Assignment Five Solutions

(1) Verify that

\[ u(re^{i\theta}) = \sum_{k=0}^{n} r^{-k} [a_k \cos(k\theta) + b_k \sin(k\theta)] \]

with \(a_k\) and \(b_k\) real constants, is harmonic in the complex plane with the origin removed.

The function can be re-written as

\[ u(re^{i\theta}) = \sum_{k=0}^{n} r^{-k} [a_k \cos(-k\theta) - b_k \sin(-k\theta)] \]

This function in question is the real part of the function

\[ f(z) = \sum_{k=0}^{n} (a_k + ib_k) z^{-k}, \]

which is analytic in the region of interest. Our desired result follows.

(2) (a) Use the mean value property of harmonic functions to prove the following special case of

the Jensen’s formula. If \(f(z)\) is analytic and free from zeros in \(|z| \leq r\) with \(r > 0\), then

\[ \log |f(0)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta. \]

If \(f(z)\) is analytic and free of zeros in \(|z| \leq r\) which is obviously simply connected, then there is an analytic branch of \(\log f(z)\) in the said region. Therefore, the real part of \(\log f(z)\) is a harmonic function in the same region. The real part in question is \(\log |f(z)|\).

Now we apply the mean value theorem for the harmonic function \(\log |f(z)|\), we get

\[ \log |f(0)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta. \]

(b) Show that if \(|z| = r\), then for all \(w\) with \(|w| < r\), we have \(\left| \frac{r^2 - \bar{w}z}{r(z - w)} \right| = 1\).

Note that \(z - w \neq 0\). We have

\[ \left| \frac{r^2 - \bar{w}z}{r(z - w)} \right| = \left| \frac{\bar{z} - \bar{w}}{\bar{r}(z - w)} \right| = \left| \frac{z(z - \bar{w})}{r(z - w)} \right| = \left| \frac{z}{r} \right| \left| \frac{z - \bar{w}}{z - w} \right| = 1. \]

(c) Note that the Jensen’s formula can be generalize to analytic functions that do vanish in \(|z| \leq r\). Let \(a_1, \ldots, a_n\) be the all zeros, counted with multiplicity, of \(f(z)\) in the interior of the disk \(D(0, r)\)(Why are there only finitely many zeros?) and supposed that \(f(z) \neq 0\) for all \(z\) with \(|z| = r\). Show that

\[ \log |f(0)| + \sum_{k=1}^{n} \log \frac{r}{|a_k|} = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta. \]
(Hint: Apply Jensen’s formula in the special case to the function
\[ F(z) = f(z) \prod_{k=1}^{n} \frac{r^2 - \bar{a}_k z}{r(z - a_k)}. \]

An analytic function that is not the constant zero function has only finitely many zeros in a compact set. Otherwise, the zeros have a cluster point and the function in question must be the constant zero function by analytic continuation. Now let
\[ F(z) = f(z) \prod_{k=1}^{n} \frac{r^2 - \bar{a}_k z}{r(z - a_k)}. \]

By the previous part, we have
\[ |F(z)| = |f(z)| \text{ if } |z| = r. \]

\( F(z) \) is now analytic function in \( |z| \leq r \) that is free of zeros. Applying Jensen’s formula in the special case, we have
\[ \log |F(0)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log |F(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(re^{i\theta})| d\theta. \]

Finally, we observe that
\[ \log |F(0)| = \log \left| \prod_{k=1}^{n} \frac{r}{a_k} \right| = \log |f(0)| + \sum_{k=1}^{n} \log \frac{r}{|a_k|}. \]

We have the desired result.

(3) Evaluate the following infinite series.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}. \]

We have
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^4} = \frac{1}{2} \text{Res} \left( \frac{\pi \csc \pi z}{z^4}, 0 \right) = -\frac{\pi}{48} \lim_{z \to 0} \frac{d^4}{dz^4} \csc \pi z \]
\[ = -\frac{\pi}{48} \frac{7\pi^3}{15} = -\frac{7\pi^4}{720}. \]

(b) \[ \sum_{n=1}^{\infty} \frac{1}{n^6}. \]

We have
\[ \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^6} = \frac{1}{2} \text{Res} \left( \frac{\pi \cot \pi z}{z^6}, 0 \right) = -\frac{\pi}{1440} \lim_{z \to 0} \frac{d^6}{dz^6} \cot \pi z = -\frac{\pi}{1440} \frac{2\pi^5}{945} = -\frac{\pi^6}{680400}. \]

(c) \[ \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{8^n}. \]

We showed in class that
\[ \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}, \text{ if } |x| < \frac{1}{4}. \]

Therefore, we have
\[ \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{8^n} = \sqrt{2}. \]
(4) Show that \( \int_0^\infty \frac{x^{a-1}}{1 + x} \, dx = \frac{\pi}{\sin a\pi} \) for all \( 0 < a < 1 \) using contour integration. (Hint: Use the “key-hole” contour.)

In the “key-hole” contour which we denote as \( C \), let \( I_1, I_2, C_R \) and \( C_L \) be the line segments in the upper half plane, lower half plane, the outer arc and inner arc, respectively. In the area the contour encloses, \( \log z \) has an analytic branch. We have

\[
\left| \int_{C_R} \frac{z^{a-1}}{1 + z} \, dz \right| = \left| \int_{C_R} \frac{e^{a\log z}}{z(1 + z)} \, dz \right| \leq 2\pi R \left( \frac{R^a}{R(R - 1)} \right).
\]

This tends to zero as \( R \) tends to infinity as \( a < 1 \). Moreover,

\[
\left| \int_{C_L} \frac{z^{a-1}}{1 + z} \, dz \right| = \left| \int_{C_L} \frac{e^{a\log z}}{z(1 + z)} \, dz \right| \leq 2\pi e^a \left( \frac{e^a}{e|\epsilon - 1|} \right) = 2\pi \frac{e^a}{|\epsilon - 1|}.
\]

This also tends to zero as \( \epsilon \) tends to zero since \( 0 < a < 1 \). On the remaining pieces, we have

\[
\lim_{R \to \infty} \int_{I_1} \frac{z^{a-1}}{1 + z} \, dz = \int_0^\infty \frac{e^{a\log x}}{x(1 + x)} \, dx = \int_0^\infty \frac{x^a}{x(1 + x)} \, dx
\]

and

\[
\lim_{R \to \infty} \int_{I_2} \frac{z^{a-1}}{1 + z} \, dz = -\int_0^\infty \frac{e^{a(\log x + 2\pi i)}}{x(1 + x)} \, dx = -e^{2\pi i} \int_0^\infty \frac{x^a}{x(1 + x)} \, dx.
\]

Therefore, we infer

\[
(1 - e^{2\pi i}) \int_0^\infty \frac{x^a}{1 + x} \, dx = \lim_{R \to \infty} \int_{C} \frac{z^{a-1}}{1 + z} \, dz = 2\pi i \text{Res} \left( \frac{z^a}{z(1 + z)}, -1 \right) = -2\pi ie^{a\pi i}.
\]

Therefore, we have

\[
\int_0^\infty \frac{x^{a-1}}{1 + x} \, dx = -\frac{2\pi ie^{a\pi i}e^{-a\pi i}}{1 - e^{2\pi i}} = \frac{-2\pi i}{2i \sin(\alpha \pi)} = \frac{\pi}{\sin(\alpha \pi)}.
\]

(5) Use contour integration to Evaluate the integral \( \int_0^\infty \frac{e^{ax}}{1 + e^x} \, dx \) for \( 0 < a < 1 \). (Hint: Consider the contour integral over the rectangle with vertices \( \pm R \) and \( \pm R + 2\pi i \), for \( R > 0 \) and large. Of course, the problem is follows trivially from the previous problem with a change of variables, but one needs to follow the instructions given.)

The integral become the same as that in the previous problem by making the change of variables \( u = e^x \). Let \( C \) be the contour that is given in the hint. Let \( I_1, I_2, I_3 \) and \( I_4 \) be line segments on the real axis, on the right half of the plane, on the upper half plane and on the left half of the plane. We have

\[
\left| \int_{I_2} \frac{e^{ax}}{1 + e^x} \, dz \right| \leq 2\pi \frac{e^{aR}}{e^R - 1}
\]

and

\[
\left| \int_{I_4} \frac{e^{ax}}{1 + e^x} \, dz \right| \leq 2\pi \frac{e^{-aR}}{e^{-R} - 1}.
\]

Both of the above tends to zero as \( R \) tends to infinity since \( 0 < a < 1 \). Moreover, we have

\[
\int_{I_3} \frac{e^{ax}}{1 + e^x} \, dz = \int_{-R}^R \frac{e^{ax}e^{2\pi i}}{1 + e^x} \, dx = -e^{2\pi i} \int_{-R}^R \frac{e^{ax}}{1 + e^{2\pi i}} \, dx.
\]
Therefore, we infer
\[
(1 - e^{a2\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \lim_{R \to \infty} \int_{C} \frac{e^{az}}{1 + e^z} \, dz = 2\pi i \text{Res} \left( \frac{e^{az}}{1 + e^z}, \pi i \right)
\]
\[
= 2\pi i \lim_{z \to \pi i} \frac{e^{az}(z - \pi i)}{1 + e^z} = -2\pi i e^{a\pi i}.
\]

Finally, we have
\[
\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} \, dx = \frac{-2\pi i e^{a\pi i}}{1 - e^{a2\pi i}} = \frac{\pi}{\sin(a\pi)}
\]
as you have already computed in the previous exercise.