One Level Density of Low-lying Zeros of Families of $L$-functions

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Density Conjecture

- The density conjecture of Katz and Sarnak suggests that the distribution of zeros near \(1/2\) of a family of \(L\)-functions is the same as that of eigenvalues near 1 of a corresponding classical compact group. This has been confirmed for various families.
  - Different families of Dirichlet \(L\)-functions (Rubenstein, Hughes-Rudnick, Miller, Özlük-Snyder, Gao)
  - \(L\)-functions with characters of the ideal class group of imaginary quadratic field (Fouvry-Iwaniec)
  - Automorphic \(L\)-functions (Iwaniec-Sarnak-Luo, Dueñez-Miller, Hughes-Miller, Ricotta-Royer)
  - Elliptic curve \(L\)-functions (Brumer, Heath-Brown, Young, Baier-Zhao)
  - Symmetric powers of \(GL(2)\) \(L\)-functions (Dueñez-Miller, Güloğlu)
Introduction

- Let $\chi$ be a primitive Dirichlet character of conductor $q$ and denote the non-trivial zeros of $L(s, \chi)$ by $1/2 + i\gamma_{\chi,j}$.
- We order these zeros as

$$
\cdots \leq \Re \gamma_{\chi,-2} \leq \Re \gamma_{\chi,-1} < 0 \leq \Re \gamma_{\chi,1} \leq \Re \gamma_{\chi,2} \leq \cdots. \tag{1}
$$

- Let $\phi$ henceforth be an even Schwartz class function with compact support and set

$$
\tilde{\gamma}_{\chi,j} = \frac{\gamma_{\chi,j}}{2\pi} \log X \quad \text{and} \quad S(\chi, \phi) = \sum_j \phi(\tilde{\gamma}_{\chi,j}). \tag{2}
$$
Quadratic Dirichlet $L$-functions

- Özluk and Snyder studied the family of quadratic Dirichlet $L$-functions $L(\chi_{8d}, \chi)$ for $d$ odd and square-free with $X \leq d \leq 2X$, where $\chi_{8d} = \left(\frac{8d}{.}\right)$ is the Kronecker symbol.
- Let $D(X)$ be the set of $d$'s described above.
- Assuming GRH for this family of $L$-functions, Özluk and Snyder proved

\[
\lim_{X \to \infty} \frac{1}{\#D(X)} \sum_{d \in D(X)} S(\chi_{8d}, \phi) = \int_{\mathbb{R}} \phi(x) \left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) \, dx,
\]

provided that the support of $\hat{\phi}$, the Fourier transform of $\phi$, is contained in the interval $(-2, 2)$.
Family of Quadratic Twists of Hecke $L$-functions

- M. O. Rubenstien studied $n$-level densities of the low-lying zeros of the families of quadratic twists of $L$-functions associated with holomorphic Hecke eigenforms.
- Let $f$ be a fixed holomorphic Hecke eigenform of level 1 and weight $k$.
- For $\Im z > 0$, $f$ has the Fourier expansion
  \[ f(z) = \sum_{n=1}^{\infty} a_f(n) n^{(k-1)/2} e(nz). \]
- Let $\chi$ be a primitive Dirichlet character of conductor $q$. The $L$-function of the twist of $f$ by $\chi$ is, for $\Re s > 1$,
  \[ L(f \times \chi, s) = \sum_{n=1}^{\infty} \frac{a_f(n)\chi(n)}{n^s} = \prod_p \left(1 - \frac{a_f(p)\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}}\right)^{-1}. \]
Family of Quadratic Twists of Hecke $L$-functions

- We denote the $j$-th zero of $L(f \times \chi_{8d}, s)$ by $1/2 + i\gamma_{f,8d,j}$ and order them in a manner similar to (1).
- Set $\tilde{\gamma}_{f,8d,j} = \gamma_{f,8d,j} 2 \log X/(2\pi)$.
- For an even Schwartz class function $\phi$ set

$$D(d, f, \phi) = \sum_{j} \phi(\tilde{\gamma}_{f,8d,j}).$$

- The result of Rubenstein asserts if $\text{supp} \left(\hat{\phi}\right) \subset (-1/2, 1/2)$.

$$\lim_{X \to +\infty} \frac{1}{\#D(X)} \sum_{d \in D(X)} D(d, f, \phi) = \int_{\mathbb{R}} \phi(x) \left(1 + \frac{\sin(2\pi x)}{2\pi x}\right) dx.$$
Our result double the size of the support of $\hat{\phi}$ under GRH. Let $\Phi_X(t)$ be a non-negative smooth function supported on $(1, 2)$ with $\Phi_X(t) = 1$ for all $t \in (1 + 1/\log X, 1 - 1/\log X)$ and $\Phi_X^{(j)}(t) \ll j U^j$ for all $j \geq 0$.

Theorem (Gao and Z.)

Suppose that GRH is true and that $\hat{\phi}(u)$ has support in $(-1, 1)$. Then

$$\lim_{X \to +\infty} \frac{1}{\#D(X)} \sum_{d \in D(X)} \Phi_X \left( \frac{d}{X} \right) D(d, f, \phi) = \int_{\mathbb{R}} \phi(x) \left( 1 + \frac{\sin(2\pi x)}{2\pi x} \right) \, dx.$$  

(3)
Let $C(X)$ denote the set of primitive cubic characters of conductor $q$ not divisible by 3 and $X \leq q \leq 2X$.

Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}$ has compact support in $(-3/7, 3/7)$.

We have

$$\lim_{X \to \infty} \frac{1}{\# C(X)} \sum_{X \leq q \leq 2X} \sum_{\chi \mod q}^* S(\chi, \phi) = \int_{\mathbb{R}} \phi(x) dx,$$

where the “$\star$” on the sum over $\chi$ means that the sum is restricted to primitive characters.
Let $K = \mathbb{Q}(\omega)$ with $\omega = \exp(2\pi i/3)$.

Let $I$ and $P$ denote the group of fractional ideals and the subgroup of principal ideals in $K$, respectively.

Given $c \in \mathbb{Z}[\omega]$. Let $I_{(c)} = \{A \in I : (A, (c)) = 1\}$ and $P_{(c)} = \{(a) \in P : a \equiv 1 \pmod{c}\}$.

The ideal ray class for $c$, $h_{(c)}$ is defined to be $h_{(c)} = I_{(c)}/P_{(c)}$.

Let $\chi_c = \left(\frac{\cdot}{c}\right)_3$ be the cubic symbol with $c$ square-free and $c \equiv 1 \pmod{9}$, regarded as a character of $h_{(c)}$. 

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Cubic Hecke $L$-functions

- The Hecke $L$-function associated with $\chi_c$ is defined to be
  \[
  L(s, \chi_c) = \sum_{0 \neq A \subset O_K} \chi_c(A)(N(A))^{-s} \text{ for } \Re s > 1,
  \]
  where $N(A)$ is the norm of $A$.

- We denote the non-trivial zeros of $L(s, \chi_c)$ as $1/2 + i\gamma_{\chi_c,j}$ and order them in a manner similar to (1).

- Let $C_{(9)}(X)$ denote the set of $\chi_c$ with $c$ square-free and $c \equiv 1 \pmod{9}$ and $X \leq N(c) \leq 2X$.

- Also define $S(\chi_c, \phi)$ in a way similar to (2).
Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}$ has compact support in $(-3/5, 3/5)$.

We have

$$
\lim_{X \to \infty} \frac{1}{\# C(9)(X)} \sum_{c \equiv 1 \mod 9}^* S(\chi_c, \phi) = \int_{\mathbb{R}} \phi(x) \, dx,
$$

where the “$*$” on the sum over $c$ means that the sum is restricted to square-free elements $c$ of $\mathbb{Z}[\omega]$.

This result was obtained by Güloğlu under GRH if $\hat{\phi}$ has support in $(-31/30, 31/30)$ while our result is unconditional.
Quartic Dirichlet $L$-functions

- Let $Q(X)$ denote the set of primitive quartic characters of odd conductor $q$ and $X \leq q \leq 2X$.
- Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}$ has compact support in $(-3/7, 3/7)$.
- We have

$$\lim_{{X \to \infty}} \frac{1}{{\#Q(X)}} \sum_{{X \leq q \leq 2X}} \sum_{{\chi \mod q \chi^4=\chi_0, \chi^2\neq \chi_0}}^* S(\chi, \phi) = \int_\mathbb{R} \phi(x)dx,$$

where the “$*$” on the sum over $\chi$ means that the sum is restricted to primitive characters.
A version of the explicit formula is used to transform $D(d, f, \phi)$, a sum over the zeros of the $L$-function, to a sum over primes and prime powers.

$$D(d, f, \phi) = \int_{\mathbb{R}} \phi(t) \left( 1 + \frac{\sin(2\pi t)}{2\pi t} \right) dt - S(f, d, x; \hat{\phi}) + O\left( \frac{\log \log 3X}{\log X} \right),$$

where

$$S(f, d, x; \hat{\phi}) = \frac{1}{\log X} \sum_{p} \frac{a_f(p) \log p}{\sqrt{p}} \left( \frac{8d}{p} \right) \hat{\phi} \left( \frac{\log p}{2 \log X} \right).$$
Quadratic Twists of Cusp form $L$-functions

Define

$$S(X, Y; \hat{\phi}, f, \Phi) = \sum_{(d,2)=1} \mu^2(d) \sum_{p \leq Y} \frac{a_f(p) \log p}{\sqrt{p}} \left( \frac{8d}{p} \right) \hat{\phi} \left( \frac{\log p}{2 \log X} \right) \Phi \left( \frac{d}{X} \right).$$

Moreover, we have

$$\mu^2(d) = \sum_{l^2|d} \mu(l) + \sum_{l > Z} \mu(l) = M_Z(d) + R_Z(d), \text{ say.}$$

To prove our Theorem, it suffices to show

$$\lim_{X \to \infty} \frac{S(X, Y; \hat{\phi}, f, \Phi)}{X \log X} = 0.$$
Quadratic Twists of Cusp form $L$-functions

- Let $S_M(X, Y; \hat{\phi}, f, \Phi)$ and $S_R(X, Y; \hat{\phi}, f, \Phi)$ be part of the sum in $S(X, Y; \hat{\phi}, f, \Phi)$ corresponding to $M_Z(d)$ and $M_R(d)$, respectively.
- $S_R(X, Y; \hat{\phi}, f, \Phi)$ can be shown to be small using GRH.
- $S_M(X, Y; \hat{\phi}, f, \Phi)$ can be further evaluated using a method of Soundararajan, leading to an expression involving quadratic Gauss sums which can further be evaluated.
- The resulting expressions can be estimated using results from Gao’s thesis.
- The condition on $\hat{\phi}$ is needed to ensure the above estimates are sufficient.
Cubic Dirichlet \( L \)-functions

- Once again, using a version of the explicit formula, the sum over zeros of the relevant \( L \)-functions is converted into sums over prime powers.
- The most important contribution will be from actual prime, as the contribution of proper prime powers can be shown to be small.
- It would be enough to prove

\[
\lim_{X \to \infty} \frac{1}{X \log X} \sum_p \frac{\log p}{p} \hat{\phi} \left( \frac{\log p}{\log X} \right) \sum_{X \leq q \leq 2X \chi \mod q} \sum^* (\chi(p) + \overline{\chi}(p)) = 0.
\]
Applying Cauchy’s inequality, it suffices to estimate

\[ \left| \sum_{p \leq Y, \ p \neq 3} \left| \sum_{X \leq q \leq 2X} \left( \sum_{\chi \ mod \ q} \chi(p) \chi^3 = \chi_0} \right) \right|^2 \right. \]

The above expression can be recast as

\[ \left| \sum_{p \leq Y, \ p \neq 3} \left( \sum_{X \leq \mathcal{N}(n) \leq 2X, \ n \equiv 1 \ mod \ 3} \left( \frac{p}{n} \right)_3 \right) \right|^2, \quad (4) \]

where the ’ on the sum over \( n \) means that the sum is restricted to square-free elements \( n \) of \( \mathbb{Z}[\omega] \).
We use a Pólya-Vinogradov type inequality, due to E. Landau, to estimate the inner-most sum of (4). The condition on $\hat{\phi}$ is needed to ensure the above estimates are sufficient. The proofs of the result for cubic Hecke $L$-functions and quartic Dirichlet $L$-functions are similar.
To widen the support of $\hat{\phi}$, one would need a mean-value estimate for cubic and quartic character sums, the analogous result of M. Jutila for the mean-value of quadratic character sums.

It would be highly interest to consider the one level density of the low-lying zeros of families of Dirichlet $L$-function for characters of orders larger than 4.

The relation between higher order residue symbols and $n$-th order primitive Dirichlet character would be more difficult to describe.