MATC34H3 COMPLEX VARIABLES I, FALL SEMESTER 2005

Assignment Four Solutions

(1) Find the Laurent expansion of the function \( \exp \left( z + \frac{1}{z} \right) \) in the domain \( 0 < |z| < \infty \).

We know that
\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},
\]
Hence, we have
\[
\exp \left( z + \frac{1}{z} \right) = \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{m=0}^{\infty} \frac{1}{m! z^m} \right).
\]
When multiplying the two series, we get a contribution for \( z^k \) with \( k \in \mathbb{Z} \) whenever \( n - m = k \) with \( m \) and \( n \) non-negative. Therefore, we have
\[
\exp \left( z + \frac{1}{z} \right) = \sum_{k=-\infty}^{\infty} \left( \sum_{m=\max(0,-k)}^{\infty} \frac{1}{m!(k+m)!} \right) z^k.
\]

(2) Evaluate the following integrals.

(a) \( \int_{|z|=2} \frac{z^3}{z^4 - 1} \, dz \).

The poles of the integrand are at \( \pm 1 \) and \( \pm i \) which are all enclosed by the circle \( |z| = 2 \). Therefore, we have
\[
\int_{|z|=2} \frac{z^3}{z^4 - 1} \, dz = 2\pi i \left[ \text{Res} \left( \frac{z^3}{z^4 - 1}, 1 \right) + \text{Res} \left( \frac{z^3}{z^4 - 1}, 1 \right) + \text{Res} \left( \frac{z^3}{z^4 - 1}, i \right) + \text{Res} \left( \frac{z^3}{z^4 - 1}, -i \right) \right]
\]
\[
= 2\pi i \left[ \frac{1}{2(1-i)(1+i)} + \frac{-1}{-2(-1-i)(-1+i)} + \frac{-i}{2(i+1)(i-1)} + \frac{i}{2(-i+1)(-i-1)} \right]
\]
\[
= \frac{2\pi i}{(1+i)(1-i)}.
\]

(b) \( \int_{|z|=1} z^n e^{2/z} \, dz \), where \( n \in \mathbb{Z} \).

We have
\[
\int_{|z|=1} z^n e^{2/z} \, dz = \int_{|z|=1} \sum_{k=0}^{\infty} \frac{2^k}{k!} z^{n-k} \, dz.
\]
The series in the integrand contains a term involving \( z^{-1} \) only if \( n+1 \geq 0 \) and in that case the residue is
\[
\frac{2^{n+1}}{(n+1)!}.
\]
Therefore the integral in question yields zero if \( n+1 < 0 \). Otherwise, it is
\[
2\pi i \frac{2^{n+1}}{(n+1)!}.
\]
(c) \[ \int_C \frac{dz}{(z-1)(z-2)^2}, \] where \( C \) is the circle \( |z - 2| = \frac{1}{2} \).

The circle is one that is centred at 2 with radius 1/2. Hence, it only encloses the double pole at \( z = 2 \). We have

\[ \text{Res} \left( \frac{1}{(z-1)(z-2)^2} \right) = \lim_{z \to 2} \frac{d}{dz} \frac{1}{z-1} = -1. \]

Therefore, the integral in question is \( -2\pi i \).

(3) Evaluate the following real integrals.

(a) \[ \int_{-\infty}^{\infty} \frac{x}{(x^2 + 4x + 13)^2} \, dx. \]

The only pole of the integrand in the upper half plane is \(-2 + 3i\). Therefore, the integral is

\[ 2\pi i \text{Res} \left( \frac{z}{(z^2 + 4z + 13)^2}, -2 + 3i \right) = 2\pi i \lim_{z \to -2+3i} \frac{d}{dz} \frac{z}{(z+2+3i)^2} = \frac{2\pi i}{54i} = -\frac{\pi}{27}. \]

(b) \[ \int_{0}^{\infty} \frac{x^2}{(x^2 + a^2)^2} \, dx, \text{ where } a > 0. \]

The integrand is an even function and has poles at \( \pm ai \). We have

\[ \int_{0}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx = \pi i \text{Res} \left( \frac{z^2 + 1}{z^4 + 1}, \exp \left( \frac{\pi i}{4} \right) \right) + \text{Res} \left( \frac{z^2 + 1}{z^4 + 1}, \exp \left( \frac{3\pi i}{4} \right) \right), \]

The above two residues are the same and,

\[ \frac{i + 1}{2(-\sqrt{2} + \sqrt{2}i)} = -\frac{i}{2\sqrt{2}}. \]

Therefore, the integral in question is

\[ \frac{\pi}{\sqrt{2}}. \]

(d) \[ \int_{0}^{\infty} \frac{x \sin x}{(x^2 + a^2)^2} \, dx, \text{ where } a > 0. \]

The integrand here is even. Note that \( x \sin x \) is an even function in \( x \). Therefore, we have

\[ \int_{0}^{\infty} \frac{x \sin x}{(x^2 + a^2)^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + a^2)^2} \, dx = \frac{1}{2} \left[ \pi i \text{Res} \left( \frac{ze^{iz}}{(z^2 + a^2)^2}, ai \right) \right] = \frac{1}{2} \left[ \pi i \lim_{z \to ai} \frac{d}{dz} \frac{ze^{iz}}{(z + ai)^2} \right] \]

\[ = \frac{1}{2} \left[ \pi i \frac{e^{-a}}{4a} \right] = \frac{\pi}{8e^{a^2}}. \]

(e) \[ \int_{0}^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} \, dx, \text{ where } a > 0 \text{ and } b > 0. \]
With the change of variables, $y = ax$, the above integral becomes
\[
\frac{1}{a} \int_{0}^{\infty} \frac{\cos y}{(y^2 + b^2)^2} \, dy = a^3 \int_{0}^{\infty} \frac{\cos y}{(y^2 + (ab)^2)^2} \, dy = \frac{a^3}{2} \int_{-\infty}^{\infty} \frac{\cos y}{(y^2 + (ab)^2)^2} \, dy
\]
\[
= \frac{a^3}{2} \Re \left[ 2\pi i \text{Res} \left( \frac{e^{iz}}{(z^2 + (ab)^2)^2}, abi \right) \right]
\]
\[
= a^3 \pi \Re \left[ i \lim_{z \to abi} \frac{e^{iz}}{z(z + abi)} \right]
\]
\[
= \pi (ab + 1) \frac{\log 2 + 3(1 + e^{5\pi i/3})}{4b^3e^{a b}}.
\]

(f) \[
\int_{0}^{\infty} \frac{dx}{x^3 + 8}.
\]

We first note that
\[
x^3 + 8 = (x + 2)(x - 2e^{\pi i/3})(x - 2e^{5\pi i/3}).
\]
We have
\[
\int_{0}^{\infty} \frac{dx}{x^3 + 8} = -\text{Res} \left( \frac{\log z}{z^3 + 8}, -2 \right) - \text{Res} \left( \frac{\log z}{z^3 + 8}, 2e^{\pi i/3} \right) - \text{Res} \left( \frac{\log z}{z^3 + 8}, 2e^{5\pi i/3} \right)
\]
\[
= -\log 2 + \pi i - \log 2 - \pi i/3
\]
\[
= \frac{\log 2 + 5\pi i/3}{4(1 + e^{\pi i/3})(1 + e^{5\pi i/3})} \frac{(1 + e^{\pi i/3})(1 + e^{5\pi i/3})}{4(1 + e^{\pi i/3})(1 + e^{5\pi i/3})}
\]
\[
= \frac{\sqrt{3} \pi - \pi i/2 - \sqrt{3} \pi /6 + 5\pi i/2 - 5\sqrt{3} \pi /6}{2\pi i}
\]
\[
= \frac{(3 + \sqrt{3}i)(3 - \sqrt{3}i)}{2\pi i}
\]
\[
= \frac{\pi}{6\sqrt{3}}.
\]

(4) If the degree of $Q$ exceeds that of $P$ by at least two, then
\[
\left| \frac{P(z)}{Q(z)} \right| \leq A \frac{1}{|z|^2},
\]
with some constant $A$ and for all $|z|$ sufficiently large. Therefore, take $R > 0$ large enough such that the above inequality hold for all $z$ with $|z| \geq R$ and the circle $|z| = R$ encloses all the zeros of $Q(z)$. Note that since $Q(z)$ is a polynomial, it has only finitely many zeros, say $z_k$’s. Therefore, we may infer that
\[
\sum_{k} \text{Res}(f(z), z_k) = 2\pi i \int_{|z|=R} f(z) \, dz \leq 2\pi 2\pi R A \frac{1}{R^2}.
\]
The right-hand side of the above has the limit zero as $R$ tends to infinity while the left-hand side is a non-negative real number majorized by the limit. We must have

$$\sum_k \text{Res}(f(z), z_k) = 0,$$

as desired.