DERIVATIONS AND BOUNDED NILPOTENCE INDEX

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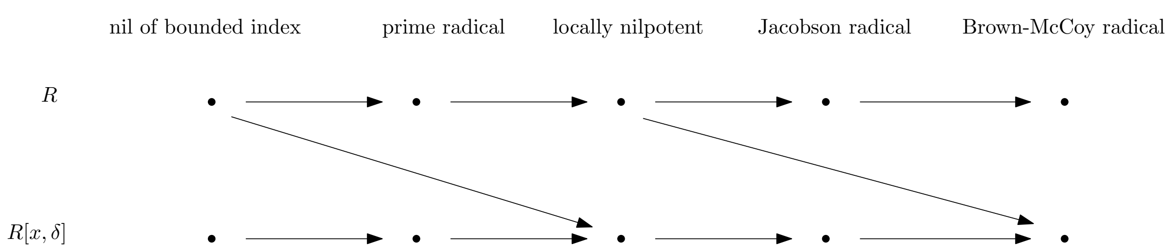
Abstract. We construct a nil ring R which has bounded index of nilpotence 2, is Wedderburn radical, and is commutative, and which also has a derivation δ for which the differential polynomial ring $R[x;\delta]$ is not even prime radical. This example gives a strong barrier to lifting certain radical properties from rings to differential polynomial rings. It also demarcates the strength of recent results about locally nilpotent PI rings.

1. Introduction

Given a ring R with a derivation δ, we can form the differential polynomial ring $R[x;\delta]$ as the set of left polynomials $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with $a_0, a_1, \ldots, a_{n-1}, a_n \in R$. Addition of such polynomials is as usual. Multiplication is also defined as usual using the associative rule, except that we need a way to convert right polynomials into left polynomials, which is accomplished by repeatedly applying the rule $xa = ax + \delta(a)$ for each $a \in R$.

Smoktunowicz and Ziembowski [5] constructed a locally nilpotent ring R with a derivation δ such that $R[x;\delta]$ is not Jacobson radical. This is in contrast to the standard polynomial ring case, where it is well known that if R is locally nilpotent, then $R[x]$ is as well. On the positive side, Bell et al. [1] recently proved that if R is a locally nilpotent PI ring, then $R[x;\delta]$ is again locally nilpotent. It is still an open question whether or not when R is a ring with no nonzero nil ideals, the Jacobson radical $J(R[x;\delta])$ is zero.

In this paper, we approach the problem of characterizing unusual behavior of the differential polynomial ring from the other end. In Section 2 we construct a nil ring R with bounded index of nilpotence 2, which is also commutative and Wedderburn radical, along with constructing an R-derivation δ for which $R[x;\delta]$ is not prime radical. Note that in this case $R[x;\delta]$ must at least be locally nilpotent, by the main result of [1], since nil rings of bounded index are PI. The following diagram describes all known implications between certain properties on R and those on $R[x;\delta]$.



We give a proof of the implication "R is locally nilpotent $\Rightarrow$ $R[x;\delta]$ is Brown-McCoy radical" using a short argument of Smoktunowicz in Section 3.

No other implications in the diagram hold in general, besides those obtained by transitivity, except possibly "R is prime radical $\Rightarrow$ $R[x;\delta]$ is locally nilpotent or Jacobson radical." The lack of any

2010 *Mathematics Subject Classification.* Primary 16N40, 16W25, Secondary 16N20, 16N60.

Key words and phrases. bounded index of nilpotence, derivation, Jacobson radical, locally nilpotent, prime radical.

additional arrow follows from the construction in this paper, the example from [5] mentioned previously, and the fact there exists a commutative Jacobson radical ring R which is not prime radical so in this case by [4, Theorem 1] even the non-skew polynomial ring $R[x]$ is not Brown-McCoy radical.

The existence of the example constructed in Section 2 is surprising. For instance, Klein [3] has shown that if R is a nil ring of bounded index, then so is the polynomial ring $R[x]$. Thus, differential polynomial rings can display behavior much different than regular polynomial rings. There is an additional interesting aspect of the construction; while the ring R we construct is all but nilpotent, we prove that the prime radical of $R[x;\delta]$ is in fact (0).

2. The Construction

Let F be a field and let $R = F\{a_0, a_1, \ldots\}$ be the non-unital F-algebra in the noncommuting variables $a_0, a_1, \ldots$ which satisfy the following relations:

(1) $\forall\ i \geq 0,\ a_i^2 = 0$,
(2) $\forall\ i, j \geq 0,\ a_i a_j + a_j a_i = 0$, and
(3) $\forall\ i, j, k \geq 0,\ 2a_i a_j a_k = 0$.

It is easy to verify that these relations are sufficient to force R to be a nil ring of bounded index 2. They are also necessary. To see this, obviously we have $a_i^2 = 0$, which gives the first relation. Next, consider $(a_i + a_j)^2 = 0$; using the first relation twice yields $a_i a_j + a_j a_i = 0$. Finally, consider

$$0 = (a_i + a_j a_k)^2 = a_i^2 + a_i a_j a_k + a_j a_k a_i + (a_j a_k)^2 = a_i a_j a_k + a_j(-a_i a_k) = a_i a_j a_k + (-a_j a_i) a_k = 2a_i a_j a_k.$$

We also note that for any element $a \in R$, the ideal $I = (a)$ is nilpotent, and so R is Wedderburn radical.

Define a function $\delta : R \to R$ first on the variables by the rule $\delta(a_i) = a_{i+1}$ for each $i \geq 0$, then extend F-linearly to all of R by using the product rule. The function δ respects each of the given relations, so gives a well-defined derivation on R. We set $S = R[x;\delta]$.

By condition (3), either all triple products are zero or the characteristic of F must be 2. In the first case, R is nilpotent of index 3 and hence so is S. Hereafter, we take F to be a field of characteristic 2. By condition (2), we then see that R is a commutative $\mathbb{F}_2$-algebra.

While the next claim will follow from our proof that S is not prime radical, since the proof is straightforward we include it here.

Lemma 2.1. *Let R and δ be as constructed above. The ring $S = R[x;\delta]$ is not nil of bounded index.*

Proof. Consider the element

$$r := \sum_{i=0}^{m-1} a_{im} x^{i+1} = a_0 x + a_m x^2 + a_{2m} x^3 + \cdots + a_{(m-1)m} x^m \in S.$$

We claim that when expanding the product r^m, the term $s := a_0 a_{m+1} a_{2m+2} \cdots a_{(m-1)m+(m-1)} x^m$ occurs with nonzero support. Indeed, note that r^m is a sum of terms of the form

$$t := a_{i_0 m} x^{i_0+1} a_{i_1 m} x^{i_1+1} \cdots a_{i_{m-1} m} x^{i_{m-1}+1}.$$

Thus, it suffices to write t as a left polynomial (in the variable x), and show that s is in the support of exactly one such term t. Clearly, after moving all x's to the right, we see that $a_{i_0 m}$ must occur in every term in t. But the subscript of $a_{i_0 m}$ has remainder 0 when dividing by m. For this term to match s, we must have $i_0 = 0$. Thus we have

$$t = a_0 x a_{i_1 m} x^{i_1+1} \cdots a_{i_{m-1} m} x^{i_{m-1}+1} = a_0 (a_{i_1 m} x + a_{i_1 m+1}) a_{i_2 m} x^{i_2+1} \cdots a_{i_{m-1} m} x^{i_{m-1}+1}.$$

The terms involving $a_0 a_{i_1 m}$ cannot match s, since s does not have two variables whose subscript are both divisible by m. The term involving $a_0 a_{i_1 m+1}$ can match s only if $i_1 = 1$ (again considering remainders). So, we may take $i_1 = 1$.

Repeating in this way, we get $i_k = k$ for all $0 \leq k \leq m-1$, and so the only term in r^m where s occurs with nonzero support is

$$t = a_0 x a_m x^2 \cdots a_{(m-1)m} x^m,$$

and it occurs exactly once in the expansion (again, by considering remainders on the subscripts). This shows that r^m is nonzero, since $s \neq 0$. Hence, S is not nil of bounded index. □

Next, to show that S is not prime radical we first establish an auxiliary result.

Lemma 2.2. *Let R and δ be as constructed above. If I is a nonzero δ-ideal of R, then for each integer $n \geq 0$ there exists some integer $m(n) > n$ for which $a_n a_{n+1} \cdots a_{m(n)} \in I$.*

Proof. First, since I is nonzero we can fix an element $a \in I$ with $a \neq 0$. Let a' be a monomial appearing with nonzero support in a, of smallest total degree. Also let $m \geq 0$ be maximal with respect to the variable a_m appearing in one of the terms in a.

Fix $S = \{i \in \mathbb{N} \ : \ 0 \leq i \leq m+1 \text{ and } \ a_i \text{ does not occur in } a'\}$, and put $a'' = \prod_{i \in S} a_i$. Notice that a'' shares a variable with each monomial in a, other than a', by the minimality of the degree of a'. Thus

$$a_0 a_1 \cdots a_{m+1} = a' a'' = a a'' \in I.$$

This establishes the base case, taking $m(0) = m+1$.

Now suppose by induction that for some $k \geq 0$ we have $r := a_k a_{k+1} \cdots a_{m(k)} \in I$, with $m(k) > k$. We may assume $m(k) = k + 2^\ell - 1$ for some sufficiently large $\ell > 0$, replacing r by the appropriate right multiple if necessary. It is an easy fact that whenever δ is a derivation on a F-algebra where $\text{char}(F) = p > 0$, then δ^{p^ℓ} is also a derivation on that algebra for any $\ell \in \mathbb{N}$. We thus compute

$$\delta^{2^\ell}(r) = \sum_{i=0}^{2^\ell - 1} (a_k \cdots a_{k+i-1}) \delta^{2^\ell}(a_{k+i})(a_{k+i+1} \cdots a_{m(k)}).$$

Thus, after multiplying by $a_{k+2^\ell+1} \cdots a_{k+2^{\ell+1}-1}$ only the $i = 0$ term is nonzero, and we get

$$a_{k+1} a_{k+2} \cdots a_{k+2^{\ell+1}-1} = \delta^{2^\ell}(r) \cdot (a_{k+2^\ell+1} a_{k+2^\ell+2} \cdots a_{k+2^{\ell+1}-1}) \in I.$$

Taking $m(k+1) = k + 2^{\ell+1} - 1 > k+1$ we are done. □

Proposition 2.3. *Let R and δ be as constructed above. If I is any nonzero δ-ideal of R, then I is not nilpotent.*

Proof. By the previous lemma, there exist elements

$$a_0 a_1 \cdots a_{m(0)}, a_{m(0)+1} a_{m(0)+2} \cdots a_{m(m(0)+1)}, \ldots \in I.$$

Any non-repeating finite product of these elements is nonzero. □

Given an arbitrary ring R, let $W(R)$ denote the Wedderburn radical of R; this is the sum of the nilpotent ideals in R. If R has a derivation δ, also define $W_\delta(R)$ to be the sum of the nilpotent δ-ideals in R. By [2], we have that $W(R[x;\delta]) = W_\delta(R)[x;\delta]$. We are now ready to prove the main result of this paper.

Theorem 2.4. *Let R and δ be as constructed above. The ring R is commutative, Wedderburn radical, and nil of bounded index 2. The ring $R[x;\delta]$ has zero prime radical.*

Proof. We've already shown that R is commutative and nil of bounded index 2. Any commutative nil ring is automatically Wedderburn radical.

Now, by the previous proposition we know $W_\delta(R) = 0$, and so $W(R[x;\delta]) = 0$. Therefore there are no nilpotent ideals in $S = R[x;\delta]$, and thus the prime radical of S is (0). □

3. Locally nilpotent to Brown-McCoy radical

As promised, we present the following nice result of Smoktunowicz (provided to us through personal communication, and given here with permission).

Proposition 3.1. *If R is a locally nilpotent ring with a derivation δ, then $R[x;\delta]$ is Brown-McCoy radical.*

Proof. Assume that R is locally nilpotent and suppose that for a derivation δ the ring $A = R[x;\delta]$ is not Brown-McCoy. Then there exists an epimorphism $\varphi : A \to S$ where S is a simple ring with 1. Let $f = a_0 + \cdots + a_n x^n \in A$ be such that $\varphi(f) = 1$. Consider the set $X := \{a_0, \ldots, a_n\} \subseteq R$. As R is locally nilpotent we have $X^n = (0)$ for some integer $n \geq 2$. As $\varphi(f) = 1$ we have $\varphi(X) \neq 0$, and so we may fix some smallest integer $k \geq 2$ for which $\varphi(X^k) = (0)$. We then have

$$0 = \varphi(X^{k-1})\varphi(f) = \varphi(X^{k-1}),$$

a contradiction. □

We end with the following open questions concerning the diagram which appeared at the beginning of the paper.

Question 3.2. If R is a prime radical ring with a derivation δ, then is $R[x;\delta]$ locally nilpotent (or even just Jacobson radical)?

4. Acknowledgements

Pace Nielsen thanks Bialystok University of Technology and Warsaw University of Technology for providing facilities for the authors during part of their collaboration. This work was partially supported by a grant from the Simons Foundation (#315828 to Pace Nielsen). Michał Ziembowski was supported by the Polish National Science Centre grant UMO-2013/09/D/ST1/03669. He also worked on this paper during a visit to Department of Mathematics, Brigham Young University, and expresses his gratitude to the University for the excellent working conditions and warm hospitality provided to him. Both authors thank the anonymous referee for comments which improved the paper.

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