Solutions to Selected Problems

Chapter 1

5. If $P$ is a polynomial in $z$ with real coefficients and $P(z) = 0$, then we have $0 = \bar{0} = \overline{P(z)} = P(\bar{z})$. The last equality comes from the result of exercise 4.c. The proof in the opposition direction is now trivial since if $P(\bar{z}) = 0$ and applying what we have proven $P(z) = P(\bar{z}) = 0$, using the result from part 4.d.

7. a. Write $z = r(\cos \theta + i \sin \theta)$. We have

$$|z^n| = |r^n(\cos n\theta + i \sin n\theta)| = |r|^n|\cos n\theta + i \sin \theta| = |r|^n,$$

upon noting that

$$|\cos n\theta + i \sin \theta| = \sqrt{\cos^2 n\theta + \sin^2 \theta} = 1.$$

b. If $z = x + iy$, then

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

c. Write $z = r(\cos \theta + i \sin \theta)$. We have

$$|\text{Re}z| = |z \cos \theta| = |z| |\cos \theta| \leq |z|.$$

The last inequality arrives from the fact that $|\cos \theta| \leq 1$. Similarly, we have

$$|\text{Im}z| = |z \sin \theta| = |z| |\sin \theta| \leq |z|.$$

Finally, by triangle inequality, we have

$$|z| = |r(\cos \theta + i \sin \theta)| \leq |r \cos \theta| + |ir \sin \theta| = |\text{Re}z| + |\text{Im}z|.$$

11. Suppose that $\zeta$ satisfies that equation $z^n - 1 = 0$ and that $\zeta \neq 1$. We have

$$0 = (\zeta - 1)(\zeta^{n-1} + \zeta^{n-2} + \cdots + 1).$$

The first factor of the above is not zero since we assume that $\zeta \neq 1$. Therefore, it must be that

$$\zeta^{n-1} + \zeta^{n-2} + \cdots + 1 = 0,$$

12. Note that by the symmetry of regular polygons, it matters not at all which particular vertex is connected with all the others. Hence it suffice to show the result connection all other vertices to the vertex 1. Let $\zeta_k$, $k = 1, \cdots, n - 1$ be the $n$-th roots of unity that are different from 1. Notes that these $\zeta_k$ are the vertices of a regular $n$-gon inscribed inside the unit circle. By exercise 11, the $\zeta_k$’s are the roots of

$$z^{n-1} + z^{n-2} + \cdots + 1 = 0$$

Hence, we have

$$z^{n-1} + z^{n-2} + \cdots + 1 \equiv \prod_{k=1}^{n-1}(z - \zeta_k),$$
from which we infer that

\[ |z^{n-1} + z^{n-2} + \cdots + 1| = \prod_{k=1}^{n-1} |z - \zeta_k|. \]

Now set \( z = 1 \) in the above, we have

\[ n = \prod_{k=1}^{n-1} |1 - \zeta_k|. \]

The right-hand side of the above is precisely the product of the distances in which we are interested.

16. We assume that polygonal lines are connected for this proof. The proof is that assumption is expectedly similar to the proof of the intermediate value theorem. Let \( S \) be a polygonally connected set. Assume that \( S \) is disconnected. Then we have disjoint open sets \( U \) and \( V \) such that \( S \subseteq U \cup V \), \( U \cap S \neq \emptyset \) and \( V \cap S \neq \emptyset \). Pick \( z_1 \in U \cap S \) and \( z_2 \in V \cap S \). Such choosing is always possible since \( U \cap S \neq \emptyset \) and \( V \cap S \neq \emptyset \) and \( z_1 \neq z_2 \) since \( U \cap V = \emptyset \). Since \( S \) is polygonally connected, there exists a polygonally line, \( L \subseteq S \), connecting \( z_1 \) and \( z_2 \). Hence we have \( L \subseteq U \cup V \), \( L \cap U \neq \emptyset \) and \( L \cap V \neq \emptyset \). The last two inequalities come from

\[ z_1 \in L \cap U, \text{ and } z_2 \in L \cap V. \]

Hence by definition, \( L \) is disconnected contradicting our assumption. Hence \( S \) must be connected.

**Chapter 2**

3. Suppose that \( P(x, y) = u(x, y) + iv(x, y) \) is an analytic polynomial that takes only imaginary values, or \( u(x, y) = 0 \) for all \( x, y \). The Cauchy-Riemann equation gives that

\[ v_x = v_y = 0. \]

This says that \( v(x, y) \) is a constant function and that \( P(x, y) \) is a constant polynomial.

**Chapter 3**

9. We see that \( u_x = 2x \) and \( u_y = 2y \). By Cauchy-Riemann equations, we must have

\[ v_y = 2x, \text{ and } v_x = -2y. \]

Therefore, we must have

\[ v(x, y) = 2xy + f(x) \text{ and } v(x, y) = -2xy + g(y), \]

for some \( f(x) \) and \( g(y) \). This gives that

\[ f(x) - g(y) = -4xy. \]

Set \( y = 0 \) in the above, we see that \( f(x) = g(0) \) for all \( x \); similarly setting \( x = 0 \), we have \( f(0) = g(y) \) for all \( y \). This gives that both \( f(x) \) and \( g(y) \) are constant functions and hence

\[ -4xy = f(x) - g(y) \]

is a constant function which is absurd. Therefore, we have the desired conclusion.
12. \[ |e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x \cos y + i \sin y = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x. \]

Chapter 4
5. We start with \(a, b \in \mathbb{C} \) and \(a \neq b\). Let \(C\) be a smooth curve connecting \(a\) to \(b\) and \(z(t)\) with \(0 \leq t \leq 1\) as its parameterization and \(z(0) = a\) and \(z(1) = b\). By proposition 4.12, we have
\[ F(b) - F(a) = F(z(0)) - F(z(1)) = \int_C F'(z) \, dz = 0, \]
since \(F'(z)\) is the constant zero function.