ON σ-NIL IDEALS OF BOUNDED INDEX OF σ-NILPOTENCE

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ABSTRACT. We investigate properties of σ-nil subsets with bounded index of σ-nilpotence, beginning with a classification of many of the possible types of nilpotence available in the σ-skewed case. In the process we introduce a σ-analog of the bounded nilradical of a ring. In many situations we completely describe the bounded nilradical for skew polynomial rings in terms of the σ-bounded nilradical in the coefficient ring. We also construct an example demonstrating that the bounded nilradical for skew polynomial rings is more complicated than one might initially guess. Further extensions are explored when one assumes additional restrictions on σ.

INTRODUCTION

Throughout this paper every ring is an associative ring with identity. We let R be a ring and σ be an automorphism of R. We use $R[x;\sigma]$ to denote the skew polynomial ring with an indeterminate x over R, subject to the skew relation $xr = \sigma(r)x$ for $r \in R$.

It is a common theme to study and classify the different radicals of $R[x;\sigma]$ via σ-ideals and natural radical ideals in R. Pearson and Stephenson [16] prove that $P(R[x;\sigma]) = (P(R) \cap P_\sigma(R)) + P_\sigma(R)xR[x;\sigma]$, where $P(R)$ and $P_\sigma(R)$ are the prime and σ-prime radicals of R. For the Jacobson radical of a skew polynomial ring, Bedi and Ram [1] show that $J(R[x;\sigma]) = (I \cap J(R)) + IxR[x;\sigma]$, and moreover, if σ is locally of finite order then $J(R[x;\sigma]) = I[x;\sigma]$, where $I = \{r \in R \mid rx \in J(R[x;\sigma])\}$. Pearson, Stephenson and Watters [17] introduce other new radicals such as the σ-nil radical, the σ-Jacobson radical, the σ-Kleinfeld radical, and the σ-Brown-McCoy radical, and show that $R[x;\sigma]$ is a σ-Jacobson ring if and only if R is a σ-Jacobson ring.

Recent work focuses on the study of prime ideals, the prime radical, and other radical properties in a general Ore extension; see for instance [5], [13], and [8]. In a different direction, Hong, et al. [7, Definition 1.1] define the σ-Wedderburn radical and the σ-Levitzki radical of rings. Using properties of these radicals, they study the Wedderburn radical, the Levitzki radical, and the upper nil radical of $R[x;\sigma]$ and $R[x,x^{-1};\sigma]$. Cheon, et al. [4] give an alternate element-wise characterization of the elements in the prime radical for skew polynomial rings.

In this paper we focus on another ideal of R. A result of Amitsur [19, Theorem 2.6.27] proves that the set $B(R)$ of all elements of R that generate one-sided ideals which are nil of bounded index is an ideal, so $B(R)$ is also the sum of all nil one-sided ideals of R of bounded index. We call $B(R)$ the bounded nilradical of R. In [9] Klein proves that $B(R)$ coincides with the set of all strongly nilpotent elements of R of bounded index, and places $B(R)$ relative to the higher Wedderburn radicals via the containments $W_1(R) \subseteq B(R) \subseteq W_2(R)$. Moreover, he proves that the sum of a finite number of nil left ideals of bounded index has bounded index, providing a new proof that $B(R)$ is an ideal.

In this paper we introduce the concept of a σ-nil ideal of R of bounded index of σ-nilpotence, and we define $B_\sigma(R)$ as the set of all elements which generate bounded σ-nil right ideals. We study these concepts, and partially characterize the structure of $B(R[x;\sigma])$. We also generalize many of Klein's results to the skew polynomial situation.

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1. σ-NILPOTENCE AND RELATED CONDITIONS

An element $a \in R$ is said to be *σ-nilpotent* if for each integer $l \geq 1$, there exists an integer $m \geq 1$ (possibly depending on l), such that

$$a\sigma^l(a)\sigma^{2l}(a)\cdots\sigma^{(m-1)l}(a) = 0.$$

Equivalently, the elements $ax^l \in R[x;\sigma]$ are nilpotent, for each $l \geq 1$. When σ is the identity function then we recapture the the usual definition of a being nilpotent.

One might notice that this definition is not left-right symmetric and ask why we do not call this *left σ-nilpotence*. The reason is that it is similarly customary to speak of *the* ring $R[x;\sigma]$ of skew polynomials, even though it is technically the ring of σ-skew left-polynomials (that is, polynomials with the coefficients written on the left). We leave it to the reader to add "left" if they so desire.

We let $\mathrm{Nil}(R)$ be the set of all nilpotent elements of R, and we let $\mathrm{Nil}_\sigma(R)$ be the set of all σ-nilpotent elements of R. It is important to note that these are only sets in general, and not usually ideals. Just as with nilpotence, there are a number of ways of relativizing the notion of σ-nilpotence to subsets.

Definition 1.1. Let $S \subseteq R$ be an arbitrary subset of R. Generalizing some standard properties found in the literature, we say that S is

$$\left.\begin{array}{l} \textit{σ-nil} \\ \textit{locally σ-nilpotent} \\ \textit{σ-nilpotent} \end{array}\right\} \text{ if } \forall\, l \geq 1 \text{ and } \forall\, T \text{ with } T \left\{\begin{array}{c} \text{a singleton subset of } S \\ \text{a finite subset of } S \\ \text{an arbitrary subset of } S \end{array}\right\}$$

then $\exists\, m \geq 1$ (possibly depending on l and T) such that $T\sigma^l(T)\sigma^{2l}(T)\cdots\sigma^{(m-1)l}(T) = 0$.

Once again, if $\sigma = \mathrm{id}$ then these properties collapse to their usual meanings. We leave it as an exercise to see that limiting T to countable subsets of S is equivalent to no restriction on T. However, we note that there are conditions which fit between the finite and countable case.

Definition 1.2. We say that S is *left σ-T-nilpotent* if for each integer $l \geq 1$ and any countable sequence $\{a_0, a_1, a_2, \ldots\}$ from S there is an integer $m \geq 1$ such that $a_0\sigma^l(a_1)\sigma^{2l}(a_2)\cdots\sigma^{(m-1)l}(a_{m-1}) = 0$. We say that S is *right σ-T-nilpotent* if for each integer $l \geq 1$ and any countable sequence $\{a_0, a_1, a_2, \ldots\}$ from S there is an integer $m \geq 1$ such that $a_{m-1}\sigma^l(a_{m-2})\cdots\sigma^{(m-1)l}(a_0) = 0$. When a subset is both left and right σ-T-nilpotent, we say it is *σ-T-nilpotent*.

In each of these definitions, the integer m may depend upon the integer $l \geq 1$, the subset T (or the countable sequence from S, in the case of the σ-T-nilpotent properties), or both. We call m the index of σ-nilpotence. Rearranging the quantified variables allows us to create new conditions. If for any of the properties $\mathcal{P}$ above the power m can be chosen so that it does not depend on ℓ, but may rely on T (or the countable sequence) then we say that S is *power-bounded* $\mathcal{P}$. On the other hand, if m can be chosen so it does not depend on T (or the countable sequence), but may rely on l, then we say S is *set-bounded* $\mathcal{P}$. If m is independent of the other two quantified variables, we simply say S is *bounded* $\mathcal{P}$, or S is $\mathcal{P}$ *of bounded index of σ-nilpotence*. When needed, we refer to the original properties as the "standard" ones. Further, if we ever take $\sigma = \mathrm{id}$ then we drop σ from the name of the properties.

There are some immediate implications among these properties. If $\mathcal{P}$ is one of the standard properties above, then we have

$$(1) \qquad \begin{array}{ccc} \text{bounded } \mathcal{P} & \Longrightarrow & \text{set-bounded } \mathcal{P} \\ \Downarrow & & \Downarrow \\ \text{power-bounded } \mathcal{P} & \Longrightarrow & \text{standard } \mathcal{P}. \end{array}$$

Similarly, if Q is an element of {bounded, set-bounded, power-bounded, standard}, then we have the implications:

$$Q\ \sigma\text{-nilpotent} \Rightarrow Q\ \sigma\text{-T-nilpotent} \Rightarrow Q\ \text{one-sided}\ \sigma\text{-T-nilpotent} \Rightarrow Q\ \text{locally}\ \sigma\text{-nilpotent} \Rightarrow Q\ \sigma\text{-nil}.$$

There are further implications among these properties. First note that any σ-nilpotent set S is set-bounded since we can take $T = S$ in the definition. On the other hand, if S is set-bounded locally σ-nilpotent with index of σ-nilpotence m (depending only on l) then S will be σ-nilpotent, using the same index of σ-nilpotence. To see this, once l is fixed, take for your finite set any collection of m elements from S.

Definition 1.3. We will be concerned with subsets $S \subseteq R$ which behave well with respect to the action of σ. Following the literature, we say that S is *σ-stable* if $\sigma(S) \subseteq S$, and that S is *σ-invariant* if $\sigma(S) = S$. Tautologically, σ-invariant sets are σ-stable. In the literature, when S is a σ-stable (left) ideal one often abbreviates by saying S is a (left) σ-ideal.

Lemma 1.4. *If S is σ-stable and σ-nilpotent, then S is power-bounded σ-nilpotent.*

Proof. Assume there exists an integer $m \geq 1$ such that $S\sigma(S)\cdots\sigma^{m-1}(S) = 0$. As $\sigma^k(S) \subseteq S$ for any $k \geq 0$ we have $S\sigma(\sigma^{k_1}(S))\sigma^2(\sigma^{k_2}(S))\cdots\sigma^{m-1}(\sigma^{k_{m-1}}(S)) = 0$ for any non-negative integers $k_1, k_2, \ldots, k_{m-1}$. Given $l \geq 1$ we let $k_i = il - i$ for each i in the range $1 \leq i \leq m-1$ and we get

$$S\sigma^l(S)\sigma^{2l}(S)\cdots\sigma^{(m-1)l}(S) = 0.$$

□

Corollary 1.5. *If S is σ-stable then the following are equivalent:*

(1) *S is σ-nilpotent,*
(2) *S is set-bounded σ-nilpotent,*
(3) *S is power-bounded σ-nilpotent,*
(4) *S is bounded σ-nilpotent,*
(5) *S is set-bounded (one-sided) σ-T-nilpotent.*
(6) *S is bounded (one-sided) σ-T-nilpotent,*
(7) *S is set-bounded locally σ-nilpotent,*
(8) *S is bounded locally σ-nilpotent.*

When σ is the identity automorphism the horizontal arrows in Diagram (1) become reversible, since the power variable l no longer matters. Furthermore, in the case that S is a one-sided ideal and σ is the identity, there is exactly one further implication among these properties not yet mentioned. Namely, it is well known that a left ideal which is nil of bounded index of nilpotence is locally nilpotent, and in fact lives in the lower nilradical (see for example [2] and [12, Exercise 10.13]).

For the rest of this paper we will concern ourselves with the situation when S is bounded σ-nil, possibly in the case when S is also a σ-invariant right ideal.

2. Adding bounded σ-nil ideals

In the rest of the paper we will need some well-known zero-divisor conditions, and new generalizations. We introduce them now.

Definition 2.1. Following the literature, a ring R is called *semicommutative*, or *zero-insertive*, if for $a, b \in R$, $ab = 0$ implies $aRb = 0$. Generalizing, we say R is *nilpotent zero-insertive* (NZI) if for $a, b \in R$ with $ab = 0$ then $arb = 0$ whenever r is nilpotent. Similarly, R is *σ-nilpotent zero-insertive* (σ-NZI) if for $a, b \in R$ with $ab = 0$ then $arb = 0$ whenever r is σ-nilpotent. Finally, R is *σ-skew σ-nilpotent zero-insertive* (σ-skew σ-NZI) if for $a, b \in R$ with $ab = 0$ and $r \in R$ an element which is σ-nilpotent, then $ar\sigma^l(b) = 0$ for all $l \geq 1$.

We will focus on these last two conditions, and how the σ-nilpotence properties behave in their presence. As the two conditions lead to dissimilar techniques, we will focus on one at a time.

2.1. σ-skew σ-NZI rings.

We begin by finding a nice class of rings which are σ-skew σ-NZI.

Definition 2.2. Following the literature, a ring R with an automorphism σ is said to be *σ-skew Armendariz* if for polynomials $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma]$ with $f(x)g(x) = 0$ then $a_i\sigma^i(b_j) = 0$ for each pair of indices (i, j).

Lemma 2.3. *If R is σ-skew Armendariz then R is σ-skew σ-NZI.*

Proof. Let $a, b \in R$ with $ab = 0$, and let $l \geq 1$. Also suppose $r \in R$ is σ-nilpotent, say

$$r\sigma^l(r)\cdots\sigma^{(m-1)l}(r) = 0.$$

If $f(x) = 1 - rx^l$ and $g(x) = 1 + rx^l + r\sigma^l(r)x^{2l} + \cdots + r\sigma^l(r)\cdots\sigma^{(m-2)l}(r)x^{(m-1)l}$ then $f(x)g(x) = 1$. Thus $(af(x))(g(x)b) = 0$. From the σ-skew Armendariz condition, using the constant term from $af(x)$ and the $rx^\ell b$-term from $g(x)b$, we have $ar\sigma^l(b) = 0$. □

Proposition 2.4. *Let R be a σ-skew σ-NZI ring, and let $S \subseteq R$. If S is σ-nilpotent, then there exists an integer $m \geq 1$ so that*

$$S\sigma^{i_1}(S)\sigma^{i_2}(S)\cdots\sigma^{i_{m-1}}(S) = 0$$

for any sequence of integers $1 \leq i_1 < i_2 < \ldots < i_{m-1}$. Thus S is power-bounded. In particular, any σ-nilpotent element in R is power-bounded σ-nilpotent.

Proof. Fix $k \geq 1$ so that

$$s_0\sigma(s_2)\sigma^2(s_4)\cdots\sigma^{k-1}(s_{2k-2}) = 0 \tag{2}$$

for any choice of k elements $s_0, s_2, s_4 \ldots, s_{2k-2} \in S$. We let $s_1, s_3, \ldots, s_{2k-3} \in S$ also be arbitrary. Set $m = 2k - 1$ and let $1 \leq i_1 < i_2 < \ldots < i_{m-1}$ be a sequence of integers. We proceed by amplifying Equation (2), by inserting terms in the correct order. From the σ-skew σ-NZI property, since $\sigma^{i_1}(s_1)$ is σ-nilpotent, we can insert it to obtain

$$0 = \Big(s_0\Big)\sigma^{i_1}(s_1)\sigma^{i_2-1}\Big(\sigma(s_2)\sigma^2(s_4)\cdots\sigma^{k-1}(s_{2k-2})\Big) = s_0\sigma^{i_1}(s_1)\sigma^{i_2}(s_2)\sigma^{i_2+1}(s_4)\cdots\sigma^{i_2+k-2}(s_{2k-2}).$$

Next, we can insert $\sigma^{i_3}(s_3)$ to obtain

$$\begin{aligned} 0 &= \Big(s_0\sigma^{i_1}(s_1)\sigma^{i_2}(s_2)\Big)\sigma^{i_3}(s_3)\sigma^{i_4-i_2-1}\Big(\sigma^{i_2+1}(s_4)\sigma^{i_2+2}(s_6)\cdots\sigma^{i_2+k-2}(s_{2k-2})\Big) \\ &= s_0\sigma^{i_1}(s_1)\sigma^{i_2}(s_2)\sigma^{i_3}(s_3)\sigma^{i_4}(s_4)\sigma^{i_4+1}(s_6)\cdots\sigma^{i_4+k-3}(s_{2k-2}). \end{aligned}$$

Continuing in this fashion, we obtain

$$s_0\sigma^{i_1}(s_1)\sigma^{i_2}(s_2)\cdots\sigma^{i_{2k-2}}(s_{2k-2}) = 0$$

which establishes the claimed result. □

It is interesting to note that one can relax the conditions on the exponents. The only necessary conditions are that $i_{2n} - i_{2n-2} \geq 2$ (where $i_0 = 0$). In particular, the exponents with odd subscripts can be chosen arbitrarily. As the proof technique in the previous proposition will be used repeatedly, we will refer to the technique as "amplifying equations" using the σ-skew σ-NZI property.

Proposition 2.5. *Let R be a σ-skew σ-NZI ring. If $S, T \subseteq R$ are bounded σ-nilpotent subsets then so is $S+T$, with index of σ-nilpotence determined by the index of σ-nilpotence for S and T. Similarly, if S and T are bounded σ-nil then so is $S+T$, with index of σ-nilpotence determined by the index for S and T.*

Proof. Using the previous proposition, there is some $k \geq 1$ so that for each $l \geq 1$ we have

$$S\sigma^l(S)\cdots\sigma^{(k-1)l}(S) = 0 = T\sigma^l(T)\cdots\sigma^{(k-1)l}(T). \tag{3}$$

Set $m = 2k-1$. Fix elements $a_0, a_1, a_2, \ldots, a_{m-1} \in S$ and $b_0, b_1, b_2, \ldots, b_{m-1} \in T$. We claim that

$$P = (a_0+b_0)\sigma^l(a_1+b_1)\sigma^{2l}(a_2+b_2)\cdots\sigma^{(m-1)l}(a_{m-1}+b_{m-1}) = 0.$$

If one expands P, a typical term is of the form

$$Q = c_0\sigma^l(c_1)\sigma^{2l}(c_2)\cdots\sigma^{(m-1)l}(c_{m-1})$$

where $c_j \in \{a_j, b_j\}$ for each $j \geq 1$. As $m = 2k-1$, at least k of the c_j's either come from S or come from T. But then one sees that Q is an amplification of an instantiation of one of the two sides of (3). (We may also need to hit that amplification with a power of σ, and then multiply on the left and the right by extra terms.) In particular, $Q = 0$. As Q was an arbitrary term, $P = 0$.

In the case when S and T are just bounded σ-nil, the above proof works by just setting all of the a's equal, and all the b's equal. □

Corollary 2.6. *Let R be a σ-skew σ-NZI ring. If a and b are σ-nilpotent then so is $a+b$.*

Lemma 2.7. *Let R be a σ-skew σ-NZI ring and let $a \in R$. If aR is σ-nil then it is bounded σ-nilpotent.*

Proof. By Proposition 2.4, there is some $k \geq 1$ so that for each $l \geq 1$ we have

$$a\sigma^l(a)\sigma^{2l}(a)\cdots\sigma^{(k-1)l}(a) = 0.$$

Amplifying this equation, using the fact that $r\sigma^j(a)s$ is σ-nilpotent for any $r, s \in R$ and $j \in \mathbb{Z}$, we have

$$0 = a(r_1\sigma^l(a)r_2)\sigma^{2l}(a)(r_3\sigma^{3l}(a)r_4)\sigma^{4l}(a)\cdots\sigma^{(2k-2)l}(a).$$

Hence

$$aR\sigma^l(aR)\sigma^{2l}(aR)\cdots\sigma^{(2k-2)l}(aR) = 0$$

for any $l \geq 1$, and thus $2k-1$ bounds the index of σ-nilpotence. □

Proposition 2.8. *Let R be a σ-skew σ-NZI ring. If $I \leq R$ is a σ-nil right ideal then it is power-bounded locally σ-nilpotent. If I is finitely generated, it is bounded σ-nilpotent.*

Proof. It suffices to show the last claim. Suppose that $\{a_1, a_2, \ldots, a_n\}$ is a generating set for I. By the previous lemma, a_iR is bounded σ-nilpotent for each $i \geq 1$. By repeated applications of Proposition 2.5, we see that $\sum_{i=1}^n a_iR$ is bounded σ-nilpotent. □

2.2. σ-NZI rings.

In this subsection we focus on the case when R is a σ-NZI ring. We no longer have that every σ-nilpotent element is power-bounded, which requires greater care from us. Also, we no longer have the ability to amplify equations. In its place we use the following result:

Lemma 2.9. *Let $k \in \mathbb{Z}_{>0}$. If $\{a_0, a_1, a_2, \ldots\}$ is a strictly increasing sequence of integers with positive density δ, then there is some integer $n = N(k, \delta)$, depending only on δ and k, so that the initial segment $\{a_0, a_1, \ldots, a_{n-1}\}$ contains a k-term arithmetic progression.*

Proof. This is Szemeredi's theorem [20]. □

Weaker forms of this theorem, such as Van der Waerden's theorem (see [6]), would also suffice. However, the way Szemeredi's theorem is stated (and its strength) makes it more convenient for our purposes.

Proposition 2.10. *Suppose that R is a σ-NZI ring. If $S, T \subseteq R$ are bounded σ-nilpotent subsets then $S+T$ is also bounded σ-nilpotent with index of σ-nilpotence determined by the index of σ-nilpotence of S and T. If we replace "σ-nilpotent" with "σ-nil" everywhere, the result still holds.*

Proof. This proof follows Proposition 2.5 closely. Fix an integer $k \geq 1$ so that for all integers $l \geq 1$ we have both $S\sigma^l(S)\cdots\sigma^{(k-1)l}(S) = 0$ and $T\sigma^l(T)\cdots\sigma^{(k-1)l}(T) = 0$. Let $s = N(k, 1/2)$, and fix elements $a_0, a_1, \ldots, a_{s-1} \in S$ and $b_0, b_1, \ldots, b_{s-1} \in T$. We claim that $P = (a_0 + b_0)\sigma^l(a_1 + b_1)\cdots\sigma^{(s-1)l}(a_{s-1} + b_{s-1})$ is zero.

If we expand P, a typical term is of the form $Q = c_1\sigma^l(c_2)\cdots\sigma^{(s-1)l}(c_s)$ where $c_j \in \{a_j, b_j\}$ for each $j \geq 1$. At least half of the c_j are either from S or T, so without loss of generality say it is S. From the definition of s, there is some k-term arithmetic progression $d_1, d_2, \ldots, d_k$, among the subscripts i with $c_i = a_i$. Write $d_2 - d_1 = t$. We have

$$Q' = \sigma^{d_1 l}(a_{d_1})\sigma^{d_2 l}(a_{d_2})\cdots\sigma^{d_k l}(a_{d_k}) = \sigma^{d_1 l}\left(a_{d_1}\sigma^{lt}(a_{d_2})\cdots\sigma^{(k-1)lt}(a_{d_k})\right) = 0.$$

Since R is σ-NZI, and the terms of Q' occur (in order) in Q, we must have $Q = 0$. As Q was arbitrary, $P = 0$.

If instead, S and T are bounded σ-nil, just make the same change as in Proposition 2.5. □

Corollary 2.11. *Let R be a σ-NZI ring. If a and b are power-bounded σ-nilpotent then so is $a+b$.*

Proposition 2.12. *Let R be a σ-NZI ring and let $a \in R$. If aR is σ-nil then aR is σ-nilpotent. In fact, if $a\sigma^{2l}(a)\sigma^{4l}(a)\cdots\sigma^{(k-1)2l}(a) = 0$ then $(aR)\sigma^l(aR)\sigma^{2l}(aR)\cdots\sigma^{(2k-1)l}(aR) = 0$. In particular, if aR is σ-nil and a is power-bounded σ-nilpotent then aR is bounded σ-nilpotent.*

Proof. Let $l \geq 1$ and fix $k \geq 1$ so that $a\sigma^{2l}(a)\cdots\sigma^{(k-1)2l}(a) = 0$. An arbitrary element of

$$(aR)\sigma^l(aR)\cdots\sigma^{(2k-1)l}(aR)$$

is of the form

$$P = ar_1\sigma^{2l}(a)r_2\sigma^{4l}(a)r_3\cdots\sigma^{(k-1)2l}(a)r_k$$

where $r_i = x_i\sigma^{(2i-1)l}(a)y_i$ for some $x_i, y_i \in R$. In particular, each r_i is σ-nilpotent as aR is σ-nil. Since R is σ-NZI, the element P must equal 0. □

Note that similar statements hold true for the two-sided ideal generated by a, not just the right ideal.

Corollary 2.13. *Suppose R is a σ-NZI ring and $I \leq R$ is a σ-nil right ideal generated by power-bounded σ-nilpotent elements. Then I is power-bounded locally σ-nilpotent.*

In the previous corollary, we do not necessarily know that I is set-bounded locally σ-nilpotent, as the index of σ-nilpotence on a finite subset may depend on that specific finite set and may, *à priori*, grow with the cardinality of that finite set.

2.3. **Both properties and generalizations.**

The results from the previous two subsections give us the following nice corollary.

Corollary 2.14. *Let R be a semicommutative or σ-skew Armendariz ring. Any finite sum of bounded σ-nil one-sided ideals is bounded σ-nil.*

The two conditions on R in the previous corollary do not imply each other, even when σ is the identity; see [3]. Moreover, we construct examples of σ-NZI rings which are not σ-skew σ-NZI, and conversely.

Example 2.15. We construct an example of a σ-skew σ-NZI ring which is not σ-NZI.

Let F be a field and let $S = F\langle a_i \,:\, i \in \mathbb{Z}\rangle$ be the polynomial ring in countably many noncommuting indeterminates. Let I be the ideal generated by monomials of the following three forms: (1) a_i^2, (2) $a_i a_j a_k$ where $i < k$, and (3) any monomial of degree ≥ 4. Let $R = S/I$, and identify each variable with its image in the quotient ring. Define an automorphism σ on R by the action $a_i \mapsto a_{i+1}$, and which is constant on F. It is easy to see that this action is defined on S, and preserves the relations in I, so is well-defined on R.

First note that each variable a_i is (power-bounded) σ-nilpotent since $a_i\sigma^l(a_i)\sigma^{2l}(a_i) = 0$. Second, $a_0^2 = 0$ but $a_0a_1a_0 \neq 0$ so R is not σ-NZI.

On the other hand, suppose that $\alpha, \beta \in R$ satisfy $\alpha\beta = 0$. Let $r \in R$ be σ-nilpotent. We will show $\alpha r\sigma^l(\beta) = 0$ for each $l \geq 1$. Write α_i for the degree i part of α, and do similarly for β and r. If α or β is zero then clearly $\alpha r\sigma^l(\beta) = 0$, so we may assume $\alpha, \beta \neq 0$. By degree considerations, $\alpha_0 = \beta_0 = 0$.

Since all monomials of degree 4 are zero and $r_0 \in Z(R)$, the only term in the product $\alpha r\sigma^l(\beta)$ which does not immediately cancel or equal zero is $\alpha_1 r_1\sigma^l(\beta_1)$. We have $\alpha_1\beta_1 = 0$, and thus either $\alpha_1 = 0$, $\beta_1 = 0$ or $\alpha_1 = \beta_1 \in Fa_i$ for some $i \in \mathbb{Z}$. In any case, $\alpha_1 r\sigma^l(\beta_1) = 0$ for every $l \geq 1$. This proves that R is σ-skew σ-NZI.

Example 2.16. We construct an example of a σ-NZI ring which is not σ-skew σ-NZI.

Let F be a field and let $R = \prod_{i\in\mathbb{Z}} F$. Notice that R is commutative. Let σ be the right-shift automorphism, which sends each coordinate to the next. Let $\alpha = (\alpha_i), \beta = (\beta_i) \in R$ with $\alpha\beta = 0$. Then $\alpha_i\beta_i = 0$ for all $i \in \mathbb{Z}$. Since F is a domain, for each i, either $\alpha_i = 0$ or $\beta_i = 0$. Hence $\alpha r\beta = 0$ for any $r \in R$. This proves that R is σ-NZI, and in fact semicommutative.

On the other hand, let

$$\alpha_i = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd.} \end{cases}$$

So α is an alternating sequence of 0's and 1's. Let $\beta = \sigma(\alpha)$. Clearly $\alpha\beta = 0$. Let

$$\gamma_i = \begin{cases} 1 & \text{if } i = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

So γ has just two nonzero coordinates, next to each other. We easily compute $\gamma\sigma^l(\gamma)\sigma^{2l}(\gamma) = 0$ for any $l \geq 1$, so γ is (power-bounded) σ-nilpotent. Finally, we compute that $(\alpha\gamma\sigma(\beta))_0 = 1$. Hence R is not σ-skew σ-NZI.

Propositions 2.5 and 2.10 show that, under mild commutativity-like conditions, the sum of two subsets which are bounded σ-nil is still bounded σ-nil. In general this seems unlikely, but there is another situation where this is true.

Proposition 2.17. *Let $I \leq R$ be a σ-stable two-sided ideal in R, and $J \subseteq R$ be a subset of R. If I and J are bounded σ-nil then so is $I + J$.*

Proof. Let $a \in I$ and $b \in J$. Fix $k \geq 1$ to be an upper-bound on the index of σ-nilpotence for elements in I and J. Given $l \geq 1$, we have

$$\alpha = (a+b)\sigma^l(a+b)\cdots\sigma^{(k-1)l}(a+b) \in I$$

since I is a two-sided ideal and σ-stable. Thus,

$$\alpha\sigma^{kl}(\alpha)\sigma^{2kl}(\alpha)\cdots\sigma^{(k-1)kl}(\alpha) = 0$$

or in other words

$$(a+b)\sigma^l(a+b)\cdots\sigma^{(k^2-1)l}(a+b)=0.$$

Thus, $I+J$ is σ-nil of bounded index at most k^2. □

More generally, one can ask:

Question 1. Let I and J be (σ-stable) right ideals of a ring R. If I and J are bounded σ-nil then is the same true for $I+J$?

It is instructive to look at a few other cases to get an idea of the complexity in this question.

Proposition 2.18. *Let I and J be σ-nilpotent right ideals of R. If J is bounded σ-nilpotent, then $I+J$ is σ-nilpotent. If, further, I is also bounded σ-nilpotent then so is $I+J$.*

Proof. Fix $t\geq 1$ so that $J\sigma^k(J)\cdots\sigma^{(t-1)k}(J)=0$ for every $k\geq 1$. Next, fix $l\geq 1$, and let $s\geq 1$ be given so that $I\sigma^l(I)\cdots\sigma^{(s-1)l}(I)=0$. Let $v=N(t,1/s)$.

Consider an arbitrary element in $(I+J)\sigma^l(I+J)\cdots\sigma^{(v-1)l}(I+J)$. It looks like a sum of terms of the form

$$Q=c_0\sigma^l(c_1)\cdots\sigma^{(v-1)l}(c_{v-1})$$

where each c_i belongs to either I or J. If s consecutive c_i's belong to I, then Q is zero. Otherwise, the $c_i's$ which belong to J are spaced with density at most $1/s$. So, from the definition of v, there is a t-term arithmetic progression on the subscripts of variables from J, say $c_{i_1},c_{i_2},\ldots,c_{i_t}$. But as J is a right ideal, we have

$$Q\in R\sigma^{i_1l}(J)\sigma^{i_2l}(J)\cdots\sigma^{i_tl}(J)=0.$$

In any case Q is zero, so the right ideal $I+J$ is σ-nilpotent.

If I is also bounded σ-nilpotent, then s is independent of l, hence so is $v=N(t,1/s)$. This proves $I+J$ is also bounded σ-nilpotent in this case. □

The following is a slight improvement on [7, Lemma 1.4(2)]

Corollary 2.19. *Let I and J be σ-nilpotent right ideals of R. If J is σ-stable then $I+J$ is σ-nilpotent.*

Proof. Use Corollary 1.5 and the previous proposition. □

One may wonder if we could weaken the conditions in the previous proposition further, by removing the hypothesis that J is bounded σ-nilpotent. The answer is no. Before we give a counter-example, we need some information on Sturmian sequences (see Chapter 2 in [14] for more detailed information).

Let $x_n=2+\lfloor(n+1)\varphi\rfloor-\lfloor(n+2)\varphi\rfloor$, where φ is the golden ratio and $n\geq 0$. The sequence $\{x_n\}$ consists of 0's and 1's. This sequence gives rise to an infinite word, called the Fibonacci word, without the subwords 11 or 000. One can alternatively describe this word by setting $A_0=0$, $A_1=01$, and recursively letting A_n be the concatenation $A_n=A_{n-1}A_{n-2}$.

We need one more bit of nomenclature. Given an arithmetic progression in the integers, $a_0,a_0+k,a_0+2k,\ldots,a_0+(t-1)k$, we call k the *jump* and t the *length.*

Lemma 2.20. *Let $\{x_n\}$ be the sequence described above, which gives rise to the Fibonacci word. If $a_0,a_1,\ldots,a_{t-1}$ is an arithmetic sequence of jump k and length t such that $x_{a_0},x_{a_1},\ldots,x_{a_{t-1}}$ is constant, then t is bounded by a function $\Xi(k)$ depending only on k.*

Proof. The function $f(z)=2+\lfloor z+\varphi\rfloor-\lfloor z+2\varphi\rfloor$ is periodic, with period 1. It is zero on an interval of length $1/\varphi$, then constantly one on an interval of length $1-1/\varphi$.

Let $k\geq 1$ be an integer, and suppose $z_0\in\mathbb{R}$ is such that $f(z_0\varphi),f((z_0+k)\varphi),\ldots,f((z_0+(t-1)k)\varphi)$ is constantly 0 or 1. This implies that $z_0\varphi,(z_0+l)\varphi,\ldots,(z_0+(t-1)k)\varphi$ are not dense modulo 1; in fact missing an interval of a length at least $\delta=1-1/\varphi$. But since φ is irrational, so is $k\varphi$. It is well known that multiples of irrational numbers are dense in the unit circle. Hence, there is some t (depending only

k) so that $\{nk\varphi \text{ modulo } 1 \ : \ 0 \leq n < t\}$ is δ-dense. This t bounds the length of any constant sequence x_{a_i} as given above. □

There are other zero-one sequences, such as the Thue-Morse sequence, which share this property (see [15]). We want to thank Anthony Quas for pointing out the straightforward proof given above, in the case of Sturmian sequences. We are now ready to prove:

Theorem 2.21. *Let I be a right ideal in R. If I is σ-nilpotent it does not need to be the case that $I + \sigma(I)$ is even σ-nil.*

We prove the theorem by constructing an example.

Example 2.22. Let F be a field and let $R = F[a_i \ : \ i \in \mathbb{Z}]/(a_i a_{i+k} a_{i+2k} \cdots a_{i+\Xi(k)k} \ : \ i \in \mathbb{Z}, k \geq 1)$ where $\Xi(k)$ is as in the previous lemma. Identify each variable a_i with its image in the quotient ring. The action $\sigma : a_i \mapsto a_{i+1}$ respects the relations, and so gives rise to an automorphism on R. It is clear that the right ideal $I = a_0 R$ is σ-nilpotent, but neither bounded σ-nilpotent nor σ-stable.

Consider the element $\alpha = a_0 + \sigma(a_0)$. Fix $t \geq 1$ and let $Q = \alpha\sigma(\alpha)\cdots\sigma^t(\alpha)$. We will show Q is nonzero. From the relations defining R, we see that Q is zero if and only if every monomial appearing in Q (after it is expanded) is zero. Set

$$b_i = \begin{cases} a_0 & \text{if } x_i = 0 \\ a_1 & \text{if } x_i = 1 \end{cases}$$

where x_i is the Sturmian sequence defined above. Then $Q' = b_0\sigma(b_1)\sigma^2(b_2)\cdots\sigma^t(b_t)$ is a monomial appearing in the expansion of Q. Simplifying, we see that the initial part of Q' looks like

$$a_0 a_2 a_2 a_3 a_5 a_5 \cdots = a_0 a_2^2 a_3 a_5^2 \cdots .$$

Let Q'' be the monomial obtained from Q' by taking the square-free part, so it starts $a_0 a_2 a_3 a_5 \cdots$. From the relations defining R, $Q' = 0$ if and only if $Q'' = 0$.

It is straightforward to see that Q'' consists of the a_i, for $0 \leq i \leq t$, such that x_i is 0 (except we might have a_{t+1} at the end if $x_t = 1$, but note that $x_{t+1} = 0$ so this causes no problems). In particular, by the previous lemma there are no length $\Xi(k)$ arithmetic progressions of jump size k in the subscripts of the variables appearing in Q''. Hence $Q'' \neq 0$.

3. The Bounded Radical

The set $B(R) = \{a \ : \ aR \text{ is nil of bounded index of nilpotence}\}$ is an ideal of R, which was first shown by Amitsur. Historically, this ideal has been written as $N(R)$, but we depart from convention. It is known that $B(R[x]) = B(R)[x]$; see for example [2, Corollary 17]. We wish to develop a σ-analog for this ideal, which has many of the same properties.

Definition 3.1. For a ring R, we let

$$B_\sigma(R) = \{a \ : \ aR \text{ is a bounded } \sigma\text{-nil right ideal}\}.$$

Note that Ra is bounded σ-nil if and only if aR is bounded σ-nil, with the index of σ-nilpotence differing by at most 1. Thus, the definition above is left-right symmetric. This also shows that $B_\sigma(R)$ is closed under multiplication on the left and right by elements of R. Further, since σ is an automorphism, aR is bounded σ-nil if and only if $\sigma(a)R$ is also (with the same index). So the set $B_\sigma(R)$ is σ-invariant. Finally, if $B_\sigma(R)$ is closed under addition then it is an ideal of R. In particular, this is true in the case when R is a (σ-skew) σ-NZI ring.

Theorem 3.2. *For any ring R we have $B(R[x;\sigma]) \subseteq (B(R) \cap B_\sigma(R)) + B_\sigma(R)xR[x;\sigma]$.*

Proof. We follow the methods in [2, Proposition 16]. Let A_m be the set of elements from R which appear as coefficients in degree m for some skew polynomial in $B(R[x;\sigma])$. Since $B(R[x;\sigma])$ is an ideal of $R[x;\sigma]$, each of the sets A_m is an ideal in R. Furthermore, as $B(R[x;\sigma])$ is closed under multiplication on the right by powers of x, we have $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$. Also note that the A_m are σ-invariant.

We claim by induction on $m \geq 0$ that $A_m \subseteq B_\sigma(R)$. More precisely, we will show that given $a \in A_m$ and a polynomial $f(x) \in B(R[x;\sigma])$ with a as the coefficient in degree m, then the right ideal aR is bounded σ-nil with index depending only on m and the index of nilpotence of $f(x)R[x;\sigma]$.

To establish the base case let $a \in A_0$, and fix some polynomial $f(x) = \sum_{i=0}^{t} a_i x^i \in B(R[x;\sigma])$ with $a_0 = a$. We have that $f(x)R[x;\sigma]$ is a nil right ideal with bounded index of nilpotence n, for some $n \geq 1$. We will show that aR is σ-nil with the same bounded index of σ-nilpotence n. Letting $r \in R$ and $l \geq 1$ then $f(x)rx^l$ is nilpotent of index n. So, looking at the coefficient of $(f(x)rx^l)^n = 0$ in degree ln, we have

$$0 = ar\sigma^l(ar)\sigma^{2l}(ar)\cdots\sigma^{(n-1)l}(ar)$$

which is what we wanted.

Now assume the inductive hypothesis for coefficients in degree smaller than s, and assume $s \geq 1$. Fix $a \in A_s$ and let $f(x) = \sum_{i=0}^{t} a_i x^i \in B(R[x;\sigma])$ with $a_s = a$. Once again, let $n \geq 1$ be the index of nilpotence for $f(x)R[x;\sigma]$. Let $r \in R$ and $l \geq 1$. As $(f(x)g(x))^n = 0$ for any $g(x) \in R[x;\sigma]$, we have

$$\left(f(x)\sigma^{-s}\left(r\sigma^l(ar)\sigma^{2l}(ar)\cdots\sigma^{(s-1)l}(ar)\right)x^{sl-s}\right)^n = 0.$$

Looking at the coefficient of the degree sln term in this expansion (which equals 0), one of the summands is of the form

$$b = ar\sigma^l(ar)\cdots\sigma^{(sn-1)l}(ar) \tag{4}$$

and all other summands are of the form $\alpha\sigma^j(a_i)\beta$ for some $\alpha, \beta \in R$, $j \geq 0$, and $i < s$. Thus, b is the negative sum of these other summands.

Notice that $\alpha\sigma^j(f(x))x^{s-1-i}\sigma^{-(s-1)}(\beta)R[x;\sigma]$ is nil with bounded index at most $n+1$. According to Klein [11, Theorem 3], the right ideal $I = \sum \alpha\sigma^j(f(x))x^{s-1-i}\sigma^{-(s-1)}(\beta)R[x;\sigma]$ is also nil of bounded index depending only on s and n, since each summand has index bounded by $n+1$ and the number of summands is bounded by a function of s and n. We let $\mu(s,n)$ be a bound on the index of I. As b equals the coefficient in degree $s-1$ of $-\sum \alpha\sigma^j(f(x))x^{s-1-i}\sigma^{-(s-1)}(\beta)$, our induction hypothesis implies that the right ideal bR is σ-nil with index bounded by a function of $s-1$ and $\mu(s,n)$, hence of s and n, say $\nu(s,n)$. In particular, we have

$$0 = b\sigma^{snl}(b)\sigma^{2snl}(b)\cdots\sigma^{(\nu(s,n)-1)snl}(b) = ar\sigma^l(ar)\sigma^{2l}(ar)\cdots\sigma^{(\nu(s,n)sn-1)l}(ar)$$

so aR is σ-nil, with index bounded by $\nu(s,n)sn$. As this bound depends only on s and n this finishes the induction.

We leave it to the reader to show that $A_0 \subseteq B(R)$. □

Corollary 3.3. *For any ring R, we have $B(R[[x;\sigma]]) \subseteq (B(R) \cap B_\sigma(R)) + B_\sigma(R)xR[[x;\sigma]]$.*

Proof. Just apply the same proof as above. □

Theorem 3.4. *If R is σ-skew σ-NZI then $B(R[x;\sigma]) = (B(R) \cap B_\sigma(R)) + B_\sigma(R)xR[x;\sigma]$.*

Proof. It suffices to show the containment $\supseteq$. Fix $f(x) \in (B(R) \cap B_\sigma(R)) + B_\sigma(R)xR[x;\sigma]$, write $f(x) = \sum_{i=0}^{n} a_i x^i$. As $a_0R \subseteq B(R)$, fix $t \geq 1$ for the bound on the index of nilpotence for elements in a_0R. By Lemma 2.7, a_iR is bounded σ-nilpotent, for each $i \geq 0$. By Proposition 2.5, $S = \sum_{i=0}^{n} a_iR$ is bounded σ-nilpotent. By Proposition 2.4, fix $m \geq 1$ so that

$$S\sigma^{i_1}(S)\sigma^{i_2}(S)\cdots\sigma^{i_{m-1}}(S) = 0$$

for any sequence of integers $1 \leq i_1 < i_2 < \ldots < i_{m-1}$.

We claim that $f(x)^{mt} = 0$. First, $f(x)^t \in S[x;\sigma]x$. So each coefficient in $f(x)^{mt}$ is a sum of terms of the form $r = s_0\sigma^{i_1}(s_1)\cdots\sigma^{i_{m-1}}(s_{m-1})$, where each $s_i \in S$, and since $f(x)^t$ has no constant term we also have $1 \leq i_1 < i_2 < \ldots < i_{m-1}$. This implies $r = 0$, hence $f(x)^{mt} = 0$. □

Proposition 3.5. *If R is a σ-skew σ-NZI ring then $B_\sigma(R)[x;\sigma] = B_\sigma(R[x;\sigma])$, where we extend σ to act trivially on x.*

Proof. Letting $f \in B_\sigma(R)[x;\sigma]$ then $fx \in B_\sigma(R)[x;\sigma]x \subseteq B(R[x;\sigma])$ by Theorem 3.4. So $fxR[x;\sigma]$ is nil of bounded index $t \geq 1$, and so is $fx^lgR[x;\sigma]$ for each $g \in R[x;\sigma]$ and any integer $l \geq 1$. Thus we have $(fx^lg)(fx^lg)\cdots(fx^lg) = 0$, and so $(gf)\sigma^l(gf)\sigma^{2l}(gf)\cdots\sigma^{tl}(gf) = 0$. This implies that $R[x;\sigma]f$ is bounded σ-nil, entailing that $f \in B_\sigma(R[x;\sigma])$.

Conversely, let $f \in B_\sigma(R[x;\sigma])$. We then have $fR[x;\sigma]$ is bounded σ-nil of index $t \geq 1$, and $R[x;\sigma]f$ is bounded σ-nil of index at most $t+1$. For any $g \in R[x;\sigma]$, $(fxg)^{t+2} = (fxg)(fxg)\cdots(fxg) = fx(gf)x(gf)\cdots(gf)xg = fx((gf)\sigma(gf)\cdots\sigma^t(gf))x^{t+1}g = 0$. Thus $fx \in B(R[x;\sigma])$. By Theorem 3.4 we have $fx \in B_\sigma(R)[x;\sigma]$. Thus each coefficient of f lives in $B_\sigma(R)$, and so $f \in B_\sigma(R)[x;\sigma]$, completing the proof. □

It turns out that Theorem 3.4 is false if we only assume R is σ-NZI. In fact, much more is true.

Example 3.6. The containment in Theorem 3.2 can be proper, even when R is commutative and reduced.

Let F be a field of characteristic 0. Let $S = F[a_i \; : \; i \in \mathbb{Z}]$ be the polynomial ring over F in countably many commuting indeterminates. Let I be the ideal generated by monomials of the form $a_{i_0}a_{i_0+k}a_{i_0+2k}$, where $i_0 \in \mathbb{Z}$, and $k \geq 1$. We set $R = S/I$, and identify each a_i with its image in the quotient ring.

Note that a monomial $m \in R$ in the letters $\{a_i\}$ is zero if and only if there is some subword of m, consisting of three of the a_i, with the indices forming an increasing arithmetic progression. (By subword, we mean a collection of letters from the word.) We let σ be the automorphism of R determined by the action $a_i \mapsto a_{i+1}$, for each $i \in \mathbb{Z}$, which fixes F.

First, we claim that $B_\sigma(R)$ is the ideal generated by the a_i. Let $\alpha \in R$ and let $l \geq 1$. If α has a nonzero constant term then $\alpha\sigma^l(\alpha)\sigma^{2l}(\alpha)\cdots\sigma^{(n-1)l}(\alpha)$ also has a nonzero constant term for every choice of $n \in \mathbb{N}$, hence is nonzero. On the other hand, if α has zero constant term we can write $\alpha = a_{i_1}\alpha_1 + a_{i_2}\alpha_2 + \cdots + a_{i_m}\alpha_m$ where $i_1, \ldots, i_m \in \mathbb{Z}$ and $\alpha_j \in R$. Notice that every element of αR is also of this form, so it suffices to show that α is σ-nilpotent with the index of σ-nilpotence depending only on m.

It suffices to find a bound N (depending only on m) so that $\beta = b_0\sigma^l(b_1)\cdots\sigma^{N-1}(b_{N-1})$ is zero, for any choice of the b_i's from $A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_m}\}$. Set $N = N(3, 1/m)$. Then in the set $\{0, 1, 2, \ldots, N-1\}$ there is a 3-term arithmetic progression, say j_1, j_2, j_3, such that $b_{j_1} = b_{j_2} = b_{j_3}$ (thinking of the elements of the set A as consisting of m possibilities), say $b_{j_1} = a_{i_0}$. But then $\sigma^{j_1}(a_{i_0})\sigma^{j_2}(a_{i_0})\sigma^{j_3}(a_{i_0}) = a_{i_0+j_1}a_{i_0+j_2}a_{i_0+j_3}$ is a subword of β, with the indices forming an arithmetic progression. So $\beta = 0$. Thus $\alpha\sigma^l(\alpha)\cdots\sigma^{(N-1)l}(\alpha) = 0$, since any monomial appearing in the support must equal zero.

Next, we claim that R is reduced. Suppose that $\alpha \in R$ with $\alpha^2 = 0$. We can order the monomials in R by degree (since I is homogeneous) and then lexicographically. If $\alpha \neq 0$ we let m be the monomial of smallest order, appearing in the support of α. By order considerations, we must have $m^2 = 0$. But then m^2 contains a subword with the variables subscripted by a 3-term arithmetic progression. This subword must also be a subword of m, and so $m = 0$, a contradiction. Thus $\alpha = 0$.

Finally, we show that $a_0xR[x;\sigma]$ does not have bounded index of nilpotence. Recursively define the sequence $z_0 = 0$, $z_1 = 1$, and $z_{k+1} = 2z_k + 1$ for each $k \geq 1$. Clearly, the sequence contains no arithmetic progression of length 3. For each $n \geq 1$ set $g_n(x) = \sum_{i=1}^{n} x^{z_i - z_{i-1} - 1}$. Then we claim the polynomial

$f_n(x) = a_0xg_n(x) \in a_0xR[x;\sigma]$ does not have nilpotence index n. Indeed, consider the coefficient of $f_n(x)^n$, in degree z_n. The monomial $a = a_{z_0}a_{z_1}\cdots a_{z_{n-1}}$ has nonzero support, since k has characteristic 0. But from how the ideal I is defined, a sum of distinct monomials (with nonzero support) is zero if and only if each monomial occurs in I. As $a \notin I$, this says $f_n(x)^n \neq 0$. This finishes the example.

(1) In the example above, we can guarantee that $B(R) = B_\sigma(R)$ if we do not guarantee that R is reduced. To do so we simply add to the defining relations of the ideal I the new relations a_i^3 for each $i \in \mathbb{Z}$.
(2) Ram proved the following in [18, Theorem 3.1]: If R is a commutative ring, σ is an automorphism of R and $r \in R$ is σ-nilpotent of bounded index, then $rxR[x;\sigma]$ is locally nilpotent. The same is true if we replace the commutativity condition with the assumption that R is σ-NZI, by the same proof *mutatis mutandis*. Note in particular that if $B_\sigma(R) \neq 0$ and R is σ-NZI then the Levitzki radical of $R[x;\sigma]$ is nonzero.
(3) If we define
$$B_{\sigma\text{-st}}(R) = \{a \ : \ \sum_{i=0}^{\infty} \sigma^i(a)R \text{ is bounded } \sigma\text{-nil}\}$$
we have $B_{\sigma\text{-st}}(R) \subseteq B_\sigma(R)$ and all of the results above still hold for this "stable" bounded σ-nilradical, with only minor changes to the proofs. Note that if $B_\sigma(R)$ is an ideal of R then these two sets are equal.

While the containment in Theorem 3.2 is not reversible, we do have the following result:

Theorem 3.7. *If a ring R is σ-NZI, then* $(B(R) \cap B_\sigma(R)) + B_\sigma(R)[x;\sigma]x \subseteq \mathrm{Nil}(R[x;\sigma])$.

Proof. Let $f(x) \in (B(R) \cap B_\sigma(R)) + B_\sigma(R)[x;\sigma]x$. Let a_0 be the constant coefficient, and fix an integer $t \geq 1$ so that $a_0^t = 0$. Replacing $f(x)$ by $f(x)^t$, we may as well assume that $a_0 = 0$. Write $f(x) = a_1x + a_2x^2 + \cdots + a_nx^n$. The right ideal $I = \sum_{i=1}^{n} a_iR$ is bounded σ-nilpotent, by Propositions 2.10 and 2.12, say with index $k \geq 1$.

In $f(x)^m$, an arbitrary element looks like $c_0\sigma^{i_1}(c_1)\cdots\sigma^{i_{m-1}}(c_{m-1})$ where each $c_i \in I$ and $1 \leq i_1 < i_2 < \ldots < i_{m-1}$ is a strictly increasing sequence of integers, with gaps of size at most $n = \deg(f)$. Letting $s = N(k, \deg(f)^{-1})$, then every monomial in every term of $f(x)^s$ is zero. □

Corollary 3.8. *Let R be a σ-NZI ring. Then $B_\sigma(R)[x;\sigma] \subseteq \mathrm{Nil}_\sigma(R[x;\sigma])$. Further, if R is bounded σ-nil then $R[x;\sigma]$ is σ-nil.*

We denote the set of all power-bounded σ-nilpotent elements of R by $\mathrm{BN}_\sigma(R)$. Similar methods to the previous theorem proves the following:

Proposition 3.9. *If R is a σ-NZI ring, then $\mathrm{BN}_\sigma(R)$ is a non-unital subring of R.*

We finish by asking an open question:

Question 2. Does there exist an ideal $I \leq R$ so that $B(R[x;\sigma]) = (B(R) \cap I) + I[x;\sigma]x$?

4. Locally finite order automorphisms

Example 3.6 shows that without further information, and even under strong conditions (such as commutativity) placed on our ring, the containment in Theorem 3.2 is not reversible. However, if we make some assumptions about the automorphism σ this situation can be rectified.

Recall that σ has *finite order* if there exists some positive integer $n \geq 1$ so that $\sigma^n = \mathrm{id}$. More generally, we say that σ is *locally of finite order* if for each $r \in R$ there exists some positive integer $n \geq 1$ so that $\sigma^n(r) = r$.

Lemma 4.1. *Let R be a ring with an automorphism σ which is locally of finite order. We have $B_\sigma(R) \subseteq B(R)$. The reverse containment holds if, further, either R is σ-NZI or σ is of finite order.*

Proof. Let $x \in B_\sigma(R)$. Fix an integer $k \geq 1$ so for each $l \geq 1$ and each $r \in R$ we have

$$xr\sigma^l(xr)\cdots\sigma^{(k-1)l}(xr) = 0.$$

Given $r \in R$, we can also fix some integer $n_r \geq 1$ so that $\sigma^{n_r}(xr) = xr$. In particular, we have

$$0 = xr\sigma^{n_r}(xr)\sigma^{2n_r}(xr)\cdots\sigma^{(k-1)n_r}(xr) = (xr)^k.$$

Since this holds for all $r \in R$, we have $x \in B(R)$.

We now prove the reverse containment. Let $x \in B(R)$, so there is some integer $k \geq 1$ so that for any $r \in R$ we have $(xr)^k = 0$. Also, we can fix some integer $n \geq 1$ so that $\sigma^n(x) = x$. Given $l \geq 1$ set $s = r\sigma^l(xr)\sigma^{2l}(xr)\cdots\sigma^{(n-1)l}(xr)$.

First assume that σ has finite order. Without loss of generality, we may assume the order is n. We have

$$0 = (xs)^k = xr\sigma^l(xr)\cdots\sigma^{(kn-1)l}(xr)$$

demonstrating that xr has a σ-nilpotence index bounded by kn.

Now, instead assume that R is σ-NZI. Without loss of generality, we can guarantee that $n \geq 2$. From $x^k = 0$ we have

$$xr\sigma^l(xr)\cdots\sigma^{(kn-1)l}(xr) = xs_1xs_2\cdots xs_k = 0$$

since $s_i = \sigma^{(i-1)nl}(s)$ is σ-nilpotent. □

Lemma 4.2. *If R is σ-skew σ-NZI and σ is locally of finite order then R is σ-NZI.*

Proof. Straightforward. □

Proposition 4.3. *If R is a σ-NZI ring and σ is locally of finite order, then*

$$B(R[x;\sigma]) = (B(R) \cap B_\sigma(R)) + B_\sigma(R)xR[x;\sigma] = B(R)[x;\sigma]$$

Proof. It suffices to show that $B_\sigma(R)[x;\sigma] \subseteq B(R[x;\sigma])$. Let $f(x) = \sum_{i=0}^m a_ix^i \in B_\sigma(R)[x;\sigma]$. We can fix a single power n so that $\sigma^n(a_i) = a_i$ for each i. Let $A = \{\sigma^j(a_i) \ : \ 0 \leq i \leq m,\ 0 \leq j \leq n-1\}$, so we have $\sigma(A) = A$. As $A \subseteq B_\sigma(R)$, we know that A is locally σ-nilpotent by Corollary 2.13. Being invariant under σ and finite, the set A is in fact nilpotent. Fix $t \geq 1$ so that $A^t = 0$. Let $A' = \{ras \ : \ r,s \in R,\ a \in A\}$, noting that A' is σ-nil.

We claim that $(f(x)g(x))^{2t} = 0$ for any $g(x) \in R[x;\sigma]$. This is because each coefficient is a sum of elements from $(AA')^t = 0$. □

Klein [10, Lemma 5] proved that every nil ideal of bounded index contains a nonzero nilpotent ideal. Applying similar methods we obtain:

Theorem 4.4. *If I be a nonzero bounded σ-nil right ideal of R then for each $l \geq 1$ there exists $0 \neq b \in I$ (possibly depending on l) such that $bR\sigma^l(b) = 0$.*

Proof. Let I be a right ideal of R which is bounded σ-nil. Fix $n \geq 1$ so that $x\sigma^k(x)\cdots\sigma^{(n-1)k}(x) = 0$ for all $x \in I$ and all $k \geq 1$.

Let $l \geq 1$ be given. For each $a \in I$, $a \neq 0$, there is some minimal integer $k_a \geq 1$ (in fact, we have $k_a \leq n-1$) such that

$$ay\sigma^l(ay)\cdots\sigma^{k_al}(a) = 0 \tag{5}$$

for all $y \in R$. Let $a_0 \in I$, $a_0 \neq 0$, be chosen so that $k_0 = k_{a_0}$ is minimal. If $k_0 = 1$ then we can take $b = a_0$.

So assume, by way of contradiction, that $k_0 \geq 2$, and fix some y_0 so that

$$a_1 = a_0y_0\sigma^l(a_0y_0)\cdots\sigma^{(k_0-1)l}(a_0) \neq 0.$$

Set

$$a_1' = a_0 y_0 \sigma^l(a_0 y_0) \cdots \sigma^{(k_0-2)l}(a_0) \neq 0.$$

We have

$$a_0 y_0 \sigma^l(a_1) = 0, \qquad a_0 y_0 \sigma^l(a_1') = a_1, \qquad a_0 y_0 \sigma^l(a_0 y_0) \sigma^{2l}(a_1') = 0. \tag{6}$$

Consider the quantity

$$A = (a_1 z + a_0 y_0) \sigma^l(a_1 z + a_0 y_0) \cdots \sigma^{(k_0-1)l}(a_1 z + a_0 y_0) \sigma^{k_0 l}(a_1')$$

where $z \in R$. Since $a_1', a_1 z + a_0 y_0 \in a_0 R$, Equation (5) says that $A = 0$. On the other hand, if we expand A, then using Equations (5) and (6) most terms are zero and we have

$$\begin{aligned} A &= a_1 z \sigma^l(a_1 z) \cdots \sigma^{(k_0-1)l}(a_0 y_0) \sigma^{k_0 l}(a_1') + a_1 z \sigma^l(a_1 z) \cdots \sigma^{(k_0-1)l}(a_1 z) \sigma^{k_0 l}(a_1') \\ &= a_1 z \sigma^l(a_1 z) \cdots \sigma^{(k_0-1)l}(a_1) + 0. \end{aligned}$$

Putting this together, we have

$$a_1 z \sigma^l(a_1 z) \cdots \sigma^{(k_0-1)l}(a_1) = 0 \tag{7}$$

for all $z \in R$. But this means that $k_{a_1} \leq k_0 - 1$, contradicting the choice of $a = a_0$ as a nonzero element minimizing k_a. □

Although the proof of the previous theorem is stated as a proof by contradiction, one can easily reframe it as a constructive proof. Starting with an arbitrary nonzero element a_0 in I, one constructs $a_1 \in a_0 R \subseteq I$ and repeats the process. It is also interesting to note that the two-sided ideal $J = RbR$ satisfies $J\sigma^l(J) = 0$.

Either by modifying the proof of the previous theorem, or applying the theorem repeatedly, we obtain:

Corollary 4.5. *Let I be a nonzero bounded σ-nil right ideal of R. Given a finite set $\{l_1, l_2, \ldots, l_k\} \subseteq \mathbb{Z}_{>0}$ of positive powers, there is a nonzero element $b \in I$ such that $bR\sigma^{l_i}(b) = 0$ for $1 \leq i \leq k$.*

Corollary 4.6. *Let σ be locally of finite order. If I is a nonzero bounded σ-nil (right) ideal of R then I contains a nonzero bounded σ-nilpotent (right) ideal of R.*

Proof. We have by Lemma 4.1, $B_\sigma(R) \subseteq B(R)$. Thus $I \subseteq B(R)$. By [10, Lemma 5], fix $0 \neq a \in I$ such that $(RaR)^2 = 0$. Also fix $k \geq 1$ so that $\sigma^k(a) = a$, and set $J = \sum_{i=1}^{k} R\sigma^i(a)R$. This is a finite sum of nilpotent ideals, so is nilpotent. It is also σ-invariant, so it is bounded σ-nilpotent (of the same index of σ-nilpotence). The sets RaR and aR are also bounded σ-nilpotent. □

Corollary 4.7. *Suppose σ has finite order, and suppose I is a nonzero bounded σ-nil right ideal of R. There exists a nonzero element $b \in I$ such that $bR\sigma^l(b) = 0$ for every $l \in \mathbb{Z}$.*

We end this section with two interesting open questions.

Question 3. If I is a nonzero bounded σ-nil ideal of R then does I contain a nonzero σ-nilpotent ideal of R?

Question 4. Is $B_\sigma(R)$ an ideal of R?

5. The σ-Wedderburn Radicals

The Wedderburn radicals are defined recursively as $W_1(R) = \sum\{I \ : \ I \leq R \text{ is a nilpotent ideal}\}$,

$$W_{\alpha+1}(R) = \{r \in R \ : \ r + W_\alpha(R) \in W_1(R/W_\alpha(R))\},$$

and for limit ordinals

$$W_\alpha(R) = \bigcup_{\beta<\alpha} W_\beta(R).$$

It is well known that these ideals stabilize to the prime radical. Klein proved that $W_1(R) \subseteq B(R) \subseteq W_2(R)$.

We define the (stable) σ-Wedderburn radicals recursively, as in [7] (but with different notation), by setting

$$W_{\sigma\text{-st}}(R) = W_{\sigma\text{-st},1}(R) = \sum\{I \ : \ I \leq R \text{ is a } \sigma\text{-nilpotent, } \sigma\text{-stable ideal}\},$$

$$W_{\sigma\text{-st},\alpha+1}(R) = \{r \in R \ : \ r + W_{\sigma\text{-st},\alpha} \in W_{\sigma\text{-st},1}(R/W_{\sigma\text{-st},\alpha}(R))\},$$

and for limit ordinals

$$W_{\sigma\text{-st},\alpha}(R) = \bigcup_{\beta<\alpha} W_{\sigma\text{-st},\beta}(R).$$

We need two more pieces of information. An ideal $I \leq R$ is *σ-semiprime* if it is σ-stable, proper in R, and given an ideal A and an integer m such that $A\sigma^n(A) \subseteq I$ for all $n \geq m$, then $A \subseteq I$. We also define $P_\sigma(R)$ as the smallest σ-semiprime ideal of R (which exists by [7, Proposition 1.3]).

Lemma 5.1. *For a ring R the following hold:*

1. $W_{\sigma\text{-}st}(R) \subseteq P_\sigma(R)$.
2. $W_{\sigma\text{-}st}(R) \subseteq B_\sigma(R)$.
3. $W_{\sigma\text{-}st,\alpha}(R) = W_\alpha(R)$ *if σ is locally of finite order.*
4. $W_{\sigma\text{-}st,1}(R) \subseteq B_\sigma(R) \subseteq W_{\sigma\text{-}st,2}(R) \subseteq P_\sigma(R)$ *if σ is locally of finite order.*

Proof. Items (1) and (3) follow from [7, Proposition 1.5 and Lemma 3.5].

To prove (2), notice that a finite sum of σ-nilpotent, σ-stable ideals is bounded σ-nilpotent, by Proposition 2.18 and Corollary 1.5. Thus, any element in $W_{\sigma\text{-st}}(R)$ generated a bounded σ-nilpotent right ideal, and thus belongs to $B_\sigma(R)$.

To prove (4), by items (2) and (3) above, and Lemma 4.1, we have the following chain of containments

$$W_{\sigma\text{-st},1}(R) \subseteq B_\sigma(R) \subseteq B(R) \subseteq W_2(R) = W_{\sigma\text{-st},2}(R) \subseteq P_\sigma(R).$$

□

Question 5. Do we always have $B_\sigma(R) \subseteq W_{\sigma\text{-st},2}(R)$? How about $B_\sigma(R) \subseteq P_\sigma(R)$?

Both Amitsur's and Klein's proofs that $B(R)$ is a two-sided ideal in R rely on the fact that $B(R) \subseteq P(R)$. Thus, this last question is perhaps where one should start in trying to answer the other open questions we have posed.

We finish by noting there are many ways to generalize the notions above. We say that a set A is *eventually 2-σ-nilpotent* if there exists an integer $m \geq 1$ such that for every $n \geq m$ we have $A\sigma^n(A) = 0$. It is easy to show that eventually 2-σ-nilpotent right ideals are bounded σ-nilpotent. One can now, in the definition of the Wedderburn radicals, replace I being σ-stable with I being eventually 2-σ-nilpotent. This new chain of Wedderburn radicals naturally stabilizes at $P_\sigma(R)$, by [16, Proposition 1.5].

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