5. We certainly have
\[ (1) \quad \sup\{a_n + b_n : n \geq m\} \leq \sup\{a_n : n \geq m\} + \sup\{b_n : n \geq m\} \]
for any \( m \in \mathbb{N} \). Now if both of the limit supremums on the right-hand side of the contention are \(+\infty\), then the contention is trivial. If the same are bounded finite, then we can take \( m \) to infinity in (1) and the contention follows. Finally if both
\[ \sup\{a_n : n \geq m\} = \sup\{b_n : n \geq m\} = -\infty, \]
Then for any \( M > 0 \), there exists \( N \in \mathbb{N} \) such that if \( m \geq M \), then
\[ \sup\{a_n : n \geq m\} \leq -M \quad \text{and} \quad \sup\{b_n : n \geq m\} \leq -M. \]
From this we can infer that, using (1),
\[ \sup\{a_n + b_n : n \geq m\} \leq -2M. \]
This is to say that
\[ \limsup_{n \to \infty} (a_n + b_n) = -\infty \]
and our contention follows.

6. (a): Applying the intermediate value theorem to the function \( f(x) = \sqrt{x} \), we see that
\[ (2) \quad \frac{1}{2\sqrt{n}} \geq a_n \geq \frac{1}{2\sqrt{n} + 1}. \]
By comparison test, we have
\[ \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + 1} \leq \sum_{n=1}^{\infty} a_n. \]
Therefore the series in question diverges.

(b): By (2), we have
\[ a_n \leq \frac{1}{2n\sqrt{n}}. \]
Now the convergence of the series in question follows from that of
\[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \]

(c): Using the root test, we have
\[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left( \sqrt[n]{n} - 1 \right) = 0 \]
as we know that \( \sqrt[n]{n} \) tends to 1 as \( n \) tends to infinity. Therefore the series in question converges.

9. (a): It is easily seen that
\[ \lim_{n \to \infty} \sqrt[n]{3} = 1 \]
using the same observation as in part (c) of exercise 6. The radius of convergence of the power series in question is the reciprocal of the above limit and hence 1.

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(b): Following Miss Chen’s suggestion, we use the ratio test here. We have
\[
\limsup_{n \to \infty} \frac{2^{n+1} z^{n+1}/(n+1)!}{2^n z^n/n!} = \limsup_{n \to \infty} \frac{2z}{n+1} = 0.
\]
From this, we can infer that the radius of convergence is infinity.

(c): We see that
\[
\lim_{n \to \infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n \to \infty} \frac{2}{n^{2/n}} = 2.
\]
Hence the radius of convergence is 1/2.

(d): We now have that
\[
\lim_{n \to \infty} \sqrt[n]{\frac{n^3}{3^n}} = \lim_{n \to \infty} \frac{n^{3/n}}{3} = \frac{1}{3}.
\]
Therefore the radius of convergence is 3.

10. The fact that \(a_n \neq 0\) for infinitely many \(n\)’s gives that \(|a_n| \geq 1\) for infinitely many \(n\)’s. From this, we can infer that
\[
\limsup_{n \to \infty} \sqrt[n]{|a_n|} \geq 1.
\]
Therefore, since the radius of convergence is the reciprocal of the above limit supremum, it cannot exceed 1.

19. First of all, we see that any \(x \in [0, 1]\) can be expressed as an infinite series
\[
\sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}
\]
with \(\alpha_n \in \{0, 1, 2\}\) for all \(n\). With \(E_n\) and \(P\) having the same meaning as in section 2.44, \(x \notin E_n\) if and only if
\[
x \in \left(\frac{k}{3^n}, \frac{k+1}{3^n}\right).
\]
The latter condition is true if and only if the \(n\)-th entry in the ternary expansion of \(x\) is 1. Therefore, \(x \in E_n\) if and only if the first \(n\) entries of its ternary expansion are free of 1 and hence \(x \in P\) if and only if all entries in its ternary expansion are free of 1. This is precisely what needs to be proved.