula and eventually identify $\xi$.

**Lemma 74.4.4** Enlarging the exceptional set, one can assume that $\xi$ is also progressively measurable. In fact, if

$$\xi_n \equiv \int_{t-1/n}^{t} \xi ds$$

is known to be progressively measurable, $\xi(t, \omega) \equiv 0$ for $t < 0$, then there exists a set of measure zero $N$ such that for $\omega \notin N, \xi(t, \omega) = \tilde{\xi}(t, \omega)$ for all $t$ off a set of measure zero and $\tilde{\xi}$ is progressively measurable.

**Proof:** Define

$$\xi_n \equiv n \int_{t-1/n}^{t} \xi ds$$

where $\xi$ is defined to be zero for $t \leq 0$. Then by what was just shown, this is progressively measurable. Also, standard approximate identity arguments verify that for each $\omega, \xi_n \to \xi$ in $V_\omega$. Next note that the set where $\xi_n$ is not a Cauchy sequence is a progressively measurable set. It equals

$$\bigcup_{n} \bigcup_{m, k, l \geq m} \{ (t, \omega) : \|\xi_l(t, \omega) - \xi_k(t, \omega)\| > \frac{1}{n} \} = S$$

Now for $p > 0$

$$\lim_{m \to \infty} P \left( \sup_{p>0} \|\xi_{m+p} - \xi_m\|_{V_\omega} > \varepsilon \right) = 0$$

This is because of the convergence of $\xi_n$ to $\xi$ in $V_\omega$. Therefore, there is a subsequence still called $\xi_n$ such that

$$P \left( \sup_{p>0} \|\xi_{n+p} - \xi_n\|_{V_\omega} > 2^{-n} \right) < 2^{-n}$$
and so there is an enlarged set of measure zero, still denoted as $N$ such that all of the above considerations hold for $\omega \notin N$ and also for $\omega \notin N$,

$$\sup_{p>0} \|\xi_{n+p} - \xi_n\|_{V_\omega} \leq 2^{-n}$$

for all $n$ large enough. Now let $S$ defined above, correspond to this particular subsequence. Let $S(\omega)$ be those $t$ such that $(t, \omega) \in S$. Then $S(\omega)$ is a set of measure zero for each $\omega \notin N$ because the above inequality implies that $t \to \xi_n(t, \omega)$ is a Cauchy sequence off a set of measure zero which by definition is $S(\omega)$. Then consider $\{\xi_n(t, \omega) X_{\omega} (t, \omega)\}$. For each $\omega$ off $N$, this converges for all $t$. Thus it converges pointwise to a function $\xi$ which must be progressively measurable. However, $t \to \xi(t, \omega)$ must also equal $t \to \xi(t, \omega)$ in $V_\omega$ by the above construction. Therefore, we can assume without loss of generality that $\xi$ is itself progressively measurable.

From the weak convergence of $u_n$ to $u$ in $V_\omega$,

$$Bu_n \to Bu \text{ weakly in } V_\omega'$$

and so

$$\langle (\lambda B + A(\omega))u_n, u_n \rangle \to \lambda Bu + \xi \text{ weakly in } V_\omega'$$

Now the above convergences and the integral equation imply that off the exceptional set $N$, for each $t$

$$Bu_n(t) \to Bu(t) \text{ weakly in } V'$$

From a generalization of standard theorems in Hilbert space, stated in Lemma 74.2.1 there exist vectors $\{e_i\} \subseteq V$ such that

$$\langle Bu_n(t), u_n(t) \rangle = \sum_{i=1}^{\infty} |\langle Bu_n(t), e_i \rangle|^2$$

Hence

$$\lim_{n \to \infty} \inf \sum_{i=1}^{\infty} \lim_{n \to \infty} \inf |\langle Bu_n(t), e_i \rangle|^2$$

$$= \sum_{i=1}^{\infty} |\langle Bu(t), e_i \rangle|^2 = \langle Bu(t), u(t) \rangle$$

(74.4.42)

Thus the above inequalities and formulas hold for a.e. $t$.

Return to the equation 74.4.41. Define the stopping time

$$\tau_p \equiv \inf \left\{ t \in [0, T] : \langle Bu, u \rangle(t) + \int_0^t \|\xi\|_{V'_\omega} \, ds > p \right\}$$
From Chapter 74 and the fact that $\xi \in \mathcal{V}_\omega$, it follows that $\tau_p = \infty$ for all $p$ large enough. Then stop the equation using this stopping time.

$$
Bu^{\tau_p}(t,\omega) - Bu_0(\omega) + \int_0^t X_{[0,\tau_p]} \xi^{\tau_p}(s,\omega) \, ds
$$

From the implicit Ito formula Theorem 74.2.3, for a.e. $t$,

$$
\frac{1}{2} \langle Bu^{\tau_p}(t), u^{\tau_p}(t) \rangle - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle X_{[0,\tau_p]} \xi^{\tau_p}, u^{\tau_p} \rangle \, ds
$$

Then letting $p \to \infty$ this yields the following formula for a.e. $t$

$$
\frac{1}{2} \langle Bu(t), u(t) \rangle - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle \lambda Bu + \xi, u \rangle \, ds = \frac{1}{2} \int_0^t \langle B\Phi, \Phi \rangle \, ds
$$

Lemma 74.4.5 It is true that

$$
\lim_{n \to \infty} \int_0^T \langle B u_n, u_n \rangle \, dt = \int_0^T \langle B u, u \rangle \, dt
$$

Proof: From Chapter 74 $Bu_n \to z$ strongly in $C\left(\mathcal{N}_\omega^C, W'\right)$. But also, for each $t, Bu_n(t) \to Bu(t)$ weakly in $V'$ and so $z(t) = Bu(t)$. This strong convergence in $C\left(\mathcal{N}_\omega^C, W'\right)$ along with the uniform norm with the weak convergence of $u_n$ to $u$ in $\mathcal{V}_\omega$ is sufficient to obtain the above limit.

You might think that

$$
\int_0^T (\Phi_n \circ J^{-1})^* Bu_n \circ J dW \to \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW
$$

but this is not entirely clear. It will be true in the case that in Chapter 41, $\alpha = 2$ and $U = W$ and this is shown later. However, it is not clearly true here unless it is also the case that $\Phi \in L^2\left(\Omega, L^\infty\left([0,T], L_2\left(Q^{1/2}U, W\right)\right)\right)$.

Lemma 74.4.6 If $\Phi \in L^2\left(\Omega, L^\infty\left([0,T], L_2\left(Q^{1/2}U, W\right)\right)\right)$ then

$$
\int_0^T (\Phi_n \circ J^{-1})^* Bu_n \circ J dW \to \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW
$$
74.4. THE GENERAL CASE

Proof:

\[
E \left( \left| \int_0^T (\Phi_n \circ J^{-1})^* B_u_n \circ J dW - \int_0^T (\Phi \circ J^{-1})^* B_u \circ J dW \right| \right)
\]

\[
\leq E \left( \left| \int_0^T (\Phi_n \circ J^{-1})^* B_u_n \circ J dW - \int_0^T (\Phi \circ J^{-1})^* B_u_n \circ J dW \right| \right)
+ E \left( \left| \int_0^T (\Phi \circ J^{-1})^* B_u_n \circ J dW - \int_0^T (\Phi \circ J^{-1})^* B_u \circ J dW \right| \right)
\]

\[
\leq \int_{\Omega} \left( \left( \int_0^T \|\Phi_n - \Phi\|_{L^2}^2 \langle B_u_n, u_n \rangle \right)^{1/2} \right) dP
+ \int_{\Omega} \left( \int_0^T \|\Phi\|_{L^2}^2 \langle B_u - B_u, u_n - u \rangle \right)^{1/2} dP
\] (74.4.44)

Consider that second term. It is no larger than

\[
\int_{\Omega} \|\Phi\|_{L^\infty([0,T], L^2)} \left( \int_0^T \langle B_u_n - B_u, u_n - u \rangle dt \right)^{1/2} dP
\]

\[
\leq \left( \int_{\Omega} \|\Phi\|_{L^\infty([0,T], L^2)}^2 \right)^{1/2} \left( \int_{\Omega} \int_0^T \langle B_u_n - B_u, u_n - u \rangle dtdP \right)^{1/2}
\]

Now consider the following. Letting the \( e_i \) be the special vectors of Lemma 31.4.2, it follows,

\[
\int_{\Omega} \int_0^T \langle B_u_n - B_u, u_n - u \rangle dtdP = \int_{\Omega} \int_0^T \sum_{i=1}^{\infty} \langle B_u_n - B_u, e_i \rangle^2 dtdP
\]

\[
= \int_{\Omega} \int_0^T \sum_{i=1}^{\infty} \lim_{p \to \infty} \langle B_u_n - B_u, e_i \rangle^2 dtdP
\]

\[
\leq \lim_{p \to \infty} \int_{\Omega} \int_0^T \sum_{i=1}^{\infty} \langle B_u_n - B_u, e_i \rangle^2 dtdP
\]

\[
= \lim_{p \to \infty} \int_{\Omega} \int_0^T \langle B_u_n - B_u, u_n - u_p \rangle dtdP \leq T^{2n}
\]

The last inequality follows from (74.4.34). Therefore, the second term in (74.4.44) is no larger than \((C(T, \Phi)/2^n)^{1/2}\) which converges to 0 as \( n \to \infty \). Now consider the
first term in (74.4.44).

\[
\int_{\Omega} \left( \left( \int_0^T \| \Phi_n - \Phi \|_{L^2}^2 \langle Bu_n, u_n \rangle \right)^{1/2} \right) dP \\
\leq \int_{\Omega} \sup_{t \in [0,T]} \langle Bu_n, u_n \rangle^{1/2} (t) \left( \left( \int_0^T \| \Phi_n - \Phi \|_{L^2}^2 dt \right)^{1/2} \right) dP \\
\leq \left( \int_{\Omega} \sup_{t \in [0,T]} \langle Bu_n, u_n \rangle (t) \right)^{1/2} \left( \int_{\Omega} \int_0^T \| \Phi_n - \Phi \|_{L^2}^2 dt \right)^{1/2}
\]

From (74.4.32)

\[
= C \left( \int_{\Omega} \int_0^T \| \Phi_n - \Phi \|_{L^2}^2 dt \right)^{1/2}
\]

which converges to 0. \( \Box \)

Return now to the equation solved by \( u_n \) in (74.4.37). Apply the Ito formula to this one. This yields for a.e. \( t \),

\[
\frac{1}{2} \langle Bu_n (t), u_n (t) \rangle - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle A (\omega) u_n, u_n \rangle ds = \frac{1}{2} \int_0^t \langle Bu_n, u_n \rangle ds \\
+ \int_0^t f (u) ds + \int_0^t (\Phi_n \circ J^{-1})^* Bu_n \circ J dW \quad (74.4.45)
\]

Assume without loss of generality that \( T \) is not in the exceptional set. If not, consider all \( T' \) close to \( T \) such that \( T' \) is not in the exceptional set.

\[
\int_0^T \langle (\lambda B + A (\omega)) u_n, u_n \rangle ds \\
= \frac{1}{2} \langle Bu_0, u_0 \rangle - \frac{1}{2} \langle Bu_n (T), u_n (T) \rangle + \int_0^T \langle f, u_n \rangle ds \\
+ \int_0^T (\Phi_n \circ J^{-1})^* Bu_n \circ J dW + \frac{1}{2} \int_0^T \langle B \Phi_n, \Phi_n \rangle ds + \int_0^T \langle \lambda Bu_n, u_n \rangle ds
\]

Now it follows from (74.4.42) applied to \( t = T \) and the above lemma that

\[
\lim sup_{n \to \infty} \int_0^T \langle (\lambda B + A (\omega)) u_n, u_n \rangle ds \\
\leq \frac{1}{2} \langle Bu_0, u_0 \rangle - \frac{1}{2} \langle Bu (T), u (T) \rangle + \int_0^T \langle f, u \rangle ds \\
+ \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW + \frac{1}{2} \int_0^T \langle B \Phi, \Phi \rangle ds + \int_0^T \langle \lambda Bu, u \rangle ds
\]
and from (74.4.3), the expression on the right equals \( \int_0^T \langle \lambda Bu + \xi, u \rangle \, ds \). Hence

\[
\lim \sup_{n \to \infty} \int_0^T \langle (\lambda B + A(\omega)) u_n, u_n \rangle \, ds \leq \int_0^T \langle \lambda Bu + \xi, u \rangle \, ds
\]

Then since \( \lambda B + A(\omega) \) is monotone and hemicontinuous, it is type \( M \) and so this requires \( A(\omega) u = \xi \).

Hence we obtain

\[
Bu(t) - Bu_0(\omega) + \int_0^t A(\omega)(s) \, ds = \int_0^t f(s, \omega) \, ds + B \int_0^t \Phi \, dW
\]

This is a solution for a given \( \omega \notin \mathcal{N} \). Also, a stopping time argument like the above and the coercivity estimates for \( A \) along with the implicit Ito formula show that \( u \in \mathcal{V} \). This yields the existence part of the following existence and uniqueness theorem.

**Theorem 74.4.7** Suppose \( \mathcal{V} \equiv L^p ([0, T] \times \Omega, \mathcal{V}) \) where \( p \geq 2 \), with the \( \sigma \) algebra of progressively measurable sets and \( \mathcal{V}_\omega \equiv L^p ([0, T], \mathcal{V}) \).

\[ \Phi \in L^2 \left( [0, T] \times \Omega, L^2 \left( Q^{1/2} U, W \right) \right) \cap L^2 \left( \Omega, L^\infty \left( [0, T], L^2 \left( Q^{1/2} U, W \right) \right) \right), \]

\[ f \in \mathcal{V}' \equiv L^p' \left( [0, T] \times \Omega, \mathcal{V}' \right) \]

and both are progressively measurable. Suppose that

\[ \lambda B + A(\omega) : \mathcal{V}_\omega \to \mathcal{V}_\omega', \lambda B + A : \mathcal{V} \to \mathcal{V}' \]

are monotone hemicontinuous and bounded where

\[
A(\omega) u(t) \equiv A(t, u(t), \omega)
\]

and \( (t, u, \omega) \to A(t, u, \omega) \) is progressively measurable. Also suppose for \( p \geq 2 \), the coercivity, and the boundedness conditions

\[
\lambda \langle Bu, u \rangle + \langle A(t, u, \omega), u \rangle_{\mathcal{V}} \geq \delta \| u \|^p_{\mathcal{V}} - c(t, \omega) \tag{74.4.46}
\]

where \( c \in L^1 ([0, T] \times \Omega) \) for all \( \lambda \) large enough. Also,

\[
\| A(t, u, \omega) \|_{\mathcal{V}} \leq k \| u \|^{p-1}_V + c^{1/p'}(t, \omega) \tag{74.4.47}
\]

also suppose the monotonicity condition for all \( \lambda \) large enough.

\[
\langle (\lambda B + A(\omega)) (u) - (\lambda B + A(\omega)) (v), u - v \rangle \geq \delta \| u - v \|_U \tag{74.4.48}
\]

Then if \( u_0 \in L^2 (\Omega, W) \) with \( u_0 \mathcal{F}_0 \) measurable, there exists a unique solution \( u(\cdot, \omega) \in \mathcal{V}_\omega \) with \( u \in \mathcal{V} \) \((L^p ([0, T] \times \Omega, \mathcal{V}) \) and progressively measurable) such that for \( \omega \) off a set of measure zero,

\[
Bu(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u(s, \omega), \omega) \, ds = \int_0^t f(s, \omega) \, ds + B \int_0^t \Phi \, dW.
\]

It is also assumed that \( V \) is a reflexive separable real Banach space.
Proof: The uniqueness assertion follows easily from the monotonicity condition.

Now we remove the assumption that \( \Phi \in L^2(\Omega, L^\infty([0,T], L^2(Q^{1/2}U,W))) \). Everything is the same except for the need for a different argument to show that

\[
\int_0^T \left( \Phi_n \circ J^{-1} \right)^* B u_n \circ J dW \to \int_0^T \left( \Phi \circ J^{-1} \right)^* B u \circ J dW.
\]

In this case we assume

\[
(\lambda B + A(\omega))(u) - (\lambda B + A(\omega))(v), u - v \geq \delta \| u - v \|^2_W
\]

Then repeating the above argument with this change yields set of measure zero, still denoted as \( N \) such that for \( \omega \notin N \)

\[
\int_0^T \| u_n - u_{n+1} \|^2_W ds \leq 2^{-n}
\]

for all \( n \) large enough. Hence for such \( \omega \), \( u_n(\cdot, \omega) \) is Cauchy in \( L^2([0,T], W) \) and in fact \( u_n(t, \omega) \) is a Cauchy sequence in \( W \). Thus \( \{ u_n(\cdot, \omega) \} \) converges in \( L^2([0,T], W) \) to \( u(\cdot, \omega) \in L^2([0,T], W) \) and by the above considerations involving continuous dependence of \( V \) into \( W \), it follows that \( u(\cdot, \omega) \) will be the same as the \( u \) from the above convergences. Now this convergence implies that in addition, for a.e. \( t \),

\[
\lim_{n \to \infty} \langle Bu_n(t, \omega) - Bu(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle = 0 \quad \text{(74.4.50)}
\]

\[
\lim_{n \to \infty} \int_0^T \langle Bu_n(t, \omega) - Bu(t, \omega), u_n(t, \omega) - u(t, \omega) \rangle dt = 0
\]

What is known from the above is that for

\[
M_n(t) = \int_0^t \left( \Phi_n \circ J^{-1} \right)^* B u_n \circ J dW
\]

there is a continuous martingale \( M \in M^*_T \) such that

\[
\lim_{n \to \infty} E \left( \sup_{t \in [0,T]} |M_n(t) - M(t)| \right) = 0 \quad \text{(74.4.51)}
\]

Define a stopping time

\[
\tau_p \equiv \inf \left\{ t : \langle Bu, u \rangle(t) + \sup_n \langle Bu_n, u_n \rangle(t) > p \right\}
\]

This is a good enough stopping time because the function used to define it as a hitting time is lower semicontinuous.

**Lemma 74.4.8** \( \int_0^T (\Phi_n \circ J^{-1})^* B u_n \circ J dW \to \int_0^T (\Phi \circ J^{-1})^* B u \circ J dW \) in probability. Also there is a further subsequence and set of measure zero such that off this set,

\[
\lim_{n \to \infty} \left( \sup_{t \in [0,T]} \left| \int_0^t (\Phi \circ J^{-1})^* B u \circ J dW - \int_0^t (\Phi_n \circ J^{-1})^* B u_n \circ J dW \right| \right) = 0.
\]
In particular, what is needed here is valid,

\[ \lim_{n \to \infty} \int_0^T (\Phi_n \circ J^{-1})^* Bu_n \circ J dW = \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW \]

**Proof:** Let \( \varepsilon > 0 \). Then define

\[ A_n = \left\{ \omega : \left| \int_0^T (\Phi_n \circ J^{-1})^* Bu_n \circ J dW - \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW \right| > \varepsilon \right\} \]

Then

\[ A_n = \bigcup_{p=1}^\infty A_n \cap (\{ \tau_p = \infty \} \setminus [\tau_{p-1} < \infty]) \]

the sets in the union being disjoint. Then \( A \cap (\{ \tau_p = \infty \}) \equiv \emptyset \)

Then as before,

\[
E \left( \left| \int_0^T X_{[0,\tau_p]} (\Phi_n \circ J^{-1})^* Bu_n \circ J dW - \int_0^T X_{[0,\tau_p]} (\Phi \circ J^{-1})^* Bu \circ J dW \right| \right)
\]

\[ \leq \int_\Omega \left( \left( \int_0^T \| \Phi_n - \Phi \|^2_{L^2} X_{[0,\tau_p]} \langle Bu_n, u_n \rangle \right)^{1/2} \right) dP
+ \int_\Omega \left( \int_0^T \| \Phi \|^2_{L^2} \langle Bu_n - Bu, u_n - u \rangle \| \Phi \|^2_{L^2} \langle Bu_n - Bu, u_n - u \rangle \right)^{1/2} dP \] (74.4.52)

Consider the second term. It is no larger than

\[ \left( \int_\Omega \int_0^T X_{[0,\tau_p]} \| \Phi \|^2_{L^2} \langle Bu_n - Bu, u_n - u \rangle \right)^{1/2} dtdP \]

Now \( t \to \langle Bu_n, u_n \rangle \) is continuous and so \( X_{[0,\tau_p]} \langle Bu_n, u_n \rangle \) \( t \leq p \). If not, then you would have \( \langle Bu_n, u_n \rangle (t) > p \) for some \( t \leq \tau_p \) and so, by continuity, there would be \( s < t \leq \tau_p \) for which \( \langle Bu_n, u_n \rangle (s) > p \) contrary to the definition of \( \tau_p \). Then \( X_{[0,\tau_p]} \langle Bu_n - Bu, u_n - u \rangle \) is bounded a.e. and also converges to 0 for a.e. \( t \leq \tau_p \) as \( n \to \infty \). Therefore, off a set of measure zero, including the set where \( t \to \| \Phi \|^2_{L^2} \) is not in \( L^1 \), the double integral converges to 0 by the dominated convergence theorem.

As to the first integral in (74.4.52), it is dominated by

\[ \int_\Omega X_{[0,\tau_p]} \sup_{t \in [0,\tau_p]} \langle Bu_n, u_n \rangle^{1/2} (t) \left( \int_0^T \| \Phi_n - \Phi \|^2_{L^2} \right)^{1/2} dP \]
\[ \leq \left( \int_{\Omega} \left( \sup_{t \in [0,T]} \langle Bu_n, u_n \rangle \right) \, dP \right)^{1/2} \left( \int_{\Omega} \int_{0}^{T} \| \Phi_n - \Phi \|_{L^2}^2 \, dt \, dP \right)^{1/2} \]

From the estimate, for a constant \( C \) independent of \( n \).

Therefore,

\[ \lim_{n \to \infty} E \left( \int_{0}^{T} X_{[0,\tau_p]} \Phi_n \circ J^{-1} \ast Bu_n \circ JdW - \int_{0}^{T} X_{[0,\tau_p]} \Phi \circ J^{-1} \ast Bu \circ JdW \right) = 0 \]

\[ P (A_n \cap [\tau_p = \infty]) \leq \frac{1}{\varepsilon} E \left( \left| \int_{0}^{T} X_{[0,\tau_p]} \Phi_n \circ J^{-1} \ast Bu_n \circ JdW - \int_{0}^{T} X_{[0,\tau_p]} \Phi \circ J^{-1} \ast Bu \circ JdW \right| \right) = 0 \]

and so

\[ \lim_{n \to \infty} P (A_n \cap [\tau_p = \infty]) = 0 \]

Then

\[ P (A_n) = \sum_{p=1}^{\infty} P (A_n \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) \]

and so from the dominated convergence theorem,

\[ \lim_{n \to \infty} P (A_n) = \sum_{p=1}^{\infty} \lim_{n \to \infty} P (A_n \cap ([\tau_p = \infty] \setminus [\tau_{p-1} < \infty])) = \sum_{p} 0 = 0. \]

There was nothing special about \( T \). The same argument holds for all \( t \) and so \( M (t) \) mentioned above has been identified as \( \int_{0}^{t} (\Phi \circ J^{-1}) \ast Bu \circ JdW \). Then from

\[ \lim_{n \to \infty} E \left( \sup_{t \in [0,T]} \left| \int_{0}^{t} (\Phi \circ J^{-1}) \ast Bu \circ JdW - \int_{0}^{t} (\Phi_n \circ J^{-1}) \ast Bu_n \circ JdW \right| \right) = 0 \]

It follows from the usual Borel Cantelli argument that there is a set of measure zero and a further subsequence such that off this set, all the above convergences happen and also

\[ \int_{0}^{T} (\Phi_n \circ J^{-1}) \ast Bu_n \circ JdW \to \int_{0}^{T} (\Phi \circ J^{-1}) \ast Bu \circ JdW \]

uniformly on \([0, T]\). \( \blacksquare \)

The rest of the argument is identical. This yields the following theorem.
Theorem 74.4.9 Suppose $V \equiv L^p ([0,T] \times \Omega, V)$ where $p \geq 2$, with the $\sigma$ algebra of progressively measurable sets and $V_\omega = L^p ([0,T], V)$.

\[
\Phi \in L^2 \left( [0,T] \times \Omega, L_2 \left( Q^{1/2} U, W \right) \right),
\]

\[
f \in V' \equiv L^p \left( [0,T] \times \Omega, V' \right)
\]

and both are progressively measurable. Suppose that $\lambda B + A (\omega) : V_\omega \rightarrow V'_\omega$, $\lambda B + A : V \rightarrow V'$

are monotone hemicontinuous and bounded where

\[
A (\omega) u (t) \equiv A (t, u (t), \omega)
\]

and $(t, u, \omega) \rightarrow A (t, u, \omega)$ is progressively measurable. Also suppose for $p \geq 2$, the coercivity, and the boundedness conditions

\[
\lambda \langle Bu, u \rangle + \langle A (t, u, \omega), u \rangle \geq \delta \| u \|_V^p - c (t, \omega)
\]

(74.4.53)

where $c \in L^1 ([0,T] \times \Omega)$ for all $\lambda$ large enough. Also,

\[
\| A (t, u, \omega) \|_V, \leq k \| u \|_V^{p-1} + c^{1/p'} (t, \omega) \quad (74.4.54)
\]

also suppose the monotonicity condition for all $\lambda$ large enough.

\[
\langle (\lambda B + A (\omega)) (u) - (\lambda B + A (\omega)) (v), u - v \rangle \geq \delta \| u - v \|^2_W \quad (74.4.55)
\]

Then if $u_0 \in L^2 (\Omega, W)$ with $u_0 F_0$ measurable, there exists a unique solution $u (\cdot, \omega) \in V_\omega$ with $u \in V (L^p ([0,T] \times \Omega, V)$ and progressively measurable) such that for $\omega$ off a set of measure zero,

\[
Bu (t, \omega) - Bu_0 (\omega) + \int_0^t A (s, u (s, \omega), \omega) ds = \int_0^t f ds + B \int_0^t \Phi dW.
\]

It is also assumed that $V$ is a reflexive separable real Banach space.

74.5 Replacing $\Phi$ With $\sigma (u)$

It is not hard to include the case where $\Phi$ is replaced with a function $\sigma (u)$. We make the following assumptions. For each $r > 0$ there exists $\lambda$ large enough that

\[
\langle \lambda B (u) + A (u) - (\lambda B (\tilde{u}) + A (\tilde{u})), u - \tilde{u} \rangle \geq r \| u - \tilde{u} \|^2_W
\]

Note that in the case where $B = I$ and there is a conventional Gelfand triple, $V, H, V'$, this kind of condition is obvious if $\lambda I + A$ is monotone for some $\lambda$. Thus this is not an unreasonable assumption to make although it is stronger than some of the assumptions used above with the integral given by $\int_0^t \Phi dW.$
As to $\sigma$ we make the following assumptions. 

$$(t,u,\omega) \in [0,T] \times W \times \Omega \rightarrow \sigma(t,u,\omega)$$ is progressively measurable into $W$

$$\| \sigma(t,u,\omega) \|_W \leq C + C \|u\|_W$$

$$\| \sigma(t,u,\omega) - \sigma(t,\hat{u},\omega) \|_{L^2(Q^{1/2}U,W)} \leq K \|u - \hat{u}\|_W$$

That is, it has linear growth and is Lipschitz.

Let $\lambda$ correspond to $r$ where $r - \|B\| K^2 > 4$. Also let $T$ be such that

$$\hat{C} e^{\lambda T} K^2 < 3$$

where $\hat{C}$ is a constant used in the Burkholder Davis Gundy inequality. This is a restriction on the size of $K$. Thus we only give a solution if $K$ is small enough. Later, this will be removed in the most interesting case. This will give a local solution valid for a fixed $T > 0$ and then the global solution can be obtained by applying this result on the succession of intervals $[0,T], [T,2T], [3T,4T], \ldots$

From Theorem 74.4.9, if $w \in W$, there exists a unique solution $u$ to

$$Bu(t,\omega) - Bu_0(\omega) + \int_0^t A(s,u(s,\omega),\omega) \, ds = \int_0^t f ds + B \int_0^t \sigma(w) \, dW.$$ 

holding in the sense described there. Let $u_i$ result from $w_i$. Then from the implicit Ito formula and the above monotonicity estimate,

$$\frac{1}{2} \langle B(u_1 - u_2), u_1 - u_2 \rangle (t) + r \int_0^t \|u_1 - u_2\|^2_W \, ds$$

$$- \lambda \int_0^t \langle B(u_1 - u_2), u_1 - u_2 \rangle \, ds$$

$$- \int_0^t \langle B\sigma(u_1) - B\sigma(u_2), \sigma(u_1) - \sigma(u_2) \rangle_{L^2} \, ds \leq M^* (t)$$

where the right side is of the form $\sup_{s \in [0,t]} |M(s)|$ where $M(t)$ is a local martingale having quadratic variation dominated by

$$C \int_0^t \|\sigma(u_1) - \sigma(u_2)\|^2 \langle B(u_1 - u_2), u_1 - u_2 \rangle \, ds \quad (74.5.56)$$

Therefore, since $M^*$ is increasing in $t$, it follows from the Lipschitz condition on $\sigma$ that

$$\frac{1}{2} \langle B(u_1 - u_2), u_1 - u_2 \rangle (t) + r \int_0^t \|u_1 - u_2\|^2_W \, ds$$

$$- \lambda \int_0^t \langle B(u_1 - u_2), u_1 - u_2 \rangle \, ds - \|B\| K^2 \int_0^t \|u_1 - u_2\|^2_W \, ds \leq M^* (t)$$
Thus, from the assumption about $r$,

$$
\sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) + 4 \int_0^t \|u_1 - u_2\|^2_W ds \\
\leq \lambda \int_0^t \langle B(u_1 - u_2), u_1 - u_2 \rangle ds + 2M^*(t)
$$

Then applying Gronwall’s inequality,

$$
\sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) + 4 \int_0^t \|u_1 - u_2\|^2_W ds \leq 2e^{\lambda T}M^*(t)
$$

Now take expectations and use the Burkholder Davis Gundy inequality. The expectation of the right side is then dominated by

$$
2\hat{C}e^{\lambda T} \left( \int_0^t \|\sigma(w_1) - \sigma(w_2)\|_{L^2} \langle B(u_1 - u_2), u_1 - u_2 \rangle ds \right)^{1/2} dP
$$

$$
\leq \left[ \int_\Omega \sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle^{1/2} \right] \left[ 2\hat{C}e^{\lambda T} \left( \int_0^t K^2 \|w_1 - w_2\|^2_W dt \right)^{1/2} dP \right]
$$

$$
\leq E \left( \frac{1}{2} \sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) \right) + \hat{C}e^{\lambda T} E \left( \int_0^t K^2 \|w_1 - w_2\|^2_W dt \right)
$$

It follows that, after adjusting constants as needed, one gets an inequality of the following form.

$$
\frac{1}{2} E \left( \sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) \right) + 4 \int_\Omega \int_0^t \|u_1 - u_2\|^2_W ds dP \leq \hat{C}e^{\lambda T} E \left( \int_0^t K^2 \|w_1 - w_2\|^2_W dt \right)
$$

This holds for every $t \leq T$ and so, from the estimate on the size of $T$, it follows that

$$
\int_0^T \int_\Omega \|u_1 - u_2\|^2_W ds dP \leq \frac{3}{4} \int_0^T \int_\Omega \|w_1 - w_2\|^2_W dt dP
$$

Therefore, there is a unique fixed point to this mapping which takes $w \in \mathcal{W}$ to $u$ the solution to the integral equation. We denote it as $u$. Thus $u$ is progressively
measurable and for \( \omega \) off a set of measure zero, we have a solution to the integral equation

\[
Bu(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u(s, \omega), \omega) \, ds = \int_0^t f ds + B \int_0^t \sigma(u) \, dW, \quad t \in [0, T]
\]

Now the same argument can be repeated for the succession of intervals mentioned above. However, you need to be careful that at \( T \), you have \( Bu(T, \omega) = B(u(T, \omega)) \) for \( \omega \) off a set of measure zero. If this is not so, you locate \( T' \) close to \( T \) for which it is so as in Lemma 71.3.1 and use this \( T' \) instead, but these are mainly technical issues. This proves the following existence and uniqueness theorem.

**Theorem 74.5.1** Suppose \( f \in V' \) is progressively measurable and that \( (t, \omega) \rightarrow \sigma(t, \omega, u(t, \omega)) \) is progressively measurable whenever \( u \) is. Suppose that

\[
\lambda B + A(\omega) : V_\omega \rightarrow V_\omega', \quad \lambda B + A : V \rightarrow V'
\]

are monotone hemicontinuous and bounded where

\[
A(\omega) u(t) \equiv A(t, u(t), \omega)
\]

and \((t, u, \omega) \rightarrow A(t, u, \omega)\) is progressively measurable. Also suppose for \( p \geq 2 \), the coercivity, and the boundedness conditions

\[
\lambda \langle Bu, u \rangle_{V'} + \langle A(t, u, \omega), u \rangle_V \geq \delta \|u\|^p_V - c(t, \omega) \quad (74.5.57)
\]

for all \( \lambda \) large enough.

\[
\|A(t, u, \omega)\|_{V'} \leq k \|u\|_{V'}^{p-1} + c^{1/p'}(t, \omega) \quad (74.5.58)
\]

where \( c \in L^1([0, T] \times \Omega) \). Also suppose the monotonicity condition that for all \( r > 0 \) there exists \( \lambda \) such that

\[
\langle (\lambda B + A(\omega)) (u) - (\lambda B + A(\omega)) (v), u - v \rangle \geq r \|u - v\|^2_W
\]

Also suppose that

\[
(t, u, \omega) \in [0, T] \times W \times \Omega \rightarrow \sigma(t, u, \omega) \text{ is progressively measurable into } W
\]

\[
\|\sigma(t, u, \omega)\|_W \leq C + C \|u\|_W
\]

\[
\|\sigma(t, u, \omega) - \sigma(t, \hat{u}, \omega)\|_{C_2(Q_{1/2}; U, W)} \leq K \|u - \hat{u}\|_W
\]

Then if \( u_0 \in L^2(\Omega, W) \) with \( u_0 \mathcal{F}_0 \text{ measurable, there exists a unique solution } u(\cdot, \omega) \in V_\omega \text{ with } u \in V \text{ (}\mathcal{F}_0 \text{ measurable) such that for } \omega \text{ off a set of measure zero,}

\[
Bu(t, \omega) - Bu_0(\omega) + \int_0^t A(s, u(s, \omega), \omega) \, ds = \int_0^t f ds + B \int_0^t \sigma(u) \, dW.
\]
In case $B$ is the Riesz map, you do not have to make any assumption on the size of $K$. Thus

$$
(Bu, u) = \|u\|_W^2
$$

The case of most interest is the usual one where $V \subseteq W = W' \subseteq V'$, the case of a Gelfand triple in which $B$ is the identity. As to $\sigma$, the assumption is made that

$$
\|\sigma(t, \omega, u_1) - \sigma(t, \omega, u_2)\|_{L_2(Q^{1/2}U, W)} \leq K \|u_1 - u_2\|_W
$$

Of course it is also assumed that whenever $u$ has values in $W$ and is progressively measurable, $(t, \omega) \to \sigma(t, \omega, u(t, \omega))$ is also progressively measurable into $L_2\left(Q^{1/2}U, W\right)$.

Letting $w_i \in L^2\left([0, T] \times \Omega, W\right)$ each $w_i$ being progressively measurable, the above assumptions imply that there exists a solution $u_i$ to the integral equation

$$
Bu_i(t, \omega) - Bu_0(\omega) + \int_0^t A(u_i) \, ds = \int_0^t f(s, \omega) \, ds + B\int_0^t \sigma(w_i) \, dW
$$

here we write $\sigma(w_i)$ for short instead of $\sigma(t, \omega, w_i)$. First, consider

$$
w \in L^2\left([0, T] \times \Omega, W\right) \cap L^\infty\left([0, T], L^2(\Omega, W)\right)
$$

and let $u$ be the solution which results from placing $w$ in $\sigma$. Then from the estimates,

$$
\langle Bu, u \rangle (t) - \langle Bu_0, u_0 \rangle + \delta \int_0^t \|u\|_V^p \, ds = 2 \int_0^t \langle f, u \rangle \, ds + C(b_3, b_4, b_5)
$$

$$
+ \lambda \int_0^t \langle Bu, u \rangle \, ds + \int_0^t \langle B\sigma(w), \sigma(w) \rangle_{L_2} \, ds + 2M^*(t)
$$

$$
\leq 2 \int_0^t \langle f, u \rangle \, ds + C(b_3, b_4, b_5) + \lambda \int_0^t \langle Bu, u \rangle \, ds + \int_0^t \left(C + C\|w\|_W^2\right) \, ds + 2M^*(t)
$$

where $M^*(t) = \sup_{s \in [0, t]} |M(s)|$ and the quadratic variation of $M$ is no larger than

$$
\int_0^t \|\sigma(w)\|^2 \langle Bu, u \rangle \, ds
$$

Then using Gronwall’s inequality, one obtains an inequality of the form

$$
\sup_{s \in [0, T]} \langle Bu, u \rangle (s) \leq C + C \left(M^*(t) + \int_0^t \|w\|_W^2 \, ds\right)
$$
where \( C = C ( u_0, f, \delta, \lambda, b_3, b_4, b_5, T ) \) and is integrable. Then take expectation. By Burkholder Davis Gundy inequality and adjusting constants as needed,

\[
E \left( \sup_{s \in [0,T]} \langle Bu, u \rangle (s) \right) \
\leq C + C \int_0^T \| w \|_W^2 \, ds \, dP
\]

and so

\[
E ( \langle Bu, u \rangle (t) ) \leq E \left( \sup_{s \in [0,T]} \langle Bu, u \rangle (s) \right) \leq C + C \int_0^T \| w \|_W^2 \, ds \, dP
\]

which implies \( u \in L^\infty ([0,T], L^2(\Omega, W)) \) and is progressively measurable.

Using the monotonicity assumption, there is a suitable \( \lambda \) such that

\[
\frac{1}{2} \left( B (u_1 - u_2), u_1 - u_2 \right)(t) + \lambda \int_0^t \| u_1 - u_2 \|_W^2 \, ds
\]

\[
- \int_0^t \left( B\sigma (u_1) - B\sigma (u_2), \sigma (u_1) - \sigma (u_2) \right) ds \leq M^* (t)
\]

where the right side is of the form \( \sup_{s \in [0,t]} | M (s) | \) where \( M (t) \) is a local martingale having quadratic variation dominated by

\[
C \int_0^t \| \sigma (w_1) - \sigma (w_2) \|^2 \langle B (u_1 - u_2), u_1 - u_2 \rangle \, ds
\]

(74.5.59)

Then by assumption and using Gronwall’s inequality, there is a constant \( C = C ( \lambda, K, T ) \) such that

\[
\langle B (u_1 - u_2), u_1 - u_2 \rangle (t) \leq C M^* (t)
\]

Then also, since \( M^* \) is increasing,

\[
\sup_{s \in [0,t]} \langle B (u_1 - u_2), u_1 - u_2 \rangle (s) \leq C M^* (t)
\]
Taking expectations and from the Burkholder Davis Gundy inequality,

\[
E \left( \sup_{s \in [0, t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) \right) \leq C \int_{\Omega} \left( \int_0^t \|\sigma(w_1) - \sigma(w_2)\|^2 \langle B(u_1 - u_2), u_1 - u_2 \rangle \right)^{1/2} dP
\]

Then it follows after adjusting constants that there exists an inequality of the form

\[
E \left( \sup_{s \in [0, t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) \right) \leq C E \left( \int_0^t \|\sigma(w_1) - \sigma(w_2)\|^2 ds \right)
\]

Hence

\[
E \left( \sup_{s \in [0, t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(t) \right) \leq CK^2 E \left( \int_0^t \|w_1 - w_2\|^2_W ds \right)
\]

Thus, for each \( t \leq T \)

\[
\int_{\Omega} \langle B(u_1 - u_2), u_1 - u_2 \rangle(t) dP \leq CK^2 E \left( \int_0^t \|w_1 - w_2\|^2_W ds \right)
\]

one can consider the map \( \psi(w) \equiv u \) as described above. Then the above inequality implies

\[
E \langle (B(\psi^n w_1 - \psi^n w_2), \psi^n w_1 - \psi^n w_2)(t) \rangle \leq CK^2 E \left( \int_0^t \|\psi^{n-1} w_1 - \psi^{n-1} w_2\|^2_W dt_1 \right)
\]

\[
= CK^2 E \left( \int_0^t \langle B(\psi^{n-1} w_1 - \psi^{n-1} w_2, \psi^{n-1} w_1 - \psi^{n-1} w_2)(t_1) \rangle dt_1 \right)
\]

\[
\leq (CK^2)^n E \left( \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \langle B(w_1 - w_2), w_1 - w_2 \rangle(t_n) dt_n \cdots dt_2 dt_1 \right)
\]

\[
= (CK^2)^n \sup_t E \langle (B(w_1 - w_2), w_1 - w_2)(t) \rangle \frac{T^n}{n!} \leq \frac{1}{2} \|w_1 - w_2\|^2_{L^\infty([0,T], L^2(\Omega, W))}
\]
CHAPTER 74. IMPLICIT STOCHASTIC EQUATIONS

provided \( n \) is sufficiently large. It follows that
\[
\| \psi_n w_1 - \psi_n w_2 \|_{L^\infty([0,T],L^2(\Omega,W))}^2 \leq \frac{1}{2} \| w_1 - w_2 \|_{L^\infty([0,T],L^2(\Omega,W))}^2
\]
for all \( n \) sufficiently large. Hence, if one begins with \( w \in L^\infty([0,T],L^2(\Omega,W)) \), the sequence of iterates \( \{ \psi_n w \}_{n=1}^\infty \) must converge to some fixed point \( u \) in \( L^\infty([0,T],L^2(\Omega,W)) \). This \( u \) is automatically in \( L^2([0,T] \times \Omega,W) \) and is progressively measurable since each of the iterates is progressively measurable.

This proves the following theorem.

**Theorem 74.5.2** Suppose \( f \in V' \) is progressively measurable and that \( (t,\omega) \to \sigma(t,u,\omega) \) is progressively measurable whenever \( u \) is. Suppose that \( B : W \to W' \) is a Riesz map.

\[
\lambda B + A(\omega) : V_\omega \to V_\omega', \quad \lambda B + A : V \to V'
\]
are monotone hemicontinuous and bounded where

\[
A(\omega) u(t) \equiv A(t,u(t),\omega)
\]

and \( (t,u,\omega) \to A(t,u,\omega) \) is progressively measurable. Also suppose for \( p \geq 2 \), the coercivity, and the boundedness conditions
\[
\lambda (Bu,u) + (A(t,u,\omega),u)_V \geq \delta \| u \|_{V'}^p - c(t,\omega) \quad (74.5.60)
\]
for all \( \lambda \) large enough.

\[
\| A(t,u,\omega) \|_{V'} \leq k \| u \|_{V'}^{p-1} + c^{1/p'}(t,\omega) \quad (74.5.61)
\]
where \( c \in L^1([0,T] \times \Omega) \). Also suppose that

\[
\| \sigma(t,u,\omega) \|_W \leq C + C \| u \|_W
\]

\[
\| \sigma(t,u,\omega) - \sigma(t,\hat{u},\omega) \|_{L^2(\Omega^1;U,W)} \leq K \| u - \hat{u} \|_W
\]

Then if \( u_0 \in L^2(\Omega,W) \) with \( u_0 \mathcal{F}_0 \) measurable, there exists a unique solution \( u(\cdot,\omega) \in V_\omega \) with \( u \in V \) \((L^p([0,T] \times \Omega,V) and progressively measurable) such that for \( \omega \) off a set of measure zero,

\[
Bu(t,\omega) - Bu_0(\omega) + \int_0^t A(s,u(s,\omega),\omega) \, ds = \int_0^t f ds + B \int_0^t \sigma(u) \, dW.
\]

74.6 Examples

Here we give some examples. The first is a standard example, the porous media equation, which is discussed well in [105]. For stochastic versions of this example, see [98]. The generalization to stochastic equations does not require the theory developed here. We will show, however, that it can be considered in terms of the theory of this paper without much difficulty using an approach proposed in [22]. These examples involve operators which are not monotone, in the usual way but they can be transformed into equations which do fit the above theory.
Example 74.6.1 The stochastic porous media equation is

\[ u_t - \Delta (u |u|^{p-2}) = f, \ u(0) = u_0, \ u = 0 \ \text{on} \ \partial U \]

where here \(U\) is a bounded open set in \(\mathbb{R}^n, n \leq 3\) having Lipschitz boundary. One can consider a stochastic version of this as a solution to the following integral equation

\[ u(t) - u_0 + \int_0^t (-\Delta) (u |u|^{p-2}) \, ds = \int_0^t \Phi dW + \int_0^t f \, ds \quad (74.6.62) \]

where here

\[ \Phi \in L^2 \left([0, T] \times \Omega, L_2 \left(Q^{1/2} U, H\right) \right) \cap L^2 \left(\Omega, L_2 \left(Q^{1/2} U, H\right) \right), \]

\(H = L^2 (U)\) and the equation holds in the manner described above in \(H^{-1}(U)\).

Assume \(p \geq 2\) and \(f \in L^2 ((0, T) \times \Omega, H)\).

One can consider this as an implicit integral equation of the form

\[ (-\Delta)^{-1} u(t) - (-\Delta)^{-1} u_0 + \int_0^t u \, |u|^{p-2} \, ds = (-\Delta)^{-1} \int_0^t \Phi dW + (-\Delta)^{-1} \int_0^t f \, ds \]

where \(-\Delta\) is the Riesz map of \(H^1_0(U)\) to \(H^{-1}(U)\). Then we can also consider \((-\Delta)^{-1}\) as a map from \(L^2(U)\) to \(L^2(U)\) as follows.

\[ (-\Delta)^{-1} f = u \] where \(-\Delta u = f, u = 0 \) on \(\partial U\).

Thus we let \(W = L^2(U) = H\) and \(V = L^p(U)\). Let \(B \equiv (-\Delta)^{-1}\) on \(L^2(U)\) as just described. Let \(A(u) = u |u|^{p-2}\). It is obvious that the necessary coercivity condition holds. In addition, there is a strong monotonicity condition which holds. Therefore, if \(u_0 \in L^2 (\Omega, L^2(U))\) and \(F_0\) measurable, Theorem \(\ref{thm:6.6.1}\) applies and we can conclude that there exists a unique solution to the integral equation \(\ref{eq:6.6.63}\) in the sense described in this theorem. Here \(u \in L^p ([0, T] \times \Omega, L^p(U))\) and is progressively measurable, the integral equation holding for all \(t\) for \(\omega\) off a set of measure zero. Since \(A\) satisfies for some \(\delta > 0\) an inequality of the form

\[ \langle Au - Av, u - v \rangle \geq \delta \|u - v\|_{L^p(U)}^p \]

it follows easily from the above methods that the solution is also unique. In fact, this follows right away from Theorem \(\ref{thm:6.6.1}\) because \((-\Delta^{-1} u, u) = \|u\|_{H^{-1}}^2\).

Also note that from the integral equation,

\[ (-\Delta)^{-1} \left( u(t) - u_0 - \int_0^t \Phi dW \right) + \int_0^t u \, |u|^{p-2} \, ds = (-\Delta)^{-1} \int_0^t f \, ds \]

and so, since \((-\Delta\) is the Riesz map on \(H^1_0(U)\), the integral equation above shows that off a set of measure zero,

\[ \int_0^t u \, |u|^{p-2} \, ds = (-\Delta)^{-1} \left( \int_0^t f \, ds - \left( u(t) - u_0 - \int_0^t \Phi dW \right) \right) \in L^2 (0, T, H^1_0(U) \cap H^2(U)) \]
by elliptic regularity results. If it were not for that stochastic integral, one could assert that $|u|^{\frac{p}{2}} u \in L^2(0, T, H^1_0(U))$. This is shown in [22]. However, it appears that no such condition can be obtained here because of the nowhere differentiability of the stochastic integral, even if more is assumed on $u_0$ and $\Phi$.

Also in this reference is a treatment of the Stefan problem. The Stefan problem involves a partial differential equation

$$u_t - \sum_i \frac{\partial}{\partial x_i} \left( k(u) \frac{\partial u}{\partial x_i} \right) = f, \text{ on } U \times [0, T] \equiv Q$$

for $(x, t) \notin S$ where $u$ is the temperature and $k(u)$ has a jump at $\sigma$ and $S$ is given by $u(x, t) = \sigma$. It is assumed that $0 < k_1 \leq k(r) \leq k_2 < \infty$ for all $r \in \mathbb{R}$. For example, its graph could be of the form

$$\begin{array}{c}
\sigma \\

k(u)(+) \\

\sigma
\end{array}$$

On $S$ there is a jump condition which is assumed to hold. Namely

$$b n_t - (k(+)u, n) - k(+)u, n = 0$$

where the sum is taken over repeated indices and $b > 0$. $u(\cdot)$ is the “limit” as $(x', t') \rightarrow (x, t) \in S$ where $(x', t') \in S_+$, $u(\cdot)$ defined similarly. Also $n$ will denote the unit normal which goes from $S_+ \equiv \{(x, t) : u(x, t) > \sigma\}$ toward $S_- \equiv \{(x, t) : u(x, t) < \sigma\}$.

$$n = (n_t, n_{x_1}, \ldots, n_{x_n})$$

In addition, there is an initial condition and boundary conditions

$$u(x, 0) = u_0(x) \notin S, \quad u(x, t) = 0 \text{ on } \partial U.$$

The idea is to obtain a variational formulation of this thing. To do this, let $K(r) \equiv \int_0^r k(s) \, ds$. Thus in the case of the above picture, the graph of $K(r)$ would look like

$$\begin{array}{c}
K(u) \\

\sigma \\

u
\end{array}$$

Now let $\beta(t)$ be a function which satisfies

$$\beta'(t) = \frac{1}{k(K^{-1}(t))} \text{ for } t \neq \tau \equiv K(\sigma)$$

and it has a jump equal to $b$ at $\tau$.

$$\begin{array}{c}
\beta(v) \\

\tau \\

v
\end{array}$$
Let \( v = K(u) \). Then for \( u \neq \sigma \), equivalently \( v \neq \tau \),

\[
v_t = K'(u) u_t = k \left( K^{-1}(v) \right) u_t
\]

and so

\[
u_t = \frac{1}{k(K^{-1}(v))} v_t = \frac{d}{dt}(\beta(v))
\]

Also,

\[
v_{,i} = K'(u) u_{,i} = k(u) u_{,i}
\]

and so

\[
u_{,i} = k(u) v_{,i}
\]

Hence

\[
(k(u) u_{,i})_{,i} = \left( k(u) \frac{1}{k(u)} v_{,i} \right)_{,i} = \Delta v
\]

Thus, off the set \( S \),

\[
\beta(v)_t - \Delta v = f
\]

Now let \( \phi \in L^2(0,T,H^1_0(U)) \) with \( \phi(x,T) = 0 \). Then assume \( S \) is sufficiently smooth that things like divergence theorem apply. Also note that \( u = \sigma \) is the same as \( v = \tau \).

\[
\int_Q (\beta(v)_t - \Delta v) \phi = \int_{S^+} (\beta(v)_t - \Delta v) \phi + \int_{S^-} (\beta(v)_t - \Delta v) \phi
\]

\[
= \int_{S^+} (\beta(v) \phi)_t - \beta(v) \phi_t - (v_{,i} \phi)_{,i} + v_{,i} \phi_{,i}
\]

\[
+ \int_{S^-} (\beta(v) \phi)_t - \beta(v) \phi_t - (v_{,i} \phi)_{,i} + v_{,i} \phi_{,i}
\]

Now using the divergence theorem, and continuing these formal manipulations, the above reduces to

\[
\int_S \beta(v(+)) \phi n_t - (v_{,i}(+)) \phi n_i + \int_{S^+} -\beta(v) \phi_t + v_{,i} \phi_{,i} - \int_{U \cap S^+} \beta(v(x,0)) \phi(x,0)
\]

\[
+ \int_S -\beta(v(-)) \phi n_t + (v_{,i}(-)) \phi n_i + \int_{S^-} -\beta(v) \phi_t + v_{,i} \phi_{,i} - \int_{U \cap S^-} \beta(v(x,0)) \phi(x,0)
\]

Combining the two integrals over \( S \) yields

\[
\int_S (bn_t - (v_{,i}(+) - v_{,i}(-)) n_i) \phi = \int_S (bn_t - (k(u) u_{,i}(+) - k(u) u_{,i}(-)) n_i) \phi = 0
\]

by assumption. Therefore, including \( f \), we obtain

\[
\int_Q -\beta(v) \phi_t + v_{,i} \phi_{,i} - \int_U \beta(v(x,0)) \phi(x,0) = \int_Q f \phi
\]
which implies, using the initial condition
\[ \int_U \beta(v_0)\phi(x,0) + \int_Q ((\beta(v))' \phi + v_i \phi_i) - \int_U \beta(v(x,0)) \phi(x,0) = \int_Q f \phi \]

Regard \( \beta \) as a maximal monotone graph and let \( \alpha(t) \equiv \beta^{-1}(t) \). Thus \( \alpha \) is single valued. It just has a horizontal place corresponding to the jump in \( \beta \). Then let \( w = \beta(v) \) so that \( v = \alpha(w) \).

Then in terms of \( w \), the above equals
\[ \int_U w_0\phi(x,0) + \int_Q (w' \phi + \alpha(w)_i \phi_i) - \int_U w(x,0) \phi(x,0) = \int_Q f \phi \]
and so this simplifies to
\[ w' - \Delta (\alpha(w)) = f, \quad w(0) = w_0 \]
where \( \alpha \) maps onto \( \mathbb{R} \) and is monotone and satisfies
\[ (\alpha(r_1) - \alpha(r_2))(r_1 - r_2) \geq 0, \quad |\alpha(r)| \leq m |r|, \]
\[ |\alpha(r_1) - \alpha(r_2)| \leq m |r_1 - r_2|, \quad \alpha(r) r \geq \delta |r|^2 \]
for some \( \delta, m > 0 \). Then \( K^{-1}(\alpha(w)) = u \) where \( u \) is the original dependent variable. Obviously, the original function \( k \) could have had more than one jump and you would handle it the same way by defining \( \beta \) to be like \( K^{-1} \) except for having appropriate jumps at the values of \( K(u) \) corresponding to the jumps in \( k \). This explains the following example.

**Example 74.6.2** It can be shown that the Stefan problem can be reduced to the consideration of an equation of the form
\[ w_t - \Delta (\alpha(w)) = f, \quad w(0) = w_0 \]
where \( \alpha : L^2(0,T, L^2(U)) \to L^2(0,T, L^2(U)) \) is monotone hemicontinuous and coercive, \( \alpha \) being a single valued function. Here \( f \) is the same which occurred in the original partial differential equation
\[ u_t - \sum_i \frac{\partial}{\partial x_i} \left( k(u) \frac{\partial u}{\partial x_i} \right) = f \]

Thus a stochastic Stefan problem could be considered in the form
\[ (-\Delta^{-1}) w(t) - (-\Delta)^{-1} w_0 + \int_0^t \alpha(w) ds = (-\Delta^{-1}) \int_0^t f ds + (-\Delta^{-1}) \int_0^t \Phi dW \]
This example can be included in the above general theory because
\[
((-\Delta^{-1}) u - (-\Delta^{-1}) v, u - v) \geq \|u - v\|^2_{V'}, \quad V' \equiv H^{-1}, \quad V = H^1_0(U)
\]
This is seen as follows. \(V, L^2(U), V'\) is a Gelfand triple. Then \(-\Delta\) is the Riesz map \(R\) from \(H^1_0(U)\) to \(H^{-1}(U)\). Then you have
\[
(y, R^{-1}y)_{L^2(U)} = \langle RR^{-1}y, R^{-1}y \rangle_{V', V} = \|R^{-1}y\|^2_V = \|y\|^2_{V'}.
\]

Next we give a simple example which is a singular and degenerate equation. This is a model problem which illustrates how the theory can be used. This problem is mixed parabolic and stochastic and nonlinear elliptic. It is a singular equation because the coefficient \(b\) can be unbounded. The existence of a solution is easy to obtain from the above theory but it does not follow readily from other methods. If \(p = 2\) it is an abstract version of stochastic heat equation which could model a material in which the density becomes vanishingly small in some regions and very large in other regions.

**Example 74.6.3** Suppose \(U\) is a bounded open set in \(\mathbb{R}^3\) and \(b(x) \geq 0, b \in L^p(U), p \geq 4\) for simplicity. Consider the degenerate stochastic initial boundary value problem
\[
b(\cdot) u(t, \cdot) - b(\cdot) u_0(\cdot) - \int_0^t \nabla : \left(|\nabla u|^{p-2} \nabla u\right) dt = b \int_0^t \Phi dW
\]
\[
u = 0 \text{ on } \partial U
\]
where \(\Phi \in L^2([0, T] \times \Omega, L^2(Q^{1/2}U, W))\) for \(W = H^1_0(U)\).

To consider this equation and initial condition, it suffices to let \(W = H^1_0(U), V = W^{1,p}_0(U), A : V \rightarrow V', \langle Au, v \rangle = \int_U |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, B : W \rightarrow W', \langle Bu, v \rangle = \int_U b(x) u(x) v(x) dx\)

Then by the Sobolev embedding theorem, \(B\) is obviously self adjoint, bounded and nonnegative. This follows from a short computation:
\[
\left| \int_U b(x) u(x) v(x) dx \right| \leq \|v\|_{L^4(U)} \left( \int_U |b(x)|^{4/3} |u(x)|^{4/3} dx \right)^{3/4}
\]
\[
\leq \|v\|_{H^1_0(U)} \left( \left( \int_U |b(x)|^4 dx \right)^{1/3} \left( \int |u(x)|^{4/3} dx \right)^{2/3} \right)^{3/4}
\]
\[
= \|v\|_{H^1_0(U)} \|b\|_{L^4(U)} \|u\|_{L^2(U)} \leq \|b\|_{L^4} \|u\|_{H^1_0} \|v\|_{H^1_0}
\]
Also for some $\delta > 0$

$$\langle Au - Av, u - v \rangle \geq \delta \| u - v \|_V^p$$

The nonlinear operator is obviously monotone and hemicontinuous. As for $u_0$, it is only necessary to assume $u_0 \in L^2(\Omega, W)$ and $\mathcal{F}_0$ measurable. Then Theorem 74.4.7 gives the existence of a solution in the sense that for a.e. $\omega$ the integral equation holds for all $t$. Note that $b$ can be unbounded and may also vanish. Thus the equation can degenerate to the case of a non stochastic nonlinear elliptic equation.

The existence theorems can easily be extended to include the situation where $\Phi$ is replaced with a function of the unknown function $u$. This is done by splitting the time interval into small sub intervals of length $h$ and retarding the function in the stochastic integral, like a standard proof of the Peano existence theorem. Then the Ito formula is applied to obtain estimates and a limit is taken.

Other examples of the usefulness of this theory will result when one considers stochastic versions of systems of partial differential equations in which there is a nonlinear coupling between a parabolic equation and a nonlinear elliptic equation. These kinds of problems occur, for example as quasistatic damage problems in which the damage parameter satisfies a parabolic equation and the balance of momentum is a nonlinear elliptic equation and the two equations are coupled in a nonlinear way.

### 74.7 Other Examples, Inclusions

The above general result can also be used as a starting point for evolution inclusions or other situations where one does not have hemicontinuous operators. Assume here that

$$\Phi \in L^\infty ([0, T] \times \Omega, L^2(\frac{Q^{1/2}}{2}U, H)).$$

We will use the following simple observation. Let $\alpha > 2$. Let $\|\Phi\|_\infty$ denote the norm in $L^\infty ([0, T] \times \Omega, L^2(\frac{Q^{1/2}}{2}U, H))$. By the Burkholder Davis Gundy inequality,

$$\int_\Omega \left( \int_s^t \Phi dW \right)^\alpha dP \leq C \int_\Omega \left( \int_s^t \|\Phi\|^2 d\tau \right)^{\alpha/2} dP$$

By the Kolmogorov Čentsov theorem, this shows that $t \to \int_0^t \Phi dW$ is Holder continuous with exponent

$$\gamma < \frac{(\alpha/2) - 1}{\alpha} = \frac{1}{2} - \frac{1}{\alpha}$$
Since $\alpha > 2$ is arbitrary, this shows that for any $\gamma < 1/2$, the stochastic integral is Holder continuous with exponent $\gamma$. This is exactly the same kind of continuity possessed by the Wiener process. We state this as the following lemma.

**Lemma 74.7.1** Let $\Phi \in L^\infty ([0,T] \times \Omega, \mathcal{L}_2 \langle Q^{1/2}U,H \rangle)$ then for any $\gamma < 1/2$, the stochastic integral $\int_0^t \Phi dW$ is Holder continuous with exponent $\gamma$.

To begin with, we consider a stochastic inclusion. Suppose, in the context of Theorem 74.4.7, that $V$ is a closed subspace of $W^{\sigma,p}(U), \sigma > 1$ which contains $C^\infty_c(U)$ where $U$ is an open bounded set in $\mathbb{R}^n$, different than the Hilbert space $U$. (In case the matrix $A$ which follows equals 0, it suffices to take $\sigma \geq 1$.) Let

$$
\sum_{i,j} a_{i,j}(x) \xi_i \xi_j \geq 0, \quad a_{ij} = a_{ji}
$$

where the $a_{i,j} \in C^1(\bar{U})$. Denoting by $A$ the matrix whose $ij^{th}$ entry is $a_{ij}$, let

$$W \equiv \left\{ u \in L^2(U) : \left( u, \ A^{1/2} \nabla u \right) \in L^2(U)^{n+1} \right\}
$$

with a norm given by

$$
\| u \|_W \equiv \left( \int_U \left( uv + \sum_{i,j} a_{i,j}(x) \partial_i u \partial_j v \right) dx \right)^{1/2}
$$

$B : W \to W'$ be given by

$$
\langle Bu, v \rangle \equiv \int_U \left( uv + \sum_{i,j} a_{i,j}(x) \partial_i u \partial_j v \right) dx
$$

so that $B$ is the Riesz map for this space. The case where the $a_{ij}$ could vanish is allowed. Thus $B$ is a positive self adjoint operator and is therefore, included in the above discussion. In this example, it will be significant that $B$ is one to one and does not vanish.

This operator maps onto $L^2(U)$ because of basic considerations concerning maximal monotone operators. This is because

$$
\langle Du, v \rangle \equiv \int_U \sum_{i,j} a_{i,j}(x) \partial_i u \partial_j v dx
$$

can be obtained as a subgradient of a convex lower semicontinuous and proper functional defined on $L^2(U)$. Therefore, the operator is maximal monotone on $L^2(U)$ which means that $I + D$ is onto. The domain of $D$ consists of all $u \in L^2(U)$ such that

$$
Du = -\sum_{i,j} \partial_j (a_{ij}(x) \partial_i u) \in L^2(U)
$$
along with suitable boundary conditions determined by the choice of $V$. It follows that if $u + Du = Bu = f \in H = L^2(U)$, then

$$u - \sum_{i,j} \partial_j (a_{ij} \partial_i u) = f$$

Therefore,

$$\|u\|^2_{L^2(U)} + \int_U \sum_{i,j} a_{ij}(x) \partial_i u \partial_j u = \|u\|^2_W = (f, u) \leq \|f\|_{L^2(U)} \|u\|_{L^2(U)} \leq \|f\|_{L^2(U)} \|u\|_W$$

which shows that the map $B^{-1} : H = L^2(U) \to W$ is continuous.

Next suppose that $\Phi \in L^\infty([0, T] \times \Omega; L^2(Q^{1/2}U, H))$. Then by continuity of the mapping $B^{-1}$, it follows that $\Psi \equiv B^{-1} \Phi$ satisfies $\Psi \in L^\infty([0, T] \times \Omega; L^2(Q^{1/2}U, W))$. Thus $\Phi = B\Psi$. In addition to this, to simplify the presentation, assume in addition that

$$\langle A(t, u, \omega) - A(t, v, \omega), u - v \rangle \geq \delta^2 \|u - v\|^p_V,$$

Also assume the uniqueness condition of Lemma 74.3.16 is satisfied. Consider the following graph.

There is a monotone Lipschitz function $J_n$ which is approximating a function with the indicated jump. For a convex function $\phi$, we denote by $\partial \phi$ its subgradient. Thus for $y \in \partial \phi(x)$

$$(y, u) \leq \phi(x + u) - \phi(x).$$

Denote the Lipschitz function as $J_n$ and the maximal monotone graph which it is approximating as $J$. Thus $J$ denotes the ordered pairs $(x, y)$ which are of the form $(0, y)$ for $|y| \leq 1$ along with ordered pairs $(x, 1), x > 0$ and ordered pairs $(x, -1)$ for $x < 0$. The graph of $J$ is illustrated in the above picture and is a maximal monotone graph. Thus $J = \partial \phi$ where $\phi(r) = |r|$. As illustrated in the graph, $J_n$ is piecewise linear.

Let $\phi_n(r) \equiv \int_0^r J_n(s) \, ds$. It follows easily that $\phi_n(r) \to \phi(r)$ uniformly on $\mathbb{R}$. Also let $h \geq 0$ be progressively measurable and uniformly bounded by $M$ and let $u_0 \in L^2(\Omega, W), u_0 \mathcal{F}_0$ measurable. Then from the above theorems, there exists a unique solution to the integral equation

$$Bu_n(t) - Bu_0 + \int_0^t A(s, u_n, \omega) \, ds + \int_0^t h(s, \omega) J_n(u_n) \, ds = B \int_0^t \Psi dW,$$
the last term equaling \( \int_0^t \Phi \, dW \). The integral equation holds off a set of measure zero and is progressively measurable.

Then from the Ito formula, one obtains, using the monotonicity of \( J_n \) an estimate in which \( C \) does not depend on \( n \)

\[
\frac{1}{2} E \langle Bu_n (t), u_n (t) \rangle - \frac{1}{2} E \langle Bu_0, u_0 \rangle + E \int_0^t \langle Au_n, u_n \rangle_V \, ds \leq C
\]

In particular, this holds for \( n = 1 \). Therefore, adjusting the constant, it follows that

\[
\int_\Omega \langle Bu_1 (t), u_1 (t) \rangle + \int_0^T \| u_1 \|_V^p \, dt \leq C (\omega)
\]

(74.7.64)

From the integral equation, it follows that, enlarging \( N \) by including countably many sets of measure zero, for \( \omega \notin N \)

\[
Bu_n (t) - Bu_1 (t) + \int_0^t A (s, u_n, \omega) - A (s, u_1, \omega) \, ds
\]

\[
+ \int_0^t h (s, \omega) J_n (u_n) - h (s, \omega) J_1 (u_1) \, ds = 0
\]

Now it is certainly true that \( |J_n (u_n) - J_1 (u_n)| \leq 2 \). Thus

\[
\int_0^t \langle h (s, \omega) J_n (u_n) - h (s, \omega) J_1 (u_1), u_n - u_1 \rangle \, ds
\]

\[
= \int_0^t \langle h (s, \omega) J_n (u_n) - h (s, \omega) J_1 (u_n), u_n - u_1 \rangle \, ds
\]

\[
+ \int_0^t \langle h (s, \omega) (J_1 (u_n) - J_1 (u_1)), u_n - u_1 \rangle \, ds
\]

\[
\geq -2M \int_0^t |u_n - u_1| \, ds
\]

Therefore, from the Ito formula and for \( \omega \notin N \),

\[
\frac{1}{2} \langle Bu_n (t) - Bu_1 (t), u_n (t) - u_1 (t) \rangle + \delta^2 \int_0^t \| u_n - u_1 \|_V^p \, ds
\]

\[
\leq \int_0^t 2M |u_n - u_1| \, ds \leq \left( 2 + \frac{1}{2} \int_0^t |u_n - u_1|^2 \, ds \right) M
\]

\[
\leq \left( 2 + \frac{1}{2} \int_0^t \langle Bu_n - Bu_1, u_n - u_1 \rangle _V \, ds \right) M
\]
where $M$ is an upper bound to $h$. Then by Gronwall’s inequality
\[
\frac{1}{2} \langle Bu_n(t) - Bu_1(t), u_n(t) - u_1(t) \rangle \leq 2Me^{MT}
\]
Hence
\[
\frac{1}{2} \langle Bu_n(t) - Bu_1(t), u_n(t) - u_1(t) \rangle + \delta^2 \int_0^t \|u_n - u_1\|_V^p \, ds \leq 2M + TM2e^{TM}
\]
It follows from (74.7.64) that for all $\omega \notin N$ and adjusting the constant,
\[
\langle Bu_n(t), u_n(t) \rangle + \int_0^T \|u_n\|_V^p \, ds \leq C(\omega)
\]
for all $n$, where $C(\omega)$ depends only on $\omega$.

For $\omega \notin N$, the above estimate implies there exists a further subsequence, still called $n$ such that
\[
Bu_n \to Bu \text{ weak * in } L^\infty(0,T,W')
\]
\[
u_n \to u \text{ weak * in } L^\infty(0,T,H)
\]
\[
u_n \to u \text{ weakly in } V
\]
From the integral equation solved and the assumption that $A$ is bounded, it can also be assumed that
\[
B\left(\left.u_n - \int_0^{(\cdot)} \Psi dW\right\rangle\right)' \to B\left(\left.u - \int_0^{(\cdot)} \Psi dW\right\rangle\right)' \text{ weakly in } V'_\omega
\]
\[
Bu(0) = Bu_0
\]
\[
Au_n \to \xi \text{ weakly in } V'_\omega
\]
It is known that $u_n$ is bounded in $V$. Also it is known that $B\left(\left.u_n - \int_0^{(\cdot)} \Psi dW\right\rangle\right)'$ is bounded in $V'_\omega$. Therefore, $B\left(\left.u_n - \int_0^{(\cdot)} \Psi dW\right\rangle\right)$ satisfies a Holder condition into $V'$. Since $\Psi$ is in $L^\infty$, $\int_0^{(\cdot)} \Psi dW$ satisfies a Holder condition, and so $Bu_n$ satisfies a Holder condition into $V'$ while $Bu_n$ is bounded in $W'_\omega$. By compactness of the embedding of $V$ into $W$, it follows that $W'$ embeds compactly into $V'$. This is sufficient to conclude that $\{Bu_n\}$ is precompact in $W'_\omega$. The proof is similar to one given by Lions. See Theorem 76.5.6. Since $B$ is the Riesz map, this implies that $\{u_n\}$ is precompact in $W_\omega$ and hence in $H_\omega$.

Therefore, one can take a further subsequence and conclude that
\[
u_n \to u \text{ strongly in } H_\omega \equiv L^2([0,T],L^2(U))
\]
Therefore, a further subsequence, still denoted by $n$ satisfies
\[
u_n(t) \to u(t) \text{ in } L^2(U) \text{ for a.e. } t
\]
We can also assume that
\[ J_n(u_n) \to \zeta \text{ weak } \ast \text{ in } L^\infty(0,T,L^\infty(U)) \]

From the integral equation solved,
\[
\left\langle B\left(u_n - \int_0^{(t)} \Psi dW\right)\right\rangle', u_n - u \rangle_{V_\omega} + \langle A(t,u_n) + h(t,\omega) J_n(u_n), u_n - u \rangle = 0 \tag{74.7.67}
\]

We claim that
\[
\int_0^t \left\langle B\left(u_n - \int_0^{(s)} \Psi dW\right)\right\rangle' - \left( B\left(u - \int_0^{(s)} \Psi dW\right)\right)', u_n - u \rangle ds \geq 0 \tag{74.7.68}
\]

The difficulty is that \( \int_0^{(t)} \Psi dW \) is only in \( W \). To see that the conclusion is so, note that it is clear from a computation that
\[
\int_0^t \left\langle \frac{1-\tau(h)}{h} \left( Bu_n - B \int_0^{(t)} \Psi dW \right) - \frac{1-\tau(h)}{h} \left( Bu - B \int_0^{(t)} \Psi dW \right) , u_n - u \right\rangle ds \geq 0
\]

Claim: The above is indeed nonnegative.

Proof: Denote by \( q(t) \) the stochastic integral, \( u_n \) as \( u \) and \( u \) as \( v \) to save notation. Then the left side of the above equals
\[
\frac{1}{h} \int_0^t \langle B(u - q) - B(v - q) , u - v \rangle ds
\]

\[
- \frac{1}{h} \int_h^t \langle B(u(s-h) - q(s-h)) - B(v(s-h) - q(s-h)), u(s) - v(s) \rangle ds
\]

\[
\geq \frac{1}{h} \int_0^t \langle B(u - q) - B(v - q) , u - v \rangle ds
\]

\[
- \frac{1}{2h} \int_h^t \langle B(u(s-h) - q(s-h)) - B(v(s-h) - q(s-h)), (u(s-h) - v(s-h)) \rangle ds
\]

\[
- \frac{1}{2h} \int_h^t \langle B(u - q) - B(v - q) , u - v \rangle ds
\]

\[
\geq \frac{1}{h} \int_0^t \langle B(u - q) - B(v - q) , u - v \rangle ds
\]

\[
- \frac{1}{2h} \int_0^{t-h} \langle B(u(s) - q(s)) - B(v(s) - q(s)), (u(s) - v(s)) \rangle ds
\]

\[
- \frac{1}{2h} \int_h^t \langle B(u - q) - B(v - q) , u - v \rangle ds
\]
\[
\begin{align*}
&= \frac{1}{h} \int_{t-h}^{t} \langle B(u-q) - B(v-q), u - v \rangle \, ds \\
&\quad + \frac{1}{h} \int_{0}^{t-h} \langle B(u-q) - B(v-q), u - v \rangle \, ds \\
&\quad - \frac{1}{2h} \int_{0}^{t-h} \langle B(u-q) - B(v-q), (u-v) \rangle \, ds \\
&\quad - \frac{1}{2h} \int_{h}^{t} \langle B(u-q) - B(v-q), (u-v) \rangle \, ds \\
&= \frac{1}{h} \int_{t-h}^{t} \langle B(u-q) - B(v-q), u - v \rangle \, ds \\
&\quad + \frac{1}{h} \int_{0}^{t-h} \langle B(u-q) - B(v-q), (u-v) \rangle \, ds \\
&\quad - \frac{1}{2h} \int_{h}^{t} \langle B(u-q) - B(v-q), (u-v) \rangle \, ds \\
&\quad - \frac{1}{2h} \int_{t-h}^{h} \langle B(u-q) - B(v-q), (u-v) \rangle \, ds \\
\end{align*}
\]

which is nonnegative as can be seen by replacing \( u - v \) with \( (u-q) - (v-q) \) and using monotonicity of \( B \).

Now pass to a limit in \( T \text{ as } h \to 0 \) to get the desired inequality. Therefore, from (6.7.67),

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \langle A(t, u_n) + h(t, \omega) J_n (u_n), u_n - u \rangle \, dt \leq 0
\]

From the above strong convergence, the left side equals

\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \langle A(t, u_n), u_n - u \rangle \, dt \leq 0
\]
74.7. OTHER EXAMPLES, INCLUSIONS

It follows that for all \( v \in V_\omega \),

\[
\int_0^T \langle A(t,u), u-v \rangle \, dt \\
\leq \liminf_{n \to \infty} \int_0^T \langle A(t,u_n), u_n-v \rangle \, dt \\
= \limsup_{n \to \infty} \left[ \int_0^T \langle A(t,u_n), u_n-u \rangle \, dt + \int_0^T \langle A(t,u_n), u-v \rangle \, dt \right] \\
\leq \int_0^T \langle \xi, u-v \rangle \, dt
\]

Since \( v \) is arbitrary, \( A(\cdot, u) = \xi \in V'_\omega \). Passing to the limit in the integral equation yields

\[
Bu(t) - Bu_0 + \int_0^t A(s,u) \, ds + \int_0^t h(s,\omega) \zeta(s,\omega) \, ds = \int_0^t \Phi dW
\]

What is \( h(s,\omega) \zeta(s,\omega) \)?

\[
\int_0^T \langle h(s,\omega) J_n(u_n(s)), v(s) - u_n(s) \rangle \, ds \leq \int_0^T h(s,\omega) (\phi_n(v) - \phi_n(u_n)) \, ds
\]

Passing to the limit and using the strong convergence described above along with the uniform convergence of \( \phi_n \) to \( \phi \),

\[
\int_0^T (h(s,\omega) \zeta(s), v(s) - u(s))_H \, ds \leq \int_0^T (h(s,\omega) (\phi(v(s)) - \phi(u(s))) \, ds
\]

Hence,

\[
\int_0^T (h(s,\omega) \zeta(s), v(s) - u(s))_H \, ds \leq 0
\]

for any choice of \( v \in H_\omega \). It follows that for a.e. \( s, h(s,\omega) \zeta(s) \in \partial_v (h(s,\omega) \phi(u(s))) \).

This has shown that for each \( \omega \notin N \), there exists a solution to the integral equation

\[
Bu(t) - Bu_0 + \int_0^t A(s,u) \, ds + \int_0^t h(s,\omega) \zeta(s,\omega) \, ds = \int_0^t \Phi dW \quad (74.7.69)
\]

where for a.e. \( s, h(s,\omega) \zeta(s,\omega) \in \partial_v (h(s,\omega) \phi(u(s))) \). Suppose you have two such solutions \((u_1, \zeta_1)\) and \((u_2, \zeta_2)\). Then

\[
Bu_1(t) - Bu_2(t) + \int_0^t A(s,u_1) - A(s,u_2) \, ds + \int_0^t h(s,\omega) (\zeta_1(s,\omega) - \zeta_2(s,\omega)) \, ds = 0
\]
Then from monotonicity of the subgradient it follows that $u_1 = u_2$. Then the two integral equations yield that for a.e. $t$

\[
\left( B \left( u_1 - \int_0^t \Psi dW \right) \right)'(t) + A(s, u_1(t)) + h(t, \omega) \zeta_1(t, \omega) = 0
\]

\[
\left( B \left( u_2 - \int_0^t \Psi dW \right) \right)'(t) + A(s, u_2(t)) + h(t, \omega) \zeta_2(t, \omega) = 0
\]

Therefore, for a.e. $t$, $h(t, \omega) \zeta_1(t, \omega) = h(t, \omega) \zeta_2(t, \omega)$. Thus the solution to the integral equation for each $\omega$ off a set of measure zero is unique.

At this point it is not clear that $(t, \omega) \mapsto u(t, \omega)$ is progressively measurable. It follows that in all of the above, we could substitute the integral equation which would contradict uniqueness.

There would exist two subsequences converging weakly to two different solutions to $u$ as $\omega$ varies. This is because the above argument shows that if $u_n$ fails to converge weakly, then there would exist two subsequences converging weakly to two different solutions to the integral equation which would contradict uniqueness.

Then from monotonicity of the subgradient it follows that $u_n \to \bar{u}$ in $L^p([0, T] \times \Omega; V)$ where the measurable sets are just the product measurable sets $B([0, T]) \times \mathcal{F}_T$. By Lemma B.3.2 for $\omega$ off a set of measure zero, $u(\cdot, \omega) = \bar{u}(\cdot, \omega)$ in $\mathcal{V}_\omega$ where $\bar{u}$ is progressively measurable. It follows that in all of the above, we could substitute $\bar{u}$ for $u$ at least for $\omega$ off a single set of measure zero. Thus $u$ can be assumed progressively measurable. The above argument along with technical details related to exponential shift considerations proves the following theorem.

**Theorem 74.7.2** In the situation of Corollary [74.7.2] where $V$ is a closed subspace of $W^{\sigma,p}(U), \sigma > 1$ and $W$ is as described above for $U$ a bounded open set, $u_0 \in L^2(\Omega, W)$, $u_0 \mathcal{F}_0$ measurable. Suppose $\lambda I + A(t, u, \omega)$ satisfies

\[
\langle \lambda I + A(t, u, \omega) - (\lambda I + A(t, v, \omega)), u - v \rangle \geq \delta^2 \|u - v\|^p_V
\]

for all $\lambda$ large enough. Also assume $\Phi \in L^\infty([0, T] \times \Omega, \mathcal{L}_2(Q^{1/2}U, H))$ with $\Phi = B\Psi$ where

\[
\Psi \in L^\infty([0, T] \times \Omega, \mathcal{L}_2(Q^{1/2}U, W'))
\]

and progressively measurable. Then there exists a unique solution to the integral equation

\[
Bu(t) - Bu_0 + \int_0^t A(s, u) ds + \int_0^t h(s, \omega) \zeta(s, \omega) ds = \int_0^t \Phi dW \quad (74.7.70)
\]

where for a.e. $s$, $h(s, \omega) \zeta(s, \omega) \in \partial_u (h(s, \omega) \phi(u(s)))$ where $\phi(r) = |r|$. The symbol $\partial_u$ is the subgradient of $\phi(u)$. Written in terms of inclusions, there exists a
set of measure zero such that off this set,
\[
\left( B \left( u - \int_0^t \Phi dW \right) \right)' + A (t, u) \in \partial_u (h(t, \omega) \phi (u(s))) \text{ a.e. } t
\]
\[u(0) = u_0\]

Note that one can replace
\[
\Phi \in L^\infty \left( [0, T] \times \Omega, L^2 \left( Q^{1/2} U, H \right) \right)
\]
with \(\Phi \in L^2 \left( [0, T] \times \Omega, L^2 \left( Q^{1/2} U, H \right) \right)\) along with an assumption that \(t \to \Phi (t, \omega)\) is continuous. This can be done by defining a stopping time \(\tau_n \equiv \inf \{ t : \| \Phi (t) \| > n \}\)

Then from the above example, there exists a solution to the integral equation off a set of measure zero
\[
Bu_n (t) - Bu_0 + \int_0^t A (s, u_n) ds + \int_0^t h (s, \omega) \zeta_n (s, \omega) ds = \int_0^{t \wedge \tau_n} \Phi dW
\]

Since \(\Phi\) is a continuous process, \(\tau_n = \infty\) for all \(n\) large enough. Hence, one can replace the above with the desired integral equation. Of course the size of \(n\) depends on \(\omega\), but we can define
\[
u(t, \omega) = \lim_{n \to \infty} u_n (t, \omega)
\]

because by uniqueness which comes from monotonicity, if for a particular \(\omega\), both \(n, k\) are sufficiently large, then \(u_n = u_k\). Thus \(u\) is progressively measurable and is the desired solution.

Next we show that the above theory can also be used as a starting point for some second order in time problems. Consider a beam which has a point mass of mass \(m\) attached to one end. Suppose for sake of illustration that the left end is clamped, \(u(0, t) = u_x (0, t) = 0\), while the right end which has the attached mass is free to move, \(u_x (1, t) = 0\), and the beam occupies the interval \([0, 1]\) in material coordinates. Then the stress is \(\sigma = -u_{xxx}\) and balance of momentum is
\[
u_{tt} = \sigma_x + f
\]

where \(f\) is a body force. Thus, letting
\[
w \in V \equiv \{ w \in H^2 (0, 1) : w (0) = w_x (0) = 0, w_x (1) = 0 \}\]

be a test function,
\[
\int_0^1 u_{tt} w dx = \sigma w|_0^1 + \int_0^1 (-\sigma) w_x dx + \int_0^1 f w dx
\]
\[
= -mu_{tt} (1, t) w (1, t) + \int_0^1 u_{xxx} w_x dx + \int_0^1 f w dx
\]
Doing another integration by parts and using the boundary conditions, it follows that an appropriate variational formulation for this problem is

\[ \int_0^1 u_{tt} w dx + m \gamma_1 u_{tt} \gamma_1 w + \int_0^1 u_{xx} w_{xx} dx = \int_0^1 f w dx \]

where here \( \gamma_1 \) is the trace map on the right end.

Letting

\[ u(t) = u_0 + \int_0^t v(s) ds, \]

where \( u(0, t) = u_0 \), we can write the above variational equation in the form

\[ (Bu)' + Au = f, \quad Bu(0) = Bu_0 \]

where we assume that \( v_0 \in W \) where \( W \) is the closure of \( V \) in \( H^1(0, 1) \) and the operators are given by

\[ B : W \to W', \quad \langle Bu, w \rangle \equiv \int_0^1 u w dx + m \gamma_1 u \gamma_1 w \]

\[ A : V \to V', \quad \langle Au, w \rangle \equiv \int_0^1 u_{xx} w_{xx} dx \]

Thus in terms of an integral equation, this would be of the form

\[ Bv(t) - Bv_0 + \int_0^t A(u) ds = \int_0^t f ds \]

This suggests a stochastic version of the form

\[ Bv(t) - Bv_0 + \int_0^t A(u) ds = \int_0^t f ds + \int_0^t \Phi dW \]

where \( \Phi \in L^\infty((0, T) \times \Omega, L^2(Q^{1/2}U, H)) \) for \( H = L^2(0, 1) \). As in the previous example, simple considerations involving maximal monotone operators imply that there exists \( \Psi \in L^\infty((0, T) \times \Omega, L^2(Q^{1/2}U, W)) \) such that \( \Phi = B\Psi \). We also assume that \( f \in V' \) and \( u_0, v_0 \) are in \( L^2(\Omega, V) \) and \( L^2(\Omega, W) \) respectively, both being \( \mathcal{F}_0 \) measurable. The above equation does not fit the general theory developed earlier because it is second order in time and is a stochastic version of a hyperbolic equation rather than a parabolic one. We consider it using a parabolic regularization which can be studied with the above general theory along with a simple fixed point argument.

Consider the approximate problem which is to find a solution to

\[ Bv(t) - Bv_0 + \varepsilon \int_0^t A(v) ds + \int_0^t A(u) ds = \int_0^t f ds + \int_0^t \Phi dW \quad (74.7.71) \]

where \( u \) is given above as an integral of \( v \). First we argue that there exists a unique solution to the above integral equation and then we pass to a limit as \( \varepsilon \to 0 \). Let \( u \in V \) be given.
From Corollary 74.4.9, there exists a unique solution $v$ to (74.7.71). Now suppose $u_1, u_2$ are two given in $\mathcal{V}$ and denote by $v_i$ the corresponding $v$ which solves the above. Then from the Ito formula or standard considerations,

$$\frac{1}{2} E \langle B (v_1 (t) - v_2 (t)), v_1 (t) - v_2 (t) \rangle + \varepsilon E \int_0^t \| v_1 - v_2 \|^2_{\mathcal{V}} \, ds \leq \varepsilon E \int_0^t \| v_1 - v_2 \|^2_{\mathcal{V}} \, ds + C_\varepsilon E \int_0^t \| u_1 - u_2 \|^2_{\mathcal{V}} \, ds$$

Now define a mapping $\theta$ from $\mathcal{V}$ to $\mathcal{V}$ as follows. Begin with $v$ and obtain $u(t) \equiv u_0 + \int_0^t v(s) \, ds \quad (74.7.72)$

Use this $u$ in (74.7.71). Then $\theta v$ is the solution to (74.7.71) which corresponds to $u$. Then the above inequality shows that

$$\int_0^t \int_\Omega \| \theta v_1 (s) - \theta v_2 (s) \|^2 dPds \leq \frac{C_\varepsilon}{\varepsilon} \int_0^t \int_\Omega \| u_1 - u_2 \|^2_{\mathcal{V}} dPds$$

$$\leq \frac{C_\varepsilon}{\varepsilon} C_T \int_0^t \int_0^s \int_\Omega \| v_1 (r) - v_2 (r) \|^2_{\mathcal{V}} dPdrds$$

It follows that a high enough power of $\theta$ is a contraction map on $L^2 (0, T, L^2 (\Omega, \mathcal{V}))$ and so there exists a unique fixed point. This yields a unique solution to the above approximate problem in which $u, v$ are related by (74.7.72).

Next we let $\varepsilon \to 0$. Index the above solution with $\varepsilon$. By the Ito formula again,

$$\frac{1}{2} E \langle B v_\varepsilon (t), v_\varepsilon (t) \rangle - \frac{1}{2} E \langle B v_0, v_0 \rangle + \varepsilon \int_0^t E \| v_\varepsilon \|^2_{\mathcal{V}} \, ds$$

$$\frac{1}{2} E \| u_\varepsilon (t) \|^2_{\mathcal{V}} - \frac{1}{2} E \| u_0 \|^2_{\mathcal{V}} = \int_0^t E \langle f, v_\varepsilon \rangle \, ds$$

Then one can obtain an estimate and pass to a limit as $\varepsilon \to 0$ obtaining the following convergences.

$$\varepsilon v_\varepsilon \to v \quad \text{strongly in } \mathcal{V}$$

$$B v_\varepsilon \to B v \quad \text{weak * in } L^\infty (0, T, L^2 (\Omega, W'))$$

$$u_\varepsilon (t) \to u (t) \quad \text{weak * in } L^\infty (0, T, L^2 (\Omega, \mathcal{V}))$$

Then one can simply pass to a limit in the approximate integral equation and obtain, thanks to linearity considerations, that

$$B v (t) - B v_0 + \int_0^t A (u) \, ds = \int_0^t f \, ds + \int_0^t \Phi dW, \quad u (t) = u_0 + \int_0^t v (s) \, ds \quad (74.7.73)$$

the equation holding in $\mathcal{V}'$. Thus for a.e. $\omega$, the above holds for a.e. $t$. It is possible to work harder and have the equation holding for all $t$. This involves using the other form of the Ito formula, estimating for a fixed $\omega$ as done above and then arguing that by uniqueness one can use a single subsequence which works independent of $\omega$. 
**Example 74.7.3** Let $u_0 \in L^2(\Omega, V)$ where $V$ is described above and let $v_0 \in L^2(\Omega, W)$ for $W$ described above. Let both of these initial conditions be progressively measurable. Also let $f \in V'$ and $\Phi \in L^\infty((0, T) \times \Omega, L^2(Q^{1/2}U, H))$. Then there exists a unique solution to the integral equation which can be written in the form

$$Bu_t(t) - Bv_0 + \int_0^t A \left( u_0 + \int_0^t u_t(r) \, dr \right) \, ds = \int_0^t f \, ds + \int_0^t \Phi \, dW$$

Note that a more standard model involves no point mass at the tip of the beam. This would be done the same way but it would not require the generalized Ito formula presented earlier. A more standard version would work.

One can find many other examples where this generalized Ito formula is a useful tool to study various kinds of stochastic partial differential equations. We have presented five examples above in which it was helpful to have the extra generality.
Chapter 75

Stochastic Inclusions

75.1 The General Context

The situation is as follows. There are spaces $V \subseteq W$ where $V, W$ are reflexive separable Banach spaces. It is assumed that $V$ is dense in $W$. Define the space for $p > 1$

$$V \equiv L^p ([0,T]; V)$$

where in each case, the $\sigma$ algebra of measurable sets will be $B([0,T])$ the Borel measurable sets. Thus, from the Riesz representation theorem,

$$V' = L^{p'} ([0,T]; V'),$$

We also assume $(\Omega, \mathcal{F}, P)$ is a complete probability space. That is, if $P(E) = 0$ and $F \subseteq E$, then $F \in \mathcal{F}$. Also

$$V \subseteq W, \ W' \subseteq V'$$

$B(\omega)$ will be a linear operator, $B(\omega): W \to W'$ which satisfies

1. $\langle B(\omega) x, y \rangle = \langle B(\omega) y, x \rangle$
2. $\langle B(\omega) x, x \rangle \geq 0$ and equals 0 if and only if $x = 0$.
3. $\omega \to B(\omega)$ is a measurable $\mathcal{L}(W,W')$ valued function.

In the above formulae, $\langle \cdot, \cdot \rangle$ denotes the duality pairing of the Banach space $W$, with its dual space. We will use this notation in the present paper, the exact specification of which Banach space being determined by the context in which this notation occurs.

For example, you could simply take $W = H = H'$ and $B$ the identity and consider a standard Gelfand triple where $H$ is a Hilbert space and $B$ equal to the identity. An interesting feature is the requirement that $B(\omega)$ be one to one. It would be interesting to include the case of degenerate $B$, but $B$ one to one includes
the case of most interest just mentioned. Also a more general set of assumptions
will allow the inclusion of this case of degenerate \( B(\omega) \) also.

We assume always that the norm on the various reflexive Banach spaces is strictly
convex.

### 75.2 Some Fundamental Theorems

The following fundamental result will be very useful. It says essentially that if
\( (Bu)' \in L^p' (0,T;V') \) and \( u \in L^p (0,T;V) \) then the map \( u \to Bu(t) \) is continuous
as a map from

\[
X \equiv \left\{ u \in L^p ([0,T];V) : (Bu)' \in L^p' ([0,T];V') \right\}
\]

having norm equal to

\[
\|u\|_X \equiv \|u\|_{L^p(0,T,V)} + \|(Bu)\|_{L^{p'}(0,T,V')}
\]

to \( W' \). There is also a convenient integration by parts formula, Theorem 31.4.3. For
convenience, the dependence of \( B \) on \( \omega \) is often suppressed. This is not a problem
because the entire approach will be to consider the situation for fixed \( \omega \).

**Theorem 75.2.1** Let \( V \subseteq W, W' \subseteq V' \) be separable Banach spaces, and let \( Y \in L^p' (0,T;V') \)
and

\[
Bu(t) = Bu_0 + \int_0^t Y(s) \, ds \in V', \quad u_0 \in W, Bu(t) = B(u(t)) \text{ for a.e. } t
\]

(75.2.1)

Thus \( Y = (Bu)' \) as a weak derivative in the sense of \( V' \) valued distributions. It
is known that \( u \in L^p (0,T,V) \) for \( p > 1 \). Then \( t \to Bu(t) \) is continuous into \( W' \)
for \( t \) off a set of measure zero \( N \) and also there exists a continuous function \( t \to \langle Bu, u \rangle (t) \) such
that for all \( t \notin N, \langle Bu, u \rangle (t) = \langle B(u(t)), u(t) \rangle, Bu(t) = B(u(t)), \) and
for all \( t, \)

\[
\frac{1}{2} \langle Bu, u \rangle (t) = \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle Y(s), u(s) \rangle ds
\]

Note that the formula shows that \( Bu_0 = Bu(0) \). Also it shows that \( t \to \langle Bu, u \rangle (t) \) is continuous. To emphasize this a little more, \( Bu \) is the name of a function. \( Bu(t) = B(u(t)) \) for a.e. \( t \) and \( t \to Bu(t) \) is continuous into \( V' \) on \([0,T]\) because of the integral equation.

**Theorem 75.2.2** In the above corollary, the map \( u \to Bu(t) \) is continuous as
a map from \( X \) to \( V' \). Also if \( Y \) denotes those \( f \in L^p ([0,T];V) \) for which \( f' \in L^p ([0,T];V) \), so that \( f \) has a representative such that \( f(t) = f(0) + \int_0^t f'(s) \, ds, \) then if \( ||f||_Y \equiv ||f||_{L^p([0,T];V)} + ||f'||_{L^p([0,T];V)} \) the map \( f \to f(t) \) is continuous.
Proof: First, why is \( u \rightarrow Bu (0) \) continuous? Say \( u, v \in X \) and say \( p \geq 2 \) first.

\[
Bu (t) - Bv (t) = Bu (0) - Bv (0) + \int_0^t (Bu)' (s) - (Bv)' (s) \, ds
\]

and so,

\[
\left( \int_0^T \| Bu (0) - Bv (0) \|_{V'}^p \, dt \right)^{1/p'} \leq \left( \int_0^T \| Bu (t) - Bv (t) \|_{V'}^p \, dt \right)^{1/p'}
\]

\[
+ \left( \int_0^T \left\| \int_0^t (Bu)' (s) - (Bv)' (s) \, ds \right\|_{V'}^p \, dt \right)^{1/p'}
\]

and so

\[
\| Bu (0) - Bv (0) \|_{V'} T^{1/p'} \leq \left( \| B \| \| u - v \|_{L_{p'} ([0, T]; V')} + T^{1/p'} \| (Bu)' - (Bv)' \|_{L_{p'} ([0, T]; V')} \right)
\]

\[
\leq C (\| B \|, T) \| u - v \|_X
\]

Thus \( u \rightarrow Bu (0) \) is continuous into \( V' \). If \( p < 2 \), then you do something similar.

\[
\left( \int_0^T \| Bu (0) - Bv (0) \|_{V'}^p \, dt \right)^{1/p} \leq \left( \int_0^T \| Bu (t) - Bv (t) \|_{V'}^p \, dt \right)^{1/p}
\]

\[
+ \left( \int_0^T \left\| \int_0^t (Bu)' (s) - (Bv)' (s) \, ds \right\|_{V'}^p \, dt \right)^{1/p}
\]

\[
\| Bu (0) - Bv (0) \|_{V'} T^{1/p} \leq \| B \| \| u - v \|_{L_p} + C (T) \| (Bu)' - (Bv)' \|_{L_{p'} ([0, T]; V')}
\]

\[
\leq C (\| B \|, T) \| u - v \|_X .
\]

However, one could just as easily have done this for an arbitrary \( s < T \) by repeating the argument for

\[
Bu (t) = Bu (s) + \int_s^t (Bu)' (r) \, dr
\]

Thus this mapping is certainly continuous into \( V' \). The last assertion is similar. ■

Also of use will be the following generalization of the Ascoli Arzela theorem.

[\text{[10], Theorem 57.5.3}]

\textbf{Theorem 75.2.3} Let \( q > 1 \) and let \( E \subseteq W \subseteq X \) where the injection map is continuous from \( W \) to \( X \) and compact from \( E \) to \( W \). Let \( S \) be defined by

\[
\left\{ u \text{ such that } \| u (t) \|_E \leq R \text{ for all } t \in [a, b], \text{ and } \| u (s) - u (t) \|_X \leq R |t - s|^{1/q} \right\}.
\]
Thus $S$ is bounded in $L^\infty (0,T,E)$ and in addition, the functions are uniformly H"older continuous into $X$. Then $S \subseteq C ([a,b];W)$ and if $\{u_n\} \subseteq S$, there exists a subsequence, $\{u_{nk}\}$ which converges to a function $u \in C ([a,b];W)$ in the following way.

$$\lim_{k \to \infty} ||u_{nk} - u||_{\infty,W} = 0.$$  

Next is a major measurable selection theorem which forms an essential part of showing the existence of measurable solutions. See Theorem 75.2.4. The following is not dependent on there being a measure but in the applications there is typically a probability measure and often a set of measure zero which occurs in a natural way so an exceptional set of measure zero is included in the statement of the theorem but it has absolutely nothing to do with a set of measure zero as will be seen by just letting the exceptional set be $\emptyset$.

**Theorem 75.2.4** Let $V$ be a reflexive separable Banach space with dual $V'$, and let $p,p'$ be such that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let the functions $t \to u_n (t,\omega)$, for $n \in \mathbb{N}$, be in $L^{p'} ([0,T];V')$ and $(t,\omega) \to u_n (t,\omega)$ be $\mathcal{B} ([0,T]) \times \mathcal{F} \equiv \mathcal{P}$ measurable into $V'$. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \not\in N$, then

$$\sup_{t \in [0,T]} \|u_n (t,\omega)\|_{V'} \leq C (\omega),$$

for all $n$. Also, suppose for each $\omega \not\in N$, each subsequence of $\{u_n\}$ has a further subsequence that converges weakly in $L^{p'} ([0,T];V')$ to $v (\cdot,\omega) \in L^{p'} ([0,T];V')$ such that the function $t \to v (t,\omega)$ is weakly continuous into $V'$.

Then, there exists a product measurable function $u$ such that $t \to u (t,\omega)$ is weakly continuous into $V'$ and for each $\omega \not\in N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)} (\cdot,\omega) \to u (\cdot,\omega)$ weakly in $L^{p'} ([0,T];V')$.

We prove the theorem in steps given below. Let $X = \prod_{k=1}^{\infty} C ([0,T])$ and note that when it is equipped with the product topology, then one can consider $X$ as a metric space using the metric

$$d (f,g) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|f_k - g_k\|}{1 + \|f_k - g_k\|},$$

where $f = (f_1, f_2, \ldots), g = (g_1, g_2, \ldots) \in X$, and the norm is the maximum norm in $C ([0,T])$. With this metric, $X$ is complete and separable.

**Lemma 75.2.5** Let $\{f_n\}$ be a sequence in $X$ and suppose that each one of the components $f_{nk}$ is bounded by $C = C (k)$ in $C^{0,1} ([0,T])$. Then, there exists a subsequence $\{f_{n_j}\}$ that converges to some $f \in X$ as $n_j \to \infty$. Thus, $\{f_n\}$ is pre-compact in $X$.

**Proof:** By the Ascoli–Arzelà theorem, there exists a subsequence $\{f_{n_1}\}$ such that the sequence of the first components $f_{n_1 k}$ converges in $C ([0,T])$. Then, taking
75.2. SOME FUNDAMENTAL THEOREMS

a subsequence, one can obtain \( \{n_2\} \) a subsequence of \( \{n_1\} \) such that both the first and second components of \( f_{n_2} \) converge. Continuing in this way one obtains a sequence of subsequences, each a subsequence of the previous one such that \( f_{n_j} \) has the first \( j \) components converging to functions in \( C([0,T]) \). Therefore, the diagonal subsequence has the property that it has every component converging to a function in \( C([0,T]) \). The resulting function is \( f \in \prod_k C([0,T]). \)

Now, for \( m \in \mathbb{N} \) and \( \phi \in V \), define \( l_m(t) \equiv \max(0, t - (1/m)) \) and \( \psi_{m,\phi} : L^p([0,T]; V') \to C([0,T]) \) by

\[
\psi_{m,\phi} u(t) = \int_0^T \langle m\phi \mathcal{X}_{l_m(t),t} (s), u(s) \rangle_V \, ds = m \int_{l_m(t)}^t \langle \phi, u(s) \rangle_V \, ds.
\]

Here, \( \mathcal{X}_{l_m(t),t} (\cdot) \) is the indicator function of the interval \( [l_m(t), t] \) and \( \langle \cdot, \cdot \rangle_V = \langle \cdot, \cdot \rangle_V \) is the duality pairing between \( V \) and \( V' \).

Let \( D = \{ \phi_r \}_{r=1}^\infty \) denote a countable dense subset of \( V \). Then the pairs \( (\phi, m) \) for \( \phi \in D \) and \( m \in \mathbb{N} \) form a countable set. Let \( (m_k, \phi_{m_k}) \) denote an enumeration of the pairs \( (m, \phi) \in \mathbb{N} \times D \). To simplify the notation, we set

\[
f_k(u)(t) = \psi_{m_k,\phi_{m_k}} (u)(t) = m_k \int_{l_{m_k}(t)}^t \langle \phi_{m_k}, u(s) \rangle_V \, ds.
\]

For fixed \( \omega \notin N \) and \( k \), the functions \( \{ t \mapsto f_k(u_j(\cdot, \omega))(t) \}_{j} \) are uniformly bounded and equicontinuous because they are in \( C^{0,1}([0,T]) \). Indeed, we have for \( \omega \notin N \),

\[
|f_k(u_j(\cdot, \omega))(t)| = \left| m_k \int_{l_{m_k}(t)}^t \langle \phi_{m_k}, u_j(s, \omega) \rangle_V \, ds \right| \leq C(\omega) \| \phi_{m_k} \|_V
\]

and for \( t \leq t' \)

\[
|f_k(u_j(\cdot, \omega))(t) - f_k(u_j(\cdot, \omega))(t')| \\
\leq |m_k \int_{l_{m_k}(t)}^t \langle \phi_{m_k}, u_j(s, \omega) \rangle_V \, ds - m_k \int_{l_{m_k}(t')}^t \langle \phi_{m_k}, u_j(s, \omega) \rangle_V \, ds| \\
\leq 2m_k |t' - t| C(\omega) \| \phi_{m_k} \|_V,
\]

By Lemma [lemma:compactness](#lemma:compactness), the set of functions \( \{ \mathcal{X}_{NC}(\omega) f(u_j(\cdot, \omega)) \}_{j=1}^\infty \) is pre-compact in \( X = \prod_k C([0,T]) \). We now define a set valued map \( \Gamma^n : \Omega \to X \) by

\[
\Gamma^n(\omega) = \bigcup_{j \geq n} \{ \mathcal{X}_{NC}(\omega) f(u_j(\cdot, \omega)) \},
\]

where the closure is taken in \( X \). Then \( \Gamma^n(\omega) \) is the closure of a pre-compact set in \( X \) and so \( \Gamma^n(\omega) \) is compact in \( X \). From the definition, a function \( f \) is in \( \Gamma^n(\omega) \) if and only if \( d(f, \mathcal{X}_{NC}(\omega) f(w_l)) \to 0 \) as \( l \to \infty \), where each \( w_l \) is one of the \( u_j(\cdot, \omega) \) for \( j \geq n \). In the topology on \( X \), this happens iff for every \( k \),

\[
f_k(t) = \lim_{l \to \infty} m_k \int_{l_{m_k}(t)}^t \langle \phi_{m_k}, \mathcal{X}_{NC}(\omega) w_l(s, \omega) \rangle_V \, ds,
\]
where the limit is the uniform limit in \( t \).

**Lemma 75.2.6** The mapping \( \omega \rightarrow \Gamma^n(\omega) \) is an \( \mathcal{F} \) measurable set-valued map with values in \( X \). If \( \sigma \) is a measurable selection, then for each \( t, \omega \rightarrow \sigma(t,\omega) \) is \( \mathcal{F} \) measurable and \( (t,\omega) \rightarrow \sigma(t,\omega) \) is \( B([0,T]) \times \mathcal{F} \) measurable.

We note that if \( \sigma \) is a measurable selection then \( \sigma(\omega) \in \Gamma^n(\omega) \), so \( \sigma = \sigma(\cdot,\omega) \) is a continuous function. To have \( \sigma \) measurable would mean that \( \sigma^{-1}(\text{open}) \in \mathcal{F} \), where the open set is in \( C([0,T]) \).

**Proof:** Let \( O \) be a basic open set in \( X \). Then \( O = \prod_{k=1}^{\infty} O_k \), where \( O_k \) is a proper open set of \( C([0,T]) \) only for \( k \in \{k_1, \ldots, k_r\} \). Thus there is a proper open set in these positions and in every other position the open set is the whole space \( C([0,T]) \).

We need to show that \( \Gamma^n(O) = \{\omega : \Gamma^n(\omega) \cap O \neq \emptyset\} \in \mathcal{F} \).

Now, \( \Gamma^n(O) = \bigcap_{k=1}^{\infty} \{\omega : \Gamma^n(\omega)_{k_i} \cap O_{k_i} \neq \emptyset\} \), so we consider whether

\[
\{\omega : \Gamma^n(\omega)_{k_i} \cap O_{k_i} \neq \emptyset\} \in \mathcal{F}. \tag{75.2.2}
\]

From the definition of \( \Gamma^n(\omega) \), this is equivalent to the condition that

\[
f_{k_i} (X_{NC}(\omega) u_j (\cdot, \omega)) = (f (X_{NC}(\omega) u_j (\cdot, \omega)))_{k_i} \in O_{k_i},
\]

for some \( j \geq n \), and so the set in (75.2.2) is of the form

\[
\bigcup_{j=n}^{\infty} \{\omega : (f (X_{NC}(\omega) u_j (\cdot, \omega)))_{k_i} \in O_{k_i}\}.
\]

Now \( \omega \rightarrow (f (X_{NC}(\omega) u_j (\cdot, \omega)))_{k_i} \) is \( \mathcal{F} \) measurable into \( C([0,T]) \) and so the above set is in \( \mathcal{F} \). To see this, let \( g \in \tilde{C}([0,T]) \) and consider the inverse image of the ball with radius \( r \) and center \( g \),

\[
B(g, r) = \{\omega : \| (X_{NC}(\omega) f (u_j (\cdot, \omega)))_{k_i} - g \|_{C([0,T])} < r\}.
\]

By continuity considerations,

\[
\| (X_{NC}(\omega) f (u_j (\cdot, \omega)))_{k_i} - g \|_{C([0,T])} = \sup_{t \in Q \setminus [0,T]} \left| (X_{NC}(\omega) f (u_j (t, \omega)))_{k_i} - g(t) \right|,
\]

which is the sup over countably many \( \mathcal{F} \) measurable functions. Thus, it is \( \mathcal{F} \) measurable. Since every open set is the countable union of such balls, it follows that the claim about \( \mathcal{F} \) measurability is valid. Hence, \( \Gamma^n(O) \) is \( \mathcal{F} \) measurable whenever \( O \) is a basic open set.
Now, $X$ is a separable metric space and so every open set is a countable union of these basic sets. Let $U \subseteq X$ be open with $U = \bigcup_{l=1}^{\infty} O_l$ where $O_l$ is a basic open set as above. Then,

$$
\Gamma^n (U) = \bigcup_{l=1}^{\infty} \Gamma^n (O_l) \in \mathcal{F}.
$$

The existence of a measurable selection follows from the standard theory of measurable multi-functions \[9, \, 63\] see \[9\] starting on Page 141 for all the necessary stuff on measurable multifunctions or Section \[141\]. If $\sigma$ is one of these measurable selections, the evaluation at $t$ is $\mathcal{F}$ measurable. Thus, $\omega \rightarrow \sigma (t, \omega)$ is $\mathcal{F}$ measurable with values in $\mathbb{R}^\infty$. Also, $t \rightarrow \sigma (t, \omega)$ is continuous, and so it follows that in fact $\sigma$ is product measurable as claimed. ■

**Definition 75.2.7** Let $\Gamma (\omega) \equiv \cap_{n=1}^{\infty} \Gamma^n (\omega)$.

**Lemma 75.2.8** $\Gamma$ is a nonempty $\mathcal{F}$ measurable set-valued function with values in compact subsets of $X$. There exists a measurable selection $\gamma$ such that $t, \omega \rightarrow \gamma (t, \omega)$ is $\mathcal{P}$ measurable. Also, for each $\omega$, there exists a subsequence, $u_{n(\omega)} (\cdot, \omega)$ such that for each $k$,

$$
\gamma_k (t, \omega) = \lim_{n(\omega) \rightarrow \infty} f (\mathcal{X}_{\mathcal{N}C} (\omega) u_{n(\omega)} (t, \omega) )_k
$$

$$
= \lim_{n(\omega) \rightarrow \infty} m_k \int_{t_{m_k}(t)}^{t} \langle \phi_{r_k}, \mathcal{X}_{\mathcal{N}C} (\omega) u_{n(\omega)} (s, \omega) \rangle \nu (ds).
$$

**Proof:** From the definition of $\Gamma (\omega) = \cap_{n=1}^{\infty} \Gamma^n (\omega)$ it follows that $\omega \rightarrow \Gamma (\omega)$ is a compact set-valued map in $X$ and is nonempty because each $\Gamma^n (\omega)$ is nonempty and compact, and the $\Gamma^n (\omega)$ are nested. We next show that $\omega \rightarrow \Gamma (\omega)$ is $\mathcal{F}$ measurable. Indeed, each $\Gamma^n$ is compact valued and $\mathcal{F}$ measurable so, if $F$ is closed,

$$
\Gamma (\omega) \cap F = \cap_{n=1}^{\infty} \Gamma^n (\omega) \cap F,
$$

and the left-hand side is not empty if and only if each $\Gamma^n (\omega) \cap F \neq \emptyset$. Thus, for $F$ closed,

$$
\{ \omega : \Gamma (\omega) \cap F \neq \emptyset \} = \cap_n \{ \omega : \Gamma^n (\omega) \cap F \neq \emptyset \},
$$

and so

$$
\Gamma^n (F) = \cap_n \Gamma^n (F) \in \mathcal{F}.
$$

The last claim follows from the theory of multi-functions, see, e.g., \[9, \, 63\] or Section \[141\]. See Proposition \[141\]. The fact that $\Gamma^n (\omega)$ is compact, $\Gamma^n$ is measurable and $\Gamma^n (U) \in \mathcal{F}$, for $U$ open, imply the strong measurability of $\Gamma^n$. \[9, \, 63\] see also Section \[141\] and also that $\Gamma^n (F) \in \mathcal{F}$. Thus, $\omega \rightarrow \Gamma (\omega)$ is a nonempty compact valued in $X$ and $\mathcal{F}$ measurable. We are using the theorem which says that when $\Gamma$ has compact values, then one can conclude that strong measurability and measurability coincide. This is why we can say that $\Gamma^n (F) \in \mathcal{F}$.

The standard theory \[9, \, 63\], Section \[141\] also guarantees the existence of an $\mathcal{F}$ measurable selection $\omega \rightarrow \gamma (\omega)$ with $\gamma (\omega) \in \Gamma (\omega)$, for each $\omega$, and also that $t \rightarrow \gamma_k (t, \omega)$ (the $k^{th}$ component of $\gamma$) is continuous. Next, we consider the product
measurability of $\gamma_k$. We know that $\omega \to \gamma_k(\omega)$ is $\mathcal{F}$ measurable into $C([0,T])$ and since pointwise evaluation is continuous, $\omega \to \gamma_k(t,\omega)$ is $\mathcal{F}$ measurable. (This is nothing more than a case of the general result that a continuous function of a measurable function is measurable.) Then, since $t \to \gamma_k(t,\omega)$ is continuous, it follows that $\gamma_k$ is a $\mathcal{P}$ measurable real valued function and that $\gamma$ is a $\mathcal{P}$ measurable $\mathbb{R}^\infty$ valued function. Since $\gamma(\omega) \in \Gamma(\omega)$, it follows that for each $n$, $\gamma(\omega) \in \Gamma^n(\omega)$. Therefore, there exists $j_n \geq n$ such that for each $\omega$,

$$d(f(\mathcal{X}_{NC}(\omega)u_{j_n}(\cdot,\omega)), \gamma(\omega)) < 2^{-n}.$$ 

Therefore, for a suitable subsequence $\{u_{n_j}(\cdot,\omega)\}$, we have

$$\gamma(\omega) = \lim_{n(\omega) \to \infty} f(\mathcal{X}_{NC}(\omega)u_{n(\omega)}(\cdot,\omega)),$$

for each $\omega$. In particular, for each $k$

$$\gamma_k(t,\omega) = \lim_{n(\omega) \to \infty} f(\mathcal{X}_{NC}(\omega)u_{n(\omega)}(t,\omega)),$$

$$= \lim_{n(\omega) \to \infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, \mathcal{X}_{NC}(\omega)u_{n(\omega)}(s,\omega) \rangle_{V} ds, \quad (75.2.3)$$

for each $t$.

Note that it is not clear that $(t,\omega) \to f(\mathcal{X}_{NC}(\omega)u_{n(\omega)}(t,\omega))$ is $\mathcal{P}$ measurable, although $(t,\omega) \to \gamma(t,\omega)$ is $\mathcal{P}$ measurable.

Now here is the proof of the theorem.

**Proof of Theorem** By assumption, there exists a further subsequence, still denoted by $n(\omega)$, such that, in addition to $\mathcal{P}$, the weak limit

$$\lim_{n(\omega) \to \infty} \mathcal{X}_{NC}(\omega)u_{n(\omega)}(\cdot,\omega) = u(\cdot,\omega)$$

exists in $L^\prime([0,T];V')$ such that $t \to u(t,\omega)$ is weakly continuous into $V'$. Then, $\mathcal{P}$ also holds for this further subsequence and in addition,

$$m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, u(s,\omega) \rangle_{V} ds$$

$$= \lim_{n(\omega) \to \infty} m_k \int_{l_{m_k}(t)}^{t} \langle \phi_{r_k}, \mathcal{X}_{NC}(\omega)u_{n(\omega)}(s,\omega) \rangle_{V} ds$$

$$= \gamma_k(t,\omega).$$

Letting $\phi \in \mathcal{D}$ be given, there exists a subsequence, denoted by $k$, such that $m_k \to \infty$ and $\phi_{r_k} = \phi$. Recall $\{m_k, \phi_{r_k}\}$ denoted an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times \mathcal{D}$. Then, passing to the limit and using the assumed continuity of $s \to u(s,\omega)$, the left-hand side of this equality converges to $\langle \phi, u(s,\omega) \rangle_{V}$ and so the right-hand side, $\gamma_k(t,\omega)$, must also converge and for each $\omega$. Since the right-hand side is a product measurable function of $(t,\omega)$, it follows that the pointwise limit is also product.
measurable. Hence, \((t, \omega) \rightarrow (\phi, u (t, \omega))_V\) is product measurable for each \(\phi \in \mathcal{D}\).

Since \(\mathcal{D}\) is a dense set, it follows that \((t, \omega) \rightarrow (\phi, u (t, \omega))_V\) is \(\mathcal{P}\) measurable for all \(\phi \in V\) and so by the Pettis theorem, \((t, \omega) \rightarrow u (t, \omega)\) is \(\mathcal{P}\) measurable into \(V'\).\]

Actually, one can say more about the measurability of the approximating sequence and in fact, we can obtain one for which \(\omega \rightarrow u_n(\omega) (t, \omega)\) is also \(\mathcal{F}\) measurable.

**Lemma 75.2.9** Suppose that \(u_n(\omega) \rightarrow u\) weakly in \(L^p ([0, T]; V')\), where \(u\) is product measurable, and \(\{u_n(\omega)\}\) is a subsequence of \(\{u_n\}\), such that there exists a set of measure zero \(N \subseteq \Omega\) and

\[
\sup_{t \in [0, T]} \|u_n (t, \omega)\|_{V'} < C (\omega) , \quad \text{for } \omega \notin N.
\]

Then, there exists a subsequence of \(\{u_n\}\), denoted as \(\{u_{k(\omega)}\}\), such that \(u_{k(\omega)} \rightarrow u\) weakly in \(L^p ([0, T]; V')\), \(\omega \rightarrow k (\omega)\) is \(\mathcal{F}\) measurable, and \(\omega \rightarrow u_{k(\omega)} (t, \omega)\) is also \(\mathcal{F}\) measurable, for each \(\omega \notin N\).

**Proof:** Assume that \(f, g \in L^p ([0, T]; V')\) and let \(\{\phi_k\}\) be a countable dense subset of \(L^p ([0, T]; V')\). Then, a bounded set in \(L^p ([0, T]; V')\) with the weak topology can be considered a complete metric space using the metric

\[
d(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{|\langle \phi_k, f - g \rangle|}{1 + |\langle \phi_k, f - g \rangle|}.
\]

Now, let \(k (\omega)\) be the first index of \(\{u_n\}\) that is at least as large as \(k\) and such that

\[
d (X_{NC} (\omega) u_{k(\omega)} , u) \leq 2^{-k}.
\]

Such an index exists because there exists a convergent sequence \(X_{NC} (\omega) u_n(\omega)\) that converge weakly to \(u\). In fact,

\[
\{\omega : k (\omega) = l\} = \bigcap_{j=1}^{l-1} \{\omega : d (u_j, u) \leq 2^{-k}\} \cap \{\omega : d (u_j, u) > 2^{-k}\}.
\]

Since \(u\) is product measurable and each \(u_j\) is also product measurable, these are all measurable sets with respect to \(\mathcal{F}\) and so \(\omega \rightarrow k (\omega)\) is \(\mathcal{F}\) measurable. Now, we have that \(X_{NC} (\omega) u_{k(\omega)} \rightarrow u\) weakly in \(L^p ([0, T]; V')\), for each \(\omega\), and each function is \(\mathcal{F}\) measurable because

\[
u_{k(\omega)} (t, \omega) = \sum_{j=1}^{\infty} X_{NC} (\omega) = j \mu_j (t, \omega),
\]

and every term in the sum is \(\mathcal{F}\) measurable.\]

Theorem [75.2.4] can be generalized in a very nice way. It is a better result because you don’t need to assume anything so strong as to have the functions bounded. One does not need any assumption that the limit is weakly continuous.
Let $\langle \cdot \rangle_X$ be pre-compact in $X$. Here, $\langle \cdot \rangle_X$.

Lemma 75.2.11 Let $V$ be a reflexive separable Banach space with dual $V'$, and let $p, p'$ be such that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let the functions $t \to u_n(t, \omega)$, for $n \in \mathbb{N}$, be in $L^p([0, T]; V) \equiv V$ and $t, \omega \to u_n(t, \omega)$ be $B([0, T]) \times F \equiv \mathcal{P}$ measurable into $V$. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then

$$\|u_n(\cdot, \omega)\|_V \leq C(\omega),$$

for all $n$. (Thus, by weak compactness, for each $\omega$, each subsequence of $\{u_n\}$ has a further subsequence that converges weakly in $V$ to $v(\cdot, \omega) \in V$. ($v$ not known to be $\mathcal{P}$ measurable).)

Then, there exists a product measurable function $u$ such that $t \to u(t, \omega)$ is in $V$ and for each $\omega \notin N$, a subsequence $u_{n(\omega)}$ such that $u_{n(\omega)}(\cdot, \omega) \to u(\cdot, \omega)$ weakly in $V$.

We prove the theorem in steps given below. Let $X = \prod_{k=1}^{\infty} C([0, T])$ and note that when it is equipped with the product topology, then one can consider $X$ as a metric space using the metric

$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|f_k - g_k\|}{1 + \|f_k - g_k\|},$$

where $f = (f_1, f_2, \ldots), g = (g_1, g_2, \ldots) \in X$, and the norm is the maximum norm in $C([0, T])$. With this metric, $X$ is complete and separable.

Lemma 75.2.11 Let $\{f_n\}$ be a sequence in $X$ and suppose that each one of the components $f_{nk}$ is bounded by $C = C(k)$ in $C^0(1/p')(0, T)$. Then, there exists a subsequence $\{f_{n_j}\}$ that converges to some $f \in X$ as $n_j \to \infty$. Thus, $\{f_n\}$ is pre-compact in $X$.

Proof: This follows right away from Tychonoff’s theorem and the compactness of the embedding of the Holder space into $C([0, 1])$.

Now, for $m \in \mathbb{N}$ and $\phi \in V'$, define $l_m(t) = \max(0, t - (1/m))$ and $\psi_{m, \phi} : V \to C([0, T])$ by

$$\psi_{m, \phi} u(t) = \int_0^T \langle m\phi \chi_{[l_m(t), t]}(s), u(s) \rangle_V \, ds = m \int_{l_m(t)}^d \langle \phi, u(s) \rangle_V \, ds.$$
Let $D = \{ \phi_r \}_{r=1}^\infty$ denote a countable subset of $V'$. Then the pairs $(\phi, m)$ for $\phi \in D$ and $m \in \mathbb{N}$ form a countable set. Let $(m_k, \phi_{r_k})$ denote an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times D$. To simplify the notation, we set

$$f_k(u)(t) \equiv \psi_{m_k, \phi_{r_k}}(u)(t) = m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u(s) \rangle_V \, ds.$$  

For fixed $\omega \notin N$ and $k$, the functions $\{ t \to f_k(u_j(\cdot, \omega))(t) \}_{j}$ are uniformly bounded and equicontinuous because they are in $C^{0,1/p}([0,T])$. Indeed, we have for $\omega \notin N$,

$$|f_k(u_j(\cdot, \omega))(t)| = \left| m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u_j(s, \omega) \rangle_V \, ds \right| \leq m \| \phi_{r_k} \|_{T^{1/p'}} \left\| \int_0^T \| u_j(s, \omega) \|_V \, ds \right\|^{1/p} \leq C(\omega) m \| \phi_{r_k} \|_V T^{1/p'}$$

and for $t \leq t'$

$$|f_k(u_j(\cdot, \omega))(t) - f_k(u_j(\cdot, \omega))(t')| \leq \left| m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, u_j(s, \omega) \rangle_V \, ds - m_k \int_{l_{m_k}(t')}^t \langle \phi_{r_k}, u_j(s, \omega) \rangle_V \, ds \right| \leq 2m_k |t' - t|^{1/p'} C(\omega) \| \phi_{r_k} \|_V.$$ 

By Lemma 75.2.1, the set of functions $\{ \chi_{NC}(\omega) f(u_j(\cdot, \omega)) \}_{j=n}^\infty$ is pre-compact in $X = \prod_k C([0,T])$. We now define a set valued map $\Gamma^n : \Omega \to X$ by

$$\Gamma^n(\omega) \equiv \cup_{j \geq n} \{ \chi_{NC}(\omega) f(u_j(\cdot, \omega)) \},$$

where the closure is taken in $X$. Then $\Gamma^n(\omega)$ is the closure of a pre-compact set in $X$ and so $\Gamma^n(\omega)$ is compact in $X$. From the definition, a function $f$ is in $\Gamma^n(\omega)$ if and only if $d(f, \chi_{NC}(\omega) f(w_j)) \to 0$ as $l \to \infty$, where each $w_j$ is one of the $u_j(\cdot, \omega)$ for $j \geq n$. In the topology on $X$, this happens if and only for each $k$,

$$f_k(t) = \lim_{l \to \infty} m_k \int_{l_{m_k}(t)}^t \langle \phi_{r_k}, \chi_{NC}(\omega) w_l(s, \omega) \rangle_V \, ds,$$

where the limit is the uniform limit in $t$.

**Lemma 75.2.12** The mapping $\omega \to \Gamma^n(\omega)$ is an $\mathcal{F}$ measurable set-valued map with values in $X$. If $\sigma$ is a measurable selection, then for each $t$, $\omega \to \sigma(t, \omega)$ is $\mathcal{F}$ measurable and $(t, \omega) \to \sigma(t, \omega)$ is $\mathcal{B}([0,T]) \times \mathcal{F}$ measurable.

We note that if $\sigma$ is a measurable selection then $\sigma(\omega) \in \Gamma^n(\omega)$, so $\sigma = \sigma(\cdot, \omega)$ is a continuous function. To have $\sigma$ measurable would mean that $\sigma^{-1}(\text{open}) \in \mathcal{F}$, where the open set is in $C([0,T])$. 


Proof: Let $O$ be a basic open set in $X$. Then $O = \prod_{k=1}^{\infty} O_k$, where $O_k$ is a proper open set of $C([0, T])$ only for $k \in \{k_1, \ldots, k_r\}$. Thus there is a proper open set in these positions and in every other position the open set is the whole space $C([0, T])$. We need to show that

$$\Gamma^n - (O) \equiv \{ \omega : \Gamma^n (\omega) \cap O \neq \emptyset \} \in \mathcal{F}.$$  

Now, $\Gamma^n - (O) = \cap_{k=1}^{\infty} \{ \omega : \Gamma^n (\omega)_{k} \cap O_{k} \neq \emptyset \}$, so we consider whether

$$\{ \omega : \Gamma^n (\omega)_{k} \cap O_{k} \neq \emptyset \} \in \mathcal{F}. \quad (75.2.4)$$

From the definition of $\Gamma^n (\omega)$, this is equivalent to the condition that

$$f_{k_i} (\mathcal{X}_{NC} (\omega) u_{j} (\cdot, \omega)) = (f (\mathcal{X}_{NC} (\omega) u_{j} (\cdot, \omega)))_{k_i} \in O_{k_i}$$

for some $j \geq n$, and so the set in [75.2.4] is of the form

$$\bigcup_{j=n}^{\infty} \{ \omega : (f (\mathcal{X}_{NC} (\omega) u_{j} (\cdot, \omega)))_{k_i} \in O_{k_i} \}.$$

Now $\omega \rightarrow (f (\mathcal{X}_{NC} (\omega) u_{j} (\cdot, \omega)))_{k_i}$ is $\mathcal{F}$ measurable into $C([0, T])$ and so the above set is in $\mathcal{F}$. To see this, let $g \in \overline{C}([0, T])$ and consider the inverse image of the ball with radius $r$ and center $g$,

$$B (g, r) = \{ \omega : \| (\mathcal{X}_{NC} (\omega) f (u_{j} (\cdot, \omega)))_{k_i} - g \|_{C([0, T])} < r \}.$$

By continuity considerations,

$$\| (\mathcal{X}_{NC} (\omega) f (u_{j} (\cdot, \omega)))_{k_i} - g \|_{C([0, T])} = \sup_{t \in [0, T]} \left| (\mathcal{X}_{NC} (\omega) f (u_{j} (t, \omega)))_{k_i} - g (t) \right|,$$

which is the sup over countably many $\mathcal{F}$ measurable functions. Thus, it is $\mathcal{F}$ measurable. Since every open set is the countable union of such balls, it follows that the claim about $\mathcal{F}$ measurability is valid. Hence, $\Gamma^n - (O)$ is $\mathcal{F}$ measurable whenever $O$ is a basic open set.

Now, $X$ is a separable metric space and so every open set is a countable union of these basic sets. Let $U \subseteq X$ be open with $U = \bigcup_{l=1}^{\infty} O_{l}$ where $O_{l}$ is a basic open set as above. Then,

$$\Gamma^n - (U) = \bigcup_{l=1}^{\infty} \Gamma^n - (O_{l}) \in \mathcal{F}.$$

The existence of a measurable selection follows from the standard theory of measurable multi-functions \[\text{[1, 13]}\] see [13] starting on Page 141 for all the necessary stuff on measurable multifunctions or Section 46. If $\sigma$ is one of these measurable selections, the evaluation at $t$ is $\mathcal{F}$ measurable. Thus, $\omega \rightarrow \sigma (t, \omega)$ is $\mathcal{F}$ measurable with values in $\mathbb{R}^{\infty}$. Also, $t \rightarrow \sigma (t, \omega)$ is continuous, and so it follows that in fact $\sigma$ is product measurable as claimed. \[\blacksquare\]
Let \( \Gamma (\omega) \equiv \cap_{n=1}^{\infty} \Gamma^n (\omega) \).

**Lemma 75.2.14** \( \Gamma \) is a nonempty \( \mathcal{F} \) measurable set-valued function with values in compact subsets of \( X \). There exists a measurable selection \( \gamma \) such that \( (t, \omega) \to \gamma(t, \omega) \) is \( \mathcal{P} \) measurable. Also, for each \( \omega \), there exists a subsequence, \( u_{n(\omega)} (\cdot, \omega) \) such that for each \( k \),

\[
\gamma_k (t, \omega) = \lim_{n(\omega) \to \infty} f (X_{N^C} (\omega) u_{n(\omega)} (t, \omega)) \cdot k
\]

\[
= \lim_{n(\omega) \to \infty} m_k \int_{l_{m_k}(t)}^{t} \phi_{rk}, X_{N^C} (\omega) u_{n(\omega)} (s, \omega) \, \psi \, ds.
\]

**Proof:** From the definition of \( \Gamma (\omega) = \cap_{n=1}^{\infty} \Gamma^n (\omega) \) it follows that \( \omega \to \Gamma (\omega) \) is a compact set-valued map in \( X \) and is nonempty because each \( \Gamma^n (\omega) \) is nonempty and compact, and the \( \Gamma^n (\omega) \) are nested. We next show that \( \omega \to \Gamma (\omega) \) is \( \mathcal{F} \) measurable. Indeed, each \( \Gamma^n \) is compact valued and \( \mathcal{F} \) measurable so, if \( F \) is closed,

\[
\Gamma (\omega) \cap F = \cap_{n=1}^{\infty} \Gamma^n (\omega) \cap F,
\]

and the left-hand side is not empty if and only if each \( \Gamma^n (\omega) \cap F \neq \emptyset \). Thus, for \( F \) closed,

\[
\{ \omega : \Gamma (\omega) \cap F \neq \emptyset \} = \cap_n \{ \omega : \Gamma^n (\omega) \cap F \neq \emptyset \},
\]

and so

\[
\Gamma^- (F) = \cap_n \Gamma^n^- (F) \in \mathcal{F}.
\]

The last claim follows from the theory of multi-functions, see, e.g., [4, 5] or Section 10. The fact that \( \Gamma^n (\omega) \) is compact, \( \Gamma^n \) is measurable and \( \Gamma^n^- (U) \in \mathcal{F} \), for \( U \) open, imply the strong measurability of \( \Gamma^n \) [4, 5] see also Section 10, and also that \( \Gamma^n^- (F) \in \mathcal{F} \). Thus, \( \omega \to \Gamma (\omega) \) is nonempty compact valued in \( X \) and \( \mathcal{F} \) measurable. We are using the theorem which says that when \( \Gamma \) has compact values, then one can conclude that strong measurability and measurability coincide. See Proposition 10.1.14. This is why we can say that \( \Gamma^n^- (F) \in \mathcal{F} \).

The standard theory [4, 5], Section 10, also guarantees the existence of an \( \mathcal{F} \) measurable selection \( \omega \to \gamma (\omega) \) with \( \gamma (\omega) \in \Gamma (\omega) \), for each \( \omega \), and also that \( t \to \gamma_k (t, \omega) \) (the \( k \)th component of \( \gamma \)) is continuous. Next, we consider the product measurability of \( \gamma_k \). We know that \( \omega \to \gamma_k (\omega) \) is \( \mathcal{F} \) measurable into \( C ([0, T]) \) and since pointwise evaluation is continuous, \( \omega \to \gamma_k (t, \omega) \) is \( \mathcal{F} \) measurable. (This is nothing more than a case of the general result that a continuous function of a measurable function is measurable.) Then, since \( t \to \gamma_k (t, \omega) \) is continuous, it follows that \( \gamma_k \) is a \( \mathcal{P} \) measurable real valued function and that \( \gamma \) is a \( \mathcal{P} \) measurable \( \mathbb{R}^\infty \) valued function. Since \( \gamma (\omega) \in \Gamma (\omega) \), it follows that for each \( n, \gamma (\omega) \in \Gamma^n (\omega) \). Therefore, there exists \( j_n \geq n \) such that for each \( \omega \),

\[
d (f (X_{N^C} (\omega) u_{j_n} (\cdot, \omega)), \gamma (\omega)) < 2^{-n}.
\]

Therefore, for a suitable subsequence \( \{ u_{n(\omega)} (\cdot, \omega) \} \), we have

\[
\gamma (\omega) = \lim_{n(\omega) \to \infty} f (X_{N^C} (\omega) u_{n(\omega)} (\cdot, \omega)),
\]
CHAPTER 75. STOCHASTIC INCLUSIONS

for each $\omega$. In particular, for each $k$

$$
\gamma_k(t, \omega) = \lim_{n(\omega) \to \infty} f(\lambda_{NC}(\omega) u_{n(\omega)}(t, \omega))_k
$$

$$
= \lim_{n(\omega) \to \infty} m_k \int_{l_{m_k}(t)} \langle \varphi_{rk}, \lambda_{NC}(\omega) u_{n(\omega)}(s, \omega) \rangle_V ds,
$$

(75.2.5)

for each $t$. ■

Note that it is not clear that $(t, \omega) \to f(\lambda_{NC}(\omega) u_{n(\omega)}(t, \omega))$ is $\mathcal{P}$ measurable, although $(t, \omega) \to \gamma(t, \omega)$ is $\mathcal{P}$ measurable.

Now here is the proof of the theorem.

Proof of Theorem 75.2.10 By assumption, there exists a further subsequence, still denoted by $n(\omega)$, such that, the weak limit

$$
\lim_{n(\omega) \to \infty} \lambda_{NC}(\omega) u_{n(\omega)}(\cdot, \omega) = v(\cdot, \omega)
$$

exists in $\mathcal{V}$. Then,

$$
m_k \int_{l_{m_k}(t)} \langle \varphi_{rk}, v(s, \omega) \rangle_V ds = \lim_{n(\omega) \to \infty} m_k \int_{l_{m_k}(t)} \langle \varphi_{rk}, \lambda_{NC}(\omega) u_{n(\omega)}(s, \omega) \rangle_V ds = \gamma_k(t, \omega), \text{ product measurable.}
$$

Letting $\phi \in \mathcal{D}$ be given, there exists a subsequence, denoted by $k$, such that $m_k \to \infty$ and $\phi_{rk} = \phi$. Recall $(m_k, \varphi_{rk})$ denoted an enumeration of the pairs $(m, \phi) \in \mathbb{N} \times \mathcal{D}$. For a given $\phi \in \mathcal{D}$ denote this sequence by $m_\phi$. Thus we have measurability of

$$(t, \omega) \to m_\phi \int_{l_{m_\phi}(t)} \langle \phi, v(s, \omega) \rangle_V ds$$

for each $\phi \in \mathcal{D}$.

Now we will be a little more careful about the countable set $\mathcal{D}$. Iterate the following. Let $\phi_1 \neq 0$. Let $\mathcal{F}$ denote linearly independent subsets of $V'$ which contain $\phi_1$ such that the elements are further apart than $1/5$. Let $\mathcal{C}$ denote a maximal chain. Thus $\cup \mathcal{C}$ is also in $\mathcal{F}$. If $W := \overline{\text{span}}(\mathcal{C})$ fails to be all of $V'$, then there would exist $\psi \notin W$ such that the distance of $\psi$ to the closed subspace $W$ is at least $1/5$. Now $\mathcal{C}, \cup \{C \cup \{\psi\}\}$ would violate maximality of $\mathcal{C}$. Hence $W = V'$. Now it follows that $\mathcal{C}$ must be countable since otherwise, $V'$ would fail to be separable. Let $\mathcal{M}$ be the rational linear combinations of $\mathcal{D}$. It must be dense in $V'$. Note that linear combinations of the $\phi_i$ are uniquely determined because none is a linear combination of the others. Now define a linear mapping on $\mathcal{M}$ which makes sense for $(t, \omega)$ on a certain set.
Definition 75.2.15 Let $E$ be those points $(t, \omega)$ such that the following limit exists for each $\phi \in \mathcal{D}$

$$
\Lambda(t, \omega) \phi \equiv \lim_{m \to \infty} m_{\phi} \int_{l_m(t)}^{t} \langle \phi, v(s, \omega) \rangle \, ds
$$

The set of points where the limit of measurable functions exists is always measurable so $E$ is a measurable set. Extend this mapping linearly. That is, for $\psi \in M, \psi \equiv \sum_i a_i \phi_i,$

$$
\Lambda(t, \omega) \psi \equiv \sum_i a_i \Lambda(t, \omega) \phi_i = \sum_i a_i \left( \lim_{m \to \infty} m_{\phi_i} \int_{l_m(t)}^{t} \langle \phi_i, v(s, \omega) \rangle \, ds \right)
$$

Thus $(t, \omega) \to \Lambda(t, \omega) \psi$ is product measurable, being the sum of limits of product measurable functions. Let $G$ denote those $(t, \omega)$ in $E$ such that there exists a constant $C(t, \omega)$ such that for all $\psi \in M,$

$$
|\Lambda(t, \omega) \psi| \leq C(t, \omega) \| \psi \|
$$

Lemma 75.2.16 $G$ is product measurable.

Proof: This follows from the formula

$$
E \cap G^C = \cap_n \cup_{\psi \in M} \{(t, \omega) \in E : |\Lambda(t, \omega) \psi| > n \| \psi \| \}
$$

which is clearly product measurable because $(t, \omega) \to \Lambda(t, \omega) \psi$ is. Thus, since $E$ is measurable, it follows that $E \cap G = G$ is also.

For $(t, \omega) \in G, \Lambda(t, \omega)$ has a unique extension to all of $V,$ the dual space of $V',$ still denoted as $\Lambda(t, \omega).$ By the Riesz representation theorem, for $(t, \omega) \in G,$ there exists $u(t, \omega) \in V,$

$$
\Lambda(t, \omega) \psi = \langle \psi, u(t, \omega) \rangle_{V', V}
$$

Thus $(t, \omega) \to X_G(t, \omega) u(t, \omega)$ is product measurable by the Pettis theorem. Let $u = 0$ off $G.$ We know $G$ is product measurable. Now we want to show that for each $\omega, \{t : (t, \omega) \in G\}$ has full measure. This involves the fundamental theorem of calculus.

Fix $\omega.$ By the fundamental theorem of calculus,

$$
\lim_{m \to \infty} m \int_{l_m(t)}^{t} v(s, \omega) \, ds = v(t, \omega) \text{ in } V
$$

for a.e. $t$ say for all $t \notin N(\omega) \subseteq [0, T].$ Of course we do not know that $\omega \to v(t, \omega)$ is measurable. However, the existence of this limit for $t \notin N(\omega)$ implies that for every $\phi \in V',$

$$
\lim_{m \to \infty} \left| m \int_{l_m(t)}^{t} \langle \phi, v(s, \omega) \rangle \, ds \right| \leq C(t, \omega) \| \phi \|
$$
for some $C(t, \omega)$. Here $m$ does not depend on $\phi$. Thus, in particular, this holds for a subsequence and so for each $t \in N(\omega), (t, \omega) \in G$ because for each $\phi \in D$,

$$
\lim_{m_\omega \to \infty} m_\omega \int_{I_{m_\omega}(t)}^t \langle \phi, v(s, \omega) \rangle ds \text{ exists and satisfies the above inequality.}
$$

Hence, for all $\psi \in M$,

$$
\Lambda(t, \omega) \psi = \langle \psi, u(t, \omega) \rangle_{V', V},
$$

where $u$ is product measurable.

Also, for $t \in N(\omega)$ and $\phi \in D$,

$$
\langle \phi, u(t, \omega) \rangle_{V', V} = \Lambda(t, \omega) \phi = \lim_{m_\phi \to \infty} m_\phi \int_{I_{m_\phi}(t)}^t \langle \phi, v(s, \omega) \rangle ds = \langle \phi, v(t, \omega) \rangle_{V', V},
$$

therefore, for all $\phi \in M$

$$
\langle \phi, u(t, \omega) \rangle_{V', V} = \langle \phi, v(t, \omega) \rangle_{V', V}
$$

and hence $u(t, \omega) = v(t, \omega)$. Thus, for each $\omega$, the product measurable function $u$ satisfies $u(t, \omega) = v(t, \omega)$ for a.e. $t$. Hence $u(\cdot, \omega) = v(\cdot, \omega)$ in $V$.

Of course a similar theorem will hold with essentially the identical proof if the functions take values in $V'$.

One can also combine the two theorems to obtain a useful result for limits of functions in $V'$ and $V$. You just let

$$
X = \prod_{k=1}^\infty C([0, T]) \times C([0, T])
$$

and let $\{\phi_k\}$ be a dense subset of $V'$ while $\{\eta_k\}$ is a dense subset of $V$. Then the mappings are given by

$$
\psi_{m, \phi} u(t) = m \int_{I_m(t)}^t \langle \phi, u(s) \rangle_{V', V} ds, \quad \psi_{m, \eta} y(t) = m \int_{I_m(t)}^t \langle \eta, y(s) \rangle_{V', V} ds
$$

and one considers for each $\phi_k, \eta_k$,

$$
\left( m_k \int_{I_{m_k}(t)}^t \langle \phi_k, u(s) \rangle_{V', V} ds, m_k \int_{I_{m_k}(t)}^t \langle \eta_k, y(s) \rangle_{V', V} ds \right)
$$

This time you need to use an enumeration of $\mathbb{N} \times V' \times V$ and in the last step, you must use a subsequence still denoted with $k$ such that $m_k \to \infty$ but $\phi_k = \phi$ and $\eta_k = \eta$ for $\phi, \eta$ two given elements of $V'$ and $V$ respectively. Then repeating the above argument, one obtains the following generalization.

**Theorem 75.2.17** Let $V$ be a reflexive separable Banach space with dual $V'$, and let $p, p'$ be such that $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let the functions $t \to u_n(t, \omega)$, for $n \in \mathbb{N}$,
be in $L^p ([0, T] ; V) \equiv V$ and $(t, \omega) \rightarrow u_n (t, \omega)$ be $B ([0, T]) \times F \equiv \mathcal{P}$ measurable into $V$. Also let the functions $t \rightarrow y_n (t, \omega)$ be in $\mathcal{V}'$ and $(t, \omega) \rightarrow y_n (t, \omega)$ is $\mathcal{P}$ measurable into $\mathcal{V}'$. Suppose there is a set of measure zero $N \subseteq \Omega$ such that if $\omega \notin N$, then

$$
\sup_{t \in [0, T]} \| y_n (t, \omega) \|_{\mathcal{V}'} + \| u_n (\cdot, \omega) \|_{\mathcal{V}} \leq C (\omega),
$$

for all $n$. (Thus, by weak compactness, for each $\omega$, each subsequence of $\{ u_n \}$ has a further subsequence that converges weakly in $V$ to $v (\cdot, \omega) \in V$. (v not known to be $\mathcal{P}$ measurable)) Suppose that each subsequence of $\{ y_n (\cdot, \omega) \}$ has a subsequence which converges weakly in $\mathcal{V}'$ to $z (\cdot, \omega) \in \mathcal{V}'$ such that the function $t \rightarrow z (t, \omega)$ is weakly continuous into $\mathcal{V}'$.

Then, there exist product measurable functions $u, y$ such that $t \rightarrow u (t, \omega)$ is in $\mathcal{V}$, $t \rightarrow y (t, \omega)$ is weakly continuous into $\mathcal{V}'$ and for each $\omega \notin N$, a subsequence of $\mathbb{N}$ denoted by $\{ n (\omega) \}$ such that $u_{n (\omega)} (\cdot, \omega) \rightarrow u (\cdot, \omega)$ weakly in $\mathcal{V}$ and $y_{n (\omega)} (\cdot, \omega)$ converges weakly to $y (\cdot, \omega)$ in $\mathcal{V}'$.

Note that the conclusion of the proposition holds if $p = 1$ and $V = L^1 ([0, T], V)$.

Here is something else about being measurable into $\mathcal{V}$ or $\mathcal{V}'$. Such functions have representatives which are product measurable.

**Lemma 75.2.18** Let $f (\cdot, \omega) \in \mathcal{V}'$. Then if $f (\cdot, \omega) \rightarrow f (\cdot, \omega)$ is measurable into $\mathcal{V}'$, it follows that for each $\omega$, there exists a representative $\hat{f} (\cdot, \omega) \in \mathcal{V}'$, $\hat{f} (\cdot, \omega) = f (\cdot, \omega)$ in $\mathcal{V}'$ such that $(t, \omega) \rightarrow \hat{f} (t, \omega)$ is product measurable. If $f (\cdot, \omega) \in \mathcal{V}'$ and $f (\cdot, \omega) \rightarrow f (\cdot, \omega)$ is product measurable, then $\omega \rightarrow f (\cdot, \omega)$ is measurable into $\mathcal{V}'$. The same holds replacing $\mathcal{V}'$ with $\mathcal{V}$.

**Proof:** If a function $f$ is measurable into $\mathcal{V}'$, then there exist simple functions $f_n$ such that $f_n \rightarrow f$ weakly in $\mathcal{V}'$. Then by Theorem 75.2.11, there exists $f (\cdot, \omega) \in \mathcal{V}'$ such that $\hat{f}$ is product measurable and a subsequence $f_{n (\omega)}$ converging weakly in $\mathcal{V}'$ to $\hat{f} (\cdot, \omega)$ for each $\omega$. Thus $f_{n (\omega)} (\omega) \rightarrow f (\omega)$ strongly in $\mathcal{V}'$ and $f_{n (\omega)} (\omega) \rightarrow \hat{f} (\omega)$ weakly in $\mathcal{V}'$. Therefore, $f (\omega) = f (\omega)$ in $\mathcal{V}'$ and so it can be assumed that if $f$ is measurable into $\mathcal{V}'$ then for each $\omega$, it has a representative $\hat{f} (\cdot, \omega)$ such that $(t, \omega) \rightarrow \hat{f} (t, \omega)$ is product measurable.

If $f$ is product measurable into $\mathcal{V}'$ and each $f (\cdot, \omega) \in \mathcal{V}'$, does it follow that $f$ is measurable into $\mathcal{V}'$? By measurability, $f (t, \omega) = \lim_{n \rightarrow \infty} \sum_{i=1}^{M} c_i \mathcal{X}_{E_i} (\omega) = \lim_{n \rightarrow \infty} \sum_{i=1}^{M} c_i \mathcal{X}_{E'_i}^n (t, \omega) = \lim_{n \rightarrow \infty} \sum_{i=1}^{M} c_i \mathcal{X}_{E'_i} (\omega)$.
\[
\lim_{n \to \infty} f_n(t, \omega) \text{ where } E_n^i \text{ is product measurable and we can assume } \|f_n(t, \omega)\|_{V'} \leq 2 \|f(t, \omega)\|. \text{ Then by product measurability, } \omega \to f_n(\cdot, \omega) \text{ is measurable into } V' \text{ because if } g \in V \text{ then }
\]

\[
\omega \to \langle f_n(\cdot, \omega), g \rangle
\]

is of the form

\[
\omega \to \sum_{i=1}^{m_n} \int_0^T \langle c^n_i X E_n^i(t, \omega), g(t) \rangle dt \text{ which is } \omega \to \sum_{i=1}^{m_n} \int_0^T \langle c^n_i, g(t) \rangle X E_n^i(t, \omega) dt
\]

and this is \( \mathcal{F} \) measurable since \( E_n^i \) is product measurable. Thus, it is measurable into \( V' \) as desired and

\[
\langle f(\cdot, \omega), g \rangle = \lim_{n \to \infty} \langle f_n(\cdot, \omega), g \rangle, \omega \to \langle f_n(\cdot, \omega), g \rangle \text{ is } \mathcal{F} \text{ measurable.}
\]

By the Pettis theorem, \( \omega \to \langle f(\cdot, \omega), g \rangle \) is measurable into \( V' \). Obviously, the conclusion is the same for these two conditions if \( V' \) is replaced with \( V \).

The following theorem is also useful. It is really a generalization of the familiar Gram Schmidt process. It is Lemma 31.4.2.

**Theorem 75.2.19** Suppose \( V, W \) are separable Banach spaces, such that \( V \) is dense in \( W \) and \( B \in L(W, W') \) satisfies

\[
\langle Bx, x \rangle \geq 0, \langle Bx, y \rangle = \langle By, x \rangle, B \neq 0.
\]

Then there exists a countable set \( \{e_i\} \) of vectors in \( V \) such that

\[
\langle Be_i, e_j \rangle = \delta_{ij}
\]

and for each \( x \in W \),

\[
\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,
\]

and also

\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i,
\]

the series converging in \( W' \). In case \( B = B(\omega) \) where \( \omega \to B(\omega) \) is measurable into \( \mathcal{L}(W, W') \), these vectors \( e_i \) will also depend on \( \omega \) and will be measurable functions of \( \omega \).

### 75.3 Preliminary Results

We use the following well known theorem [32]. It is Theorem 31.7.6.
Theorem 75.3.1 Let \( E \subseteq F \subseteq G \) where the injection map is continuous from \( F \) to \( G \) and compact from \( E \) to \( F \). Let \( p \geq 1 \), let \( q > 1 \), and define

\[
S \equiv \{ u \in L^p ([a, b], E) : \text{for some } C, \| u(t) - u(s) \|_G \leq C |t - s|^{1/q} \text{ and } \| u \|_{L^p([a, b], E)} \leq R \}.
\]

Thus \( S \) is bounded in \( L^p ([a, b], E) \) and Holder continuous into \( G \). Then \( S \) is precompact in \( L^p ([a, b], F) \). This means that if \( \{ u_n \}_{n=1}^\infty \subseteq S \), it has a subsequence \( \{ u_{n_k} \} \) which converges in \( L^p ([a, b], F) \).

We recall the following theorem which is proved in [30] and earlier, Theorem 23.5.2 for what will suffice here.

Theorem 75.3.2 If \( A \) and \( B \) are pseudo monotone and bounded then \( A + B \) is also pseudo monotone and bounded.

Also the following result, found in [32] is well known.

Theorem 75.3.3 If a single valued map, \( A : X \to X' \) is monotone, hemicontinuous, and bounded, then \( A \) is pseudo monotone. Furthermore, the duality map, \( J^{-1} : X \to X' \) which satisfies \( \langle J^{-1} f, f \rangle = \| f \|^2, \| J^{-1} f \|_X = \| f \|_X \) is strictly monotone hemicontinuous and bounded. So is the duality map \( F : X \to X' \) which satisfies \( \| F f \|_X = \| f \|^{p-1}_X, \langle F f, f \rangle = \| f \|_X^p \) for \( p > 1 \).

The following fundamental result will be of use in what follows. There is somewhat more in this than will be needed. In this paper, \( B \) is a possibly degenerate operator satisfying only the following:

\[
B \in \mathcal{L}(W, W'), \quad \langle Bu, u \rangle \geq 0, \langle Bu, v \rangle = \langle Bv, u \rangle \quad (75.3.6)
\]

where here \( V \subseteq W \) and \( V \) is dense in \( W \). In the case where \( B = B(\omega) \), we will assume for the sake of simplicity that

\[
B(\omega) = k(\omega) B, \quad k(\omega) \geq 0, k \text{ being } \mathcal{F} \text{ measurable}
\]

Allowing \( B \) to depend on \( \omega \) introduces some technical considerations so if there is no interest in this, simply assume \( B \) is independent of \( \omega \). This includes all cases of most interest.

Lemma 75.3.4 Suppose \( V, W \) are separable Banach spaces such that \( V \) is dense in \( W \) and \( B \in \mathcal{L}(W, W') \) satisfies

\[
\langle Bx, x \rangle \geq 0, \quad \langle Bx, y \rangle = \langle By, x \rangle, B \neq 0.
\]

Then there exists a countable set \( \{ e_i \} \) of vectors in \( V \) such that

\[
\langle Be_i, e_j \rangle = \delta_{ij}
\]
and for each \( x \in W \),
\[
\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,
\]
and also
\[
Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle B e_i,
\]
the series converging in \( W' \). If \( B = B(\omega) \) and \( B \) is \( \mathcal{F} \) measurable into \( \mathcal{L}(W,W') \) and if the \( e_i = e_i(\omega) \) are as described above, then these \( e_i \) are measurable into \( V \). If \( t \to B(t,\omega) \) is \( C^1([0,T], \mathcal{L}(W,W')) \) and if for each \( w \in W \),
\[
\langle B'(t,\omega) w, w \rangle \leq k_{w,\omega}(t) \langle B(t,\omega) w, w \rangle
\]
Where \( k_{w,\omega} \in L^1([0,T]) \), then the vectors \( e_i(t) \) can be chosen to also be right continuous functions of \( t \).

The following has to do with the values of \( Bu \) and gives an integration by parts formula.

**Corollary 75.3.5** Let \( V \subseteq W, W' \subseteq V' \) be separable Banach spaces, and \( B \in \mathcal{L}(W,W') \) is nonnegative and self adjoint. Also suppose \( t \to B(u(t)) \) has a weak derivative \( (Bu)' \in L^p(0,T,V') \) for \( u \in L^p(0,T,V) \). Then there is a continuous function denoted as \( t \to Bu(t) \) which equals \( B(u(t)) \) a.e. \( t \). Say for \( t \notin N \). Suppose \( Bu(0) = Bu_0, u_0 \in W \). Then
\[
Bu(t) = Bu_0 + \int_0^t (Bu)'(s) \, ds \text{ in } V' \tag{75.3.7}
\]
Then \( t \to Bu(t) \) is in \( C(N^C, W') \) and also for such \( t \),
\[
\frac{1}{2} \langle Bu(t), u(t) \rangle = \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle (Bu)'(s), u(s) \rangle \, ds
\]
There exists a continuous function \( t \to \langle Bu, u(t) \rangle(t) \) which equals the right side of the above for all \( t \) and equals \( \langle B(u(t)), u(t) \rangle \) off \( N \). This also satisfies
\[
\sup_{t \in [0,T]} \langle Bu, u(t) \rangle(t) \leq C \left( \|Bu\|_{L^p(0,T,V')} \cdot \|u\|_{L^p(0,T,V)} \right)
\]
This also makes it easy to verify continuity of pointwise evaluation of \( Bu \). Let \( Lu = (Bu)' \).
\[
\begin{align*}
  & u \in D(L) \equiv X \equiv \left\{ u \in L^p(0,T,V) : Lu \equiv (Bu)' \in L^p(0,T,V') \right\} \\
  & \|u\|_X \equiv \max \left( \|u\|_{L^p(0,T,V)}, \|Lu\|_{L^p(0,T,V')} \right) \tag{75.3.8}
\end{align*}
\]
Since \( L \) is closed, this \( X \) is a Banach space.

Then the following theorem is obtained.
Theorem 75.3.6 Say \((Bu)' \in L^{p'}(0,T,V')\) so

\[ Bu(t) = Bu(0) + \int_0^t (Bu)'(s) \, ds \text{ in } V' \]

the map \(u \mapsto Bu(t)\) is continuous as a map from \(X\) to \(V'\). Also, if \(Y\) denotes those \(f \in L^{p'}([0,T],V)\) for which \(f' \in L^{p'}([0,T],V)\), so that \(f\) has a representative such that \(f(t) = f(0) + \int_0^t f'(s) \, ds\), then if \(\|f\|_Y \equiv \|f\|_{L^{p'}([0,T],V)} + \|f'\|_{L^{p'}([0,T],V')}\), the map \(f \mapsto f(t)\) is continuous.

Also one can obtain the following for \(p > 1\).

Proposition 75.3.7 Let

\[ X = \left\{ u \in L^p(0,T,V) \equiv V : Lu \equiv (Bu)' \in L^{p'}(0,T,V') \right\} \]

where \(V\) is a reflexive Banach space. Let a norm on \(X\) be given by

\[ \|u\|_X = \max(\|u\|_V, \|Lu\|_{V'}) \]

Then there is a continuous function \(t \mapsto \langle Bu, v \rangle(t)\) such that

\[ \langle Bu, v \rangle(t) = \langle B(u(t)), v(t) \rangle \]

a.e. \(t\) such that

\[ \sup_{t \in [0,T]} |\langle Bu, v \rangle(t)| \leq C \|u\|_X \|v\|_X \]

and if \(K : X \to X'\)

\[ \langle Ku, v \rangle \equiv \int_0^T \langle Lu, v \rangle \, ds + \langle Bu, v \rangle(0) \]

Then \(K\) is continuous and linear and

\[ \langle Ku, u \rangle = \frac{1}{2}[\langle Bu, u \rangle(T) + \langle Bu, u \rangle(0)] \]

If \(u \in X\) and \(Bu(0) = 0\) then there exists a sequence \(\{u_n\}\) such that \(\|u_n - u\|_X \to 0\) but \(u_n(t) = 0\) for all \(t\) close to \(0\).

### 75.4 Measurable Approximate Solutions

The main result in this section is the following theorem. Its proof follows a method due to Brezis and Lions [82] adapted to the case considered here where the operator is set valued. In this theorem, we let \(F : V \to V'\) be the duality map \(\langle Fu, u \rangle = \|u\|^p, \|Fu\| = \|u\|^{p-1}\) for \(p > 1\).
As above, \( Lu = (Bu)' \). In addition to this, define \( \Lambda \) to be the restriction of \( L \) to those \( u \in X \) which have \( Bu(0) = 0 \). Thus

\[
D(\Lambda) = \{ u \in X : Bu(0) = 0 \}
\]

Then one can show that \( \Lambda^* \) is monotone. It is not hard to see that this should be the case. Let \( v \in D(\Lambda^*) \) and suppose it is smooth. Then

\[
\int_0^T \langle \Lambda u, v \rangle dt = \langle Bu(T), v(T) \rangle - \int_0^T \langle Bu', v \rangle dt
\]

and so, if \( \int_0^T \langle \Lambda u, v \rangle dt \leq C \| u \|_V \), then we need to have \( v(T) = 0 \) and \( \Lambda^* v = -Bv' \). Now it is just a matter of doing the computations to verify that

\[
\langle \Lambda^* v, v \rangle \geq 0.
\]

**Lemma 75.4.1** Let \( K \) and \( L \) be as in Proposition 75.3.7. Then for each \( f \in V' \) and \( u_0 \in W \), there exists a unique \( u \in X \) such that

\[
\langle Ku, v \rangle + Fu = \langle f, v \rangle + \langle Bu(0), u_0 \rangle
\]

for all \( v \in X \). Also, the mapping which takes \( (f, u_0) \) to this solution is demicontinuous in the sense that if \( f_n \to f \) strongly in \( V' \) and \( u_0 \to u_0 \) in \( W \), then \( u_n \to u \) weakly in \( V \).

**Proof:** Let \( J^{-1} \) be the duality map mentioned above and define \( H_\varepsilon : X \to X' \) by

\[
\langle H_\varepsilon (u), v \rangle = \varepsilon \langle Lv, J^{-1} Lu \rangle + \langle Fu, v \rangle + \langle Ku, v \rangle
\]

for all \( v \in X \). Then \( H_\varepsilon \) is pseudo monotone because it is monotone, bounded, and hemicontinuous. This follows from Theorem 75.3.4 and 75.3.5. It is also easy to see that \( H_\varepsilon \) is coercive.

\[
\frac{\langle H_\varepsilon (u), u \rangle}{\| u \|_X} = \varepsilon \frac{\| Lu \|_V^2}{\| u \|_X} + \frac{\| u \|_V^2}{\| u \|_X} + \frac{1}{2} \left[ \langle Bu(T), u(T) \rangle + \langle Bu, u_0 \rangle \right] \frac{1}{\| u \|_X}
\]

If not, then there is \( \| u_n \|_X \to \infty \) but for some \( M \),

\[
\varepsilon \frac{\| Lu_n \|_V^2}{\| u_n \|_X} + \frac{\| u_n \|_V^2}{\| u_n \|_X} + \frac{1}{2} \left[ \langle Bu_n(T), u_n(T) \rangle + \langle Bu_n, u_0 \rangle \right] \frac{1}{\| u_n \|_X} \leq M
\]

Then one of \( \| u_n \|_V \) or \( \| Lu_n \|_V \) is unbounded. Either way, a contradiction is obtained. Thus \( H_\varepsilon \) is coercive bounded, and pseudomonotone. It follows that it maps onto \( X' \).

There exists \( u_\varepsilon \in X \) such that for all \( v \in X \),

\[
\varepsilon \langle Lv, J^{-1} Lu_\varepsilon \rangle + \langle Fu_\varepsilon, v \rangle + \langle Ku_\varepsilon, v \rangle = \langle f, v \rangle + \langle Bu(0), u_0 \rangle.
\]
75.4. MEASURABLE APPROXIMATE SOLUTIONS

In (75.4.11), let $v = \varepsilon$. Using the inequality,

$$\langle \langle Bv(0), u_0 \rangle \rangle \leq \langle \langle Bv, v \rangle \rangle^{1/2} \langle \langle Bu_0, u_0 \rangle \rangle^{1/2} \leq \frac{1}{2} \langle \langle Bv, v \rangle \rangle^{1/2} + \frac{1}{2} \langle \langle Bu_0, u_0 \rangle \rangle^{1/2},$$

it follows that

$$\langle Fu_\varepsilon, u_\varepsilon \rangle + \frac{1}{2} \langle \langle Bu_\varepsilon, u_\varepsilon \rangle \rangle (T) + \langle Bu_\varepsilon, u_\varepsilon \rangle (0) \leq \|f\|_V \mathcal{V} (u_\varepsilon) + \frac{1}{2} \langle \langle Bu_0, u_0 \rangle \rangle + \langle Bu_\varepsilon, u_\varepsilon \rangle (0) + \frac{1}{2} \langle \langle Bu_0, u_0 \rangle \rangle.$$

Thus,

$$\|u_\varepsilon\|_V + \frac{1}{2} \langle \langle Bu_\varepsilon, u_\varepsilon \rangle \rangle (T) \leq \frac{1}{2} \langle \langle Bu_0, u_0 \rangle \rangle + \|f\|_V \|u_\varepsilon\|_V,$$

which implies that there exists a constant $C$ independent of $\varepsilon$ such that

$$\|u_\varepsilon\|_V \leq C. \quad (75.4.11)$$

Now let $v \in D(\Lambda)$. Thus $v \in X$ and $Bv(0) = 0$ so the last term of (75.4.11) equals 0. The term, $\langle Bu_\varepsilon, v \rangle (0)$ found in the definition of $\langle Ku_\varepsilon, v \rangle$ also equals 0. This follows from

$$\langle Bu_\varepsilon, v \rangle (0) = \lim_{n \to \infty} \langle Bu_\varepsilon, v_n \rangle (0) = 0,$$

where $v_n = 0$ near 0 and converges to $v$ in $X$ by Proposition (75.4.11). Therefore, for $v \in D(\Lambda)$, a dense subset of $\mathcal{V}$,

$$\varepsilon \langle \Lambda v, J^{-1} Lu_\varepsilon \rangle + \langle Fu_\varepsilon, v \rangle + \langle Lu_\varepsilon, v \rangle = \langle f, v \rangle.$$

It follows that $J^{-1} Lu_\varepsilon \in D(\Lambda^*)$ and so for all $v \in D(\Lambda)$,

$$\varepsilon \langle \Lambda^* J^{-1} Lu_\varepsilon, v \rangle + \langle Fu_\varepsilon, v \rangle + \langle Lu_\varepsilon, v \rangle = \langle f, v \rangle. \quad (75.4.12)$$

Since $D(\Lambda)$ is dense in $\mathcal{V}$, this equation holds for all $v \in \mathcal{V}$ and so in particular, it holds for $v = J^{-1} Lu_\varepsilon$. Therefore,

$$- \|Fu_\varepsilon\|_{\mathcal{V}'} \|Lu_\varepsilon\|_{\mathcal{V}'} + \|Lu_\varepsilon\|_{\mathcal{V}'}^2 \leq \|f\|_{\mathcal{V}'} \|Lu_\varepsilon\|_{\mathcal{V}'} \quad (75.4.13)$$

It follows from (75.4.12) that $\|Lu_\varepsilon\|_{\mathcal{V}'}$ is bounded independent of $\varepsilon$. Therefore, there exists a sequence $\varepsilon \to 0$ such that

$$u_\varepsilon \to u \text{ in } \mathcal{V}, \quad (75.4.14)$$

$$Ku_\varepsilon \to Ku \text{ in } X', \quad (75.4.15)$$

$$Fu_\varepsilon \to u^* \text{ in } \mathcal{V}', \quad (75.4.16)$$

$$Bu_\varepsilon (0) \to Bu (0) \text{ in } W'. \quad (75.4.17)$$
In (75.4.10) replace \( v \) with \( u_x - u \). Using \( J^{-1} \) is monotone,

\[
\varepsilon (L u_x - L u, J^{-1} L u) + (F u_x + Ku_x, u_x - u) \\
\leq (f, u_x - u) + (B (u_x - u) (0), u_0)
\]

(75.4.18)

Formula (75.4.17) applied to the last term of (75.4.18) implies

\[
\limsup_{\varepsilon \to 0} (F u_x + Ku_x, u_x - u) \leq 0.
\]

(75.4.19)

By pseudomonotonicity,

\[
\liminf_{\varepsilon \to 0} (F u_x + Ku_x, u_x - u) \geq (F u + Ku, u - v)
\]

so \( \lim_{\varepsilon \to 0} (F u_x + Ku_x, u_x - u) = 0 \) and so

\[
\langle u^* + Ku, u - v \rangle
\]

\[
\liminf_{\varepsilon \to 0} (\langle F u_x + Ku_x, u_x - u \rangle + \langle F u_x + Ku_x, u - v \rangle) =
\]

\[
\liminf_{\varepsilon \to 0} \langle F u_x + Ku_x, u_x - v \rangle \geq \langle F u + Ku, u - v \rangle
\]

and so \( u^* = F u \) and from (75.4.10),

\[
\langle Ku, v \rangle + \langle F u, v \rangle = \langle f, v \rangle + \langle B (0), u_0 \rangle
\]

(75.4.20)

Thus for every \( v \in X \),

\[
\int_0^T \langle (B u)' , v \rangle ds + \langle Bu, v \rangle (0) + \int_0^T \langle F u, v \rangle ds = \int_0^T \langle f, v \rangle ds + \langle B (0), u_0 \rangle
\]

So let \( v \) be smooth and equal to 0 except for \( t \in [0, \delta] \) and equals \( v_0 \) at 0. Then as \( \delta \to 0 \), the integrals become increasingly small and so

\[
\langle Bu (0), v_0 \rangle = \langle Bv_0, u_0 \rangle = \langle Bu_0, v_0 \rangle
\]

and since \( v_0 \) is arbitrary in \( V \), then it follows that \( Bu (0) = Bu_0 \). Thus this has provided a solution \( u \) to the system

\[
(Bu)' + Fu = f, \quad Bu (0) = Bu_0, \quad u \in X
\]

It remains to consider the assertion about continuity. First note that the solution to the above initial value problem is unique due to the strict monotonicity of \( F \). In fact, if there are two solutions, \( u, w \), then

\[
\frac{1}{2} \| Bu (t) - Bw (t) \|^2_W + \int_0^t \langle F u - F w, u - w \rangle ds = 0
\]

and so, in particular, \( \langle F u - F w, u - w \rangle_{\mathcal{V}', \mathcal{V}} = 0 \) which implies \( u = w \) in \( \mathcal{V} \).
Let \( u \) be the solution which goes with \((f, u_0)\) and let \( u_n \) denote the solution which goes with \((f_n, u_{0n})\) where it is assumed that \( f_n \to f \) in \( V' \) and \( u_{0n} \to u_0 \) in \( W \). It is desired to show that \( u_n \to u \) weakly in \( V \). First note that the \( u_n \) are bounded in \( V \) because
\[
\frac{1}{2} \langle Bu_n, u_n \rangle (T) - \frac{1}{2} \langle Bu_{0n}, u_{0n} \rangle + \int_0^T \|u_n\|^p_V ds = \int_0^T \langle f_n, u_n \rangle ds \leq \|f_n\|_{V'} \|u_n\|_V
\]
and this clearly implies that \( \|u_n\|_V \) is indeed bounded. Thus if this sequence fails to converge weakly to \( u \), it must be the case that there is a subsequence, still denoted as \( u_n \) which converges weakly to \( w \neq u \) in \( V \). Then by the fact that \( F \) is bounded, there is an estimate of the form
\[
\|u_n\|_V + \|Lu_n\|_{V'} \leq C
\]
Thus, a further subsequence satisfies
\[
u_n \to w \text{ weakly in } V
\]
\[
Lu_n \to Lw \text{ weakly in } V',
\]
\[
F u_n \to \xi \text{ weakly in } V'
\]
then
\[
\int_0^T \langle (B(u_n - w))', u_n - w \rangle dt = \frac{1}{2} \langle B(u_n - w), (u_n - w) \rangle (T) - \frac{1}{2} \langle B(u_n - w), (u_n - w) \rangle (0) \geq -\frac{1}{2} \langle B(u_{0n} - u_0), u_{0n} - u_0 \rangle
\]
It follows
\[
\langle Lw, u_n - w \rangle_V + \langle Fu_n, u_n - w \rangle_{V'} - \frac{1}{2} \langle B(u_{0n} - u_0), u_{0n} - u_0 \rangle \leq \langle f_n, u_n - w \rangle
\]
and so \( \limsup_{n \to \infty} \langle Fu_n, u_n - w \rangle \leq 0 \). Then as before, \( \xi = Fw \) and one obtains
\[
(Bw)' + Fw = f, \quad Bw(0) = Bu_0, \ w \in X
\]
contradicting uniqueness. Hence \( u_n \to u \) weakly as claimed. \( \blacksquare \)

Now suppose \((\Omega, \mathcal{F})\) is a measurable space and \( B = B(\omega) \) and is measurable into \( \mathcal{L}(W, W') \) and \( f : [0, T] \times \Omega \to V' \) is product measurable, \( \mathcal{B}([0, T]) \times \mathcal{F} \) measurable where \( \mathcal{B}([0, T]) \) denotes the Borel sets. Also, it is assumed that for each \( \omega, f(\cdot, \omega) \in V' \). The following lemma ties together these ideas. It is Lemma 75.4.1 proved above. It is stated here for convenience.

**Lemma 75.4.2** Let \( f(\cdot, \omega) \in V' \). Then if \( \omega \to f(\cdot, \omega) \) is measurable into \( V' \), it follows that for each \( \omega \), there exists a representative \( \tilde{f}(\cdot, \omega) \in V', \tilde{f}(\cdot, \omega) = f(\cdot, \omega) \) in \( V' \) such that \( (t, \omega) \to \tilde{f}(t, \omega) \) is product measurable. If \( f(\cdot, \omega) \in V' \) and \( (t, \omega) \to f(t, \omega) \) is product measurable, then \( \omega \to f(\cdot, \omega) \) is measurable into \( V' \). The same holds replacing \( V' \) with \( \mathcal{V} \).
Now consider the initial value problem
\[
\begin{align*}
(B (\omega) u (\cdot, \omega))' + F u (\cdot, \omega) &= f (\cdot, \omega), \\
B (\omega) u (0, \omega) &= B (\omega) u_0 (\omega), \ u (\cdot, \omega) \in X
\end{align*}
\]
where we also assume \( u_0 \) is \( \mathcal{F} \) measurable into \( W \). From Lemma 75.4.2, \( \omega \mapsto (f (\cdot, \omega), u_0 (\omega)) \) is measurable into \( V' \times W \). That is,
\[
(f, u_0)^{-1} (U) = \{ \omega : (f (\cdot, \omega), u_0 (\omega)) \in U \} \in \mathcal{F}
\]
for \( U \) an open set in \( V' \times W \). From Lemma 75.4.2, the map \( \Phi_\omega \) which takes \( (f, u_0) \) to the solution \( u \) is demicontinuous. We desire to argue that \( u \) is measurable into \( V \). In doing so, it is easiest to assume that \( B \) does not depend on \( \omega \). However, the dependence on \( \omega \) can be included using the approximation assumption for \( B (\omega) \) mentioned earlier.

Letting \( f_n (\cdot, \omega) \to f (\cdot, \omega) \) where \( f_n \) is a simple function and \( u_{0n} (\omega) \to u_0 (\omega) \) where \( u_{0n} \) is also a simple function, it follows that
\[
\Phi_\omega (f_n (\cdot, \omega), u_{0n} (\omega)) \to \Phi_\omega (f (\cdot, \omega), u_0 (\omega)) = u
\]
weakly.

**Lemma 75.4.3** Suppose \( f (\cdot, \omega) \in V' \) for each \( \omega \) and that \( (t, \omega) \to f (t, \omega) \) is product measurable into \( V' \). Also \( u_0 \) is \( \mathcal{F} \) measurable into \( W \) and
\[
B (\omega) = k (\omega) B, \ k (\omega) \geq 0, \ k \text{ measurable}
\]
Then for each \( \omega \in \Omega \), there exists a unique solution \( u (\cdot, \omega) \) in \( V \) satisfying
\[
\begin{align*}
(B (\omega) u (\cdot, \omega))' + F u (\cdot, \omega) &= f (\cdot, \omega), \\
B (\omega) u (0, \omega) &= B (\omega) u_0 (\omega), \ u (\cdot, \omega) \in X
\end{align*}
\]
This solution has a representative which satisfies \( (t, \omega) \to u (t, \omega) \) is product measurable into \( V \).

**Proof:** Let \( B_n (\omega) \equiv k_n (\omega) B \) where \( \{ k_n (\omega) \} \) is an increasing sequence of simple functions converging pointwise to \( k (\omega) \). Replace \( B (\omega) \) with \( B_n (\omega) \). Then define
\[
\langle K_n u, v \rangle \equiv \int_0^T \langle L_n u, v \rangle \ dt + \langle Bu, v \rangle (0)
\]
where \( L_n \) is defined as
\[
L_n u = (B_n (\omega) u)'
\]
for \( B_n \) having values in \( \mathcal{L} (W, W') \) such that \( B_n (\omega) \to B (\omega) \) and each of these is self adjoint and nonnegative. Let \( u_n \) be the solution to the above initial value problem
\[
\langle K_n u_n, v \rangle + F u_n = \langle f_n, v \rangle + \langle B v (0), u_0 \rangle
\]
in which \( u_{0n} \) and \( f_n \) are simple functions converging to \( u_0 \) and \( f \) in \( W \) and \( \mathcal{V}' \) respectively for each \( \omega \). Thus these have constant values in \( \mathcal{V}' \) or \( W \) on finitely many measurable subsets of \( \Omega \). Since \( B_n \) is constant on measurable sets, it follows that \( u_n (\cdot, \omega) \) is also a constant element of \( \mathcal{V} \) on each of finitely many measurable sets. Hence \( u_n (\cdot, \omega) \) is measurable into \( \mathcal{V} \). Then fixing \( \omega \), and letting \( v = u_n \),

\[
\frac{1}{2} [ \langle Bu_n, u_n \rangle (T) + \langle Bu_n, u_n \rangle (0) ] + \int_0^T ||u_n||_{\mathcal{V}}^2 \, ds = \int_0^T (f_n, u_n) \, ds + \langle Bv(0), u_{0n} \rangle
\]

Thus, since \( F \) is bounded, one obtains an inequality of the form

\[
||u_n||_{\mathcal{V}} + \|(B_n (\omega) u_n)'\|_{\mathcal{V}'} \leq C
\]

Then there is a subsequence such that

\[
u_n \to u \text{ weakly in } \mathcal{V}
\]

\[
B_n (\omega) u_n \to B (\omega) u \text{ weak * in } L^\infty ([0, T]; \mathcal{V}')
\]

\[
B_n (\omega) u_n \to B (\omega) u \text{ weakly in } \mathcal{V}'
\]

\[
Fu_n \to \xi \text{ weakly in } \mathcal{V}'
\]

\[
(B_n (\omega) u_n)' \to (B (\omega) u)' \text{ weakly in } \mathcal{V}'
\]

Also, suppressing the dependence on \( \omega \),

\[
(B_n u_n) (t) = B_n u_{0n} + \int_0^t (B_n u)' (s) \, ds
\]

and so in fact,

\[
(B_n u_n) (t) \to (Bu) (t) \text{ in } \mathcal{V}' \text{ for each } t
\]

Also,

\[
\langle (B_n u_n)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}} + \langle Fu_n, u_n - u \rangle_{\mathcal{V}', \mathcal{V}} = \langle f_n, u_n - u \rangle_{\mathcal{V}', \mathcal{V}}
\]

\[
\langle k_n (\omega) (Bu_n)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}} + \langle Fu_n, u_n - u \rangle_{\mathcal{V}', \mathcal{V}} = \langle f_n, u_n - u \rangle_{\mathcal{V}', \mathcal{V}}
\]

Thus, by monotonicity,

\[
\langle k_n (\omega) (Bu_n)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}} = \langle k (\omega) (Bu_n)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}}
\]

\[
+ \langle (k_n (\omega) - k (\omega)) (Bu_n)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}}
\]

\[
\geq \langle (k (\omega))(Bu)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}} + \langle (k_n (\omega) - k (\omega)) (Bu_n)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}}
\]

The last term in the above expression converges to 0 due to the convergence of \( k_n (\omega) \) to \( k (\omega) \). Thus

\[
\langle (B (\omega) u)', u_n - u \rangle_{\mathcal{V}', \mathcal{V}} + \langle Fu_n, u_n - u \rangle_{\mathcal{V}', \mathcal{V}} \leq \langle f_n, u_n - u \rangle_{\mathcal{V}', \mathcal{V}}
\]
and so
\[ \limsup_{n \to \infty} \langle Fu_n, u_n - u \rangle_{V', V} \leq 0 \]

Then as before, one can conclude that \( Fu = \xi \). Then passing to the limit gives the desired solution to the equation, this for each \( \omega \). However, by uniqueness, it follows that if \( \bar{u} \) is the solution to the evolution equation of Lemma 75.4.1 then for each \( \omega, u = \bar{u} \) in \( V \). Also this \( u \) just obtained is measurable into \( V \) thanks to the Pettis theorem. Therefore, \( \bar{u} \) can be modified on a set of measure zero for each fixed \( \omega \) to equal \( u \) a function measurable into \( V \). Hence there exists a solution to the evolution equation of this lemma \( u \) which is measurable into \( V \). By the Lemma 75.4.2, it follows that there is a representative for \( u \) which is product measurable into \( V \).

### 75.5 The Main Result

The main result is an existence theorem for product measurable solutions to the system
\[
\begin{align*}
(B(\omega)(\cdot, \omega))' + u^*(\cdot, \omega) &= f(\cdot, \omega) \text{ in } V' \\
B(\omega)u(0, \omega) &= B(\omega)u_0(\omega)
\end{align*}
\]
(75.5.22)

where \( u^*(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \). It is Theorem 75.5.6 below. First are some assumptions.

Here \( I \) will denote a subinterval of \([0, T]\), of the form \( I = [0, \hat{T}] \), \( \hat{T} \leq T \), and \( V_I \equiv L^p(I, V) \) with similar things defined analogously. We assume only that \( p > 1 \).

**Definition 75.5.1** For \( X \) a reflexive Banach space, we say \( A : X \to \mathcal{P}(X') \) is pseudomonotone and bounded if the following hold.

1. The set \( Au \) is nonempty, closed and convex for all \( u \in X \). \( A \) takes bounded sets to bounded sets.
2. If \( u_i \to u \) weakly in \( X \) and \( u_i^* \in Au_i \) is such that
   \[ \limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle \leq 0, \]  
   (75.5.23)
   then, for each \( v \in X \), there exists \( u^*(v) \in Au \) such that
   \[ \liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle \geq \langle u^*(v), u - v \rangle. \]
   (75.5.24)

Now suppose the following for the operator \( A(\cdot, \omega) \). \( A(\cdot, \omega) : V_I \to V_I' \) for each \( I \) a subinterval of \([0, T]\) and
\[ A(\cdot, \omega) : V_I \to \mathcal{P}(V_I') \text{ is bounded}, \]
(75.5.25)
75.5. **THE MAIN RESULT**

If, for $u \in \mathcal{V}$,

$$u^* \mathcal{X}_{[0,T]} \in A \left( \mathcal{X}_{[0,T]} \mid \omega \right)$$

for each $\hat{T}$ in an increasing sequence converging to $T$, then

$$u^* \in A (u, \omega) \tag{75.5.26}$$

Assume the specific estimate

$$\sup \left\{ \| u^* \|_{\mathcal{V}_t} : u^* \in A (u, \omega) \right\} \leq a (\omega) + b (\omega) \| u \|_{\mathcal{V}_t}^{-1} \tag{75.5.27}$$

where $a (\omega), b (\omega)$ are nonnegative. Also assume the coercivity condition:

$$\lim_{\| u \|_{\mathcal{V}} \to \infty} \frac{\inf_{u \in \mathcal{X}_r} \left\{ 2 \langle u^*, u \rangle_{\mathcal{V}_t} + \langle Bu, u \rangle (T) : u^* \in A (u, \omega) \right\}}{\| u \|_{\mathcal{V}}} = \infty, \tag{75.5.28}$$

or alternatively the following specific estimate valid for each $t \leq T$ and for some $\lambda (\omega) \geq 0$,

$$\inf \left( \int_{0}^{t} \langle u^*, u \rangle + \lambda (\omega) \langle Bu, u \rangle \, dt : u^* \in A (u, \omega) \right) \geq \delta (\omega) \int_{0}^{t} \| u \|_{\mathcal{V}_t}^p \, ds - m (\omega) \tag{75.5.29}$$

where $m (\omega)$ is some nonnegative constant, $\delta (\omega) > 0$. Note that the estimate is a coercivity condition on $\lambda B + A$ rather than on $A$ but is more specific than 75.5.28.

Let $U$ be a Banach space dense in $\mathcal{V}$ and that if $u_i \to u$ in $\mathcal{V}_t$ and $u_i^* \in A (u_i)$ with $u_i^* \rightharpoonup u^*$ in $\mathcal{V}_t$ and $(Bu_i)' \to (Bu)'$ in $\mathcal{U}_I'$, $\rightharpoonup$ denoting weak convergence, then if

$$\lim_{i \to \infty} \sup_{t} \langle u_i^*, u_i - u \rangle_{\mathcal{V}_t, \mathcal{V}_t} \leq 0$$

it follows that for all $v \in \mathcal{V}_t$, there exists $u^* (v) \in Au$ such that

$$\lim_{i \to \infty} \inf \langle u_i^*, u_i - v \rangle_{\mathcal{V}_t, \mathcal{V}_t} \geq \langle u^* (v), u - v \rangle_{\mathcal{V}_t, \mathcal{V}_t} \tag{75.5.30}$$

where $r > \max (p, 2)$, and we replace $p$ with $r$ and $I$ an arbitrary subinterval of the form $[0, T], T < T$, for $[0, T]$, and $U$ for $\mathcal{V}$ where indicated. Here

$$\mathcal{U}_I \equiv L^r (I; U)$$

Note that we are not assuming $A$ is pseudomonotone, just that it satisfies a similar limit condition. Typically, this limit condition holds because of a use of the compact embedding of theorem 69.3.4 or similar result and it does not matter whether $U$ is a small subset of $V$ as long as it is dense in $V$.

Here is an alternate limit condition. Let $U$ be a Banach space dense in $\mathcal{V}$ and that if $u_i \to u$ in $\mathcal{V}_t$ and $u_i^* \in A (u_i)$ with $u_i^* \rightharpoonup u^*$ in $\mathcal{V}_t$ and $t \to Bu_i (t)$ is continuous and

$$\sup_{i} \sup_{t \neq s} \frac{\| Bu_i (t) - Bu_i (s) \|_{\mathcal{U}_t}}{|t - s|^{\alpha}} \leq C \tag{75.5.31}$$
then if
\[ \limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{V'_i, V_i} \leq 0 \]  
(75.5.32)
it follows that for all \( v \in V_i \), there exists \( u^*(v) \in Au \) such that
\[ \liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle_{V'_i, V_i} \geq \langle u^*(v), u - v \rangle_{V'_i, V_i} \]  
(75.5.33)
This alternate condition is implied by (75.5.30) but the conditions under which either condition holds are likely to depend on some sort of compactness which will be useable for either limit condition. Technically if you assume this alternate condition, you are assuming more, but I don’t have any examples to show that it would be actually assuming more.

For \( \omega \to u(\cdot, \omega) \) measurable into \( V \),
\[ \omega \to A(u(\cdot, \omega), \omega) \text{ has a measurable selection into } V'. \]  
(75.5.34)
This last condition means there is a function \( \omega \to u^*(\omega) \) which is measurable into \( V' \) such that \( u^*(\omega) \in A(u(\cdot, \omega), \omega) \). This is assured to take place if the following standard measurability condition is satisfied for all \( O \) open in \( V' \):
\[ \{ \omega : A(u(\cdot, \omega), \omega) \cap O \neq \emptyset \} \in \mathcal{F} \]  
(75.5.35)
See for example [II], [I]. Our assumption is implied by this one but they are not equivalent. Thus what is considered here is more general than an assumption that \( \omega \to A(u(\cdot, \omega), \omega) \) is set valued measurable.

Note that this condition would hold if \( u \to A(t, u, \omega) \) is bounded and pseudomonotone as a single valued map from \( V \) to \( V' \) and \( (t, \omega) \to A(t, u, \omega) \) is measurable into \( V' \). One would use the demicontinuity of \( u \to A(\cdot, u, \omega) \) which comes from the pseudo monotone and bounded assumption and consider a sequence of simple functions \( u_n(t, \omega) \to u(t, \omega) \) in \( V \) for \( u \) measurable, each \( u(\cdot, \omega) \) being in \( V \), Then the measurability of \( A(t, u_n, \omega) \) would attach to \( A(t, u, \omega) \) in the limit. More generally, here is a useful lemma. It is about preserving the existence of a measurable representative under the assumption that the values are closed and convex.

**Lemma 75.5.2** Suppose \( \omega \to A(u, \omega) \) has a measurable selection in \( V' \) for \( u \) a given element of \( V \) not dependent on \( \omega \) and for each \( \omega \), \( A(u, \omega) \) is a closed bounded, convex set in \( V' \). Also suppose \( u \to A(u, \omega) \) is upper semicontinuous from the strong topology of \( V \) to the weak topology of \( V' \). That is, if \( u_n \to u \) in \( V \) strongly, then if \( O \) is a weakly open set containing \( A(u, \omega) \), it follows that \( A(u_n, \omega) \in O \) for all \( n \) large enough. Then whenever \( u \) is measurable into \( V \), it follows that there is a measurable selection for \( \omega \to A(u(\omega), \omega) \) into \( V' \).

**Proof:** Let \( \omega \to u(\omega) \) be measurable into \( V \) and let \( u_n(\omega) \to u(\omega) \) in \( V \) where \( u_n \) is a simple function
\[ u_n(\omega) = \sum_{k=1}^{m_n} c_k^n A E_k^\omega \ (\omega), \text{ the } E_k^\omega \text{ disjoint, } \Omega = \cup_k E_k^\omega, \]
The Main Result

75.5. THE MAIN RESULT

each $c_k^n$ being in $V$. Then by assumption, there is a measurable selection for $\omega \to A(c_k^n, \omega)$ denoted as $\omega \to y_k^n(\omega)$. Thus $\omega \to y_k^n(\omega)$ is measurable into $V'$ and $y_k^n(\omega) \in A(c_k^n, \omega)$ for all $\omega \in \Omega$. Then consider

$$y^n(\omega) = \sum_{k=1}^{m_n} y_k^n(\omega) X_{E_k^n}(\omega)$$

It is measurable and for $\omega \in E_k^n$ it equals $y_k^n(\omega) \in A(c_k^n, \omega) = A(u_n(\omega), \omega)$. Thus $y^n$ is a measurable selection of $\omega \to A(u_n(\omega), \omega)$. By the estimates, for each $\omega$ these $y^n(\omega)$ lie in a bounded subset of $V'$. The bound might depend on $\omega$ of course. By Theorem 75.5.2 and Lemma 46.3.1 there is a measurable into $V'$ function $\omega \to y(\omega)$ and a subsequence for each $\omega$, $y^{n(\omega)}(\omega)$ such that $y^{n(\omega)}(\omega) \to y(\omega)$ weakly in $V'$. By the Pettis theorem, $y$ is measurable into $V'$. Where is $y(\omega)$? If $y(\omega) \notin A(u(\omega), \omega)$, then there would exist $z(\omega) \in V$ such that $\langle y(\omega), z(\omega) \rangle > r > \langle w, z \rangle$ for all $w \in A(u(\omega), \omega)$. Let $O = \{w \in V' \text{ such that } r > \langle w, z \rangle \}$. Then $O$ contains $A(u(\omega), \omega)$ and is a weakly open set. It follows from the upper semicontinuity assumption that $y^{n(\omega)}(\omega) \in O$ for all $n(\omega)$ large enough. Thus $r > \langle y^{n(\omega)}(\omega), z \rangle$. But by weak convergence,

$$\lim_{n(\omega) \to \infty} \langle y^{n(\omega)}(\omega), z \rangle = \langle y(\omega), z \rangle$$

contradicting $y(\omega) \notin A(u(\omega), \omega)$. Hence $y(\omega) \in A(u(\omega), \omega)$ and $\omega \to y(\omega)$ is a measurable selection. $\blacksquare$

In fact, this is just a special case of a general result in the next theorem. It says essentially that having a measurable selection is preserved when going from constant to measurable functions. In this theorem, $V$ is a reflexive separable Banach space. This is difficult to show for measurable multifunctions. It is Theorem 86.3.1 and was proved earlier. A proof is given here also.

Theorem 75.5.3 Suppose $\omega \to A(u, \omega)$ has a measurable selection in $V'$ for $u$ a given element of $V$ not dependent on $\omega$ and for each $\omega$, $A(u, \omega)$ is a closed, convex set in $V'$ and $A(\cdot, \omega)$ is bounded. Also suppose $\omega \to A(u, \omega)$ is upper semicontinuous from the strong topology of $V$ to the weak topology of $V'$. That is, if $u_n \to u$ in $V$ strongly, then if $O$ is a weakly open set containing $A(u, \omega)$, it follows that $A(u_n, \omega) \in O$ for all $n$ large enough. Then whenever $\omega \to A(u, \omega)$ is measurable into $V$, it follows that there is a measurable selection for $\omega \to A(u(\omega), \omega)$ into $V'$.

Proof: Let $\omega \to u(\omega)$ be measurable into $V$ and let $u_n(\omega) \to u(\omega)$ in $V$ where $u_n$ is a simple function

$$u_n(\omega) = \sum_{k=1}^{m_n} c_k^n X_{E_k^n}(\omega), \text{ the } E_k^n \text{ disjoint, } \Omega = \cup_k E_k^n,$$

each $c_k^n$ being in $V$. We can assume that $\|u_n(\omega)\| \leq 2 \|u(\omega)\|$ for all $\omega$. Then by assumption, there is a measurable selection for $\omega \to A(c_k^n, \omega)$ denoted as $\omega \to \omega \to y_k^n(\omega)$.
Thus $\omega \rightarrow y^n_k(\omega)$ is measurable into $V'$ and $y^n_k(\omega) \in A(c^n_k, \omega)$ for all $\omega \in \Omega$. Then consider

$$y^n(\omega) = \sum_{k=1}^{m_n} y^n_k(\omega) X_{E^n_k}(\omega)$$

It is measurable and for $\omega \in E^n_k$ it equals $y^n_k(\omega) \in A(c^n_k, \omega) = A(u_n(\omega), \omega)$. Thus $y^n$ is a measurable selection of $\omega \rightarrow A(u_n(\omega), \omega)$. By the estimates, for each $\omega$ these $y^n(\omega)$ lie in a bounded subset of $V'$. The bound might depend on $\omega$ of course.

Now let $\{z_i\}$ be a countable dense subset of $V$. Then let $X = \prod_{i=1}^{\infty} \mathbb{R}$. It is a Polish space. Let

$$f(y^j)(\omega) = \prod_{i=1}^{\infty} \langle y^j(\omega), z_i \rangle$$

$$\Gamma_n(\omega) = \bigcup_{k \geq n} f(y^k)(\omega),$$

the closure taken in $X$. Now $y^k(\omega) \in A(u_k(\omega), \omega)$ and so by assumption, since $\|u_k(\omega)\| \leq 2\|u(\omega)\|$ it is bounded in $V'$, this for each $\omega$.

Thus the components of $f(y^j)(\omega)$ lie in a compact subset of $\mathbb{R}$, this for each $\omega$. It follows from Tychanoff’s theorem that $\Gamma_n(\omega)$ is a compact subset of the Polish space $X$.

**Claim:** $\Gamma_n$ is measurable into $X$.

**Proof of claim:** It is necessary to show that $\Gamma_{n}^{-}(U) \equiv \{\omega : \Gamma_n(\omega) \cap U \neq \emptyset\}$ is measurable whenever $U$ is open. It suffices to verify this for $U$ a basic open set in the topology of $X$. Thus let $U = \prod_{i=1}^{\infty} O_i$ where $O_i$ is a proper open subset of $\mathbb{R}$ only for $i \in \{j_1, \cdots, j_n\}$. Then

$$\Gamma_n^{-}(U) = \bigcup_{j \geq n} \cap_{i=1}^{j} \{\omega : \langle y^j(\omega), z_i \rangle \in O_i, \}$$

which is a measurable set thanks to $y^j$ being measurable.

In addition to this, $\Gamma_n(\omega)$ is compact, as explained above. Therefore, $\Gamma_n$ is also strongly measurable meaning $\Gamma_n^{-}(F)$ is measurable for all $F$ closed. Now let $F(\omega) \equiv \cap_{n=1}^{\infty} \Gamma_n(\omega)$. It is a nonempty closed subset of $X$ and if $F$ is closed in $X$,

$$\Gamma^{-}(F) = \cap_{n=1}^{\infty} \Gamma_n^{-}(F)$$

a measurable set since each $\Gamma_n^{-}(F)$ is measurable. Thus $\Gamma$ is a measurable multifunction and so it has a measurable selection $\omega \rightarrow z(\omega)$. Thus by definition, for each $i$, $z_i(\omega) = \lim_{n(\omega) \rightarrow \infty} \langle y^n(\omega), z_i \rangle$ for some subsequence indexed by $n(\omega)$. The sequence $\{y^n(\omega)\}$ is bounded in $V'$ and so there is a subsequence still denoted as $\{y^n(\omega)\}$ which converges weakly to $y(\omega)$. Thus $z_i(\omega) = \langle y(\omega), z_i \rangle$ for each $i$. Since $\omega \rightarrow z_i(\omega)$ is measurable, it follows from density of the $\{z_i\}$ that $y$ is weakly, hence strongly measurable, this by the Pettis theorem. Now $y(\omega) = \lim_{n(\omega) \rightarrow \infty} y^n(\omega, \omega)$. But

$$y^n(\omega) \in A(u_n(\omega), \omega)$$

which is a convex closed set for which $u \rightarrow A(u, \omega)$ is upper semicontinuous and $u_n(\omega) \rightarrow u$ so in fact, $y(\omega) \in A(u(\omega), \omega)$. This is the claimed measurable selection.
75.5. THE MAIN RESULT

Note that we are not assuming that $\omega \rightarrow A(u, \omega)$ is measurable, only that it has a measurable selection and of course the upper semicontinuity and that the values are closed and convex. Also note that $\omega \rightarrow \Gamma(\omega)$ is measurable so there is a dense subset of measurable functions $\{x_k(\omega)\}$ each being measurable into $X$. However, we don’t know much about $\Gamma(\omega)$ other than it is measurable into $X$.

Also $\mathcal{V}$ could be replaced with $L^p(I, V)$ where $I$ is any interval and nothing changes.

The condition leading to 75.5.26 will typically be satisfied. For example, suppose $u^* \in A(u, \omega)$ means that

$$u^*(t) = A \left( t, u(t), \int_0^t u(s) \, ds, \omega \right)$$

for a.e. $t$, where $A$ has values in $\mathcal{P}(V')$. Then to say that $u^* \mathcal{K}_{[0, \tilde{T}]} \in A \left( u \mathcal{K}_{[0, \tilde{T}]} , \omega \right)$ for each $\tilde{T}$ in an increasing sequence converging to $T$ would imply the above holding for a.e. $t$. While the above is the typical situation one would expect to see, the following proposition is also interesting.

**Proposition 75.5.4** Suppose $A(\cdot, \omega) : \mathcal{V} \rightarrow \mathcal{P}(V')$ is upper semicontinuous from the strong topology of $\mathcal{V}$ to the weak topology of $\mathcal{V}'$ and has closed convex values. Then if

$$u^* \mathcal{K}_{[0, \tilde{T}]} \in A \left( u \mathcal{K}_{[0, \tilde{T}]} , \omega \right)$$

for each $\tilde{T}$ in an increasing sequence converging to $T$, then

$$u^* \in A(u, \omega)$$

(75.5.36)

**Proof**: Let $\tilde{T}_n \uparrow T$ such that $u^* \mathcal{K}_{[0, \tilde{T}_n]} \in A \left( u \mathcal{K}_{[0, \tilde{T}_n]} , \omega \right)$. Then if $u^* \notin A(u, \omega)$, there exists $z \in \mathcal{V}$ such that

$$\langle u^*, z \rangle > r > \langle u^*, z \rangle$$

for all $u^* \in A(u, \omega)$. Now $u^* \mathcal{K}_{[0, \tilde{T}_n]} \rightarrow u^*$ in $V'$ and $u \mathcal{K}_{[0, \tilde{T}_n]} \rightarrow u$ in $\mathcal{V}$. Letting $O$ be the weakly open set, $\{z^*: \langle z^*, z \rangle < r \}$, it follows that this $O$ is a weakly open set which contains $A(u, \omega)$. Hence, by upper semicontinuity, $\langle u^* \mathcal{K}_{[0, \tilde{T}_n]} , z \rangle < r$ for all $n$ large enough. Hence, passing to a limit, one obtains $\langle u^*, z \rangle > r \geq \langle u^*, z \rangle$, a contradiction. Thus $u^* \in A(u, \omega)$. ■

Let $r > \max(p, 2)$. Let $\mathcal{U}$ and $\mathcal{U}_f$ be defined by analogy with $\mathcal{V}$ and $\mathcal{V}_f$ where $\mathcal{U} \equiv L^r([0, T], U)$. Here $U$ is a Hilbert space which is dense in $V$ and embedds compactly into $V$, $\|u\|_U \geq \|u\|_V$. Also let $F : U \rightarrow U'$ be the duality map for $r$. Thus

$$\|Fu\|_{V'} = \|u\|_{V'}^{r-1}, \quad \langle Fu, u \rangle = \|u\|_V^r.$$

Also define the following notation for small positive $h$.

$$\tau_h g (t) \equiv \begin{cases} g(t-h) & \text{if } t > h \\ 0 & \text{if } t \leq h \end{cases}$$
Let \( \omega \to u_0(\omega) \) be \( F \) measurable into \( W \). Also let \( \omega \to f(\cdot, \omega) \) be \( F \) measurable into \( \mathcal{V}' \), \( \omega \to B(\omega) \) measurable into \( \mathcal{L}(W, \mathcal{W}') \). Now let \( u_h \) for \( h > 0 \) and small, be the unique solution to the initial value problem

\[
(B(\omega) u_h(\cdot, \omega))' + \varepsilon F u_h(\cdot, \omega) = f(\cdot, \omega) - u_h(\cdot, \omega) \quad \text{in} \quad \mathcal{U}',
\]

\[
Bu_h(0, \omega) = Bu_0(\omega)
\]

where \( u_h^* \in A(\tau_h u, \omega) \) is a \( F \) measurable selection into \( \mathcal{V}' \). Since \( F \) is monotone bounded and hemicontinuous, there is no problem with it being pseudomonotone from \( X_r \) to \( X_r' \). Such a solution exists on \([0, h]\) by the above reasoning. Let this solution be denoted by \( u_1 \). Then use it to define a solution to the evolution equation on \([0, 2h]\) called \( u_2 \). By uniqueness, these coincide on \([0, h]\). Then use \( u_2 \) to extend to a solution on \([0, 3h]\) called \( u_3 \). Then \( u_3 = u_2 \) on \([0, 2h]\). Continue this way to obtain a solution valid on \([0, T]\). By Lemma 75.3.4, this solution may be assumed to be measurable into \( \mathcal{U}' \). One gets this by using the lemma on a succession of intervals \([0, h], [0, 2h], \) and so forth.

Now acting on \( u_k \) and suppressing the dependence on \( \omega \) in most places, it follows from the assumed estimates that

\[
\frac{1}{2} \langle Bu_h, u_h \rangle(T) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \varepsilon \int_0^T \|u_h\|_{\mathcal{V}'} ds \\
\leq \left( \int_0^T \|f\|_{\mathcal{V}'} ds \right)^{1/p'} \left( \int_0^T \|u_h\|_V^p \right)^{1/p} \\
+ \int_0^T (a + b \|\tau_h u_h\|_{\mathcal{V}'}^{p-1}) \|u_h\|_{\mathcal{V}'} ds \\
\leq \|f\|^p_{\mathcal{V}'} + \|u_h\|^p_{\mathcal{V}} + \|u_h\|^p_{\mathcal{V}'} + aT^{1/p'} + b \|u_h\|^p_{\mathcal{V}'}
\]

which is of the form

\[
\leq C \left( \|f\|^p_{\mathcal{V}'} + a \langle \omega \rangle T \right) + (2 + b) \|u_k\|^p_{\mathcal{V}}
\]

Now here is where it is good that \( p < r \).

\[
\|u_h\|^p_{\mathcal{V}'} \leq \int_0^T \frac{1}{\delta} \|u_h\|_{\mathcal{U}}^p ds \\
\leq \left( \int_0^T \delta^{-r/p} \|u_h\|_{\mathcal{U}}^r ds \right)^{p/r} \left( \int_0^T \frac{1}{\delta^{r/(r-p)}} 1^{r-r/p} \right)^{(r-p)/r} \\
\leq \frac{1}{\delta^{r/(r-p)}} T^{(r-p)/r} \frac{T^{(r-p)/r}}{r} (r-p) + \frac{p\delta^{r/p} \|u_h\|_{\mathcal{U}}^r}{r}
\]

Thus this has shown

\[
\frac{1}{2} \langle Bu_h, u_h \rangle(T) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \varepsilon \|u_h\|^r_{\mathcal{U}} \\
\leq C \left( \|f\|^p_{\mathcal{V}'} + a \langle \omega \rangle T \right) + \frac{1}{\delta^{r/(r-p)}} T^{(r-p)/r} \frac{T^{(r-p)/r}}{r} (r-p) + \frac{p\delta^{r/p} \|u_h\|_{\mathcal{U}}^r}{r}
\]
Then for \( \delta \) small enough, depending on \( \varepsilon \),
\[
\frac{p\delta^p}{r} < \frac{\varepsilon}{2}
\]
And so the inequality ending at \( 75.5.38 \) yields
\[
\langle Bu_h, u_h \rangle (T) + \varepsilon \| u_h \|_{\mathcal{U}}^2 \leq C \left( \| f \|_{\mathcal{V}'}^2, (a(\omega) + 1) T \varepsilon \right) + \langle Bu_0, u_0 \rangle
\]
From \( 75.5.37 \) and the boundedness of the various operators, \( (B(\omega) u_h (\cdot, \omega))^' \) is bounded in \( \mathcal{U}' \). Thus, summarizing these estimates yields
\[
\| (B(\omega) u_h (\cdot, \omega))^' \|_{\mathcal{U}'} + \| u_h \|_{\mathcal{U}} + \| u_h^* \|_{\mathcal{V}} \leq C \quad (75.5.39)
\]
where \( C \) does not depend on \( h \) although it does depend on \( \varepsilon \) and of course on \( \omega \). Then one can get a subsequence, still denoted with \( h \) such that as \( h \to 0 \),
\[
u_h \to u \text{ weakly in } \mathcal{U} \quad (75.5.40)
\]
\[
\tau_h u_h \to u \text{ weakly in } \mathcal{U} \quad (75.5.41)
\]
\[
(B(\omega) u_h (\cdot, \omega))^' \to (B(\omega) u)^' \text{ weakly in } \mathcal{U}' \quad (75.5.42)
\]
\[
u_h \to u \text{ strongly in } \mathcal{V} \quad (75.5.43)
\]
\[
u_h^* \to u^* \text{ weakly in } \mathcal{V}' \quad (75.5.44)
\]
\[
F u_h \to \xi \in \mathcal{U}' \quad (75.5.45)
\]
\[
Bu_0 (0, \omega) = Bu_0 (\omega) \quad (75.5.46)
\]
The fourth of these comes from a use of Theorem \[75.3.1\]. We need to argue that \( u^* \in A(u, \omega) \). From the equation and initial conditions of \( 75.5.37 \), it follows from monotonicity conditions and the observation that \( \mathcal{V}' \) is contained in \( \mathcal{U}' \) that
\[
\langle (B(\omega) u_h (\cdot, \omega))^', u_h - u \rangle + \langle \varepsilon F u_h (\cdot, \omega), u_h - u \rangle + \langle u_h^* (\cdot, \omega), u_h - u \rangle = \langle f (\cdot, \omega), u_h - u \rangle
\]
and so
\[
\langle (B(\omega) u (\cdot, \omega))^', u_h - u \rangle + \langle \varepsilon F u_h (\cdot, \omega), u_h - u \rangle_{\mathcal{U}' \mathcal{U}} + \langle u_h^* (\cdot, \omega), u_h - u \rangle_{\mathcal{V}' \mathcal{V}} \leq \langle f (\cdot, \omega), u_h - u \rangle_{\mathcal{U}' \mathcal{U}}
\]
by the strong convergence of \( 75.5.43 \), it follows that the third term converges to 0 as \( h \to 0 \). This is because the estimate \( 75.5.27 \) implies that the \( u_h^* \) are bounded, and then the strong convergence gives the desired result. Hence
\[
\limsup_{h \to 0} \langle \varepsilon F u_h (\cdot, \omega), u_h - u \rangle_{\mathcal{U}' \mathcal{U}} \leq 0
\]
and since \( F \) is monotone and hemicontinuous, it follows that in fact,
\[
\lim_{h \to 0} \langle \varepsilon F u_h (\cdot, \omega), u_h - u \rangle_{\mathcal{U}' \mathcal{U}} = 0
\]
so for $v \in \mathcal{U}$ arbitrary,
\[
\langle \varepsilon \xi, u - v \rangle = \liminf_{h \to 0} \left( \langle \varepsilon F u_h (\cdot, \omega), u - u_h \rangle_{\mathcal{U}' \mathcal{U}} + \langle \varepsilon F u_h (\cdot, \omega), u_h - v \rangle_{\mathcal{U}' \mathcal{U}} \right)
= \liminf_{h \to 0} \langle \varepsilon F u_h (\cdot, \omega), u_h - v \rangle_{\mathcal{U}' \mathcal{U}} \geq \langle \varepsilon F u, u - v \rangle
\]
and so, since $v$ is an arbitrary element of $\mathcal{U}$, it follows that $\xi = F(u)$.

Now consider the other term involving $u^*_h$. Recall that $u^*_h \in A(\tau_h u_h, \omega)$.
\[
\| \tau_h u_h - u_h \|_V \leq \| \tau_h u_h - \tau_h u \|_V + \| \tau_h u - u \|_V \leq \| u_h - u \|_V + \| \tau_h u - u \|_V
\]
and both of these on the right converge to 0 thanks to continuity of translation and separation theorems. Therefore,
\[
\lim_{h \to 0} \langle u^*_h (\cdot, \omega), \tau_h u_h - u \rangle_{\mathcal{V}' \mathcal{V}} = 0.
\]
It follows that
\[
\langle u^*, u - v \rangle_{\mathcal{V}' \mathcal{V}} = \liminf_{h \to 0} \left( \langle u^*_h (\cdot, \omega), u - \tau_h u_h \rangle_{\mathcal{V}' \mathcal{V}} + \langle u^*_h (\cdot, \omega), \tau_h u_h - v \rangle \right)
\geq \liminf_{h \to 0} \langle u^*_h (\cdot, \omega), \tau_h u_h - v \rangle \geq \langle u^* (v), u - v \rangle
\]
where $u^* (v) \in A(u, \omega)$. Then it follows that $u^* \in A(u, \omega)$ because if not, then by separation theorems, there would exist $v$ such that
\[
\langle u^*, u - v \rangle_{\mathcal{V}' \mathcal{V}} < \langle w^*, u - v \rangle_{\mathcal{V}' \mathcal{V}}
\]
for all $w^* \in A(u, \omega)$ which contradicts the above inequality. Thus, passing to the limit in (75.5.47)
\[
(B(u(\cdot, \omega)))' + \varepsilon F(u(\cdot, \omega)) + u^* = f(\cdot, \omega) \quad \text{in } \mathcal{U}',
Bu(0, \omega) = Bu_0 (\omega) \quad \text{in } \mathcal{U}.
\]
Here $u^* \in A(u, \omega)$. Of course nothing is known about the measurability of $u^*$, $u$. All that has been obtained in the above is a solution for each fixed $\omega$. However, each of the functions $u_h, u^*_h$ is measurable. Also we have the estimate (75.5.53). By Theorem (75.4.1), there are functions $\hat{u}(\cdot, \omega), \hat{u}^*(\cdot, \omega)$ and a subsequence with subscript $h(\omega)$ such that the following weak convergences in $\mathcal{V}$ and $\mathcal{V}'$ take place
\[
u_h(\omega)(\cdot, \omega) \rightharpoonup \hat{u}(\cdot, \omega), \quad u^*_h(\omega)(\cdot, \omega) \rightharpoonup \hat{u}^*(\cdot, \omega)
\]
such that the functions $(t, \omega) \rightarrow \hat{u}(t, \omega), (t, \omega) \rightarrow \hat{u}^*(t, \omega)$ are product measurable into $\mathcal{V}$ and $\mathcal{V}'$ respectively. The above argument shows that for each $\omega$, there is a further subsequence, still denoted with subscript $h(\omega)$ such that $u_h(\omega)(\cdot, \omega) \rightarrow u(\cdot, \omega)$ weakly in $\mathcal{V}$ and $u^*_h(\omega)(\cdot, \omega) \rightarrow u^*(\cdot, \omega)$ weakly in $\mathcal{V}'$ such that $(u(\cdot, \omega), u^*(\cdot, \omega))$ is a solution to the evolution equation for each $\omega$. By uniqueness of limits, $u(\cdot, \omega) = \hat{u}(\cdot, \omega)$, similar for $\hat{u}^*$. Thus this solution which is defined for each $\omega$ has a representative for each $\omega$ such that the resulting functions of $t, \omega$ are product measurable into $\mathcal{V}, \mathcal{V}'$ respectively. This proves the following lemma.
Lemma 75.5.5 For each \( \varepsilon > 0 \) there exists a solution to
\[
(B(\omega)u(\cdot,\omega))' + \varepsilon Fu(\cdot,\omega) + u^*(\cdot,\omega) = f(\cdot,\omega) \quad \text{in} \ U',
\]
\[
Bu(0,\omega) = Bu_0(\omega)
\]
\[
u^*(\cdot,\omega) \in A(u(\cdot,\omega),\omega)
\]
this solution satisfies \((t,\omega) \rightarrow u(t,\omega)\) is product measurable into \( V \). Also \((t,\omega) \rightarrow u^*(t,\omega), \quad (t,\omega) \rightarrow B(\omega)u(t,\omega)\) are product measurable into \( V' \) and \( W' \) respectively.

Next it is desired to remove the regularizing term \( \varepsilon Fu \). This will involve another use of Theorem 75.2.10. Denote by \( u_\varepsilon \) the solution to the above lemma. Then act on \( u_\varepsilon \) on both sides. This yields
\[
\frac{1}{2} \langle Bu_\varepsilon , u_\varepsilon \rangle (T) - \frac{1}{2} \langle Bu_0 , u_0 \rangle + \varepsilon \int_0^T \langle Fu_\varepsilon , u_\varepsilon \rangle \, ds
\]
\[
+ \int_0^T \langle u^*_\varepsilon , u_\varepsilon \rangle \, ds = \int_0^T \langle f , u_\varepsilon \rangle \, ds
\]
(75.5.50)

Then by the coercivity assumption,
\[
\lim_{\|u\|_V \to \infty} \inf_{u^* \in X_r} \{ 2\langle u^* , u \rangle + \langle Bu , u \rangle (T) : u^* \in A(u,\omega) \} = \infty
\]

it follows that
\[
\varepsilon \langle Fu_\varepsilon , u_\varepsilon \rangle_{U',U} + \|u_\varepsilon\|_V \leq C(u_0,f)
\]
(75.5.51)

where the constant on the right does not depend on \( \varepsilon \). Then
\[
\varepsilon Fu_\varepsilon \to 0 \quad \text{strongly in} \ U'
\]
this follows because from properties of the duality map,
\[
\langle \varepsilon Fu_\varepsilon , v \rangle \leq \varepsilon \langle Fu_\varepsilon , u_\varepsilon \rangle^{1/r'} \langle Fv , v \rangle^{1/r}
\]
\[
= \varepsilon^{1/r'} \langle Fu_\varepsilon , u_\varepsilon \rangle^{1/r'} \varepsilon^{1/r} \|v\|_U \leq C \varepsilon^{1/r} \|v\|_U
\]

Then since \( A \) is bounded, there is a constant \( C \) independent of \( \varepsilon \) such that
\[
\|u^*_\varepsilon\|_V + \|Bu_\varepsilon\|_{U'} + \|u_\varepsilon\|_V \leq C
\]
(75.5.52)

It follows there is a subsequence, still denoted with \( \varepsilon \) such that
\[
u^*_\varepsilon \to u^* \quad \text{weakly in} \ V',
\]
(75.5.53)
\[
(Bu_\varepsilon)' \to (Bu)' \quad \text{weakly in} \ U',
\]
(75.5.54)
\[
u_\varepsilon \to u \quad \text{weakly in} \ V.
\]
(75.5.55)
Also
\[
\frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (T) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \langle u_\varepsilon^*, u_\varepsilon \rangle + \varepsilon \langle Fu_\varepsilon, u_\varepsilon \rangle = \langle f, u_\varepsilon \rangle
\]

Assume \( T \) is such that
\[
\langle Bu_\varepsilon, u_\varepsilon \rangle (T) = \langle B(u_\varepsilon^*(T)), u_\varepsilon(T) \rangle, \ Bu_\varepsilon(T) = B(u_\varepsilon(T))
\]

for all \( \varepsilon \) in the sequence converging to 0 and also \( Bu_\varepsilon(T) = B(u(T)), \langle Bu, u \rangle (T) = \langle B(u(T)), u(T) \rangle \). If not, carry out the argument for \( \hat{T} \) close to \( T \) for which this condition does hold. We have the integral equation
\[
Bu_\varepsilon(t) - Bu_0 + \int_0^t u_\varepsilon^* \, ds + \int_0^t \varepsilon Fu_\varepsilon \, ds = \int_0^t f \, ds
\]

and so \( Bu_\varepsilon(t) \) converges to \( Bu(t) \) in \( U' \) weakly. This follows right away from the convergence of \( (Bu_\varepsilon)' \) in the above. Also from the above equation,
\[
Bu(t) - Bu_0 + \int_0^t u^* \, ds = \int_0^t f \, ds
\]

Thus
\[
Bu(0) = Bu_0
\]
\[
(Bu)' + u^* = f \text{ in } U'
\]

Since \( V' \subseteq U' \),
\[
\frac{1}{2} \langle Bu, u \rangle (t) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle u^*, u \rangle_{V', V} \, ds = \int_0^t \langle f, u \rangle \, ds
\]

Also
\[
\frac{1}{2} \langle Bu_\varepsilon, u_\varepsilon \rangle (t) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \langle u_\varepsilon^*, u_\varepsilon \rangle_{V', V} \, ds
\]
\[
+ \int_0^t \langle \varepsilon Fu_\varepsilon, u_\varepsilon \rangle \, ds = \int_0^t \langle f, u_\varepsilon \rangle \, ds \quad (75.5.56)
\]

Now let \( \{e_i\} \) be the vectors of Lemma 75.3.4 where these are in \( U \). Thus
\[
\langle Bu_\varepsilon, u_\varepsilon \rangle (T) = \sum_{i=1}^{\infty} \langle B(u_\varepsilon(T)), e_i \rangle^2
\]
Hence, by Fatou’s lemma,

\[
\liminf_{\varepsilon \to 0} \langle B u_\varepsilon, u_\varepsilon \rangle (T) = \liminf_{\varepsilon \to 0} \sum_{i=1}^{\infty} \langle B (u_\varepsilon (T)), e_i \rangle^2 \\
\geq \sum_{i=1}^{\infty} \liminf_{\varepsilon \to 0} \langle B (u_\varepsilon (T)), e_i \rangle^2 \\
= \sum_{i=1}^{\infty} \liminf_{\varepsilon \to 0} \langle B u_\varepsilon (T), e_i \rangle^2 \\
= \liminf_{\varepsilon \to 0} \langle B u (T), e_i \rangle^2 \\
= \langle B (u (T)), u (T) \rangle = \langle B u, u \rangle (T)
\]

From (75.5.56) letting \( t = T \),

\[
\limsup_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon \rangle_{\mathcal{V}', \mathcal{V}} \leq \limsup_{\varepsilon \to 0} \left( \langle f, u_\varepsilon \rangle + \frac{1}{2} \langle B u_0, u_0 \rangle - \frac{1}{2} \langle B u_\varepsilon, u_\varepsilon \rangle (T) \right) \\
\leq \langle f, u \rangle_{\mathcal{V}', \mathcal{V}} + \frac{1}{2} \langle B u_0, u_0 \rangle - \frac{1}{2} \langle B u, u \rangle (T) = \langle u^*, u \rangle_{\mathcal{V}', \mathcal{V}}
\]

It follows that

\[
\limsup_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon - u \rangle \leq \langle u^*, u \rangle_{\mathcal{V}', \mathcal{V}} - \langle u^*, u \rangle_{\mathcal{V}', \mathcal{V}} = 0
\]

and so

\[
\liminf_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon - v \rangle \geq \langle u^* (v), u - v \rangle
\]

for any \( v \in \mathcal{V} \) where \( u^* (v) \in A (u, \omega) \). In particular for \( v = u \). Hence

\[
\liminf_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon - u \rangle \geq \langle u^* (v), u - u \rangle = 0 \geq \limsup_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon - u \rangle
\]

showing that \( \lim_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon - u \rangle = 0 \). Thus

\[
\langle u^*, u - v \rangle \geq \liminf_{\varepsilon \to 0} (\langle u^*_\varepsilon, u_\varepsilon - u \rangle + \langle u^*_\varepsilon, u_\varepsilon - v \rangle) \\
= \liminf_{\varepsilon \to 0} \langle u^*_\varepsilon, u_\varepsilon - v \rangle \geq \langle u^* (v), u - v \rangle
\]

This implies \( u^* \in A (u, \omega) \) because if not, then by separation theorems, there exists \( v \in \mathcal{V} \) such that for all \( w^* \in A (u, \omega) \),

\[
\langle u^*, u - v \rangle < \langle w^*, u - v \rangle
\]

contrary to what was shown above. Thus this obtains

\[
Bu (t) - Bu_0 + \int_0^t u^* \, ds = \int_0^t f \, ds
\]
where \( u^* \in A(u, \omega) \). In case \( Bu^*_\varepsilon(T) \neq B(u^*_\varepsilon(T)) \), you do the same argument for \( \hat{T} < T \) where \( Bu^*_\varepsilon(\hat{T}) = B(u^*_\varepsilon(\hat{T})) \) for all \( \varepsilon \) and for \( u \). Then the above argument shows that \( u^* \chi_{[0, \hat{T}]} \in A\left(\chi_{[0, \hat{T}]} u, \omega\right) \). This being true for every such \( \hat{T} < T \) implies that it holds on \([0, T]\) and shows part of the following theorem which is the main result.

**Theorem 75.5.6** Let the conditions on \( A \) hold and assume that it holds on \([0, T]\) implies that it holds on \([0, T]\) and shows part of the following theorem which is the main result.

Let \( u_0 \) be \( \mathcal{F} \) measurable into \( W \), and let \( f \) be product measurable into \( V' \), \( f(\cdot, \omega) \in V' \). Then there exists a solution to the following evolution inclusion

\[
(B(\omega) u(\cdot, \omega))' + u^*(\cdot, \omega) = f(\cdot, \omega) \quad \text{in } V' \\
B(\omega) u(0, \omega) = B(\omega) u_0(\omega)
\]

where \( u^*(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \). In addition to this, \( (t, \omega) \rightarrow u(t, \omega) \) is product measurable into \( V \) and \( (t, \omega) \rightarrow u^*(t, \omega) \) is product measurable into \( V' \).

In place of the coercivity condition assume the coercivity condition involving both \( B \) and \( A \) given in \( 75.5.25 \). Then

\[
(B(\omega) u(\cdot, \omega))' + u^*(\cdot, \omega) = f(\cdot, \omega) \quad \text{in } \mathcal{U}' \\
B(\omega) u(0, \omega) = B(\omega) u_0(\omega)
\]

Thus the following holds in \( V' \)

\[
(B(\omega) u(\cdot, \omega))(t) - B(\omega) u_0(\omega) + \int_0^t u^*(\cdot, \omega) \, ds = \int_0^t f(s, \omega) \, ds \\
(Bu)' \quad \in \quad V'
\]

**Proof of Theorem 75.5.6**: First consider the claim about replacing the coercivity condition. Returning to \( 75.5.24 \), one obtains by integrating up to \( t \) and adding \( \lambda \int_0^t \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle \, ds \) to both sides,

\[
\frac{1}{2} \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle(t) - \frac{1}{2} \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle(0) + \varepsilon \int_0^t \langle Fu^*_\varepsilon, u^*_\varepsilon \rangle \, ds
\]

\[
+ \int_0^t \langle u^*_\varepsilon, u^*_\varepsilon \rangle ds + \lambda \int_0^t \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle ds = \int_0^t \langle f, u^*_\varepsilon \rangle ds + \lambda \int_0^t \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle ds
\]

Then from the estimate \( 75.5.24 \),

\[
\frac{1}{2} \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle(t) - \frac{1}{2} \langle Bu^*_\varepsilon, u^*_\varepsilon \rangle(0) + \varepsilon \int_0^t \langle Fu^*_\varepsilon, u^*_\varepsilon \rangle \, ds
\]
75.5. **THE MAIN RESULT**

\[ + \delta (\omega) \int_0^t \| u_\varepsilon \|^p_V \, ds - m (\omega) = \int_0^t \langle f, u_\varepsilon \rangle \, ds + \lambda \int_0^t \langle B u_\varepsilon, u_\varepsilon \rangle \, ds \quad (75.5.61) \]

From this, it is a routine use of Gronwall’s inequality to obtain the estimate

\[ \varepsilon \langle Fu_\varepsilon, u_\varepsilon \rangle \leq C (u_0, f, \lambda, \omega) \quad (75.5.62) \]

Then the rest of the argument is the same. You obtain the following in \( U' \).

\[ B (\omega) u (t, \omega) - B (\omega) u_0 (\omega) + \int_0^t u^* (\cdot, \omega) \, ds = \int_0^t f (s, \omega) \, ds \]

Since all terms but the first are in \( V' \), the equation holds in \( V' \). Also, the equation in (75.5.58) shows that \((B (\omega) u (\cdot, \omega))' \in V'\).

It only remains to show that there is a product measurable solution. The above argument has shown that there exists a solution for each \( \omega \). This is another application of Theorem 75.2.10. For the sequence defined in the convergences 75.5.53-75.5.55, there is an estimate 75.5.52. Therefore, the conditions of this theorem hold and there exists a subsequence denoted with \( u_\varepsilon (\omega) \) such that

\[ u_\varepsilon (\omega) \rightarrow \hat{u} (\cdot, \omega) \text{ weakly in } V, \]

\[ u^*_\varepsilon (\omega) \rightarrow \hat{u}^* (\cdot, \omega) \text{ weakly in } V' \]

where the \( \hat{u} \) and \( \hat{u}^* \) are product measurable. Now the above argument shows that for each \( \omega \) there exists a further subsequence, still denoted with \( \varepsilon (\omega) \) such that convergence to a solution to the evolution inclusion is obtained \((u (\cdot, \omega), u^* (\cdot, \omega))\). Then by uniqueness of limits, \( \hat{u} (\cdot, \omega) = u (\cdot, \omega) \) in \( V \), similar for \( u^* \) and \( \hat{u}^* \). Hence there is a solution to the above evolution problem which satisfies the claimed product measurability.

One can give a very interesting generalization of the above theorem.

**Theorem 75.5.7** In the context of Theorem 75.5.4, let \( q (t, \omega) \) be a product measurable function into \( V \) such that \( t \rightarrow q (t, \omega) \) is continuous, \( q (0, \omega) = 0 \).

Then, there exists a solution \( u \) of the integral equation

\[ B u (t, \omega) + \int_0^t z (s, \omega) \, ds = \int_0^t f (s, \omega) \, ds + B u_0 (\omega) + B q (t, \omega), \]

where \((t, \omega) \rightarrow u (t, \omega)\) is product measurable. Moreover, for each \( \omega \), \( B u (t, \omega) = B (u (t, \omega)) \) for a.e. \( t \) and \( z (\cdot, \omega) \in A (u (\cdot, \omega)) \) for a.e. \( t \), \( z \) is product measurable into \( V' \). Also, for each \( a \in [0, T] \),

\[ B u (t, \omega) + \int_a^t z (s, \omega) \, ds = \int_a^t f (s, \omega) \, ds + B u (a, \omega) + B q (t, \omega) - B q (a, \omega) \]

**Proof:** Define a stopping time

\[ \tau_r \equiv \inf \{ t : |q (t, \omega)| > r \} \]
Then this is the first hitting time of an open set by a continuous random variable and so it is a valid stopping time. Then for each \( r \), let
\[
A_r (\omega, w) \equiv A (\omega, w + q^{\tau_r} (\cdot, \omega)) ,
\]
where the notation means \( q^{\tau_r} (t) \equiv q (t \land \tau_r) \). Then, since \( q^{\tau_r} \) is uniformly bounded, all of the necessary estimates and measurability for the solution to the above corollary hold for \( A_r \) replacing \( A \). Therefore, there exists a solution \( w_r \) to the inclusion
\[
(Bw_r)' (\cdot, \omega) + A_r (w_r (\cdot), \omega) \ni f (\cdot, \omega), \quad Bw_r (0, \omega) = Bu_0 (\omega)
\]
Now for fixed \( \omega \), \( q^{\tau_r} (t, \omega) \) does not change for all \( r \) large enough. This is because it is a continuous function of \( t \) and so is bounded on the interval \([0, T]\). Thus, for \( r \) large enough and fixed \( \omega \),
\[
q^{\tau_r} (t, \omega) = q (t, \omega).
\]
Thus, we obtain
\[
\langle Bw_r (t, \omega), w_r (t, \omega) \rangle + \int_0^t \| w_r (s, \omega) \|^p_V ds \leq C (\omega) \quad (75.5.63)
\]
Now, as before one can pass to a limit involving a subsequence, as \( r \to \infty \) and obtain a solution to the integral equation
\[
Bw (t, \omega) - Bu_0 (\omega) + \int_0^t z (s, \omega) ds = \int_0^t f (s, \omega) ds
\]
where \( z (s, \omega) \in A (s, \omega, w (s, \omega) + q (s, \omega)) \) for a.e. \( s \) and \( z \) is product measurable. Then an application of Theorem 75.2.10 shows that there exists a solution \( w \) to this integral equation for each \( \omega \) which also has \( (t, \omega) \to w (t, \omega) \) product measurable and \( (t, \omega) \to z (t, \omega) \) product measurable. Now just let \( u (t, \omega) = w (t, \omega) + q (t, \omega) \).

The last claim follows from letting \( t = a \) in the top equation and then subtracting this from the top equation with \( t > a \).}

## 75.6 Variational Inequalities

We have some good theorems above in the context of 75.5.25 - 75.5.28, 75.5.30 - 75.5.34 and \( B \) satisfies 75.3.6 and assume, if it depends on \( \omega \), it is of the form
\[
B (\omega) = k (\omega) B, \quad k (\omega) \geq 0, \quad k \text{ measurable}
\]
Now this will be used to consider variational inequalities.

Let \( K \) be a closed convex subset of \( V \) containing 0. Let \( P : V \to V' \) be an operator of penalization. Thus \( P = 0 \) on \( K \) and is monotone and demicontinuous and nonzero off \( K \).
\[
Pu = F (u - \text{proj}_K (u))
\]
where \( F \) is the duality map such that \( \langle Fu, u \rangle = \| u \|^2, \| Fu \| = \| u \| \). Then \( A (\cdot, \omega) + nP \) satisfies the conditions for Theorem 75.5 assuming \( A (\cdot, \omega) \) satisfies the conditions of this theorem. Then by Theorem 75.5 there exists a solution \( u_n \), such
that \((t, \omega) \to u_n(t, \omega), (t, \omega) \to u^*_n(t, \omega)\) are product measurable, and for each \(\omega\),

\[
(Bu_n)' + u^*_n(\cdot, \omega) + nP(u_n(\cdot, \omega)) = f(\cdot, \omega) \quad \text{in } V' \\
Bu_n(0, \omega) = 0
\]

(75.6.64)

Here \(B\) is as described in that theorem. Using 0 \(\in K\) and monotonicity of \(P\), the estimates for \(A\) lead to an estimate of the form

\[
\|u_n(\cdot, \omega)\|_V + \|u^*_n(\cdot, \omega)\|_{V'} \leq C(\omega)
\]

Then there is a subsequence

\[
u_n \to u \text{ weakly in } V \\
u^*_n \to u^* \text{ weakly in } V' \\
Pu_n \to \xi \text{ weakly in } V'
\]

Let \(\Lambda\) denote those \(v \in V\) such that \((Bv)' \in V'\) and \(Bv(0) = 0\). Then for \(v \in \Lambda\),

\[
\langle (Bu_n)', u_n - v \rangle + \langle u^*_n(\cdot, \omega), u_n - v \rangle + n \langle P(u_n(\cdot, \omega)), u_n - v \rangle = \langle f(\cdot, \omega), u_n - v \rangle
\]

Thus by monotonicity considerations,

\[
\langle (Bu_n)', u_n - v \rangle + \langle u^*_n(\cdot, \omega), u_n - v \rangle + n \langle P(u_n(\cdot, \omega)), u_n - v \rangle \leq \langle f(\cdot, \omega), u_n - v \rangle
\]

(\(*\))

It follows that

\[
\limsup_{n \to \infty} \langle P(u_n(\cdot, \omega)), u_n - v \rangle \leq 0 \\
\limsup_{n \to \infty} \langle P(u_n(\cdot, \omega)), u_n - u \rangle \leq \langle -\xi, u - v \rangle
\]

Now, since \(\Lambda\) is dense, \(v\) can be chosen as close as desired to \(u\) and hence

\[
\limsup_{n \to \infty} \langle P(u_n(\cdot, \omega)), u_n - u \rangle \leq 0
\]

Since \(P\) is monotone, in fact the limit exists in the above. Therefore, for any \(v \in \Lambda\) and \(*\),

\[
\liminf_{n \to \infty} \langle (P(u_n(\cdot, \omega)), u_n - v) \rangle \geq \langle Pu, u - v \rangle \geq 0
\]

and so

\[
\langle Pu, u - v \rangle \leq 0
\]

for all \(v \in \Lambda\). It follows that \(Pu = 0\) and so \(u \in K\).

Now for \(v \in \Lambda \cap K\), monotonicity considerations imply

\[
\langle (Bu)'', u_n - v \rangle + \langle u^*_n(\cdot, \omega), u_n - u \rangle + \langle u^*_n(\cdot, \omega), u - v \rangle \leq \langle f(\cdot, \omega), u_n - v \rangle
\]
Then
\[ (u_n^* (\cdot, \omega), u_n - u) \leq (f (\cdot, \omega), u_n - v) - (Bu')', u_n - v) - (u_n^* (\cdot, \omega), u - v) \] (75.6.55)

Then
\[ \limsup_{n \to \infty} (u_n^* (\cdot, \omega), u_n - u) \leq (f (\cdot, \omega), u - v) + (Bu')', v - u) + (u^* (\cdot, \omega), v - u) \]

We assume the existence of a regularizing sequence. If \( u \in K \) there exists \( u_i \to u \) weakly in \( V \) such that
\[ \limsup_{i \to \infty} (Bu_i)', u_i - u) \leq 0 \]

In the above inequality, let \( v = u_i \)
\[ \limsup_{n \to \infty} (u_n^* (\cdot, \omega), u_n - u) \leq (f (\cdot, \omega), u - u_i) + (Bu_i)', u_i - u) + (u^* (\cdot, \omega), u_i - u) \]

Then take \( \limsup_{i \to 0} \) of both sides to obtain
\[ \limsup_{n \to \infty} (u_n^* (\cdot, \omega), u_n - u) \leq 0. \]

Now assume the usual limit condition holds for \( A (\cdot, \omega) \). In practice, this typically means \( A (\cdot, \omega) \) will be single valued, monotone and hemicontinuous because there is no control on the time derivative. However, we will go ahead and assume just that the limit condition holds. This would also take place if \( A (\cdot, \omega) \) were defined on \( V \) and maximal monotone, for example. Then for every \( v \in V \),
\[ \liminf_{n \to \infty} (u_n^* (\cdot, \omega), u_n - v) \geq (u^* (v), u - v) \]
where \( u^* (v) \in A (u, \omega) \). In particular, this holds for \( v = u \) and so
\[ \liminf_{n \to \infty} (u_n^* (\cdot, \omega), u_n - u) \geq 0 \geq \limsup_{n \to \infty} (u_n^* (\cdot, \omega), u_n - u) \]
showing the the limit exists. Then
\[ (u^* (v), u - v) \leq \liminf_{n \to \infty} (u_n^* (\cdot, \omega), u_n - v) \]
\[ = \liminf_{n \to \infty} (u_n^* (\cdot, \omega), u_n - u) + (u_n^* (\cdot, \omega), u - v) \]
\[ = (u^*, u - v) \]

and since this is true for all \( v \in V \) it follows that \( u^* \in A (u (\cdot, \omega), \omega) \) since otherwise, separation theorems would give a contradiction. If \( u^* \) were not in \( A (u (\cdot, \omega), \omega) \) there would exist \( v \) such that for all \( z^* \in A (u, \omega) \),
\[ (z^*, u - v) > (u^*, u - v) \]
conclude that for every \( v \in K \) such that \((Bv') \in V'\), \(Bv(0) = 0\),
\[
\langle (Bv'), u - v \rangle + \langle u^*, u - v \rangle \leq \langle f(\cdot, \omega), u - v \rangle
\]
where \( u^* \in A(u, \omega) \).

This has proved the first part of the following theorem which gives measurable solutions to a variational inequality.

**Theorem 75.6.1** Suppose \( A(\cdot, \omega) \) is monotone hemicontinuous bounded and single valued and coercive as a map from \( V \) to \( V' \). Suppose also that for \( \omega \rightarrow u(\omega) \) measurable into \( V \), it follows that \( \omega \rightarrow A(u(\omega), \omega) \) is measurable into \( V' \). Let \( K \) be a closed convex subset of \( V \) containing 0 and let \( B \in L(W, W') \) be self adjoint and nonnegative as above. Let there be a regularizing sequence \( \{u_i\} \) for each \( u \in K \) satisfying \( Bu_i(0) = 0 \), \((Bu_i') \in V'\), \( u_i \in K\),
\[
\lim \sup_{i \rightarrow \infty} \langle (Bu_i'), u_i - u \rangle \leq 0
\]
Then for each \( \omega \), there exists a solution to
\[
\langle (Bv'), u - v \rangle + \langle A(u(\cdot, \omega), \omega), u(\cdot, \omega) - v \rangle \leq \langle f(\cdot, \omega), u - v \rangle
\]
valid for all \( v \in K \) such that \((Bv') \in V'\), \(Bv(0) = 0\), and \( (t, \omega) \rightarrow u(t, \omega) \) is \( B([0, T]) \times F \) measurable.

**Proof:** This follows from Theorem 75.2.10. This is because there is an estimate of the right sort for the measurable functions \( u_n(\cdot, \omega) \) and \( u_n^*(\cdot, \omega) \) and the above argument which shows that a subsequence has a convergent subsequence which converges appropriately to a solution. 

You can have \( K = K(\omega) \). There would be absolutely no change in the above theorem. You just need to have the operator of penalization satisfy \( \omega \rightarrow P(u(\omega), \omega) = F(u(\omega) - \text{proj}_{K(\omega)} u(\omega)) \) is measurable into \( V' \) provided \( \omega \rightarrow u(\omega) \) is measurable into \( V \). What are the conditions on the set valued \( \omega \rightarrow K(\omega) \) which will cause this to take place?

**Lemma 75.6.2** Let \( \omega \rightarrow K(\omega) \) be measurable into \( V \). Then \( \omega \rightarrow \text{proj}_{K(\omega)} u(\omega) \) is also measurable into \( V \) if \( \omega \rightarrow u(\omega) \) is measurable.

**Proof:** It follows from standard results on measurable multi-functions that there is a countable collection \( \{w_n(\omega)\} \), \( \omega \rightarrow w_n(\omega) \) being measurable and \( w_n(\omega) \in K(\omega) \) for each \( \omega \) such that for each \( \omega \), \( K(\omega) = \bigcup_n w_n(\omega) \). Let
\[
d_n(\omega) \equiv \min \{||u(\omega) - w_k(\omega)||, k \leq n\}
\]
Let \( u_1(\omega) \equiv w_1(\omega) \). Let
\[
u_2(\omega) = w_1(\omega)
\]
on the set
\[ \{ \omega : \| u(\omega) - w_1(\omega) \| < \{ \| u(\omega) - w_2(\omega) \| \} \} \]
and
\[ u_2(\omega) \equiv w_2(\omega) \] off the above set.
Thus \[ \| u_2(\omega) - u(\omega) \| = d_2. \]
Let
\begin{align*}
u_3(\omega) &= w_1(\omega) \quad \text{on} \quad \left\{ \omega : \| u(\omega) - w_1(\omega) \| < \| u(\omega) - w_j(\omega) \|, j = 2, 3 \right\} = S_1 \\
u_3(\omega) &= w_2(\omega) \quad \text{on} \quad S_1 \cap \left\{ \omega : \| u(\omega) - w_1(\omega) \| < \| u(\omega) - w_j(\omega) \|, j = 3 \right\}
\end{align*}
Thus \[ \| u_3(\omega) - u(\omega) \| = d_3. \]
Continue this way, obtaining \( u_n(\omega) \) such that \( \| u_n(\omega) - u(\omega) \| = d_n(\omega) \) and \( u_n(\omega) \in K(\omega) \) with \( u_n \) measurable. Thus, in effect one picks the closest of all the \( u_k(\omega) \) for \( k \leq n \) as the value of \( u_n(\omega) \) and \( u_n \) is measurable and by density in \( K(\omega) \) of \( \{ u_n(\omega) \} \) for each \( \omega \), \( \{ u_n(\omega) \} \) must be a minimizing sequence for
\[ \lambda(\omega) \equiv \inf \{ \| u(\omega) - z \| : z \in K(\omega) \} \]
Then it follows that \( u_n(\omega) \to \text{proj}_{K(\omega)} u(\omega) \) weakly in \( V \). Here is why: Suppose it fails to converge to \( \text{proj}_{K(\omega)} u(\omega) \). Since it is minimizing, it is a bounded sequence. Thus there would be a subsequence, still denoted as \( u_n(\omega) \) which converges to some \( q(\omega) \neq \text{proj}_{K(\omega)} u(\omega) \). Then
\[ \lambda(\omega) = \lim_{n \to \infty} \| u(\omega) - u_n(\omega) \| \geq \| u(\omega) - q(\omega) \| \]
because convex and lower semicontinuous is weakly lower semicontinuous. But this implies \( q(\omega) = \text{proj}_{K(\omega)} (u(\omega)) \) because the projection map is well defined thanks to strict convexity of the norm used. This is a contradiction. Hence \( \text{proj}_{K(\omega)} u(\omega) = \lim_{n \to \infty} u_n(\omega) \) and so is a measurable function. It follows that \( \omega \to P(u(\omega), \omega) \) is measurable into \( V \).

The following corollary is now immediate.

**Corollary 75.6.3** Suppose \( A(\cdot, \omega) \) is monotone hemi-continuous bounded, single valued, and coercive as a map from \( V \) to \( V' \). Suppose also that for \( \omega \to u(\omega) \) measurable into \( V \), it follows that \( \omega \to A(u(\omega), \omega) \) is measurable into \( V' \). Let \( K(\omega) \) be a closed convex subset of \( V \) containing \( 0 \) and \( \omega \to K(\omega) \) is a set valued measurable multifunction. Let \( B \in \mathcal{L}(W, W') \) be self adjoint and nonnegative as above. Let there be a regularizing sequence \( \{ u_i \} \) for each \( u \in K \) satisfying \( Bu_i(0) = 0, (Bu_i)' \in V', u_i \in K \),
\[ \lim \sup_{i \to \infty} \langle (Bu_i)' , u_i - u \rangle \leq 0 \]
Then for each \( \omega \), there exists a solution to
\[ \langle (Bv)' , u - v \rangle + \langle A(u(\cdot, \omega), \omega) , u(\cdot, \omega) - v \rangle \leq \langle f(\cdot, \omega) , u(\cdot, \omega) - v \rangle \]
valid for all \( v \in K(\omega) \) such that \( (Bv)' \in V' \), \( Bv(0) = 0 \), and \( (t, \omega) \to u(t, \omega) \), is \( B([0, T]) \times \mathcal{F} \) measurable.
75.7. PROGRESSIVELY MEASURABLE SOLUTIONS

**Proof:** The proof is identical to the above. One obtains a measurable solution to Eq. 75.6.64 in which $P$ is replaced with $P(\cdot, \omega)$. Then one proceeds in exactly the same steps as before and finally uses Theorem 75.2.10 to obtain the measurability of a solution to the variational inequality. □

What does it mean for $u(\omega) \in \mathcal{K}(\omega)$ for each $\omega$? It means that there is a sequence of the $w_n \{w_n(\omega)\}$ such that each $w_n$ is measurable into $V$ implying that for each $\omega$ there is a representative $t \rightarrow w_n(t, \omega)$ such that the resulting $(t, \omega) \rightarrow w_n(t, \omega)$ is product measurable and $\|u(\cdot, \omega) - w_n(\cdot, \omega)\|_V \rightarrow 0$. Thus there is no reason to think that $(t, \omega) \rightarrow u(t, \omega)$ is product measurable. The message of the above corollary says that nevertheless, there is a measurable solution to the variational inequality.

75.7 Progressively Measurable Solutions

In the context of uniqueness of the evolution initial value problem for fixed $\omega$, one can prove theorems about progressively measurable solutions fairly easily. First is a definition of the term progressively measurable.

**Definition 75.7.1** Let $F_t$ be an increasing in $t$ set of $\sigma$ algebras of sets of $F$. Thus each $F_t$ is a $\sigma$ algebra and if $s \leq t$, then $F_s \subseteq F_t$. This set of $\sigma$ algebras is called a filtration. A set $S \subseteq [0, T] \times \Omega$ is called progressively measurable if for every $t \in [0, T]$,

$$S \cap [0, t] \times \Omega \in \mathcal{B}([0, t]) \times F_t$$

Denote by $P$ the progressively measurable sets. This is a $\sigma$ algebra of subsets of $[0, T] \times \Omega$. A function $g$ is progressively measurable if $X_{[0, t]}g$ is $\mathcal{B}([0, t]) \times F_t$ measurable for each $t$.

Let $A$ satisfy the bounded condition 75.5.25, the condition on subintervals 75.5.26, the specific boundedness estimate 75.5.27, the specific coercivity estimate involving $B$ and $A$ in 75.5.29, and the limit condition 75.5.30. In place of the condition on the existence of a measurable selection 75.5.34, we will assume the following condition.

**Condition 75.7.2** For each $t \leq T$, if $\omega \rightarrow u(\cdot, \omega)$ is $F_t$ measurable into $\mathcal{V}_{[0, t]}$, then there exists a $F_t$ measurable selection of $A(u(\cdot, \omega), \omega)$ into $\mathcal{V}_{[0, t]}$.

Note that $u(\cdot, \omega)$ is in $\mathcal{V}_{[0, t]}$ so $u(t, \omega) \in V.$

In this section, we assume that $\omega \rightarrow B(\omega)$ is $\mathcal{F}_0$ measurable into $\mathcal{L}(W, W')$.

The theorem to be shown is the following.

**Theorem 75.7.3** Assume the above conditions, 75.5.24, 75.5.29, 75.5.30, and the Condition 75.7.2. Let $u_0$ be $\mathcal{F}_0$ measurable and $\omega \rightarrow B(\omega)$ also $\mathcal{F}_0$ measurable and $(t, \omega) \rightarrow A_{[0, t]}(t, f(t, \omega))$ is $\mathcal{B}([0, t]) \times F_t$ product measurable into
CHAPTER 75. STOCHASTIC INCLUSIONS

There exists a solution to the integral equation on \([0, T]\) for each \(t\). Also assume that for each \(\omega\), there is at most one solution to the evolution equation

\[
(B(\omega) u(t, \omega))(t) - B(\omega) u_0(\omega) + \int_0^t u^*(t, \omega) \, ds = \int_0^t f(s, \omega) \, ds,
\]

\[
u^*(t, \omega) \in A(u(t, \omega), \omega)
\]

for \(t \in [0, T]\) for each \(T \leq T\). Then there exists a unique solution \((u(\cdot, \omega), u^*(\cdot, \omega))\) in \(\mathcal{V} \times \hat{\mathcal{V}}\) to the above integral equation for each \(\omega\). This solution satisfies \((t, \omega) \to (u(t, \omega), u^*(t, \omega))\) is progressively measurable into \(\mathcal{V} \times \hat{\mathcal{V}}\).

**Proof:** Let \(\mathcal{T}\) denote subsets of \((0, T]\) which contain \(T\) such that for \(S \in \mathcal{T}\), there exists a solution \(u_S\) for each \(\omega\) to the above integral equation on \([0, T]\) such that \((t, \omega) \to X_{[0, s]}(t) u_S(t, \omega)\) is \(\mathcal{B}(\mathcal{L}, \omega) \times \mathcal{F}_s\) measurable for each \(s \in S\). Then \(\{T\} \in \mathcal{T}\). If \(S, S'\) are in \(\mathcal{T}\), then \(S \subseteq S'\) will mean that \(S \subseteq S'\) and also \(u_S(t, \omega) = u_{S'}(t, \omega)\) in \(\mathcal{V}\) for all \(t \in S\), similar for \(u^*_S\) and \(u^*_{S'}\). Note that equality must hold in \(\mathcal{V}\) by uniqueness. Now let \(C\) denote a maximal chain. Is \(\cup C = S_{\infty}\) all of \((0, T)\)?

What is \(u_{S_{\infty}}\)? Define \(u_{S_{\infty}}(t, \omega)\) the common value of \(u_S(t, \omega)\) for all \(S \in C\), which contain \(t \in S_{\infty}\). If \(s \in S_{\infty}\), then it is in some \(S \in C\) and so the product measurability condition holds for this \(s\). Thus \(S_{\infty}\) is a maximal element of the partially ordered set. Is \(S_{\infty}\) all of \((0, T)\)? Suppose \(\hat{s} \notin S_{\infty}, T > \hat{s} > 0\).

From Theorem 75.3.14 there exists a solution to the integral equation on \([0, \hat{s}]\) called \(u_1\) such that \((t, \omega) \to u_1(t, \omega)\) is \(\mathcal{B}(\mathcal{L}, \mathcal{F}_s)\) measurable, similar for \(u^*_1\). By the same theorem, there is a solution on \([0, T]\), \(u_2\) which is \(\mathcal{B}(\mathcal{L}, \mathcal{F}_T)\) measurable. Now by uniqueness, \(u_2(\cdot, \omega) = u_1(\cdot, \omega)\) in \(\mathcal{V}\), similar for \(u^*_2\). Therefore, no harm is done in re-defining \(\hat{u}_2\) on \([0, \hat{s}]\) so that \(u_2(t, \omega) = u_1(t, \omega)\) for all \(t \in [0, \hat{s}]\), similar for \(u^*_2\). Denote these functions as \(\hat{u}, \hat{u}^*\). By uniqueness, \(u_{S_{\infty}}(\cdot, \omega) = \hat{u}(\cdot, \omega)\) in \(L^p(\mathcal{L}, \mathcal{F}_s, V)\). Thus no harm is done by re-defining \(\hat{u}(s, \omega)\) to equal \(u_{S_{\infty}}(s, \omega)\) for \(s < \hat{s}\) and \(\hat{u}(\hat{s}, \omega)\) at \(\hat{s}\). As to \(s > \hat{s}\) also re define \(\hat{u}(s, \omega) \equiv u_{S_{\infty}}(s, \omega)\) for such \(s\). By uniqueness, the two are equal in \(\mathcal{V}\) and so no change occurs in the solution of the integral equation. Now \(S_{\infty}\) was not maximal after all. \(S_{\infty} \cup \{\hat{s}\}\) is larger.

This contradiction shows that in fact, \(S_{\infty} = (0, T]\). \(\blacksquare\)

**Theorem 75.7.4** Assume the above conditions, \(\{10.5.24, 10.5.25, 10.5.26, 10.5.27, 10.5.30\}\), and the Condition 75.7.2. Let \(u_0\) be \(\mathcal{F}_0\) measurable and \(\omega \to B(\omega)\) also \(\mathcal{F}_0\) measurable and \((t, \omega) \to X_{[0, t]}(t) f(t, \omega)\) is \(\mathcal{B}(\mathcal{L}, \mathcal{F}_t)\) product measurable into \(\mathcal{V}'\) for each \(t\).

\[
B(\omega) = k(\omega) B, \ k(\omega) \geq 0, k \text{ measurable}.
\]

Also let \(t \to q(t, \omega)\) be continuous and \(q\) is progressively measurable into \(\mathcal{V}\). Suppose there is at most one solution to

\[
Bu(t, \omega) + \int_0^t z(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega) + Bq(t, \omega), \quad (75.7.66)
\]

for each \(\omega\). Then the solution \(u\) to the above integral equation is progressively measurable and so is \(z\). Moreover, for each \(\omega\), both \(Bu(t, \omega) = B(u(t, \omega))\) a.e. \(t\).
75.8. ADDING A QUASI-BOUNDED OPERATOR

and \(z(\cdot, \omega) \in A(u(\cdot, \omega), \omega)\). Also, for each \(a \in [0, T]\),

\[
Bu(t, \omega) + \int_a^t z(s, \omega) \, ds = \int_a^t f(s, \omega) \, ds + Bu(a, \omega) + Bq(t, \omega) - Bq(a, \omega)
\]

**Proof:** By Theorem 75.5.7 there exists a solution to 75.7.66 which is \(\mathcal{B}([0, T]) \times \mathcal{F}_T\) measurable. Now, as in the proof of Theorem 75.5.7 one can define a new operator

\[
A_r(w, \omega) 
\]

where \(\tau_r\) is the stopping time defined there. Then, since \(q\) is progressively measurable, the progressively measurable condition is satisfied for this new operator. Hence by Theorem 75.7.3 there exists a unique solution \(w_r\) which is progressively measurable to the integral equation

\[
Bw_r(t, \omega) + \int_0^t z_r(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega)
\]

where \(z_r(\cdot, \omega) \in A_r(w(\cdot, \omega) + q^\tau(\cdot, \omega), \omega)\). Then as in Theorem 75.7.3 you can let \(r \to \infty\) and eventually \(q^\tau(\cdot, \omega) = q(\cdot, \omega)\). Then, passing to a limit, it follows that for a given \(\omega\), there is a solution to

\[
Bw(t, \omega) + \int_0^t z(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega)
\]

which is progressively measurable because \(w(\cdot, \omega) = \lim_{r \to \infty} w_r(\cdot, \omega)\) in \(V\) each \(w_r\) being progressively measurable. Note how uniqueness for fixed \(\omega\) is important in this argument. Recall that

\[
\tau_r \equiv \inf \{t : |q(t, \omega)| > r\}
\]

By continuity, eventually, for a given \(\omega, \tau_r = \infty\) and so no further change takes place in \(q^\tau(\cdot, \omega)\) for that \(\omega\). By uniqueness, the same is true of the solution \(w_r(\cdot, \omega)\) and so pointwise convergence takes place for the \(w_r\). Without uniqueness holding, this becomes very unclear. Thus for each \(T < T, \omega \to w(\cdot, \omega)\) is measurable into \(V_{[0, T]}\).

Then by Lemma 75.4.2, \(w\) has a representative in \(V\) for each \(\omega\) such that the resulting function satisfies \((t, \omega) \to \mathcal{X}([0, T]) (t) w(t, \omega)\) is \(\mathcal{B}([0, T]) \times \mathcal{F}_T\) measurable into \(V\). Thus one can assume that \(w\) is progressively measurable. Now as in Theorem 75.5.7, Define \(u = w + q\).

The last claim follows from letting \(t = a\) in the top equation and then subtracting this from the top equation with \(t > a\). □

75.8 Adding A Quasi-bounded Operator

Recall the following conditions for the various operators.
Bounded and coercive conditions

\[ A(\cdot, \omega), A(\cdot, \omega) : \mathcal{V}_I \to \mathcal{V}'_I \] for each \( I \) a subinterval of \([0, T] \)

\[ I = \left[ 0, \hat{T} \right], \hat{T} \leq T \]

\[ A(\cdot, \omega) : \mathcal{V}_I \to \mathcal{P}(\mathcal{V}'_I) \] is bounded, \( (75.8.67) \)

If, for \( u \in \mathcal{V} \),

\[ u^* \mathcal{X}_{[0, \hat{T}]} \in A \left( u \mathcal{X}_{[0, \hat{T}]}, \omega \right) \]

for each \( \hat{T} \) in an increasing sequence converging to \( T \), then

\[ u^* \in A(u, \omega) \] \( (75.8.68) \)

Assume the specific estimate

\[
\sup \left\{ \| u^* \|_{\mathcal{V}'_I} : u^* \in A(u, \omega) \right\} \leq a(\omega) + b(\omega) \| u \|_{\mathcal{V}_I}^{p-1} \]

\( (75.8.69) \)

where \( a(\omega), b(\omega) \) are nonnegative. Also assume the following coercivity estimate valid for each \( t \leq T \) and for some \( \lambda(\omega) \geq 0 \),

\[
\inf \left( \int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle Bu, u \rangle \, dt : u^* \in A(u, \omega) \right) \geq \delta(\omega) \int_0^t \| u \|_{\mathcal{V}}^p \, ds - m(\omega) \]

\( (75.8.70) \)

where \( m(\omega) \) is some nonnegative constant, \( \delta(\omega) > 0 \).

Limit condition

Let \( U \) be a Banach space dense in \( \mathcal{V} \) and that if \( u_i \rightharpoonup u \) in \( \mathcal{V}_I \) and \( u_i^* \in A(u_i) \) with \( u_i^* \rightharpoonup u^* \) in \( \mathcal{V}'_I \) and \((Bu_i)' \rightharpoonup (Bu)'\) in \( \mathcal{U}'_I \), \(-\) denoting weak convergence, then if

\[
\lim \sup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{\mathcal{V}'_I, \mathcal{V}_I} \leq 0
\]

it follows that for all \( v \in \mathcal{V}_I \), there exists \( u^*(v) \in Au \) such that

\[
\lim_{i \to \infty} \langle u_i^*, u_i - v \rangle_{\mathcal{V}'_I, \mathcal{V}_I} \geq \langle u^*(v), u - v \rangle_{\mathcal{V}'_I, \mathcal{V}_I}
\]

\( (75.8.71) \)

where \( r > \max(p, 2) \), and we replace \( p \) with \( r \) and \( I \) an arbitrary subinterval of the form \( [0, \hat{T}], \hat{T} < T \), for \([0, T] \), and \( U \) for \( V \) where indicated. Here

\[ \mathcal{U}_r \equiv L^r(I; U) \]

Typically, \( U \) is compactly embedded in \( V \).

Measurability condition
For \( \omega \to u (\cdot, \omega) \) measurable into \( \mathcal{V} \),
\[
\omega \to A (u (\cdot, \omega), \omega) \text{ has a measurable selection into } \mathcal{V}'. \tag{75.8.72}
\]
This last condition means there is a function \( \omega \to u^* (\omega) \) which is measurable into \( \mathcal{V}' \) such that \( u^* (\omega) \in A (u (\cdot, \omega), \omega) \).

As for the operator \( B \) it is either independent of \( \omega \) and is a nonnegative self adjoint operator mapping \( \mathcal{W} \) to \( \mathcal{W}' \) or else it is of the form \( k (\omega) B \) where \( k \geq 0 \) and is measurable.

We will assume here that \( p > 1 \). Then the following main result was obtained above. It is Theorem 75.5.1.

**Theorem 75.8.1** If 75.8.67 - 75.8.72 and \( B \) as described above, let \( q (t, \omega) \) be a product measurable function into \( \mathcal{V} \) such that \( t \to q (t, \omega) \) is continuous, \( q (0, \omega) = 0 \). Then, there exists a solution \( u \) of the integral equation
\[
Bu (t, \omega) + \int_0^t z (s, \omega) \, ds = \int_0^t f (s, \omega) \, ds + Bu_0 (\omega) + Bq (t, \omega),
\]
where \( (t, \omega) \to u (t, \omega) \) is measurable. Moreover, for each \( \omega \), \( Bu (t, \omega) = B (u (t, \omega)) \) for a.e. \( t \) and \( z (\cdot, \omega) \in A (\omega, u (\cdot, \omega)) \) for a.e. \( t \). Also, for each \( a \in [0, T] \),
\[
Bu (t, \omega) + \int_a^t z (s, \omega) \, ds = \int_a^t f (s, \omega) \, ds + Bu (a, \omega) + Bq (t, \omega) - Bq (a, \omega)
\]

The idea here is to show that everything works as well if a suitable unbounded maximal monotone operator is added in. The result is interesting but not as interesting as it might be. This is because the maximal monotone operator must be quasi bounded. Still it is interesting to note that the above holds for some unbounded operators. This has been pointed out in the case where there are no stochastic effects in a recent paper [51]. This generalizes this result by considering the measurability of solutions and allowing for possibly degenerate leading operator \( B \).

To begin with assume \( q = 0 \).

Now let \( G : D (G) \subseteq \mathcal{V} \to \mathcal{P} (\mathcal{V}') \) be maximal monotone. Also assume that \( 0 \in D (G) \). Then you have
\[
\langle u^*, u \rangle \geq \langle g^*, u \rangle \text{ if } u^* \in Gu
\]
for every \( g^* \in G (0) \). Hence
\[
\langle u^*, u \rangle \geq - |G (0)| \| u \|_{\mathcal{V}} \text{ if } u^* \in Gu \tag{75.8.73}
\]
where \( |G (0)| = \inf \{ \| y^* \|_{\mathcal{V}} : y^* \in G (0) \} \).

There is a standard way of approximating \( G \) with bounded demicontinuous operators which is reviewed next. It is all in Barbu [12]. See Section 23.7.4. Since \( G \) is maximal monotone,
\[
0 \in F (x_\mu - x) + \mu^{p-1} G (x_\mu)
\]
where $F$ is a duality map for $p$, the one used in the above theorem. Barbu uses only $p = 2$ but it works just as well for arbitrary $p > 1$ with the minor modifications used here. To see this, you consider $\hat{G}(y) \equiv G(x + y)$. Then $\hat{G}$ is also maximal monotone and so there exists a solution to

$$0 \in F(\hat{x}) + \mu^{p-1}\hat{G}(\hat{x}) = F(\hat{x}) + \mu^{p-1}G(x + \hat{x})$$

Now let $x_\mu = x + \hat{x}$ so $\hat{x} = x_\mu - x$. Hence

$$0 \in F(x_\mu - x) + \mu^{p-1}Gx_\mu$$

The symbol $\limsup_{n,n \to \infty} a_{mn}$ means $\lim_{N \to \infty} (\sup_{m \geq N, n \geq N} a_{mn})$.

**Lemma 75.8.2** Suppose $\limsup_{n,n \to \infty} a_{mn} \leq 0$. Then $\limsup_{m \to \infty} (\limsup_{n \to \infty} a_{mn}) \leq 0$.

**Proof:** Suppose the first inequality. Then for $\varepsilon > 0$, there exists $N$ such that if $n, m$ are both as large as $N$, then $a_{mn} \leq \varepsilon$. Thus $\sup_{n \geq N} a_{mn} \leq \varepsilon$ provided $m \geq N$ also. Hence for such $m$,

$$\lim_{n \to \infty} \left( \sup_{n \geq N} a_{mn} \right) \leq \varepsilon$$

for each $m \geq N$. It follows $\limsup_{m \to \infty} (\limsup_{n \to \infty} a_{mn}) \leq \varepsilon$. Since $\varepsilon$ is arbitrary, this proves the lemma. ■

Then here is a simple observation.

**Lemma 75.8.3** Let $G : D(G) \subseteq X \to \mathcal{P}(X')$ where $X$ is a Banach space be maximal monotone and let $v_n \in Gu_n$ and

$$u_n \to u, \ v_n \to v \text{ weakly.}$$

Also suppose that

$$\limsup_{m,n \to \infty} \langle v_n - v_m, u_n - u_m \rangle \leq 0$$

or

$$\limsup_{n \to \infty} \langle v_n - v, u_n - u \rangle \leq 0$$

Then $[u, v] \in G(G)$ and $\langle v_n, u_n \rangle \to \langle v, u \rangle$.

**Proof:** By monotonicity,

$$0 \geq \limsup_{m,n \to \infty} \langle v_n - v_m, u_n - u_m \rangle$$

and so

$$\lim_{m,n \to \infty} \langle v_n - v_m, u_n - u_m \rangle = 0$$
Suppose then that \( \langle v_n, u_n \rangle \) fails to converge to \( \langle v, u \rangle \). Then there is a subsequence, still denoted with subscript \( n \) such that \( \langle v_n, u_n \rangle \to \mu \neq \langle v, u \rangle \). Let \( \varepsilon > 0 \). Then there exists \( M \) such that if \( n, m > M \), then

\[
|\langle v_n, u_n \rangle - \mu| < \varepsilon, |\langle v_n - v_m, u_n - u_m \rangle| < \varepsilon
\]

Then if \( m, n > M \),

\[
|\langle v_n - v_m, u_n - u_m \rangle| = |\langle v_n, u_n \rangle + \langle v_m, u_m \rangle - \langle v_n, u_m \rangle - \langle v_m, u_n \rangle| < \varepsilon
\]

Hence it is also true that

\[
|2\mu - (\langle v, u \rangle + \langle v, u \rangle)| < 3\varepsilon
\]

Now take a limit first with respect to \( n \) and then with respect to \( m \) to obtain

\[
|2\mu - (\langle v, u \rangle + \langle v, u \rangle)| < 3\varepsilon
\]

Since \( \varepsilon \) is arbitrary, \( \mu = \langle v, u \rangle \) after all. Hence the claim that \( \langle v_n, u_m \rangle \to \langle v, u \rangle \) is verified. Next suppose \([x, y] \in \mathcal{G}(G) \) and consider

\[
\langle v - y, u - x \rangle = \langle v, u \rangle - \langle v, x \rangle - \langle y, u \rangle + \langle y, x \rangle
\]

\[
= \lim_{n \to \infty} (\langle v_n, u_n \rangle - \langle v_n, x \rangle - \langle y, u_n \rangle + \langle y, x \rangle)
\]

\[
= \lim_{n \to \infty} (v_n - y, u_n - x) \geq 0
\]

and since \([x, y] \) is arbitrary, it follows that \( v \in Gu \).

Next suppose \( \lim_{n \to \infty} \langle v_n - v, u_n - u \rangle \leq 0 \). It is not known that \([u, v] \in \mathcal{G}(G) \).

\[
\lim_{n \to \infty} \sup \langle v_n, u_n \rangle - \langle v_n, u_n \rangle - \langle v_n, u \rangle + \langle v, u \rangle \leq 0
\]

\[
\lim_{n \to \infty} \langle v_n, u_n \rangle - \langle v, u \rangle \leq 0
\]

Thus \( \lim_{n \to \infty} \langle v_n, u_n \rangle \leq \langle v, u \rangle \). Now let \([x, y] \in \mathcal{G}(G) \)

\[
\langle v - y, u - x \rangle = \langle v, u \rangle - \langle v, x \rangle - \langle y, u \rangle + \langle y, x \rangle
\]

\[
\geq \lim_{n \to \infty} \sup \langle v_n, u_n \rangle - \langle v_n, x \rangle - \langle y, u_n \rangle + \langle y, x \rangle
\]

\[
\geq \lim_{n \to \infty} \inf \langle v_n - y, u_n - x \rangle \geq 0
\]

Hence \([u, v] \in \mathcal{G}(G) \). Now

\[
\lim_{n \to \infty} \sup \langle v_n - v, u_n - u \rangle \leq 0 \leq \lim_{n \to \infty} \inf \langle v_n - v, u_n - u \rangle
\]

the second coming from monotonicity and the fact that \( v \in Gu \). Therefore,

\[
\lim_{n \to \infty} \langle v_n - v, u_n - u \rangle = 0
\]

which shows that \( \lim_{n \to \infty} \langle v_n, u_n \rangle = \langle v, u \rangle \).

Similar reasoning implies
Lemma 75.8.4 Suppose $A$ is a set valued operator, $A : X \rightarrow \mathcal{P}(X)$ and $u_n^* \in Au_n$. Suppose also that $u_n \rightarrow u$ weakly and $u_n^* \rightarrow u^*$ weakly. Suppose also that

$$\lim_{m,n \to \infty} \langle u_n^* - u_m^*, u_n - u_m \rangle \leq 0$$

Then one can conclude that

$$\lim_{n \to \infty} \langle u_n^*, u_n - u \rangle \leq 0$$

Proof: It is assumed that

$$\lim_{m,n \to \infty} \left( \langle u_n^*, u_n \rangle + \langle u_m^*, u_m \rangle - \left( \langle u_n^*, u_m \rangle + \langle u_m^*, u_n \rangle \right) \right) \leq 0$$

Then is it the case that $\lim_{n \to \infty} \langle u_n^*, u_n - u \rangle \leq \langle u^*, u \rangle$? Let $\mu$ equal $\lim_{n \to \infty} \langle u_n^*, u_n \rangle$. Then in the above, it implies

$$(2\mu - \left( \langle u_n^*, u_m \rangle + \langle u_m^*, u_n \rangle \right)) < \varepsilon$$

whenever $m, n$ large enough. Thus taking $\lim_{n \to \infty} \lim_{m \to \infty}$ of the above, you get

$$(2\mu - \left( \langle u^*, u \rangle + \langle u^*, u \rangle \right)) < \varepsilon$$

Thus you at least need $\mu \leq \langle u^*, u \rangle$. That is, $\lim_{n \to \infty} \langle u_n^*, u_n \rangle \leq \langle u^*, u \rangle$. Hence

$$\lim_{n \to \infty} \langle u_n^*, u_n - u \rangle = \lim_{n \to \infty} \langle u_n^*, u_n \rangle - \langle u^*, u \rangle \leq \langle u^*, u \rangle - \langle u^*, u \rangle = 0$$

Definition 75.8.5 Let $x_\mu$ just defined be denoted by $J_\mu x$ and define also

$$G_\mu (x) = -\mu^{-(p-1)}F (x_\mu - x).$$

This $x_\mu$ is defined as follows.

$$0 \in F (x_\mu - x) + \mu^{p-1}Gx_\mu$$

Later, we will write $J_\mu u$ for $u_\mu$. Thus

$$0 = F (J_\mu u - u) + \mu^{p-1}z_\mu, z_\mu \in G (J_\mu u)$$

Also from this definition,

$$G_\mu (u) = -\mu^{-(p-1)}F (J_\mu u - u) = z_\mu \in G (J_\mu u)$$

Then there are some things which can be said about these operators.

Theorem 75.8.6 The following hold. Here $V$ is a reflexive Banach space with strictly convex norm. $G : D(G) \rightarrow \mathcal{P}(V')$ is maximal monotone. Then

1. $J_\mu$ and $G_\mu$ are bounded single valued operators defined on $V$. Bounded means they take bounded sets to bounded sets. Also $G_\mu$ is a monotone operator.
2. \( G_\mu, J_\mu \) are demicontinuous. That is, strongly convergent sequences are mapped to weakly convergent sequences.

3. For every \( x \in D(G) \), \( \|G_\mu(x)\| \leq \|Gx\| \equiv \inf \{ \|y^*\| : y^* \in Gx \} \). For every \( x \in \text{conv}(D(G)) \), it follows that \( \lim_{\mu \to 0} J_\mu(x) = x \). The new symbol means the closure of the convex hull. It is the closure of the set of all convex combinations of points of \( D(G) \).

Then \( A(\cdot, \omega) + G_\mu \) will be bounded and have the same limit properties as \( A(\cdot, \omega) \). As to measurability, \( G \) and hence \( G_\mu \) do not depend on \( \omega \) and so the measurability condition will hold.

What about the estimates? We need to consider the estimates. Recall what these were:

\[
\sup \left\{ \|u^*\|_{V'} : u^* \in A(u, \omega) \right\} \leq a(\omega) + b(\omega) \|u\|^{p-1}_{V'}
\]

(75.8.74)

where \( a(\omega), b(\omega) \) are nonnegative. Also assume the following coercivity estimate valid for each \( t \leq T \) and for some \( \lambda(\omega) \geq 0 \),

\[
\inf \left( \int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle Bu, u \rangle \, dt : u^* \in A(u, \omega) \right) \geq \delta(\omega) \int_0^t \|u\|^p \, ds - m(\omega)
\]

(75.8.75)

where \( m(\omega) \) is some nonnegative constant, \( \delta(\omega) > 0 \).

The coercivity is not too bad. This is because \( G_\mu \) is monotone and \( 0 \in D(G) \).

Therefore,

\[
\langle G_\mu u, u \rangle = \langle G_\mu u - G_\mu(0), u \rangle \geq 0
\]

so

\[
(G_\mu u, u) \geq - |G(0)| \|u\| \geq \frac{\delta(\omega)}{2} \|u\|^p - \tilde{m}(\omega)
\]

so the coercivity condition \( (75.8.75) \) will end up holding for \( A + G_\mu \). However, more needs to be considered for the growth condition.

From the definition of \( u_\mu \), there exists \( z_\mu \in Gu_\mu \)

\[
0 = F(u_\mu - u) + \mu^{p-1} z_\mu
\]

Then from the choice of \( F \), it is also the duality map from \( V \) to \( V' \) corresponding to \( p > 2 \).

\[
0 = \langle F(u_\mu - u), u_\mu \rangle_{V', V} + \mu^{p-1} \langle z_\mu, u_\mu \rangle_{V', V} \\
\geq \langle F(u_\mu - u), u_\mu \rangle_{V', V} - \mu^{p-1} |G(0)| \|u_\mu\|_V \\
= \|u_\mu - u\|^p_V - \mu \|u_\mu\|^p - \mu |G(0)| \|u_\mu\| \\
\geq \frac{1}{p} \|u_\mu - u\|^p - \frac{1}{p} \|u_\mu\|^p - \mu |G(0)| \|u_\mu\|
Thus
\[ 0 \geq \|u_\mu\| - \|u\|_p^p - \|u\|_p^p - p\mu |G(0)| \|u_\mu\| \]
This requires that there is some constant $C$ such that $\|u_\mu\| \leq C \|u\| + C$. The details follow.

Let $a = \|u_\mu\|, b = \|u\|$. Then they are both positive and
\[ 0 \geq |a - b|^p - b^p - \alpha a \]
where $\alpha = p\mu |G(0)|$. Want to say $a \leq Cb + C$ for some $C$. This is the conclusion of the following lemma.

**Lemma 75.8.7** Suppose $0 \geq |a - b|^p - b^p - \alpha a$ for $a, b \geq 0$ and $\alpha > 0$. Then there exists a constant $C$ such that
\[ a \leq Cb + C \]

**Proof:** If $b \geq a$, then there is nothing to show. Therefore, it suffices to show that the desired inequality holds for $a > b$. Thus from now on, $a > b$.
\[ 0 \geq (a - b)^p - b^p - \alpha a \]
Suppose $a > nb + n$. Let $x = b/a$. Then for $x \in [0, 1]$,
\[
0 \geq (1 - x)^p - x^p - \alpha \frac{1}{ap^{-1}} \\
\geq (1 - x)^p - x^p - \alpha \frac{1}{(nb + n)p^{-1}} \\
\geq (1 - x)^p - x^p - \alpha \frac{1}{(n)p^{-1}}
\]
Now for all $n$ large enough, the right side is a decreasing function of $x$ which is positive at $x = 0$ and negative at $x = 1$. Thus $x$ corresponds to the place where this function is negative. Taking a limit as $n \to \infty$, it follows that we must have
\[ x \geq \delta, \delta \in (0, 1) \]
It is where $(1 - x)^p - x^p = 0$. Thus $x = \frac{\delta}{a} \geq \delta$. Then, since $a > nb + n$,
\[ \frac{1}{\delta} b \geq a > nb + n \]
Now this is a contradiction when $n$ is taken increasingly large. Hence, for large enough $n, a \leq nb + n$.

It follows that $\|u_\mu\| \leq C \|u\| + C$ for some $C$. Hence,
\[
\|G_\mu u\| \leq \frac{1}{\mu p^{-1}} \|u_\mu - u\|_p^{p-1} \leq \frac{1}{\mu p^{-1}} (\|u_\mu\| + \|u\|)^{p-1} \\
\leq \frac{2^{p-2}}{\mu p^{-1}} (\|u_\mu\|^{p-1} + \|u\|^{p-1})
\]
75.8. ADDING A QUASI-BOUNDED OPERATOR

\[
\begin{align*}
    &\leq \frac{2^{p-2}}{\mu^{p-1}} \left( (C \|u\| + C)^{p-1} + \|u\|^{p-1} \right) \\
    &\leq \frac{2^{p-2}}{\mu^{p-1}} \left( 2^{p-2} \left( C^{p-1} \|u\|^{p-1} + C^{p-1} \right) + \|u\|^{p-1} \right) \\
    &\leq C_\mu \|u\|^{p-1} + C_\mu
\end{align*}
\]

This is the case that \( p \geq 2 \). The case that \( p > 1 \) but \( p < 2 \) is easier. In this case,

\[
\frac{1}{\mu^{p-1}} (\|u_\mu\| + \|u\|)^{p-1} \leq \frac{1}{\mu^{p-1}} \left( \|u_\mu\|^{p-1} + \|u\|^{p-1} \right)
\]

A similar inequality holds. Thus the necessary growth condition is obtained for \( G_\mu \) and consequently, the necessary growth condition remains valid for \( G_\mu + A \). It was noted earlier that the coercivity estimate continues to hold.

It follows that there exists a solution to the integral equation

\[
Bu(t, \omega) + \int_0^t z(s, \omega) \, ds + \int_0^t G_\mu(u(s, \omega)) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega)
\]

where \( z(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \) which has the measurability described above. That is, both \( u \) and \( z \) are product measurable. Then acting on \( u \mathcal{X}_{[0,t]} \) and using the estimates valid for \( \lambda \) large enough, one can get an estimate of the form

\[
\frac{1}{2} \langle Bu(t), u(t) \rangle - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \|u(s)\|_V^p \, ds + \int_0^t \langle G_\mu u, u \rangle \, ds \leq \lambda \int_0^t \langle Bu, u \rangle \, ds + C(f)
\]  

(75.8.76)

Now \( G_\mu \) is monotone and so,

\[
\langle G_\mu u, u \rangle = \langle G_\mu u - G_\mu 0, u \rangle + \langle G_\mu (0), u \rangle \geq \langle G_\mu (0), u \rangle \geq - |G(0)| \|u\|
\]

It follows easily from standard manipulations and (75.8.76) that \( \|u\|_V \) is bounded independent of \( \mu \). That is, there is a constant \( C \) independent of \( \mu \) such that

\[
\|u\|_V \leq C
\]  

(75.8.77)

The details follow. The above inequality \((75.8.76)\) implies that by acting on \( u \mathcal{X}_{[0,t]} \),

\[
\frac{1}{2} \langle Bu(t), u(t) \rangle - \frac{1}{2} \langle Bu_0, u_0 \rangle + \int_0^t \|u(s)\|_V^p \, ds - \int_0^t |G(0)| \|u\|_V \, ds \leq \lambda \int_0^t \langle Bu, u \rangle \, ds + C(f)
\]

Then by Gronwall’s inequality and adjusting constants,

\[
\langle Bu(t), u(t) \rangle + \int_0^t \|u(s)\|_V^p \, ds \leq C(u_0, f, \lambda) + C(\lambda) \int_0^t |G(0)| \|u\|_V \, ds
\]  

(75.8.78)

so it is clear that there is an inequality of the form

\[
\sup_{t \in [0,T]} \langle Bu(t), u(t) \rangle + \int_0^T \|u(s)\|_V^p \, ds \leq C(u_0, f, \lambda)
\]
Then returning to 75.8.76, all terms are bounded except \( \int_0^T \langle G_\mu u, u \rangle \, dt \), so this term must also be bounded for \( t = T \) also. Thus

\[
\left| \int_0^T \langle G_\mu u, u \rangle \, dt \right| \leq C
\]

where \( C \) is independent of \( \mu \). We denote by \( u_\mu \) the solution to the above equation.

Here is the definition of quasi-bounded.

**Definition 75.8.8** A set valued operator \( G \) is quasi-bounded if whenever \( x \in D(G) \) and \( x^* \in Gx \) are such that

\[
|\langle x^*, x \rangle|, \|x\| \leq M,
\]

it follows that \( \|x^*\| \leq K_M \). Bounded would mean that if \( \|x\| \leq M \), then \( \|x^*\| \leq K_M \).

Here you only know this if there is another condition.

**Assumption 75.8.9** \( G : D(G) \to P(V') \) is quasi-bounded and maximal monotone.

By Proposition 23.7.23 an example of a quasi-bounded operator is a maximal monotone operator \( G \) for which \( 0 \in \text{int} (D(G)) \).

Now \( G_\mu u_\mu \in G J_\mu u_\mu \) as noted above. Therefore, there exists \( g_\mu \in G (J_\mu u_\mu) \) such that

\[
C \geq \langle G_\mu u_\mu, u_\mu \rangle_{V', V} = \langle g_\mu, u_\mu \rangle_{V', V} = \langle g_\mu, J_\mu u_\mu \rangle_{V', V} + \langle g_\mu, u_\mu - J_\mu u_\mu \rangle_{V', V} \quad (75.8.79)
\]

\[
\geq - |G(0)| \|J_\mu u_\mu\|_V + \left\langle - \frac{1}{\mu^{p-1}} F \left(J_\mu u_\mu - u_\mu\right), u_\mu - J_\mu u_\mu \right\rangle_{V', V} \\
= - |G(0)| \|J_\mu u_\mu\|_V + \frac{1}{\mu^{p-1}} \|J_\mu u_\mu - u_\mu\|_V^p \quad (75.8.80)
\]

Thus the fact that \( \|u_\mu\| \) is bounded independent of \( \mu \) implies that \( \|J_\mu u_\mu\| \) is also bounded and that in fact \( \|u_\mu - J_\mu u_\mu\|_V \to 0 \) as \( \mu \to 0 \). This follows from consideration of the last line of the above formula. Note also that

\[
\langle g_\mu, u_\mu - J_\mu u_\mu \rangle_{V', V} = \frac{1}{\mu^{p-1}} \|J_\mu u_\mu - u_\mu\|_V^p \text{ is bounded.} \quad (75.8.81)
\]

Then from 75.8.76, it follows that \( \langle g_\mu, J_\mu u_\mu \rangle_{V', V} \) is bounded. By the assumption that \( G \) is quasi-bounded, \( g_\mu \) must also be bounded.

Then we have shown

\[
Bu_\mu (t, \omega) + \int_0^t z_\mu (s, \omega) \, ds + \int_0^t g_\mu (s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega) \quad (75.8.82)
\]

where

\[
\|g_\mu\|_{V'} + \|z_\mu\|_{V'} + \sup_{t \in [0, T]} \langle Bu_\mu, u_\mu \rangle (t) + \|J_\mu u_\mu\|_V + \|u_\mu\|_V + \|B u_\mu \|_{V'} \leq C \quad (75.8.83)
\]
The last term in the sum being bounded follows from the integral equation and the fundamental theorem of calculus along with the boundedness $f,g,z$. In addition to this, the estimate \((75.8.80)\) implies

$$\lim_{\mu \to 0} \| J_\mu u_\mu - u_\mu \|_V = 0. \quad (75.8.84)$$

There is a subsequence, $\mu \to 0$ still denoted as $\mu$ such that

$$g_\mu \to g \text{ weakly in } V' \quad (75.8.85)$$
$$z_\mu \to z \text{ weakly in } V' \quad (75.8.86)$$
$$u_\mu \to u \text{ weakly in } V \quad (75.8.87)$$
$$J_\mu u_\mu \to u \text{ weakly in } V \quad (75.8.88)$$
$$\langle Bu_\mu \rangle' \to (Bu)' \text{ weakly in } V' \quad (75.8.89)$$
$$Bu_\mu (t) \to Bu (t) \text{ weakly in } V' \quad (75.8.90)$$

Now consider two of these for $\mu$ and $\nu$. Subtract and act on $u_\mu - u_\nu$. Then one obtains

$$\langle Bu_\mu - Bu_\nu, u_\mu - u_\nu \rangle + \int_0^T \langle z_\mu - z_\nu, u_\mu - u_\nu \rangle + \int_0^T \langle g_\mu - g_\nu, u_\mu - u_\nu \rangle = 0 \quad (75.8.91)$$

Consider that last term for $t = T$. It equals

$$\int_0^T \langle G_\mu u_\mu - G_\nu u_\nu, u_\mu - u_\nu \rangle \geq 0 \quad (75.8.92)$$
$$\int_0^T \langle G_\mu u_\mu - G_\nu u_\nu, J_\mu u_\mu - J_\nu u_\nu \rangle + \int_0^T \langle g_\mu - g_\nu, u_\mu - J_\mu u_\mu \rangle$$
$$+ \int_0^T \langle g_\mu - g_\nu, u_\nu - J_\nu u_\nu \rangle + \varepsilon (\mu, \nu)$$

where

$$|\varepsilon (\mu, \nu)| \leq \left( \int_0^T (\|g_\mu\| + \|g_\nu\|)^p \right)^{1/p'} \left( \int_0^T (\|u_\mu - J_\mu u_\mu\| + \|u_\nu - J_\nu u_\nu\|)^p \right)^{1/p}$$
$$\leq 2C (\|u_\mu - J_\mu u_\mu\|_V + \|u_\nu - J_\nu u_\nu\|_V)$$

Adjusting constants and using \((75.8.81)\)

$$\leq C \left( \mu^{(1-(1/p))} + \nu^{(1-(1/p))} \right)$$
Thus
\[ \int_0^T \langle G_\mu u_\mu - G_\nu u_\nu, u_\mu - u_\nu \rangle = \int_0^T \langle g_\mu - g_\nu, J_\mu u_\mu - J_\nu u_\nu \rangle + \epsilon(\mu, \nu) \]
where \( \lim_{\mu, \nu \to 0} \epsilon(\mu, \nu) = 0 \). It follows from
\[ \text{Lemma 75.8.91} \]
\[ \lim \sup_{\mu, \nu \to 0} \left( \int_0^T \langle z_\mu - z_\nu, u_\mu - u_\nu \rangle \, ds + \epsilon(\mu, \nu) \right) \]
\[ = \lim \sup_{\mu, \nu \to 0} \left( \int_0^T \langle z_\mu - z_\nu, u_\mu - u_\nu \rangle \, ds \right) \leq 0 \]
From Lemma \[ \text{75.8.4} \]
\[ \lim \inf_{\mu \to 0} \langle z_\mu, u_\mu - u \rangle_{V', V} \leq 0 \]
By the limit condition for \( A(u, \omega) \), for each \( v \in V \), there exists \( z(v) \in Au \) such that
\[ \lim \inf_{\mu \to 0} \langle z_\mu, u_\mu - v \rangle = \lim \inf_{\mu \to 0} \left( \langle z_\mu, u_\mu - u \rangle + \langle z_\mu, u - v \rangle \right) \]
\[ = \langle z, u - v \rangle \geq \langle z(v), u - v \rangle \]
Since \( A(u, \omega) \) is convex and closed, separation theorems imply that \( z \in Au \). Return to the equation solved.
\[ (Bu_\mu) \, + z_\mu + g_\mu = f \]
Then act on \( u_\mu - u \) and use monotonicity arguments to write
\[ \langle (Bu_\mu)', u_\mu - u \rangle_{V', V} + \langle z_\mu, u_\mu - u \rangle_{V', V} + \langle g_\mu, u_\mu - u \rangle_{V', V} \leq \langle f, u_\mu - u \rangle_{V', V} \]
Then it was shown above that
\[ 0 \geq \lim \sup_{\mu \to 0} \langle z_\mu, u_\mu - u \rangle_{V', V} \geq \lim \inf_{\mu \to 0} \langle z_\mu, u_\mu - u \rangle_{V', V} \geq \langle z(u), u - u \rangle_{V', V} = 0 \]
and so, from \[ \text{Theorem 75.8.2} \]
\[ \lim_{\mu \to 0} \langle g_\mu, u_\mu - u \rangle_{V', V} = \lim_{\mu \to 0} \langle g_\mu, J_\mu u_\mu - u \rangle_{V', V} = 0 \]
and so
\[ \lim_{\mu \to 0} \langle g_\mu, J_\mu u_\mu \rangle_{V', V} = \langle g, u \rangle_{V', V} \]
Now let \( [a, b] \in \mathcal{G}(G) \). Then
\[ \langle b - g, a - u \rangle = \lim_{\mu \to 0} \langle b - g_\mu, a - J_\mu u_\mu \rangle \geq 0 \]
because \( g_\mu \in G(J_\mu u_\mu) \). Since \( G \) is maximal monotone, it follows that \( [u, g] \in \mathcal{G}(G) \).
This has shown that for each \( \omega \) fixed, and every sequence of solutions to the integral equation \( \{u_\mu\} \), each function \( \{Bu_\mu\} \) being product measurable by Theorem
Adding a Quasi-Bounded Operator

There exists a subsequence which converges to a solution \( u \) to the integral equation. In particular, \( t \to Bu(t) \) is weakly continuous into \( V' \). Then by the fundamental measurable selection theorem, Theorem 75.8.11, there exists a product measurable function \( \bar{u}(t, \omega) \) with values in \( V \) weakly continuous in \( t \) and a sequence depending on \( \omega \), \( \{ u_{\mu(\omega)} \} \) such that for each \( \omega \), \( \lim_{\mu(\omega) \to 0} u_{\mu(\omega)}(\cdot, \omega) = \bar{u}(\cdot, \omega) \) weakly in \( V \). However, from the above argument, for each \( \omega \), there is a further subsequence, still denoted with subscript \( \mu(\omega) \) such that in \( V' \),

\[
\lim_{\mu(\omega) \to 0} u_{\mu(\omega)}(\cdot, \omega) = u(\cdot, \omega)
\]

where \( u \) is a solution to the integral equation. Since \( u(\cdot, \omega) = \bar{u}(\cdot, \omega) \) in \( V \) it follows that these must be equal a.e. and hence \( (t, \omega) \to u(t, \omega) \) is product measurable. This proves the following theorem.

**Theorem 75.8.10** Suppose 75.8.4 - 75.8.7 and \( B \) as described above and \( u_0 \) is \( F \) measurable. Also let \( G : D(G) \subseteq V \to P(V') \) be maximal monotone and quasi-bounded.

Then, there exists a solution \( u \) of the integral equation

\[
Bu(t, \omega) + \int_0^t z(s, \omega) \, ds + \int_0^t g(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega),
\]

where \( (t, \omega) \to u(t, \omega) \) is product measurable \( (t, \omega) \to z(t, \omega) \) also. Moreover, for each \( \omega \), \( Bu(t, \omega) = B(u(t, \omega)) \) a.e. \( t \) and \( z(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \), and \( g(\cdot, \omega) \in G(u(\cdot, \omega)) \) for each \( \omega \).

Note that in the case of most interest where you have a Gelfand triple and \( B \) is the identity, the fundamental theorem of calculus implies easily that \( \omega \to z(s, \omega) + g(s, \omega) \) is measurable for a.e. \( s \). One can also generalize to the following in which a measurable \( q(t, \omega) \) is added.

**Corollary 75.8.11** Suppose 75.8.4 - 75.8.7 and \( B \) as described above and \( u_0 \) is \( F \) measurable. Also let \( G : D(G) \subseteq V \to P(V') \) be maximal monotone and quasi-bounded. Let \( (t, \omega) \to q(t, \omega) \) be product measurable into \( V' \) and let \( t \to q(t, \omega) \) be continuous, \( q(0, \omega) = 0 \). Then, there exists a solution \( u \) of the integral equation

\[
Bu(t, \omega) + \int_0^t z(s, \omega) \, ds + \int_0^t g(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega) + Bq(t, \omega),
\]

where \( (t, \omega) \to u(t, \omega), (t, \omega) \to z(t, \omega), g(t, \omega) \) are product measurable. Moreover, for each \( \omega \), \( Bu(t, \omega) = B(u(t, \omega)) \) a.e. \( t \) and \( z(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \) for a.e. \( t \), and \( g(\cdot, \omega) \in G(u(\cdot, \omega)) \) for each \( \omega \).

**Proof:** Define a stopping time

\[
\tau_n(\omega) = \inf \{ t : q(t, \omega) > n \}
\]
Then let \( \tilde{A}(\cdot, \omega) \equiv A(q^n(\cdot, \omega) + w, \omega) \). Then \( \tilde{A} \) satisfies the same properties as \( A \) and so there exists a solution to the integral equation

\[
Bw_n(t, \omega) + \int_0^t z_n(s, \omega) \, ds + \int_0^t g_n(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega)
\]

where \( w_n, z_n, g_n \) are product measurable, \( z_n(\cdot, \omega) \in \tilde{A}(w_n(\cdot, \omega), \omega) \) a.e. \( t \). By continuity of \( t \to q(t, \omega) \), \( \tau_n = \infty \) for all \( n \) sufficiently large and so \( q(t, \omega) = q^n(t, \omega) \). As before, for each \( \omega \), one obtains the convergences \( 75.8.85 - 75.8.90 \) as \( n \to \infty \). As before, for each \( \omega, z(\cdot, \omega) \in \tilde{A}(w(\cdot, \omega), \omega) \) a.e. where \( t \to w(t, \omega) \) is the function to which \( w_n(\cdot, \omega) \) converges weakly. Note that the estimates allowing this to happen are dependent on \( \omega \). However, one can apply Theorem \( 75.2.10 \) as before and obtain a solution to

\[
Bw(t, \omega) + \int_0^t z(s, \omega) \, ds + \int_0^t g(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega)
\]

such that \( w, z, g \) are product measurable into \( V \) or \( V' \) and \( z(\cdot, \omega) \in \tilde{A}(w(\cdot, \omega), \omega) \). Now let \( u(t, \omega) = w(t, \omega) + q(t, \omega) \) to obtain the existence of the desired solution in the corollary. \( \blacksquare \)
Chapter 76

A Different Approach

76.1 Summary Of The Problem

The situation is as follows. There are spaces $V \subseteq W$ where $V, W$ are reflexive separable Banach spaces. It is assumed that $V$ is dense in $W$. Define the space for $p > 1$

$$V \equiv L^p ([0, T]; V)$$

where in each case, the $\sigma$ algebra of measurable sets will be $B([0, T])$ the Borel measurable sets. Thus, from the Riesz representation theorem,

$$V' = L^{p'} ([0, T]; V'),$$

We also assume $(\Omega, \mathcal{F})$ is a measurable space. No measure is needed. Also

$$V \subseteq W, \ W' \subseteq V', \ V \text{ dense in } W,$$

$B(t)$ will be a linear operator, $B(t) : W \rightarrow W'$ which satisfies

1. $\langle B(t)x, y \rangle = \langle B(t)y, x \rangle$

2. $\langle B(t)x, x \rangle \geq 0$

3. $B \in C^1 ([0, T]; \mathcal{L}(W, W'))$ so in particular, the time derivative is bounded.

In the above formulae, $\langle \cdot , \cdot \rangle$ denotes the duality pairing of the Banach space $W$, with its dual space. We will use this notation in the present paper, the exact specification of which Banach space being determined, by the context in which this notation occurs.

For example, you could simply take $W = H = H'$ and $B$ the identity and consider a standard Gelfand triple where $H$ is a Hilbert space and $B$ equal to the identity.
The product measurable sets are those in the smallest σ algebra which contains the measurable rectangles $B \times A$ where $B \in \mathcal{B}([0,T])$, $A \in \mathcal{F}$. The paper is about the existence of product measurable solutions to the system

$$(Bu(\cdot,\omega))' + u^*(\cdot,\omega) = f(\cdot,\omega) \text{ in } \mathcal{V}'$$

$Bu(0,\omega) = Bu_0(\omega)$

$$u^*(\cdot,\omega) \in A(u(\cdot,\omega),\omega).$$

(76.1.1)

(76.1.2)

The evolution inclusion is well understood for fixed $\omega$. However, we will show the existence of a solution $u$, $u^*$ such that $(t,\omega) \mapsto u(t,\omega)$ and $(t,\omega) \mapsto u^*(t,\omega)$ are product measurable in this solution. Essentially, we show the existence of a measurable selection in the set of solutions. There are no assumptions made on the measurable space. It is just a set with a σ algebra of subsets. Essentially this involves showing that the usual limit processes preserve measurability in some sense. The main theorems in this paper are essentially measurable selection theorems for the set of solutions to these implicit inclusions.

To begin with, we will assume $p \geq 2$. The reason for this is that we want to consider

$$\int_0^T \langle B' u, u \rangle \, dt$$

and this won’t make sense unless $p \geq 2$. However, this restriction is not necessary if $B$ is a constant operator, as we show in the succeeding section.

### 76.1.1 General Assumptions On $A$

The case $A(u,\omega)$ for $u \in \mathcal{V}$ given by $A(u,\omega)(t) = A(u(t),\omega)$ is included as a special case. In addition to this commonly used situation, we are including the case where $A(u,\omega)(t)$ depends on past values of $u(s)$ for $s \leq t$. This makes our theory useful in situations where the problem is second order in $t$. The following definition is the standard one.

**Definition 76.1.1** For $X$ a reflexive Banach space, we say $A : X \to \mathcal{P}(X')$ is pseudomonotone and bounded if the following hold.

1. The set $Au$ is nonempty, closed and convex for all $u \in X$. $A$ takes bounded sets to bounded sets.

2. If $u_i \to u$ weakly in $X$ and $u_i^* \in Au_i$ is such that

$$\limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle \leq 0,$$

(76.1.3)

then, for each $v \in X$, there exists $u^*(v) \in Au$ such that

$$\liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle \geq \langle u^*(v), u - v \rangle.$$

(76.1.4)
We will assume in this section in which $B$ is time dependent that $p \geq 2$. The specific assumptions on $A(\cdot, \omega)$ are described next.

- **growth estimate**
  Assume the specific estimate
  \[
  \sup \{ \| u^* \|_{V'} : u^* \in A(u, \omega) \} \leq a(\omega) + b(\omega) \| u \|_{V'}^{p-1}
  \]  
  where $a(\omega), b(\omega)$ are nonnegative.

- **coercivity estimate**
  Also assume the coercivity condition: valid for each $t \leq T$,
  \[
  \inf \left( \int_0^t \langle u^*, u \rangle ds : u^* \in A(u, \omega) \right)
  + \frac{1}{2} \int_0^t \langle B' u, u \rangle ds \geq \delta(\omega) \int_0^t \| u \|_V^p ds - m(\omega)
  \]  
  where $m(\omega)$ is some nonnegative constant, $\delta(\omega) > 0$. In fact, it is often enough to assume the left side is given by
  \[
  \inf \left( \int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle B u, u \rangle ds : u^* \in A(u, \omega) \right)
  \]  
  for some $\lambda(\omega)$ by using a suitable exponential shift argument and changing the dependent variable. We will sometimes denote weak convergence by $\rightharpoonup$.

- **limit condition**
  If $u_i \rightharpoonup u$ in $V$ and $(Bu_i)' \rightharpoonup (Bu)'$ in $V'$, $u_i^* \in A(u_i)$, $\rightharpoonup$ denoting weak convergence, then if
  \[
  \limsup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{V', V} \leq 0
  \]  
  it follows that for all $v \in V$, there exists $u^*(v) \in Au$ such that
  \[
  \liminf_{i \to \infty} \langle u_i^*, u_i - v \rangle_{V', V} \geq \langle u^*(v), u - v \rangle_{V', V}
  \]  

- **measurability condition**
  For $\omega \to u(\cdot, \omega)$ measurable into $V$,
  \[
  \omega \to A(u(\cdot, \omega), \omega) \text{ has a measurable selection into } V'.
  \]  
  This last condition means there is a function $\omega \to u^*(\omega)$ which is measurable into $V'$ such that $u^*(\omega) \in A(u(\cdot, \omega), \omega)$. This is assured to take place if the following standard measurability condition is satisfied for all $O$ open in $V'$:
  \[
  \{ \omega : A(u(\cdot, \omega), \omega) \cap O \neq \emptyset \} \in \mathcal{F}
  \]
CHAPTER 76. A DIFFERENT APPROACH

See for example [52], [4] or the chapter on measurable multifunctions Chapter 46. Our assumption is implied by this one but they are not equivalent. Thus what is considered here is more general than an assumption that \( \omega \to A(u(\cdot, \omega), \omega) \) is set valued measurable.

Note that this condition would hold if \( u \to A(t, u, \omega) \) is bounded and pseudomonotone as a single valued map from \( V \) to \( V' \) and \( (t, \omega) \to A(t, u, \omega) \) is product measurable into \( V' \) for each \( u \). One would use the demicontinuity of \( u \to A(t, u, \omega) \) which comes from a pseudo monotone and bounded assumption and consider a sequence of simple functions \( u_n(\cdot, \omega) \to u(\cdot, \omega) \) in \( V \) for \( u \) measurable, each \( u_n(\cdot, \omega) \) being in \( V \). Then the measurability of \( A(t, u_n, \omega) \) would attach to \( A(t, u, \omega) \) in the limit. In the situation where \( A(\cdot, \omega) \) satisfies a suitable upper semicontinuity condition, it is enough to assume only that \( \omega \to A(u, \omega) \) has a measurable selection for each \( u \in V \). This is a straightforward exercise in approximating with simple functions and then using upper semicontinuity instead of continuity.

We assume always that the norm on the various reflexive Banach spaces is strictly convex.

76.1.2 Preliminary Results

We use the following well known theorem [52]. It is stated here for the situation in which a Holder condition is given rather than a bound on weak derivatives. See Theorem 51.7.4 on Page 1213.

**Theorem 76.1.2** Let \( E \subseteq F \subseteq G \) where the injection map is continuous from \( F \) to \( G \) and compact from \( E \) to \( F \). Let \( p \geq 1 \), let \( q > 1 \), and define

\[
S = \{ u \in L^p([a,b], E) : \text{for some } C, \|u(t) - u(s)\|_G \leq C |t - s|^{1/q} \}
\]

and \( \|u\|_{L^p([a,b], E)} \leq R \).

Thus \( S \) is bounded in \( L^p([a,b], E) \) and Holder continuous into \( G \). Then \( S \) is precompact in \( L^p([a,b], F) \). This means that if \( \{u_n\}_{n=1}^\infty \subseteq S \), it has a subsequence \( \{u_{n_k}\} \) which converges in \( L^p([a,b], F) \). The same conclusion can be drawn if it is known instead of the Holder condition that \( \|u\|_{L^1([a,b], X)} \) is bounded.

Next are some measurable selection theorems which form an essential part of showing the existence of measurable solutions. They are not dependent on there being a measure but in the applications of most interest to us, there is typically a probability measure. First is a basic selection theorem for a set of limits. See Lemma 46.2.2 on Page 1580.

**Theorem 76.1.3** Let \( U \) be a separable reflexive Banach space. Suppose there is a sequence \( \{u_j(\omega)\}_{j=1}^\infty \) in \( U \), where \( \omega \to u_j(\omega) \) is measurable and for each \( \omega \),

\[
\sup_j \|u_j(\omega)\|_U < \infty.
\]
76.1. SUMMARY OF THE PROBLEM

Then there exists a function \( \omega \to u(\omega) \) with values in \( U \) such that \( \omega \to u(\omega) \) is measurable, and a subsequence \( n(\omega) \), depending on \( \omega \), such that

\[
\lim_{n(\omega) \to \infty} u_{n(\omega)}(\omega) = u(\omega) \quad \text{weakly in } U.
\]

Next is a specialization to the situation where the Banach space is a function space. The proof is in [72]. This gives a result on product measurability. It is Theorem 76.1.4 on Page 2739.

**Theorem 76.1.4** Let \( V \) be a reflexive separable Banach space with dual \( V' \), and let \( p, p' \) be such that \( p > 1 \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Let the functions \( t \to u_n(t, \omega) \), for \( n \in \mathbb{N} \), be in \( L^p([0, T]; V) \equiv V \) and \( (t, \omega) \to u_n(t, \omega) \) be \( B([0, T]) \times F \equiv P \) measurable into \( V \). Suppose

\[
\|u_n(\cdot, \omega)\|_V \leq C(\omega),
\]

for all \( n \). (Thus, by weak compactness, for each \( \omega \), each subsequence of \( \{u_n\} \) has a further subsequence that converges weakly in \( V \) to \( u(\cdot, \omega) \in V \). (\( u \) not known to be \( P \) measurable))

Then, there exists a product measurable function \( u \) such that \( t \to u(t, \omega) \) is in \( V \) and for each \( \omega \) a subsequence \( u_{n(\omega)} \) such that \( u_{n(\omega)}(\cdot, \omega) \to u(\cdot, \omega) \) weakly in \( V \).

Next is what it means to be measurable into \( V \) or \( V' \). Such functions have representatives which are product measurable.

**Lemma 76.1.5** Let \( f(\cdot, \omega) \in V' \). Then if \( \omega \to f(\cdot, \omega) \) is measurable into \( V' \), it follows that for each \( \omega \), there exists a representative \( \hat{f}(\cdot, \omega) \in V' \), \( \hat{f}(\cdot, \omega) = f(\cdot, \omega) \) in \( V' \) such that \( (t, \omega) \to \hat{f}(t, \omega) \) is product measurable. If \( f(\cdot, \omega) \in V' \) and \( (t, \omega) \to f(t, \omega) \) is product measurable, then \( \omega \to f(\cdot, \omega) \) is measurable into \( V' \). The same holds replacing \( V' \) with \( V \).

**Proof:** If a function \( f \) is measurable into \( V' \), then there exist simple functions \( f_n \)

\[
\lim_{n \to \infty} \|f_n(\omega) - f(\omega)\|_{V'} = 0, \quad \|f_n(\omega)\|_{V'} \leq 2\|f(\omega)\|_{V'} = C(\omega)
\]

Now one of these simple functions is of the form

\[
\sum_{i=1}^{M} c_i \chi_{E_i}(\omega)
\]

where \( c_i \in V' \). Therefore, there is no loss of generality in assuming that \( c_i(t) = \sum_{j=1}^{N} d_{ij} \chi_{F_j}(t) \) where \( d_{ij} \in V' \). Hence we can assume each \( f_n \) is product measurable into \( B(V') \times F \). Then by Theorem 76.1.4 there exists \( \hat{f}(\cdot, \omega) \in V' \) such that \( \hat{f} \) is product measurable and a subsequence \( f_{n(\omega)} \) converging weakly in \( V' \) to \( \hat{f}(\cdot, \omega) \) for each \( \omega \). Thus \( f_{n(\omega)}(\omega) \to f(\omega) \) strongly in \( V' \) and \( f_{n(\omega)}(\omega) \to \hat{f}(\omega) \) weakly in \( V' \). Therefore, \( \hat{f}(\omega) = f(\omega) \) in \( V' \) and so it can be assumed that if \( f \) is measurable
into $V'$ then for each $\omega$, it has a representative $\hat{f}(\omega)$ such that $(t, \omega) \to \hat{f}(t, \omega)$ is product measurable.

If $f$ is product measurable into $V'$ and each $f(\cdot, \omega) \in V'$, does it follow that $f$ is measurable into $V'$? By measurability, $f(t, \omega) = \lim_{n \to \infty} \sum_{i=1}^{m_n} c_{n,i} X_{E_{n,i}}(t, \omega) = \lim_{n \to \infty} f_n(t, \omega)$ where $E_{n,i}$ is product measurable and we can assume $\|f_n(t, \omega)\|_{V'} \leq 2 \|f(t, \omega)\|$. Then by product measurability, $\omega \to f_n(\cdot, \omega)$ is measurable into $V'$ because if $g \in V$ then

$$\omega \to \langle f_n(\cdot, \omega), g \rangle$$

is of the form

$$\omega \to \sum_{i=1}^{m_n} \int_0^T \langle c_{n,i} X_{E_{n,i}}(t, \omega), g(t) \rangle dt$$

and this is $\mathcal{F}$ measurable since $E_{n,i}$ is product measurable. Thus, it is measurable into $V'$ as desired and

$$\langle f(\cdot, \omega), g \rangle = \lim_{n \to \infty} \langle f_n(\cdot, \omega), g \rangle, \omega \to \langle f_n(\cdot, \omega), g \rangle$$

is $\mathcal{F}$ measurable.

By the Pettis theorem, $\omega \to \langle f(\cdot, \omega), g \rangle$ is measurable into $V'$. Obviously, the conclusion is the same for these two conditions if $V'$ is replaced with $V$.  

The following theorem is also useful. It is really a generalization of the familiar Gram Schmidt process. See Lemma 31.4.2 on Page 1171.

**Theorem 76.1.6** Suppose $V, W$ are separable Banach spaces, such that $V$ is dense in $W$ and $B \in \mathcal{L}(W, W')$ satisfies

$$\langle Bx, x \rangle \geq 0, \langle Bx, y \rangle = \langle By, x \rangle, B \neq 0.$$

Then there exists a countable set $\{e_i\}$ of vectors in $V$ such that

$$\langle Be_i, e_j \rangle = \delta_{ij}$$

and for each $x \in W$,

$$\langle Bx, x \rangle = \sum_{i=1}^{\infty} |\langle Bx, e_i \rangle|^2,$$

and also

$$Bx = \sum_{i=1}^{\infty} \langle Bx, e_i \rangle Be_i,$$

the series converging in $W'$. In case $B = B(\omega)$ where $\omega \to B(\omega)$ is measurable into $\mathcal{L}(W, W')$, these vectors $e_i$ will also depend on $\omega$ and will be measurable functions of $\omega$. In particular, we could let $\omega = t$ with the Lebesgue measurable sets.

The following result, found in [82] is well known. See Section 23.2 on Page 813.
Theorem 76.1.7 If a single valued map, \( A : X \to X' \) is monotone, hemicontinuous, and bounded, then \( A \) is pseudo monotone. Furthermore, the duality map, \( J^{-1} : X \to X' \) which satisfies \( \langle J^{-1} f, f \rangle = \|f\|^2, \|J^{-1} f\|_X = \|f\|_X \) is strictly monotone hemicontinuous and bounded. So is the duality map \( F : X \to X' \) which satisfies \( \|Ff\|_X = \|f\|_X^{p-1}, \langle Ff,f \rangle = \|f\|^p_0 \) for \( p > 1 \).

The following fundamental result will be of use in what follows. There is somewhat more in this than will be needed. \( B \) is a possibly degenerate operator satisfying only the following:

\[
B \in \mathcal{L}(W,W'), \langle Bu, u \rangle \geq 0, \langle Bu, v \rangle = \langle Bv, u \rangle
\]  

(76.1.10)

where here \( V \subseteq W \) and \( V \) is dense in \( W \).

Also one can obtain the following for \( p \geq 2 \). It is an integration by parts formula. See Theorem 31.6.4 on Page 126.

Proposition 76.1.8 Let \( p \geq 2 \) in what follows. For \( u, v \in X \), the following hold. If \( B \) is time independent, then it is not necessary to assume \( p \geq 2 \). It is enough to assume \( p > 1 \).

1. \( t \to \langle B(t)u(t), v(t) \rangle_{W', W} \) equals an absolutely continuous function a.e., denoted by \( \langle Bu, v \rangle (\cdot) \).
2. \( \langle Lu(t), u(t) \rangle = \frac{1}{2} [(Bu,u)'(t) + \langle B'(t)u(t), u(t) \rangle] \) a.e. \( t \)
3. \( |\langle Bu, v \rangle(t)| \leq C \|u\|_X \|v\|_X \) for some \( C > 0 \) and for all \( t \in [0, T] \).
4. \( t \to B(t)u(t) \) equals a function in \( C(0, T; W'; W) \) a.e., denoted by \( Bu(\cdot) \).
5. \( \sup \{\|Bu(t)\|_{W'}, t \in [0, T] \} \leq C\|u\|_X \) for some \( C > 0 \).

If \( K : X \to X' \) is given by

\[
\langle Ku, v \rangle_{X', X} = \int_0^T \langle Lu(t), v(t) \rangle dt + \langle Bu, v \rangle(0),
\]

then

6. \( K \) is linear, continuous and weakly continuous.

7. \( \langle Ku, u \rangle = \frac{1}{2} [\langle Bu, u \rangle(T) + \langle Bu, u \rangle(0)] + \frac{1}{2} \int_0^T \langle B'(t)u(t), u(t) \rangle dt. \)

8. If \( Bu(0) = 0 \), for \( u \in X \), there exists \( u_n \to u \) in \( X \) such that \( u_n(t) \) is 0 near 0. A similar conclusion could be deduced at \( T \) if \( Bu(T) = 0 \).

Fussing with \( p \geq 2 \) is necessary only because of the consideration of

\[
\int_0^T \langle B'(t)u(t), u(t) \rangle dt.
\]

If \( p < 2 \), this term might not make sense. The last assertion about approximation makes possible the following corollary.
Corollary 76.1.9 If \( Bu(0) = 0 \) for \( u \in X \), then \( \langle Bu, u \rangle(0) = 0 \). The converse is also true. An analogous result will hold with 0 replaced with \( T \).

Proof: Let \( u_n \to u \) in \( X \) with \( u_n(t) = 0 \) for all \( t \) close enough to 0. For \( t \) off a set of measure zero consisting of the union of sets of measure zero corresponding to \( u_n \) and \( u \),

\[
\langle Bu_n, u_n \rangle(t) = \langle B(t)u_n(t), u_n(t) \rangle, \quad \langle Bu, u \rangle(t) = \langle B(t)u(t), u(t) \rangle,
\]

\[
\langle B(u - u_n), u \rangle(t) = \langle B(t)(u(t) - u_n(t)), u(t) \rangle
\]

\[
\langle Bu_n, u - u_n \rangle(t) = \langle B(t)u_n(t), u(t) - u_n(t) \rangle
\]

Then, considering such \( t \),

\[
\langle B(t)u(t), u(t) \rangle - \langle B(t)u_n(t), u_n(t) \rangle = \langle B(t)(u(t) - u_n(t)), u(t) \rangle + \langle B(t)u_n(t), u(t) - u_n(t) \rangle
\]

Hence from Theorem [76.1.8]

\[
|\langle B(t)u(t), u(t) \rangle - \langle B(t)u_n(t), u_n(t) \rangle| \leq C \|u - u_n\|_X (\|u\|_X + \|u_n\|_X)
\]

Thus if \( n \) is sufficiently large,

\[
|\langle B(t)u(t), u(t) \rangle - \langle B(t)u_n(t), u_n(t) \rangle| < \varepsilon
\]

So let \( n \) be fixed and this large and now let \( t_k \to 0 \) to obtain

\[
\langle B(t_k)u_n(t_k), u_n(t_k) \rangle = 0
\]

for \( k \) large enough. Hence

\[
\langle Bu, u \rangle(0) = \lim_{k \to \infty} \langle B(t_k)u(t_k), u(t_k) \rangle < \varepsilon
\]

Since \( \varepsilon \) is arbitrary, \( \langle Bu, u \rangle(0) = 0 \).

Next suppose \( \langle Bu, u \rangle(0) = 0 \). Then letting \( v \in X \), with \( v \) smooth,

\[
\langle Bu(0), v(0) \rangle = \langle Bu, v \rangle(0) = \langle Bu, u \rangle^{1/2}(0) \langle Bu, v \rangle^{1/2}(0) = 0
\]

and it follows that \( Bu(0) = 0 \). \( \blacksquare \)

Note also that this shows that if \( (Bu)' \in L^p(0, T; V') \) as well as \( (Bu)' \), then there is a continuous function

\[
t \to \langle B(u + v), u + v \rangle(t)
\]

which equals \( \langle B(u(t) + v(t)), u(t) + v(t) \rangle \) for a.e. \( t \) and so, defining

\[
\langle Bu, v \rangle(t) \equiv \langle (Bu, u)(t) + (Bu, v)(t) - (Bu + u + v)(t) \rangle^{1/2}
\]
76.1. SUMMARY OF THE PROBLEM

It follows that \( t \to \langle Bu, v \rangle(t) \) is continuous and equals \( \langle B(u(t)), v(t) \rangle \) a.e. \( t \).

This also makes it easy to verify continuity of pointwise evaluation of \( Bu \).

Let \( Lu = (Bu)' \).

\[
\langle \xi, \phi v \rangle_{V', V} = \lim_{n \to \infty} \int_0^T \langle (Bu_n)', \phi v \rangle = \lim_{n \to \infty} - \int_0^T \langle Bu_n, \phi' v \rangle \quad (76.1.12)
\]

We can take a subsequence, still denoted with \( n \) such that \( u_n(t) \to u(t) \) pointwise a.e. Also

\[
\int_0^T |\langle Bu_n, \phi' v \rangle|^p \leq \int_0^T \|u_n\|_{V'}^p dt \leq C(\|\phi'\|_V, v)
\]

and these terms on the right are uniformly bounded by the assumption that \( u_n \) is bounded in \( V \). Therefore, by the Vitali convergence theorem, and using the subsequence just described, we can pass to the limit in \( 76.1.12 \).

Since \( \xi = (Bu)' \) because this is what is meant by \( (Bu)' \). Hence \( L \) is indeed closed and \( X \) is a Banach space. It is also a reflexive Banach space because it is isometric to a closed subspace of the reflexive Banach space \( V \times V' \). Also, the following is useful. See Theorem 31.4.7 on Page 1185.

**Theorem 76.1.10** If \( Y \) denotes those \( f \in L^p([0, T]; V) \) for which \( f' \in L^p([0, T]; V') \), so that \( f \) has a representative such that \( f(t) = f(0) + \int_0^t f'(s) ds \) a.e. \( t \), then if \( f \) has a representative such that \( f(t) = f(0) + \int_0^t f'(s) ds \) a.e. \( t \), then the map \( f \to f(t) \) is continuous in the sense that \( \|f(t)\| \leq C(\|f\|_Y) \).

We also have the following general theory about existence of measurable solutions to elliptic problems. First are conditions which a nonlinear set valued map should satisfy. In what follows, \( X \) denotes a reflexive Banach space with dual \( X' \), \( (\Omega, F) \) is a measurable space, and \( A(\cdot, \omega) : X \to P(X') \), for \( \omega \in \Omega \), denotes a set valued operator. We make the following assumptions on such an operator:

...
• **H$_1$ Measurability condition.** For each $u \in X$, there is a measurable selection $z(\omega)$ such that
  \[ z(\omega) \in A(u, \omega). \]

• **H$_2$ Values of $A$.** $A(\cdot, \omega) : X \to P(X')$ has bounded, closed, nonempty, and convex values. $A(\cdot, \omega)$ maps bounded sets to bounded sets.

• **H$_3$ Limit conditions, $A(\cdot, \omega)$ is pseudomonotone:**
  
  If $u_n \to u$ and $\limsup_{n \to \infty} \langle z_n, u_n - u \rangle \leq 0$, for $z_n \in A(u_n, \omega),$ then for each $v$, there exists $z(v) \in A(u, \omega)$ such that
  \[ \liminf_{k \to \infty} \langle z_n, u_n - v \rangle \geq \langle z(v), u - v \rangle. \]

In our use of the above, the space $X$ will be a space of functions defined on $[0, T]$ to be described more later.

We note that for a fixed $\omega$, the operator $A(\cdot, \omega)$ described earlier is set-valued, bounded and pseudomonotone as a map from $X$ to $P(X)$. Moreover, the sum of two of such operators is set-valued, bounded and pseudomonotone, Theorem 46.5.2 below. The limit condition $H_3$ implies that $A(\cdot, \omega)$ is upper-semicontinuous from the strong topology to the weak topology. This can be used to show that when $\omega \to u(\omega)$ is measurable, then $A(u(\omega), \omega)$ has a measurable selection assuming only that $\omega \to A(u, \omega)$ has a measurable selection for fixed $u \in X$. Here is a well known result on the sum of pseudomonotone operators. See Theorem 23.5.1 on Page 836.

**Theorem 76.1.11** Assume that $A$ and $B$ are set-valued, bounded and pseudomonotone operators. Then, their sum is also a set-valued, bounded and pseudomonotone operator. Moreover, if $u_n \to u$ weakly, $z_n \to z$, $z_n \in A(u_n)$, $w_n \to w$ weakly with $w_n \in A(u_n)$, and
  \[ \limsup_{n \to \infty} \langle z_n + w_n, u_n - u \rangle \leq 0, \]

then,
  \[ \liminf_{n \to \infty} \langle z_n + w_n, u_n - w \rangle \geq \langle z(v) + w(v), u - v \rangle, \]

for $z(v) \in A(u), w(v) \in B(u)$ and in fact, $z \in A(u)$ and $w \in B(u)$.

We now state our result on measurable solutions to general elliptic variational inequalities that may contain sums of set-valued, bounded and pseudomonotone operators. Then the following is proved in [?]. See also Theorem 16.6.3 on Page 530.

**Theorem 76.1.12** Let $\omega \to K(\omega)$ be a measurable set-valued function, where $K(\omega) \subset V$ is convex, closed and bounded. Let the operators $A(\cdot, \cdot)$ and $B(\cdot, \cdot)$
satisfy assumptions $H_1 - H_3$. Finally, let $\omega \to f(\omega)$ be measurable with values in $V'$.

Then, there exists a measurable function $\omega \to u(\omega) \in K(\omega)$ such that $\omega \to w^A(\omega)$, and $\omega \to w^B(\omega)$ with $w^A(\omega) \in A(u(\omega), \omega)$ and $w^B(\omega) \in B(u(\omega), \omega)$, and

$$\langle f(\omega) - (w^A(\omega) + w^B(\omega)), z - u(\omega) \rangle \leq 0,$$

for all $z \in K(\omega)$.

If it is only known that $K(\omega)$ is closed and convex, the same conclusion holds true if it is also known that for some $z(\omega) \in K(\omega)$, $A(\cdot, \omega) + B(\cdot, \omega)$ is coercive, that is

$$\lim_{\|v\| \to \infty} \inf \{ \langle z^*, v - z \rangle \|v\| : z^* \in (A(\cdot, \omega) + B(\cdot, \omega)) \} = \infty.$$

Instead of two operators, one could have the sum of finitely many with the same conclusions.

### 76.2 Measurable Solutions To Evolution Inclusions

The main result in this section is Theorem 76.2.2 below. It gives an existence theorem for many evolution inclusions. We are assuming that $A(\cdot, \omega) : V \to P(V')$ satisfies some conditions presented earlier: These are 76.1.1-76.1.1. Then we can regard $A(\cdot, \omega)$ as a set valued pseudomonotone map from $X$ to $P(X')$. It is clear that $A(u, \omega)$ is a closed convex set in $X'$ because if $z^*_n \to z^*$ in $X'$, $z^*_n \in A(u, \omega)$, then $z^* \in A(u, \omega)$ because a subsequence, still denoted as $z^*_n$ converges weakly to some $w^* \in A(u, \omega)$ in $V'$. Since $X$ is dense in $V$, this requires $w^* = z^*$. The necessary limit conditions for pseudomonotone are nothing more than the assumed conditions in 76.1.1. Also, we will assume in this section that $p \geq 2$. This restriction is necessary because of the desire to consider time dependent $B$ and the assumption 76.1.6 which involves a term $\int_0^T \langle B'(u), u \rangle$ which might not make sense if $p$ were only larger than 1. If $B$ were not time dependent, this assumption would not be necessary and the argument given here would continue to be valid. We essentially show this is the case in the following section in which we also consider a more general coercivity condition than 76.1.1, but one can see that there is no change in the argument and it is in fact simpler if we assume $B$ is constant.

As above, $K : X \to X'$ can be defined as

$$\langle Ku, v \rangle \equiv \int_0^T \langle Lu, v \rangle \, ds + \langle Bu, v \rangle(0)$$

Note that from Proposition 76.1.8, if $v \in X$ and $Bv(0) = 0$, then

$$\langle Ku, v \rangle = \int_0^T \langle Lu, v \rangle \, ds \quad (76.2.13)$$
This is because, from the Cauchy Schwarz inequality and continuity of \( \langle Bu, u \rangle (\cdot) \),
\[
    \langle Bu, v \rangle (0) \leq \langle Bu, u \rangle^{1/2} (0) \langle Bv, v \rangle^{1/2} (0)
\]
and if \( Bv (0) = 0 \), then from Corollary 76.1.14, \( \langle Bv, v \rangle^{1/2} (0) = 0 \). From the above Proposition 76.1.2, this operator \( K \) is hemicontinuous and bounded and monotone as a map from \( X \) to \( X' \). Thus \( K + A (\cdot, \omega) \) is a set valued pseudomonotone map for which we can apply Theorem 76.1.12 and obtain existence theorems for measurable solutions to variational inequalities right away, but we want to obtain solutions to an evolution equation in which \( K (\omega) = V \) and the above theorem does not apply because the sum of these two operators is not coercive. Therefore, we consider another operator which, when added, will result in coercivity. Let \( J : \mathcal{Y} \to \mathcal{Y}' \) be the duality map for 2. Thus \( \|Ju\|_{\mathcal{Y}'} = \|u\|_{\mathcal{Y}} \) and \( \langle Ju, u \rangle = \|u\|^{2}_{\mathcal{Y}} \). Then \( J^{-1} : \mathcal{Y}' \to \mathcal{Y} \) also satisfies \( \langle f, J^{-1} f \rangle = \|f\|^{2}_{\mathcal{Y}'} \).

The main result in this section is based on methods due to Brezis [21] and Lions [32] adapted to the case considered here where the operator is set valued, and we consider measurability. We define the operator \( M : X \to X' \) by
\[
    \langle Mu, v \rangle \equiv \langle Lv, J^{-1} Lu \rangle_{\mathcal{Y}', \mathcal{Y}}\quad \text{where as above, } Lu = (Bu)' .
\]
Then let \( f \) be measurable into \( \mathcal{Y}' \). Thus, in particular, \( f (\omega) \in \mathcal{Y}' \) for each \( \omega \). Consider the approximate problem and a solution \( u_{\varepsilon} \) to
\[
    \varepsilon Mu_{\varepsilon} (\omega) + Ku_{\varepsilon} (\omega) + w^{*}_{\varepsilon} (\omega) = f (\omega) + g (\omega) , \quad w^{*}_{\varepsilon} (\omega) \in A (u_{\varepsilon} (\omega), \omega) .
\]
Where \( g (\omega) \in X' \) is given by
\[
    \langle g (\omega), v \rangle \equiv \langle Bu (0), u_{0} (\omega) \rangle
\]
where \( u_{0} (\omega) \) is a given function measurable into \( W \). Now for \( u \in X \), we let
\[
    A (u, \omega) = \varepsilon Mu + Ku + A (u, \omega)
\]
Then by the assumptions on \( A (\cdot, \omega) \), there is \( u^{*} (\omega) \) for which \( \omega \to u^{*} (\omega) \) is measurable into \( \mathcal{Y}' \), hence measurable into \( X' \). Therefore, \( \omega \to A (u, \omega) \) has a measurable selection, namely \( \varepsilon Mu + Ku + u^{*} (\omega) \) and so condition 76.1.14 is verified.

By Theorem 76.1.12, a solution to 76.2.14 will exist with both \( u_{\varepsilon} \) and \( w^{*}_{\varepsilon} \) measurable if we can argue that the sum of the operators \( \varepsilon M + K + A (\cdot, \omega) \) is coercive, since this is the sum of pseudomonotone operators. From 76.1.1
\[
    \inf \left( \int_{0}^{T} \langle u^{*}, u \rangle ds : u^{*} \in A (u, \omega) \right) + \frac{1}{2} \int_{0}^{T} \langle B' u, u \rangle \geq \delta (\omega) \int_{0}^{T} \|u\|^{2}_{\mathcal{Y}} ds - m (\omega)
\]
and so routine considerations show that \( \varepsilon M + K + A (\cdot, \omega) \) does indeed satisfy a suitable coercivity estimate for each positive \( \varepsilon \). Thus we have the following existence theorem for approximate solutions.
Lemma 76.2.1 Let $f$ be measurable into $\mathcal{V}'$ and let $A$ satisfy the conditions -76.1.1-76.1.11. Then for $K$ and $M$ defined as above, it follows there exist measurable $u_\varepsilon$ and $w^*_\varepsilon$ satisfying 76.2.14.

Note this implies that, suppressing dependence on $\omega$, $\langle Bu_\varepsilon, v \rangle(0) = \langle Bu(0), u_0 \rangle$

for all $v \in X$. Thus, letting $v$ be a smooth function with values in $V$

$\langle Bu_\varepsilon(0), v(0) \rangle = \langle Bu_0, v(0) \rangle$

Since $V$ is dense in $W$, this requires $Bu_\varepsilon(0) = Bu_0$.

Now define $\Lambda$ to be the restriction of $L$ to those $u \in X$ which have $Bu(0) = 0$.

Thus by Corollary 76.1.9,

$D(\Lambda) = \{ u \in X : Bu(0) = 0 \} = \{ u \in X : \langle Bu, u \rangle(0) = 0 \}$

and if $v \in D(\Lambda), u \in X$, then as noted earlier,

$\langle Ku, v \rangle = \int_0^T \langle Lu, v \rangle ds$

Also, one can show an estimate for $\Lambda^*$.

You can define $D(T) \equiv \{ u \in \mathcal{V} : u' \in \mathcal{V}, u(T) = 0 \}$ and let $Tu = -Bu'$. Then

$\langle Tu, u \rangle = -\int_0^T \langle Bu', u \rangle = -\langle Bu, u \rangle |_0^T + \int_0^T \langle (Bu)', u \rangle$

$= \langle Bu, u \rangle(0) + \int_0^T \langle B'u, u \rangle + \int_0^T \langle Bu', u \rangle$

and so we obtain

$2 \langle Tu, u \rangle \geq \int_0^T \langle B'u, u \rangle$ (76.2.15)

Then one shows that $T^* = \Lambda$ and that the graph of $\Lambda^*$ is the closure of the graph of $T$ thus showing that $\Lambda^*$ also satisfies an inequality like 76.2.15 for $u \in D(\Lambda^*)$.

From 76.2.15, we have

$\varepsilon \langle Lv, J^{-1}Lu_\varepsilon \rangle_{\mathcal{V}', \mathcal{V}} + \langle Ku_\varepsilon(\omega), v \rangle_{\mathcal{X}', \mathcal{X}} + \langle w^*_\varepsilon(\omega), v \rangle_{\mathcal{V}', \mathcal{V}}$

$= \langle f(\omega), v \rangle_{\mathcal{V}', \mathcal{V}} + \langle g(\omega), v \rangle_{\mathcal{V}', \mathcal{V}}$

If we restrict to $v \in D(\Lambda)$ so $Bu(0) = 0$, then it reduces to

$\varepsilon \langle Lv, J^{-1}Lu_\varepsilon \rangle_{\mathcal{V}', \mathcal{V}} + \langle Lu_\varepsilon(\omega), v \rangle_{\mathcal{V}', \mathcal{V}} + \langle w^*_\varepsilon(\omega), v \rangle_{\mathcal{V}', \mathcal{V}} = \langle f(\omega), v \rangle_{\mathcal{V}', \mathcal{V}}$

and so $J^{-1}Lu_\varepsilon \in D(\Lambda^*)$. Thus, since $D(\Lambda)$ is dense in $\mathcal{V}$, it follows that

$\varepsilon \Lambda^* J^{-1}Lu_\varepsilon + Lu_\varepsilon + w^*_\varepsilon = f$ in $\mathcal{V}'$
Then act on $J^{-1}Lu_\varepsilon$ on both sides in the above. This yields for some $C$ dependent on $B'$ an inequality of the following form.

$$-\varepsilon C \|Lu_\varepsilon\|^2 + \|Lu_\varepsilon\|^2 + \langle w_\varepsilon^*, J^{-1}Lu_\varepsilon \rangle \leq \langle f, J^{-1}Lu_\varepsilon \rangle \quad (76.2.16)$$

Also, acting on both sides of (76.2.14) with $u_\varepsilon$ and using the formula for $\langle Ku, u \rangle$,

$$\varepsilon \langle Lu_\varepsilon, J^{-1}Lu_\varepsilon \rangle + \frac{1}{2} \langle (Bu_\varepsilon, u_\varepsilon) (T) + (Bu_\varepsilon, u_\varepsilon) (0) \rangle$$

$$+ \frac{1}{2} \int_0^T \langle B' (t) u_\varepsilon (t) , u_\varepsilon (t) \rangle dt + \langle w_\varepsilon^*, u_\varepsilon \rangle_{V' \times V} = \langle f, u_\varepsilon \rangle + \langle Bu_\varepsilon (0) , u_0 (\omega) \rangle = \langle f, u_\varepsilon \rangle + \langle Bu_0 (\omega) , u_0 (\omega) \rangle$$

It follows easily from the coercivity condition (76.2.16) that $u_\varepsilon$ is bounded in $V$ and consequently $w_\varepsilon^*$ is bounded in $V'$, this from the growth estimate (76.2.17). Now from (76.2.17), it also follows that $\|Lu_\varepsilon\|_{V'}$ is bounded for small $\varepsilon$. Thus

$$\|Lu_\varepsilon (\omega)\|_{V'} + \|u_\varepsilon (\omega)\|_{V'} + \|w_\varepsilon^* (\omega)\|_{V'} \leq C (\omega) < \infty,$$

$C (\omega)$ independent of small $\varepsilon$. By Theorem (6.2.11), there is a subsequence $\varepsilon (\omega) \rightarrow 0$ such that

$$(Lu_\varepsilon (\omega), u_\varepsilon (\omega), w_\varepsilon^* (\omega), Bu_\varepsilon (\omega) (0)) \rightarrow (Lu (\omega), u (\omega), \xi (\omega), Bu (\omega) (0)) \quad (76.2.17)$$

in $V' \times V \times V' \times V'$ weakly and $\omega \rightarrow (Lu (\omega), u (\omega), \xi (\omega))$ is measurable into $V' \times V \times V'$. It follows that $Bu (\omega) (0) = Bu_0 (\omega)$ because each $Bu_\varepsilon (\omega) (0) = Bu_0 (\omega)$. Note that this also shows that $Ku_\varepsilon \rightarrow Ku$ in $X'$. Thus, suppressing the dependence on $\omega$, use (6.2.13) to act on $u_\varepsilon - u$ and obtain

$$\varepsilon \langle Lu_\varepsilon - Lu, J^{-1}Lu_\varepsilon \rangle + \langle Ku_\varepsilon - u_\varepsilon, u \rangle + \langle w_\varepsilon^*, u \rangle = \langle f, u_\varepsilon - u \rangle + \langle g, u_\varepsilon - u \rangle$$

Using monotonicity of $J^{-1},$

$$\varepsilon \langle Lu_\varepsilon - Lu, J^{-1}Lu \rangle + \langle Ku_\varepsilon - u_\varepsilon - u \rangle + \langle w_\varepsilon^*, u_\varepsilon - u \rangle \leq \langle f, u_\varepsilon - u \rangle + \langle g, u_\varepsilon - u \rangle$$

Now $(Bu_\varepsilon - Bu) (0) = 0$. Therefore, $u_\varepsilon - u \in D (\Lambda)$ and so

$$\varepsilon \langle \Lambda^* J^{-1}Lu, u_\varepsilon - u \rangle + \langle Ku_\varepsilon - u_\varepsilon - u \rangle + \langle w_\varepsilon^*, u_\varepsilon - u \rangle \leq \langle f, u_\varepsilon - u \rangle + \langle g, u_\varepsilon - u \rangle$$

Recall that $K$ is monotone, bounded and hemicontinuous. In fact, it is monotone and linear. Hence, $K + A$ is pseudomonotone. Then from the above,

$$\limsup_{\varepsilon \rightarrow 0} \langle Ku_\varepsilon + w_\varepsilon^*, u_\varepsilon - u \rangle \leq 0$$

Now these weak convergences in (76.2.17) include the weak convergence of $u_\varepsilon$ to $u$ in $X$. Thus, since $K + A (\cdot, \omega)$ is pseudomonotone as a map from $X$ to $\mathcal{P} (X')$, for every $v \in X$, there exists $w^* (v) \in K (u) + A (u, \omega)$ such that

$$\liminf_{\varepsilon \rightarrow 0} \langle Ku_\varepsilon + w_\varepsilon^*, u_\varepsilon - v \rangle \geq \langle w^* (v), u - v \rangle$$
In particular, this holds if \( v = u \) which shows that
\[
\lim_{\varepsilon \to 0} \langle K(u_{\varepsilon}) + w^*_\varepsilon, u_{\varepsilon} - u \rangle = 0.
\]

It follows then that for \( v \in X \),
\[
\langle \xi + Ku, u - v \rangle = \lim_{\varepsilon \to 0} \langle w^*_\varepsilon + Ku_{\varepsilon}, u - v \rangle
\]
\[
= \lim_{\varepsilon \to 0} \left[ \langle w^*_\varepsilon + Ku_{\varepsilon}, u_{\varepsilon} - u \rangle + \langle w^*_\varepsilon + Ku_{\varepsilon}, u_{\varepsilon} - v \rangle \right]
\]
\[
\geq \liminf_{\varepsilon \to 0} \langle w^*_\varepsilon + Ku_{\varepsilon}, u_{\varepsilon} - v \rangle \geq \langle w^*(v), u - v \rangle
\]

Since \( v \) is arbitrary, separation theorems imply that
\[
\xi(\omega) + Ku(\omega) \equiv w^*(\omega) + Ku(\omega) \in A(u(\omega), \omega) + Ku(\omega).
\]

Then passing to the limit in (76.2.14), we have
\[
Ku(\omega) + w^*(\omega) = f(\omega) + g(\omega) \text{ in } X',
\]
\[
w^*(\omega) \in A(u(\omega), \omega), \quad Bu(\omega)(0) = Bu_0(\omega) \quad (76.2.18)
\]
and \( Lu, w^*, u \) are all measurable into the appropriate spaces. This implies for each \( v \in X \),
\[
\int_0^T \langle Lu, v \rangle + \langle Bu, v \rangle(0) + \int_0^T \langle w^*, v \rangle = \int_0^T \langle f, v \rangle + \langle Bu(0), u_0 \rangle
\]

In particular, letting \( v = u \),
\[
\int_0^T \langle Lu, u \rangle + \langle Bu, u \rangle(0) + \int_0^T \langle w^*, u \rangle = \int_0^T \langle f, v \rangle + \langle Bu(0), u_0 \rangle
\]
\[
= \int_0^T \langle f, v \rangle + \langle Bu_0, u_0 \rangle
\]

Thanks to (76.2.18), this shows that \( \langle Bu, u \rangle(0) = \langle Bu_0, u_0 \rangle \). Also it follows from Theorem 76.2.1.

This has proved the following theorem.

**Theorem 76.2.2** Let \( p \geq 2 \) and let \( A \) satisfy (76.1.1) and let \( f \) be measurable into \( V' \) and let \( u_0 \) be measurable into \( W \). Then there exists a solution to (76.2.18) such that \( Lu, w^*, u \) are all measurable. We also have for \( u \) this solution that for fixed \( \omega, \langle Bu_0, u_0 \rangle = \langle Bu, u \rangle(0) \).

We also have the following corollary which gives measurable solutions to periodic problems. Of course there is no uniqueness for such periodic problems so this is another place where our theory is applicable. In this corollary, we assume for the sake of simplicity that \( B(t) = B \) a constant. Thus, it is not necessary to assume \( p \geq 2 \) in the following corollary.
**Corollary 76.2.3** Let $A$ satisfy \[76.1.1\], $p > 1$, and let $f$ be measurable into $V'$. Then there exists a solution to

$$Lu(\omega) + w^*(\omega) = f(\omega), Bu(0,\omega) = Bu(T,\omega), w^*(\omega) \in A(u(\omega),\omega)$$

such that $Lu, w^*, u$ are all measurable.

**Proof:** Define $\Lambda$ as the restriction of $L$ to the space $\{u \in D(L) : Bu(0) = Bu(T)\}$. This enables periodic conditions. Let $D(T) \equiv \{v \in V : v' \in V$ and $v(T) = v(0)\}$, $Tv = -Bv'$.

Then consider $T^*$. If $u \in D(\Lambda), v \in D(T)$,

$$-\int_0^T \langle Bu', u \rangle = -\int_0^T \langle Bu, v \rangle + \int_0^T \langle (Bu)', v \rangle$$

and so, since the boundary term vanishes, this shows that $D(\Lambda) \subseteq D(T^*)$ and that $T^* = \Lambda$ on $D(\Lambda)$.

Next let $u \in D(T^*)$. By definition, this means that

$$|\langle Tv, u \rangle| \leq C_u \|v\|_V \tag{*}$$

So let $v \in C_c^\infty([0,T];V)$.

$$\langle Tv, u \rangle = -\int_0^T \langle Bu', u \rangle = -\int_0^T \langle Bu, v' \rangle$$

From the Riesz representation theorem, there exists a unique $(Bu)'$ such that the above equals $\int_0^T \langle (Bu)', v \rangle$ and by density of $C_c^\infty([0,T];V)$ this shows $T^*u = (Bu)' = Lu$. Thus $T^* = L$ on $D(T^*)$ and in particular $(Bu)' \in V'$. It remains to consider the boundary conditions. For $u \in D(T^*)$ and $v \in D(T)$,

$$\langle Tv, u \rangle = -\int_0^T \langle Bu', u \rangle = -\langle Bu, v \rangle |_0^T + \int_0^T \langle (Bu)', v \rangle$$

The boundary term is of the form

$$\langle Bu(0) - Bu(T), v(0) \rangle$$

If $*$ is to hold for all $v \in D(T)$ we must have $Bu(0) = Bu(T)$. If the difference is $\xi \neq 0$, you would need to have

$$|\langle \xi, v(0) \rangle| \leq C_u \|v\|_V$$

for all $v \in D(T)$. So pick $v \in D(T)$ such that $|\langle \xi, v(0) \rangle| = \delta > 0$ and consider a piecewise linear function $\psi_n$ which is one at 0 and $T$ but zero on $[1/n, T - (1/n)]$. Then if $v_n = \psi_n v$, the left side is $\delta$ for all $n$ but the right converges to 0.
This shows that $D(T^*) = D(\Lambda)$ and $T^* = \Lambda$. Now it follows that $T^{**} = \Lambda^*$ and so $\Lambda^*$ is monotone because $T$ is and the graph of $\Lambda^*$ is the closure of the graph of $T$. Indeed,

$$\int_0^T \langle -Bu', v \rangle = \int_0^T \langle Bu', v \rangle$$

so $\langle Tv, v \rangle = 0$. The same is true of $\Lambda^*$.

Now in this case we let $X$ be the same as before. Then we consider the approximate problem for $u_\varepsilon$ given by

$$\varepsilon \langle Lv, J^{-1}(Lu_\varepsilon) \rangle + \langle Lu_\varepsilon(\omega), v \rangle_{\mathcal{V}'\mathcal{V}} +$$

$$\frac{1}{2} \langle Bu, v \rangle (0) - \frac{1}{2} \langle Bu, v \rangle (T) + \langle w^*_\varepsilon(\omega), v \rangle = \langle f(\omega), v \rangle, w^*_\varepsilon(\omega) \in A(u_\varepsilon(\omega), \omega)$$

Then using monotonicity of $\Lambda^*$ and $\Lambda$ as before, one obtains the existence of a measurable solution. To see that the necessary monotonicity holds, note that

$$\langle Lu, u \rangle_{\mathcal{V}'\mathcal{V}} + \frac{1}{2} \langle Bu, u \rangle (0) = (f(\omega), v)$$

This follows from \ref{1} and \ref{2}. Indeed, these imply that

$$\langle Lu, u \rangle_{\mathcal{V}'\mathcal{V}} + \langle Bu, u \rangle (0) = \frac{1}{2} (\langle Bu, u \rangle (T) + \langle Bu, u \rangle (0))$$

and so the above follows. The rest of the argument is similar to that used to prove Theorem 76.2.2. At the end you will obtain that

$$\frac{1}{2} \langle Bu, v \rangle (0) - \frac{1}{2} \langle Bu, v \rangle (T) = 0$$

which will require that $Bu(T) = Bu(0)$ since $v$ is arbitrary.

76.3 Relaxed Coercivity Condition

This section is devoted to proving Theorem 76.3.2 below. It includes a more general coercivity condition and uses a slightly modified limit condition. Also, it removes the restriction that $p \geq 2$, which was made because of the terms involving $B'$. However, we will specialize to the case where $B$ does not depend on $t$. It seems that this will be necessary because if one is required to consider $\langle B'u, u \rangle$ then this won’t make sense unless $p \geq 2$. In what follows $p > 1$.

Let $U$ be dense in $V$ with the embedding compact, $U$ being a separable reflexive Banach space. It is always possible to get such a space, (In fact, it can be assumed a Hilbert space.) but in applications of most interest to us, it can be obtained by Sobolev embedding theorems. We will let $r > \max (2, p)$ and $\mathcal{U}_r = L^r([0, T]; U)$. 

Also, for $I = [0, \hat{T}]$, $\hat{T} < T$, we will denote as $\mathcal{V}_I$ the space $L^p(I; V)$ with a similar usage of this notation in other situations. If $u \in \mathcal{V}$, then we will always consider $u \in \mathcal{V}_I$ also by simply considering its restriction to $I$. With this convention, it is clear that if $u$ is measurable into $\mathcal{V}$ then it is also measurable into $\mathcal{V}_I$.

Then the modified conditions on $A : \mathcal{V}_I \to \mathcal{P}(\mathcal{V}_I')$ are as follows for $A(u, \omega)$ a convex closed set in $\mathcal{V}_I'$ whenever $u \in \mathcal{V}_I$.

- **growth estimate**
  Assume the specific estimate for $u \in \mathcal{V}_I$.

  \[
  \sup \left\{ \|u^*\|_{\mathcal{V}_I'} : u^* \in A(u, \omega) \right\} \leq a(\omega) + b(\omega) \|u\|_{\mathcal{V}_I}^{p-1}
  \] 

  (76.3.19)

  where $a(\omega), b(\omega)$ are nonnegative.

- **coercivity estimate**
  Also assume the coercivity condition: valid for each $t \leq T$ and for some $\lambda(\omega) \geq 0$,

  \[
  \inf \left( \int_0^t \langle u^*, u \rangle + \lambda(\omega) \langle Bu, u \rangle \, ds : u^* \in A(u, \omega) \right)
  \geq \delta(\omega) \int_0^t \|u\|_{\mathcal{V}_I}^p \, ds - m(\omega)
  \] 

  (76.3.20)

  where $m(\omega)$ is some nonnegative constant for fixed $\omega$, and $\delta(\omega) > 0$. No uniformity in $\omega$ is necessary.

- **Limit conditions**
  Let $U$ be a Banach space dense and compact in $V$ and that if $u_i \rightharpoonup u$ in $\mathcal{V}_I$ and $u_i^* \in A(u_i, \omega)$ with $(Bu_i)' \to (Bu)'$ weakly in $\mathcal{U}_I'$, then if

  \[
  \lim \sup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{\mathcal{V}_I', \mathcal{V}_I} \leq 0
  \] 

  (76.3.21)

  it follows that for all $v \in \mathcal{V}_I$, there exists $u^*(v) \in Au$ such that

  \[
  \lim \inf_{i \to \infty} \langle u_i^*, u_i - v \rangle_{\mathcal{V}_I', \mathcal{V}_I} \geq \langle u^*(v), u - v \rangle_{\mathcal{V}_I', \mathcal{V}_I}
  \] 

  (76.3.22)

  You typically obtain this kind of thing from Theorem 76.1.2 applied to lower order terms along with some sort of compactness of the embedding of $V$ into $W$.

- **measurability condition**
  For $\omega \to u(\cdot, \omega)$ measurable into $\mathcal{V}$,

  \[
  \omega \to A(\mathcal{X}_I u(\cdot, \omega), \omega) \text{ has a measurable selection into } \mathcal{V}_I'.
  \] 

  (76.3.23)
This condition means there is a function \( \omega \rightarrow u^*(\omega) \) which is measurable into \( V'_I \) such that \( u^*(\omega) \in A(X_Iu(\cdot,\omega),\omega) \). This is assured to take place if the following standard measurability condition is satisfied for all \( O \) open in \( V'_I \):

\[
\{ \omega : A(X_Iu(\cdot,\omega),\omega) \cap O \neq \emptyset \} \in F
\]

(76.3.24)

A sufficient condition for this condition is that \( \omega \rightarrow A(u(\cdot,\omega)) \) has a measurable selection into \( V'_I \) for any \( \omega \rightarrow u(\cdot,\omega) \) measurable into \( V \) and if \( u^* \in A(u(\cdot,\omega),\omega) \), then \( X_Iu^* \in A(X_Iu(\cdot,\omega),\omega) \), and this is typical of what we will always consider, in which the values of \( u^* \) are dependent on the earlier values of \( u \) only.

Let \( F \) be the duality map for \( r \). Thus

\[
\langle Fu,u \rangle = ||u||^r, \quad ||Fu|| = ||u||^{r-1}
\]

and is a demicontinuous map. Let \( X \) be those \( u \in \mathcal{U}_r \) such that \( (Bu)' \in \mathcal{U}'_r \) with a convenient norm given by \( \max \left( ||u||_{\mathcal{U}_r}, \left| |(Bu)'||_{\mathcal{U}'_r} \right| \right) \). Then if we let \( \mathcal{U}_I \) play the role of \( \mathcal{V}_I \) in Theorem 76.2.2, we obtain the following lemma as a corollary of this theorem.

**Lemma 76.3.1** Let \( A \) satisfy \([6.3.6][6.3.7]\) and let \( f \) be measurable into \( V' \) and let \( u_0 \) be measurable into \( W \). Then for \( \varepsilon > 0 \), there exists a solution to

\[
Lu + \varepsilon Fu + u^* = f, \quad Bu(0,\omega) = Bu_0(\omega)
\]

(76.3.25)

such that \( Lu, u^*, u \) are all measurable into \( \mathcal{U}'_r, \mathcal{U}'_r, \) and \( \mathcal{U}_r \) respectively, \( u^*(\omega) \in A(u,\omega) \). In other terms, for \( v \in X = \{ u \in \mathcal{U}_r : Lu \in \mathcal{U}'_r \} \)

\[
\int_0^T \langle Lu,v \rangle + \varepsilon \int_0^T \langle Fu,v \rangle + \int_0^T \langle u^*,v \rangle +
\]

\[
\langle Bu,v \rangle(0) = \int_0^T \langle f,v \rangle + \langle Bv(0),u_0 \rangle
\]

(76.3.26)

**Proof:** Using easy estimates and the definition that \( r > \max (p,2) \), it is routine to show that the earlier coercivity condition holds for \( \varepsilon F + A(\cdot,\omega) \). Indeed, we have the following from the above assumptions.

\[
\inf \left( \int_0^t \langle u^*,u \rangle + \lambda(\omega) \langle Bu,u \rangle \, ds : u^* \in A(u,\omega) \right)
\]

\[
\geq \delta(\omega) \int_0^t ||u||_V^p \, ds - m(\omega)
\]

Thus,

\[
\inf \left( \int_0^t \langle u^*,u \rangle : u^* \in A(u,\omega) \right) \geq \delta(\omega) \int_0^t ||u||_V^p \, ds
\]
−C_B\lambda(\omega) \int_0^t \eta \left( \frac{1}{\eta} \right) ||u||^2_V - m(\omega)

Then the right side is no smaller than

\begin{align*}
&\quad -C_B\lambda(\omega) \int_0^t \left( \frac{1}{\eta} \right) \eta ||u||^2_V - m(\omega) \\
\geq &\quad -C_B\lambda(\omega) \eta^{r/2} \int_0^t ||u||^r_{U'} - C_B\lambda(\omega) T \left( \frac{1}{\eta} \right)^{r/(r-2)} - m(\omega)
\end{align*}

Then picking \( \eta \) small enough, we obtain \( C_B\lambda(\omega) \eta^{r/2} < \varepsilon/2 \).

Both operators \( \varepsilon F \) and \( A(\cdot, \omega) \) are pseudomonotone as maps from \( X \) to \( \mathcal{P}(X') \) where \( X \) is defined in terms of \( U_rI \) as before. Therefore, the existence of the measurable solution is obtained.

Denoting with \( u_\epsilon \) the above solution, suppose \( Lu_\epsilon \to Lu \) weakly in \( U' \) with \( Lu = (Bu)' \in V' \) and \( u_\epsilon \to u \) weakly in \( V \) and \( u_\epsilon^* \to u^* \) in \( V' \), \( \varepsilon Fu_\epsilon \to 0 \) strongly in \( U_rI \).

Then passing to the limit in (76.3.25) we obtain

\[ Lu \in V' \]

and

\[ Bu_\epsilon (t) = Bu_0 + \int_0^t Lu_\epsilon (s) \, ds \]

The weak convergence of \( Lu_\epsilon \) implies that \( Bu_\epsilon (t) \to Bu (t) \) in \( U' \). Thus

\[ Bu (t) = Bu_0 + \int_0^t Lu (s) \, ds \]

and so \( Bu(0) = Bu_0 \). We will show that there exist suitable subsequences such that the kind of convergence just described will hold.

Using the equation to act on \( u \) in (76.3.26) or in (76.3.25) we obtain from the assumed coercivity condition the following for fixed \( \omega \),

\[ \frac{1}{2} \langle Bu, u \rangle (t) - \frac{1}{2} \langle Bu, u \rangle (0) + \varepsilon \int_0^t ||u||^r_{U'} \, ds + \delta \langle \omega \rangle \int_0^t ||u||^p_V \, ds - m(\omega) \leq \lambda(\omega) \int_0^t \langle Bu, u \rangle (s) \, ds + \int_0^t \langle f, u \rangle (s) \, ds \]

From Gronwall’s inequality, one obtains an estimate of the form

\[ \langle Bu, u \rangle (t) + \varepsilon \int_0^T ||u||^r_{U'} \, ds + \int_0^T ||u||^p_V \, ds \leq C(f, \omega) \]
where the constant depends only on the indicated quantities. It follows from this and the definition of the duality map $F$ that if $u_\varepsilon$ is the solution to Lemma 76.3.1, then $\varepsilon F u_\varepsilon \to 0$ strongly in $\mathcal{U}'$. Also, the estimates for $A$ and the above estimate implies that $Lu_\varepsilon$ is bounded in $\mathcal{U}'$. Thus we have an inequality of the form

$$
(B u_\varepsilon, u_\varepsilon) (t) + \varepsilon \int_0^T ||u_\varepsilon||_{\mathcal{U}}' ds + ||u_\varepsilon||_{\mathcal{V}}' + ||Lu_\varepsilon||_{\mathcal{U}'} + ||u_\varepsilon^*||_{\mathcal{V}'} \leq C(f, \omega)
$$

Of course each of these $u_\varepsilon, u_\varepsilon^*$ are measurable into $\mathcal{V}$ and $\mathcal{U}'$ respectively. By density considerations, $u_\varepsilon^*$ is also measurable into $\mathcal{V}'$. It follows from Theorem 76.1.3 that there exists $(u, u^*)$ which is measurable into $\mathcal{V} \times \mathcal{V}'$ and a sequence with $\varepsilon(\omega)$ such that as $\varepsilon(\omega) \to 0$, $(u_\varepsilon(\omega), u_\varepsilon^*(\omega)) \to (u(\omega), u^*(\omega))$ in $\mathcal{V} \times \mathcal{V}'$. Then, taking a further subsequence, we can obtain the following convergences for fixed $\omega$.

$$u_\varepsilon(\omega) \to u(\omega) \text{ weakly in } \mathcal{V}$$

$$u_\varepsilon^*(\omega) \to u^*(\omega) \text{ weakly in } \mathcal{V}'$$

$$Lu_\varepsilon(\omega) \to Lu \text{ weakly in } \mathcal{U}'$$

These convergences continue to hold for $\mathcal{V}$ and $\mathcal{U}'$ replaced with $\mathcal{V}_I$ and $\mathcal{U}'_I$ and we simply consider the restrictions of the functions to $I$. The problem here is that we do not know that $u$ is in $\mathcal{U}_I$. This is why it is necessary to take a little different approach.

Letting $\sigma > 0$, there exists $\hat{T}(\omega) > T - \sigma$ such that for each $\varepsilon(\omega)$ in that sequence,

$$
(B u_\varepsilon(\omega), u_\varepsilon(\omega))(\hat{T}) = \langle B(u_\varepsilon(\hat{T})), u_\varepsilon(\hat{T}) \rangle, \quad Bu_\varepsilon(\hat{T}) = B(u_\varepsilon(\hat{T}))
$$

for all $\varepsilon(\omega)$ in the sequence converging to 0 and also

$$Bu(\hat{T}) = B(u(\hat{T})), \quad \langle Bu, u \rangle(\hat{T}) = \langle B(u(\hat{T})), u(\hat{T}) \rangle.$$

Now let $\{e_i\}$ be the vectors of Theorem 76.1.6 where these are in $U$. Thus for $\hat{T}$,

$$
(B u_\varepsilon, u_\varepsilon)(\hat{T}) = \langle Bu_\varepsilon(\hat{T}), u_\varepsilon(\hat{T}) \rangle = \sum_{i=1}^{\infty} \langle B(u_\varepsilon(\hat{T})), e_i \rangle^2
$$
Hence, by Fatou’s lemma,

$$\liminf_{\varepsilon \to 0} \langle Bu_\varepsilon, u_\varepsilon \rangle \left( \hat{T} \right) = \liminf_{\varepsilon \to 0} \sum_{i=1}^{\infty} \left\langle B \left( u_\varepsilon \left( \hat{T} \right) \right), e_i \right\rangle^2$$

$$\geq \sum_{i=1}^{\infty} \liminf_{\varepsilon \to 0} \left\langle B \left( u_\varepsilon \left( \hat{T} \right) \right), e_i \right\rangle^2$$

$$= \sum_{i=1}^{\infty} \left\langle Bu_\varepsilon \left( \hat{T} \right), e_i \right\rangle^2$$

$$= \sum_{i=1}^{\infty} \left\langle Bu_\varepsilon \left( \hat{T} \right), e_i \right\rangle^2$$

$$= \left\langle B \left( u_\varepsilon \left( \hat{T} \right) \right), u_\varepsilon \left( \hat{T} \right) \right\rangle = \left\langle Bu, u \right\rangle \left( \hat{T} \right)$$

(76.3.29)

Then by 76.3.25, we can obtain

$$\frac{1}{2} \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle \left( \hat{T} \right) - \frac{1}{2} \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle \left( 0 \right) +$$

$$\int_0^{\hat{T}} \varepsilon \left\langle Fu_\varepsilon, u_\varepsilon \right\rangle dt + \int_0^{\hat{T}} \left\langle u_\varepsilon^*, u_\varepsilon \right\rangle = \int_0^{\hat{T}} \left\langle f, u_\varepsilon \right\rangle$$

(76.3.30)

From what was shown above, \( \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle \left( 0 \right) = \left\langle Bu_0, u_0 \right\rangle \). Now passing to the limit as \( \varepsilon \to 0 \),

$$Lu + u^* = f$$

in \( U'_r \). But every term is in \( V^\prime \) except the first and so it is also in \( V' \). Also, we know that \( \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle \left( 0 \right) = \left\langle Bu_0, u_0 \right\rangle \) also. Then the integration by parts formula yields

$$\frac{1}{2} \left\langle Bu, u \right\rangle \left( \hat{T} \right) - \frac{1}{2} \left\langle Bu_0, u_0 \right\rangle + \int_0^{\hat{T}} \left\langle u^*, u \right\rangle dt = \int_0^{\hat{T}} \left\langle f, u \right\rangle dt$$

which shows

$$\int_0^{\hat{T}} \left\langle u^*, u \right\rangle dt = \int_0^{\hat{T}} \left\langle f, u \right\rangle dt - \frac{1}{2} \left\langle Bu, u \right\rangle \left( \hat{T} \right) + \frac{1}{2} \left\langle Bu_0, u_0 \right\rangle$$

Then from 76.3.30 and the lower semicontinuity shown in 76.3.26, it follows that

$$\limsup_{\varepsilon \to 0} \int_0^{\hat{T}} \left\langle u_\varepsilon^*, u_\varepsilon \right\rangle \leq \int_0^{\hat{T}} \left\langle f, u \right\rangle dt + \frac{1}{2} \left\langle Bu_0, u_0 \right\rangle - \liminf_{\varepsilon \to 0} \frac{1}{2} \left\langle Bu_\varepsilon, u_\varepsilon \right\rangle \left( \hat{T} \right)$$

$$\leq \int_0^{\hat{T}} \left\langle f, u \right\rangle dt + \frac{1}{2} \left\langle Bu_0, u_0 \right\rangle - \frac{1}{2} \left\langle Bu, u \right\rangle \left( \hat{T} \right) = \int_0^{\hat{T}} \left\langle u^*, u \right\rangle dt$$
76.3. RELAXED COERCIVITY CONDITION

Thus we have $u_\varepsilon \to u$ weakly in $V_I$ and $(Bu_\varepsilon)' \to (Bu)'$ weakly in $U'_I$,

$$
\limsup_{\varepsilon \to 0} \int_0^\hat{T} \langle u_\varepsilon^*, u_\varepsilon - u \rangle \leq \int_0^\hat{T} \langle u^*, u \rangle - \int_0^\hat{T} \langle u^*, u \rangle = 0
$$

Therefore, by the limit condition 76.3, for any $v \in V$

$$
\liminf_{\varepsilon \to 0} \int_0^\hat{T} \langle u_\varepsilon^*, u_\varepsilon - v \rangle \geq \int_0^\hat{T} \langle u^* (v), u - v \rangle , \text{ some } u^* (v) \in A(u, \omega)
$$

In particular, this holds for $u$ and so, in fact, $\int_0^\hat{T} \langle u_\varepsilon^*, u_\varepsilon - u \rangle$ converges to 0. Therefore,

$$
\int_0^\hat{T} \langle u^*, u - v \rangle = \lim_{\varepsilon \to 0} \int_0^\hat{T} \langle u_\varepsilon^*, u - v \rangle \\
\geq \liminf_{\varepsilon \to 0} \left( \int_0^\hat{T} \langle u_\varepsilon^*, u - u_\varepsilon \rangle + \int_0^\hat{T} \langle u_\varepsilon^*, u_\varepsilon - v \rangle \right) \\
\geq \int_0^\hat{T} \langle u^* (v), u - v \rangle , \text{ some } u^* (v) \in A(u, \omega)
$$

since $v$ is arbitrary, this shows from separation theorems that $u^* (\omega) \in A(u (\omega), \omega)$ in $V'_I$.

This has proved the following theorem in which a more general coercivity condition is used.

**Theorem 76.3.2** Suppose the conditions on $A$ 76.3 - 76.3. Also let $u_0$ be measurable into $W$ and $f$ measurable into $V'$. Let $B \in \mathcal{L}(W,W')$ be nonnegative and self adjoint as described above. Let $\sigma > 0$ be small. Then there exist functions $u, u^*$ measurable into $V_{[0,T-\sigma]} \times V'_I$ such that $u^* (\omega) \in A \left( X_{[0,T-\sigma]} u (\omega), \omega \right)$ for each $\omega$ and for $t \leq T - \sigma$, for each $\omega$,

$$
Bu (t) - Bu_0 + \int_0^t u^* (s) \, ds = \int_0^t f (s) \, ds
$$

Note that if for a given $\omega$ there is a unique solution to the evolution equation, then we can obtain the solution on $(0, T)$. However, $\sigma$ was totally arbitrary so it seems like there is not much difference between the above and the optimum solution. However, one could also index the above solutions relative to $\sigma$, take an appropriate extension of each on $(T - \sigma, T)$ and get similar estimates and pass to a limit as above as $\sigma \to 0$ and thereby obtain a measurable solution valid on $(0, T)$. This time, it will be clear that $Lu, Lu_\sigma$ are both in $V'$ so a monotonicity condition will hold for $L$ without the delicate argument given above which caused a smaller interval to be considered. Thus the following corollary will hold if enough additional details are considered. The issue does not seem sufficiently significant to justify the consideration of these details.
Corollary 76.3.3 In the situation of Theorem 76.3.2 there exists the same kind of measurable solution valid on \((0, T)\). This time, \(u^*, u\) are measurable into \(V\) and \(V'\) respectively.

One can give a very interesting generalization of Theorem 76.3.4.

Theorem 76.3.4 In the context of Theorem 76.3.3 let \(q(t, \omega)\) be a product measurable function into \(V\) such that \(t \to q(t, \omega)\) is continuous, \(q(0, \omega) = 0\).

Then for each small \(\sigma\), there exists a solution \(u\) of the integral equation

\[
Bu(t, \omega) + \int_0^t u^*(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega) + Bq(t, \omega), \quad t \leq T - \sigma
\]

where \((t, \omega) \to u(t, \omega)\) is product measurable. Moreover, for each \(\omega\), \(Bu(t, \omega) = B(u(t, \omega))\) for a.e. \(t\) and \(u^*(\cdot, \omega) \in A(u^*(\cdot, \omega), \omega)\) for a.e. \(t\), \(u^*\) is product measurable into \(V'\). Also, for each \(a \in [0, T - \sigma]\),

\[
Bu(t, \omega) + \int_0^t u^*(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu(a, \omega) + Bq(t, \omega) - Bq(a, \omega)
\]

Proof: Define a stopping time

\[
\tau_r(\omega) \equiv \inf \{ t : |q(t, \omega)| > r \}
\]

Then this is the first hitting time of an open set by a continuous random variable and so it is a valid stopping time. Then for each \(r\), let

\[
A_r(\omega, w) \equiv A(\omega, w + q^r(\cdot, \omega))
\]

where the notation means \(q^r(t) \equiv q(t \land \tau_r)\). Then, since \(q^r\) is uniformly bounded, all of the necessary estimates and measurability for the solution to the above corollary hold for \(A_r\) replacing \(A\). Therefore, there exists a solution \(w_r\) to the inclusion

\[
(Bw_r)'(\cdot, \omega) + A_r(w_r(\cdot, \omega), \omega) \ni f(\cdot, \omega), \quad Bw_r(0, \omega) = Bu_0(\omega), \quad t \in [0, T - \sigma/2]
\]

Now for fixed \(\omega\), \(q^r(t, \omega)\) does not change for all \(r\) large enough. This is because it is a continuous function of \(t\) and so is bounded on the interval \([0, T - \sigma/2]\). Thus, for \(r\) large enough and fixed \(\omega\), \(q^r(t, \omega) = q(t, \omega)\). Thus, we obtain

\[
\langle Bw_r(t, \omega), w_r(t, \omega) \rangle + \int_0^t ||u_r(s, \omega)||_V^2 \, ds \leq C(\omega) \quad (76.3.31)
\]

Now, as before in the proof of Theorem 76.3.4 one can pass to a limit involving a subsequence, as \(r(\omega) \to \infty\) and obtain a solution to the integral equation

\[
Bu(t, \omega) - Bu_0(\omega) + \int_0^t u^*(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds, \quad t \in [0, T - \sigma]
\]

where \(u^*(\omega) \in A(w(s, \omega) + q(s, \omega), \omega)\) and \(u^*, w\) are measurable into \(V'_{[0, T - \sigma]}\).

Now let \(u(t, \omega) = w(t, \omega) + q(t, \omega)\).

The last claim follows from letting \(t = a\) in the top equation and then subtracting this from the top equation with \(t > a\). \(\blacksquare\)
76.4 Progressively Measurable Solutions

In the context of uniqueness of the evolution initial value problem for fixed \( \omega \), one can prove theorems about progressively measurable solutions fairly easily.

First is a definition of the term progressively measurable.

**Definition 76.4.1** Let \( F_t \) be an increasing in \( t \) set of \( \sigma \) algebras of sets of \( \Omega \) where \( (\Omega, F) \) is a measurable space. Thus each \( F_t \) is a \( \sigma \) algebra and if \( s \leq t \), then \( F_s \leq F_t \). This set of \( \sigma \) algebras is called a filtration. A set \( S \subseteq [0, T] \times \Omega \) is called progressively measurable if for every \( t \in [0, T] \),

\[
S \cap [0, t] \times \Omega \in B([0, t]) \times F_t
\]

Denote by \( P \) the progressively measurable sets. This is a \( \sigma \) algebra of subsets of \([0, T] \times \Omega \). A function \( g \) is progressively measurable if \( \mathcal{X}_{[0, t]} g \) is \( B([0, t]) \times F_t \) measurable for each \( t \).

Let \( A \) satisfy the conditions\( 76.3.1 \) - \( 76.3.3 \) but the last condition will be modified as follows.

**Condition 76.4.2** For each \( t \leq T \), if \( \omega \rightarrow u(\cdot, \omega) \) is \( F_t \) measurable into \( \mathcal{V}_{[0, t]} \), then there exists a \( F_t \) measurable selection of \( A\left(\mathcal{X}_{[0, t]} u(\cdot, \omega), \omega\right) \) into \( \mathcal{V}'_{[0, t]} \).

Note that \( u(\cdot, \omega) \) is in \( \mathcal{V}_{[0, t]} \) so \( u(t, \omega) \in V \).

The theorem to be shown is the following.

**Theorem 76.4.3** Assume the above conditions, \( 76.3 \) - \( 76.3.3 \), and \( 76.4.2 \). Let \( u_0 \) be \( F_0 \) measurable and \( (t, \omega) \rightarrow \mathcal{X}_{[0, t]}(t) f(t, \omega) \) is \( B([0, t]) \times F_t \) product measurable into \( V' \) for each \( t \). Also assume that for each \( \omega \), there is at most one solution \((u, u^*)\) to the evolution equation

\[
Bu(\omega)(t) - Bu_0(\omega) + \int_0^t u^*(\cdot, \omega) ds = \int_0^t f(s, \omega) ds,
\]

\( u^*(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \)

for \( t \in [0, T] \). Then there exists a unique solution \((u(\cdot, \omega), u^*(\cdot, \omega))\) in \( \mathcal{V}_{[0, T]} \times \mathcal{V}'_{[0, T]} \) to the above integral equation for each \( \omega, t \in (0, T) \). This solution satisfies \((t, \omega) \rightarrow (u(t, \omega), u^*(t, \omega))\) is progressively measurable into \( V \times V' \).

**Proof:** First note that Theorem \( 76.4.2 \) there exists a solution on \([0, T - \sigma]\) for each small \( \sigma > 0 \). Then by uniqueness, there exists a solution on \((0, T)\). Let \( T \) denote subsets of \((0, T - \sigma]\) which contain \( T - \sigma \) such that for \( S \in T \), there exists a solution \( u_S \) for each \( \omega \) to the above integral equation on \([0, T - \sigma]\) such that \((t, \omega) \rightarrow \mathcal{X}_{[0, s]}(t) u_S(t, \omega) \) is \( B([0, s]) \times F_S \) measurable for each \( s \in S \). Then \( T - \sigma \in T \). If \( S, S' \) are in \( T \), then \( S \leq S' \) will mean that \( S \subseteq S' \) and also \( u_S(t, \omega) = u_{S'}(t, \omega) \) in \( V \) for all \( t \in S \), similar for \( u_S^* \) and \( u_{S'}^* \). Note how we are considering a particular representative of a function in \( \mathcal{V}_{[0, T - \sigma]} \) and \( \mathcal{V}'_{[0, T - \sigma]} \) because
of the pointwise condition. Now let $\mathcal{C}$ denote a maximal chain. Is $\cup \mathcal{C} = S_{\infty}$ all of $(0, T - \sigma)$? What is $u_{S_{\infty}}$? Define $u_{S_{\infty}}(t, \omega)$ the common value of $u_{S}(t, \omega)$ for all $S$ in $\mathcal{C}$, which contain $t \in S_{\infty}$. If $s \in S_{\infty}$, then it is in some $S \in \mathcal{C}$ and so the product measurability condition holds for this $s$. Thus $S_{\infty}$ is a maximal element of the partially ordered set. Is $S_{\infty}$ all of $(0, T - \sigma)$? Suppose $\dot{s} \notin S_{\infty}, T - \sigma > \dot{s} > 0$.

From Theorem 76.4.3 there exists a solution to the integral equation 76.4.3 on $[0, \dot{s}]$ called $u_{1}$ such that $(t, \omega) \mapsto u_{1}(t, \omega)$ is $B((0, \dot{s}]) \times F_{\dot{s}}$ measurable, similar for $u_{1}^{*}$. By the same theorem, there is a solution on $[0, T - \sigma]$, $u_{2}$ which is $B((0, T - \sigma]) \times F_{[0, T - \sigma]}$ measurable. Now by uniqueness, $u_{2}(\cdot, \omega) = u_{1}(\cdot, \omega)$ in $\mathcal{V}_{[0, \dot{s}]}$, similar for $u_{1}^{*}$. Therefore, no harm is done in re-defining $u_{2}$, $u_{2}^{*}$ on $[0, \dot{s}]$ so that $u_{2}(t, \omega) = u_{1}(t, \omega)$, for all $t \in [0, \dot{s}]$, similar for $u^{*}$. Denote these functions as $\bar{u}, \bar{u}^{*}$. By uniqueness, $u_{S_{\infty}}(\cdot, \omega) = \bar{u}(\cdot, \omega)$ in $L^{p}((0, \dot{s}], V)$. Thus no harm is done by re-defining $\bar{u}(s, \omega)$ to equal $u_{S_{\infty}}(s, \omega)$ for $s < \dot{s}$ and $u_{1}(\dot{s}, \omega)$ at $\dot{s}$. As to $s > \dot{s}$ also re define $\bar{u}(s, \omega) \equiv u_{S_{\infty}}(s, \omega)$ for such $s$. By uniqueness, the two are equal in $\mathcal{V}_{[\dot{s}, T - \sigma]}$ and so no change occurs in the solution of the integral equation. Now $S_{\infty}$ was not maximal after all. $S_{\infty} \cup \{\dot{s}\}$ is larger. This contradiction shows that in fact, $S_{\infty} = (0, T - \sigma)$. Thus there exists a unique progressively measurable solution to 76.4.3 on $[0, T - \sigma]$ for each small $\sigma$. Thus we can simply use uniqueness to conclude the existence of a unique progressively measurable solution on $[0, T)$. $
$

**Theorem 76.4.4** Assume the above conditions, 76.2, 76.3 and 76.4.3. Let $u_{0}$ be $\mathcal{F}_{t}$ measurable and $(t, \omega) \mapsto X_{[0, \dot{t}]}(t) f(t, \omega)$ is $B([0, \dot{t}]) \times \mathcal{F}_{t}$ product measurable into $V'$ for each $t \in [0, T - \sigma]$. Also let $t \mapsto q(t, \omega)$ be continuous and $q$ is progressively measurable into $V$. Suppose there is at most one solution to

$$Bu(t, \omega) + \int_{0}^{t} u^{*}(s, \omega) ds = \int_{0}^{t} f(s, \omega) ds + Bu_{0}(\omega) + Bq(t, \omega), \quad (76.4.33)$$

for each $\omega$. Then the solution $u$ to the above integral equation is progressively measurable and so is $u^{*}$. Moreover, for each $\omega$, $u^{*}(\cdot, \omega) \in A\{u(\cdot, \omega), \omega\}$. Also, for each $a \in [0, T]$

$$Bu(\omega)(t) + \int_{0}^{t} u^{*}(s, \omega) ds = \int_{0}^{t} f(s, \omega) ds + Bu(\omega)(a) + Bq(t, \omega) - Bq(a, \omega)$$

**Proof:** By Theorem 76.4.3 there exists a solution to 76.4.3 which is $B((0, T - \sigma]) \times F_{T - \sigma}$ measurable. Since this is true for all $\sigma > 0$, there exists a unique $B((0, \dot{T}]) \times F_{\dot{T}}$ measurable solution for each $\dot{T} < T$. Now, as in the proof of Theorem 76.4.3 one can define a new operator

$$A_{\tau_{r}}(w, \omega) \equiv A(\omega, w + q_{r}^{\tau_{r}}(\cdot, \omega))$$

where $\tau_{r}$ is the stopping time defined there. Then, since $q$ is progressively measurable, the progressively measurable condition is satisfied for this new operator. Hence by Theorem 76.4.3 there exists a unique solution $w$ which is progressively measurable to the integral equation

$$Bw_{r}(t, \omega) + \int_{0}^{t} u_{r}^{*}(s, \omega) ds = \int_{0}^{t} f(s, \omega) ds + Bu_{0}(\omega)$$
where \( u_r^*(\cdot, \omega) \in \mathcal{A}_r(w(\cdot, \omega), \omega) \). Then you can let \( r \to \infty \) and eventually \( q^r(\cdot, \omega) = q(\cdot, \omega) \). Thus there is a solution to

\[
Bw(t, \omega) + \int_0^t u^*(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + Bu_0(\omega)
\]

which is progressively measurable because \( w(\cdot, \omega) = \lim_{r \to \infty} w_r(\cdot, \omega) \) in \( V \) each \( w_r \) being progressively measurable. Uniqueness is needed in passing to the limit. Thus for each \( \hat{T} < T, \omega \to w(\cdot, \omega) \) is measurable into \( V_{[0, \hat{T}]} \). Then by Lemma 76.1.5, \( w \) has a representative in \( V_{[0, \hat{T}]} \) for each \( \omega \) such that the resulting function satisfies \( (t, \omega) \to X_{[0, \hat{T}]}(t) w(t, \omega) \) is \( B \left( [0, \hat{T}] \right) \times \mathcal{F}_{\hat{T}} \) measurable into \( V \). Thus one can assume that \( w \) is progressively measurable. Now as in Theorem 76.3.4, Define \( u = w + q \). It follows by uniqueness that there exists a unique progressively measurable solution to (0, T).

The last claim follows from letting \( t = a \) in the top equation and then subtracting this from the top equation with \( t > a \). ■
Chapter 77

Including Stochastic Integrals

You can include stochastic integrals in the above formulation. In this section and from now on, we will assume that $W$ is a Hilbert space because the stochastic integrals featured here will have values in $W$ and the version of the stochastic integral to be considered here will be the Ito integral. Here is a brief review of this integral.

Let $U$ be a separable real Hilbert space and let $Q : U \to U$ be self adjoint and nonnegative. Also $H$ will be a separable real Hilbert space. $L_2 (Q^{1/2}U, H)$ will denote the Hilbert Schmidt operators which map $Q^{1/2}U$ to $H$. Here $Q^{1/2}U$ is the Hilbert space which has an inner product given by

$$(y, z) \equiv (Q^{-1/2}y, Q^{-1/2}z)$$

where $Q^{-1/2}y$ denotes $x$ such that $Q^{1/2}x = y$ and out of all such $x$, this is the one which has the smallest norm. It is like the Moore Penrose inverse in linear algebra.

Then one can define a stochastic integral

$$\int_0^t \Phi dW$$

where $\Phi \in L^2 ([0, T] \times \Omega; L_2 (Q^{1/2}U, H))$ where here $\Phi$ is progressively measurable with respect to the filtration $\mathcal{F}_t$. This filtration will be

$$\mathcal{F}_t = \cap_{t > r} \sigma (W (r) - W (s) : 0 \leq s \leq r \leq p)$$

The horizontal line indicates completion. The symbol

$$\sigma (W (r) - W (s) : 0 \leq s \leq r \leq p)$$

indicates the smallest $\sigma$ algebra for which all those increments are measurable. Here $W (t)$ is a Wiener process which has values in $U_1$, some other Hilbert space,
CHAPTER 77. INCLUDING STOCHASTIC INTEGRALS

maybe $H$. There is a Hilbert Schmidt operator $J \in L_2 \left(Q^{1/2}U, U_1 \right)$ such that $W(t) = \sum_{i=1}^{\infty} \psi_i(t) J e_i$ where here the $\psi_i$ are independent real Wiener processes. You could take $U, U_1$ to both be $H$. This is following [38]. Then the stochastic integral has the following properties.

1. $\int_0^t \Phi dW$ is a martingale with respect to $\mathcal{F}_t$ with values in $H$, equal to 0 when $t = 0$.

2. One has the Ito isometry

$$E \left( \left\| \int_0^t \Phi dW \right\|_H^2 \right) = \int_0^t \| \Phi \|_{L_2}^2 \, ds$$

3. One can localize as follows. For $\tau$ a stopping time,

$$\int_0^{t \wedge \tau} \Phi dW = \int_0^t \chi_{[0, \tau]} \Phi dW$$

4. One can also generalize to the case where $\Phi$ is only progressively measurable and instead of being in $L^2 \left([0, T] \times \Omega; L_2 \left(Q^{1/2}U, H \right) \right)$, you have only that

$$P \left( \int_0^T \| \Phi(t) \|_{L_2}^2 \, dt < \infty \right) = 1$$

This is done by using an appropriate sequence of stopping times called a localizing sequence. More generally a local martingale is a stochastic process $M(t)$ adapted to the filtration for which there is a localizing sequence of stopping times $\{ \tau_n \}$ such that $\lim_{n \to \infty} \tau_n = \infty$ and $M^{\tau_n}$ is a martingale. Local martingales will occur in the estimates which are encountered in what follows.

5. Denoting by $M(t)$ the stochastic integral, $M(t) = \int_0^t \Phi dW$, the quadratic variation is given by

$$[M](t) = \int_0^t \| \Phi \|_{L_2}^2 \, ds$$

6. We will also need a part of the Burkholder Davis Gundy inequality [41], Theorem 61.4.4 which in terms of this stochastic integral is of the form

$$\int_{\Omega} M^* dP \leq C E \left( \left( \int_0^T \| \Phi \|_{L_2}^2 \, ds \right)^{1/2} \right), \quad C \text{ some constant}$$

where $M(t)$ is the above stochastic integral and

$$M^* \equiv \sup \{ \| M(t) \|_H : t \in [0, T] \}$$
Now let $\Phi \in L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2}U, W))$. Let an orthonormal basis for $Q^{1/2}U$ be $\{g_i\}$ and an orthonormal basis for $W$ be $\{f_i\}$. Then $\{f_i \otimes g_i\}$ is an orthonormal basis for $\mathcal{L}_2 (Q^{1/2}U, W)$. Hence,

$$\Phi = \sum_i \sum_j \Phi_{ij} f_i \otimes g_j$$

where $f_i \otimes g_j (y) \equiv (g_j, y)_{Q^{1/2}U} f_i$. Let $E$ be a separable real Hilbert space which is dense in $V$. Then without loss of generality, one can assume that the orthonormal basis for $W$ are all vectors in $E$. Thus the orthogonal projection of $\Phi$ onto the closed subspace span $\{\{f_i \otimes g_i\}, i, j \leq n\}$ given by

$$\Phi_n = \sum_{i=1}^n \sum_{j=1}^n \Phi_{ij} f_i \otimes g_j$$

Then $\Phi_n \in L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2}U, E))$ and also

$$\lim_{n \to \infty} \|\Phi_n - \Phi\|_{L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2}U, W))} = 0$$

and $\int_0^t \Phi_n dW$ is continuous and progressively measurable into $E$ hence into $V$. We can take a subsequence such that $\|\Phi_n - \Phi\|_{L^2 ([0, T] \times \Omega; \mathcal{L}_2 (Q^{1/2}U, W))} < 2^{-n}$ and this will be assumed whenever convenient.

Note that if $P_n$ is the orthogonal projection onto span $\{f_1, \cdots, f_n\}$, then

$$|P_n \Phi (y)|_W = \left| P_n \sum_i \sum_j \Phi_{ij} f_i \otimes g_j (y) \right|_W$$

$$= \left| P_n \sum_i \sum_j \Phi_{ij} f_i (y, g_j) \right|_W$$

$$= \sum_{i=1}^n \sum_{j=1}^n \Phi_{ij} f_i (y, g_j)$$

$$\geq \sum_{i=1}^n \sum_{j=1}^n \Phi_{ij} f_i (y, g_j) = |\Phi_n (y)|_W$$

Thus

$$\left| \int_s^t \Phi_n dW \right|_W \leq \left| \int_s^t P_n \Phi dW \right|_W = \left| P_n \int_s^t \Phi dW \right|_W \leq \left| \int_s^t \Phi dW \right|_W.$$

The following corollary will be useful.
Corollary 77.0.5 Let $\Phi_n$ be as described above. Then

$$\|\Phi_n(t,\omega)\|_{L^2(Q^{1/2}U,W)} \leq \|\Phi(t,\omega)\|_{L^2(Q^{1/2}U,W)}$$

where $\|\Phi_n(t,\omega)\|_{L^2(Q^{1/2}U,W)} \uparrow \|\Phi(t,\omega)\|_{L^2(Q^{1/2}U,W)}$

$\Phi \in L^\alpha(\Omega;L^\infty([0,T],L^2(Q^{1/2}U,W))) \cap L^2([0,T] \times \Omega, L^2(Q^{1/2}U,W))$

where $\alpha > 2$. Then off a set of measure zero, the stochastic integrals $\int_0^t \Phi_n dW$ satisfy

$$\sup_n \sup_{t \neq s} \frac{\left| \int_s^t \Phi_n dW \right|}{|t-s|^{\gamma}} < C(\omega), \gamma < 1/2, \gamma = \left(\frac{\alpha}{2}\right) - 1$$

Proof: Let, $\alpha > 2$. As explained above, $|\int_s^t \Phi_n dW| \leq |\int_s^t \Phi dW|$. Thus by the Burkholder Davis Gundy inequality,

$$\sup_n \left| \int_s^t \Phi_n dW \right| \leq \int_s^t \Phi dW$$

$$\int_\Omega \left( \left\| \int_s^t \Phi dW \right\| \right)^{\alpha} dP \leq C \int_\Omega \left( \int_s^t \|\Phi\|^2 d\tau \right)^{\alpha/2} dP$$

$$\leq C \int_\Omega \|\Phi\|_{L^\infty([0,T],L^2(Q^{1/2}U,W))}^\alpha |t-s|^{\alpha/2}$$

$$\leq C \|\Phi\|_{L^\infty(\Omega;L^\infty([0,T],L^2(Q^{1/2}U,W))))} |t-s|^{\alpha/2}$$

$$= C |t-s|^{\alpha/2}$$

Then by the Kolmogorov Čentsov theorem, for $\gamma$ as given,

$$E \left( \sup_{0 \leq s < t \leq T} \sup_n \frac{\left| \int_s^t \Phi_n dW \right|}{(t-s)^{\gamma}} \right) \leq E \left( \sup_{0 \leq s < t \leq T} \frac{\left| \int_s^t \Phi dW \right|}{(t-s)^{\gamma}} \right) \leq C$$

where $\gamma < \beta/\alpha$ where, $\beta + 1 = \alpha/2$. Thus for $\gamma < \left(\frac{\alpha/2}{\alpha}\right) - 1$,

$$\sup_n \sup_{0 \leq s < t \leq T} \frac{\left| \int_s^t \Phi_n dW \right|}{(t-s)^{\gamma}} \leq C(\omega)$$

for all $\omega$ off a set of measure zero. ■

Recall the following conditions for the various operators.

**Bounded and coercive conditions**
\[ A(\cdot, \omega) : V_I \to V'_I \] for each \( I \) a subinterval of \([0, T]\) \( I = [0, \bar{T}] \), \( \bar{T} \leq T \)

\[ A(\cdot, \omega) : V_I \to \mathcal{P}(V'_I) \] is bounded, \(\text{(77.0.1)}\)

If, for \( u \in V \),

\[ u^* X_{[0, T]} \in A(u X_{[0, T]}, \omega) \]

for each \( \bar{T} \) in an increasing sequence converging to \( T \), then

\[ u^* \in A(u, \omega) \quad\text{(77.0.2)}\]

Assume the specific estimate

\[ \sup \left\{ \| u^* \|_{V'_I} : u^* \in A(u, \omega) \right\} \leq a(\omega) + b(\omega) \| u \|_{V_I}^{p-1} \quad\text{(77.0.3)}\]

where \( a(\omega), b(\omega) \) are nonnegative. Also assume the following coercivity estimate valid for each \( t \leq T \) and for some \( \lambda(\omega) \geq 0 \),

\[ \inf \left( \int_0^t \langle \lambda(\omega) Bu + u^* - \lambda(\omega) Bv + v^*, u - v \rangle dt : u^* \in A(u, \omega) \right) \geq \delta(\omega) \int_0^t \| u \|_{V_I}^p ds - m(\omega) \quad\text{(77.0.4)}\]

where \( m(\omega) \) is some nonnegative constant, \( \delta(\omega) > 0 \).

**Monotonicity**

It will also be assumed that \( \lambda(\omega) B + A \) is monotone in the sense that

\[ \int_0^t \langle \lambda(\omega) Bu + u^* - \lambda(\omega) Bv + v^*, u - v \rangle ds \geq 0 \]

for a suitable choice of \( \lambda(\omega) \) whenever \( u^* \in A(u, \omega), v^* \in A(v, \omega) \).

**Limit condition**

Let \( U \) be a Banach space dense in \( V \) and that if \( u_i \rightharpoonup u \) in \( V_I \) and \( u_i^* \in A(u_i) \) with \( u_i^* \rightharpoonup u^* \) in \( V'_I \) and \( t \to Bu_i(t) \) is continuous and

\[ \sup \sup \frac{\| Bu_i(t) - Bu_i(s) \|_{V'} }{|t - s|^\alpha} \leq C \quad\text{(77.0.5)}\]

then if

\[ \lim \sup_{i \to \infty} \langle u_i^*, u_i - u \rangle_{V'_I, V_I} \leq 0 \quad\text{(77.0.6)}\]

it follows that for all \( v \in V_I \), there exists \( u^*(v) \in Au \) such that

\[ \lim \inf_{i \to \infty} \langle u_i^*, u_i - v \rangle_{V'_I, V_I} \geq \langle u^*(v), u - v \rangle_{V'_I, V_I} \quad\text{(77.0.7)}\]

As to \( B(\omega) \), it is \( k(\omega) B \) where \( B \in \mathcal{L}(W, W') \) and is self adjoint and nonnegative where \( k \) is \( \mathcal{F}_0 \) measurable.
Progressively measurable condition

**Condition 77.0.6** For each $t \leq T$, if $\omega \rightarrow u (\cdot, \omega)$ is $\mathcal{F}_t$ measurable into $\mathcal{V}_{[0,t]}$, then there exists a $\mathcal{F}_t$ measurable selection of $A (u (\cdot, \omega), \omega)$ into $\mathcal{V}_{[0,t]}$.

Then there is a theorem. It was Theorem 75.7.4 which gave existence and uniqueness of progressively measurable solutions $u$ to the integral equation.

**Theorem 77.0.7** Assume the above conditions, along with the progressive measurability condition 77.0.6. Let $u_0$ be $\mathcal{F}_0$ measurable and $\omega \rightarrow B (\omega)$ also $\mathcal{F}_0$ measurable and $(t, \omega) \rightarrow X_{[0,t]} (t)$ $f (t, \omega)$ is $\mathcal{B} ([0, t]) \times \mathcal{F}_t$ product measurable into $V'$ for each $t$.

$$B (\omega) = k (\omega) B, \quad k (\omega) \geq 0, \quad k \text{ measurable.}$$

Also let $t \rightarrow q (t, \omega)$ be continuous and $q$ is progressively measurable into $V$. Suppose there is at most one solution to

$$Bu (t, \omega) + \int_0^t z (s, \omega) \, ds = \int_0^t f (s, \omega) \, ds + Bu_0 (\omega) + Bq (t, \omega), \quad (77.0.8)$$

for each $\omega$. Then the solution to the above integral equation $u$ is progressively measurable. Moreover, for each $\omega$, both $Bu (t, \omega) = B (u (t, \omega))$ for a.e. $t$ and $z (t, \omega) \in A (u (t, \omega), \omega)$ for a.e. $t$. Also, for each $a \in [0, T]$,

$$Bu (t, \omega) + \int_a^t z (s, \omega) \, ds = \int_a^t f (s, \omega) \, ds + Bu (a, \omega) + Bq (t, \omega) - Bq (a, \omega)$$

Letting $q (t) = \int_0^t \Phi_n dW$ defined above with the filtration also being the one obtained from the Wiener process, this implies the following theorem. The $\sigma$ algebra of progressively measurable sets will be denoted by $\mathcal{P}$.

**Theorem 77.0.8** Assume the above conditions, along with the progressive measurability condition 77.0.6. Also assume there is at most one solution to 77.0.4 where

$$q (t, \cdot) = \int_0^t \Phi_n dW$$

Then there exists a $\mathcal{P}$ measurable $u_n$ such that also $z_n$ is progressively measurable

$$Bu_n (t, \omega) - Bu_0 (\omega) + \int_0^t z_n (s, \omega) \, ds = \int_0^t f (s, \omega) \, ds + B \int_0^t \Phi_n dW$$

where for each $\omega$, $z_n (\cdot, \omega) \in A (u_n (\cdot, \omega), \omega)$. The function $Bu_n (t, \omega) = B (u_n (t, \omega))$ for a.e. $t$.

This gives an existence theorem for the inclusion of a stochastic integral. However, it is desired to get a similar result for $\Phi$ rather than $\Phi_n$. Next is the Ito formula which is useable because of the progressive measurability of $u_n, z_n$. This formula applies to the following situation.
Situation 77.0.9 Let $X$ have values in $V$ and satisfy the following
\[ BX(t) = BX_0 + \int_0^t Y(s) \, ds + B \int_0^t Z(s) \, dW(s), \tag{77.0.9} \]

$X_0 \in L^2(\Omega; W)$ and is $F_0$ measurable, where $Z$ is $L_2(Q^{1/2}U, W)$ progressively measurable and
\[ \|Z\|_{L^2([0,T] \times \Omega, L_2(Q^{1/2}U, W))} < \infty. \]

This is what is needed to define the stochastic integral in the above formula.
Assume $X, Y$ satisfy
\[ BX, Y \in K' \equiv L^{p'}([0,T] \times \Omega; V'), \]

the $\sigma$ algebra of measurable sets defining $K'$ will be the progressively measurable sets. Here $1/p' + 1/p = 1, p > 1$.

Also the sense in which the equation holds is as follows. For a.e. $\omega$, the equation holds in $V'$ for all $t \in [0, T]$. Thus we are considering a particular representative $X$ of $K$ for which this happens. Also it is only assumed that $BX(t) = B(X(t))$ for a.e. $t$. Thus $BX$ is the name of a function having values in $V'$ for which $BX(t) = B(X(t))$ for a.e. $t$, all $t \notin N$ a set of measure zero. Assume that $X$ is progressively measurable also and $X \in L^p([0,T] \times \Omega, V)$.

Then in the above situation, we obtain the following integration by parts formula which is called the Ito formula. This particular version is presented in Theorem 71.7.2 and is a generalization of work of Krylov. A proof of the case of a Gelfand triple in which $B = I$ is in [18].

**Theorem 77.0.10** In Situation 77.0.9, for $\omega$ off a set of measure zero, for every $t \in N^c$, the measure of $N$ equalling $0$,

\[ \langle BX(t), X(t) \rangle = \langle BX_0, X_0 \rangle + \int_0^t 2 \langle Y(s), X(s) \rangle \, ds + \int_0^t \langle BZ, Z \rangle_{L_2} \, ds + 2M(t) \tag{77.0.10} \]

where $M(t)$ is a stochastic integral and a local martingale equal to $0$ when $t = 0$.
Also, there exists a unique continuous, progressively measurable function denoted as $\langle BX, X \rangle$ such that it equals $\langle BX(t), X(t) \rangle$ for a.e. $t$ and $\langle BX, X \rangle(t)$ equals the right side of the above for all $t$. In addition to this,

\[ E(\langle BX, X \rangle(t)) = \]

\[ E(\langle BX_0, X_0 \rangle) + E\left( \int_0^t (2 \langle Y(s), X(s) \rangle + \langle BZ, Z \rangle_{L_2}) \, ds \right) \tag{77.0.11} \]
Also the quadratic variation of \( M(t) \) in \((77.0.14)\) is dominated by

\[
C \int_0^t \|Z\|^2_{L_2} \|BX\|^2_W \, ds \tag{77.0.12}
\]

for a suitable constant \( C \). Also \( t \to BX(t) \) is continuous with values in \( W' \) for \( t \in N^C_\omega \). In fact, this martingale can be written as

\[
\int_0^t (Z \circ J^{-1})^* BX \circ J dW
\]

That ugly integral displayed above can be written in the form

\[
\int_0^t \langle BX, dN \rangle
\]

where \( N(t) = \int_0^t Z(s) \, dW \).

Now we consider the meaning of the symbol \( \langle BZ, Z \rangle_{L_2^*} \). You begin with a complete orthonormal set \( \{g_k\} \) in \( Q^{1/2,U} \). Then to say that \( Z \) has values in \( L_2(Q^{1/2,U};W) \) is to say that

\[
\sum_j \sum_i (Z(g_i),e_j)^2 = \sum_i \|Z(g_i)\|^2_W < \infty
\]

where \( \{e_j\} \) is an orthonormal basis in \( W \). You can let it be the one used earlier where each is actually in \( V \) or even in \( E \). Then the symbol means

\[
(R^{-1}BZ,Z)_{L_2}
\]

where \( R \) is the Riesz map from the Hilbert space \( W \) to its dual space. Thus it equals

\[
\sum_i (R^{-1}BZ(g_i),Z(g_i))_W = \sum_i (BZ(g_i),Z(g_i))_W
\]

so it is seen to be nonnegative.

Now apply this Ito formula to Theorem \((77.0.8)\) in which we make the assumptions there on \( \|u_0\| \in L^2(\Omega) \) and that \( f \in L^{p'}([0,T] \times \Omega;V') \) where the \( \sigma \) algebra is \( \mathcal{P} \) the progressively measurable \( \sigma \) algebra, and

\[
\Phi \in L^2\left(\Omega,L^2([0,T],L_2(Q^{1/2,U};W))\right)
\]

which implies the same is true of \( \Phi_n \). This yields, from the assumed estimates, an expression of the form where \( \delta > 0 \) is a suitable constant.

\[
\frac{1}{2} \langle Bu_n, u_n \rangle(t) - \frac{1}{2} \langle Bu_0, u_0 \rangle + \delta \int_0^t \|u_n(s)\|_V^p \, ds
\]

\[
\leq \lambda \int_0^t \langle Bu_n, u_n \rangle(s) \, ds + \int_0^t \langle f, u_n \rangle_{V',V} \, ds + \int_0^t c(s,\omega) \, ds
\]

\[
+ \int_0^t \langle B\Phi_n, \Phi_n \rangle_{L_2} \, ds + M_n(t) \tag{77.0.13}
\]
where \( c \in L^1([0,T] \times \Omega) \). Then taking expectations or using that part of the Ito formula,

\[
\frac{1}{2} E(\langle Bu_n, u_n \rangle(t)) + \delta E\left(\int_0^T \|u_n(s)\|_V^p \, ds\right)
\]

\[
\leq \lambda \int_0^t E(\langle Bu_n, u_n \rangle(s)) \, ds + \int_0^t E\left(\langle f, u_n \rangle_{V', V} \right) ds + C(\Phi, u_0)
\]

Then by Gronwall’s inequality and some simple manipulations,

\[
E(\langle Bu_n, u_n \rangle(t)) + E\left(\int_0^T \|u_n(s)\|_V^p \, ds\right) \leq C(T, f, u_0, \Phi)
\]

Then using obvious estimates and Gronwall’s inequality in (77.0.13), this yields an inequality of the form

\[
\langle Bu_n, u_n \rangle(t) - \langle Bu_0, u_0 \rangle + \int_0^t \|u_n(s)\|_V^p \, ds \leq C(f, \lambda, c) + \|B\| \int_0^t \|\Phi_n\|_{L^2_2}^2 \, ds + M_n^*(t)
\]

where the random variable \( C(f, \lambda, c) \) is nonnegative and is integrable. Now \( t \to M_n^*(t) \) is increasing as is the integral on the right. Hence it follows that, modifying the constants,

\[
\sup_{s \in [0,t]} \langle Bu_n, u_n \rangle(s) + \int_0^t \|u_n(s)\|_V^p \, ds \leq C(f, \lambda, c, u_0) + 2 \|B\| \int_0^t \|\Phi_n\|_{L^2_2}^2 \, ds + 2M_n^*(t)
\]  

(77.0.14)

Next take the expectation of both sides and use the Burkholder Davis Gundy inequality along with the description of the quadratic variation of the martingale \( M_n(t) \). This yields

\[
E\left(\sup_{s \in [0,t]} \langle Bu_n, u_n \rangle(s) \right) + E\left(\int_0^t \|u_n(s)\|_V^p \, ds\right) \leq C(f, \lambda, c, \Phi) + C \int_\Omega \left(\int_0^t \|Bu_n\|_{w'}^p \|\Phi_n\|_{L^2_2}^2 \, ds\right)^{1/2} \, dP
\]

Now \( \|Bw\| = \sup_{\|v\| \leq 1} \langle Bw, v \rangle \leq \langle Bw, w \rangle^{1/2} \). Also \( \int_0^t \|\Phi_n\|_{L^2_2}^2 \, ds \leq \int_0^T \|\Phi\|_{L^2_2}^2 \, ds \) and so the above inequality implies

\[
E\left(\sup_{s \in [0,t]} \langle Bu_n, u_n \rangle(s) \right) + E\left(\int_0^t \|u_n(s)\|_V^p \, ds\right) \leq C(f, \lambda, c, \Phi) + C \int_\Omega \sup_{s \in [0,t]} \langle Bu_n, u_n \rangle^{1/2}(s) \left(\int_0^t \|\Phi\|_{L^2_2}^2 \right)^{1/2} \, dP
\]
Then adjusting the constants yields

\[
\frac{1}{2} E \left( \sup_{s \in [0,T]} \langle Bu_n, u_n \rangle (s) \right) + E \left( \int_0^T \| u_n (s) \|_V^p \, ds \right) \\
\leq C + \int_{\Omega} \int_0^T \| \Phi \|_{L_2}^2 \, dt \, dP \equiv C
\]  

(77.0.15)

If needed, you could use a stopping time to be sure that \( E \left( \sup_{s \in [0,T]} \langle Bu_n, u_n \rangle (s) \right) < \infty \) and then let it converge to \( \infty \).

From the integral equation,

\[
Bu_n (t) - Bu_m (t) + \int_0^t z_n - z_m ds = B \int_0^t (\Phi_n - \Phi_m) \, dW
\]

Then using the monotonicity assumption and the Ito formula,

\[
\frac{1}{2} \langle Bu_n - Bu_m, u_n - u_m \rangle (t) \leq \int_0^t \langle Bu_n - Bu_m, u_n - u_m \rangle \, ds \\
+ \int_0^t \langle B (\Phi_n - \Phi_m), \Phi_n - \Phi_m \rangle \, ds + \int_0^t (\Phi_n - \Phi_m) \circ J^{-1} B (u_n - u_m) \circ JdW
\]

and so, from Gronwall’s inequality, there is a constant \( C \) which is independent of \( m, n \) such that

\[
\langle Bu_n - Bu_m, u_n - u_m \rangle (t) \leq CM_{nm} (t) \leq CM_{nm}^* (T) + C \int_0^T \| \Phi_n - \Phi_m \|_{L_2}^2 \, ds
\]

where \( M_{nm} \) refers to that local martingale on the right. Thus also

\[
\sup_{t \in [0,T]} \langle Bu_n - Bu_m, u_n - u_m \rangle (t) \leq CM_{nm} (t) \leq CM_{nm}^* (T) + C \int_0^T \| \Phi_n - \Phi_m \|_{L_2}^2 \, ds
\]

(77.0.16)

Taking the expectation and using the Burkholder Davis Gundy inequality again in a similar manner to the above,

\[
E \left( \sup_{t \in [0,T]} \langle Bu_n - Bu_m, u_n - u_m \rangle (t) \right) \leq C \int_{\Omega} \int_0^T \| \Phi_n - \Phi_m \|_{L_2}^2 \, dt \, dP
\]

Now the right side converges to 0 as \( m, n \to \infty \) and so there is a subsequence, denoted with the index \( k \) such that whenever \( m > k \),

\[
E \left( \sup_{t \in [0,T]} \langle Bu_k - Bu_m, u_k - u_m \rangle (t) \right) \leq \frac{1}{2^k}
\]
Note how this implies

\[ \int_{\Omega} \int_{0}^{T} \langle Bu_k - Bu_m, u_k - u_m \rangle \, dt \, dP \leq \frac{T}{2^k} \]  \hspace{1cm} (77.0.17)

Then consider the martingales \( M_k(t) \) considered earlier. One of these is of the form

\[ M_k = \int_{0}^{t} (\Phi_k \circ J^{-1})^* Bu_k \circ J \, dW \]

Then by the Burkholder Davis Gundy inequality and modifying constants as appropriate,

\[ E \left( (M_k - M_{k+1})^* \right) \leq C \int_{\Omega} \left( \int_{0}^{T} \left\| (\Phi_k \circ J^{-1})^* Bu_k - (\Phi_{k+1} \circ J^{-1})^* Bu_{k+1} \right\|^2 \, dt \right)^{1/2} \, dP \]

\[ \leq C \int_{\Omega} \left( \int_{0}^{T} \left\| \Phi_k - \Phi_{k+1} \right\|^2 \langle Bu_k, u_k \rangle + \left\| \Phi_{k+1} \right\|^2 \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle \, dt \right)^{1/2} \, dP \]

\[ \leq C \int_{\Omega} \left( \int_{0}^{T} \left\| \Phi_k - \Phi_{k+1} \right\|^2 \langle Bu_k, u_k \rangle \, dt \right)^{1/2} \]

\[ + C \int_{\Omega} \left( \int_{0}^{T} \left\| \Phi_{k+1} \right\|^2 \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle \, dt \right)^{1/2} \, dP \]

\[ \leq C \int_{\Omega} \left( \sup_{t} \langle Bu_k, u_k \rangle^{1/2} \left( \int_{0}^{T} \left\| \Phi_k - \Phi_{k+1} \right\|^2 \, dt \right)^{1/2} \, dP \right) \]

\[ + C \int_{\Omega} \left( \sup_{t} \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle^{1/2} \left( \int_{0}^{T} \left\| \Phi_k - \Phi_{k+1} \right\|^2 \, dt \right)^{1/2} \, dP \right) \]

\[ \leq C \left( \int_{\Omega} \sup_{t} \langle Bu_k, u_k \rangle \, dP \right)^{1/2} \left( \int_{\Omega} \int_{0}^{T} \left\| \Phi_k - \Phi_{k+1} \right\|^2 \, dt \, dP \right)^{1/2} \]

\[ + C \left( \int_{\Omega} \sup_{t} \langle Bu_k - Bu_{k+1}, u_k - u_{k+1} \rangle \, dP \right)^{1/2} \left( \int_{\Omega} \int_{0}^{T} \left\| \Phi_{k+1} \right\|^2 \, dtdP \right)^{1/2} \]

From the above inequality, and after adjusting the constants, the above is no larger than an expression of the form \( C \left( \frac{1}{2} \right)^{k/2} \) which is a summable sequence. Then

\[ \sum_{k} \int_{\Omega} \sup_{t \in [0, T]} |M_k(t) - M_{k+1}(t)| \, dP < \infty \]
Thus \( \{M_k\} \) is a Cauchy sequence in \( M^1_T \) and so there is a continuous martingale \( M \) such that
\[
\lim_{k \to \infty} E \left( \sup_t |M_k(t) - M(t)| \right) = 0
\]
Taking a further subsequence if needed, one can also have
\[
P \left( \sup_t |M_k(t) - M(t)| > \frac{1}{k} \right) \leq \frac{1}{2k}
\]
and so by the Borel Cantelli lemma, there is a set of measure zero such that off this set, \( \sup_t |M_k(t) - M(t)| \) converges to 0. Hence for such \( \omega \), \( M_{\omega}^k(T) \) is bounded independent of \( k \). Thus for \( \omega \) off a set of measure zero, \( 77.0.14 \) implies that for such \( \omega \),
\[
\sup_{s \in [0,T]} \langle Bu_r, u_r \rangle (s) + \int_0^T \|u_r(s)\|V^p_s \ ds \leq C(\omega)
\]
where \( C(\omega) \) does not depend on the index \( r \), this for the subsequence just described which will be the sequence of interest in what follows. Using the boundedness assumption for \( A \), one also obtains an estimate of the form
\[
\sup_{s \in [0,T]} \langle Bu_r, u_r \rangle (s) + \int_0^T \|u_r(s)\|V^p_s \ ds + \int_0^T \|z_r\|V^p_s \ ds \leq C(\omega) \quad (77.0.18)
\]
The idea here is to take weak limits converging to a function \( u \) and then identify \( z(\cdot,\omega) \) as being in \( A(u,\omega) \) but this will involve a difficulty. It will require a use of the above Ito formula and this will need \( u \) to be progressively measurable. By uniqueness, it would seem that this could be concluded by arguing that one does not need to take a subsequence due to uniqueness but the problem is that we won’t know the limit of the sequence is a solution unless we use the Ito formula. This is why we make the extra assumption that for \( z_i(\cdot,\omega) \in A(u_i,\omega) \) and for all \( \lambda \) large enough,
\[
\langle \lambda Bu_1 + z_1 - (\lambda Bu_2 + z_2), u_1 - u_2 \rangle \geq \delta \|u_1 - u_2\|^2_{A}, \quad \alpha \geq 1 \quad (77.0.19)
\]
where here \( \hat{V} \) will be a Banach space such that \( V \) is dense in \( \hat{V} \) and the embedding is continuous. As mentioned, this is not surprising in the case of most interest where there is a Gelfand triple and \( B = I \). Then using the integral equation for \( r = p, q, p < q \) along with the conclusion of the Ito formula above,
\[
E \left( \langle B(u_n - u_m), u_n - u_m \rangle (t) \right) + E \left( \int_0^t \|u_n - u_m\|^2_{V} \ ds \right) \leq E \left( \int_0^t \|B\| \|\Phi_n - \Phi_m\|^2_{L^2} \ ds \right) \equiv e(m,n)
\]
Hence, the right side converges to 0 as \( m, n \to \infty \) from the dominated convergence theorem. In particular,
\[
E \left( \int_0^T \|u_n - u_m\|^2_{\hat{V}} \ ds \right) \leq E \left( \int_0^T \|B\| \|\Phi_n - \Phi_m\|^2_{L^2} \ ds \right) \equiv e(m,n) \quad (77.0.20)
\]
Then also
\[
P \left( \int_0^T \| u_n - u_m \| \overline{V}^\alpha \ ds > \lambda \right) \leq \frac{e(m,n)}{\lambda}
\]
and so there exists a subsequence, denoted by \( r \) such that
\[
P \left( \int_0^T \| u_r - u_{r+1} \| \overline{V}^\alpha \ ds \leq 2^{-r} \right) < 2^{-r}
\]
Thus, by the Borel Cantelli lemma, there is a further enlarged set of measure zero, still denoted as \( N \) such that for \( \omega \notin N \)
\[
\int_0^T \| u_r - u_{r+1} \| \overline{V}^\alpha \ ds \leq 2^{-r}
\]
for all \( r \) large enough. Hence, by the usual proof of completeness, for these \( \omega \),
\[
\{ u_r (\cdot, \omega) \}
\]
is Cauchy in \( L^\alpha \left( [0, T], \overline{V} \right) \) and also \( u_r (t, \omega) \) converges to some \( u(t, \omega) \) pointwise in \( \overline{V} \) for a.e. \( t \). In addition, from \( \text{Lemma 74.3.4} \) these functions are a Cauchy sequence in \( L^\alpha \left( [0, T] \times \Omega; \overline{V} \right) \) with respect to the \( \sigma \) algebra of progressively measurable sets. Thus from Lemma \( \text{74.3.4} \), it can be assumed that for \( \omega \) off the set of measure zero, \( (t, \omega) \to u(t, \omega) \) is progressively measurable. From now on, this will be the sequence or a further subsequence. For \( \omega \notin N \), a set of measure zero and from \( \text{Lemma 74.0.18} \), there is a further subsequence for which the following convergences occur as \( r \to \infty \).
\[
u_r \to u \text{ weakly in } V
\]
\[
B(u_r) \to B(u) \text{ weakly in } V'
\]
\[
z_r \to z \text{ weakly in } V'
\]
\[
\left( B \left( u_r - \int_0^{(1)} \Phi_r dW \right) \right)' \to \left( B \left( u - \int_0^{(1)} \Phi dW \right) \right)'
\text{ weakly in } V'
\]
\[
\int_0^{(1)} \Phi_r dW \to \int_0^{(1)} \Phi dW \text{ uniformly in } C([0, T]; W)
\]
\[
Bu_r(t) \to Bu(t) \text{ weakly in } V'
\]
\[
Bu(0) = Bu_0,
\]
\[
Bu(t) = B(u(t)) \text{ a.e. } t
\]
In addition to this, we can choose the subsequence such that
\[
\sup_r \sup_{t \neq s} \frac{\left\| \int_s^t \Phi_r dW \right\|}{|t-s|^\gamma} < C(\omega) < \infty
\]
This is thanks to Corollary \ref{cor:boundedness}. The boundedness of the operator $A$, in particular the given estimates, imply that $z_r$ is bounded in $L^p([0,T] \times \Omega, V')$. Thus a subsequence can be obtained which yields weak convergence of $z_r$ in $L^p([0,T] \times \Omega, V')$ and then Lemma \ref{lemma:weak_convergence} may be applied to conclude that off a set of measure zero, $z$ is progressively measurable.

The claim \ref{claim:continuity_evaluation} and \ref{claim:convergence} follow from the continuity of the evaluation map defined on $X$, Theorem \ref{thm:evaluation_continuity}. The claim in \ref{claim:psi_convergence} follows from \ref{claim:continuity_evaluation} and the convergence \ref{lemma:weak_convergence}. To see this, let $\psi \in C^\infty_c(0,T)$.

\[
\int_0^T Bu(t) \psi(t) dt = \lim_{r \to \infty} \int_0^T Bu_r(t) \psi(t) dt
\]

Since this is true for all such $\psi$, it follows that $Bu(t) = B(u(t))$ for a.e. $t$. Passing to a limit in the integral equation yields the following for $\omega$ off a set of measure zero,

\[
Bu(t,\omega) - Bu_0(\omega) + \int_0^t z(s,\omega) ds = \int_0^t f(s,\omega) ds + B \int_0^t \Phi_n dW
\]

In the following claim, assume $\Phi \in L^2\left(\Omega, L^\infty\left([0,T], \mathcal{L}_2\left(Q^{1/2}V_1, W\right)\right)\right)$

**Claim:** $\lim_{r \to \infty} \int_0^T (\Phi_r \circ J^{-1})^* Bu_r \circ J dW = \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW$ off a set of measure zero.

**Proof of claim:**

\[
E\left(\left| \int_0^T (\Phi_r \circ J^{-1})^* Bu_r \circ J dW - \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW \right| \right)
\]

\[
\leq E\left(\left| \int_0^T (\Phi_r \circ J^{-1})^* Bu_r \circ J dW - \int_0^T (\Phi \circ J^{-1})^* Bu_r \circ J dW \right| \right)
\]

\[
+ E\left(\left| \int_0^T (\Phi \circ J^{-1})^* Bu_r \circ J dW - \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW \right| \right)
\]

Then, by the Burkholder Davis Gundy inequality,

\[
\leq \int_\Omega \left(\int_0^T \|\Phi_r - \Phi\|^2 (Bu_r, u_r)\right)^{1/2} dP
\]

\[
+ \int_\Omega \left(\int_0^T \|\Phi\|^2 (Bu_r - Bu, u_r - u)\right)^{1/2} dP
\]

\[
\leq \int_\Omega \sup_t (Bu_r(t), u_r(t))^{1/2} \left(\int_0^T \|\Phi_r - \Phi\|^2 dt\right)^{1/2} dP
\]
\[ + \int_{\Omega} \| \Phi_n \|_{L^\infty([0,T],\mathcal{L}_2)} \left( \int_0^T (Bu_r - Bu, u_r - u) \right)^{1/2} dP \]

\[ \leq \left( \int_{\Omega} \sup_t \langle Bu_r(t), u_r(t) \rangle \, dP \right)^{1/2} \left( \int_{\Omega} \int_0^T \| \Phi_r - \Phi \|^2 \, dt \right)^{1/2} \]

\[ + \left( \int_{\Omega} \| \Phi_n \|^2_{L^\infty([0,T],\mathcal{L}_2)} \right)^{1/2} \left( \int_{\Omega} \int_0^T \langle Bu_r - Bu, u_r - u \rangle \, dt \, dP \right)^{1/2} \]

\[ (77.0.30) \]

Letting the \( e_i \) be the special vectors of Theorem 77.0.19,

\[ \int_{\Omega} \int_0^T (Bu_r - Bu, u_r - u) \, dtdP = \int_{\Omega} \int_0^T \sum_i \langle Bu_r - Bu, e_i \rangle^2 \, dtdP \]

\[ = \int_{\Omega} \int_0^T \sum_i \lim_{p \to \infty} \langle Bu_r - Bu_p, e_i \rangle^2 \, dtdP \]

\[ \leq \lim_{p \to \infty} \int_{\Omega} \int_0^T \sum_i \langle Bu_r - Bu_p, e_i \rangle^2 \, dtdP \]

\[ = \lim_{p \to \infty} \int_{\Omega} \int_0^T \sum_i \langle Bu_r - Bu_p, e_i \rangle^2 \, dtdP \]

\[ = \lim_{p \to \infty} \int_{\Omega} \int_0^T \langle Bu_r - Bu_p, u_r - u_p \rangle \, dtdP \]

Now by 77.0.17, the last expression is no larger than \( T/2^r \) and so

\[ \int_{\Omega} \int_0^T (Bu_r - Bu, u_r - u) \, dtdP \leq \frac{T}{2^r} \]

Then, from 77.0.17,

\[ E \left( \left\| \int_0^T (\Phi_r \circ J^{-1})^* Bu_r \circ JdW - \int_0^T (\Phi \circ J^{-1})^* Bu \circ JdW \right\| \right) \]

\[ \leq \left( \int_{\Omega} \sup_t \langle Bu_r(t), u_r(t) \rangle \, dP \right)^{1/2} \left( \int_{\Omega} \int_0^T \| \Phi_r - \Phi \|^2 \, dt \right)^{1/2} + C \left( \frac{T}{2^r} \right)^{1/2} \]

\[ \leq C \left( \int_{\Omega} \int_0^T \| \Phi_r - \Phi \|^2 \, dt \right)^{1/2} + C \left( \frac{T}{2^r} \right)^{1/2} < C2^{-r} + C \left( \frac{T}{2^r} \right)^{1/2} \]

which clearly converges to 0 as \( r \to \infty \). Since the right side is summable, one obtains also pointwise convergence. This proves the claim.
From the above considerations using the space $\hat{V}$, it follows that this $u$ is the same as the one just obtained in the sense that for $\omega$ off $N$, the two are equal for a.e. $t$. Thus we take $u$ to be this common function. Hence there is a set of measure zero such that $(t, \omega) \to X_N \circ u(t, \omega)$ is progressively measurable in the above convergences. Also, this shows that we are taking $u \in L^p ([0, T] \times \Omega; V)$. From the measurability of $u_r$, $u$, we can obtain a dense countable subset \{t_k\} and an enlarged set of measure zero $N$ such that for $\omega \notin N, Bu(t_k, \omega) = B(u(t_k, \omega))$ and $Bu_r(t_k, \omega) = B(u_r(t_k, \omega))$ for all $t_k$ and $r$. This uses the same argument as in Lemma 77.0.31.

It remains to verify that \( z(\cdot, \omega) \in A(u(\cdot, \omega), \omega) \). It follows from the above considerations that the Ito formula above can be used at will. Assume that for a given $\omega \notin N, Bu(T, \omega) = B(u(T, \omega))$, similar for $Bu_r$. If not, just do the following argument for all $T'$ close to $T$, letting $T'$ be in the dense subset just described. Then from the integral equation solved, and letting \{\( e_i \)\} be the special set described in Theorem 77.0.24 and suppressing the dependence on $\omega$,

\[
\sum_{i=1}^{\infty} \langle Bu_r(T), e_i \rangle^2 - \sum_{i=1}^{\infty} \langle Bu_0, e_i \rangle + 2 \int_0^T \langle z_r, u_r \rangle \, ds
\]

\[
= 2 \int_0^T \langle f, u_r \rangle \, ds + 2 \int_0^T (\Phi_r \circ J^{-1})^* Bu_r \circ J dW
\]

Thus also

\[
2 \int_0^T \langle z_r, u_r \rangle \, ds = - \sum_{i=1}^{\infty} \langle Bu(T), e_i \rangle^2 + \sum_{i=1}^{\infty} \langle Bu_0, e_i \rangle + 2 \int_0^T \langle f, u \rangle \, ds + 2 \int_0^T (\Phi_r \circ J^{-1})^* Bu_r \circ J dW
\]

(77.0.31)

A similar formula to (77.0.31) holds for $u$. Thus

\[
2 \int_0^T \langle z, u \rangle \, ds = - \sum_{i=1}^{\infty} \langle Bu(T), e_i \rangle^2 + \sum_{i=1}^{\infty} \langle Bu_0, e_i \rangle + 2 \int_0^T \langle f, u \rangle \, ds + 2 \int_0^T (\Phi \circ J^{-1})^* Bu \circ J dW
\]

It follows from (77.0.24) and the other convergences that

\[
\limsup_{r \to \infty} \int_0^T \langle z_r, u_r \rangle \, ds \leq \int_0^T \langle z, u \rangle \, ds
\]

Hence

\[
\limsup_{r \to \infty} \langle z_r, u_r - u \rangle_{\mathcal{V}' \mathcal{V}} \leq 0
\]

Now from the limit condition, for any $v \in \mathcal{V}$, there exists a $z(v) \in A(u(\cdot, \omega), \omega)$ such that

\[
\langle z, u - v \rangle_{\mathcal{V}' \mathcal{V}} \geq \liminf_{r \to \infty} (\langle z_r, u_r - u \rangle + \langle z_r, u - v \rangle)
\]

\[
\geq \liminf_{r \to \infty} (\langle z_r, u_r - v \rangle) \geq \langle z(v), u - v \rangle
\]
The reason the limit condition applies is the estimate and the convergence which shows that
\[ B \left( u_r - \int_0^{(s)} \Phi_r dW \right) \]
satisfy a Hölder condition into \( V' \). Then the estimate implies that the \( Bu_r \) are bounded in a Hölder norm and so the same is true of the \( Bu \).

Thus the situation of the limit condition is obtained. Then it follows from separation theorems and the fact that \( A(u(\cdot,\omega),\omega) \) is closed and convex that \( z(\cdot,\omega) \in A(u(\cdot,\omega),\omega) \). This has proved the following Theorem.

**Theorem 77.0.11** Assume the above conditions, along with the progressive measurability condition. Also assume there is at most one solution to where
\[ q(t,\cdot) \equiv \int_0^t \Phi dW \]
Then there exists a \( \mathcal{P} \) measurable \( u \) such that also \( z \) is progressively measurable
\[ Bu(t,\omega) - Bu_0(\omega) + \int_0^t z(s,\omega) \, ds = \int_0^t f(s,\omega) \, ds + B \int_0^t \Phi dW \]
where for each \( \omega \), \( z(\cdot,\omega) \in A(u(\cdot,\omega),\omega) \). The function \( Bu(t,\omega) = B(u(t,\omega)) \) for a.e. \( t \). Here
\[ \Phi \in L^\alpha \left( [0,T] \times \Omega, \mathcal{L}_2 \left( Q^{1/2}U,W \right) \right) \cap L^2 \left( \Omega, L^\infty \left( [0,T], \mathcal{L}_2 \left( Q^{1/2}U,W \right) \right) \right), \alpha > 2 \]
and \( u_0 \in L^2(\Omega,W), f \in L^p'([0,T] \times \Omega; V') \).

### 77.1 Replacing \( \Phi \) With \( \sigma(u) \)

This has an easy generalization to the case where \( \Phi(t,\omega) \) is replaced with \( \sigma(t,\omega,u) \) for \( u \) the solution to the integral equation. In this generalization, we assume that
\[ (Bu,u) = \|u\|^2_W \]
For example, it could be \( W = W' \) and \( B = I \) which is the usual case of a Gelfand triple. We will strengthen the monotonicity assumption to for \( z_1 \in A(u_1,\omega) \),
\[ (\lambda Bu_1 + z_1 - (\lambda Bu_2 + z_2), u_1 - u_2) \geq \delta \| u_1 - u_2 \|^2_W \quad (77.1.32) \]
The case of most interest is the usual one where \( V \subseteq W = W' \subseteq V' \), the case of a Gelfand triple in which \( B \) is the identity. As to \( \sigma \), the assumption is made that
\[ \| \sigma(t,\omega,u) \|_W \leq C + C \| u \|_W \]
\[ \| \sigma(t,\omega,u_1) - \sigma(t,\omega,u_2) \|_{\mathcal{L}_2(Q^{1/2}U,W)} \leq K \| u_1 - u_2 \|_W \]
Of course it is also assumed that whenever \( u \) has values in \( W \) and is progressively measurable, \( (t, \omega) \to \sigma(t, \omega, u(t, \omega)) \) is also progressively measurable into \( \mathcal{L}_2(Q^{1/2}U, W) \).

Letting \( w_i \in L^2([0, T] \times \Omega, W) \cap L^2(\Omega, L^\infty([0, T], W)) \) each \( w_i \) being progressively measurable, the above assumptions and Theorem 77.11, there exists a solution \( u_i \) to the integral equation

\[
Bu_i(t, \omega) - Bu_0(\omega) + \int_0^t z_i(s, \omega) \, ds = \int_0^t f(s, \omega) \, ds + B \int_0^t \sigma(w_i) \, dW
\]

here we write \( \sigma(w_i) \) for short instead of \( \sigma(t, \omega, w_i) \). Then from the estimates,

\[
\langle Bu, u \rangle(t) - \langle Bu_0, u_0 \rangle + \delta \int_0^t \|u\|^2_W \, ds = 2 \int_0^t \langle f, u \rangle \, ds + C(b_3, b_4, b_5)
\]

\[
+ \lambda \int_0^t \langle Bu, u \rangle \, ds + \int_0^t \langle B\sigma(w), \sigma(w) \rangle_{\mathcal{L}_2} \, ds + 2M^*(t)
\]

\[
\leq 2 \int_0^t \langle f, u \rangle \, ds + C(b_3, b_4, b_5) + \lambda \int_0^t \langle Bu, u \rangle \, ds + \int_0^t \left( C + \|w\|^2_W \right) \, ds + 2M^*(t)
\]

where \( M^*(t) = \sup_{s \in [0, t]} |M(s)| \) and the quadratic variation of \( M \) is no larger than

\[
\int_0^t \|\sigma(w)\|^2 \langle B(u), u \rangle \, ds
\]

Then using Gronwall’s inequality, one obtains an inequality of the form

\[
\sup_{s \in [0, T]} \langle Bu, u \rangle(s) \leq C + C \left( M^*(t) + \int_0^t \|w\|^2_W \, ds \right)
\]

where \( C = C(u_0, f, \delta, \lambda, b_3, b_4, b_5, T) \) and is integrable. Then take expectation. By Burkholder Davis Gundy inequality and adjusting constants as needed,

\[
\mathbb{E}\left( \sup_{s \in [0, T]} \langle Bu, u \rangle(s) \right)
\]

\[
\leq C + C \int_{\Omega} \int_0^T \|w\|^2_W \, dsdP + C \int_{\Omega} \left( \int_0^T \|\sigma(w)\|^2 \langle B(u), u \rangle \, ds \right)^{1/2} \, dP
\]

\[
\leq C + C \int_{\Omega} \int_0^T \|w\|^2_W \, dsdP + C \int_{\Omega} \sup_{s \in [0, T]} \langle Bu, u \rangle^{1/2}(s) \left( \int_0^T \|\sigma(w)\|^2 \, ds \right)^{1/2} \, dP
\]

\[
\leq C + C \int_{\Omega} \int_0^T \|w\|^2_W \, dsdP + \frac{1}{2} \mathbb{E}\left( \sup_{s \in [0, T]} \langle Bu, u \rangle(s) \right) + C \int_{\Omega} \int_0^T \left( C + \|w\|^2_W \right)
\]
Thus

\[
E \left( \langle Bu, u \rangle (t) \right) \leq E \left( \sup_{s \in [0, T]} \langle Bu, u \rangle (s) \right) \leq C + C \int_{\Omega} \int_{0}^{T} \|w\|_{W}^{2} \, ds \, dP
\]

and so

\[
\|u\|_{L^{2}(\Omega, L^{\infty}([0, T], W))}^{2} \leq C + C \int_{\Omega} \int_{0}^{T} \|w\|_{W}^{2} \, ds \, dP
\]

which implies \(u \in L^{2} (\Omega, L^{\infty} ([0, T], W))\) and is progressively measurable.

Using the monotonicity assumption, there is a suitable \(\lambda\) such that

\[
\frac{1}{2} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle (t) + r \int_{0}^{t} \|u_{1} - u_{2}\|_{W}^{2} \, ds
\]

\[
- \lambda \int_{0}^{t} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle \, ds
\]

\[
- \int_{0}^{t} \langle B \sigma (u_{1}) - B \sigma (u_{2}), \sigma (u_{1}) - \sigma (u_{2}) \rangle_{L^{2}} \, ds \leq M^{*} (t)
\]

where the right side is of the form \(\sup_{s \in [0, t]} |M (s)|\) where \(M (t)\) is a local martingale having quadratic variation dominated by

\[
C \int_{0}^{t} \|\sigma (w_{1}) - \sigma (w_{2})\|^{2} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle \, ds
\]

Then by assumption and using Gronwall’s inequality, there is a constant \(C = C (\lambda, K, T)\) such that

\[
\langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle (t) \leq CM^{*} (t)
\]

Then also, since \(M^{*}\) is increasing,

\[
\sup_{s \in [0, t]} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle (s) \leq CM^{*} (t)
\]

Taking expectations and from the Burkholder Davis Gundy inequality,

\[
E \left( \sup_{s \in [0, t]} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle (s) \right)
\]

\[
\leq C \int_{\Omega} \left( \int_{0}^{t} \|\sigma (w_{1}) - \sigma (w_{2})\|^{2} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle \right)^{1/2} \, dP
\]

\[
\leq C \int_{\Omega} \sup_{s \in [0, t]} \langle B (u_{1} - u_{2}), u_{1} - u_{2} \rangle^{1/2} (s) \left( \int_{0}^{t} \|\sigma (w_{1}) - \sigma (w_{2})\|^{2} \right)^{1/2} \, dP
\]
Then it follows after adjusting constants that there exists an inequality of the form

\[
E \left( \sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(s) \right) \leq CE \left( \int_0^t \| \sigma(w_1) - \sigma(w_2) \|_{L_2}^2 \, ds \right)
\]

Hence

\[
E \left( \sup_{s \in [0,t]} \langle B(u_1 - u_2), u_1 - u_2 \rangle(t) \right) \leq CK^2 E \left( \int_0^t \| w_1 - w_2 \|_W^2 \, ds \right)
\]

Thus, for each \( t \leq T \)

\[
\int_{\Omega} \langle B(u_1 - u_2), u_1 - u_2 \rangle(t) \, dP \leq CK^2 E \left( \int_0^t \| w_1 - w_2 \|_W^2 \, ds \right)
\]

one can consider the map \( \psi(w) \equiv u \) as described above. The function \( t \to \langle B(\psi^n w_1 - \psi^n w_2), \psi^n w_1 - \psi^n w_2 \rangle(t) \) is continuous. So there exists \( t(\omega) \) where the maximum occurs. Then let \( t \) be this \( t(\omega) \). Then the above inequality implies

\[
E \left( \| \psi^n w_1 - \psi^n w_2 \|_{L_\infty([0,T],W)}^2 \right)
\]

\[
= E \left( \langle B(\psi^n w_1 - \psi^n w_2), \psi^n w_1 - \psi^n w_2 \rangle(t) \right)
\]

\[
\leq CK^2 E \left( \int_0^t \| \psi^{n-1} w_1 - \psi^{n-1} w_2 \|_W^2 \, dt_1 \right)
\]

\[
= CK^2 E \left( \int_0^t \langle B(\psi^{n-1} w_1 - \psi^{n-1} w_2), \psi^{n-1} w_1 - \psi^{n-1} w_2 \rangle(t_1) \, dt_1 \right)
\]

\[
\leq (CK^2)^2 E \left( \int_0^t \int_0^{t_1} \langle B(\psi^{n-2} w_1 - \psi^{n-2} w_2), \psi^{n-2} w_1 - \psi^{n-2} w_2 \rangle(t_2) \, dt_2 \, dt_1 \right)
\]

\[
\cdots \leq (CK^2)^n E \left( \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \langle B(w_1 - w_2), w_1 - w_2 \rangle(t_n) \, dt_n \cdots dt_2 \, dt_1 \right)
\]

\[
\leq (CK^2)^n E \left( \sup_t \langle B(w_1 - w_2), w_1 - w_2 \rangle(t) \right) \left( \frac{T^n}{n!} \right) \leq \frac{1}{2} \| w_1 - w_2 \|_{L_2([0,T],W)}^2
\]

provided \( n \) is sufficiently large. It follows that

\[
\| \psi^n w_1 - \psi^n w_2 \|_{L_2([\Omega,L_\infty([0,T],W)])}^2 \leq \frac{1}{2} \| w_1 - w_2 \|_{L_2([\Omega,L_\infty([0,T],W)])}^2
\]

for all \( n \) sufficiently large. Hence, if one begins with \( w \in L^2([\Omega,L_\infty([0,T],W)]) \cap L^2([0,T] \times \Omega,W) \), the sequence of iterates \( \{ \psi^n w \}_{n=1}^\infty \) must converge to some fixed point \( u \in L^2(\Omega,L_\infty([0,T],W)) \). This \( u \) is automatically in \( L^2([0,T] \times \Omega,W) \) and is progressively measurable since each of the iterates is progressively measurable.
Appendix A

The Hausdorff Maximal Theorem

The purpose of this appendix is to prove the equivalence between the axiom of choice, the Hausdorff maximal theorem, and the well-ordering principle. The Hausdorff maximal theorem and the well-ordering principle are very useful but a little hard to believe; so, it may be surprising that they are equivalent to the axiom of choice. First it is shown that the axiom of choice implies the Hausdorff maximal theorem, a remarkable theorem about partially ordered sets.

A nonempty set is partially ordered if there exists a partial order, $\prec$, satisfying

$x \prec x$

and

if $x \prec y$ and $y \prec z$ then $x \prec z$.

An example of a partially ordered set is the set of all subsets of a given set and $\subseteq$. Note that two elements in a partially ordered sets may not be related. In other words, just because $x$, $y$ are in the partially ordered set, it does not follow that either $x \prec y$ or $y \prec x$. A subset of a partially ordered set, $\mathcal{C}$, is called a chain if $x$, $y \in \mathcal{C}$ implies that either $x \prec y$ or $y \prec x$. If either $x \prec y$ or $y \prec x$ then $x$ and $y$ are described as being comparable. A chain is also called a totally ordered set. $\mathcal{C}$ is a maximal chain if whenever $\mathcal{C}'$ is a chain containing $\mathcal{C}$, it follows the two chains are equal. In other words $\mathcal{C}$ is a maximal chain if there is no strictly larger chain.

Lemma A.0.1 Let $\mathcal{F}$ be a nonempty partially ordered set with partial order $\prec$. Then assuming the axiom of choice, there exists a maximal chain in $\mathcal{F}$.

Proof: Let $\mathcal{X}$ be the set of all chains from $\mathcal{F}$. For $\mathcal{C} \in \mathcal{X}$, let

$S_\mathcal{C} = \{x \in \mathcal{F} \text{ such that } \mathcal{C} \cup \{x\} \text{ is a chain strictly larger than } \mathcal{C}\}$.
If $S_C = \emptyset$ for any $C$, then $C$ is maximal. Thus, assume $S_C \neq \emptyset$ for all $C \in \mathcal{X}$. Let $f(C) \in S_C$. (This is where the axiom of choice is being used.) Let

$$g(C) = C \cup \{f(C)\}.$$ 

Thus $g(C) \supseteq C$ and $g(C) \setminus C = \{f(C)\} = \{\text{a single element of } \mathcal{F}\}$. A subset $T$ of $\mathcal{X}$ is called a tower if

$$\emptyset \in T,$$

$$C \in T \text{ implies } g(C) \in T,$$

and if $S \subseteq T$ is totally ordered with respect to set inclusion, then

$$\cup S \in T.$$ 

Here $S$ is a chain with respect to set inclusion whose elements are chains.

Note that $\mathcal{X}$ is a tower. Let $T_0$ be the intersection of all towers. Thus, $T_0$ is a tower, the smallest tower. Are any two sets in $T_0$ comparable in the sense of set inclusion so that $T_0$ is actually a chain? Let $C_0$ be a set of $T_0$ which is comparable to every set of $T_0$. Such sets exist, $\emptyset$ being an example. Let

$$B \equiv \{D \in T_0 : D \supseteq C_0 \text{ and } f(C_0) \notin D\}.$$ 

The picture represents sets of $B$. As illustrated in the picture, $D$ is a set of $B$ when $D$ is larger than $C_0$ but fails to be comparable to $f(C_0)$. Thus there would be more than one chain ascending from $C_0$ if $B \neq \emptyset$, rather like a tree growing upward in more than one direction from a fork in the trunk. It will be shown this can’t take place for any such $C_0$ by showing $B = \emptyset$.

This will be accomplished by showing $\tilde{T}_0 \equiv T_0 \setminus B$ is a tower. Since $T_0$ is the smallest tower, this will require that $\tilde{T}_0 = T_0$ and so $B = \emptyset$.

**Claim:** $\tilde{T}_0$ is a tower and so $B = \emptyset$.

**Proof of the claim:** It is clear that $\emptyset \in \tilde{T}_0$ because for $\emptyset$ to be contained in $B$ it would be required to properly contain $C_0$ which is not possible. Suppose $D \in \tilde{T}_0$. The plan is to verify $g(D) \in \tilde{T}_0$.

Case 1: $f(D) \in C_0$. If $D \subseteq C_0$, then since both $D$ and $\{f(D)\}$ are contained in $C_0$, it follows $g(D) \subseteq C_0$ and so $g(D) \notin B$. On the other hand, if $D \supseteq C_0$, then since $D \in \tilde{T}_0$, $f(C_0) \in D$ and so $g(D)$ also contains $f(C_0)$ implying $g(D) \notin B$. These are the only two cases to consider because $C_0$ is comparable to every set of $T_0$. 
Case 2: $f(D) \notin C_0$. If $D \subseteq C_0$ it can’t be the case that $f(D) \notin C_0$ because if this were so, $g(D)$ would not compare to $C_0$.

\[
\begin{array}{c}
D \quad C_0 \\
\text{f(C)} \quad \text{f(D)}
\end{array}
\]

Hence if $f(D) \notin C_0$, then $D \supseteq C_0$. If $D = C_0$, then $f(D) = f(C_0) \in g(D)$ so $g(D) \notin B$. Therefore, assume $D \supseteq C_0$. Then, since $D$ is in $T_0$, $f(C_0) \in D$ and so $f(C_0) \in g(D)$. Therefore, $g(D) \in T_0$.

Now suppose $S$ is a totally ordered subset of $T_0$ with respect to set inclusion. Then if every element of $S$ is contained in $C_0$, so is $\cup S$ and so $\cup S \in T_0$. If, on the other hand, some chain from $S$, $C$, contains $C_0$ properly, then since $C \notin B$, $f(C_0) \in C \subseteq \cup S$ showing that $\cup S \notin B$ also. This has proved $T_0$ is a tower and since $T_0$ is the smallest tower, it follows $T_0 = T_0$. This has shown roughly that no splitting into more than one ascending chain can occur at any $C_0$ which is comparable to every set of $T_0$. Next it is shown that every element of $T_0$ has the property that it is comparable to all other elements of $T_0$. This is done by showing that these elements which possess this property form a tower.

Define $T_1$ to be the set of all elements of $T_0$ which are comparable to every element of $T_0$. (Recall the elements of $T_0$ are chains from the original partial order.)

Claim: $T_1$ is a tower.

Proof of the claim: It is clear that $\emptyset \in T_1$ because $\emptyset$ is a subset of every set. Suppose $C_0 \in T_1$. It is necessary to verify that $g(C_0) \in T_1$. Let $D \in T_0$ (Thus $D \subseteq C_0$ or else $D \supseteq C_0$) and consider $g(C_0) = C_0 \cup \{f(C_0)\}$. If $D \subseteq C_0$, then $D \subseteq g(C_0)$ so $g(C_0)$ is comparable to $D$. If $D \supseteq C_0$, then $D \supseteq g(C_0)$ by what was just shown ($B = \emptyset$). Hence $g(C_0)$ is comparable to $D$. Since $D$ was arbitrary, it follows $g(C_0)$ is comparable to every set of $T_0$. Now suppose $S$ is a chain of elements of $T_0$ and let $D$ be an element of $T_0$. If every element in the chain, $S$ is contained in $D$, then $\cup S$ is also contained in $D$. On the other hand, if some set, $C$, from $S$ contains $D$ properly, then $\cup S$ also contains $D$. Thus $\cup S \in T_1$ since it is comparable to every $D \in T_0$.

This shows $T_1$ is a tower and proves therefore, that $T_0 = T_1$. Thus every set of $T_0$ compares with every other set of $T_0$ showing $T_0$ is a chain in addition to being a tower.

Now $\cup T_0, g(\cup T_0) \in T_0$. Hence, because $g(\cup T_0)$ is an element of $T_0$, and $T_0$ is a chain of these, it follows $g(\cup T_0) \subseteq \cup T_0$. Thus

$$\cup T_0 \supseteq g(\cup T_0) \supseteq \cup T_0,$$

a contradiction. Hence there must exist a maximal chain after all. This proves the lemma.

If $X$ is a nonempty set, $\leq$ is an order on $X$ if

$$x \leq x,$$
and if \( x, y \in X \), then

either \( x \leq y \) or \( y \leq x \)

and

if \( x \leq y \) and \( y \leq z \) then \( x \leq z \).

\( \leq \) is a well order and say that \((X, \leq)\) is a well-ordered set if every nonempty subset of \( X \) has a smallest element. More precisely, if \( S \neq \emptyset \) and \( S \subseteq X \) then there exists an \( x \in S \) such that \( x \leq y \) for all \( y \in S \). A familiar example of a well-ordered set is the natural numbers.

**Lemma A.0.2** The Hausdorff maximal principle implies every nonempty set can be well-ordered.

**Proof:** Let \( X \) be a nonempty set and let \( a \in X \). Then \( \{a\} \) is a well-ordered subset of \( X \). Let

\[
F = \{S \subseteq X : \text{there exists a well order for } S\}.
\]

Thus \( F \neq \emptyset \). For \( S_1, S_2 \in F \), define \( S_1 \prec S_2 \) if \( S_1 \subseteq S_2 \) and there exists a well order for \( S_2 \), \( \leq_2 \) such that

\[
(S_2, \leq_2) \text{ is well-ordered}
\]

and if

\[
y \in S_2 \setminus S_1 \text{ then } x \leq_2 y \text{ for all } x \in S_1,
\]

and if \( \leq_1 \) is the well order of \( S_1 \) then the two orders are consistent on \( S_1 \). Then observe that \( \prec \) is a partial order on \( F \). By the Hausdorff maximal principle, let \( C \) be a maximal chain in \( F \) and let

\[
X_\infty = \bigcup C.
\]

Define an order, \( \leq_1 \), on \( X_\infty \) as follows. If \( x, y \) are elements of \( X_\infty \), pick \( S \in C \) such that \( x, y \) are both in \( S \). Then if \( \leq_1 \) is the order on \( S \), let \( x \leq y \) if and only if \( x \leq_1 y \). This definition is well defined because of the definition of the order, \( \prec \). Now let \( U \) be any nonempty subset of \( X_\infty \). Then \( S \cap U \neq \emptyset \) for some \( S \in C \). Because of the definition of \( \leq_1 \), if \( y \in S_2 \setminus S_1, S_1 \in C \), then \( x \leq y \) for all \( x \in S_1 \). Thus, if \( y \in X_\infty \setminus S \) then \( x \leq y \) for all \( x \in S \) and so the smallest element of \( S \cap U \) exists and is the smallest element in \( U \). Therefore \( X_\infty \) is well-ordered. Now suppose there exists \( z \in X \setminus X_\infty \). Define the following order, \( \leq_1 \), on \( X_\infty \cup \{z\} \).

\[
x \leq_1 y \text{ if and only if } x \leq y \text{ whenever } x, y \in X_\infty
\]

\[
x \leq_1 z \text{ whenever } x \in X_\infty.
\]

Then let

\[
\tilde{C} = \{S \in C \text{ or } X_\infty \cup \{z\}\}.
\]

Then \( \tilde{C} \) is a strictly larger chain than \( C \) contradicting maximality of \( C \). Thus \( X \setminus X_\infty = \emptyset \) and this shows \( X \) is well-ordered by \( \leq \). This proves the lemma.

With these two lemmas the main result follows.
Theorem A.0.3 The following are equivalent.

The axiom of choice

The Hausdorff maximal principle

The well-ordering principle.

Proof: It only remains to prove that the well-ordering principle implies the axiom of choice. Let $I$ be a nonempty set and let $X_i$ be a nonempty set for each $i \in I$. Let $X = \cup \{X_i : i \in I\}$ and well order $X$. Let $f(i)$ be the smallest element of $X_i$. Then

$$f \in \prod_{i \in I} X_i.$$ 

A.1 The Hamel Basis

A Hamel basis is nothing more than the correct generalization of the notion of a basis for a finite dimensional vector space to vector spaces which are possibly not of finite dimension.

Definition A.1.1 Let $X$ be a vector space. A Hamel basis is a subset of $X$, $\Lambda$ such that every vector of $X$ can be written as a finite linear combination of vectors of $\Lambda$ and the vectors of $\Lambda$ are linearly independent in the sense that if $\{x_1, \ldots, x_n\} \subseteq \Lambda$ and

$$\sum_{k=1}^{n} c_k x_k = 0$$

then each $c_k = 0$.

The main result is the following theorem.

Theorem A.1.2 Let $X$ be a nonzero vector space. Then it has a Hamel basis.

Proof: Let $x_1 \in X$ and $x_1 \neq 0$. Let $\mathcal{F}$ denote the collection of subsets of $X$, $\Lambda$ containing $x_1$ with the property that the vectors of $\Lambda$ are linearly independent as described in Definition A.1.1 partially ordered by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathcal{C}$ Let $\Lambda = \cup \mathcal{C}$. Since $\mathcal{C}$ is a chain, it follows that if $\{x_1, \ldots, x_n\} \subseteq \mathcal{C}$ then there exists a single $\Lambda' \subseteq \mathcal{C}$ containing all these vectors. Therefore, if

$$\sum_{k=1}^{n} c_k x_k = 0$$

it follows each $c_k = 0$. Thus the vectors of $\Lambda$ are linearly independent. Is every vector of $X$ a finite linear combination of vectors of $\Lambda$?
Suppose not. Then there exists \( z \) which is not equal to a finite linear combination of vectors of \( \Lambda \). Consider \( \Lambda \cup \{ z \} \). If

\[
    cz + \sum_{k=1}^{m} c_k x_k = 0
\]

where the \( x_k \) are vectors of \( \Lambda \), then if \( c \neq 0 \) this contradicts the condition that \( z \) is not a finite linear combination of vectors of \( \Lambda \). Therefore, \( c = 0 \) and now all the \( c_k \) must equal zero because it was just shown \( \Lambda \) is linearly independent. It follows \( C \cup \{ \Lambda \cup \{ z \} \} \) is a strictly larger chain than \( C \) and this is a contradiction. Therefore, \( \Lambda \) is a Hamel basis as claimed. This proves the theorem.

### A.2 Exercises

1. Zorn’s lemma states that in a nonempty partially ordered set, if every chain has an upper bound, there exists a maximal element, \( x \) in the partially ordered set. \( x \) is maximal, means that if \( x \prec y \), it follows \( y = x \). Show Zorn’s lemma is equivalent to the Hausdorff maximal theorem.

2. Show that if \( Y, Y_1 \) are two Hamel bases of \( X \), then there exists a one to one and onto map from \( Y \) to \( Y_1 \). Thus any two Hamel bases are of the same size.

3. Using the Baire category theorem of the chapter on Banach spaces show that any Hamel basis of a Banach space is either finite or uncountable.

4. Consider the vector space of all polynomials defined on \([0,1]\). Does there exist a norm, \( ||·|| \) defined on these polynomials such that with this norm, the vector space of polynomials becomes a Banach space (complete normed vector space)?
Bibliography


2789


BIBLIOGRAPHY


[80] Levinson, N. and Redheffer, R. Complex Variables, Holden Day, Inc. 1970


Index

$C^1$ functions, 127
$C^k$, 697
$C_\infty$, 392
$C_m$, 392
$F_\sigma$ sets, 208
$G_\delta$, 421
$G_\delta$ sets, 208
$L^1$, weak sequential compactness, 619
$L^1_{loc}$, 918
$L^p$, compactness, 387
$L^p$ (i.e., $L^p(\Omega)$), 379
$L^p (\Omega; X)$, 655
$L^\infty$, 400
$L^p_{loc}$, 1276
$\Delta$ regular, 964
$\sigma$ algebra, 207
(1,p) extension operator, 1357
a density result, 2479
Abel’s formula, 91
Abel’s theorem, 1661
absolutely continuous, 923
adapted, 1997, 2109
adjugate, 82
Alexander subbasis theorem, 368
algebra of sets, 311
analytic continuation, 1741, 1802
analytic function, 1741
Analytic functions, 1741
analytic sets, 1657
application of Bair theorem, 365
approximate identity, 365
apriori estimates, 192
area formula, 2479
at most countable, 40
atlas, 1041, 1320
automorphic function, 1878
axiom of choice, 25
axiom of extension, 25
axiom of specification, 25
axiom of unions, 25
axioms for a norm, 12
Banach Alaoglu theorem, 1407
Banach space, 391, 410
Banach Steinhaus theorem, 124
barycenter, 1041
basis of a topology, 1041
basis of module of periods, 1867
Bessel’s inequality, 526
Bifurcation point, 784
Big Picard theorem, 1066
Binet Cauchy formula, 1044
Binet Cauchy theorem, 1325
Blaschke products, 1847
Bloch’s lemma, 1793
block matrix, 55
Bochner integrable, 636
Bochner covering theorem, 1107
Bochner’s inequality, 924
Borel sets, 127
boundary, 1741
bounded, 410
boundary of a set, 127
boundary value, 1847
boundary value problem, 784
Braille notation, 2796
Braille symbols, 2796
Brownian motion, 1927
Buck’s theorem, 421
Betti numbers, 421
INDEX

fixed point, 706
fixed point theorem, 706
convergence in measure, 220
convex, 410
set, 507
convex functions, 398
convex and lower semicontinuous, 889
convex function
subgradient, 1236
convex hull, 195, 410
convex lower semicontinuous
approximation, 898
domain and domain of subgradient, 898
convex sets, 195
convolution, 393, 1102
convolution of measures, 1950
Coordinates, 59
countable, 30
counting zeros, 1700
covariation, 2178
cowlick, 752
Cramer's rule, 82
cycle, 1677
cylinder sets, 2078
cylindrical set, 1916, 1943
Darboux, 56
Darboux integral, 56
definition of $L^p$, 383
degenerate inclusions with extra stochastic integrals added, 73
degenerate operators identities, 38
degree
definition, 1941
homotopy invariance, 426, 427
Leray Schauder, 771
on components, 691
properties, 698
semicontinuous, 806, 822, 2601
density of continuous functions in $L^p$, 388
derivative
chain rule, 687
continuous, 685
continuous Gateaux derivatives, 691
Frechet, 685
Gateaux, 685, 691
generalized partial, 685
higher order, 685, 691
matrix, 685
partial, 685
second, 691
well defined, 685
derivatives, 684, 685
alternating property, 76
cofactor expansion, 79
expansion along row (column), 3
matrix inverse formula, 81
product, 78
transpose, 76
differentiable, 684
continuous, 685
continuous partials, 699
differential equations, 721
global existence, 490
differential form, 1045
derivative, 1049
integral, 1047
differential forms, 1046
differentiation
Radon measures, 931, 1079
dilations, 1778
Dini derivates, 931, 957
directional derivatives, 931, 957
distance
compact set and closed set, 73
distribution, 931, 1048, 1110
distribution function, 931, 956, 1110, 1900
distributional derivative, 1154, 1443, 2428
divergence
of cofactor matrix, 73
divergence theorem, 1024
DL, 2209
domain
subgradient of convex function, 1236
dominated convergence theorem, 241, 657
Doob
approximation lemma, 2290
INDEX

Doob Dynkin lemma, 1913
Doob estimate, 2001
Doob Meyer decomposition, 2217
Doob’s submartingale estimate, 2119
doubly periodic, 1864
dual space, 818
duality map
  strictly monotone, 2609
demicontinuous, 815
inequality property, 815
projection map, 822
strict monotone inequality, 819
strictly monotone, 816

duality mapping, 813

application of duality, 814

properties, 817

Eberlein Smulian theorem, 450
Egoroff theorem, 217, 229

elementary factors, 1831

elementary function, 2188, 2285

integral, 461

elementary functions, 808

elementary set, 377

elementary sets, 303

elliptic, 1864

entire, 1666

epsilon net, 146, 152

equality of mixed partial derivatives, 134, 702

equi integrable, 622, 2209

equicontinuous, 156

equivalence class, 92

equivalence relation, 92

ergodic, 124, 685

individual ergodic theorem, 451

Erling’s lemma, 1209

essential singularity, 1672

essentially more slowly, 973

Euler’s theorem, 1859

events, 1909

exchanged theorem, 61

exponential growth, 1104

extended complex plane, 1631

extension theorem, 2607

Fatou’s lemma, 482

filtration, 484

filtration
  normal, 484

finite intersection property, 150, 171

finite measure space, 408

first hitting time, 2130

fixed point property, 116, 681

fixed point theorem
  Cariste, 116
  Kakutani, 116, 1842

fixed points
  measurability, 1688

Fourier series
  uniform convergence, 461

Fourier transform $L^1$, 1093

fractional linear transformations, 1778, 1780

mapping three points, 1781

fractional powers
  sectorial operator, 1781

fractional spaces
  reflexive, 1781

Frechet derivative, 1704, 1705

Fredholm alternative, 550

Fredholm operator
  Banach space, 1707

Fresnel integrals, 1733

Fubini’s theorem, 408, 417, 418

Bochner integrable functions, 1618

function, 48

uniformly continuous, 171

function element, 1718, 1892

functional equations, 1688

functions of Wiener processes, 4379
INDEX

fundamental theorem of algebra, 1667
fundamental theorem of calculus, 55, 919
  general Radon measures, 1069
  Radon measures, 1074
  Gamma function, 398, 992
  Gamma function, 1853
  Gateaux derivative, 689, 691
  gauge function, 429
  Gauss’s formula, 1854
  Gaussian measure, 2059, 2069
  Gelfand triple, 1159, 1448, 2430
  general spherical coordinates, 349
  generalized gradient
    pseudomonotone, 843
  generalized gradients, 846
  generalized subgradient
    upper semicontinuity, 988
  Gerschgorin’s theorem, 1704
  good lambda inequality, 281, 2181
  Gram determinant, 518
  Gram matrix, 518
  Gram Schmidt process, 92
  great Picard theorem, 1850
  Hadamard three circles theorem, 1695
  Hahn Banach theorem, 430
  Hamel basis, 2787
  Hardy Littlewood maximal function, 728
  Hardy's inequality, 399
  harmonic functions, 1652
  Haursdorff measures, 190, 193
  Hausdorff and Lebesgue measure, 190, 193
  Hausdorff dimension, 190
  Hausdorff maximal principle, 190, 193
  Hausdorff maximal theorem, 2078
  Hausdorff measure
    Lipschitz function of measure zero, 1171
    translation invariant, 1183
  Hausdorff measures, 2074
  Hausdorff metric, 172
  Hausdorff space, 1177
  Heine Borel, 26
  Heine Borel theorem, 1107
  hemicontinuous, 2684, 2777
  Hermitian, 113
  higher order derivative
    multilinear form, 977
  higher order derivatives
    implicit function theorem, 149
    inverse function theorem, 149
  Hilbert Schmidt operators, 728, 2087
  Hilbert Schmidt theorem, 528, 649
  Hilbert space, 505
  hitting this before that, 4157
  Holder inequality
    backwards, 149
  Holder space
    compact embedding, 149
    not separable, 149
  Holder spaces, 149
  Holder’s inequality, 149
  homotopic, 737
  homotopic to a point, 1823
  homotopy, 737
  Hormander condition, 1114
  implicit function theorem, 135, 138, 139, 707, 709
  higher order derivatives, 149
  implicit inclusion, 1478
  increasing function
    differentiability, 149
  independent events, 1909
  independent random vectors, 1910
  independent sigma algebras, 1910
  indicator function, 217
  inequality
    Morrey, 937
  infinite products, 1827
  inner product, 12
  inner product properties, 12
  inner product space, 190
  inner regular, 1109, 2482
  inner regular measure, 1109
INDEX

integration
integration with respect to a martingale, 2188
integration by parts, 2674
integration with respect to martingales
Ito isometry, 2192
interior point, 103
interpolation inequalities, 1373
interpolation inequality, 1543
invariance of domain, 208, 218
Brouwer fixed point theorem, 361
inverse
left inverse, 84
right inverse, 84
inverse function theorem, 138, 139, 709, 726
higher order derivatives, 710
inverses and determinants, 81
inversions, 1778
invertible maps, 81
different spaces, 707
isodiametric inequality, 985, 989
isosogonal, 1652, 1777
isolated singularity, 1671
isometric, 66
Ito isometry, 2192, 2288
Ito representation theorem, 2368
James map, 435
Jensen’s formula, 1844
Jensens inequality, 398, 1996
Jordan curve theorem, 783
Jordan Separation theorem, 756
Jordan separation theorem, 758
Kakutani fixed point theorem, 827, 1596
Kolmogorov Centsov theorem, 2101
Kolmogorov extension theorem, 371
Polish Space, 1906
Kolmogorov zero one law, 1919
Kolmogorov’s inequality, 1920
Lagrange multipliers, 140, 141
Laplace expansion, 79
Laplace transform, 1105
Laurent series, 1722
law, 2054
Lebesgue
set, 925
Lebesgue decomposition, 714
Lebesgue measure, 926
Lebesgue point, 914
Leray Schauder alternative, 1581
Leray Schauder degree, 707, 941
properties, 772, 773
Levy’s theorem, 2228, 2280
limit
continuity, 858
infinite limits, 858
limit of a function, 107, 851
limit point, 107, 851
limit points, 107, 851
limits
combinations of functions, 852
independent random variables, 1104, 2027
limits and continuity, 858
linear combination, 85, 77
linearly dependent, 77
linearly independent, 77
Lions, 1476
Liouville theorem, 1894
Lipschitz, 68, 110, 117
functions, 852
Lipschitz boundary, 1104, 1497
Lipschitz continuous, 1104
Lipschitz function
integral of its derivative, 859
Lipschitz manifold, 1104
Lipschitz maps
extension, 1104, 1105, 1495
little o notation, 857
little Picard theorem, 1894
local martingale, 477, 1896
local martingale
stochastic integral, 478
local submartingale, 474
localization
elementary functions, 461
general case, 461
stochastically square integrable functions, 461
localizing sequence, 471
locally bounded, \textit{INDEX}
locally compact, \textit{INDEX}
locally compact, \textit{INDEX}
locally convex topological vector space, \textit{INDEX}
locally finite, \textit{INDEX}, \textit{INDEX}, \textit{INDEX}

logarithm
branch of logarithm, \textit{INDEX}
lower semicontinuous, \textit{INDEX}
equivalent conditions, \textit{INDEX}
Lusin’s theorem, \textit{INDEX}
Lyapunov Schmidt procedure, \textit{INDEX}

manifold, \textit{INDEX}
manifolds
boundary, \textit{INDEX}
interior, \textit{INDEX}
orientable, \textit{INDEX}
radon measure, \textit{INDEX}
smooth, \textit{INDEX}
surface measure, \textit{INDEX}
mapping
compact, \textit{INDEX}

Marcinkiewicz interpolation, \textit{INDEX}
martingale, \textit{INDEX}
quadratic variation, \textit{INDEX}
martingales
equivintegrable, \textit{INDEX}

matrix
left inverse, \textit{INDEX}
lower triangular, \textit{INDEX}
non defective, \textit{INDEX}
normal, \textit{INDEX}
right inverse, \textit{INDEX}
upper triangular, \textit{INDEX}
maximal function
Radon measures, \textit{INDEX}
maximal monotone
approximation, \textit{INDEX}, \textit{INDEX}
coercive, \textit{INDEX}
equivalent conditions, \textit{INDEX}
interior of domain, \textit{INDEX}
linear map, \textit{INDEX}
onto, \textit{INDEX}
pseudomonotone, \textit{INDEX}
subgradient, \textit{INDEX}
sum of two of them, \textit{INDEX}, \textit{INDEX}
sum with duality map, \textit{INDEX}
sum with monotone and hemicontinuous, \textit{INDEX}

maximal monotone
Banach space, \textit{INDEX}
maximal monotone operator, \textit{INDEX}
sum with a subgradient, \textit{INDEX}
maximum modulus theorem, \textit{INDEX}
McShane’s lemma, \textit{INDEX}
mean ergodic theorem, \textit{INDEX}
mean value inequality, \textit{INDEX}, \textit{INDEX}, \textit{INDEX}
mean value theorem, \textit{INDEX}, \textit{INDEX}, \textit{INDEX}
for integrals, \textit{INDEX}
measurable, \textit{INDEX}
Borel, \textit{INDEX}
multifunctions, \textit{INDEX}, \textit{INDEX}
measurable function, \textit{INDEX}
pointwise limits, \textit{INDEX}
measurable functions
Borel, \textit{INDEX}
combinations, \textit{INDEX}
measurable rectangle, \textit{INDEX}, \textit{INDEX}
measurable representative, \textit{INDEX}
measurable selection, \textit{INDEX}, \textit{INDEX}, \textit{INDEX}
from set of limits, \textit{INDEX}
measurable selection theorem, \textit{INDEX}
improved version, \textit{INDEX}
limits of measurable functions, \textit{INDEX}
measurable sets, \textit{INDEX}, \textit{INDEX}
measurable solutions to degenerate inclusions, \textit{INDEX}
measurable solutions to inclusions with quasibounded operator, \textit{INDEX}
measure space, \textit{INDEX}
isomorphic, \textit{INDEX}
regular, \textit{INDEX}
separable, \textit{INDEX}
Mellin transformations, \textit{INDEX}
meromorphic, \textit{INDEX}
Merten’s theorem, \textit{INDEX}
metric space, \textit{INDEX}
Cauchy sequence, \textit{INDEX}
closed set, \textit{INDEX}
compact, \(145\)
compactly of product space, \(150\)
conditions for compactness, \(146\)
distance to a set, \(144\)
distance to set is continuous, \(145\)
extreme value theorem, \(149\)
limit points, \(143\)
open balls open, \(143\)
open set, \(143\)
sequentially compact, \(145\)
subsequences of Cauchy sequence, \(144\)
Tietze extension, \(159\)
uniform continuity, \(149\)
uniform continuity on compact set, \(149\)

Meyer Serrin theorem, \(1330\)
Mihlin’s theorem, \(1126\)
min max theorem, \(852\)
Minkowski functional, \(461\)
Minkowski inequality, \(385\)
integrals, \(385\)
Minkowski inequality backwards, \(385\)
Minkowski’s inequality, \(859\)
minor, \(79\)
Mittag Leffler, \(1735\)
mixed partial derivatives, \(133\), \(701\)
modification, \(1085\)
module of periods, \(1862\)
mollifier, \(393\)
monotone, \(805\), \(2599\)
Banach space, \(800\)
Hilbert space, \(788\)
Muntz’s first theorem, \(523\)
Muntz’s second theorem, \(523\)

n simplex, \(145\)
natural, \(2401\), \(2401\)
Nemytskii operator, \(856\)
Neumann series, \(704\), \(108\)
Neumann series, \(1740\)
non equal mixed partials example, \(702\)
nonlinear Fubini’s theorem, \(1314\)
normal, \(1958\), \(1981\), \(2054\)
normal family of functions, \(1783\)
normal filtration, \(2109\)
normal topological space, \(108\)
not pseudomonotone, \(864\)
nowhere differentiable functions, \(856\)
nuclear operator, \(560\)
numerical range, \(1753\)

obstacle problem
non-constant obstacle, \(1602\)
one point compactification, \(170\), \(201\)
on open cover, \(166\)
open mapping theorem, \(1201\), \(1257\)
on open set, \(108\)
open sets, \(171\)
operator norm, \(1201\), \(1241\)
operator of penalization demicontinuous, \(560\)
operators closed range, \(1252\)
optional sampling theorem, \(1601\)
order, \(1857\)
order of a pole, \(1676\)
order of a zero, \(1676\)
order of an elliptic function, \(1864\)
orientable manifold, \text{1043}
orientation, \text{1047}

orthonormal set, \text{524}
orientation, \text{1047}

outer measure, \text{219}

outer regular, \text{1902}

outer regular measure, \text{260}

pap97, \text{1571}

paracompact space, \text{403}

partial derivative, \text{128}

partial derivatives, \text{688}

continuous, \text{699}

partial order, \text{32}

partition, \text{41}

partition of unity, \text{268}

metric space, \text{407}

penalization operators, \text{821}

period parallelogram, \text{1864}

permutation, \text{73}

Phragmen Lindelof theorem, \text{1693}

pivot space, \text{1159}

Polish space, \text{1556}

polynomial, \text{181}

positive, \text{551}

positive and negative parts of a measure, \text{951}

positive definite, \text{2050}

positive definite functions, \text{2050}

positive linear functional, \text{206}

power series

analytic functions, \text{1004}

power set, \text{28}

precompact, \text{1042}

predictable, \text{2114}

primitive, \text{1049}

principal branch of logarithm, \text{1691}

principal ideal, \text{1691}

probability space, \text{105}

product formula, \text{755}

product measure, \text{693}

product rule, \text{1278}

product topology, \text{105}

progressively measurable, \text{2403}

composition, \text{2403}

integral of, \text{2403}

progressively measurable solutions to inclusions, \text{2719}

progressively measurable solutions with noise, \text{2720}

progressively measurable version, \text{2719}

projection in Hilbert space, \text{205}

projection map, \text{2403}

Prokhorov's theorem, \text{2039}

proper, \text{889}

properties of integral properties, \text{53}

pseudo continuous, \text{2051}

psuedogradient, \text{789}

pseudomonotone, \text{2599}

bounded, \text{807}

demicontinuous, \text{814}

generalized gradient, \text{832}

generalized perturbation, \text{806}

L pseudomonotone, \text{905}

modified bounded, \text{833}

modified, \text{834}

monotone and hemicontinuous, \text{805}

perturbation, \text{805}

set valued, \text{849}

single valued, \text{806}

something which isn't, \text{806}

sum, \text{836}

sum of, \text{837}

sum with densely defined max monotone, \text{874}

sum with maximal monotone, \text{874}

type M, \text{807}

variational inequality, \text{811}
INDEX

variational inequality, sum two operators, 1094
pseudomonotone
onto, 843
pseudomonotone operator, 833, 834, 868, 1599
set valued, 831, 833, 834, 1599

Q Wiener process, 2483
quadratic variation, 2221
convergence in probability, 2401
fantastic properties, 2401
quasi-bounded, 868, 912
quotient space, 1504

Rademacher’s theorem, 1284, 1287
Radon measure, 257
Radon Nikodym
Radon measures, 1059
Radon Nikodym derivative, 577
Radon Nikodym property, 577
Radon Nikodym Theorem
σ finite measures, 576
finite measures, 576
Radon Nikodym theorem
Radon Measures, 1058
random variable, 1058, 1090
independent, 1090
real Schur form, 56
recognizing a martingale
stopping times, 1039
refinement of a cover, 108
reflexive Banach Space, 1039
reflexive Banach space, 1039
region, 1039
regular measure, 261
regular measure space, 257
regular topological space, 1039
regular values, 1039
relative topology, 1041
removable singularity, 1075
representation of martingales, 1071
residue, 1175
resolvent, 561
resolvent set, 1775, 1776
retract
Banach space, 1129
closed and convex set, 1129
retraction onto boundary
not for a ball, 1751
Riemann criterion, 111
Riemann integrable, 41
Riemann integral, 41
Riemann sphere, 1694
Riemann Stieltjes integral, 51
Riesz map, 511
Riesz representation theorem, 571
C₀ (X), 571
Hilbert space, 511
locally compact Hausdorff space, 501
Riesz Representation theorem
C⁰ (X), 571
Riesz representation theorem Lᵖ finite measures, 589
Riesz representation theorem Lᵖ σ finite case, 599, 675
Riesz representation theorem for L¹ finite measures, 587
right polar decomposition, 77
Rouche’s theorem, 1717
Runge’s theorem, 1586

Sard’s lemma, 751
Sard’s theorem, 1551
scalars, 51
scale of Banach spaces, 1770
Schaefer fixed point theorem, 1580
Schauder fixed point
approximate fixed point, 1581, 1582
Schauder fixed point theorem, 1580, 1581
degree theory, 1578
Schottky’s theorem, 1581
Schröder Bernstein theorem, 49
Schwartz class, 1577
Schwarz formula, 1571
Schwarz reflection principle, 1587
Schwarz’s lemma, 1571
INDEX

second derivative, 696
sections of open sets, 698
sectorial, 1743
self adjoint, 72, 551
semigroup
  adjoint, 571
  contraction
    bounded, 558
    generator, 556
    growth estimate, 557
  Hille Yosida theorem, 561
strongly continuous, 556
seminorms, 465
separability of $C(H)$, 2038
separated, 171
separation theorem, 104
sequential compactness, 33, 619
$L^1$, 138
$L^1$, 618
sequential compactness in $L^1$, 138
sequential weak* compactness, 448
set valued functions
  measurability, 1007, 2437
set valued map
  locally bounded, 304
sets, 4
sgn, 71
uniqueness, 85
Shannon sampling theorem, 1107
sign of a permutation, 63
Simon, 1474
simple function, 224, 627
simple functions, 212
simplex, 195
simplices, 195
singular values, 737
Skorokhod's theorem, 2043
Sobolev Space
  embedding theorem, 1106
  equivalent norms, 1105
Sobolev space, 1106
Sobolev spaces, 1107
space of continuous martingales, 2160
  Hilbert space, 410
span, 47, 69
spectral radius, 174
spectral theory
  Banach space, 150
  compact operator, 152
  Sperner's lemma, 105
  Steiner symetrisation, 157
step functions
  approximation result, 2291
  stereographic projection, 1152, 1154
Stirling’s formula, 1855
stochastic integral, 2285
stochastic integral, 2314
  definition, 2301
  linear transformation, 2320
  main result, 2301
  quadratic variation, 2316
  stochastically square integrable, 2315
stochastic process, 2168
stochastically square integrable, 2419
Stokes theorem, 1053
Stone’s theorem, 406
strong law of large numbers, 1926
strong topology, 171
stopped martingale, 2154
stopped process, 2157
stopped submartingale, 2148
stopping time, 2002, 2006, 2122, 2129
strict convexity, 102
strictly convex, 813
strong law of large numbers, 1926, 2030
strongly measurable, 129
subbasis, 108
subgradient
  convex lower semicontinuous, 301
  maximal monotone, 816
  surjective, 599
subgradients
  sum, 599
submartingale, 1497
submartingale convergence theorem, 2401
  continuous case, 2417
subspace, 38
sum
maximal monotone and pseudomonotone, 874
sum of pseudomonotone operators, 874
sums
independent random variables, 1920, 1924, 2021
supermartingale, 1924
support of a function, 209
Suslin space, 1557
symmetric derivative
existence, 1080
measurable, 1080
upper and lower, 1079
symmetric sets, 745
tail event, 1919
Tietze extension theorem, 104
tight, 104, 107, 407
measures, 107
topological degree
definition, 108

topological space, 107
topological vector space dual, 467
total variation, 585, 923
totally bounded set, 146
totally ordered set, 572
trace, 543
trajectories, 207
translation invariant, 325
translations, 1778
triangle inequality, 65, 381
triangulation, 196
labeling vertices, 196
small diameter, 196
trivial, 66
Tychonoff fixed point theorem, 807, 2601
type M, 868, 2678

demicontinuous, 810
demicontinuous, 810
sum of linear and type M, 807
surjective, 810
uniform boundedness theorem, 88
uniform continuity, 107
uniform continuity and compactness, 108
uniform contraction principle, 108
uniform convergence, 108, 1809
uniform convergence and continuity, 108
uniform convexity, 108
uniformly bounded, 108, 1809
uniformly continuous, 108
uniformly equicontinuous, 108, 1809
uniformly integrable, 403
uniformly integrable
unimodular transformations, 1866
uniqueness of limits, 682
upcrossing,
upper and lower sums, 192
upper semi continuity
set valued map, 2061
upper semicontinuity
set valued map, 2061
upper semicontinuous
onto, 2061, 2062
upper semicontinuous composition, 2062
Urysohn’s lemma, 2062

variational inequalities with respect to
set valued map, 2715

variational inequality, 585
non-constant convex set, 1581
sum of set valued pseudomonotone
maps, 1584

vector measures, 852
Vector valued distributions, 1152, 1153, 2426
vector valued function
limit theorems, 1367
version, 2097
Vitali
convergence theorem, 248
Vitali
convergence theorem, 248
Vitali convergence theorem, 248
Vitali cover, 1077
Vitali covering
closed balls, 1078
Vitali covering theorem, 928
Vitali coverings, 928
Vitali theorems, 1368
Vitali theorem, 1368
volume of unit ball, 991
weak * convergence, 1271
weak convergence, 1980
weak convergence of measures, 1271
weak derivative, 928, 1274
a function, 930
weak derivatives, 928
weak derivatives in Lp
differentiability, 1271
weak topology, 928, 1271
weak* measurable, 1980
weak* topology, 1980
weak* topology, 1980
weakly measurable, 1980
Weierstrass
Stone Weierstrass theorem, 186
Weierstrass M test, 1871
Weierstrass P function, 1871
well ordered sets, 2786
Wiener process, 2221, 2231, 2270
Wiener process in Hilbert Space, 2247, 2272
winding number, 1674
Wronskian, 91, 534
Yankov von Neumann Aumann, 1563, 1565
Young’s inequality, 379, 625, 1520
zeta function, 1855