RESEARCH AND INTUITION

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The following is a well-known formula, but the quote, as quoted in [2], is perhaps not so well-known.

$$\int_{-\infty}^{\infty} e^{-x^2} \mathrm{d}x = \sqrt{\pi}. \tag{1}$$

> A mathematician is one to whom that (1) is as obvious as that twice two makes four is to you. Liouville was a mathematician.
>
> – Sir William Thomson, First Baron Kelvin of Largs.

One cannot help but feel good for being a mathematician after reading the above quote. After all, if a man such as Lord Kelvin, a well-known Englishman who was primarily a physicist, spoke well of mathematicians and more in particular a French mathematician, then we indeed all have great cause to rejoice at our own profession. Deriving (1) takes nothing more than a little knowledge in multi-variable calculus and some *ad hoc* proofs require even less, although a truly rigorous proof needs some convergence arguments in the execution. But the ideas behind (1) are not deep.

When I was in graduate school, I have always been amazed and impressed with the vast amount of mathematical knowledge my thesis adviser, Professor Henryk Iwaniec, has and how he can put complicated mathematical theorem into simple words. In brief, I enjoyed his lectures because he revealed not only mathematical theorems, but also mathematical intuitions behind the proofs. He made mathematical ideas stand out like "a simple and clear-cut constellation," not some star hidden in "a scattered cluster in the Milky Way" veiled behind the enunciation of the statement and proof of the theorems([1]). Indeed, mathematical ideas have less to do with the formality of the proofs, but rather the intuitions based on which on the proofs are executed which is the key for making mathematical discoveries. Having such intuitions is generally the mark of a good mathematician, and was perhaps partially the reason Lord Kelvin made the above remark. For a mathematician like Joseph Liouville, it was the mathematical intuitions he had that made (1) as simple and clear as $2 \times 2 = 4$.

It is generally believed that intuition and rigor are two integral parts of mathematics. Intuition tells us where to go and rigor gets us there. Hence, it is in this sense that mathematical intuition serves as guiding light in problem solving and having it or not is, for the most part, the touchstone for mathematical sophistication. We, as mathematical educators, must be judged by a higher set of standards than our students and we would find ourselves severely handicapped if our mathematical sophistication is the bare minimum of what is needed to carry out our teaching duties. We all had the experience of looking back at the problems in elementary school while in high school and realizing that the arithmetic problems that troubled us before have become matters of triviality. It is not because we have been constantly practicing these arithmetic problems throughout our four

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years of high school that enables us to do them with ease, but rather because we are working at higher and higher levels of mathematical stratum. Indeed, we have grown over those years and gained much mathematical sophistication and hence the problems in fourth grade aren't so hard any more because we know exactly what to do when we see them. Hence to be able to accomplish the similar feat at a university level and solve problems and explain concepts well, we must raise the bar of mathematical sophistication even higher and set it at original mathematical research. It is indeed an oxymoron, but it takes experience to do things a priori, id est doing things with intuition. Mathematical intuition, the best indicator of mathematical sophistication, takes practice to attain and such practice, for us, is mathematical research.

It is through this that I believe mathematical research reinforces teaching, but not in a direct way. I have yet to find any direct significant relevance my research in the theory of exponential and character sums would have in any course I have taught thus far. But research does help me gain mathematical sophistication, and that gain facilitates deeper understanding of the material I must teach. One of the luxuries that a mathematician has in his work is that he never has to verify his motivations but only his results. It is this luxury that sometimes does a disservice to mathematics and makes much of the best of mathematics appear *ad hoc* and almost magical to novice such as our students. This phenomenon is present in both university mathematics and higher mathematics. I have been asked by a curious cadet what the motivations of the definition of the determinant of a matrix were. That definition is truly quite *ad hoc* upon first glace and the motivations are quite well-concealed. Indeed, definitions are mere assignments of names and we can call whatever thing whatever name we want. But if such assignments are to be meaningful at all, we must be somewhat motivated to make the assignments. To fully explain the motivation of the definition of the determinant of a matrix, one is unlikely to do a satisfactory job without some deeper understanding and intuition in the theories of matrices and linear operators. That kind of understanding and intuition is generally attained through mathematical research.

Often I hear my cadets asking questions such as "is that the final answer?" or "what does that(the solution to a mathematical model) tell us?" We certainly should not expect our cadets to have the kind of mathematical sophistication found in research mathematicians, and some computations tend to be lengthy such that it is easy to lose directions along the way and hence the two questions above are perhaps somewhat excusable. But I believe that it is precisely the lack of mathematical intuition that prompts the two questions from cadets. My response to the first question has often been "What was the question we started with? Have we addressed it yet? If so, then we have the final answer; otherwise, no." In the similar manner, my response to the second question has been "What does our mathematical model represent? What do you think the solution to that model should represent?" It is certainly not the most courteous thing to do to answer questions with questions. But at least it is hoped that they will put some thought into the problem solving process.

I feel that having my students gain mathematical sophistication or intuition is among the most important tasks I have as a mathematical educator. As noted in the last paragraph, such intuition is very much needed by many of my students. They seem to be lost in the middle of problem-solving and as mentioned before mathematical intuition is to serve the very purpose of telling one where to go. Consequently, to impart the kind of intuition that is so needed by my students, it is imperative that I must have a higher level of mathematical sophistication myself and I have pointed out earlier such sophistication is attained through mathematical research.

However, I believe that the reinforcement in the other direction is not as strong, at least I have not seen in any significant way how my teaching has helped me in my research yet. I feel that it is simply because that the material taught in early undergraduate mathematical education(all the courses I have taught are freshmen or sophomore courses) hardly raises up to scholarship standards that are commonly recognized in mathematical research. The closest among my encounters of having teaching reinforce research was being reminded the techniques of converting a repeating decimals into a fraction of integers while teaching a pre-calculus class and later realizing such techniques were precisely what were needed to re-prove a lemma of Gauss. But of course, that was certainly NOT original research and neither was it a deep lemma by today's standards. It is certainly conceivable that teaching can reinforce research, but my experiences have been that the reinforcement goes essentially only one way.

References

[1] G. H. Hardy, *A Mathematician's Apology*, First Edition, Cambridge University Press, London, 1969. Forward by C. P. Snow.
[2] M. Spivak, *Calculus on Manifolds*, Addision-Wesley, New York, 1965.

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