BOOTSTRAPPING THE BOUNDED NILRADICAL

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Abstract. We give a new characterization of the bounded nilradical. Using the interplay between this and previous characterizations, we prove that the bounded nilradical is a graded ideal if the ring is graded. This allows us to completely determine the bounded nilradical of skew polynomial and skew Laurent polynomial rings in terms of information in the coefficient ring.

1. Basic facts

Given a ring R, we let $P(R)$ denote the *prime radical* of R. In the literature this radical also goes by the names the Baer radical, the lower nilradical, and the Baer-McCoy radical. One can define it as the intersection of all prime ideals in R, or alternatively as the smallest semiprime ideal of R. It is also characterized as the lower radical determined by the class of nilpotent ideals. More specifically, for ordinals α make the following definitions (following [7]): Define

$$N_1(R) = \sum\{I \ : \ I \text{ is a nilpotent ideal of } R\}.$$

If α is a successor ordinal, say of β, let

$$N_\alpha(R) = \{r \in R \ : r + N_\beta(R) \in N_1(R/N_\beta(R))\}.$$

If α is a limit ordinal let

$$N_\alpha(R) = \bigcup_{\beta<\alpha} N_\beta(R).$$

As proven by Levitzki, this sequence stabilizes to the prime radical $P(R)$.

We find two other well known observations of Levitzki of prime importance. First, recall that a sequence $a_0, a_1, a_2, \ldots$ in R where $a_{i+1} \in a_i R a_i$ (for each $i \geq 0$) is called an *m-sequence*. An element a is *strongly nilpotent* if each m-sequence starting with a is eventually zero. The prime radical consists of exactly the strongly nilpotent elements. Second, $P(R)$ is *locally nilpotent*, meaning that finite subsets of $P(R)$ are nilpotent. In particular it follows that $P(R)$ is a nil ideal.

We now define the central object of study in this paper, the set

$$B(R) = \{a \in R \ : \ aR \text{ is nil of bounded index}\} \tag{1}$$

which we call the *bounded nilradical*. Note that aR is nil of bounded index if and only if Ra is nil of bounded index, so the definition of $B(R)$ is left-right symmetric. Clearly $N_1(R) \subseteq B(R)$. Further, suppose $B(R) \neq 0$, so there exists a nil one-sided ideal $I \neq 0$ of bounded index of nilpotence $n \geq 2$. Fix $a \in I$, $a \neq 0$, with index of nilpotence n. A clever computation of Klein [5, Lemma 5] shows that $Ra^{n-1}R \neq 0$ is a nilpotent ideal. Since $N_1(R/P(R)) = 0$ this proves that $B(R) \subseteq P(R)$ (which was first shown by Levitzki [9]). In particular, we see that the bounded nilradical is not properly a Kurosh-Amitsur radical (in fact, $B(R/B(R))$ can be nonzero), but rather gives rise to $P(R)$ just as $N_1(R)$ does (once we know $B(R)$ is an ideal).

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How closely related are $B(R)$ and $P(R)$? Leveraging a result of Amitsur [1, Theorem 1], Klein proved that $N_1(R) \subseteq B(R) \subseteq N_2(R)$ and constructed examples showing that any combination of the containments may be proper or equalities [6]. Klein further showed that $B(R)$ has another characterization, namely as the set

$$B(R) = \{a \in R \ : \ m\text{-sequences starting with } a \text{ have bounded length}\} \tag{2}$$

thus shedding more light on the relationship between $B(R)$ and $P(R)$. In fact, the proof tells us more (even if we don't assume R has 1). If there exists an integer $n \geq 0$ so that every m-sequence starting with a has bounded length n, then aR is nil of bounded index at most $2^{n+1} - 1$. Conversely, if aR is nil of bounded index n then all m-sequences starting with a have bounded length as a function depending only on n [6, Proof of Theorem 1] (where the function has an unknown growth rate). Thus, results about strongly nilpotent elements whose m-sequences have bounded lengths translate into results about nil one-sided ideals of bounded index.

It is easy to see that $B(R)$ is closed under left and right multiplication from R. However, $B(R)$ is also closed under addition and is thus an ideal. This was first observed by Amitsur [11, Theorem 2.6.27]. His proof is not only clever but reveals another connection between the prime radical and the bounded nilradical, so we reproduce it here.

Let R be a ring and let $S = \prod_{r \in R} R$. In other words, S is the direct product of $|R|$ copies of R. We embed R into S diagonally and view R as a subring of S. Let $\alpha = (r)_{r \in R}$ be the generic R-tuple with r in the rth coordinate. If $x \in B(R)$ then $x\alpha = (xr)_{r \in R}$ is nilpotent. In fact xS is nil with the same bounded index as xR, so $x \in P(S)$. If $x \in R \setminus B(R)$ then $x\alpha$ is not nilpotent, so $x \notin P(S)$. This proves that $B(R) = R \cap P(S)$ is an ideal.

Decades later Klein improved this result by showing that if aR and bR are nil of bounded index, then $aR + bR$ is not only nil of bounded index but the bound depends only on the nilpotence index of aR and bR. (There is a way to leverage Amitsur's original proof to also imply this result.) The proof makes essential use of the fact that $B(R) \subseteq P(R)$ is locally nilpotent. There is still no easy proof, using only characterization (1), showing that $B(R) \trianglelefteq R$ (i.e., $B(R)$ is an ideal in the ring R). Furthermore, both known proofs that $B(R)$ is an ideal rely inherently on the containment $B(R) \subseteq P(R)$. If one delves into the proof of characterization (2), one finds that Klein uses the containment $B(R) \subseteq P(R)$, and hence it seems possible there may be an easy proof that $B(R)$ is an ideal simply using characterization (2). We will realize this goal after introducing a bit of notation.

Set $a^{(0)} = a$ and inductively define

$$a^{(n+1)}(x_1, x_2, \ldots, x_{n+1}) = a^{(n)}(x_1, x_2 \ldots, x_n) x_{n+1} a^{(n)}(x_1, x_2, \ldots, x_n).$$

If the elements x_i are clear from context, or arbitrary, we may drop them from the notation and simply write $a^{(n)}$. The sequence $a^{(0)}, a^{(1)}, a^{(2)}, \ldots$ is an m-sequence starting with a. We define $\sigma(a)$ to be the greatest $n \geq 0$ such that there exists an n-term m-sequence of nonzero elements starting with a (and set it to infinity if no maximum exists). Thus characterization (2) can be rephrased as

$$B(R) = \{a \in R \ : \ \sigma(a) < \infty\}.$$

Given $a \in B(R)$ and $r \in R$, an easy computation as in [6] shows that $\sigma(ra), \sigma(ar) \leq \sigma(a)$. Also, $\sigma(ara) \leq \sigma(a)$ with equality if and only if $a = 0$. We will use these facts freely in the proof below.

Proposition 1. *Given $a, b \in R$, if $\sigma(a) < \infty$ and $\sigma(b) < \infty$ then $\sigma(a + b) < \infty$. Additionally, $\sigma(a + b)$ is bounded in terms of $\sigma(a)$ and $\sigma(b)$.*

Proof. We work by induction on $n = \sigma(b)$. If $n = 0$ then $b = 0$ and we have $\sigma(a + b) = \sigma(a) < \infty$. Fix $n \geq 1$, and suppose by induction that if $c, d \in R$ with $\sigma(c) < \infty$ and $\sigma(d) < n$ then $\sigma(c + d)$ is bounded as a function of $\sigma(c)$ and $\sigma(d)$. Set $m = \sigma(a)$ and expand $A = (a + b)^{(m+1)}(x_1, x_2, \ldots, x_{m+1})$.

We have $(a+b)^{(m+1)} = (a+b)^{(m)}x_{m+1}(a+b)^{(m)}$. If we further expand $(a+b)^{(m)}$ then we see that $(a+b)^{(m)} \in RbR$ since the only term without b is $a^{(m)} = 0$. Thus $(a+b)^{(m+1)} \in RbRbR$ and we write

$$A = (a+b)^{(m+1)} = \sum_{i\in I} r_i b s_i b t_i$$

for some indexing set I, whose size is bounded as a function of m. From the fact that $\sigma(r_i b s_i b t_i) < n$ we can apply our inductive hypothesis to A, $|I|-1$ times, and we are done. □

With these results in place we mention a result that doesn't appear in the literature, but follows easily from ideas already found there.

Proposition 2. *For any ring R, we have $B(\mathbb{M}_n(R)) = \mathbb{M}_n(B(R))$.*

Proof. Let $S = \mathbb{M}_n(R)$, and let $E_{i,j}$ denote the usual matrix unit (with 1 in the (i,j)-position, and zeros elsewhere). We first show the containment $\subseteq$. Fix $A = (a_{i,j}) \in B(\mathbb{M}_n(R))$ and let $r \in R$. There exists an integer $n \geq 1$ so that $\sum_{i,j} SAE_{i,j}$ is nil of bounded index n. As $rE_{1,i}AE_{j,1} = ra_{i,j}E_{1,1}$, we have $(ra_{i,j})^n E_{1,1} = (ra_{i,j}E_{1,1})^n = 0$. Thus $a_{i,j} \in B(R)$.

For the converse, first notice that for $a \in B(R)$ the left ideal $SaE_{i,j} = \sum_{\ell=1}^{n} RaE_{\ell,j}$ is nil of bounded index at most one larger than the index for Ra. In particular, given $A = (a_{i,j}) \in \mathbb{M}_n(B(R))$ it belongs to the left ideal $\sum_{i,j} Sa_{i,j}E_{i,j} \subseteq B(S)$. Alternatively, one could follow the methods in [12]. □

2. New characterizations

What is $B(R[x])$? The answer is, unsurprisingly, $B(R)[x]$. As noted by Rowen, this follows by an easy Vandermonde matrix argument in the case when R is a faithful algebra over an infinite domain. In fact, if I is a nil ideal of bounded index $n \geq 1$ then $I[x]$ is nil of the same index in that case; more generally Klein proved for a general ring that $I[x]$ is nil of index at most $n!$, and subsequently improved this bound slightly [5, Theorem 9]. His proof involves a clever induction on n coupled with characterizing polynomial identities that must be satisfied by elements in $B(R[x])$. We rederive those polynomial identities here as they will be useful to us, but we also continue to assume the results of [5].

Recall that a *partition* of $n \geq 1$ (of *length* k) is a sequence of non-increasing positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. Given a partition $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ for n, define $P_\lambda(x_1, x_2, \ldots, x_k)$ to be the sum of all monomials in the noncommuting variables $x_1, x_2, \ldots, x_k$ where x_m occurs exactly λ_m times. For example,

$$P_{\{3,2\}}(x,y) = x^3y^2 + x^2yxy + x^2y^2x + xyx^2y + xyxyx + xy^2x^2 + yx^3y + yx^2yx + yxyx^2 + y^2x^3.$$

These polynomials are intricately tied to the coefficients of powers of polynomials.

Lemma 3. *If $g(x) = \sum_{j=0}^{m} r_j x^j \in R[x]$ is a polynomial and $n \geq 1$ is an integer then*

$$g(x)^n = \sum_{\ell=0}^{mn} \left(\sum_{\text{partitions } \lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_k\} \text{ of } n} \left(\sum_{\substack{J=\{j_1,j_2,\ldots,j_k\}\subseteq\{0,1,\ldots,m\},\ |J|=k, \\ \lambda_1 j_1+\lambda_2 j_2+\cdots+\lambda_k j_k=\ell, \\ \text{if } \lambda_p=\lambda_{p+1} \text{ then } j_p<j_{p+1}}} P_\lambda(r_{j_1}, r_{j_2}, \ldots, r_{j_k}) \right) \right) x^\ell.$$

The proof of the previous lemma is left to the reader as it is straightforward. Note that the condition "if $\lambda_p = \lambda_{p+1}$ then $j_p < j_{p+1}$" is present merely to prevent over-counting terms.

Lemma 4. *Let $n \geq 1$ and let λ be a partition of n of length k. Given $\{r_1, r_2, \ldots, r_k\} \subseteq R$ and $a \in R$ then there exists a polynomial $f(x) \in R[x]$ so that $P_\lambda(ar_1, ar_2, \ldots, ar_k)$ is a coefficient of $(af(x))^n$.*

Proof. First we introduce a bit of notation. Given a polynomial $g(x) \in R[x]$ write $g(x)[t]$ for the coefficient of x^t in $g(x)$.

Let $n \geq 1$, let $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a partition of n, and fix elements $\{r_1, r_2, \dots, r_k\} \subseteq R$. Set $f(x) = \sum_{j=1}^{k} r_j x^{N^j}$ where $N > n$. As λ is a partition for n, notice in particular that each component satisfies $\lambda_p \leq n < N$. For any reordering $J = \{j_1, j_2, \dots, j_k\}$ of $\{N, N^2, \dots, N^k\}$, the quantity $\lambda_1 j_1 + \lambda_2 j_2 + \cdots + \lambda_k j_k = \ell$ uniquely determines both the ordered set J and the partition λ, since these two sets are exactly the information needed to write ℓ as an integer in base-N notation. Hence, by the previous lemma, if we fix $\ell = \sum_{p=1}^{k} \lambda_p N^p$ then $(af(x))^n[\ell] = P_\lambda(ar_1, ar_2, \dots, ar_k)$. □

Theorem 5. *Given a ring R,*

$$B(R) = \{a \ : \ \exists\ n \geq 1,\ \forall \text{ partition } \lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\} \text{ of } n,\ aR \text{ satisfies } P_\lambda(x_1, x_2, \dots, x_k)\}. \tag{3}$$

Proof. The containment $\supseteq$ is clear, taking the trivial partition $\lambda = \{n\}$. Conversely, given $a \in B(R)$ let $n \geq 1$ be the index of nilpotence for $aR[x]$, which exists by [5]. The previous lemma then says that $P_\lambda(ar_1, ar_2, \dots, ar_k) = 0$ for every choice of partition $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ of n and every choice of elements $r_1, r_2, \dots, r_k \in R$. □

Characterization (3) is exactly what is needed to guarantee $aR[x]$ is nil of bounded index $n \geq 1$. One may now wonder what happens if we try to repeat this process. If we guarantee that $aR[x]$ satisfies (3), do we get even more conditions on elements in R? The answer is no, as this reduces to forcing $aR[x, y]$ to be nil of bounded index. The analysis of the case of two variables is exactly the same as the case of one variable.

However, there are still surprising ways in which characterizations (1) and (3) interact. We now use this interaction to bootstrap Klein's result about polynomials to say a little more about the bounded nilradical.

Theorem 6. *If R is a $\mathbb{Z}$-graded ring then $B(R)$ is a $\mathbb{Z}$-graded ideal.*

Proof. Given $a \in B(R)$ if $a = 0$ there is nothing to show. So assume $a \neq 0$ and write $a = \sum_{j \in \mathbb{Z},\ s \leq j \leq t} a_j$ where a_j belongs to grade j, and $a_s, a_t \neq 0$ are the components in the smallest and largest grades, respectively. We know that $B(R)$ is an ideal, and so if we can show that $a_t \in B(R)$ then $a - a_t \in B(R)$ has fewer graded components. Thus, repeating the argument finishes the theorem.

We now show $a_t \in B(R)$. By (3), fix $n \geq 1$ so that for every partition $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ of n we know aR satisfies $P_\lambda(x_1, x_2, \dots, x_k)$. Let $r \in R$ be arbitrary. We can write $r = \sum_{j=\ell}^{m} r_j$, where r_j lives in grade j. We can think of r as the polynomial $\sum_{j=\ell}^{m} r_j x^{j-\ell}$ specialized to $x = 1$. Thus Lemma 3 implies

$$(a_t r)^n \in \sum_{\substack{\text{partitions } \lambda \text{ of } n \text{ of length } k \\ \{j_1, j_2, \dots, j_k\} \subseteq \mathbb{Z}}} P_\lambda(a_t r_{j_1}, a_t r_{j_2}, \dots, a_t r_{j_k}) R.$$

We claim that each summand above is zero. To see this, note that if we expand the left-hand side of the equality

$$P_\lambda(ar_{j_1}, ar_{j_2}, \dots, ar_{j_k}) = 0$$

the term of largest grade is $P_\lambda(a_t r_{j_1}, a_t r_{j_2}, \dots, a_t r_{j_k})$, and hence must be zero. Thus $(a_t r)^n = 0$ for all $r \in R$, and therefore $a_t \in B(R)$ by (1), which finishes the theorem. □

We end this section with two further interesting observations. First, recall that we obtained characterization (3) by viewing (1) over polynomial rings. Similarly one can view (2) over polynomial rings and come up with the new characterization for $B(R)$. This characterization is complicated to state and unwieldy, but may be of independent interest. To begin, one views $P = a^{(n)}(x_1, x_2, \dots, x_n)$ as a polynomial in the variables $x_1, x_2, \dots, x_n$. The variable x_j occurs 2^{n-j} times in P. Given a partition λ_j of 2^{n-j} let k_j be the length of λ_j and let $X_j = \{x_{j,1}, x_{j,2}, \dots, x_{j,k_j}\}$ be a set of noncommuting variables.

We transform the polynomial $P = a^{(n)}(x_1, x_2, \ldots, x_n)$ into a new polynomial Q which is the sum over all possible replacements of the 2^{n-j} copies of x_j in P by the variables in X_j with respective frequencies given by λ_j, for all j simultaneously. The new condition is then that there is some $n \geq 1$ so that for all collections of partitions as above, R satisfies Q.

Second, given $a \in B(R)$ suppose we define $\tilde{\sigma}(a)$ to be the minimal integer n satisfying characterization (3). Just as in Proposition 1, given $a, b \in B(R)$ we have $\tilde{\sigma}(a+b)$ is bounded as a function of $\tilde{\sigma}(a)$ and $\tilde{\sigma}(b)$. Furthermore, $\tilde{\sigma}(a)$ is bounded as a function of the maximum index of nilpotence for an element in aR (and conversely).

3. Skew polynomial and Laurent polynomial rings

Given an automorphism σ on R, we let $R[x;\sigma]$ denote the skew polynomial ring subject to the skewing condition $xr = \sigma(r)x$. Similarly, we let $R[x, x^{-1};\sigma]$ denote the skew Laurent polynomial ring. Many of the usual zero-divisor conditions have been successfully generalized to the skew setting. For example, following the literature one says an element $a \in R$ is *σ-nilpotent* if for every $l \geq 1$ (called the *power variable*) there exists $n \geq 1$ (called the *index*, which can depend on l) such that

$$a\sigma^l(a)\sigma^{2l}(a)\cdots\sigma^{(n-1)l}(a) = 0.$$

This notion was introduced in [4] and has been used effectively to study radical properties in skew rings (see [2] and [8]). We also know $P(R[x;\sigma]) = (P(R) \cap P_\sigma(R)) + xP_\sigma(R)[x;\sigma]$, where $P_\sigma(R) \leq R$ is the *σ-prime radical* of Pearson and Stephenson [10].

In [3], the bounded nilradical was generalized to the following set

$$B_\sigma(R) = \{a \ : \ aR \text{ is a bounded } \sigma\text{-nil right ideal}\},$$

where a subset of R is *bounded σ-nil* if every element is σ-nilpotent with the same index $n \geq 1$ (with the index independent of the power variable $l \geq 1$). It was further shown that

$$B(R[x;\sigma]) \subseteq (B(R) \cap B_\sigma(R)) + xB_\sigma(R)[x;\sigma].$$

However, examples were constructed showing that the containment above is proper in general. Furthermore, it is not known if $B_\sigma(R)$ is even closed under addition. Thus, $B_\sigma(R)$ somehow does not generalize the bounded nilradical in the right way, unless one assumes special properties for R or σ.

Using Theorem 6 we characterize the bounded nilradical of $R[x;\sigma]$ completely. In the process, we arrive at the correct generalization of $B(R)$ to the skew polynomial case. We do this in a two steps.

Proposition 7. *Given a ring R, there exists an ideal $I \leq R$ so that $B(R[x;\sigma]) = (B(R) \cap I) + xI[x;\sigma]$.*

Proof. As $R[x;\sigma]$ is a graded ring, graded by degree, by Theorem 6 we know $B(R[x;\sigma])$ is a graded ideal. Thus, we can write $B(R[x;\sigma]) = I_0 + I_1x + I_2x^2 + \cdots$ where I_k is a subset of R. Clearly each I_k is a σ-invariant ideal of R as σ extends to an automorphism of $R[x;\sigma]$ and thus fixes $B(R[x;\sigma])$. Further, $I_0 \subseteq I_1 \subseteq I_2 \subseteq \ldots$ as $B(R[x;\sigma])$ is closed under multiplication by x.

Suppose that $a \in I_k$ for some $k \geq 1$, so that $ax^k \in B(R[x;\sigma])$. We then see that $(axR[x;\sigma])^k \subseteq ax^kR[x;\sigma]$ is nil of bounded index, hence so is $axR[x;\sigma]$. Thus $ax \in B(R[x;\sigma])$ and therefore $a \in I_1$. This proves that $I_1 = I_2 = \cdots$.

We already have $I_0 \subseteq I_1$. Further, the containment $I_0 \subseteq B(R)$ follows by considering what happens to constant polynomials. Conversely, fix $a \in I_1 \cap B(R)$. We will prove that $a \in I_0$, or in other words given $f(x) \in R[x;\sigma]$ we show that $af(x)$ is nilpotent with index depending only on a. As aR is nil of some bounded index $n_1 \geq 1$, we know that $(af(x))^{n_1}$ has zero constant term. Thus $(af(x))^{n_1} \in axR[x;\sigma]$, which has bounded index $n_2 \geq 1$. This implies that $af(x)$ is nilpotent of index at most $n = n_1 + n_2$. Taking $I = I_1$ we are done. □

All that remains is to characterize the ideal I above. To do that, we apply the process used to obtain characterization (3). Let $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a partition for $n \geq 1$, and let $\ell = \{\ell_1, \ell_2, \dots, \ell_k\}$ be a sequence of integers. Let $P_\lambda^\ell(x_1, x_2, \dots, x_k)$ be the sum of all monomials in the noncommuting variables $x_1, x_2, \dots, x_k$ occurring with degrees λ, as before, but with $x_{j_1}x_{j_2}\cdots x_{j_n}$ replaced by

$$x_{j_1}\sigma^{\ell_{j_1}}(x_{j_2})\sigma^{\ell_{j_1}+\ell_{j_2}}(x_{j_3})\cdots\sigma^{\sum_{t=1}^{n-1}\ell_{j_t}}(x_{j_n}).$$

For example $P_{\{2,1\}}(x,y) = x^2y + xyx + yx^2$ while

$$P_{\{2,1\}}^{\{1,5\}}(x,y) = x\sigma(x)\sigma^2(y) + x\sigma(y)\sigma^6(x) + y\sigma^5(x)\sigma^6(x).$$

Theorem 8. *Let J be the set of elements $a \in R$ where there exists an integer $n \geq 1$, so that for every partition $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ of n and for every sequence of positive integers $\ell = \{\ell_1, \ell_2, \dots, \ell_k\}$ the right ideal aR satisfies $P_\lambda^\ell(x_1, x_2, \dots, x_k)$. We have*

$$B(R[x;\sigma]) = (B(R) \cap J) + xJ[x;\sigma]$$

and J is an ideal of R.

Proof. Let $I \leq R$ be the ideal from the previous proposition, so $B(R[x;\sigma]) = (B(R)\cap I) + xI[x;\sigma]$. We wish to prove $I = J$. We begin with the inclusion $\supseteq$. Fix $a \in J$. Given $f(x) \in R[x;\sigma]$ we can write $axf(x) = \sum_{j=0}^{s} ar_jx^{j+1}$ for some elements $r_j \in R$. We compute

$$(axf(x))^n \in \sum_{\substack{\text{partitions } \lambda=\{\lambda_1,\lambda_2,\dots,\lambda_k\} \text{ of } n\\ \{j_1,j_2,\dots,j_k\}\subseteq\{0,1,\dots,s\}}} P_\lambda^{\{j_1+1,j_2+1,\dots,j_k+1\}}(ar_{j_1}, ar_{j_2}, \dots, ar_{j_k})R[x;\sigma].$$

As $a \in J$, there is some $n \geq 0$ (independent of $f(x)$) so that each summand is zero. Thus $axR[x;\sigma]$ has bounded index of nilpotence, and hence $a \in I$.

Conversely, let $a \in I$. Fix $n \geq 1$ so that $axR[x;\sigma]$ satisfies characterization (3) of the bounded nilradical. Let $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a partition of n, let $\ell = \{\ell_1, \ell_2, \dots, \ell_k\}$ be a sequence of positive integers, and let $\{r_1, r_2, \dots, r_k\} \subseteq R$. As (3) holds, using the elements $x_i = ar_ix^{\ell_i} \in axR[x;\sigma]$ (for $1 \leq i \leq k$) we have

$$0 = P_\lambda(ar_1x^{\ell_1}, ar_2x^{\ell_2}, \dots, ar_kx^{\ell_k}) = \left(P_\lambda^\ell(ar_1, ar_2, \dots, ar_k)\right)x^{\lambda_1\ell_1+\lambda_2\ell_2+\cdots+\lambda_k\ell_k}.$$

Thus $a \in J$. ☐

Theorem 9. *Let J be the set of elements $a \in R$ where there exists $n \geq 1$, so that for every partition $\lambda = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ of n and for every sequence of integers $\ell = \{\ell_1, \ell_2, \dots, \ell_k\}$ the right ideal aR satisfies $P_\lambda^\ell(x_1, x_2, \dots, x_k)$. We have*

$$B(R[x, x^{-1};\sigma]) = J[x;\sigma]$$

and J is an ideal of R.

Proof. Use the same methods as before, but notice that now one can multiply by negative powers of x so that every grade in the graded ideal $B(R[x, x^{-1};\sigma])$ is equal. The rest follows by the previous proofs, *mutatis mutandis.* ☐

4. Alternate directions

Many of the techniques in this paper apply broadly to radicals and other ideal-theoretic formations. In particular, we can characterize some graded radicals (such as the Levitski radical) in skew polynomial rings. In a forthcoming work, these facts and techniques are generalized further.

There are many other constructions for which one can study the bounded nilradical. For example:

Question: If G is a group, can we describe $B(R[G])$ in a simple manner?

If G is an ordered group many of the techniques above should be applicable.

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