QUESTION 1.

(a) Show that \( \sqrt{3} \) is irrational. \hspace{1cm} (10 marks)

Proof. Suppose that \( \sqrt{3} \) is rational and \( \sqrt{3} = p/q \) with integers \( p \) and \( q \) not both divisible by 3. We get the relation

\[ p^2 = 3q^2 \]

from which we infer that \( p^2 \) is divisible by 3. Hence \( p \) itself is divisible by 3, as 3 is a prime number and \( p = 3k \) for some integer \( k \). Therefore, we must have

\[ 9k^2 = 3q^2 \]

which yields \( 3k^2 = q^2 \)

and \( q^2 \) is a multiple of 3. Now 3 must also divide \( q \). This is a contradiction as we assumed that \( p \) and \( q \) are not both divisible by 3. \( \square \)

(b) Let \( \{p_n\} \) be a sequence in a metric space. Give the definition of the convergence of this sequence. \hspace{1cm} (5 marks)

The sequence \( \{p_n\} \) is said to converge if there exists a points \( p \) in the metric spaces with the following property. For \( \varepsilon > 0 \), there exists a natural number \( N \) such that if \( n \geq N \), then \( d(p_n, p) < \varepsilon \).

(c) Show that the base of the natural logarithm, \( e \) as defined below, is irrational. You may assume the sequence converges.

\[ e = \lim_{n \to \infty} \left( \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right). \]

\hspace{1cm} (15 marks)

Proof. Let

\[ e_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}. \]

We then have, if \( n \) is a natural number,

\[ 0 < e - e_n = \lim_{N \to \infty} \left( \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots + \frac{1}{(n+N)!} \right) \]

\[ = \frac{1}{(n+1)!} \lim_{N \to \infty} \left( 1 + \frac{1}{(n+1)!} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+2)(n+3)\cdots(n+N)} \right) \]

\[ \leq \frac{1}{(n+1)!} \lim_{N \to \infty} \left( 1 + \frac{1}{(n+1)!} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+2)^{N-1}} \right) \]

\[ = \frac{1}{(n+1)!} \lim_{N \to \infty} \frac{1 - \left( \frac{1}{n+1} \right)^N}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)!} \frac{n+1}{n} = \frac{1}{n \times n!}. \]

Collecting this information, we get

\[ 0 < e - e_n < \frac{1}{n \times n!}. \]

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Now if $e$ is a rational number, then $e = p/q$ with natural numbers $p$ and $q$. We have, from (1),

$$0 < e - e_q < \frac{1}{q \times q!} \quad \text{which gives } 0 < q!(e - e_q) < \frac{1}{q}.$$  

We note here that $e \times q!$ is certainly a natural number and so is $q! \times e_q = q! \times \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$.

Hence $q!(e - e_q)$ is the difference between two natural numbers and hence an integer. But this quantity lies strictly between zero and one as in shown in (2) which is certainly an absurdity. Hence $e$ must be irrational.  

\end{proof}

Total: 30 marks

**QUESTION 2.**

(i) Define the Cantor set. (7 marks)

Let

$$E_n = \bigcup_{k=1}^{3^n-1} \left[ k \times 3^{-n}, k + 1 \times 3^{-n} \right].$$

The Cantor set, $E$, is defined to be

$$E = \bigcap_{n=1}^{\infty} E_n.$$  

(ii) Show that the Cantor set is perfect. (13 marks)

\begin{proof}
Each $E_n$ is a finite union of closed intervals and hence closed. $E$ is an intersection of closed sets and hence is also closed. If $x \in E$, then $x \in E_n$ for all $n$. Therefore, let $I_n$ be the closed interval of $E_n$ such that $x \in I_n$ and let $x_n$ be the endpoint of $I_n$ with $x_n \neq x$.

Note that $I_n$ is an interval of length $3^{-n}$. Hence we have

$$|x - x_n| < \frac{1}{3^n}.$$  

We certainly have $x_n \neq x$ and the size of their difference can be made smaller than any $\epsilon > 0$ by taking $n$ large enough, larger than $-\log_3 \left( \min \left\{ \epsilon, \frac{1}{2} \right\} \right)$.  

\end{proof}

Total: 20 marks

**QUESTION 3.**

Let $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2 + s_n}$ for $n = 1, 2, 3, \ldots$.

(i) Show that $\{s_n\}$ is an monotonically increasing sequence. (Hint: Use induction.) (5 marks)

\begin{proof}
We need to show $s_{n+1} \geq s_n$ for all $n$. This is easy for $n = 1$ as

$$s_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = s_1.$$  

Assuming the contention holds for $n = k - 1$. Then we have

$$s_{k+1} = \sqrt{2 + s_k} \geq \sqrt{2 + s_{k-1}} = s_k$$

where the inequality above follows from the induction hypothesis. The proof is completed.  

\end{proof}
(ii) Show that \( s_n \leq 2 \) for all \( n \). (Hint: Use induction again.) (5 marks)

**Proof.** Once again, the case for \( n = 1 \) is easily true as \( s_1 = \sqrt{2} \leq 2 \). Assuming the contention hold for \( n = k - 1 \), then
\[
s_k = \sqrt{2 + s_{k-1}} \leq \sqrt{2 + 2} = 2,
\]
where the inequality above follows from the induction hypothesis. \( \square \)

(iii) Now does the sequence converge at all? Compute this limit.

**Hint:** If the limit is \( L \), then taking \( n \) to \( \infty \) on both sides of
\[
s_{n+1} = \sqrt{2 + s_n}
\]
gives the relation
\[
L = \sqrt{2 + L}.
\]

**Proof.** Following the hint, we have the limit \( L \) is a solution to the equation \( L^2 - L - 2 = 0 \). This equation has solutions \( L = 2 \) and \( L = -1 \). It is obvious that the limit in question must be non-negative. Hence it must be that
\[
\lim_{n \to \infty} s_n = 2.
\]
\( \square \)

(10 marks)

**Total:** 20 marks

**QUESTION 4.**

(a) Let \( f \) be a uniformly continuous function from a metric space \( X \) to another metric space \( Y \) and \( g \) a uniformly continuous function from the metric space \( Y \) to another metric space \( Z \). Show that \( g \circ f \) is a uniformly continuous function from \( X \) to \( Z \). (10 marks)

**Proof.** Let \( \varepsilon > 0 \) be given. Since \( g \) is uniformly continuous, then there exists \( \delta_1 > 0 \) such that
\[
d_Z(g((y_1), g(y_2))) < \varepsilon
\]
so long as \( d_Y(y_1, y_2) \leq \delta_1 \). Since now \( f \) is also uniformly continuous, then there exists \( \delta > 0 \) such that
\[
d_Y(f(x_1), f(x_2)) < \delta_1
\]
provided that \( d(x_1, x_2) \leq \delta \). From these conditions, we infer that if the distance between \( x_1 \) and \( x_2 \) in \( X \) does not exceed \( \delta \), then we will have
\[
d_Z(g(f(x_1)), g(f(x_2))) < \varepsilon.
\]
This shows that \( g \circ f \) is a uniformly continuous function from \( X \) to \( Z \). \( \square \)

(b) Consider function \( f \) defined on the real numbers as
\[
f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational;} \\ 1/n, & \text{if } x = m/n \text{ with } m, n \text{ in lowest terms}. \end{cases}
\]
Show that \( f \) is continuous at all irrational numbers. (10 marks)
Proof. Let \( x_0 \) be an irrational real number and an \( \varepsilon > 0 \) be given. Let \( N \) be large enough such that \( N \times \varepsilon > 1 \) or equivalently \( 1/N < \varepsilon \). Now let \( q \) be closest rational number to \( x_0 \) among the rational numbers with denominators not exceeding \( N \), when written in lowest terms. Such a \( q \) certainly can be found as, there are only finitely many rational numbers with denominators not exceeding \( N \) in the interval \([x_0 - 1, x_0 + 1]\). Note that \( |q - x_0| \) cannot be zero as \( x_0 \) itself is not a rational number. Let \[ \delta = \frac{1}{2}|q - x_0|. \]

By this construction, if \( r = s/t \) is a rational number with integers \( s \) and \( t \) in lowest terms and \( |r - x_0| \leq \delta \), then \( t > N \). Therefore \[ f(r) = \frac{1}{t} < \frac{1}{N} < \varepsilon. \]

But if \( v \) is irrational, \( f(v) = 0 \). Summarizing, for any \( \varepsilon > 0 \), there can be found a \( \delta \) such that \[ |f(x) - f(x_0)| = |f(x) - 0| = |f(x)| < \varepsilon, \]
provided \( |x - x_0| \leq \delta \).

This is to say that \( f \) is continuous at the arbitrary irrational number \( x_0 \) and this completes the proof. \( \square \)

(c) Suppose that \( f \) is a real-valued function on \((−\infty, \infty)\). \( x \) is called a fixed point of \( f \) if \( f(x) = x \). Show that if \( f \) is differentiable on \((−\infty, \infty)\) and \( f'(t) \neq 1 \) for all \( t \). Prove that \( f \) has at most one fixed point.

Proof. Suppose that \( x_1 \) and \( x_2 \) are two fixed points of \( f \) with \( x_1 \neq x_2 \) and consider \( g(x) = f(x) - x \). Clearly \( g \) is also differentiable on \((−\infty, \infty)\) and \( g(x_1) = g(x_2) = 0 \). By Rolle’s theorem, there must be an \( x_3 \) which lies between \( x_1 \) and \( x_2 \) with \( g'(x_3) = 0 \). However, \( g'(x) = f'(x) - 1 \) and hence \( g'(x) \neq 0 \) for all \( x \). Therefore, there can be at most one fixed point of \( f \). This completes the proof. \( \square \)

(10 marks)

Total: 30 marks