Quantum Rings of Singularities

Natalie Wilde

March 14, 2008
Introduction

Background
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- Definitions: singularity, quasi-homogeneous polynomial, symmetry group
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- Mirror Symmetry Conjecture.
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- Vector space structure.
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Theory

▶ Vector space structure.
  ▶ Singularity $Z_{11}$
Why do we care about this?

Singularity Theory appears in physics with applications to String Theory. Landau-Ginzburg model associated with a singularity. However...

Initial interest: Solving the Witten equation:

\[ \bar{\partial} u_i + \frac{\partial W}{\partial u_i} = 0 \]

\( W \) is a quasi-homogeneous polynomial

\( u_i \) is the solution that we are looking for satisfying the Witten Equation
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What is a singularity?

A point \((a_1,...,a_n)\) on a curve \(f(x_1,...,x_n) = 0\) is singular if the partial derivatives of \(f\) with respect to each \(x_j\) are zero at the point \((a_1,...,a_n)\).

Example
\[x^3 - x^2 + y^2 = 0\] where \[3x^2 - 2x, 2y\] are the partial derivatives of \(f\) with respect to \(x\) and \(y\), respectively.
What is a singularity?

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Example \(x^3 - x^2 + y^2 = 0\) where \(0^3 - 0^2 + 0^2 = 0\)
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Example \(x^3 - x^2 + y^2 = 0\) where \(0^3 - 0^2 + 0^2 = 0\)

\[
\frac{\partial f}{\partial x}(x, y) = 3x^2 - 2x, \quad \frac{\partial f}{\partial y}(x, y) = 2y
\]
Homogeneous vs. quasi-homogeneous polynomials

Homogeneous polynomial:
A polynomial where each term has the same total degree.

Example:
\[ f(x, y) = x^2 + 2xy + y^2 \]

Quasi-homogeneous Polynomial (weighted homogeneous):
A polynomial \( W \in \mathbb{C}[x_1, \ldots, x_n] \) where there exist unique \( q_1, \ldots, q_n \in \mathbb{Q} > 0 \) such that for any \( \lambda \in \mathbb{C} \),
\[ W(\lambda^{q_1}x_1, \ldots, \lambda^{q_n}x_n) = \lambda^{q_1+\ldots+q_n} W(x_1, \ldots, x_n) \]

Example:
\[ Z_{11} : x^3y + y^5 + ax^4y = 1, q_y = \frac{1}{5}, q_x = \frac{4}{15} \]
\[ (\lambda^{\frac{4}{15}}x)^3(\lambda^{\frac{1}{5}}y) + (\lambda^{\frac{1}{5}}y)^5 + a(\lambda^{\frac{4}{15}}x)^4(\lambda^{\frac{1}{5}}y) = \lambda x^3y + \lambda y^5 + \lambda^{\frac{16}{15}}axy \]
\( Z_{11} \) is not quasi-homogeneous unless \( a = 0 \).
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\[ W(\lambda q_1 x_1, \ldots, \lambda q_n x_n) = \lambda W(x_1, \ldots, x_n) \]

Example:

\[ Z_{11}: \quad x^3 y + y^5 + ax^4 y = 15, \quad q_y = 15, \quad q_x = 4 \]

\[ (\lambda^{15} y)^3 (\lambda^{4} x) + (\lambda^{15} y)^5 + a (\lambda^{4} x)^4 y = \lambda x^3 y + \lambda y^5 + \lambda^{16} 15 ax^4 y = \lambda x^3 y + \lambda y^5 + \lambda^{16} 15 ax^4 y \]

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**Example**

\[Z_{11} = x^3 y + y^5 + axy^4\]

\[q_y = \frac{1}{5}, \quad q_x = \frac{4}{15}\]

\[\lambda^4 \frac{1}{15} x^3 (\lambda \frac{1}{5} y) + (\lambda \frac{1}{5} y)^5 + a \lambda^4 \frac{4}{15} (\lambda \frac{1}{5} y)^4 = \lambda x^3 y + \lambda y^5 + \lambda^16 \frac{1}{15} axy^4\]

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- **Example** $f(x, y) = x^2 + 2xy + y^2$

**Quasi-homogeneous Polynomial** *(weighted homogeneous)*:

$Z_{11}^1: x^3y + y^5 + ax^4y^4 = λ^{15}x^3(λ^{15}y) + (λ^{15}y)^5 + a(λ^{15}x)(λ^{15}y)^4 = λx^2y + λy^5 + λ^{16^{15}}axy^4$

$Z_{11}$ is not quasi-homogeneous unless $a=0$. 

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**Quantum Rings of Singularities**
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\( Z_{11} \) is not quasi-homogeneous unless \( a=0 \).
Symmetry Group

The Symmetry Group is defined to be $G = \langle J \rangle$.

Example:

$\mathbb{Z}_{11}^2$: $x^3 y^5 + y^5 x = 1^{15} = 3^{15}, \quad q x = 4^{15}$.

$J = \begin{pmatrix} e^{2\pi i q_{11}^4} & 0 \\ 0 & e^{2\pi i q_{11}^3} \end{pmatrix} = \begin{pmatrix} \xi_{11}^4 & 0 \\ 0 & \xi_{11}^3 \end{pmatrix}$.

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Example \( \mathbb{Z}_{11} \):

\[ x^3 y^5 q = 1 \]

\[ q x = 4 \]

\[ J = \begin{pmatrix}
  e^{2\pi i q_1} & 0 \\
  \vdots & \ddots \\
  0 & e^{2\pi i q_n}
\end{pmatrix} = \begin{pmatrix}
  \xi & 0 \\
  0 & \xi^n
\end{pmatrix} \]
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Example

\[ Z_{11} : x^3 y + y^5 \quad q_y = \frac{1}{5} = \frac{3}{15}, \quad q_x = \frac{4}{15} \]
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Example

$Z_{11} : x^3 y + y^5 \quad q_y = \frac{1}{5} = \frac{3}{15}, \quad q_x = \frac{4}{15}$

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    0 & e^{2\pi i \frac{3}{15}}
\end{pmatrix} = \begin{pmatrix}
    \xi_{15}^4 & 0 \\
    0 & \xi_{15}^3
\end{pmatrix}$$
Mirror Symmetry Conjecture

Two models
1. Orbifold Landau-Ginzburg model
2. FJR model (Fan, Jarvis, Ruan)

Ruan's Mirror Symmetry Conjecture
If $W$ is a singularity, the FJR model corresponding to $W$ is isomorphic to the Orbifold Landau-Ginzburg model $W$.

The Witten's original conjecture of this symmetry for simple singularities has been proven (Fan, Jarvis, Ruan)
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The Milnor ring of a singularity

For the duration of this consider $W(x_1, \ldots, x_n)$ to be a quasi-homogeneous polynomial with a singularity.

Milnor ring of $W$:

$$M_W = \mathbb{C}[x_1, \ldots, x_n] / (\partial W)$$

Example $Z_{11}$:

$$x^3y + y^5$$

$$M_{Z_{11}} = \mathbb{C}[x, y] / (3x^2y, x^3 + 5y^4)$$

Finding a basis by hand can be tedious. Even Merril wrote a code to find a basis for the Milnor ring.
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For the duration of this consider $W(x_1, \ldots, x_n)$ to be quasi-homogeneous polynomial with a singularity.

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Finding a basis by hand can be tedious. Even Merril wrote a code to find a basis for the Milnor ring.
First, determine the vector space structure of $H$. Recall that $J = \begin{pmatrix} e^{2\pi i q_1} & 0 & \cdots & 0 \\ e^{2\pi i q_2} & e^{2\pi i q_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i q_n} \end{pmatrix}$. The $J_k$ action fixes subspaces of $\mathbb{C}^n$. Consider the Milnor ring of $W$ restricted to these subspaces. The invariants of these Milnor rings under the action of $J$ form a basis for the vector space.
First, determine the vector space structure of $\mathcal{H}_W$. 

Recall that 

$$J = \begin{pmatrix} e^{2\pi i q_1} & 0 & \cdots & 0 \\ 0 & e^{2\pi i q_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i q_n} \end{pmatrix}.$$ 

$J$ action fixes subspaces of $\mathbb{C}^n$. 

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Vector space structure of $Z_{11}$

Recall $Z_{11}$:

\[ x^3 y + y^5, \]

\[ q_x = 4^{15}, \quad q_y = 3^{15}, \]

\[ J = \left( \begin{array}{ccc} e^{8\pi i} & 0 \cr 0 & e^{6\pi i} \end{array} \right) \]

The $J_k$ action for $0 \leq k < 15$ fixes subspaces of $C^2$:

\[
\text{Fix}(J_k) = \begin{cases} 
C^2_k = 0 & \text{else} \\
C_y^k = 5, 10 & \text{other wise} 
\end{cases}
\]

$Z_{11} \mid C^2 = Z_{11}$, $Z_{11} \mid C_y = y^5$, and $Z_{11} \mid \{0\} = \{0\}$. 

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Recall $Z_{11} : x^3 y + y^5$, $q_x = \frac{4}{15}$, $q_y = \frac{3}{15}$, $J = \begin{pmatrix} e^{\frac{8\pi i}{15}} & 0 \\ 0 & e^{\frac{6\pi i}{15}} \end{pmatrix}$
Vector space structure of $Z_{11}$

Recall $Z_{11}: x^3 y + y^5$, $q_x = \frac{4}{15}$, $q_y = \frac{3}{15}$, $J = \left( \begin{array}{cc} e^{\frac{8\pi i}{15}} & 0 \\ 0 & e^{\frac{6\pi i}{15}} \end{array} \right)$

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$$\text{Fix}(J^k) = \begin{cases} \mathbb{C}^2 & k = 0 \\ \mathbb{C}_y & k = 5, 10 \\ \{0\} & \text{otherwise} \end{cases}$$
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$Z_{11}|_{\mathbb{C}^2} = Z_{11}$, $Z_{11}|_{\mathbb{C}_y} = y^5$, and $Z_{11}|_{\{0\}} = \{0\}$
Vector space structure of $Z_{11}$ continued

\[ Z_{11}|_{\mathbb{C}^2} = Z_{11}, \quad Z_{11}|_{\mathbb{C}y} = y^5, \text{ and } Z_{11}|_{\{0\}} = \{0\} \]
Vector space structure of $Z_{11}$ continued

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For a basis of the vector space consider
Vector space structure of $\mathbb{Z}_{11}$ continued

$\mathbb{Z}_{11}|_{\mathbb{C}^2} = \mathbb{Z}_{11}$, $\mathbb{Z}_{11}|_{\mathbb{C}^y} = y^5$, and $\mathbb{Z}_{11}|_{\{0\}} = \{0\}$

For a basis of the vector space consider

$$\begin{pmatrix} M_{\mathbb{Z}_{11}} e_0 \oplus M_{y^5} e_5 \oplus M_{y^5} e_{10} \bigoplus_{5 \nmid i} \mathbb{C} e_i \end{pmatrix} <J>$$
Vector space structure of $Z_{11}$ continued

$Z_{11}|_{\mathbb{C}^2} = Z_{11}$, $Z_{11}|_{\mathbb{C}y} = y^5$, and $Z_{11}|_{\{0\}} = \{0\}$

For a basis of the vector space consider

$$
\begin{pmatrix}
M_{Z_{11}} e_0 \oplus M_{y^5} e_5 \oplus M_{y^5} e_{10} \bigoplus_{5 \nmid i} \mathbb{C} e_i
\end{pmatrix}^{<J>}
$$

$M_{Z_{11}} = \langle 1, x, x^2, x^3, x^4, y, y^2, y^3, xy, xy^2, xy^3 \rangle$

$M_{y^5} = \langle 1, y, y^2, y^3, y^4 \rangle$
Vector space structure of $Z_{11}$ continued

$Z_{11}|_{\mathbb{C}^2} = Z_{11}$, $Z_{11}|_{\mathbb{C}^y} = y^5$, and $Z_{11}|_{\{0\}} = \{0\}$

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$$
\left( M_{Z_{11}} e_0 \oplus M_{y^5} e_5 \oplus M_{y^5} e_{10} \bigoplus \mathbb{C} e_i \right)^{<J>}
$$

$M_{Z_{11}} = \langle 1, x, x^2, x^3, x^4, y, y^2, y^3, xy, xy^2, xy^3 \rangle$

$M_{y^5} = \langle 1, y, y^2, y^3, y^4 \rangle$

Invariant elements: $M_{Z_{11}}$ has $x^2$, $M_{y^5}$ has none.
Vector space structure of $Z_{11}$ continued

$Z_{11}|_{\mathbb{C}^2} = Z_{11}$, $Z_{11}|_{\mathbb{C}y} = y^5$, and $Z_{11}|_{\{0\}} = \{0\}$

For a basis of the vector space consider

$$\left( M_{Z_{11}} e_0 \oplus M_{y^5} e_5 \oplus M_{y^5} e_{10} \bigoplus \mathbb{C} e_i \right) \langle J \rangle$$

$M_{Z_{11}} = \langle 1, x, x^2, x^3, x^4, y, y^2, y^3, xy, xy^2, xy^3 \rangle$

$M_{y^5} = \langle 1, y, y^2, y^3, y^4 \rangle$

Invariant elements: $M_{Z_{11}}$ has $x^2$, $M_{y^5}$ has none.
Although this only gave the vector space structure for $\mathbb{Z}_{11}$ more presentations about the quantum ring of a singularity will be given tomorrow at BYU’s Spring Research Conference.

We hope to determine the quantum ring for many singularities to help with direction on how to prove Ruan’s conjecture.