IDEMPOTENT LIFTING AND RING EXTENSIONS

ALEXANDER J. DIESL, SAMUEL J. DITTMER, AND PACE P. NIELSEN

Abstract. We answer multiple open questions concerning lifting of idempotents that appear in the literature. Most of the results are obtained by constructing explicit counter-examples. For instance, we provide a ring R for which idempotents lift modulo the Jacobson radical $J(R)$, but idempotents do not lift modulo $J(\mathbb{M}_2(R))$. Thus, the property "idempotents lift modulo the Jacobson radical" is not a Morita invariant. We also prove that if I and J are ideals of R for which idempotents lift (even strongly), then it can be the case that idempotents do not lift over $I + J$. On the positive side, if I and J are enabling ideals in R, then $I + J$ is also an enabling ideal. We show that if $I \trianglelefteq R$ is (weakly) enabling in R, then $I[t]$ is not necessarily (weakly) enabling in $R[t]$ while $I[\![t]\!]$ is (weakly) enabling in $R[\![t]\!]$. The latter result is a special case of a more general theorem about completions. Finally, we give examples showing that conjugate idempotents are not necessarily related by a string of perspectivities.

1. Introduction

In ring theory it is useful to be able to lift properties of a factor ring of R back to R itself. This is often accomplished by restricting to a nice class of rings. Indeed, certain common classes of rings are defined precisely in terms of such lifting properties. For instance, semiperfect rings are those rings R for which $R/J(R)$ is semisimple and idempotents lift modulo the Jacobson radical.

Recall that *idempotents lift modulo an ideal* $I \trianglelefteq R$ if whenever $x + I \in R/I$ is idempotent, then there exists an idempotent $e \in R$ such that $x - e \in I$. It is well known that when idempotents lift modulo the Jacobson radical then there are additional properties which follow. For instance, idempotents *lift strongly*, meaning that (using the notation above) we can choose $e \in xR$, see [7, Theorem 21.28]. Further, countable (but not arbitrary!) orthogonal sets of idempotents can be lifted orthogonally modulo the Jacobson radical, see [7, Proposition 21.25].

Recently Alkan, Nicholson, and Özcan discovered that strong lifting is a consequence of a property which splits from, and is independent of, idempotents lifting. Call an ideal $I \trianglelefteq R$ *enabling* in R if whenever we have $x \equiv e \pmod{I}$ for elements $x, e \in R$ with $e^2 = e$, then there exists an idempotent $f \in xR$ such that $e \equiv f \pmod{I}$. It turns out that the Jacobson radical is always enabling [1, Proposition 5], even when idempotents do not lift. Using this fact, Camillo and the third author were able to generalize many of the standard theorems in the literature by replacing the hypothesis "idempotents lift" with weaker assumptions [2].

In this paper we answer two of the three open questions about idempotent lifting and enabling which appear in [9], both in the negative. We also answer two of the three open questions appearing (at the end of Section 1) in [1], one in the negative and one in the positive. Most of these answers consist of producing explicit counter-examples. However, we also prove that a weakened form of one of the questions has a positive answer. We construct additional examples answering other questions found in the literature.

In this paper $J(R)$ will always denote the Jacobson radical of a ring R, and $\mathrm{idem}(R)$ is the set of idempotents. We write $I \trianglelefteq R$ to signify that I is a two-sided ideal of R. By $\mathbb{M}_n(R)$ we mean the ring of $n \times n$ matrices over R, and we will write $e_{i,j}$ for the matrix units in this ring.

2010 *Mathematics Subject Classification.* Primary 16U99, Secondary 16D90, 16N20, 16S85.

Key words and phrases. enabling, idempotents, Jacobson radical, lifting, localization, perspective summands.

2. Lifting modulo $J(R)$ is not a Morita invariant

Our first construction is of a ring which demonstrates that "idempotent lifting modulo the Jacobson radical" is not a Morita invariant, answering the question raised in [9, p. 798, footnote 1] in the negative.

Theorem 2.1. *If idempotents lift modulo the Jacobson radical $J(S)$ of some ring S, idempotents do not necessarily lift modulo $J(\mathbb{M}_2(S))$ in the ring $\mathbb{M}_2(S)$.*

Proof. For an arbitrary field k, take $R = k[a, b, c]$ where a, b, c are commuting indeterminates. Let $I = (a^2 - a + bc)$ which is an ideal of R. Let S be the subring of the field of fractions $\mathrm{Frac}(R) = k(a, b, c)$ consisting of those fractions f/g with $f, g \in R$ and $g \in 1 + I$.

Note that $S/IS \cong R/I$. In R/I, the relation $a^2 = a - bc$ along with the commuting relations form a complete reduction system, and the images of the monomials $ab^k c^\ell$ and $b^k c^\ell$, for $k, \ell \geq 0$, form a k-basis for R/I. Define a function φ from the monomials in R to $\mathbb{N} \times \mathbb{N}$ by setting $\varphi(a^j b^k c^\ell) = (j + 2k, j + 2\ell)$. (More informally, one can think of this map as treating $a = b^{\frac{1}{2}} c^{\frac{1}{2}}$ for the purposes of computing degrees for monomials.) If $r \in R$ contains no monomial which is divisible by a^2, we define the *φ-degree* of r to be the maximum of $\varphi(r')$ for monomials r' in the support of r, with respect to the lexicographical ordering on $\mathbb{N} \times \mathbb{N}$. We then define the φ-degree of an element in R/I by taking its unique preimage in R written in reduced form (repeatedly replacing occurrences of a^2 by $a - bc$) and taking the φ-degree in R.

We claim that this degree for R/I is additive. In other words, $\varphi(\overline{rs}) = \varphi(\overline{r}) + \varphi(\overline{s})$ for any $r, s \in R \setminus \{0\}$. To see this, first we may assume each monomial in r and s is not divisible by a^2 (writing them in their reduced forms). Let r' be the monomial in r so that the φ-degree of r is just $\varphi(r')$, and similarly pick a monomial s' in the support of s such that the φ-degree of s is $\varphi(s')$. Writing $r' = a^{j_1} b^{k_1} c^{\ell_1}$ and $s' = a^{j_2} b^{k_2} c^{\ell_2}$, we see that $r's'$ is already reduced if $j_1 + j_2 \leq 1$, otherwise it reduces to $-b^{k_1+k_2+1} c^{\ell_1+\ell_2+1} + ab^{k_1+k_2} c^{\ell_1+\ell_2}$, of which the first monomial has the larger φ-degree. In either case, no other pair of monomials from r and s produces a monomial of φ-degree equal to $(2k_1 + 2k_2 + j_1 + j_2, 2\ell_1 + 2\ell_2 + j_1 + j_2)$, even after reduction, and so that monomial does not cancel.

The fact that R/I has no nontrivial units nor idempotents is now immediate, by considering φ-degree. In particular, R/I is semiprimitive. Since $1 + IS \subseteq \mathrm{U}(S)$, but $S/IS \cong R/I$ is semiprimitive, we have that $IS = J(S)$. Since $S/IS \cong R/I$ has no nontrivial idempotents, certainly idempotents lift modulo $J(S)$.

The element

$$X = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \in \mathbb{M}_2(S)$$

is an idempotent modulo $J(\mathbb{M}_2(S))$. Indeed, the ideal I was chosen exactly to force this fact. We finish by showing that X does not lift modulo $J(\mathbb{M}_2(S))$ to an idempotent in $\mathbb{M}_2(S)$.

Suppose, by way of contradiction, that X does lift. Such a lift must be of the form

$$E = \begin{pmatrix} u & v \\ w & x \end{pmatrix} \in \mathbb{M}_2(S)$$

where $u - a, v - b, w - c, x - (1 - a) \in IS$. As $E^2 = E$, we find that $u^2 = u - vw$.

Taking common denominators, write $u = u'/(1 + z)$, $v = v'/(1 + z)$, and $w = w'/(1 + z)$ for some $u', v', w', z \in R$ with $z \in I$. From $u^2 = u - vw$ we obtain

$$u'(1 - u' + z) = v'w'. \tag{2.2}$$

As R is a UFD, the factorizations into irreducibles on both sides of (2.2) must match. Further, as $a \equiv u \equiv u' \pmod{IS}$ we have $u' \equiv a \pmod{I}$. In particular, u' is non-constant, and so it has a nontrivial irreducible factor d which is not congruent to a constant modulo I. Without loss of generality, we may assume v' also has d as a factor. Write $u' = du''$ and $v' = dv''$.

Passing to the factor ring R/I, we have

$$\overline{a} = \overline{u}' = \overline{du''}, \qquad \overline{b} = \overline{v'} = \overline{dv''}.$$

The φ-degree of $\overline{a}$ is $(1,1)$, the φ-degree of $\overline{b}$ is $(2,0)$, and $\overline{d}$ does not have $(0,0)$ as its φ degree, hence the only option for the φ-degree of $\overline{d}$ is $(1,0)$. But there is no element in R/I with φ-degree $(1,0)$, yielding the needed contradiction. This proves that X does not lift modulo the Jacobson radical to an idempotent. □

3. Sums of lifting and enabling ideals

It is easy to construct an example of two ideals $L_1, L_2 \trianglelefteq R$ for which idempotents lift, and yet idempotents do not lift modulo $L_1 + L_2$. However, if we replace "lifting" with "enabling" we get a positive result.

Proposition 3.1. *If L_1 and L_2 are enabling ideals in a ring R, then $L_1 + L_2$ is an enabling ideal in R.*

Proof. Let $e^2 = e \in R$ be an idempotent, and let $x \in R$ satisfy $x \equiv e \pmod{L_1 + L_2}$. We may choose some $w \in L_1$ with $x + w \equiv e \pmod{L_2}$.

Since L_2 is enabling, there is some $r \in R$ such that $e' = xr + wr$ is an idempotent, and $e' \equiv x + w \equiv e \pmod{L_2}$. Note that, since $L_2 \subseteq L_1 + L_2$, we have further that $e' \equiv e \equiv x \pmod{L_1 + L_2}$.

Now $e' = xr + wr \equiv xr \pmod{L_1}$ because $w \in L_1$. Since L_1 is enabling, there exists an idempotent $f \in xrR \subseteq xR$ such that $f \equiv e' \pmod{L_1}$. But then $f \equiv e' \equiv x \pmod{L_1 + L_2}$, and thus $L_1 + L_2$ is enabling as desired. □

If L_1 and L_2 are strongly lifting (i.e. both lifting and enabling), then this does not in general imply that $L_1 + L_2$ will be lifting. For example, take R to be the subring of the rational function field over $\mathbb{Z}$ given by all polynomials of the form $P(t)/Q(t)$, with $P(t), Q(t) \in \mathbb{Z}[t]$ and $(6, Q(0)) = 1$.

Let I be an ideal of R. If $I + (t) = R$, then we can write $P_1(t)/Q_1(t) + tP_2(t)/Q_2(t) = 1$ with $P_1(t) \in I$. Hence $P_1(0)Q_2(0) = Q_1(0)$. But then $P_1(t)/Q_1(t) \in \mathrm{U}(R)$, and $I = R$. So if I is maximal, we have $(t) \subseteq I$, and we can compute directly that the only maximal ideals of R are $(2,t)$ and $(3,t)$, so that $J(R) = (6,t)$.

Now let $L_1 = (t)$ and $L_2 = (t-6)$, so that $L_1, L_2 \subseteq J(R)$, and thus these ideals are enabling. Furthermore, $R/L_1 \cong R/L_2 \cong \mathbb{Z}_{(6)}$ (the subring of $\mathbb{Q}$ with denominators relatively prime to 6). Hence R, R/L_1, and R/L_2 are domains, and so each of these rings have as their only idempotents 0 and 1. This implies that idempotents lift strongly modulo L_1 and L_2. However, $L_1 + L_2 = (6,t) = J(R)$, so that $R/(L_1 + L_2) \cong \mathbb{Z}/6\mathbb{Z}$. The ring $\mathbb{Z}/6\mathbb{Z}$ has four idempotents $\overline{0}, \overline{1}, \overline{3}, \overline{4}$, and hence idempotents do not lift modulo $L_1 + L_2$. This answers the question posed in [9, p. 799] in the negative.

4. Enabling over polynomials

It turns out that enabling does not pass naturally to ideals in polynomial rings, answering the second of the three questions in [1, p. 1882] in the negative. Y. Zhou pointed out to us that this example was independently found in [3], but we include it here for completeness.

Proposition 4.1. *If $I \trianglelefteq R$ is an enabling ideal, then this does not necessarily imply that $I[t] \trianglelefteq R[t]$ is an enabling ideal.*

Proof. Take $R = \mathbb{Z}_{(6)}$, the subring of $\mathbb{Q}$ with denominators relatively prime to 6. Fix $I = (6)$ which is the Jacobson radical of R, and hence enabling. The ring R is a domain, thus $\mathrm{Nil}(R) = 0$, and so by Amitsur's Theorem $J(R[t]) = (0)$.

But $R[t]$ is a domain, since R is a domain, and thus $I[t]$ is idempotent-free. By [1, Proposition 6] it follows that $I[t]$ is not enabling, because $I[t] \not\subseteq J(R[t]) = 0$. □

Recall that an ideal $I \trianglelefteq R$ is *weakly enabling* if whenever we have $1-x \in I$ then there exists an idempotent $f \in xR$ with $f - x \in I$. We note that Proposition 4.1 remains true if we replace enabling by weakly enabling, using the same proof, because an ideal in a ring with only trivial idempotents is enabling if and only if it is weakly enabling by [1, Proposition 34].

5. Enabling over power series

Given the results of the last section, it is perhaps surprising that the corresponding question about power series has a *positive* solution, answering the third question in [1, p. 1882]. We can simplify much of the discussion in this section by working more generally with completions, of which power series rings are the prototypical example.

Lemma 5.1. *Assume $J, K \trianglelefteq R$ are ideals of R, and fix $x \in R$ with $x^2 \equiv x \pmod{K}$. Given $n \in \mathbb{Z}_{>0}$ and an element $f_n \in xR$, satisfying $x - f_n \in K$ and $f_n^2 - f_n \in xJ^n$, then there exists $f_{n+1} \in xR$ with:*

(1) $x - f_{n+1} \in K$,
(2) $f_{n+1} - f_n \in xJ^n$,
(3) $f_{n+1}^2 - f_{n+1} \in xJ^{n+1}$.

Proof. Put $f_{n+1} := f_n^2 + 2(f_n^2 - f_n^3) \in xR$. Since $f_n^2 \equiv x^2 \equiv x \equiv f_n \pmod{K}$, we compute

$$x - f_{n+1} = x - f_n^2 - 2f_n(f_n - f_n^2) \in K$$

and hence (1) holds. Similarly, we can verify (2) by noting

$$f_{n+1} - f_n = (f_n - f_n^2)(2f_n - 1) \in xJ^n.$$

Finally, to verify (3) we factor to find

$$f_{n+1}^2 - f_{n+1} = (3f_n^2 - 2f_n^3)^2 - (3f_n^2 - 2f_n^3) = (f_n - f_n^2)^2(4f_n^2 - 4f_n - 3) \in (xJ^n)^2 \subseteq xJ^{n+1}$$

since $n \geq 1$. □

This lemma acts as an inductive machine which "lifts" the enabling property through completions.

Theorem 5.2. *Assume $J \trianglelefteq R$ is an ideal for which R is J-adically complete, and assume $K \trianglelefteq R$ is an arbitrary ideal. Let $\pi : R \to R/J$ be the natural projection. If $\pi(K) \trianglelefteq R/J$ is an enabling ideal of R/J and K is closed in the J-adic topology, then K is an enabling ideal of R.*

Proof. We write $\overline{R} := R/J$, and use bar notation for the image of π. Fix $x \in R$, $e \in \mathrm{idem}(R)$ with $e - x \in K$. Thus $\overline{e} - \overline{x} \in \overline{K}$. From the enabling hypothesis on $\overline{K}$, fix $r \in R$ such that $\overline{xr} \in \mathrm{idem}(\overline{R})$ and

$$\overline{x} - \overline{xr} \in \overline{K}. \tag{5.3}$$

As R is J-adically complete, it is a well-known fact that idempotents lift strongly modulo J. (Lifting follows from [7, Theorem 21.31], while enabling follows from the fact that J is contained in the Jacobson radical by [7, Remark 21.30].) Thus, we may as well assume that $xr \in \mathrm{idem}(R)$. Using (5.3), we can write $x = xr + y + z$ for some $y \in J$ and $z \in K$. Our next step is to modify y so it is a right multiple of x. To this end, we find

$$x = x - x^2 + x^2 = (x - x^2) + x(xr + y + z) = (x - x^2)(1 - r) + xr + xy + xz.$$

Put $f_1 := xr + xy$ and $w := (x - x^2)(1 - r) + xz$. Using the fact that $x^2 \equiv e^2 = e \equiv x \pmod{K}$ we have $x - f_1 = w \in K$. From $(xr)^2 = xr$ we have

$$f_1^2 - f_1 = (xr + xy)^2 - (xr + xy) = xrxy + xyxr + xyxy - xy \in xJ.$$

Applying Lemma 5.1 recursively, for each $n \geq 1$ we obtain elements $f_n \in xR$ satisfying the appropriate compatibility conditions allowing us to define $f := \lim_{n\to\infty} f_n \in R$. Since K is closed in the J-adic topology, by Lemma 5.1(1) we have $x - f = \lim_{n\to\infty}(x - f_n) \in K$. Further, by part (3) of the lemma,

$f^2 = f$. So it suffices to prove that $f \in xR$. This follows from the fact that, by Lemma 5.1(2), we can write $f_n = xs_n$ for some J-adically compatible $s_n \in R$, and putting $s = \lim_{n\to\infty} s_n \in R$ we have $f = xs$. □

There are two important special cases of this theorem.

Corollary 5.4. *If $I \trianglelefteq R$ is an enabling ideal of R, then $I[\![t]\!] \trianglelefteq R[\![t]\!]$ is an enabling ideal of $R[\![t]\!]$.*

Proof. Apply Theorem 5.2 to the ring $R[\![t]\!]$, taking $J = (t)$ and $K = I[\![t]\!]$. □

Corollary 5.5. *Let R and S be rings, and let ${}_RV_S$ be an R-S-bimodule. Let $T = \begin{pmatrix} R & V \\ 0 & S \end{pmatrix}$ be a split-null extension. Finally, let $I_1 \trianglelefteq R$, $I_2 \trianglelefteq S$, and let $W \subseteq V$ be an R-S-submodule.*

The ideal $K := \begin{pmatrix} I_1 & W \\ 0 & I_2 \end{pmatrix}$ is enabling in T if and only if I_1 is enabling in R and I_2 is enabling in S.

Proof. The forward direction is a slight strengthening of [1, Proposition 17], so for completeness we include the details here. Fix $x_1 \in R$, $e_1 \in \mathrm{idem}(R)$, $x_2 \in S$, and $e_2 \in \mathrm{idem}(S)$ with $e_1 - x_1 \in I_1$ and $e_2 - x_2 \in I_2$. Then we have $\mathrm{diag}(e_1, e_2) - \mathrm{diag}(x_1, x_2) \in K$. By hypothesis there exists an element

$$t = \begin{pmatrix} p & w \\ 0 & q \end{pmatrix} \in \mathrm{diag}(x_1, x_2)T \cap \mathrm{idem}(T)$$

with $t - \mathrm{diag}(x_1, x_2) \in K$. We easily see that $p^2 = p \in x_1R$, $p - x_1 \in I_1$, $q^2 = q \in x_2S$, and $q - x_2 \in I_2$.

For the converse apply Theorem 5.2 to the ring T, taking J to be the ideal of strictly upper-triangular matrices. Since $J^2 = 0$, the ring T is J-adically complete and K is closed in the J-adic topology. □

Using the previous corollary and a straightforward induction, we obtain the following result, which was also left open in [1, p. 1882].

Corollary 5.6. *An ideal $I \trianglelefteq R$ is an enabling ideal of R if and only if $\mathbb{T}_n(I)$ is an enabling ideal of $\mathbb{T}_n(R)$ (the ring of $n \times n$ upper-triangular matrices) for every $n \geq 1$.*

Note that the results of this section remain true if we replace "enabling" by "weakly enabling" everywhere, using the same proofs except with the obvious changes when needed.

6. Perspectivity vs. conjugation

We now change course in order to answer a question raised in [2]. To motivate the problem, we need to first recall some standard equivalence relations one can place on the set of idempotents in a ring.

Definition 6.1. Let $e, f \in R$ be two idempotents.

(1) These idempotents are *conjugate* if there exists a unit $u \in R$ with $f = u^{-1}eu$, and we write $e \sim f$ in this case.
(2) These idempotents are *isomorphic* if there exist $a, b \in R$ such that $e = ab$ and $f = ba$, and we write $e \cong f$ in this case.
(3) These idempotents have *isomorphic complements* if $1 - e$ and $1 - f$ are isomorphic, and we write $e \cong' f$ in this case.
(4) These idempotents are *right associate* (respectively *left associate*) if there exists a unit $u \in R$ with $f = eu$ (respectively $f = ue$), and we write $e \sim_r f$ (respectively $e \sim_\ell f$).

There are some immediate connections between these properties.

Lemma 6.2. *Let $e, f \in R$ be idempotents.*

(1) *The relations $e \cong f$ and $e \cong' f$ hold if and only if $e \sim f$ holds.*
(2) *We have $e \cong f$ if and only if $eR \cong fR$.*

(3) *The following are all equivalent:*
   (a) $e \sim_r f$,
   (b) $eR = fR$,
   (c) $ef = f$ *and* $fe = e$,
   (d) $(1-e) \sim_\ell (1-f)$.

(4) *If* $e \sim_r f$ *or* $e \sim_\ell f$*, then* $e \sim f$*. Moreover, if* $e \sim_r f$ *and* $e \sim_\ell f$ *both hold, then* $e = f$.

Proof. For a proof of (2) see [7, Proposition 21.20], and for (3) see [8, Exercise 21.4]. The other parts are straightforward and left to the reader. □

Recall that two direct summands $A, B \subseteq^{\oplus} M$ (of some module M) are *perspective* exactly when there exists a common direct sum complement C. In other words, $A \oplus C = B \oplus C = M$. Since direct summands are associated with idempotents (in the endomorphism ring), we might expect by Lemma 6.2(3) that left and right associativity of idempotents is connected to perspectivity in some way. The following lemma tells us exactly how they are related in the right regular module R_R.

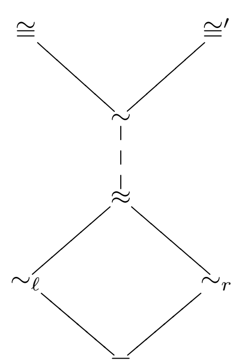
Lemma 6.3. *Given two idempotents* $e, f \in R$*, the direct summands* eR *and* fR *in* R_R *are perspective if and only if there exist idempotents* $g, h \in R$ *with* $e \sim_r g \sim_\ell h \sim_r f$.

Proof. (⇒): Assume eR and fR are perspective, say with common direct sum complement C_R. Let g be the idempotent with $gR = eR$ and $(1-g)R = C$. Similarly, define h as the idempotent satisfying $hR = fR$ and $(1-h)R = C$. We then have $e \sim_r g \sim_\ell h \sim_r f$.

(⇐): Given the chain $e \sim_r g \sim_\ell h \sim_r f$, we compute directly that $(1-g)R = (1-h)R$ is a common direct sum complement to both eR and fR in R_R. □

While in general the union of two equivalence relations is not an equivalence relation, after closing the union under transitivity it does become an equivalence relation. We will write $\approx$ for the transitive closure of the union of the two relations $\sim_\ell$ and $\sim_r$. In other words, given two idempotents $e, f \in R$ we will write $e \approx f$ exactly when $e = e_0 \sim_\ell e_1 \sim_r e_2 \sim_\ell \cdots \sim_r e_{2n} = f$ for some idempotents $e_0, e_1, \ldots, e_{2n} \in R$ and some $n \in \mathbb{N}$. By an easy application of the previous lemma, we see that $\approx$ is also the transitive closure of the perspectivity relation on direct summands in R_R (or, by left-right symmetry, in ${}_RR$).

We have the following partial lattice among the relations we have introduced so far.



This begs the question of whether or not $\sim$ is the same as $\approx$. Equivalently, given $R_R = A \oplus A' = B \oplus B'$ with $A \cong B$ and $A' \cong B'$, is it possible to start with $A \oplus A'$ and by successively replacing one summand at a time (in other words, replacing a direct sum decompositions with a perspective decomposition) get to $B \oplus B'$? The answer is no in general, but there are many situations where the answer to this question is yes. To formalize these claims we introduce a new definition.

Definition 6.4 (Perspectivity length)**.** The *perspectivity length* of a ring R, written $\mathrm{pl}(R)$, is the smallest integer $n \geq 0$ such that for any two idempotents $e, f \in R$ with $e \approx f$, then given a chain $e = e_0 \sim_* e_1 \sim_* \cdots \sim_* e_m = f$ with m minimal, we must have $m \leq n$. (By the symbol $\sim_*$ we mean either $\sim_\ell$ or $\sim_r$ in each instance). In case no such smallest integer exists, we set $\mathrm{pl}(R) = \infty$.

Here, and throughout the remainder of the paper, whenever talking about *chains* we will either be referring to chains of the form $e = e_0 \sim_* e_1 \sim_* \cdots \sim_* e_m = f$ as above, or to the corresponding chains of direct sum decompositions $(e_0R \oplus (1-e_0)R) \to (e_1R \oplus (1-e_1)R) \to \cdots \to (e_mR \oplus (1-e_mR))$. (Here, an arrow means that we not only have an equality $e_iR \oplus (1-e_i)R = e_{i+1}R \oplus (1-e_{i+1})R$, but in fact one of the summands is unchanged.)

Proposition 6.5. *A ring R has $\mathrm{pl}(R) = 0$ if and only if R is abelian.*

Proof. Let $e \in R$ be an idempotent. By [8, Exercise 21.4], we have $e \sim_r f$ if and only if $f = e + ex(1-e)$ for some $x \in R$. We will use this fact freely below.

($\Rightarrow$): Assume $\mathrm{pl}(R) = 0$. Since $e \sim_r e + ex(1-e)$ we must have $ex(1-e) = 0$ (else we have a minimal length 1 chain). Thus $eR(1-e) = 0$, and similarly $(1-e)Re = 0$. This proves that e is central, hence R is abelian.

($\Leftarrow$): Assume R is abelian. Thus $eR(1-e) = (1-e)Re = 0$. Hence the only idempotent connected by $\approx$ to e is e itself. Since e is arbitrary, we have $\mathrm{pl}(R) = 0$. □

Proposition 6.6. *A ring R has $\mathrm{pl}(R) = 1$ if and only if R is not abelian and for each idempotent $e \in R$ either $eR(1-e) = 0$ or $(1-e)Re = 0$.*

Proof. This is similar to the proof above. □

Notice that there do exist rings with $\mathrm{pl}(R) = 1$, such as the ring of upper-triangular matrices $\left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right)$ over any nonzero ring k with only trivial idempotents.

Theorem 6.7. *Perspectivity of direct summands in R_R is transitive if and only if $\mathrm{pl}(R) \leq 3$ and any chain of length 3 can be written starting with $\sim_r$ (or, equivalently, with $\sim_\ell$).*

Proof. ($\Rightarrow$): Suppose first that we have a chain of the form $e_0 \sim_\ell e_1 \sim_r e_2 \sim_\ell e_3$. By Lemma 6.3 we see that e_0R is perspective to e_2R (since they are connected by the chain $e_0 \sim_r e_0 \sim_\ell e_1 \sim_r e_2$). Similarly, e_2R is perspective to e_3R. By hypothesis perspectivity is transitive, and so e_0 is connected to e_3 by a length 3 chain starting with $\sim_r$, again appealing to Lemma 6.3.

That chains of length ≥ 4 can be reduced to chains of length ≤ 3 follows from the fact that (as was done in the previous paragraph) chains can be broken up into strings of perspectivities, and then we can apply transitivity and Lemma 6.3.

($\Leftarrow$) : Suppose that eR is perspective to fR and fR is perspective to gR, for idempotents $e, f, g \in R$. We then have $e \approx g$ (by another application of Lemma 6.3). By hypothesis, e and g are connected by a chain of length 3 starting with $\sim_r$ (increasing the length of the chain if necessary). By Lemma 6.3, eR is perspective to gR.

Finally, the parenthetical remark follows by noticing that if e is connected to f by a chain of length 3, then $(1-e)$ is connected to $(1-f)$ by a chain of length 3, but with each instance of $\sim_\ell$ replaced by $\sim_r$ (and vice versa). □

Proposition 6.8. *Let F be a field. If $R = \mathbb{M}_n(F)$ with $n > 1$, then $\mathrm{pl}(R) = 3$ and any chain of length 3 can be written starting with $\sim_r$ (or with $\sim_\ell$).*

Proof. First, it takes a simple computation to show that idempotents $e_{1,1}$ and $e_{2,2}$ are connected by the chain

$$e_{1,1} \sim_\ell (e_{1,1} + e_{2,1}) \sim_r (e_{1,2} + e_{2,2}) \sim_\ell e_{2,2}.$$

Next suppose, by way of contradiction, that there is a smaller chain connecting these idempotents. Without loss of generality (by left-right symmetry) we would have $e_{1,1} \sim_\ell g \sim_r e_{2,2}$ for some idempotent $g \in R$. But the set of all idempotents left associate to $e_{1,1}$ are of the form $e_{1,1} + (1 - e_{1,1})xe_{1,1}$ (for some $x \in R$), while the set of right associate idempotents to $e_{2,2}$ are of the form $e_{2,2} + e_{2,2}y(1 - e_{2,2})$ (for some $y \in R$). These two sets do not intersect, since the first is annihilated on the right by $e_{2,2}$, but the second set is sent to $e_{2,2}$ under right multiplication by $e_{2,2}$. This proves that $\mathrm{pl}(R) \geq 3$.

The final statement follows from the previous theorem and the well-known fact that perspectivity in R_R is transitive (for this specific ring). For example, see [4, Theorem 5.12] for a much stronger statement. □

To prove that $\approx$ is not the same relation as $\sim$, we will make use of the following key example, which shows that $\mathrm{pl}(R)$ can be infinite. This also answers [2, Question 2.7] in the negative. We thank S. Carnahan for providing ideas leading to this example, and giving an alternate proof of the following:

Proposition 6.9. *For the ring $R = \mathbb{M}_2(\mathbb{Z})$ we have $\mathrm{pl}(R) = \infty$.*

Proof. Let S be the set of non-trivial idempotents R. The elements of S are characterized as the rank 1 matrices with trace 1. There is a nice bijection $\mathrm{PSL}_2(\mathbb{Z}) \to S$ via the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } ad - bc = 1 \qquad \longmapsto \qquad \begin{pmatrix} d \\ c \end{pmatrix} \begin{pmatrix} a & -b \end{pmatrix} = \begin{pmatrix} ad & -bd \\ ac & -bc \end{pmatrix} \text{ with } ad - bc = 1$$

noting that the latter matrices are precisely those with rank 1 and trace 1. (The matrices on the left are only defined up to a factor of $\pm I$. This corresponds to the two possible factorizations on the right.)

Under this identification, passing between (left or right) associates in S corresponds to left multiplying in $\mathrm{PSL}_2(\mathbb{Z})$ by matrices of the form

$$S(n) := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad S(n)^t := \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.$$

So it suffices to show that writing elements of $\mathrm{PSL}_2(\mathbb{Z})$ in the form $S(n_1)S(n_2)^t \cdots S(n_{2k-1})S(n_{2k})^t$, then k cannot be bounded. This is well-known, so we only sketch the short proof here.

Taking

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

it happens that $A^2 = B^3 = -I$, $(AB)^n = S(n)$, and $(AB^{-1})^n = (-1)^n S(n)^t$. Working modulo $\pm I$, the images of A and B generate $\mathrm{PSL}_2(\mathbb{Z})$ as a free product of cyclic groups of orders two and three, see [10, Example III, p. 171] for more details. Elements in such a free product can be written uniquely as words in the form $AB^{\pm 1}AB^{\pm 1}A \cdots AB^{\pm 1}A$, possibly with the leading or terminal A's dropped, and where the exponents on the B's are independent. We will be interested in the number of *sign changes* in the exponents of the B's (say, when reading the word from left to right). If we multiply such a word (either on the left or the right) by $S(n) = (AB)^n$ and write it in its reduced form (by repeatedly removing copies of A^2 and B^3), then there are *at most* two new sign changes (but it is also possible for there to be fewer sign changes). The same statement is true for $S(n)^t$.

Thus, any chain of left and right associates of length $\leq k$ corresponds to a word in the free product with at most $2k$ sign changes. As $X_k := (S(1)S(1)^t)^{k+1} \equiv (ABAB^{-1})^{k+1} \pmod{\pm I}$ has $2k + 1$ sign changes among the exponents, we see that the idempotents corresponding to $I, X_k \in \mathrm{PSL}_2(\mathbb{Z})$ are connected by a chain, no shorter than $k + 1$ steps long. (Additional improvements on the lengths of chains are available using geometric and number-theoretic methods, but we won't pursue that here.) □

With this example in place, we are ready to prove that conjugation is strictly stronger than $\approx$.

Theorem 6.10. *The relations $\approx$ and $\sim$ are, in general, different.*

Proof. Let R be a ring with $\mathrm{pl}(R) = \infty$. For each integer $k \geq 1$ we fix a pair of idempotents (e_k, f_k) with $e_k \approx f_k$ for which there exists no chain of length $\leq k$ connecting them. Let $S = \prod_{i=1}^{\infty} R$ be the direct product of countably many copies of R. We will show that $\approx$ does not agree with $\sim$ in S.

The elements $e := (e_1, e_2, \ldots), f := (f_1, f_2, \ldots) \in S$ are idempotents. Further $e \sim f$, since $e_i \sim f_i$ for each $i \geq 1$. Suppose, by way of contradiction, that $e \approx f$; say, the two idempotents are connected by a chain of length n. Restricting this chain to the nth coordinate would give a chain of length n connecting e_n to f_n, yielding the needed contradiction. □

This still leaves open the possibility that $\approx$ and $\sim$ are equal on any ring with $\mathrm{pl}(R) < \infty$. There is a cryptic remark in [6, p. 15], which claims the existence of "an example (due to G. Bergman) of a regular ring with perspectivity transitive" that has $\approx$ different from $\sim$. By Theorem 6.7, such a ring necessarily has perspectivity length ≤ 3. Bergman was unable to recall this example, but fortunately K. Goodearl was able to supply enough of the communication for us to reproduce a version of the example here.

Theorem 6.11. *There exists a ring S with perspectivity transitive, and with conjugate idempotents e, f satisfying $e \not\approx f$.*

Proof. Let F be a field, and let R be the subring of the column-finite matrices of F which are eventually banded, in the sense that there is a row after which all diagonals are constant. Thus, elements of R are of the form

$$\begin{pmatrix} * & * & * & \cdots & \\ \vdots & \vdots & \vdots & \cdots & \\ * & * & * & \cdots & \\ a_1 & a_2 & a_3 & \cdots & \\ & a_1 & a_2 & a_3 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

This is the opposite ring of a well-known example due to Bergman, see [6, Example 1], [5, Example 4.26], and [11, Example 3.1]. Let P be the (non-unital) subring of R consisting of matrices with only finitely many nonzero rows.

We put $S = \left(\begin{smallmatrix} R & P \\ P & R \end{smallmatrix}\right)$. It is a straightforward calculation to show that idempotents in S are exactly the matrices of the form

$$E := \begin{pmatrix} \begin{pmatrix} X_0 & X_1 \\ 0 & D_1 \end{pmatrix} & \begin{pmatrix} Y_0 & Y_1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} Z_0 & Z_1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} W_0 & W_1 \\ 0 & D_2 \end{pmatrix} \end{pmatrix}$$

where $X_0, Y_0, Z_0, W_0 \in \mathbb{M}_n(F)$ (for some sufficiently large $n \in \mathbb{N}$), $D_1, D_2 \in \{0, I\}$,

$$A := \begin{pmatrix} X_0 & Y_0 \\ Z_0 & W_0 \end{pmatrix} \tag{6.12}$$

is an idempotent in $\mathbb{M}_{2n}(F)$, and

$$\begin{aligned} X_0X_1 + X_1D_1 + Y_0Z_1 &= X_1, & Z_0X_1 + Z_1D_1 + W_0Z_1 &= Z_1, \\ X_0Y_1 + Y_0W_1 + Y_1D_2 &= Y_1, & Z_0Y_1 + W_0W_1 + W_1D_2 &= W_1. \end{aligned} \tag{6.13}$$

Consider the following types of idempotents:

$$\text{Type I} = \begin{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \ \text{Type II} = \begin{pmatrix} \begin{pmatrix} 0_k & 0 \\ 0 & I \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \ \text{Type III} = \begin{pmatrix} I & 0 \\ 0 & \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}.$$

We claim that every idempotent can be brought, using a chain of length 3 starting with $\sim_r$, to one of these types or their complements. We will sketch the details in the case where $D_1 = 0$ and $D_2 = I$; the other cases are similar.

By Proposition 6.8 we know that there exists a chain

$$A \sim_r A_1 \sim_\ell A_2 \sim_r D$$

where D is a diagonal matrix with only 1's and 0's. (The number of 0's and 1's in D is fixed, but their placement along the diagonal is arbitrary.) Writing

$$A_1 = \begin{pmatrix} X_0' & Y_0' \\ Z_0' & W_0' \end{pmatrix},$$

put

$$E_1 := \begin{pmatrix} \begin{pmatrix} X_0' & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} Y_0' & Y_1' \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} Z_0' & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} W_0' & W_1' \\ 0 & I \end{pmatrix} \end{pmatrix}$$

where $Y_1' = -X_0'Y_1 - Y_0'W_1 + Y_1$ and $W_1' = -Z_0'Y_1 - W_0'W_1 + W_1$. One checks directly that E_1 is idempotent, $EE_1 = E_1$, and $E_1E = E$, using (6.13) and the equalities $AA_1 = A_1$ and $A_1A = A$ (which follow from $A \sim_r A_1$, by Lemma 6.2(3)). Thus, again by Lemma 6.2(3), we have $E \sim_r E_1$. Writing

$$A_2 = \begin{pmatrix} X_0'' & Y_0'' \\ Z_0'' & W_0'' \end{pmatrix},$$

put

$$E_2 := \begin{pmatrix} \begin{pmatrix} X_0'' & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} Y_0'' & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} Z_0'' & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} W_0'' & 0 \\ 0 & I \end{pmatrix} \end{pmatrix}.$$

Since $A_1 \sim_\ell A_2$, one similarly shows that $E_1 \sim_\ell E_2$. Finally, if $D = \operatorname{diag}(D_1', D_2')$ then putting $E_3 = \operatorname{diag}(D_1', D_1, D_2', D_2)$ we have $E_2 \sim_r E_3$. Taking n large enough, we can force E_3 to have the appropriate form by choosing D_1', D_2' correctly.

Next we claim that the matrices of types I, II, and III, and their complements, are distinct representatives for the equivalence classes of idempotents under $\approx$. To prove this, we will describe an invariant (under $\approx$) for each class. For type I, the invariant is simply the rank of the matrix (thought of as acting on $F^{(\omega)} \times F^{(\omega)}$). For the complements of type I, the invariant is the dimension of the null space. For any idempotent E which neither has finite rank nor finite nullity, we note that (taking n large enough) the number

$$\operatorname{rank}(A) - \operatorname{nullity}(A) - n$$

is an invariant of E; here A and $n \in \mathbb{N}$ are as in (6.12). We see that this invariant distinguishes among the matrices of types II and III. Finally, to distinguish these matrices from their complements, we just look to see whether the infinite identity matrix I appears in the upper left corner, or the lower right corner (which property does not change under multiplication by a unit in S).

We thus have that $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \not\approx \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}$. Finally, it is easy to see that these two specific idempotents are conjugate in S. $\square$

7. Unanswered questions

We finish this paper by listing some of the questions we were not able to answer. One occurs in [9] on page 800, and asks:

Question 7.1. If $I \trianglelefteq R$ is an exchange ideal, and idempotents lift modulo I, is I an enabling ideal?

The remaining open question in [1] is the first of the three questions posed on page 1882, and asks:

Question 7.2. If $I \trianglelefteq R$ is an enabling ideal of R, is $\mathbb{M}_n(I) \trianglelefteq \mathbb{M}_n(R)$ an enabling ideal of $\mathbb{M}_n(R)$?

Another question, implicitly found in [1], is the following:

Question 7.3. Is there an example of an ideal which is weakly enabling but not enabling?

8. Acknowledgements

We thank D. Khurana and Y. Zhou for comments which improved the presentation of the material in the paper, S. Carnahan for ideas which led to Proposition 6.9, and K. Goodearl for providing notes which helped us recreate G. Bergman's example. We also thank the anonymous referee, whose comments led to improvements in the paper. This work was partially supported by a grant from the Simons Foundation (#315828 to Pace Nielsen).

References

1. M. Alkan, W. K. Nicholson, and A. Ç. Özcan, *Strong lifting splits*, J. Pure Appl. Algebra **215** (2011), no. 8, 1879–1888. MR 2776430 (2012c:16012)
2. Victor P. Camillo and Pace P. Nielsen, *Half-orthogonal sets of idempotents*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 965–987. MR 3430355
3. Weixing Chen and Shuying Cui, *On strong lifting splits*, 2011 International Conference on Multimedia Technology (ICMT), July 2011, pp. 2235–2237.
4. Shelly Garg, Harpreet K. Grover, and Dinesh Khurana, *Perspective rings*, J. Algebra **415** (2014), 1–12. MR 3229504
5. K. R. Goodearl, *von Neumann Regular Rings*, second ed., Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1991. MR 1150975 (93m:16006)
6. David Handelman, *Perspectivity and cancellation in regular rings*, J. Algebra **48** (1977), no. 1, 1–16. MR 0447329 (56 #5642)
7. T. Y. Lam, *A First Course in Noncommutative Rings*, second ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR 1838439 (2002c:16001)
8. ———, *Exercises in Classical Ring Theory*, second ed., Problem Books in Mathematics, Springer-Verlag, New York, 2003. MR 2003255 (2004g:16001)
9. W. K. Nicholson and Y. Zhou, *Strong lifting*, J. Algebra **285** (2005), no. 2, 795–818. MR 2125465 (2006f:16006)
10. Derek J. S. Robinson, *A course in the theory of groups*, second ed., Graduate Texts in Mathematics, vol. 80, Springer-Verlag, New York, 1996. MR 1357169 (96f:20001)
11. Janez Šter, *Corner rings of a clean ring need not be clean*, Comm. Algebra **40** (2012), no. 5, 1595–1604. MR 2924469

Department of Mathematics, Wellesley College, Wellesley, MA 02481, United States
E-mail address: adiesl@wellesley.edu

Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, USA
E-mail address: samuel.dittmer@math.ucla.edu

Department of Mathematics, Brigham Young University, Provo, UT 84602, USA
E-mail address: pace@math.byu.edu