We construct a nil ring $R$ which has bounded index of nilpotence 2, is Wedderburn radical, and is commutative, and which also has a derivation $\delta$ for which the differential polynomial ring $R[x; \delta]$ is not even prime radical. This example gives a strong barrier to lifting certain radical properties from rings to differential polynomial rings. It also demarcates the strength of recent results about locally nilpotent PI rings.

1. Introduction

Given a ring $R$ with a derivation $\delta$, we can form the differential polynomial ring $R[x; \delta]$ as the set of left polynomials $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $a_0, a_1, \ldots, a_{n-1}, a_n \in R$. Addition of such polynomials is as usual. Multiplication is also defined as usual using the associative rule, except that we need a way to convert right polynomials into left polynomials, which is accomplished by repeatedly applying the rule $xa = ax + \delta(a)$ for each $a \in R$.

Smoktunowicz and Ziembowski [5] constructed a locally nilpotent ring $R$ with a derivation $\delta$ such that $R[x; \delta]$ is not Jacobson radical. This is in contrast to the standard polynomial ring case, where it is well known that if $R$ is locally nilpotent, then $R[x]$ is as well. On the positive side, Bell et al. [1] recently proved that if $R$ is a locally nilpotent PI ring, then $R[x; \delta]$ is again locally nilpotent. It is still an open question whether or not when $R$ is a ring with no nonzero nil ideals, the Jacobson radical $J(R[x; \delta])$ is zero.

In this paper, we approach the problem of characterizing unusual behavior of the differential polynomial ring from the other end. In Section 2 we construct a nil ring $R$ with bounded index of nilpotence 2, which is also commutative and Wedderburn radical, along with constructing an $R$-derivation $\delta$ for which $R[x; \delta]$ is not prime radical. Note that in this case $R[x; \delta]$ must at least be locally nilpotent, by the main result of [1]; since nil rings of bounded index are PI. The following diagram describes all known implications between certain properties on $R$ and those on $R[x; \delta]$.

We give a proof of the implication “$R$ is locally nilpotent $\Rightarrow R[x; \delta]$ is Brown-McCoy radical” using a short argument of Smoktunowicz in Section 3.

No other implications in the diagram hold in general, besides those obtained by transitivity, except possibly “$R$ is prime radical $\Rightarrow R[x; \delta]$ is locally nilpotent or Jacobson radical.” The lack of any

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additional arrow follows from the construction in this paper, the example from [5] mentioned previously, and the fact there exists a commutative Jacobson radical ring $R$ which is not prime radical so in this case by [4] Theorem 1 even the non-skew polynomial ring $R[x]$ is not Brown-McCoy radical.

The existence of the example constructed in Section 2 is surprising. For instance, Klein [3] has shown that if $R$ is a nil ring of bounded index, then so is the polynomial ring $R[x]$. Thus, differential polynomial rings can display behavior much different than regular polynomial rings. There is an additional interesting aspect of the construction; while the ring $R$ we construct is all but nilpotent, we prove that the prime radical of $R[x;\delta]$ is in fact (0).

2. THE CONSTRUCTION

Let $F$ be a field and let $R = F\{a_0, a_1, \ldots\}$ be the non-unital $F$-algebra in the noncommuting variables $a_0, a_1, \ldots$ which satisfy the following relations:

(1) $\forall i \geq 0$, $a_i^2 = 0$,
(2) $\forall i, j \geq 0$, $a_i a_j + a_j a_i = 0$, and
(3) $\forall i, j, k \geq 0$, $2a_i a_j a_k = 0$.

It is easy to verify that these relations are sufficient to force $R$ to be a nil ring of bounded index 2. They are also necessary. To see this, obviously we have $a_i^2 = 0$, which gives the first relation. Next, consider $(a_i + a_j)^2 = 0$; using the first relation twice yields $a_i a_j + a_j a_i = 0$. Finally, consider

$$0 = (a_i + a_j a_k)^2 = a_i^2 + a_i a_j a_k + a_j a_i a_k + (a_j a_k)^2 = a_i a_j a_k + a_j (-a_i a_k) = a_i a_j a_k + (-a_i a_k) = 2a_i a_j a_k.$$

We also note that for any element $a \in R$, the ideal $I = \langle a \rangle$ is nilpotent, and so $R$ is Wedderburn radical.

Define a function $\delta : R \to R$ first on the variables by the rule $\delta(a_i) = a_{i+1}$ for each $i \geq 0$, then extend $F$-linearly to all of $R$ by using the product rule. The function $\delta$ respects each of the given relations, so gives a well-defined derivation on $R$. We set $S = R[x;\delta]$.

By condition (3), either all triple products are zero or the characteristic of $F$ must be 2. In the first case, $R$ is nilpotent of index 3 and hence so is $S$. Hereafter, we take $F$ to be a field of characteristic 2. By condition (2), we then see that $R$ is a commutative $F_2$-algebra.

While the next claim will follow from our proof that $S$ is not prime radical, since the proof is straightforward we include it here.

**Lemma 2.1.** Let $R$ and $\delta$ be as constructed above. The ring $S = R[x;\delta]$ is not nil of bounded index.

**Proof.** Consider the element

$$r := \sum_{i=0}^{m-1} a_{im} x^{i+1} = a_0 x + a_m x^2 + a_{2m} x^3 + \cdots + a_{(m-1)m} x^m \in S.$$

We claim that when expanding the product $r^m$, the term $s := a_0 a_{m+1} a_{2m+2} \cdots a_{(m-1)m+(m-1)} x^m$ occurs with nonzero support. Indeed, note that $r^m$ is a sum of terms of the form

$$t := a_{i_0m} x^{i_0+1} a_{i_1m} x^{i_1+1} \cdots a_{i_{m-1}m} x^{i_{m-1}+1}.$$

Thus, it suffices to write $t$ as a left polynomial (in the variable $x$), and show that $s$ is in the support of exactly one such term $t$. Clearly, after moving all $x$'s to the right, we see that $a_{i_0m}$ must occur in every term in $t$. But the subscript of $a_{i_0m}$ has remainder 0 when dividing by $m$. For this term to match $s$, we must have $i_0 = 0$. Thus we have

$$t = a_0 x a_{i_1m} x^{i_1+1} \cdots a_{i_{m-1}m} x^{i_{m-1}+1} = a_0 (a_{i_1m} x + a_{i_1m+1}) a_{i_2m} x^{i_2+1} \cdots a_{i_{m-1}m} x^{i_{m-1}+1}.$$

The terms involving $a_0 a_{i_1m}$ cannot match $s$, since $s$ does not have two variables whose subscript are both divisible by $m$. The term involving $a_0 a_{i_1m+1}$ can match $s$ only if $i_1 = 1$ (again considering remainders). So, we may take $i_1 = 1$. 

Repeating in this way, we get \( i_k = k \) for all \( 0 \leq k \leq m-1 \), and so the only term in \( r^m \) where \( s \) occurs with nonzero support is
\[
t = a_0 x a_m x^2 \cdots a_{(m-1)m} x^m,
\]
and it occurs exactly once in the expansion (again, by considering remainders on the subscripts). This shows that \( r^m \) is nonzero, since \( s \neq 0 \). Hence, \( S \) is not nil of bounded index.

Next, to show that \( S \) is not prime radical we first establish an auxiliary result.

**Lemma 2.2.** Let \( R \) and \( \delta \) be as constructed above. If \( I \) is a nonzero \( \delta \)-ideal of \( R \), then for each integer \( n \geq 0 \) there exists some integer \( m(n) > n \) for which \( a_n a_{n+1} \cdots a_{m(n)} \in I \).

**Proof.** First, since \( I \) is nonzero we can fix an element \( a \in I \) with \( a \neq 0 \). Let \( a' \) be a monomial appearing with nonzero support in \( a \), of smallest total degree. Also let \( m \geq 0 \) be maximal with respect to the variable \( a_m \) appearing in one of the terms in \( a \).

Fix \( S = \{ i \in \mathbb{N} : 0 \leq i \leq m+1 \text{ and } a_i \text{ does not occur in } a' \} \), and put \( a'' = \prod_{i \in S} a_i \). Notice that \( a'' \) shares a variable with each monomial in \( a \), other than \( a' \), by the minimality of the degree of \( a' \). Thus \[
a_0 a_1 \cdots a_{m+1} = a' a'' = aa'' \in I.
\]
This establishes the base case, taking \( m(0) = m + 1 \).

Now suppose by induction that for some \( k \geq 0 \) we have \( r := a_k a_{k+1} \cdots a_{m(k)} \in I \), with \( m(k) > k \). We may assume \( m(k) = k + 2^\ell - 1 \) for some sufficiently large \( \ell > 0 \), replacing \( r \) by the appropriate right multiple if necessary. It is an easy fact that whenever \( \delta \) is a derivation on a \( F \)-algebra where \( \text{char}(F) = p > 0 \), then \( \delta^p \) is also a derivation on that algebra for any \( \ell \in \mathbb{N} \). We thus compute
\[
\delta^p(r) = \sum_{i=0}^{2^\ell-1} (a_k \cdots a_{k+i-1}) \delta^p(a_k+i+1) a_{k+i+1} \cdots a_{m(k)}.
\]
Thus, after multiplying by \( a_{k+2^\ell+1} \cdots a_{k+2^\ell+1-1} \) only the \( i = 0 \) term is nonzero, and we get
\[
a_{k+1} a_{k+2} \cdots a_{k+2^\ell+1-1} = \delta^p(r) \cdot (a_{k+2^\ell+1} a_{k+2^\ell+2} \cdots a_{k+2^\ell+1-1}) \in I.
\]
Taking \( m(k+1) = k + 2^\ell+1 - 1 > k + 1 \) we are done.

**Proposition 2.3.** Let \( R \) and \( \delta \) be as constructed above. If \( I \) is any nonzero \( \delta \)-ideal of \( R \), then \( I \) is not nilpotent.

**Proof.** By the previous lemma, there exist elements
\[
a_0 a_1 \cdots a_{m(0)}, a_{m(0)+1} a_{m(0)+2} \cdots a_{m(m(0)+1)}, \ldots \in I.
\]
Any non-repeating finite product of these elements is nonzero.

Given an arbitrary ring \( R \), let \( W(R) \) denote the Wedderburn radical of \( R \); this is the sum of the nilpotent ideals in \( R \). If \( R \) has a derivation \( \delta \), also define \( W_\delta(R) \) to be the sum of the nilpotent \( \delta \)-ideals in \( R \). By [2], we have that \( W(R[x; \delta]) = W_\delta(R)[x; \delta] \). We are now ready to prove the main result of this paper.

**Theorem 2.4.** Let \( R \) and \( \delta \) be as constructed above. The ring \( R \) is commutative, Wedderburn radical, and nil of bounded index 2. The ring \( R[x; \delta] \) has zero prime radical.

**Proof.** We’ve already shown that \( R \) is commutative and nil of bounded index 2. Any commutative nil ring is automatically Wedderburn radical.

Now, by the previous proposition we know \( W_\delta(R) = 0 \), and so \( W(R[x; \delta]) = 0 \). Therefore there are no nilpotent ideals in \( S = R[x; \delta] \), and thus the prime radical of \( S \) is \( (0) \).
3. Locally nilpotent to Brown-McCoy radical

As promised, we present the following nice result of Smoktunowicz (provided to us through personal communication, and given here with permission).

**Proposition 3.1.** If \( R \) is a locally nilpotent ring with a derivation \( \delta \), then \( R[x; \delta] \) is Brown-McCoy radical.

**Proof.** Assume that \( R \) is locally nilpotent and suppose that for a derivation \( \delta \) the ring \( A = R[x; \delta] \) is not Brown-McCoy. Then there exists an epimorphism \( \varphi : A \to S \) where \( S \) is a simple ring with 1. Let \( f = a_0 + \cdots + a_n x^n \in A \) be such that \( \varphi(f) = 1 \). Consider the set \( X := \{a_0, \ldots, a_n\} \subseteq R \). As \( R \) is locally nilpotent we have \( X^n = (0) \) for some integer \( n \geq 2 \). As \( \varphi(f) = 1 \) we have \( \varphi(X) \neq 0 \), and so we may fix some smallest integer \( k \geq 2 \) for which \( \varphi(X^k) = (0) \). We then have

\[
0 = \varphi(X^{k-1}) \varphi(f) = \varphi(X^{k-1}),
\]

a contradiction. \( \square \)

We end with the following open questions concerning the diagram which appeared at the beginning of the paper.

**Question 3.2.** If \( R \) is a prime radical ring with a derivation \( \delta \), then is \( R[x; \delta] \) locally nilpotent (or even just Jacobson radical)?

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